On nonlinear effects in multiphase WKB analysis for the nonlinear Schrodinger equation

Rémi Carles

ON NONLINEAR EFFECTS IN MULTIPHASE WKB ANALYSIS FOR THE NONLINEAR SCHRÖDINGER EQUATION

RÉMI CARLES

Abstract. We consider the Schrödinger equation with an external potential and a cubic nonlinearity, in the semiclassical limit. The initial data are sums of WKB states, with smooth phases and smooth, compactly supported initial amplitudes, with disjoint supports. We show that according to the size of the initial data, a superposition principle may or may not hold. Surprisingly, it holds for large data, like in the linear case, but not for smaller ones. The proof relies on WKB analysis: for large data, we use the theory known in the case of a single initial WKB state, and properties of compressible Euler equations, while for smaller data, nonlinear interactions are present at leading order.

1. Introduction

1.1. **Setting.** We consider the cubic defocusing Schrödinger equation on $\mathbb{R}^d$, $d \geqslant 1$, in the semiclassical régime

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + |u^\varepsilon|^2 u^\varepsilon. \tag{1.1}$$

The potential $V = V(x)$ is supposed real-valued, smooth, and at most quadratic:

$$V \in C^\infty(\mathbb{R}^d;\mathbb{R}), \quad \partial^\alpha V \in L^\infty(\mathbb{R}^d), \quad \forall \alpha \in \mathbb{N}^d,\ |\alpha| \geqslant 2. \tag{1.2}$$

Typical examples are $V = 0$, V linear ($V(x) = E\cdot x$), V harmonic ($V(x) = \omega^2|x|^2$). As initial data, we consider the sum of WKB states of size $\mathcal{O}(1)$, so we are in a supercritical case in terms of WKB analysis:

$$u^\varepsilon(0,x) = u_0^\varepsilon(x) := \sum_{j=1}^N \alpha_j(x) e^{i\varphi_j(x)/\varepsilon}. \tag{1.3}$$

We refer to [5, Chapter 1] for the reason why this setting is supercritical in terms of WKB analysis. Essentially, the evolution of the phase describing the rapid oscillation is given by an eikonal equation which involves the leading order amplitude, and a standard application of the WKB asymptotic expansion leads to systems which are not closed, no matter how many correcting terms are considered.

Assumption 1.1. *The phases are smooth and real-valued, $\varphi_j \in C^\infty(\mathbb{R}^d;\mathbb{R})$. The initial amplitudes are smooth and compactly supported: $\alpha_j \in C_0^\infty(\mathbb{R}^d;\mathbb{C})$, with pairwise disjoint supports,*

$$\operatorname{supp}\alpha_{j_1} \cap \operatorname{supp}\alpha_{j_2} = \emptyset, \quad j_1 \neq j_2.$$

Work supported by Centre Henri Lebesgue, program ANR-11-LABX-0020-0.

The case $N = 1$, referred to as *monokinetic case*, is well understood for *short time*, as we recall below, in the sense that the asymptotic behavior of u^ε as $\varepsilon \to 0$ is described precisely, locally in time on some interval independent of ε. The large time behavior is, in general, unknown; the one-dimensional case, with $V = 0$, is an exception, since it is completely integrable, see e.g. [13, 23]. In the case $V \equiv 0$, the leading order asymptotic description involves the compressible Euler equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + v \cdot \nabla v + \nabla \rho = 0. \end{cases} \tag{1.4}$$

This equation is quasilinear, while (1.1) is semilinear (the nonlinear term is viewed as a perturbation when solving the Cauchy problem). In Section 2, we recall how to justify, in this case, the existence of a WKB approximation of the form

$$u^\varepsilon(t,x) = \left(\mathsf{a}(t,x) + \varepsilon \mathsf{a}_1(t,x) + \ldots + \varepsilon^k \mathsf{a}_k(t,x)\right) e^{i\phi(t,x)/\varepsilon} + \mathcal{O}\left(\varepsilon^{k+1}\right),$$

in $L^\infty([0,T]; L^2 \cap L^\infty(\mathbb{R}^d))$, for all $k \geqslant 0$, for some $T > 0$ independent of ε. We choose to measure errors in $L^2 \cap L^\infty$ in the spatial norm, in order to avoid to introduce ε-dependent norms when derivatives are involved, due to rapid oscillations. This time T can be taken as the lifespan of the smooth solution to the Euler equation (1.4) with suitable initial data. When $N \geqslant 2$, the new question arising is the nonlinear interaction of the WKB states. As the problem is supercritical, even a formal computation is a delicate issue: if we plug an approximate solution of the form

$$u^\varepsilon_{\rm app}(t,x) = \sum_{j=1}^M b_j(t,x) e^{i\phi_j(t,x)/\varepsilon}$$

into (1.1), how do we choose M (possibly infinite), and which equations must be satisfied by the amplitudes b_j and the phases ϕ_j? Surprisingly enough, it turns out that as long as the solutions of the Euler equations, involved in the description of each individual initial WKB states, are smooth, there is no interaction, at arbitrary order in terms of powers of ε.

Remark 1.2 (Infinitely many states). The case $N = \infty$ may also be addressed, under suitable assumptions on the growth in space of the phases ϕ_j compared to the size of the support of α_j, as $j \to \infty$. More precisely, as will be clear from the proof of the main result, we can consider the case $N = \infty$ provided that we may find cutoff functions χ_j so that

$$\phi_0 = \sum_{j=1}^\infty \varphi_j \chi_j \in H^\infty(\mathbb{R}^d) := \cap_{s>0} H^s(\mathbb{R}^d),$$

or at least in a weaker form if $\phi_0 \in H^s(\mathbb{R}^d)$ for some $s > 2 + d/2$. Another constraint, in this case, is that we have to find a common lower bound for the lifespan of all the approximate solutions (ϕ_j, a_j) considered below, an aspect which is obvious when N is finite, since we consider the minimum of a finite set.

1.2. **Main results.** The nonlinear evolution of each initial WKB state will play a crucial role:

$$i\varepsilon \partial_t u_j^\varepsilon + \frac{\varepsilon^2}{2}\Delta u_j^\varepsilon = V u_j^\varepsilon + |u_j^\varepsilon|^2 u_j^\varepsilon \quad ; \quad u_{j|t=0} = \alpha_j e^{i\varphi_j/\varepsilon}. \tag{1.5}$$

Under our assumptions, for fixed initial data, we know that:

- If $d \leqslant 3$, the equation is energy-subcritical, and for fixed $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d))$, and it is smooth. See e.g. [8].
- If $d = 4$, the equation is energy-critical: the above conclusion is known to remain when $V = 0$ ([25]), when V is an isotropic quadratic potential ([14]), or when V is harmonic at infinity ([12]).
- If $d \geqslant 5$, the equation is energy-supercritical: only a local in time smooth solution is known to exist by classical theory.

