ON THE EXISTENCE OF WEAK SOLUTION OF THE KINETIC FOKKER-PLANCK EQUATION IN A BOUNDED DOMAIN WITH ABSORBING BOUNDARY

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ON THE EXISTENCE AND ASYMPTOTIC BEHAVIOR OF WEAK SOLUTION OF THE KINETIC FOKKER-PLANCK EQUATION IN A BOUNDED DOMAIN WITH ABSORBING BOUNDARY

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Abstract. We study here the kinetic Fokker-Planck equation in a bounded domain with absorbing boundary. We show the existence and the uniqueness of a weak solution to the initial-boundary problem and we associate it to a positive compact strongly continuous semigroup. Then, we establish that the weak solution satisfies the balanced exponential growth, meaning both exponential decay toward zero and a thermalization phenomenon. Eventually, we show that it has a trace in the sense of the Green formula.

Introduction

We discuss here the existence, the unicity and the asymptotic behavior of weak solutions of the kinetic Fokker-Planck equation in a bounded domain with absorbing boundary. Let an integer $d \geq 2$ and let Ω be a bounded, open and smooth subset of $\mathbb{R}^d$. As its boundary is smooth, we can define for any $q \in \partial\Omega$ the outer normal vector n_q. Hence, we can decompose $\partial\Omega = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$ with:

$$\Gamma^+ = \big\{(q,p) \in \partial\Omega \times \mathbb{R}^d \,|p \cdot n_q > 0\,\big\},$$
$$\Gamma^- = \big\{(q,p) \in \partial\Omega \times \mathbb{R}^d \,|p \cdot n_q < 0\,\big\},$$
$$\Gamma^0 = \big\{(q,p) \in \partial\Omega \times \mathbb{R}^d \,|p \cdot n_q = 0\,\big\}.$$

Let $\sigma \in \mathbb{R}_+^*,\ \gamma \in \mathbb{R}$ and $F \in C^\infty\big(\mathbb{R}^d, \mathbb{R}^d\big)$, we introduce the operator:

$$\mathcal{L} = -p \cdot \nabla_q - F(q) \cdot \nabla_p + \gamma \nabla_p \cdot (p\cdot) + \frac{\sigma^2}{2}\Delta_p, \tag{1}$$

and its formal adjoint in $L^2(\mathrm{d}x)$

$$\mathcal{L}^* = p \cdot \nabla_q + F(q) \cdot \nabla_p - \gamma p \cdot \nabla_p + \frac{\sigma^2}{2}\Delta_p. \tag{2}$$

Following [10, 11], we denote $D := \Omega \times \mathbb{R}^d,\ \overline{D} = \overline{\Omega} \times \mathbb{R}^d$ and $\partial D := \partial\Omega \times \mathbb{R}^d$. The variable $x \in D$ represents the couple $(q,p) \in D$ and $\mathrm{d}x = \mathrm{d}q\mathrm{d}p$. We will use preferably this notation except when the two variables q and p do not have symmetrical roles, as in the beginning of subsection 2.1.

The so-called kinetic Fokker-Planck equation is the following initial-boundary problem for $f \equiv f(t,x)$:

$$\begin{cases} \partial_t f & = \mathcal{L}f \text{ on } (0,T) \times D,\ T > 0, \\ f(t,x) & = 0 \text{ on } (0,T) \times \Gamma^-, \\ f(0,x) & = f^{in}(x) \text{ on } D. \end{cases} \tag{3}$$

It models the collective behavior of particles evolving inside Ω with velocity in $\mathbb{R}^d$, with initial distribution f^{in}, submitted both to friction and to an exterior force F with a thermal bath respectively encoded by the friction term $\gamma\nabla_p \cdot (p\cdot)$, the exterior term $F(q) \cdot \nabla_p \cdot$ and the term of diffusion in velocity $\frac{\sigma^2}{2}\Delta_p$. The quantity $f(t,x)\mathrm{d}t\mathrm{d}x \equiv f(t,q,p)\mathrm{d}t\mathrm{d}q\mathrm{d}p$ represents the number of particles at time interval $[t, t+\mathrm{d}t]$, in the infinitesimal volume $B(q,\mathrm{d}q) \times B(p,\mathrm{d}p) \subset \Omega \times \mathbb{R}^d$, where q is the spatial position and p the velocity. The

homogeneous boundary condition $f(t,x)=0$ on $(0,T)\times\Gamma^-$ means that whenever a particle coming from the interior of Ω hits the boundary, it exits and disappears forever.

In the case of $\Omega=\mathbb{R}^d$, the theory of the kinetic Fokker-Planck equation is well-established (see for instance [7, 16] and references therein). The bounded case is more challenging because of the trace issue in kinetic theory due to the presence of singularities at the boundary. There are several possible strategies for dealing with, for more details, we refer to the enlightening discussion in Stéphane Mischler's "habilitation à diriger les recherches" p. 29-32. In the specific case of the Fokker-Planck equation, among the approaches, two stand out. The first is to define the trace using a renormalized Green's formula. This strategy has the merit of naturally extending to the non-linear case, see [12]. The second strategy is to bypass the trace issue by define a weaker notion of solution with a smaller space of test functions canceling on the problematic part of the boundary. The tricky part of this strategy is that the space of test functions must nevertheless be sufficiently rich to allow us to derive a sufficiently "good" theory of the weaker solutions. Such a strategy is followed in [4] that we recall now in more detail:

Let $T>0$, define the cylinder domain $Q_T:=[0,T[\times\overline{D}$ and the following space of test functions:

$$\mathcal{D}_{Q_T}:=\left\{\phi\in C_c^\infty(Q_T)\;\middle|\;\phi(t,x)=0 \text{ whenever } x\in\Gamma^+\cup\Gamma^0\right\} \tag{4}$$

We recall that $C_c^\infty(Q_T)$ denotes the space of smooth functions with compact support in Q_T. In [4], it is denoted by $C_0^\infty(Q_T)$ without being explicitly defined but the context implies the compact support in velocity. Indeed, the definition of a weak solution in [4] (see formula below) implicitly requires that $(\partial_t\phi+\mathcal{L}^*\phi)$ belongs to $L^\infty(Q_T)$ whereas a smooth function only "vanishing at infinity" may not have its derivative essentially bounded.

