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# ON THE EXISTENCE OF WEAK SOLUTION OF THE KINETIC FOKKER-PLANCK EQUATION IN A BOUNDED DOMAIN WITH ABSORBING BOUNDARY

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## INTRODUCTION

In the present paper, we discuss the existence and unicity of weak solutions of the kinetic Fokker-Planck equation in a bounded domain with absorbing boundary. More precisely, let an integer  $d \geq 2$  and let  $\Omega$  be a bounded, open and smooth subset of  $\mathbb{R}^d$ . As its boundary is smooth, we can define for any  $q \in \partial\Omega$  the outer normal vector  $n_q$ . Thus, we can decompose  $\partial\Omega = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$  with:

$$\begin{aligned}\Gamma^+ &= \{(q, p) \in \partial\Omega \times \mathbb{R}^d \mid p \cdot n_q > 0\}, \\ \Gamma^- &= \{(q, p) \in \partial\Omega \times \mathbb{R}^d \mid p \cdot n_q < 0\}, \\ \Gamma^0 &= \{(q, p) \in \partial\Omega \times \mathbb{R}^d \mid p \cdot n_q = 0\}.\end{aligned}$$

Let  $\sigma \in \mathbb{R}_+^*$ ,  $\gamma \in \mathbb{R}$  and  $F \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ , we introduce the operator:

$$(1) \quad \mathcal{L} = -p \cdot \nabla_q - F(q) \cdot \nabla_p + \gamma \nabla_p \cdot (p \cdot) + \frac{\sigma^2}{2} \Delta_p,$$

and its formal adjoint in  $L^2(dx)$

$$(2) \quad \mathcal{L}^* = p \cdot \nabla_q + F(q) \cdot \nabla_p - \gamma p \cdot \nabla_p + \frac{\sigma^2}{2} \Delta_p.$$

Following [8, 9], we denote  $D := \Omega \times \mathbb{R}^d$ ,  $\bar{D} = \bar{\Omega} \times \mathbb{R}^d$  and  $\partial D := \partial\Omega \times \mathbb{R}^d$ . The variable  $x \in D$  represents the couple  $(q, p) \in D$  and  $dx = dqdp$ . We will use preferably this notation except when the two variables  $q$  and  $p$  do not have symmetrical roles, such as the beginning of subsection 2.1.

The so-called kinetic Fokker-Planck equation is the following initial-boundary problem  $f \equiv f(t, x)$ :

$$(3) \quad \begin{cases} \partial_t f &= \mathcal{L}f \text{ on } (0, T) \times D, \quad T > 0, \\ f(t, x) &= 0 \text{ on } (0, T) \times \Gamma^-, \\ f(0, x) &= f^{in}(x) \text{ on } D. \end{cases}$$

It models the collective behavior of particles evolving inside  $\Omega$  with velocity in  $\mathbb{R}^d$ , with initial distribution  $f^{in}$ , submitted to an exterior force  $F$  with a thermal bath encoded by the term of diffusion in velocity  $\frac{\sigma^2}{2} \Delta_p$  and the friction term  $\gamma \nabla_p \cdot (p \cdot)$ . More precisely, the quantity  $f(x)dx \equiv f(t, q, p)dtdqdp$  represents the number of particles at time interval  $[t, t + dt]$ , in the volume  $B(q, dq) \subset \Omega$  and with velocities in  $B(p, dp) \subset \mathbb{R}^d$ . The homogeneous boundary condition  $f(t, x) = 0$  on  $(0, T) \times \Gamma^-$  means that whenever a particle coming from the interior of  $\Omega$  hits the boundary, it exits and disappears forever.