In the cases where the global existence of a smooth solution is not known, the local existence time might go to zero as $\varepsilon \to 0$, so the existence of a smooth solution on a time interval independent of $\varepsilon > 0$ is already a nontrivial step. The description of the solutions u_j^ε as $\varepsilon \to 0$ on some time interval $[0, T_j]$ independent of ε was evoked above, and is recalled in Sections 2 (case $V = 0$) and 3 (V satisfying (1.2)). Our main result is the following nonlinear superposition principle:

Theorem 1.3. *Let $d \geqslant 1$, V satisfying* (1.2)*, and initial data satisfying Assumption 1.1. There exists $T^* > 0$ independent of $\varepsilon \in]0, 1]$ such that* (1.1)-(1.3) *has a unique solution $u^\varepsilon \in C([0, T^*]; H^\infty(\mathbb{R}^d))$. In addition,*

$$\sup_{t\in[0,T^*]} \left\| u^\varepsilon(t) - \sum_{j=1}^N u_j^\varepsilon(t) \right\|_{L^2\cap L^\infty} = \mathcal{O}\left(\varepsilon^k\right), \quad \forall k > 0,$$

where u_j^ε is the solution of (1.5).

This result is actually a corollary of a detailed WKB analysis, as well as a property of finite speed of propagation for the compressible Euler equation, discovered initially in [17]. The key feature of our setting then is the compact, disjoint supports of the initial amplitudes α_j. More precisely, in the case $V = 0$, as long as WKB analysis is valid for each u_j^ε in (1.5), u_j^ε remains supported in (essentially) $\operatorname{supp} \alpha_j$ up to $\mathcal{O}(\varepsilon^\infty)$: all the amplitude terms of the WKB expansion (at leading order as well as correctors at arbitrary order) remain compactly supported, and amplitudes associated with $u_{j_1}^\varepsilon$ and $u_{j_2}^\varepsilon$, respectively, with $j_1 \neq j_2$, do not interact. In the case $V \not\equiv 0$, u_j^ε remains supported in $\operatorname{supp} \alpha_j$ evolving according to the classical flow generated by V, up to $\mathcal{O}(\varepsilon^\infty)$. In other words, we recover the same phenomenon as in the linear case (see e.g. [18, 24]), even though the régime associated to (1.1) is strongly nonlinear. In particular, the initial modes cannot interact at a "visible" order before WKB analysis for at least one of the u_j^ε's ceases to be valid, that is, before the solution of the corresponding Euler equation (1.4) breaks down. Recent progress on this precise question, [20, 21, 3] (see also [19] for a relation with the nonlinear Schrödinger equation), suggest that the expected scenario is rather that of an implosion: the conclusion of Theorem 1.3 might remain valid even after WKB has ceased to be valid.

Surprisingly enough, the conclusion of Theorem 1.3 is false in the weakly nonlinear case (which amounts to multiplying the nonlinearity in (1.1) by ε, leaving (1.3) unchanged):

Proposition 1.4. *Let $d \geqslant 1$ and $T > 0$. There exist $k_1, k_2 \in \mathbb{R}^d$, and $\alpha_1, \alpha_2 \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports, such that the solution to*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon|u^\varepsilon|^2 u^\varepsilon \quad ; \quad u_0^\varepsilon(x) = \alpha_1(x)e^{ik_1\cdot x/\varepsilon} + \alpha_2(x)e^{ik_2\cdot x/\varepsilon}, \tag{1.6}$$

satisfies

$$\liminf_{\varepsilon\to 0}\sup_{t\in[0,T]}\left\| u^\varepsilon(t)-\sum_{j=1}^{2}u_j^\varepsilon(t)\right\|_{L^p}>0,\quad \forall p\in[2,\infty],$$

where each u_j^ε *solves*

$$i\varepsilon\partial_t u_j^\varepsilon+\frac{\varepsilon^2}{2}\Delta u_j^\varepsilon=\varepsilon|u_j^\varepsilon|^2u_j^\varepsilon\quad;\quad u_j^\varepsilon(0,x)=\alpha_j(x)e^{ik_j\cdot x/\varepsilon}.$$

Setting $v^\varepsilon=\sqrt{\varepsilon}u^\varepsilon$, we see that (1.6) is equivalent to

$$i\varepsilon\partial_t v^\varepsilon+\frac{\varepsilon^2}{2}\Delta v^\varepsilon=|v^\varepsilon|^2v^\varepsilon\quad;\quad v^\varepsilon(0,x)=\sqrt{\varepsilon}\sum_{j=1}^{2}\alpha_j(x)e^{ik_j\cdot x/\varepsilon}.$$

In other words, v^ε solves (1.1) with $V=0$, and initial data (1.3) have been multiplied by $\sqrt{\varepsilon}$, which is another way to see that nonlinear effects are attenuated compared to (1.1)-(1.3) (small data). This shows that the superposition result stated in Theorem 1.3 is actually the consequence of strong nonlinear effects.

1.3. **Content.** In Section 2, we recall the WKB construction introduced in [11], and emphasize the finite speed of propagation which appears in our framework. In Section 3, we explain how to adapt the previous approach to the case where V satisfies (1.2) and is not necessarily trivial. In Section 4, we complete the proof of Theorem 1.3. Proposition 1.4 is established in Section 5. In an appendix, we propose an alternative (statement and) proof of Proposition 1.4, in the case $d\geqslant 2$ with $N=3$, showing that there are indeed several obstructions for Theorem 1.3 to be valid in the weakly nonlinear case.

2. The monokinetic case without potential

In this section, we consider (1.1)-(1.3) in the monokinetic $N=1$, with slightly different notations for future reference:

$$i\varepsilon\partial_t u^\varepsilon+\frac{\varepsilon^2}{2}\Delta u^\varepsilon=|u^\varepsilon|^2u^\varepsilon\quad;\quad u^\varepsilon_{|t=0}=a_0e^{i\phi_0/\varepsilon}. \tag{2.1}$$

In view of the setting of this paper, we assume $a_0,\phi_0\in C_0^\infty(\mathbb{R}^d)$. In particular, $a_0,\phi_0\in H^\infty(\mathbb{R}^d)$. We first consider the case $V\equiv 0$, then introduce the main ideas that make it possible to incorporate a subquadratic potential V.