In [4] a function $f\in L^\infty\big([0,T];L^1(D)\big)$ is called a weak solution of (3) if it verifies [4, formula (1.7)]:

$$\int_D f(\partial_t\phi+\mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}x+\int_D f^{in}(x)\phi(0,x)\mathrm{d}x=0,\ \forall\phi\in\mathcal{D}_{Q_T}.$$

It is easy to check that a classical solution of (3) would be also such a weak solution. However, we emphasize that, as the space of test functions is smaller, the notion of weak solution is therefore weaker that the usual. In [4], it is claimed to have established the existence of a weak solution provided that f^{in} satisfies a few conditions [4, Theorem 1.1]. But the proof of a crucial lemma [4, Lemma 2.3] has a gap which turns out difficult to fill (see the discussion in Appendix A in the first version of [2]). We should also mention the work of Hwang et alii [9, 8] where they follow a similar strategy and, in addition, prove some results of Hölder-regularity of the solution on $\overline{D}$ and in the case of 1-D shows the exponential decay of the solution toward zero [9, Theorem 1.3]. Yet, this result about the asymptotic behavior in long time limit has not been extended in the multidimensional case studied in [8].

Meanwhile, in a series of articles, [10, 11], the initial-boundary problem dual problem $\mathcal{L}^*$, which is more significant from a probabilistic point of view, is studied. With probabilistic tools, the authors establish the existence of classical solutions with several interesting properties, such as its regularity, the existence of a strongly continuous semigroup, its compactness and so on.

The aim of the present paper is to take the same approach as in [4] while relying on [10, 11] to deduce by duality not only a theory of weaker solution, but also some properties such a compact positive strongly continuous semigroup which allows us to get some more precise results about the asymptotic behavior whatever the dimension of the space, thus extending the results of [9, 8]. Indeed, we retrieve the exponential decay toward zero shown in [9] but we show moreover a thermalization phenomenon. Before stating more precisely the results, we introduce the notations.

We first recall that a real valued function f defined on a local compact space E vanishing "at infinity" is defined by

$$\forall \varepsilon > 0, \exists \text{ compact } K_\varepsilon \subset E : \sup_{x \in K_\varepsilon^c} |f(x)| \leq \varepsilon.$$

We denote by $C_0^\infty(E)$ the set of smooth functions vanishing at infinity on E. We introduce the following space:

$$\mathcal{D}_{\tilde{Q}_T} := \left\{ \phi \in C_0^\infty\left(\tilde{Q}_T\right) \middle| \phi \in L^1(D; L^\infty((0,T))) \text{ and } (\partial_t + \mathcal{L}^*)\phi \in L^\infty(Q_T) \right\},$$

with

$$\tilde{Q}_T := [0,T[\times \left(\overline{\Omega} \setminus \left(\Gamma^+ \cup \Gamma^0\right)\right) \times \mathbb{R}^d.$$

Notice that by extension by continuity, we have $\phi \in C^b\left([0,T] \times \overline{D}\right)$ with $\phi(T,\cdot) = 0$ and $\phi = 0$ on $\Gamma^+ \cup \Gamma^0$ for any $\phi \in \mathcal{D}_{\tilde{Q}_T}$ wich implies $\mathcal{D}_{Q_T} \subset \mathcal{D}_{\tilde{Q}_T}$ with $\mathcal{D}_{Q_T}$ defined in (4) and in [4]. We now give a new definition of weak solution.

Definition 1. *A function $f \in L^\infty\left([0,T]; L^1(D)\right)$ is said to be a weak solution of (3) with initial condition f^{in} if it satisfies*

$$\int_D f(\partial_t \phi + \mathcal{L}^* \phi) \mathrm{d}t\mathrm{d}x + \int_D f^{in}(x)\phi(0,x)\mathrm{d}x = 0, \ \forall \phi \in \mathcal{D}_{\tilde{Q}_T}.$$

We can now state the first main result:

Theorem 2 (Existence and uniqueness)**.** *For any $f^{in} \in L^1(D)$, there exists a unique weak solution $f \in L^\infty\left([0,T]; L^1(D)\right)$ of (3) with initial condition f^{in} in the sense of Definition 1.*

The existence and the uniqueness of the weak solution together with the properties derived in [10, 11] implies the existence of a strongly continuous semigroup with some nice properties:

Theorem 3. *Let $f^{in} \in L^1(D)$, then:*

1. *(Semigroup property) There exists a positive, irreducible and immediately compact C_0 semigroup $(P_t)_{t \geq 0}$ such that for any $f^{in} \in L^1(D)$, $t \mapsto P_t f^{in}$ is the unique weak solution of (3) with initial condition f^{in} in the sense of Definition 1.*
2. *(Regularity) For any $t > 0$, we have $P_t f^{in} \in C^\infty(D) \cap C^b\left(\overline{D}\right)$*
3. *(Balanced exponential growth), let denotes the spectral bound of the operator $\mathcal{L}$*

$$s(\mathcal{L}) := \sup \{ Re\lambda \,|\, \lambda \in \sigma(\mathcal{L}) \}$$

then $s(\mathcal{L}) \in \sigma(\mathcal{L})$ with $s(\mathcal{L}) < 0$ and up to a multiplicative constant, there exists unique nonnegative functions φ and $\psi \in C^b\left(\overline{D}\right) \cap L^1(D)$ such that

$$\mathcal{L}\varphi = s(\mathcal{L})\varphi, \ \mathcal{L}^*\psi = s(\mathcal{L})\psi.$$

Moreover, normalizing ψ such that $\int_D \psi\varphi = 1$, we have

$$\lim_{t \to +\infty} \left|\!\left|\!\left| e^{-s(\mathcal{L})t} P_t - \varphi \otimes \psi \right|\!\right|\!\right|_{L^1(D)} = 0,$$

where

$$\forall f \in L^1(D) \ (\varphi \otimes \psi)(f) = \left(\int_D \psi f \right) \varphi,$$

and $|||\cdot|||_{L^1(D)}$ denotes the norm operator in $L^1(D)$.