In the case of  $\Omega = \mathbb{R}^d$ , the theory of the kinetic Fokker-Planck equation is well-established. The bounded case is another matter because of the trace issue in kinetic theory. For more details, we refer to the enlightening discussion in Stéphane Mischler's "habilitation à diriger les recherches" p. 29-32. Let us just recall here that a possible strategy is to bypass the trace issue by define a weaker notion of solution with a

smaller space of test functions canceling on the problematic part of the boundary. This strategy is followed in [5] that we recall now in more detail:

Let  $T > 0$ , define the cylinder domain  $Q_T := [0, T[ \times \overline{D}$  and the following space of test functions:

$$\mathcal{D}_{Q_T} := \{ \phi \in C_c^\infty(Q_T) \mid \phi(t, x) = 0 \text{ whenever } x \in \Gamma^+ \cup \Gamma^0 \}$$

We recall that  $C_c^\infty(Q_T)$  denotes the space of smooth functions with compact support in  $Q_T$ . In [5], it is denoted by  $C_0^\infty(Q_T)$  without being explicitly defined but the context implies the compact support in velocity. Indeed, the definition of a weak solution in [5] (see below) implicitly requires that  $(\partial_t \phi + \mathcal{L}^* \phi)$  belongs to  $L^\infty(Q_T)$  whereas a smooth function "vanishing at infinity" may not have its derivative essentially bounded.

In [5] a function  $f \in L^\infty([0, T]; L^1(D))$  is called a weak solution of (3) if it verifies [5, formula (1.7)]:

$$\int_D f(\partial_t \phi + \mathcal{L}^* \phi) dt dx + \int_D f^{in}(x) \phi(0, x) dx = 0, \quad \forall \phi \in \mathcal{D}_{Q_T}.$$

It is easy to check that a classical solution of (3) would be also such a weak solution. However, we emphasize that, as the space of test functions is smaller, the notion of weak solution is therefore weaker than the usual. In [5], it is claimed to have established the existence of a weak solution provided that  $f^{in}$  satisfies a few conditions [5, Theorem 1.1]. But the proof of a crucial lemma [5, Lemma 2.3] has a gap which turns out difficult to fill (see the discussion in Appendix A in the first version of [2]). Carrillo's paper is sometimes cited decisively for other results such as [10] where the authors extend Carrillo's theory to the case of reflexive boundary with a test function space calibrated to bypass the trace issue in the same spirit as Carrillo's approach. To the best of my knowledge, I am not aware of any work that rigorously proves the existence and a fortiori the uniqueness of a weak solution. Meanwhile, in a series of articles, [8, 9], the existence of classical solutions with several interesting properties to the initial-boundary problem associated to  $\mathcal{L}^*$  is established with probabilistic tools. The aim of the present paper is to deduce by duality a theory of weak solution from these articles.

We first recall that a real valued function  $f$  defined on a local compact space  $E$  vanishing "at infinity" is defined by

$$\forall \varepsilon > 0, \exists \text{ compact } K_\varepsilon \subset E : \sup_{x \in K_\varepsilon} |f(x)| \leq \varepsilon.$$

We denote by  $C_0^\infty(E)$  the set of smooth functions vanishing at infinity on  $E$ .

We introduce the following space:

$$\mathcal{D}_{\tilde{Q}_T} := \{ \phi \in C_0^\infty(\tilde{Q}_T) \mid \phi \in L^1(D; L^\infty((0, T))) \text{ and } (\partial_t + \mathcal{L}^*)\phi \in L^\infty(Q_T) \},$$

with

$$\tilde{Q}_T := [0, T[ \times (\overline{\Omega} \setminus (\Gamma^+ \cup \Gamma^0)) \times \mathbb{R}^d.$$

Notice that by extension by continuity, we have  $\phi \in C^b([0, T] \times \overline{D})$  with  $\phi(T, \cdot) = 0$  and  $\phi = 0$  on  $\Gamma^+ \cup \Gamma^0$  for any  $\phi \in \mathcal{D}_{\tilde{Q}_T}$  which implies  $\mathcal{D}_{Q_T} \subset \mathcal{D}_{\tilde{Q}_T}$ . We now give a new definition of weak solution.