We recall the main steps to the construction introduced in [11] (see also [5, Section 4.2]). The idea introduced in [11] consists in writing the solution to (2.1) as

$$u^\varepsilon(t,x)=a^\varepsilon(t,x)e^{i\phi^\varepsilon(t,x)/\varepsilon}, \tag{2.2}$$

with a^ε complex-valued and ϕ^ε real-valued, solving

$$\begin{cases}\partial_t\phi^\varepsilon+\dfrac{1}{2}|\nabla\phi^\varepsilon|^2+|a^\varepsilon|^2=0,\quad \phi^\varepsilon_{|t=0}=\phi_0,\\ \partial_t a^\varepsilon+\nabla\phi^\varepsilon\cdot\nabla a^\varepsilon+\dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon=i\dfrac{\varepsilon}{2}\Delta a^\varepsilon,\quad a^\varepsilon_{|t=0}=a_0.\end{cases} \tag{2.3}$$

The key remark is that this leads to a symmetric hyperbolic system, perturbed be a skew-symmetric term. The hyperbolic system appears when considering the unknown

$$U^\varepsilon = \begin{pmatrix} \operatorname{Re} a^\varepsilon \\ \operatorname{Im} a^\varepsilon \\ \nabla\phi^\varepsilon \end{pmatrix} = \begin{pmatrix} \operatorname{Re} a^\varepsilon \\ \operatorname{Im} a^\varepsilon \\ v^\varepsilon \end{pmatrix}.$$

Considering the gradient of the first equation in (2.3), the system can be written

$$\partial_t U^\varepsilon + \sum_{j=1}^d A_j(U^\varepsilon)\partial_j U^\varepsilon = \varepsilon L U^\varepsilon, \tag{2.4}$$

with

$$A(U^\varepsilon,\xi) = \sum_{j=1}^d A_j(U^\varepsilon)\xi_j = \begin{pmatrix} v^\varepsilon\cdot\xi & 0 & \frac{1}{2}\operatorname{Re} a^\varepsilon\, {}^t\xi \\ 0 & v^\varepsilon\cdot\xi & \frac{1}{2}\operatorname{Im} a^\varepsilon\, {}^t\xi \\ 2\operatorname{Re} a^\varepsilon\,\xi & 2\operatorname{Im} a^\varepsilon\,\xi & v^\varepsilon\cdot\xi \mathrm{I}_d \end{pmatrix},$$

and

$$L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & 0_{d\times d} & \end{pmatrix}.$$

To be precise, the system is made symmetric thanks to the constant symmetrizer

$$S = \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & \frac{1}{4}\mathrm{I}_d \end{pmatrix}.$$

Once v^ε is known, one recovers ϕ^ε by integrating in time the first equation in (2.3),

$$\phi^\varepsilon(t,x) = \phi_0(x) - \frac{1}{2}\int_0^t |v^\varepsilon(s,x)|^2 ds - \int_0^t |a^\varepsilon(s,x)|^2 ds,$$

and since $\partial_t(v^\varepsilon - \nabla\phi^\varepsilon) = 0$, $v^\varepsilon = \nabla\phi^\varepsilon$. Assuming that $a_0, \nabla\phi_0 \in H^s(\mathbb{R}^d)$ for s large (we will always assume $a_0,\phi_0 \in C_0^\infty(\mathbb{R}^d)$ in the forthcoming applications), the limit $\varepsilon\to 0$ leads to an asymptotic expansion of the form

$$\phi^\varepsilon \sim \phi + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \dots, \quad a^\varepsilon \sim a + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \dots$$

The leading order term is obtained by simply setting $\varepsilon = 0$ in (2.4):

$$\begin{cases} \partial_t\phi + \dfrac{1}{2}|\nabla\phi|^2 + |a|^2 = 0, \quad \phi_{|t=0} = \phi_0, \\ \partial_t a + \nabla\phi\cdot\nabla a + \dfrac{1}{2}a\Delta\phi = 0, \quad a_{|t=0} = a_0. \end{cases} \tag{2.5}$$

Working with the intermediary unknown $v = \nabla\phi$, we get a system of the form

$$\partial_t U + \sum_{j=1}^d A_j(U)\partial_j U = 0,$$

and we infer the following result from [17]:

Proposition 2.1. *Let $a_0, \phi_0 \in C_0^\infty(\mathbb{R}^d)$, with* $\operatorname{supp} a_0, \operatorname{supp}\phi_0 \subset K$. *There exists $T_* > 0$ and a unique solution $(\phi, a) \in C([0,T_*]; H^\infty(\mathbb{R}^d))^2$ to (2.5). Moreover, (ϕ,a) remains compactly supported for $t\in[0,T_*]$, and*

$$\operatorname{supp}\phi(t,\cdot),\ \operatorname{supp} a(t,\cdot) \subset K.$$

The first part of the statement is a consequence of classical theory for symmetric hyperbolic systems (see e.g. [2, 16]). The property stated that initial compactly supported condition lead to a zero speed of propagation is due to the structure of this hyperbolic system, and is well understood from the simplest model of the Burgers equation

$$\partial_t u + u\partial_x u = 0, \quad u_{|t=0} = u_0 \in C_0^\infty(\mathbb{R}).$$

Suppose we have a smooth solution on some time interval $[0, T_*]$. In particular,

$$\int_0^{T_*} \|\partial_x u(t)\|_{L^\infty} dt < \infty.$$

We have directly, for all $(t, x) \in [0, T_*] \times \mathbb{R}$,

$$|\partial_t u(t, x)| \leqslant \|\partial_x u(t)\|_{L^\infty} |u(t, x)|.$$

Gronwall lemma then shows that if $u_0(x_0) = 0$, then $u(t, x_0) = 0$ for all $t \in [0, T_*]$, hence the zero speed of propagation for smooth solutions. As the matrix $A(U, \xi)$ is linear in U, the result follows in the setting of (2.5). Note that to prove this zero speed of propagation, we do not invoke the symmetry of A: it was used in order to get Sobolev estimates (which ensure that $U \in L^1([0, T_*]; W^{1,\infty})$), but only the fact that it is (at least) linear in U is used at this stage. We then have, for the same T_* as in Proposition 2.1:

Proposition 2.2. *Let $a_0, \phi_0 \in C_0^\infty(\mathbb{R}^d)$. There exists $T_* > 0$ independent of $\varepsilon \in]0, 1]$ such that for all $s \geqslant 0$, there exists $C = C(s)$ such that*

$$\|\phi^\varepsilon - \phi\|_{L^\infty([0,T_*];H^s(\mathbb{R}^d))} + \|a^\varepsilon - a\|_{L^\infty([0,T_*];H^s(\mathbb{R}^d))} \leqslant C\varepsilon.$$

To infer the pointwise description of u^ε at leading order, we must in addition know ϕ^ε up to $o(\varepsilon)$, which is achieved by considering the linearization of (2.5). At the next step of the WKB expansion, we find that

$$\|\phi^\varepsilon - \phi - \varepsilon\phi^{(1)}\|_{L^\infty([0,T_*];H^s(\mathbb{R}^d))} + \|a^\varepsilon - a - \varepsilon a^{(1)}\|_{L^\infty([0,T_*];H^s(\mathbb{R}^d))} \leqslant C\varepsilon^2,$$

where the first corrector $(\phi^{(1)}, a^{(1)})$ solves the system:

$$\begin{cases} \partial_t\phi^{(1)} + \nabla\phi\cdot\nabla\phi^{(1)} + 2\operatorname{Re}\left(\overline{a}a^{(1)}\right) = 0, \\ \partial_t a^{(1)} + \nabla\phi\cdot\nabla a^{(1)} + \nabla\phi^{(1)}\cdot\nabla a + \dfrac{1}{2}a^{(1)}\Delta\phi + \dfrac{1}{2}a\Delta\phi^{(1)} = \dfrac{i}{2}\Delta a, \\ \phi^{(1)}_{|t=0} = 0 \quad ; \quad a^{(1)}_{|t=0} = 0. \end{cases}$$