Remark 1. *The balanced exponential growth naturally implies the exponential decay toward zero, but also a thermalization phenomenon: the rescaled C^0-semigroup converges uniformly toward a positive rank-one projector. In other words, before disappearing, the population of particles converges towards the same profile whatever the initial condition.*

About the spectral bound, it satisfies a minmax theorem which could be useful for estimates:

Corollary 4. *(Collatz-Wielandt characterization) Denote Q the set of almost everywhere positive functions in $L^1(D)$ and Σ_0 a subfamily of the Lebesgue algebra $\sigma(D)$ such that for any $E \in \Sigma_0$, $\int_D \mathbb{1}_E < +\infty$ and $\{\mathbb{1}_E \,|E \in \Sigma_0\}$ is weak-$\star$ total in $L^\infty(D)$ then for any $t > 0$*

$$e^{s(\mathcal{L})t} = \sup_{f\in Q} \inf_{E\in\Sigma_0} \frac{\int_E P_t f}{\int_E f} = \inf_{f\in Q} \sup_{E\in\Sigma_0} \frac{\int_E P_t f}{\int_E f}$$

As for the trace, since $f(t,\cdot) \in C^b\big(\overline{D}\big)$, we can define $\gamma f \in C^b(\partial D)$ and $\gamma_\pm f := \gamma f_{|\pm(p\cdot n_q)>0}$ but we have more:

Theorem 5. *Let γf the trace of f defined as above, then $\gamma f \in L^1\big(\partial\Omega \times \mathbb{R}^d; |p\cdot n_q| \mathrm{d}\sigma_{\partial\Omega}\mathrm{d}p\big)$ where $\mathrm{d}\sigma_{\partial\Omega}$ denotes the Lebesgue measure on $\partial\Omega$. And for any $t > 0$ and for any $\phi \in C^\infty(D) \cap C\big(\overline{D}\big)$ we have*

$$\int_D (\mathcal{L}f)\phi = \int_D f\mathcal{L}^*\phi + \int_{\partial D} \gamma f \phi (n_q \cdot p)\mathrm{d}p\mathrm{d}\sigma_{\partial\Omega}.$$

As the proofs of Theorem 2 and Theorem 3 rely heavily upon [10, 11] and upon the theory of Banach lattice, for sake of completeness, we will first recall the mots importants points of [10, 11] and of the theory of Banach lattice in Section 1. Then, we will establish Theorem 2, Theorem 3 and Corollary 4 in Section 2. After that, the proof of Theorem 5 will be given in Section 3.

1. A few recall about Banach lattices and probabilistic investigations of the kinetic Fokker-Planck equation

First, we recall the main elements of the theory of Banach lattice and a few of useful theorems around positive operators and semigroups of operators in a Banach lattice. Then, we recall and synthesize the main results of [10, 11].

1.1. **Banach lattices.**

First, we underline that $L^1(D)$ is a ordered set with the partiel order defined by

$$0 \le f \text{ if and only if } 0 \le f(x) \text{ a.e. on } D.$$

This ordered space is a Banach lattice, meaning that the ordering is compatible with the vector structure and is such that

$$\forall f, g \in L^1(D), \text{ if } |g| \le |f|, \text{ then } \|g\|_{L^1(D)} \le \|g\|_{L^1(D)}.$$

A semigroup $\mathcal{S} \equiv (S_t)_{t\ge 0}$ is said to be positive if it preserves positivity, i.e.

$$f \ge 0 \Rightarrow S_t f \ge 0, \text{for each } t \ge 0.$$

A closed subspace I of $L^1(D)$ is said to be an (order) ideal if

$$\forall (f,g) \in L^1(D) \times I, \ |f| \le |g| \Rightarrow f \in I.$$

The null subspace $\{0\}$ and $L^1(D)$ are ideals. A semigroup $\mathcal{S}$ is said to be irreducible if and only if the only ideals that are invariant for $\mathcal{S}$ are $\{0\}$ and $L^1(D)$. That is equivalent to the following property (for instance [15, Proposition III.8.3 p.186]):

$$\forall 0 \le f \in L^1(D), 0 \le g \in L^\infty(D) \text{ with } f \ne 0, g \ne 0, \exists t > 0 : \int_D g S_t f > 0.$$

In others words, an irreducible semigroup spreads the support of any nonnegative function through the entire space. A nonnegative function $f \in L^1(D)$ is a quasi-interior point if it is a.e. strictly positive on D. In $L^\infty(D)$ that is also a Banach lattice, a non negative function g is a quasi-interior point if there exists $\delta > 0$ such that $\delta \mathbb{1}_D \le g$ a.e. The ground bound of a C_0-semigroup $\mathcal{S}$ is

$$\omega_0(\mathcal{S}) := \inf_{\omega\in\mathbb{R}} \left\{\exists M \ge 0 : \|S_t\| \le M e^{\omega t}\right\}$$

It is known [1, Proposition 12.1 p. 181] that

$$\omega_0(\mathcal{S}) = \lim_{t\to} \frac{\log \|S_t\|}{t} = \inf_{t\geq 0} \frac{\log \|S_t\|}{t},$$

and that for any $t > 0$, the spectral radius of S_t is equal to:

$$r(S_t) = e^{t\omega_0(\mathcal{S})}. \tag{5}$$

and let A an unbounded operator, we recall its spectral bound is

$$s(A) := \sup \{\text{Re } \mu \,|\mu \in \sigma(A)\}$$

where $\sigma(A)$ is the spectrum of A. If $\mathcal{S}$ is a positive C_0-semigroup on $L^1(D)$ then its growth bound coincides with the spectral bound of its generator (see for instance [1, Theorem 12.17 p.193]). Moreover we have:

Lemma 6. *Let A be a positive, compact and irreducible operator on $L^1(D)$*

(1) *then $r(A) > 0$ and there exists $u \in L^1(D)$ and $v \in L^\infty(D)$ strictly positive a.e., unique up to a multiplicative constant such that*

$$\begin{cases} Au &= r(A)u, \\ A^{'}v &= r(A)v. \end{cases}$$

(2) *Besides, let $\mu > 0$ such there exists nonzero nonnegative $w \in L^1(D)$ with $Aw = \mu w$ then $\mu = r(A)$ and $w = u$ up to a multiplicative constant.*

Proof. The first point is classical in Banach lattice theory and is known as De Pagter's theorem [5]. The second point is almost immediate, but despite my best efforts, I have not found citable results in the literature. For the sake of completeness, I provide the evidence. As w is nonnegative and v strictly positive a.e., we have $\int_{\Omega\times\mathbb{R}^d} vw > 0$ Keeping that in mind, $Aw = \mu w$ implies

$$\int_D Awv = \mu \int wv$$

thus, as $\int_D vw > 0$ is strictly positive

$$r(A) = \mu,$$

and the equality of w with u is a direct consequence of the uniqueness of the eigenvector associated to $r(A)$. □

The theory of the asymptotic behavior of positive C_0-semigroups in Banach lattice is very rich. The following proposition is a synthesis of [17, Propositions 2.3, 2.4, 2.5 and Remark 2.2] and for the precise form of P, see Proposition III.8.5 p. 189 in [15].