**Definition 1.** *A function  $f \in L^\infty([0, T]; L^1(D))$  is said to be a weak solution of (3) with initial condition  $f^{in}$  if it satisfies*

$$\int_D f(\partial_t \phi + \mathcal{L}^* \phi) dt dx + \int_D f^{in}(x) \phi(0, x) dx = 0, \quad \forall \phi \in \mathcal{D}_{\tilde{Q}_T}.$$

We can now state the first main result:

**Theorem 2** (Existence and uniqueness). *For any  $f^{in} \in L^1(D)$ , there exists a unique weak solution  $f \in L^\infty([0, T]; L^1(D))$  of (3) with initial condition  $f^{in}$  in the sense of Definition 1.*

The existence and the uniqueness of the weak solution together with the properties derived in [8, 9] implies the existence of a strongly continuous semigroup with nice properties:

**Theorem 3.** *Let  $f^{in} \in L^1(D)$ , then:*

- (1) (*Semigroup property*) *There exists a positive, irreducible and immediately compact  $C_0$  semigroup  $(P_t)_{t \geq 0}$  such that for any  $f^{in} \in L^1(D)$ ,  $t \mapsto P_t f^{in}$  is the unique weak solution of (3) with initial condition  $f^{in}$  in the sense of Definition 1.*
- (2) (*Regularity*) *For any  $t > 0$ , we have  $P_t f^{in} \in C^\infty(D) \cap C^b(\overline{D})$*
- (3) (*Balanced exponential growth*), *let denotes the spectral bound of the operator  $\mathcal{L}$*

$$s(\mathcal{L}) := \sup \{ \operatorname{Re} \lambda \mid \lambda \in \sigma(\mathcal{L}) \}$$

*then  $s(\mathcal{L}) \in \sigma(\mathcal{L})$  with  $s(\mathcal{L}) < 0$  and up to a multiplicative constant, there exists unique nonnegative functions  $\varphi$  and  $\psi \in C^b(\overline{D}) \cap L^1(D)$  such that*

$$\mathcal{L}\varphi = s(\mathcal{L})\varphi, \quad \mathcal{L}^*\psi = s(\mathcal{L})\psi.$$

*Moreover, normalizing  $\psi$  such that  $\int_D \psi \varphi = 1$ , we have*

$$\lim_{t \rightarrow +\infty} \left\| \left\| e^{-s(\mathcal{L})t} P_t - \varphi \otimes \psi \right\| \right\|_{L^1(D)} = 0,$$

*where*

$$\forall f \in L^1(D) \quad (\varphi \otimes \psi)(f) = \left( \int_D \psi f \right) \varphi,$$

*and  $\|\cdot\|_{L^1(D)}$  denotes the norm operator in  $L^1(D)$ .*

As  $f(t, \cdot) \in C^b(\overline{D})$ , we can define  $\gamma f \in C^b(\partial D)$  and  $\gamma_\pm f := \gamma f|_{\pm(p \cdot n_q) > 0}$  but we have more:

**Theorem 4 (Trace).** *Let  $\gamma f$  the trace of  $f$  defined as above, then  $\gamma f \in L^1(\partial\Omega \times \mathbb{R}^d; |p \cdot n_q| d\sigma_{\partial\Omega} dp)$  where  $d\sigma_{\partial\Omega}$  denotes the Lebesgue measure on  $\partial\Omega$ . And for any  $t > 0$  and for any  $\phi \in C^\infty(D) \cap C(\overline{D})$  we have*

$$\int_D (\mathcal{L}f)\phi = \int_D f \mathcal{L}^*\phi + \int_{\partial D} \gamma f \phi(n_q \cdot p) dp d\sigma_{\partial\Omega}.$$

**Remark 1.** *The results on the existence and integrability of the traces are all the more remarkable that in general it is not true in mathematical kinetic theory that even for  $f \in L^1$  such that  $q \cdot \nabla_p f \in L^1$ :*

$$\gamma f \in L^1(\Omega \times \mathcal{V}; |n_q \cdot p| dp dq)$$

*see for instance [3] for a counter-example.*

As the proofs of Theorem 2 and Theorem 3 rely heavily upon [8, 9] and upon the theory of Banach lattice, for sake of completeness, we will first recall the most important points of [8, 9] and of the theory of Banach lattice in Section 1. Then, we will establish Theorem 2 and Theorem 3 in Section 2. After that, the proof will be given in Section 3.