At higher order $k \geqslant 2$, the corrector $(\phi^{(k)}, a^{(k)})$ is given by:

$$\begin{cases} \partial_t\phi^{(k)} + \nabla\phi\cdot\nabla\phi^{(k)} + 2\operatorname{Re}\left(\overline{a}a^{(k)}\right) = F_k\left(\left(\nabla\phi^{(\ell)}\right)_{1\leqslant\ell\leqslant k-1}, \left(a^{(\ell)}\right)_{1\leqslant\ell\leqslant k-1}\right), \\ \partial_t a^{(k)} + \nabla\phi\cdot\nabla a^{(k)} + \nabla\phi^{(k)}\cdot\nabla a + \dfrac{1}{2}a^{(k)}\Delta\phi + \dfrac{1}{2}a\Delta\phi^{(k)} = \dfrac{i}{2}\Delta a^{(k-1)} \\ \qquad\qquad - \displaystyle\sum_{1\leqslant\ell\leqslant k-1}\nabla\phi^{(\ell)}\cdot\nabla a^{(k-\ell)} - \dfrac{1}{2}\sum_{1\leqslant\ell\leqslant k-1} a^{(k-\ell)}\Delta\phi^{(\ell)}, \\ \phi^{(k)}_{|t=0} = 0 \quad ; \quad a^{(k)}_{|t=0} = 0, \end{cases}$$

for some function F_k which is a polynomial in its arguments, without constant term, and whose precise expression is unimportant here. The left hand side is always the linearization of the left hand side of (2.5) about (ϕ, a), and the right hand side depends on previous correctors. We infer (see [11, 5]), by induction:

Proposition 2.3. *Let $a_0, \phi_0 \in C_0^\infty(\mathbb{R}^d)$. Let $T_* > 0$ given by Proposition 2.1. For all $k \geqslant 1$, there exists a unique solution $(\phi^{(k)}, a^{(k)}) \in C([0,T_*]; H^\infty(\mathbb{R}^d))^2$ to the above system, and for all $s \geqslant 0$, there exists $C = C(k,s)$ such that*

$$\begin{aligned}\left\|\phi^\varepsilon - \phi - \varepsilon\phi^{(1)} - \ldots - \varepsilon^k \phi^{(k)}\right\|_{L^\infty([0,T_*];H^s(\mathbb{R}^d))}&\\ + \left\|a^\varepsilon - a - \varepsilon a^{(1)} - \ldots - \varepsilon^k a^{(k)}\right\|_{L^\infty([0,T_*];H^s(\mathbb{R}^d))} &\leqslant C\varepsilon^{k+1}.\end{aligned}$$

In addition, if $\operatorname{supp} a_0, \operatorname{supp}\phi_0 \subset K$, then $(\phi^{(k)}, a^{(k)})$ remains compactly supported for $t \in [0, T_*]$, and

$$\operatorname{supp}\phi^{(k)}(t,\cdot),\ \operatorname{supp} a^{(k)}(t,\cdot) \subset K.$$

The support property is a consequence of the same argument as in the proof of Proposition 2.1. Using the embedding $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $s > d/2$, we also deduce from the above error estimate the bound, for $k \geqslant 1$:

$$\left\| u^\varepsilon - \left(\sum_{\ell=0}^{k-1} \varepsilon^\ell a^{(\ell)}\right) \exp\left(\frac{i}{\varepsilon}\sum_{\ell=0}^{k}\varepsilon^\ell \phi^{(\ell)}\right)\right\|_{L^\infty([0,T_*];L^2\cap L^\infty(\mathbb{R}^d))} = \mathcal{O}\left(\varepsilon^k\right), \tag{2.6}$$

with the convention $(\phi^{(0)}, a^{(0)}) = (\phi, a)$. The standard form of WKB expansions,

$$u^\varepsilon(t,x) = \left(\mathtt{a}(t,x) + \varepsilon \mathtt{a}_1(t,x) + \ldots + \varepsilon^k \mathtt{a}_k(t,x)\right) e^{i\phi(t,x)/\varepsilon} + \mathcal{O}_{L^\infty_{T_*}(L^2\cap L^\infty)}\left(\varepsilon^{k+1}\right),$$

is then obtained by setting

$$\mathtt{a} = a e^{i\phi^{(1)}}, \quad \mathtt{a}_1 = a^{(1)} e^{i\phi^{(1)}} + ia\phi^{(2)} e^{i\phi^{(1)}}, \quad \text{etc.}$$

Remark 2.4 (Higher order nonlinearities). If instead of (2.1), one considers

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma} u^\varepsilon \quad ; \quad u^\varepsilon_{|t=0} = a_0 e^{i\phi_0/\varepsilon},$$

with $\sigma \geqslant 2$ an integer, then the justification of WKB analysis requires a different approach. We refer to [1, 9] for two different proofs, which show that the conclusions of the propositions stated in this section remain valid.

Remark 2.5 (Focusing nonlinearity). If instead of (2.1), one considers a cubic focusing nonlinearity,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = -|u^\varepsilon|^2 u^\varepsilon \quad ; \quad u^\varepsilon_{|t=0} = a_0 e^{i\phi_0/\varepsilon},$$

then the analogue of (2.5) is no longer hyperbolic, but elliptic. Working with analytic initial data (ϕ_0, a_0) is then necessary in order to solve (2.5) ([15, 22]), and this is a framework where nonlinear WKB analysis is fully justified ([10, 26]). However, analyticity is incompatible with an initial compact support.

3. The monokinetic case with a potential

We consider the same framework as in the previous section, now with a potential:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + |u^\varepsilon|^2 u^\varepsilon \quad ; \quad u^\varepsilon_{|t=0} = a_0 e^{i\phi_0/\varepsilon}. \tag{3.1}$$

As noticed in [4], it is possible to adapt the above WKB analysis in the presence of an external potential satisfying (1.2) by simply mixing the standard approach followed in the linear case (see e.g. [24]) and Grenier's method.