Lemma 7. *Let $(S_t)_{t\geq 0}$ a irreducible, positive and compact semigroup on a Banach lattice E, then there exists a positive projection of rank one, $P = x_0 \otimes x_0^{'}$ where x_0 is, up to a constant, the eigenvalue associated to $s(A)$ and $x_0^{'} \in E^{'}$ such that $x_0^{'}(x_0) = 1$, and constants $\delta > 0$ and $M \geq 1$ such that*

$$\left|\left|\left| e^{-ts(A)}S_t - P \right|\right|\right|_{L^1(D)} \leq Me^{-\delta t}.$$

1.2. **A brief recalling of the results of Tony Lelièvre and alii.** We recall here the theory developed by Tony Lelièvre, Mouad Ramil, and Julien Reygnier in a probabilistic framework for the kinetic Fokker-Planck in a bounded domain with absorbing [10, 11]. From here, we use there notations except the ones of the operators $\mathcal{L}$ and $\mathcal{L}^*$. Indeed, in [10, 11], $\mathcal{L}$ (respectively, $\mathcal{L}^*$) corresponds to $\mathcal{L}^*$ (respectively $\mathcal{L}$) in the present paper, because the operator $\mathcal{L}^*$ defined in (2).

Let $(X_t^x = (q_t^x, p_t^x))_{t\geq 0}$ the Langevin process starting at $x \in D$ at $t = 0$ defined by the following SDE:

$$\begin{cases} \mathrm{d}q_t^x & = p_t^x \mathrm{d}t, \\ \mathrm{d}p_t^x & = F(q_t^x)\mathrm{d}t - \gamma p_t^x \mathrm{d}t + \sigma \mathrm{d}B_t, \\ (q_0^x, p_0^x) & = x, \end{cases}$$

where $(B_t)_{t\geq 0}$ is a $d-$dimensional Brownian motion. Define also the first exit time from D of the process $(X_t^x)_{t\geq 0}$:

$$\tau_\partial^x := \inf\{t > 0 : X_t^x \notin D\}.$$

We denote the following unbounded operator:

$$\mathcal{L}_\gamma^* \equiv p \cdot \nabla_q + F(q) \cdot \nabla_p - \gamma p \cdot \nabla_p + \frac{\sigma^2}{2}\Delta_p.$$

It is the operator defined in (2) with the subscript γ in order to emphasize its dependence on the parameter γ. It is the infinitesimal generator of the Langevin process whereas (3) describes the evolution of Langevin process absorbed at ∂D. That explains why in a probabilistic framework such as in [10, 11], the initial problem with $\mathcal{L}_\gamma^*$ is most often studied while in mathematical physics, it is (3) that arouses the most interest.

That being said, the first main result [10, Theorem 2.10] is then

Theorem 8. *Let $\gamma \in \mathbb{R}$ and let $f \in C^b(D \cup \Gamma^-)$ and define*

$$u_\gamma : (t, x) \mapsto \mathbb{E}\Big[\mathbb{1}_{\tau_\partial^x > t} f(X_t^x)\Big]$$

then

(1) *(Initial and boundary values) the function u satisfies :*

$$u_\gamma(0, x) = \begin{cases} f(x) & \text{if } x \in D \cup \Gamma^-, \\ 0 & \text{if } x \in \Gamma^+ \cup \Gamma^0, \end{cases}$$

and

$$\forall t > 0, \forall x \in \Gamma^+ \cup \Gamma^0, u(t, x) = 0.$$

(2) *(Continuity) we have :*

$$u_\gamma \in C^b\big((\mathbb{R}_+ \times \overline{D}) \setminus (\{0\} \times (\Gamma^- \cup \Gamma^0))\big)$$

(3) *(Interior regularity) $u \in C^\infty(\mathbb{R}_+^*)$ and, for all $t > 0$, $x \in D$*

$$\partial_t u_\gamma = \mathcal{L}_\gamma^* u_\gamma$$

Moreover, $f \mapsto \Big((t, x) \mapsto \mathbb{E}\Big[\mathbb{1}_{\tau_\partial^x > t} f(X_t^x)\Big]\Big)$ is a semigroup with nice properties including the existence of a transition kernel. We sum up the main results (specially Theorem 2.7 and Theorem 2.9 in [11] and Theorem 2.14 in [10]).

Theorem 9. *Let $\gamma \in \mathbb{R}$, there exists*

$$(t, x, y) \mapsto p_t^\gamma(x, y) \in C^\infty(\mathbb{R}_+^* \times D \times D) \cap C(\mathbb{R}_+^* \times \overline{D} \times \overline{D})$$

which satisfies for all $t > 0$

- $p_t^\gamma(x,y) > 0$ *for all* $x \notin \Gamma^+ \cup \Gamma^0$ *and* $y \notin \Gamma^- \cup \Gamma^0$
- $p_t^\gamma(x,y) = 0$ *if* $x \in \Gamma^+ \cup \Gamma^0$ *or* $y \in \Gamma^- \cup \Gamma^0$

Moreover, for $f \in C^b\big(\overline{D}\big)$*, the functions* u_γ, v_γ *defined by*

$$\forall t > 0, \forall x \in D, u_\gamma(t,x) := \int_D p_t^\gamma(x,y) f(y) \mathrm{d}y, \ v_\gamma(t,x) := \int_D p_t^\gamma(y,x) f(y) \mathrm{d}y$$

are in $C^\infty\big(\mathbb{R}_+^* \times D\big)$ *and satisfy*

$$\begin{cases} \partial_t u_\gamma & = \mathcal{L}_\gamma^* u_\gamma, \\ u_\gamma & = 0 \text{ on } \Gamma^+, \\ u_\gamma(0,\cdot) = f; \end{cases} \tag{6}$$

and

$$\begin{cases} \partial_t v_\gamma & = \mathcal{L}_\gamma v_\gamma, \\ v_\gamma & = 0 \text{ on } \Gamma^-, \\ v_\gamma(0,\cdot) = f. \end{cases} \tag{7}$$

Besides, the solution of (6) *satisfies Maximum principle. For any* $t > 0$*, the operator*

$$Q_t^\gamma f : x \in \overline{D} \mapsto \mathbb{E}\Big[\mathbb{1}_{\tau_\partial^x > t} f(X_t^x)\Big] = \int p_t^\gamma(y,x) f(y) \mathrm{d}y$$

is well-defined and forms a positive and irreducible semigroup of compact operators $(Q_t^\gamma)_{t>0}$ *on* $L^n(D)$ *to* $L^n(D)$ *and* $C^b\big(\overline{D}\big)$ *to* $C^b\big(\overline{D}\big)$*. Moreover, it maps* $L^n(D)$ *into* $C^b\big(\overline{D}\big)$ *continuously for any* $n \in [1, \infty]$*.*