## 1. OF BANACH LATTICES AND PROBABILISTIC INVESTIGATIONS OF THE KINETIC FOKKER-PLANCK EQUATION

First, we recall the main elements of the theory of Banach lattice and a few of useful theorems around positive operators and semigroups of operators in a Banach lattice. Then, we recall and synthesize the main results of [8, 9].

1.1. **Banach lattices.** First, we underline that  $L^1(D)$  is a ordered set with the partial order defined by

$$0 \leq f \text{ if and only if } 0 \leq f(x) \text{ a.e. on } D.$$

This ordered space is a Banach lattice, meaning that the ordering is compatible with the vector structure and is such that

$$\forall f, g \in L^1(D), \text{ if } |g| \leq |f|, \text{ then } \|g\|_{L^1(D)} \leq \|f\|_{L^1(D)}.$$

A semigroup  $\mathcal{S} \equiv (S_t)_{t \geq 0}$  is said to be positive if it preserves positivity, i.e.

$$f \geq 0 \Rightarrow S_t f \geq 0, \text{ for each } t \geq 0.$$

A closed subspace  $I$  of  $L^1(D)$  is said to be an (order) ideal if

$$\forall (f, g) \in L^1(D) \times I, |f| \leq |g| \Rightarrow f \in I.$$

The null subspace  $\{0\}$  and  $L^1(D)$  are ideals. A semigroup  $\mathcal{S}$  is said to be irreducible if and only if the only ideals that are invariant for  $\mathcal{S}$  are  $\{0\}$  and  $L^1(D)$ . That is equivalent to the following property (for instance [12, Proposition III.8.3 p.186]):

$$\forall 0 \leq f \in L^1(D), 0 \leq g \in L^\infty(D) \text{ with } f \neq 0, g \neq 0, \exists t > 0 : \int_D g S_t f > 0.$$

In others words, an irreducible semigroup spreads the support of any nonnegative function through the entire space. A nonnegative function  $f \in L^1(D)$  is a quasi-interior point if it is a.e. strictly positive on  $D$ . In  $L^\infty(D)$  that is also a Banach lattice, a non negative function  $g$  is a quasi-interior point if there exists  $\delta > 0$  such that  $\delta \mathbb{1}_D \leq g$  a.e. The ground bound of a  $C_0$ -semigroup  $\mathcal{S}$  is

$$\omega_0(\mathcal{S}) := \inf_{\omega \in \mathbb{R}} \{ \exists M \geq 0 : \|S_t\| \leq M e^{\omega t} \}$$

It is known [1, Proposition 12.1 p. 181] that

$$\omega_0(\mathcal{S}) = \lim_{t \rightarrow \infty} \frac{\log \|S_t\|}{t} = \inf_{t \geq 0} \frac{\log \|S_t\|}{t},$$

and that for any  $t > 0$ , the spectral radius of  $S_t$  is equal to:

$$(4) \quad r(S_t) = e^{t\omega_0(\mathcal{S})}.$$

and let  $A$  an unbounded operator, we recall its spectral bound is

$$s(A) := \sup \{ \operatorname{Re} \mu \mid \mu \in \sigma(A) \}$$

where  $\sigma(A)$  is the spectrum of  $A$ . If  $\mathcal{S}$  is a positive  $C_0$ -semigroup on  $L^1(D)$  then its growth bound coincides with the spectral bound of its generator (see for instance [1, Theorem 12.17 p.193]).