3.1. **Linear case.** The eikonal equation associated to

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon \quad ; \quad u^\varepsilon_{|t=0} = a_0, \tag{3.2}$$

that is, without initial rapid oscillation, reads:

$$\partial_t\phi_{\rm eik} + \frac{1}{2}|\nabla\phi_{\rm eik}|^2 + V = 0 \quad ; \quad \phi_{{\rm eik}|t=0} = 0. \tag{3.3}$$

This eikonal equation is solved by introducing the classical trajectories, solving

$$\dot{x}(t,y) = \xi(t,y), \quad x(0,y) = y \quad ; \quad \dot{\xi}(t,y) = -\nabla V\left(x(t,y)\right), \quad \xi(0,y) = 0. \tag{3.4}$$

As V is at most quadratic, from (1.2), the above system has a unique, global, smooth solution, and in addition.

$$\nabla_y x(t,y) = {\rm I}_d + \mathcal{O}(t),$$

uniformly in $y \in \mathbb{R}^d$, for any matricial norm on $\mathbb{R}^{d\times d}$. Therefore, the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t,y),$$

remains non-zero and bounded on some time interval $[0,T]$ with $T > 0$. Since we also have, by uniqueness in ordinary differential equations,

$$\nabla\phi_{\rm eik}\left(t, x(t,y)\right) = \xi(t,y),$$

for any smooth solutions to (3.3), the global inversion theorem implies the following result (see also [5, Proposition 1.9]):

Lemma 3.1. *Let V satisfying (1.2). There exists $T > 0$ and a unique solution $\phi_{\rm eik} \in C^\infty\left([0,T]\times\mathbb{R}^d\right)$ to (3.3). In addition, this solution is at most quadratic in space: $\partial_x^\alpha\phi_{\rm eik} \in L^\infty([0,T]\times\mathbb{R}^d)$ as soon as $|\alpha| \geqslant 2$. There exists $C > 1$ such that the Jacobi determinant satisfies:*

$$\frac{1}{C} \leqslant J_t(y) \leqslant C, \quad \forall (t,y) \in [0,T]\times\mathbb{R}^d.$$

Since the above relations imply

$$\dot{x}(t,y) = \nabla\phi_{\rm eik}\left(t, x(t,y)\right),$$

we infer the classical formula

$$\partial_t J_t(y) = J_t(y)\Delta\phi_{\rm eik}\left(t, x(t,y)\right). \tag{3.5}$$

3.2. **Introducing the nonlinearity.** As noticed in [4], the approach presented in the case $V = 0$ for the nonlinear case can be adapted by changing the representation (2.2) to

$$u^\varepsilon(t,x) = a^\varepsilon(t,x)e^{i\phi_{\rm eik}(t,x)/\varepsilon + i\phi^\varepsilon(t,x)/\varepsilon},$$

and requiring

$$\begin{cases} \partial_t\phi^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + |a^\varepsilon|^2 = 0, \\ \partial_t a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} + \frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \\ \quad \phi^\varepsilon_{|t=0} = \phi_0 \quad ; \quad a^\varepsilon_{|t=0} = a_0. \end{cases} \tag{3.6}$$

The new terms compared to (2.3) involve $\phi_{\rm eik}$, and since $\phi_{\rm eik}$ is at most quadratic in space, it turns out that they can be estimated like (semilinear) perturbative terms (using commutator estimates for the transport part). The natural limit for (3.6) when $\varepsilon \to 0$ is given by

$$\begin{cases} \partial_t \phi + \nabla\phi_{\rm eik}\cdot\nabla\phi + \dfrac{1}{2}|\nabla\phi|^2 + |a|^2 = 0,\\ \partial_t a + \nabla\phi_{\rm eik}\cdot\nabla a + \nabla\phi\cdot\nabla a + \dfrac{1}{2}a\Delta\phi_{\rm eik} + \dfrac{1}{2}a\Delta\phi = 0,\\ \quad \phi_{|t=0} = \phi_0 \quad ; \quad a_{|t=0} = a_0. \end{cases} \tag{3.7}$$

The following result is a consequence of [4]:

Proposition 3.2. *Let $a_0, \phi_0 \in C_0^\infty(\mathbb{R}^d)$, V satisfying (1.2), and T, $\phi_{\rm eik}$ given by Lemma 3.1. There exists $0 < T_* \leqslant T$ independent of $\varepsilon \in]0,1]$ such that (3.6) has a unique solution $(\phi^\varepsilon, a^\varepsilon) \in C([0,T_*]; H^\infty(\mathbb{R}^d))^2$, (3.7) has a unique solution $(\phi, a) \in C([0,T_*]; H^\infty(\mathbb{R}^d))^2$, and for all $s \geqslant 0$, there exists $C = C(s)$ such that*

$$\|\phi^\varepsilon - \phi\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} + \|a^\varepsilon - a\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} \leqslant C\varepsilon.$$

The correctors $\left(\phi^{(j)}, a^{(j)}\right)_{j\geqslant 1}$ as obtained in the same fashion as in Section 2. The only difference is that the operator ∂_t is replaced by

$$\partial_t + \nabla\phi_{\rm eik}\cdot\nabla + \frac{1}{2}\Delta\phi_{\rm eik}.$$

3.3. **Finite speed of propagation: following the classical trajectories.** In order to prove that if $a_0 \in C_0^\infty(\mathbb{R}^d)$, the solution to (3.2) remains compactly supported in the support of a_0 transported by the classical flow (3.4), it is standard to introduce the following change of unknown function (e.g. [24, 5]):

$$A(t,y) := \sqrt{J_t(y)}a\left(t, x(t,y)\right),$$

where a solves the transport equation

$$\partial_t a + \nabla\phi_{\rm eik}\cdot\nabla a + \frac{1}{2}a\Delta\phi_{\rm eik} = 0 \quad ; \quad a_{|t=0} = a_0.$$

as given by WKB analysis. Indeed, using (3.5), we easily check that A is constant in time, $\partial_t A = 0$. Correctors $(a^{(k)})_{k\geqslant 1}$ in the (linear) WKB analysis solve the equation

$$\partial_t a^{(k)} + \nabla\phi_{\rm eik}\cdot\nabla a^{(k)} + \frac{1}{2}a^{(k)}\Delta\phi_{\rm eik} = \frac{i}{2}\Delta a^{(k-1)} \quad ; \quad a^{(k)}_{|t=0} = 0,$$

with the convention $a^{(0)} = a$. Setting

$$A^{(k)}(t,y) := \sqrt{J_t(y)}a^{(k)}\left(t, x(t,y)\right),$$

we infer that

$$\operatorname{supp} A^{(k)}(t,\cdot) \subset \operatorname{supp} a_0, \quad \forall t \in [0,T],\ \forall k \geqslant 0,$$

where T is given by Lemma 3.1. Thus, for $t \in [0,T]$, up to $\mathcal{O}(\varepsilon^\infty)$, u^ε remains compactly supported, in the support of a_0 transported by the classical flow.