Eventually, we recall here a special case of Theorem 2.13 in [11], the last property being a consequence of the irreducibility of the semigroup (see Lemma 4.3 in [10])

Theorem 10. *Under assumptions above, there exists* $\lambda_\gamma > 0$ *such that* $\lambda_\gamma + \gamma d > 0$ *and positive functions* $\varphi_\gamma, \psi_\gamma \in C^b\big(\overline{D}\big) \cap L^1(D)$*, unique up to multiplicative constants such*

$$\begin{cases} \mathcal{L}\varphi_\gamma + \lambda_\gamma \varphi_\gamma & = 0 \\ \varphi_\gamma & = 0 \text{ on } \Gamma^- \cup \Gamma^0 \end{cases} \tag{8}$$

and

$$\begin{cases} \mathcal{L}^*\psi_\gamma + \lambda_\gamma \psi_\gamma & = 0 \\ \psi_\gamma & = 0 \text{ on } \Gamma^+ \cup \Gamma^0. \end{cases} \tag{9}$$

2. Proof of Theorem 2 and Theorem 3

We establish here Theorem 2 and Theorem 3. The proof of Theorem 2 is split into several steps.

2.1. **A semigroup for the kinetic Fokker-Planck equation.** In order to use the results of [10, 11] recalled above, first notice that if $u_{-\gamma}$ is the solution associated to $\mathcal{L}_{-\gamma}^*$ as in Theorems 8 and 9 then it is $C^\infty\big(\mathbb{R}_+^* \times D\big)$ and thus we can derive it in classical sense. It is then easy to check that the function

$$v(t,q,p) := e^{d\gamma t} u_{-\gamma}(t,q,-p)$$

solves (3) in the classical sense. That remark motivates the introduction of the following operator defined on the set of measurable functions $L^0(D)$

$$(Sf)(q,p) := f(q,-p) \ \forall f \in L^0(D).$$

The operator S is obviously bounded on $L^n(D)$ and $C^b\big(\overline{D}\big)$, moreover it is an involution, which imply that the operators defined by

$$P_t^\gamma := e^{d\gamma t} S Q_t^{-\gamma} S, \tag{10}$$

with Q_t^γ defined in Theorem 9, form a semigroup, which, together with Theorem 2.8 in [11], implies the following theorem:

Theorem 11. *The operators $(P_t^\gamma)_{t>0}$ form a positive and irreducible semigroup of compact operators on $L^n(D)$ for any $n \in [1,\infty]$ and $C^b\big(\overline{D}\big)$. Moreover, it maps $L^n(D)$ into $C^b\big(\overline{D}\big)$. And for any $f \in L^n(D)$, $v_\gamma := P_t^\gamma f$ is in $C^\infty\big(\mathbb{R}_+^* \times D\big)$ and satisfies*

$$\begin{cases} \partial_t v_\gamma & = \mathcal{L}_\gamma v_\gamma \text{ on } D, \\ v_\gamma & = 0 \text{ on } \Gamma^-, \\ v_\gamma(0,\cdot) & = f. \end{cases}$$

Moreover, we have the identity for any $f \in C^b\big(\overline{D}\big)$ and $v_\gamma = P_t^\gamma f$

$$v_\gamma = \int_D p_t^\gamma(y,x) f(y) \mathrm{d}y.$$

where p_t^γ is the transition kernel defined in Thorem 9.

From here on, we drop the subscript γ whenever it is obvious from the context what the value of γ is, P_t for P_t^γ and so on. We now show that P_t is a substochastic semigroup:

Lemma 12. *For any $t > 0$*

$$\|\!|P_t|\!\|_{L^1(D)} \leq 1.$$

As consequence, $\forall f \in L^1(D),\ t \mapsto P_t f \in L^\infty\big(\mathbb{R}_+; L^1(D)\big)$.

Proof. By density, it suffices to show that for any $t > 0$ and for any $f \in C^b\big(\overline{D}\big)$, we have

$$\|P_t f\|_{L^1(D)} \leq \|f\|_{L^1(D)}.$$

By Theorem 11 we have for any $f \in C^b\big(\overline{D}\big)$

$$P_t f = \int_D p_t^\gamma(y,x) f(y) \mathrm{d}y$$

thus for any $g \in L^\infty(D)$ by Fubini-Tonelli Theorem

$$\begin{aligned} \int_D P_t f g \mathrm{d}x &= \int_D \int_D p_t(y,x) f(y) g(x) \mathrm{d}y \mathrm{d}x \\ &= \int_D f(y) \bigg(\int_D p_t(y,x) g(x) \mathrm{d}x \bigg) \mathrm{d}y \end{aligned}$$

Thus by Maximum principle applied to $\int_D p_t(y,x) g(x) \mathrm{d}x$ that is solution of (6) by Theorem 9, we have for any $g \in L^\infty(D)$

$$\left| \int_D P_t f g \right| \leq \|f\|_{L^1(D)} \|g\|_{L^\infty(D)}$$

which implies by Hahn-Banach Theorem:

$$\|P_t f\|_{L^1(D)} \leq \|f\|_{L^1(D)}$$

which is the desired result. As for the eigenvalue ϕ we have the following immediate lemma: □

Lemma 13. *Let $\lambda > 0$ and φ be defined as in Theorem 10, then*

$$P_t \varphi = e^{-\lambda t} \varphi, \forall t > 0.$$

We can now show that $(P_t)_{t\geq 0}$ is a strongly continuous semigroup:

Proposition 14. *The operators $(P_t)_{t\geq 0}$ with $P_0 = I$ and $(P_t)_{t>0}$ defined above form a C_0-semigroup of $L^1(D)$. As a consequence, the semigroup $(Q_t)_{t\geq 0}$ associated to the adjoint Cauchy problem is also strongly continuous on $\mathbb{R}_+$ in $L^1(D)$.*

Proof. It is already known that $(P_t)_{t>0}$ is strongly continuous on $\mathbb{R}_+^*$ by Theorem 9. It remains to show that it is strongly continuous at $t = 0$. By Proposition I.5.3 p. 38 in [6], it suffices to show that there exist $\delta > 0,\ M \geq 1$, and a dense subset $\mathcal{D} \subset L^1(D)$ such that

(1) $\|P_t\|_{L^1(D)} \leq M$ for all $t \in [0, \delta]$,
(2) $\lim_{t\to 0^+} P_t g = g$ strongly in $L^1(D)$ for all $g \in \mathcal{D}$.