**Lemma 5.** *Let  $A$  be a positive, compact and irreducible operator on  $L^1(D)$*

- (1) *then  $r(A) > 0$  and there exists  $u \in L^1(D)$  and  $v \in L^\infty(D)$  strictly positive a.e., unique up to a multiplicative constant such that*

$$\begin{cases} Au &= r(A)u, \\ A'v &= r(A)v. \end{cases}$$

- (2) *Besides, let  $\lambda > 0$  such there exists nonnegative  $w \in L^1(D)$  with  $Aw = \lambda w$  then  $\lambda = r(A)$  and  $w = u$  up to a multiplicative constant.*

*Proof.* The first point is classical in Banach lattice theory and is known as De Pagter's theorem [6]. The second point is almost immediate, but despite my best efforts, I have not found citable results in the literature. For the sake of completeness, I provide the evidence. As  $w$  is nonnegative and  $v$  strictly positive a.e., we have  $\int_{\Omega \times \mathbb{R}^d} vw > 0$ . Keeping that in mind,  $Aw = \mu w$  implies

$$\int_D Awv = \mu \int_D wv$$

thus, as  $\int_D vw > 0$  is strictly positive

$$r(A) = \mu,$$

and the equality of  $w$  with  $u$  is a direct consequence of the uniqueness of the eigenvector associated to  $r(A)$ .  $\square$

The theory of the balanced growth is very rich. The following proposition is a special case, it is a synthesis of Propositions 2.3 and 2.4 and 2.5 and Remark 2.2 in [13]) and for the precise form of  $P$ , see Proposition III.8.5 p. 189 in [12].

**Lemma 6.** *Let  $(S_t)_{t \geq 0}$  a irreducible, positive and compact semigroup on a Banach lattice  $E$ , then there exists a positive projection of rank one,  $P = x_0 \otimes x'_0$  where  $x_0$  is, up to a constant, the eigenvalue associated to  $s(A)$  and  $x'_0 \in E'$  such that  $x'_0(x_0) = 1$ , and constants  $\delta > 0$  and  $M \geq 1$  such that*

$$\left\| e^{-ts(A)} S_t - P \right\|_{L^1(D)} \leq M e^{-\delta t}.$$

**1.2. A brief recalling of the results of Tony Lelièvre and alii.** We recall here the theory developed by Tony Lelièvre, Mouad Ramil, and Julien Reygnier in a probabilistic framework for the kinetic Fokker-Planck in a bounded domain with absorbing [8, 9]. From here, we use there notations except the ones of the operators  $\mathcal{L}$  and  $\mathcal{L}^*$ . Indeed, in [8, 9],  $\mathcal{L}$  (respectively,  $\mathcal{L}^*$ ) corresponds to  $\mathcal{L}^*$  (respectively  $\mathcal{L}$ ) in the present paper, because the operator  $\mathcal{L}^*$  defined in (2).

Let  $(X_t^x = (q_t^x, p_t^x))_{t \geq 0}$  the Langevin process starting at  $x \in D$  at  $t = 0$  defined by the following SDE:

$$\begin{cases} dq_t^x &= p_t^x dt, \\ dp_t^x &= F(q_t^x) dt - \gamma p_t^x dt + \sigma dB_t, \\ (q_0^x, p_0^x) &= x, \end{cases}$$

where  $(B_t)_{t \geq 0}$  is a  $d$ -dimensional Brownian motion. Define also the first exit time from  $D$  of the process  $(X_t^x)_{t \geq 0}$ :

$$\tau_\partial^x := \inf \{t > 0 : X_t^x \notin D\}.$$

We denote the following unbounded operator:

$$\mathcal{L}_\gamma^* \equiv p \cdot \nabla_q + F(q) \cdot \nabla_p - \gamma p \cdot \nabla_p + \frac{\sigma^2}{2} \Delta_p.$$

It is the operator defined in (2) with the subscript  $\gamma$  in order to emphasize its dependence on the parameter  $\gamma$ . It is the infinitesimal generator of the Langevin process whereas (3) describes the evolution of Langevin process absorbed at  $\partial D$ . That explains why in a probabilistic framework such as in [8, 9], the initial problem with  $\mathcal{L}_\gamma^*$  is most often studied while in mathematical physics, it is (3) that arouses the most interest.