In the nonlinear case, we check that the same argument remains valid. Consider $\phi_{\rm eik}$ solution to (3.3), and (ϕ, a) solving (3.7). The natural adaptation of the above computation consists in showing that if $\phi_0, a_0 \in C_0^\infty(\mathbb{R}^d)$, the new unknown (ψ, A), defined by

$$A(t,y) := \sqrt{J_t(y)}a\left(t, x(t,y)\right), \quad \psi(t,y) := \phi\left(t, x(t,y)\right), \tag{3.8}$$

enjoys a zero speed of propagation. Note that in view of Proposition 3.2, we already know that $\phi, a \in C([0,T_*];H^\infty(\mathbb{R}^d))$, so it suffices to check that (ψ, A) solves a system for which the argument presented on the toy model of Burgers equation in Section 2 remains valid. Introducing

$$M(t,y) = \nabla_y x(t,y) \in \mathbb{R}^{d\times d},$$

whose determinant is by definition $J_t(y)$, we find:

$$\partial_t \psi + \frac{1}{2}\left\langle M^{-1}\nabla\psi, M^{-1}\nabla\psi\right\rangle + \frac{1}{J_t(y)}|A|^2 = 0, \quad \psi_{|t=0} = \phi_0,$$
$$\partial_t A = -\sqrt{J_t(y)}\left(\nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi\right)(t,x(t,y)), \quad A_{|t=0} = a_0.$$

We do not express the right hand side of the last equation in terms of (ψ, A): differentiating the first equation with respect to y, the bounds stated in Proposition 3.2 make it possible to infer an inequality of the form

$$|\partial_t\nabla\psi(t,y)| + |\partial_t A(t,y)| \lesssim |\nabla\psi(t,y)| + |A(t,y)|, \quad (t,y)\in[0,T_*]\times\mathbb{R}^d.$$

Therefore, if $\operatorname{supp}\phi_0, \operatorname{supp} a_0 \subset K$, then $\operatorname{supp}\nabla\psi(t,\cdot), \operatorname{supp} A(t,\cdot) \subset K$ for all $t\in[0,T_*]$. Integrating in time the equation solved by ψ, we conclude to the zero speed of propagation for (ψ, A). Arguing like in Section 2 for the correctors, we have:

Proposition 3.3. *Let $\phi_0, a_0 \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi_0, \operatorname{supp} a_0 \subset K$. There for any $t\in[0,T_*]$, where T_* is given by Proposition 3.2,*

$$\operatorname{supp}\psi(t,\cdot), \operatorname{supp} A(t,\cdot) \subset K,$$

where ψ and A are related to ϕ and a through (3.8). *The same is true for the correctors $(\psi^{(k)}, A^{(k)})_{k\geqslant 1}$ corresponding to the next terms $(\phi^{(k)}, a^{(k)})_{k\geqslant 1}$ in the asymptotic expansion in* (3.6).

Remark 3.4 (Special potentials). In the case where V is linear in x or isotropic quadratic, explicit formulas allow to bypass the above arguments. If $V(x) = E\cdot x$ for some (constant) $E\in\mathbb{R}^d$ and u^ε solves (1.1), then

$$v^\varepsilon(t,x) = u^\varepsilon\left(t, x - \frac{t^2}{2}E\right)e^{i\left(tE\cdot x - \frac{t^3}{3}|E|^2\right)/\varepsilon}$$

solves (2.1). If $V(x) = \frac{\omega^2}{2}|x|^2$, $\omega>0$, then

$$w^\varepsilon(t,x) = \frac{1}{\left(1+(\omega t)^2\right)^{d/4}}u^\varepsilon\left(\frac{\arctan(\omega t)}{\omega}, \frac{x}{\sqrt{1+(\omega t)^2}}\right)e^{i\frac{\omega^2 t}{1+(\omega t)^2}\frac{|x|^2}{2\varepsilon}}$$

solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = (1+t^2)^{d/2-1}|w^\varepsilon|^2 w^\varepsilon \quad ; \quad w^\varepsilon_{|t=0} = a_0 e^{i\phi_0/\varepsilon}.$$

If $d=2$ (the cubic nonlinearity is L^2-critical), we recover exactly (2.1). Otherwise, a (smooth) time dependent factor has appeared, which obviously does not change the conclusion of Propositions 2.1 and 2.3. The case of a potential with the opposite sign is obtained by changing ω to $i\omega$ in the formulas. See e.g. [5, Section 11.2] and references therein regarding these changes of unknown functions. For such potentials, the classical trajectories given by (3.4) are computed explicitly, and we can check directly the conclusions of Proposition 3.3.

4. Separation of states

We complete the proof of Theorem 1.3, by proving the nonlinear superposition. For $1 \leqslant j \leqslant N$, let $\chi_j \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$, $0 \leqslant \chi_j \leqslant 1$, with

$$\chi_j \equiv 1 \text{ on } \operatorname{supp} \alpha_j, \quad \operatorname{supp} \chi_{j_1} \cap \operatorname{supp} \chi_{j_2} = \emptyset \text{ if } j_1 \neq j_2.$$

We set

$$a_0(x) = \sum_{j=1}^N \alpha_j(x), \quad \phi_0(x) = \sum_{j=1}^N \varphi_j(x)\chi_j(x).$$

Then $a_0, \phi_0 \in C_0^\infty(\mathbb{R}^d)$, ϕ_0 is real-valued, and

$$u_0^\varepsilon(x) = a_0(x) e^{i\phi_0(x)/\varepsilon}.$$

We can then resume the analysis from the monokinetic case as presented in Sections 2 and 3, with the same notations. Let $\phi_{\rm eik}$ be given by Lemma 3.1 (it does not depend on the initial data, but only on V). The WKB analysis for each u_j^ε, solution to (1.5), involves the following system:

$$\begin{cases} \partial_t \phi_j + \nabla \phi_{\rm eik} \cdot \nabla \phi_j + \dfrac{1}{2}|\nabla \phi_j|^2 + |a_j|^2 = 0, \quad \phi_{j|t=0} = \varphi_j \chi_j, \\ \partial_t a_j + \nabla \phi_{\rm eik} \cdot \nabla a + \dfrac{1}{2} a \Delta \phi_{\rm eik} + \nabla \phi_j \cdot \nabla a_j + \dfrac{1}{2} a_j \Delta \phi_j = 0, \quad a_{j|t=0} = \alpha_j. \end{cases} \tag{4.1}$$

To simplify the discussion, suppose first that $V=0$, hence $\phi_{\rm eik}=0$. Each solution to (4.1) remains smooth on some time interval $[0,T_j]$ for some $0 < T_j \leqslant T$, and, on this time interval, enjoys a zero speed of propagation. As a consequence of Proposition 2.1, we have

$$\phi = \sum_{j=1}^N \phi_j, \quad a = \sum_{\jmath=1}^N a_j,$$

since nonlinear terms containing two indices $j_1 \neq j_2$ involve two functions whose supports are disjoint. Also, for all $k \geqslant 1$, the correctors satisfy

$$\phi^{(k)} = \sum_{j=1}^N \phi_j^{(k)}, \quad a = \sum_{\jmath=1}^N a_j^{(k)}.$$

Set

$$T^* = \min\left(T_*, T_1, \dots, T_N\right).$$

As we have, in view of (2.6), in $L^\infty([0,T^*]; L^2 \cap L^\infty)$, for any $k>0$,

$$\begin{aligned} u^\varepsilon - \left(\sum_{\ell=0}^{k-1} \varepsilon^\ell a^{(\ell)}\right) \exp\left(\frac{i}{\varepsilon}\sum_{\ell=0}^{k} \varepsilon^\ell \phi^{(\ell)}\right) &= \mathcal{O}\left(\varepsilon^k\right), \\ u_j^\varepsilon - \left(\sum_{\ell=0}^{k-1} \varepsilon^\ell a_j^{(\ell)}\right) \exp\left(\frac{i}{\varepsilon}\sum_{\ell=0}^{k} \varepsilon^\ell \phi_j^{(\ell)}\right) &= \mathcal{O}\left(\varepsilon^k\right), \quad j = 1, \dots, N, \end{aligned}$$

we obtain Theorem 1.3 in the case $V=0$. In the case where V is not trivial, we just have to resume the above arguments by replacing the functions (ϕ, a) (possibly with indices and/or superscripts) with (ψ, A), as defined by the change of unknown function (3.8), which involves only V (see (3.4)), and not the initial data.