By Lemma 12, we know that $\|P_t\|_{L^1(D)} \leq 1$. It remains to establish (2) for $\mathcal{D} = C_c(D)$. Let $g \in C_c(D)$ that we assume nonnegative and denote $h(t, x) := P_t g(x)$ for any $x \in D$, we know that $h \in C(\mathbb{R}_+ \times D)$, thus for any $t \in [0, T]$, $x \mapsto h(t, x)$ is measurable. In the same way, for every $x \in D$, $t \mapsto h(t, x)$ is continuous. Let φ defined in Theorem 10 and Lemma 13. As $g \in C_c(D)$ and $\varphi > 0$ everywhere, there exists $C > 0$ such that

$$g \leq C\varphi \text{ a.e. on } D$$

and by positivity of P_t and Lemma 13, we have

$$h \leq Ce^{-\lambda t}\varphi \leq C\varphi \text{ a.e. on } D, \forall t \geq 0$$

with $\varphi \in L^1(D)$. In others words, h is dominated on $[0, T]$ by a integrable function on D which implies that:

$$t \to \int_D h(t, x)\mathrm{d}x \text{ is continuous on } [0, T].$$

As h is nonnegative, Schaffé's lemma implies:

$$\|P_t g - g\|_{L^1(D)} \to 0 \text{ as } t \to 0^+ \text{ for any nonnegative } g \in C_c(D) \tag{11}$$

Now consider $g \in C_c(D)$ that can change sign and let $g = g_+ - g_-$ with $g_\pm \in C_c(D)$ and $g_\pm \geq 0$. We have

$$\|P_t g - g\|_{L^1(D)} \leq \|P_t g_+ - g_+\|_{L^1(D)} + \|P_t g_- - g_-\|_{L^1(D)}$$

thus by (11)

$$\|P_t g - g\|_{L^1(D)} \to 0 \text{ as } t \to 0^+ \ \forall g \in C_c(D),$$

Consequently, $(P_t)_{t\geq 0}$ is a C_0-semigroup of $L^1(D)$.

As for $(Q_t)_{t\geq 0}$, it is a direct consequence of (10), i.e.

$$Q_t^\gamma = e^{d\gamma t} S P_t^{-\gamma} S.$$

□

Let us make also an important remark

Remark 2. *For any $t > 0$, the kernel p_t satisfies a Gaussian upper-bound (see* [10, Corollary 2.22] *or* [11, Theorem 2.7]*) that is established by studying the Langevin process on the space $\mathbb{R}^{2d}$ after extending the driving force over the whole space. The authors have deduced an inequality valid on the whole space which they then truncate on $\Omega \times \mathbb{R}^d$. As the two functions are continuous, the proof of the authors implies that the inequality is valid on $\overline{\Omega} \times \mathbb{R}^d$ and thus on $\partial\Omega \times \mathbb{R}^d$.*

The recall being done, for the sake of readability, we split the proof of Theorem 2 into several steps. First we show that for any $f^{in} \in L^1\big(\Omega \times \mathbb{R}^d\big)$, $t \mapsto P_t f^{in}$ belongs to the good function space.

Lemma 15. *Let $f^{in} \in L^1(D)$ and $f := P_t f^{in}$ with $(P_t)_{t>0}$ defined in Theorems 9 and 8. Then $f \in L^\infty\big([0, T]; L^1(D)\big)$.*

Proof. It is simply a direct consequence that for any $t>0$, we have $\left\|P_t f^{in}\right\|_{L^1(D)} \leq \left\|f^{in}\right\|_{L^1(D)}$. □

We now show that for any smooth initial condition, the solution in the sense of Theorems 9 and 8 is also a weak solution.

Lemma 16. *Let $f^{in} \in C^b(D \cup \Gamma^+)$ and $u := P_t f^{in}$ with $(P_t)_{t>0}$ defined in Theorems 9 and 8, then for any $\phi \in \mathcal{D}_{Q_T}$ we have*

$$\int_{Q_T} u(\partial_t \phi + \mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}q\mathrm{d}p + \int_D f^{in}(q,p)\phi(0,q,p)\mathrm{d}q = 0.$$

In others words, u is a weak solution of (3)

Proof. We cannot apply integration by parts (Green Theorem) to the cylindric domain Q_T because f is not C^1 on Q_T. Therefore, we introduce $\varepsilon \in (0,T)$ and the subset:

$$Q_{\varepsilon,T} := [\varepsilon, T] \times D.$$

Let $f^{in} \in C^b(D \cup \Gamma^+)$ and let u be defined in Theorem 8. We have $u \in C^b\big(\overline{Q}_{\varepsilon,T}\big) \cap C^\infty(Q_{\varepsilon,T})$, so that by Green Theorem applied to the cylindric domain (see Theorem 5.5 p. 47 in [13]), we have for any $\phi \in \mathcal{D}_{\tilde{Q}_T}$

$$\begin{aligned} 0 &= \iint_{Q_{\varepsilon,T}} (\partial_t u - \mathcal{L}u)\phi \mathrm{d}t\mathrm{d}x \\ &= \iint_{Q_{\varepsilon,T}} (-\partial_t \phi - \mathcal{L}^*\phi) u \mathrm{d}t\mathrm{d}x - \iint_D u(\varepsilon,x)\phi(\varepsilon,x)\mathrm{d}x \end{aligned}$$

We know that, since $u \in C^b\big(\big(\mathbb{R}_+ \times \overline{D}\big) \setminus \big(\{0\} \times \big(\Gamma^- \cup \Gamma^0\big)\big)\big)$,

$$u(\varepsilon,\cdot)\phi(\varepsilon,\cdot) \to f^{in}\phi(0,\cdot) \text{ as } \varepsilon \to 0^+ \text{ a.e. on } \Omega \times \mathbb{R}^d.$$

Moreover

$$|u(\varepsilon,\cdot)\phi(\varepsilon,\cdot)| \leq \left\|f^{in}\right\|_{L^\infty(D)} \left(\sup_{\varepsilon \geq 0} \phi(\varepsilon,\cdot)\right).$$