That being said, the first main result [8, Theorem 2.10] is then

**Theorem 7.** *Let  $\gamma \in \mathbb{R}$  and let  $f \in C^b(D \cup \Gamma^-)$  and define*

$$u_\gamma : (t, x) \mapsto \mathbb{E} \left[ \mathbb{1}_{\tau_\partial^x > t} f(X_t^x) \right]$$

then

(1) (Initial and boundary values) the function  $u$  satisfies :

$$u_\gamma(0, x) = \begin{cases} f(x) & \text{if } x \in D \cup \Gamma^-, \\ 0 & \text{if } x \in \Gamma^+ \cup \Gamma^0, \end{cases}$$

and

$$\forall t > 0, \forall x \in \Gamma^+ \cup \Gamma^0, u(t, x) = 0.$$

(2) (Continuity) we have :

$$u_\gamma \in C^b((\mathbb{R}_+ \times \overline{D}) \setminus (\{0\} \times (\Gamma^- \cup \Gamma^0)))$$

(3) (Interior regularity)  $u \in C^\infty(\mathbb{R}_+^*)$  and, for all  $t > 0$ ,  $x \in D$

$$\partial_t u_\gamma = \mathcal{L}_\gamma^* u_\gamma$$

Moreover,  $f \mapsto \left( (t, x) \mapsto \mathbb{E} \left[ \mathbb{1}_{\tau_\partial^x > t} f(X_t^x) \right] \right)$  is a semigroup with nice properties including the existence of a transition kernel. We sum up the main results (specially Theorem 2.7 and Theorem 2.9 in [9] and Theorem 2.14 in [8]).

**Theorem 8.** Let  $\gamma \in \mathbb{R}$ , there exists

$$(t, x, y) \mapsto p_t^\gamma(x, y) \in C^\infty(\mathbb{R}_+^* \times D \times D) \cap C(\mathbb{R}_+^* \times \overline{D} \times \overline{D})$$

which satisfies for all  $t > 0$

- $p_t^\gamma(x, y) > 0$  for all  $x \notin \Gamma^+ \cup \Gamma^0$  and  $y \notin \Gamma^- \cup \Gamma^0$
- $p_t^\gamma(x, y) = 0$  if  $x \in \Gamma^+ \cup \Gamma^0$  or  $y \in \Gamma^- \cup \Gamma^0$

Moreover, for  $f \in C^b(\overline{D})$ , the functions  $u_\gamma, v_\gamma$  defined by

$$\forall t > 0, \forall x \in D, u_\gamma(t, x) := \int_D p_t^\gamma(x, y) f(y) dy, \quad v_\gamma(t, x) := \int_D p_t^\gamma(y, x) f(y) dy$$

are in  $C^\infty(\mathbb{R}_+^* \times D)$  and satisfy

$$(5) \quad \begin{cases} \partial_t u_\gamma & = \mathcal{L}_\gamma^* u_\gamma, \\ u_\gamma & = 0 \text{ on } \Gamma^+, \\ u_\gamma(0, \cdot) & = f; \end{cases}$$

and

$$(6) \quad \begin{cases} \partial_t v_\gamma & = \mathcal{L}_\gamma v_\gamma, \\ v_\gamma & = 0 \text{ on } \Gamma^-, \\ v_\gamma(0, \cdot) & = f. \end{cases}$$

Besides, the solution of (5) satisfies Maximum principle. For any  $t > 0$ , the operator

$$Q_t^\gamma f : x \in \overline{D} \mapsto \mathbb{E} \left[ \mathbb{1}_{\tau_\partial^x > t} f(X_t^x) \right] = \int p_t^\gamma(y, x) f(y) dy$$

is well-defined and forms a positive and irreducible semigroup of compact operators  $(Q_t^\gamma)_{t>0}$  on  $L^n(D)$  to  $L^n(D)$  and  $C^b(\overline{D})$  to  $C^b(\overline{D})$ . Moreover, it maps  $L^n(D)$  into  $C^b(\overline{D})$  continuously for any  $n \in [1, \infty]$ .