5. Weakly nonlinear case

In this section, we prove Proposition 1.4. Instead of (1.1)-(1.3), we consider the weakly nonlinear case,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon|u^\varepsilon|^2u^\varepsilon \quad ; \quad u_0^\varepsilon(x) = \sum_{j=1}^N \alpha_j(x)e^{i\varphi_j(x)/\varepsilon},$$

or, equivalently, (1.1) with initial data (1.3) multiplied by $\sqrt{\varepsilon}$. When $d \geqslant 2$, the creation of new WKB terms is possible by resonant interactions, provided that $N \geqslant 3$. The one-dimensional cubic case is special, as there are no nontrivial resonances, see [6]. In order to present an argument including the cubic one-dimensional case, we propose a proof which does not use the creation of, e.g., a fourth term out of three.

To prove Proposition 1.4, in agreement with the assumptions of this statement, it suffices to set $N = 2$, and consider linear phases,

$$\varphi_j(x) = k_j \cdot x.$$

WKB analysis in the monokinetic case $N = 1$ leads to the hierarchy

$$\begin{aligned}
&\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \quad ; \quad \phi(0,x) = k\cdot x,\\
&\partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = -i|a|^2a \quad ; \quad a(0,x) = \alpha(x).
\end{aligned}$$

The eikonal equation is solved explicitly,

$$\phi(t,x) = k\cdot x - \frac{|k|^2}{2}t.$$

As $\Delta\phi = 0$, the initial amplitude α is transported along the vector k with a phase self-modulation:

$$a(t,x) = \alpha(x-tk)e^{-it|\alpha(x-tk)|^2}.$$

In the case $N = 2$, no new WKB term is created, but interactions between the two modes lead to a modification of the phase modulation. As computed in [6, Section 3], we find

$$\begin{aligned}
a_1(t,x) &= \alpha_1(x-tk_1)e^{-i\left(2\int_0^t|\alpha_2(x+(\tau-t)k_1-\tau k_2)|^2d\tau + t|\alpha_1(x-tk_1)|^2\right)},\\
a_2(t,x) &= \alpha_2(x-tk_2)e^{-i\left(2\int_0^t|\alpha_1(x+(\tau-t)k_2-\tau k_1)|^2d\tau + t|\alpha_2(x-tk_2)|^2\right)}.
\end{aligned}$$

The leading order nonlinear interactions between the two modes correspond to the integrals in time in the exponentials. In addition, we have (see [5, 6])

$$\sup_{t\in[0,T]}\left\|u^\varepsilon(t) - a_1(t)e^{i\phi_1(t)/\varepsilon} - a_2(t)e^{i\phi_2(t)/\varepsilon}\right\|_{L^2\cap L^\infty} = \mathcal{O}(\varepsilon),$$

for some $T > 0$ independent of ε, depending only on $\|\hat\alpha_1\|_{L^1\cap L^2} + \|\hat\alpha_2\|_{L^1\cap L^2}$. The conclusion in Proposition 1.4 then follows from the property:

$$\sup_{t\in[0,T]}\left\|\alpha_1(\cdot - tk_1)\left(e^{-2i\int_0^t|\alpha_2(\cdot+(\tau-t)k_1-\tau k_2)|^2d\tau} - 1\right)\right\|_{L^p} > 0,$$

or, equivalently,

$$\sup_{t\in[0,T]}\left\|\alpha_1\sin\left(\int_0^t|\alpha_2(\cdot+\tau(k_1-k_2))|^2d\tau\right)\right\|_{L^p} > 0.$$

Now let $\alpha \in C_0^\infty(\mathbb{R}^d)$ supported in the ball centered at the origin, of radius 1, and set

$$\alpha_1(x) = \alpha(x), \quad \alpha_2(x) = \alpha(x + 3e_1),$$

where $(e_1, \dots, e_d)$ is the canonical basis of $\mathbb{R}^d$. Setting $k_2 - k_1 = \lambda e_1$, we can choose $\lambda > 0$ sufficiently large so the argument of the sine function is not zero on the support of α for $t \in [0, T]$, hence the result.

Appendix A. Weakly nonlinear case and creation of a new term

In this appendix, we prove that in the weakly nonlinear case, if $d \geqslant 2$, then nonlinear interactions may lead to the creation of a new WKB term, which is a stronger phenomenon than that used in the proof of Proposition 1.4. Consider

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon|u^\varepsilon|^2 u^\varepsilon \quad ; \quad u_0^\varepsilon(x) = \sum_{j=1}^{N} \alpha_j(x) e^{i\varphi_j(x)/\varepsilon},$$

with now $N = 3$, and $d \geqslant 2$. The one-dimensional cubic case is special, as there are no nontrivial resonances, see [6]. Again, we consider linear phases,

$$\varphi_j(x) = k_j \cdot x.$$

Recall ([6, Lemma 2.2]) that the resonant set is defined by

$$\mathrm{Res}(n) = \{(j, \ell, m), \quad k_j - k_\ell + k_m = k_n,\ |k_j|^2 - |k_\ell|^2 + |k_m|^2 = |k_n|^2\}.$$

is characterized as follows: $(k_j, k_\ell, k_m) \in \mathrm{Res}(n)$ when the endpoints of the vectors k_j, k_ℓ, k_m, k_n form four corners of a nondegenerate rectangle with k_ℓ and k_n opposing each other, or when this quadruplet corresponds to one of the following two degenerate cases: $(k_j = k_n, k_m = k_\ell)$ or $(k_j = k_\ell, k_m = k_n)$. Note that we always have

$$\{(j, j, n),\ ((n, j, j),\ a_j \not\equiv 0\} \subset \mathrm{Res}(n), \tag{A.1}$$

where a_j is the amplitude associated with the phase

$$\phi_j(t, x) = k_j \cdot x - \frac{|k_j|^2}{2} t.$$

In order for the nonlinearity to create a term associated with a phase ϕ_4, out of three phases associated with wave numbers k_1, k_2 and k_3, we must have

$$k_4 := k_2 - k_1 + k_3, \quad |k_4|^2 = |k_2|^2 - |k_1|^2 + |k_3|^2.$$

This resonant condition is equivalent to the following conditions:

$$(k_1 - k_2) \cdot (k_1 - k_3) = 0,$$

and the endpoints of k_1, k_2 and k_3 are not aligned (the case of alignment corresponds to the set on the left in (A.1)); this is possible with pairwise different k_1, k_2, k_3 and $k_4 \notin \{k_1, k_2, k_3\}$ provided that $d \geqslant 2$, see [6] (or [5, Section 2.6]). For instance if $d = 2$, we can choose, for $\lambda > 0$,

$$k_1 = \lambda(1, 1), \quad k_2 = \lambda(1, 0), \quad k_3 = \lambda(0, 1), \quad \text{hence } k_4 = (0, 0).$$