As $(\sup_\varepsilon \phi(\varepsilon,\cdot)) \in L^1\big(\Omega \times \mathbb{R}^d\big)$ since it is in $\mathcal{D}_{\tilde{Q}_T}$, we can apply dominated convergence to get:

$$\iint_D u(\varepsilon,x)\phi(\varepsilon,x)\mathrm{d}x \to_{\varepsilon \to 0^+} \iint_D f^{in}\phi \mathrm{d}x.$$

In the same way, we have

$$\iint_{Q_{\varepsilon,T}} (\partial_t \phi + \mathcal{L}^*\phi) u \mathrm{d}t\mathrm{d}x \to_{\varepsilon \to 0^+} \iint_{Q_T} (\partial_t \phi + \mathcal{L}^*\phi) u \mathrm{d}t\mathrm{d}x$$

Hence u satisfies for all $\phi \in \mathcal{D}_{\tilde{Q}_T}$

$$\iint_{Q_T} (\partial_t \phi + \mathcal{L}^*\phi) u \mathrm{d}t\mathrm{d}x + \iint_D f^{in}(x)\phi(0,x)\mathrm{d}x = 0$$

which is the desired result. □

Let now extend the result above to the case of $f^{in} \in L^1(D)$.

Lemma 17. *Let $f^{in} \in L^1(D)$ and for any $t>0$, $f(t,\cdot) := P_t f^{in}$ then $f \in L^\infty\big([0,T]; L^1(D)\big)$ and for any $\phi \in \mathcal{D}_{Q_T}$ we have*

$$\int_{Q_T} f(\partial_t \phi + \mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}q\mathrm{d}p + \int_D f^{in}(q,p)\phi(0,q,p)\mathrm{d}q = 0.$$

In others words, f is a weak solution of (3)

Proof. Let $f^{in} \in L^1(D)$. We know that for any $t > 0$, $P_t \in \mathcal{L}(L^p(D))$. Keeping it in mind, take $f_n^{in} \in C_0^\infty(D)$ such that $f_n^{in} \to_{n\to+\infty} f^{in}$ in $L^1(D)$. Then, denoting $f_n := P_t f_n^{in}$ we have by Lemma 16 for any $\phi \in \mathcal{D}_{\tilde{Q}_T}$ and for any $n \in \mathbb{N}$:

$$\int_{Q_T} f_n(\partial_t\phi + \mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}q\mathrm{d}p + \int_D f_n^{in}(q,p)\phi(0,q,p)\mathrm{d}q = 0. \tag{12}$$

By Lebesgue partial converse theorem (see Theorem 4.9 p. 94 in [3]), we can moreover assume that, up to a subsequence, there exists $g \in L^1(D)$ such that:

$$\left|f_n^{in}(x)\right| \leq g \text{ a.e. in } x \in D,$$

and

$$f_n^{in} \to f^{in} \text{ a.e. in } x \in D.$$

That implies immediately:

$$\int_D f_n^{in}(q,p)\phi(0,x)\mathrm{d}x \to \int_D f^{in}(x)\phi(0,x)\mathrm{d}x \text{ as } n \to +\infty, \tag{13}$$

and

$$\forall (t,x) \in [0,T]\times D,\ \int_D p_t(y,x)f_n^{in}\mathrm{d}y \to_{n\to+\infty,} \int_D p_t(y,x)f^{in}\mathrm{d}y. \tag{14}$$

Besides, observe that

$$0 \leq f_n(t,x) \leq \int_D f_n^{in}(y)p_t(y,x)\mathrm{d}y \leq \int_D g(y)p_t(y,x)\mathrm{d}y$$

with $\int_D g(y)p_t(y,x)\mathrm{d}y \in L^1([0,T]\times D)$ and by (14)

$$\int_D f_n^{in}(y)p_t(y,x)\mathrm{d}y \to \int_D f^{in}(y)p_t(y,x)\mathrm{d}y \text{ a.e. in } (t,x) \in [0,T]\times D$$

as $n \to +\infty$. Therefore by dominated convergence we have

$$f_n \to f \text{ in } L^1([0,T]\times D) - \text{strong}$$

as $n \to +\infty$. That implies:

$$\int_{Q_T} f_n(\partial_t\phi + \mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}q\mathrm{d}p \to \int_{Q_T} f(\partial_t\phi + \mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}q\mathrm{d}p \tag{15}$$

as $n \to +\infty$. Applying (13) and (15) to (12) shows that f is a weak solution to the initial boundary problem (3) with f^{in} as initial condition.

As for the uniqueness of the weak solution, as the problem is linear, it suffices to show that if $f \in L^1(Q_T)$ is such that

$$\int_{Q_T} f(\partial_t\phi + \mathcal{L}^*\phi)\mathrm{d}t\mathrm{d}q\mathrm{d}p = 0,\ \forall \phi \in \mathcal{D}_{\tilde{Q}_T},$$

then

$$f = 0.$$

We remark that if for any $\psi \in C_c((0,T)\times D)$, there exists $\phi \in \mathcal{D}_{\tilde{Q}_T}$ solution of the backward problem

$$\begin{cases} \partial_t\phi + \mathcal{L}^*\phi & = \psi \\ \phi_{|\Gamma^0\cup\Gamma^-} & = 0 \\ \phi(T,\cdot) & = 0 \end{cases} \tag{16}$$

that would imply that $f = 0$ in distributional sense and thus $f = 0$ in $L^1(Q_T)$. This leads us to consider $\tilde{\phi}$ solution of the forward problem:

$$\begin{cases} \partial_t \tilde{\phi} - \mathcal{L}^* \tilde{\phi} & = \psi \\ \tilde{\phi}_{|\Gamma^0 \cup \Gamma^-} & = 0 \\ \tilde{\phi}(0, \cdot) & = 0 \end{cases} \tag{17}$$

It is obvious that $t \mapsto \tilde{\phi}(T-t, \cdot)$ is solution of (16). As $(Q_t)_{t \geq 0}$ is strongly continuous by Proposition 14, we have by Duhamel formula [6, Corollary III.1.7 p. 161]

$$\tilde{\phi} = \int_0^t Q_{t-s} \psi(s) \mathrm{d}s$$

thus

$$\begin{aligned} \phi &= \int_0^{T-t} Q_{T-t-s} \psi(s) \mathrm{d}s \\ &= \int_t^T Q_{T-s} \psi(s-t) \mathrm{d}s \end{aligned}$$

It remains to show that $\phi \in \mathcal{D}_{\tilde{Q}_T}$ to conclude. By linearity, we can assume that ψ is nonnegative. That being said, first, by the properties of the function p_t (Theorems 9 and 8 above), we have $\phi \in C_0^\infty\big(\tilde{Q}_T\big)$. Besides, we have obviously $(\partial_t + \mathcal{L}^*)\phi \in L^\infty(Q_T)$. Eventually, let ψ_λ being the positive eigenvector associated to $\mathcal{L}^*$, see (9) in Theorem 10, with eigenvalue λ and $\int_D \psi_\lambda = 1$. Let K compact of $(0,T) \times D$ such that supp $\psi \subset K$. As There ψ_λ is positive everywhere on D and constant in time, there exists $c > 0$ such that $\psi_\lambda \geq c$ for any $(t,x) \in K$. Therefore, there exists $C > 0$ such that

$$\psi \leq C \psi_\lambda \text{ everywhere on } (0,T) \times D.$$

As $(Q_t)_{t>0}$ are positive operators, the properties of ψ_λ and the inequality above imply

$$Q_{t-s} \psi(s) \leq C e^{-\lambda(t-s)} \psi_\lambda.$$

Integrating the inequality above in time $s \in [0,t]$ leads to:

$$\tilde{\phi} \leq \frac{C}{\lambda}\big(1 - e^{-\lambda t}\big)\psi_\lambda,$$

thus

$$0 \leq \phi \leq \frac{C}{\lambda}\Big(1 - e^{-\lambda(T-t)}\Big)\psi_\lambda$$

which implies that $\phi \in L^1(D; L^\infty((0,T)))$. Consequently, $\phi \in \mathcal{D}_{\tilde{Q}_T}$ which is the desired conclusion. □

2.2. **Proof of Theorem 3.** Now, we establish Theorem 3

(1) By Theorem 2, for any $f^{in} \in L^1(D)$, there exists a unique weak solution f in sense of Definition 1, thus we have $f(t, \cdot) = P_t f^{in}$ with $(P_t)_{t \geq 0}$ defined in (10). By Theorem 11, Lemma 12 and Proposition 14, it is a positive, irreducible, contractive and compactC_0- semigroup.
(2) By Theorem 8 with the property of semigroup, we have for any $f^{in} \in L^1(D)$, $\forall t > 0$, $P_t f^{in} \in C^\infty(D) \cap C^b\big(\overline{D}\big)$.
(3) With the properties of the semigroup established above, the balanced growth law of $(P_t)_{t \geq 0}$ is a direct consequence of Lemma 7 with the eigenvectors identified in Theorem 10.

Now we give the proof of Corollary 4

Proof. We know that

$$r(P_t) = e^{\omega_0 t}$$

where ω_0 is the growth bound of the semigroup (Proposition 2.2 p. 251 in [6]). As P_t is a positive C_0-semigroup in $L^1(D)$, it is equal to the spectral bound of its generator (Theorem VI.1.15 p. 358 in [6]). As P_t is moreover irreducible and compact, the result is then a direct consequence of Theorem 1 in [14]. □

3. A Trace theory for the kinetic Fokker-Planck equation

We now establish a trace theory for the kinetic Fokker-Planck equation. First, using the regularity of the weak solution, we can easily define its trace:

Lemma 18. *Let f be the weak solution with initial condition $f^{in} \in L^1(D)$. The trace of f on $\mathbb{R}_+^* \times \partial D$ is well defined with $\gamma f := f_{|\partial D}$ and $\gamma_\pm := \gamma f_{|\pm(p\cdot n_q)>0}$. Moreover we have*

$$\gamma_- f = 0 \text{ and } \gamma_+ \in L^1\big(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| \mathrm{d}q\mathrm{d}p\big).$$

Proof. By Theorem 9, we know that for any $t > 0$, P_t maps continuously $L^1(D)$ into $C^b\big(\overline{D}\big)$. Thus we can define $\gamma f := f_{\partial\Omega\times\mathbb{R}^d}$ and $\gamma_\pm := \gamma f_{|\pm(p\cdot n_q)>0}$.

Besides, by semigroup property, we have for any $t > 0$, $P_t f^{in} = P_{t/2}\big(P_{t/2} f^{in}\big)$ with $P_{t/2} f^{in} \in C^b\big(\overline{D}\big)$ which implies by Theorem 8 that $\gamma_- f = 0$.

Now show that $\gamma_+ f \in L^1\big(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| \mathrm{d}q\mathrm{d}p\big)$. As for any $t > 0$, $\gamma_+ f \in C^b(\partial D)$, it is measurable. Now we have by the semigroup property and by continuity:

$$\gamma_+ f(t,x) = \int_D p_{t/2}(x,y) f\left(\frac{t}{2}, y\right) \mathrm{d}y$$

with $y \mapsto f\big(\frac{t}{2}, y\big) \in C^b(\partial D)$, thus

$$|\gamma_+ f(t,x)| \le \left\| f\left(\frac{t}{2}, \cdot\right)\right\|_{C^b(\partial D)} \int_D p_{t/2}(x,y)\mathrm{d}y.$$

The Gaussian estimates on $p_{t/2}$ (see [10, Corollary 2.22] and Remark 2) implies that

$$x \in \partial D \mapsto \int_D p_{t/2}(x,y)\mathrm{d}y \in \ L^1\big(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| \mathrm{d}\sigma_{\partial\Omega}\mathrm{d}p\big),$$

hence the desired result. □

The existence of the trace and the regularity of the weak solution implies that the integration by parts, also known as Green's formula, is satisfied:

Lemma 19. *Let f be the weak solution with initial condition $f^{in} \in L^1(D)$ and let γf be its trace. Then, it satisfies Green's formula, meaning that for any $t > 0$ and for any $\phi \in C^\infty(D) \cap C\big(\overline{D}\big)$ we have*

$$\int_D (\mathcal{L} f)\phi = \int_D f \mathcal{L}^* \phi + \int_{\partial D} \gamma f \phi (n_q \cdot p) \mathrm{d}p\mathrm{d}\sigma_{\partial\Omega}.$$

Proof. By unicity of the weak solution, we know that for any $t > 0$, $f = P_t f^{in}$ and thus $f \in C^\infty(D) \cap C\big(\overline{D}\big)$ with $\gamma f \in L^1\big(\partial\Omega \times \mathbb{R}^d; |n_q \cdot| \mathrm{d}\sigma_{\partial\Omega}\mathrm{d}p\big)$ by Lemma 18. Hence, the classical theorem of Green's formula (Integration by parts) [13, Th 5.5 p. 47] applies here and leads to the desired result. □

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