In higher dimension, we simply complete each vector by zero coordinates. Then a new term, associated with the phase ϕ_4 may be created by nonlinear resonance. Because of the geometric characterization of resonances, no other term can be created apart from this one, since we have completed a rectangle. The creation is effective only if the associated amplitude does not remain zero. The equation for the corresponding amplitude is

$$\partial_t a_4 + k_4 \cdot \nabla a_4 = -i \sum_{(j,\ell,m)\in \mathrm{Res}(4)} a_j \bar{a}_\ell a_m, \quad a_{4|t=0} = 0.$$

More generally, the term a_n solves

$$\partial_t a_n + k_n \cdot \nabla a_n = -i \sum_{(j,\ell,m)\in \mathrm{Res(n)}} a_j \bar{a}_\ell a_m, \quad a_{n|t=0} = \alpha_n.$$

If we assume that the mode 4 is not effectively created, that is $a_4 \equiv 0$, then the inclusion (A.1) is actually an equality, and

$$\partial_t a_j + k_j \cdot \nabla a_j = -2i \sum_{k=1}^{3} |a_k|^2 a_j + i|a_j|^2 a_j, \quad j = 1, 2, 3.$$

hence

$$a_j(t, x) = \alpha_j(x - tk_j)e^{-iS_j(t,x)}, \quad j = 1, 2, 3,$$

for some explicit real-valued phase, whose expression is irrelevant here (see [6, Section 3.1] for the formula). Given any $T > 0$, we may choose $\alpha_1, \alpha_2, \alpha_3$ compactly supported, with disjoint supports, k_1, k_2, k_3 like above, so that

$$a_2 \bar{a}_1 a_{3|t=T/2} \not\equiv 0.$$

This shows that the term a_4 is actually created, in the sense that a_4 does not remain trivial on $[0, T]$. The error estimate proved in [7] (see also [5, Section 2.6]) yields

$$\sup_{t\in[0,T]} \left\| u^\varepsilon(t) - \sum_{j=1}^{4} a_j(t) e^{i\psi_j(t)/\varepsilon} \right\|_{L^2\cap L^\infty} = \mathcal{O}(\varepsilon),$$

hence again the conclusion of Proposition 1.4

References

[1] T. Alazard and R. Carles. Supercritical geometric optics for nonlinear Schrödinger equations. *Arch. Ration. Mech. Anal.*, 194(1):315–347, 2009.

[2] S. Alinhac and P. Gérard. *Pseudo-differential operators and the Nash-Moser theorem*, volume 82 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. Translated from the 1991 French original by Stephen S. Wilson.

[3] T. Buckmaster, G. Cao-Labora, and J. Gómez-Serrano. Smooth imploding solutions for 3D compressible fluids. Preprint, archived at `https://arxiv.org/abs/2208.09445`.

[4] R. Carles. WKB analysis for nonlinear Schrödinger equations with potential. *Comm. Math. Phys.*, 269(1):195–221, 2007.

[5] R. Carles. *Semi-classical analysis for nonlinear Schrödinger equations: WKB analysis, focal points, coherent states*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2nd edition, xiv+352 p. 2021.

[6] R. Carles, E. Dumas, and C. Sparber. Multiphase weakly nonlinear geometric optics for Schrödinger equations. *SIAM J. Math. Anal.*, 42(1):489–518, 2010.

[7] R. Carles, E. Dumas, and C. Sparber. Geometric optics and instability for NLS and Davey-Stewartson models. *J. Eur. Math. Soc. (JEMS)*, 14(6):1885–1921, 2012.

[8] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[9] D. Chiron and F. Rousset. Geometric optics and boundary layers for nonlinear Schrödinger equations. *Comm. Math. Phys.*, 288(2):503–546, 2009.

[10] P. Gérard. Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire. In *Séminaire sur les Équations aux Dérivées Partielles, 1992–1993*, pages Exp. No. XIII, 13. École Polytech., Palaiseau, 1993.

[11] E. Grenier. Semiclassical limit of the nonlinear Schrödinger equation in small time. *Proc. Amer. Math. Soc.*, 126(2):523–530, 1998.

[12] C. Jao. Energy-critical NLS with potentials of quadratic growth. *Discrete Contin. Dyn. Syst.*, 38(2):563–587, 2018.

[13] S. Jin, C. D. Levermore, and D. W. McLaughlin. The semiclassical limit of the defocusing NLS hierarchy. *Comm. Pure Appl. Math.*, 52(5):613–654, 1999.

[14] R. Killip, M. Visan, and X. Zhang. Energy-critical NLS with quadratic potentials. *Comm. Partial Differential Equations*, 34(10-12):1531–1565, 2009.

[15] N. Lerner, T. Nguyen, and B. Texier. The onset of instability in first-order systems. *J. Eur. Math. Soc. (JEMS)*, 20(6):1303–1373, 2018.

[16] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[17] T. Makino, S. Ukai, and S. Kawashima. Sur la solution à support compact de l'équations d'Euler compressible. *Japan J. Appl. Math.*, 3(2):249–257, 1986.

[18] V. P. Maslov and M. V. Fedoryuk. *Semi-classical approximation in quantum mechanics. Transl. from the Russian by J. Niederle and J. Tolar*, volume 7 of *Math. Phys. Appl. Math.* Kluwer Academic Publishers, Dordrecht, 1981.

[19] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel. On blow up for the energy super critical defocusing nonlinear Schrödinger equations. *Invent. Math.*, 227(1):247–413, 2022.

[20] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel. On the implosion of a compressible fluid I: Smooth self-similar inviscid profiles. *Ann. of Math. (2)*, 196(2):567–778, 2022.

[21] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel. On the implosion of a compressible fluid II: Singularity formation. *Ann. of Math. (2)*, 196(2):779–889, 2022.

[22] G. Métivier. Remarks on the well-posedness of the nonlinear Cauchy problem. In *Geometric analysis of PDE and several complex variables*, volume 368 of *Contemp. Math.*, pages 337–356. Amer. Math. Soc., Providence, RI, 2005.

[23] P. D. Miller. On the generation of dispersive shock waves. *Phys. D*, 333:66–83, 2016.

[24] D. Robert. *Autour de l'approximation semi-classique. (Around semiclassical approximation)*, volume 68 of *Prog. Math.* Birkhäuser, Cham, 1987.

[25] E. Ryckman and M. Visan. Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$. *Amer. J. Math.*, 129(1):1–60, 2007.

[26] L. Thomann. Instabilities for supercritical Schrödinger equations in analytic manifolds. *J. Differential Equations*, 245(1):249–280, 2008.

Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
Email address: `Remi.Carles@math.cnrs.fr`