Eventually, we recall here a special case of Theorem 2.13 in [9], the last property being a consequence of the irreducibility of the semigroup (see Lemma 4.3 in [8])

**Theorem 9.** *Under assumptions above, there exists  $\lambda_\gamma > 0$  such that  $\lambda_\gamma + \gamma d > 0$  and positive functions  $\varphi_\gamma, \psi_\gamma \in C^b(\overline{D}) \cap L^1(D)$ , unique up to multiplicative constants such*

$$(7) \quad \begin{cases} \mathcal{L}\varphi_\gamma + \lambda_\gamma\varphi_\gamma & = 0 \\ \varphi_\gamma & = 0 \text{ on } \Gamma^- \cup \Gamma^0 \end{cases}$$

and

$$(8) \quad \begin{cases} \mathcal{L}^*\psi_\gamma + \lambda_\gamma\psi_\gamma & = 0 \\ \psi_\gamma & = 0 \text{ on } \Gamma^+ \cup \Gamma^0. \end{cases}$$

## 2. PROOF OF THEOREM 2 AND THEOREM 3

We establish here Theorem 2 and Theorem 3. The proof of Theorem 2 is split into several steps.

**2.1. A semigroup for the kinetic Fokker-Planck equation.** In order to use the results of [8, 9] recalled above, first notice that if  $u_{-\gamma}$  is the solution associated to  $\mathcal{L}_{-\gamma}^*$  as in Theorems 7 and 8 then it is  $C^\infty(\mathbb{R}_+^* \times D)$  and thus we can derive it in classical sense. It is then easy to check that the function

$$v(t, q, p) := e^{d\gamma t} u_{-\gamma}(t, q, -p)$$

solves (3) in the classical sense. That remark motivates the introduction of the operator defined on the set of measurable functions  $L^0(D)$

$$(Sf)(q, p) := f(q, -p) \quad \forall f \in L^0(D).$$

The operator  $S$  is obviously bounded on  $L^n(D)$  and  $C^b(\overline{D})$ , moreover we have  $S^2 = I$  which imply that the operators defined by

$$(9) \quad P_t^\gamma := e^{d\gamma t} S Q_t^{-\gamma} S,$$

with  $Q_t^\gamma$  defined in Theorem 8, form a semigroup, which, together with Theorem 2.8 in [9], implies the following theorem:

**Theorem 10.** *The operators  $(P_t^\gamma)_{t \geq 0}$  form a positive and irreducible semigroup of compact operators on  $L^n(D)$  for any  $n \in [1, \infty]$  and  $C^b(\overline{D})$ . Moreover, it maps  $L^n(D)$  into  $C^b(\overline{D})$ . And for any  $f \in L^n(D)$ ,  $v_\gamma := P_t^\gamma f$  is in  $C^\infty(\mathbb{R}_+^* \times D)$  and satisfies*

$$\begin{cases} \partial_t v_\gamma & = \mathcal{L}_\gamma v_\gamma \text{ on } D, \\ v_\gamma & = 0 \text{ on } \Gamma^-, \\ v_\gamma(0, \cdot) & = f. \end{cases}$$

Moreover, we have the identity for any  $f \in C^b(\overline{D})$  and  $v_\gamma = P_t^\gamma f$

$$v_\gamma = \int_D p_t^\gamma(y, x) f(y) dy.$$

where  $p_t^\gamma$  is the transition kernel defined in Theorem 8.

From here on, we drop the subscript  $\gamma$  whenever it is obvious from the context what the value of  $\gamma$  is,  $P_t$  for  $P_t^\gamma$  and so on. We now show that  $P_t$  is a substochastic semigroup:

**Lemma 11.** *For any  $t > 0$*

$$\|P_t\|_{L^1(D)} \leq 1.$$

As consequence,  $\forall f \in L^1(D)$ ,  $t \mapsto P_t f \in L^\infty(\mathbb{R}_+; L^1(D))$ .



*Proof.* By density, it suffices to show that for any  $t > 0$  and for any  $f \in C^b(\overline{D})$ , we have

$$\|P_t f\|_{L^1(D)} \leq \|f\|_{L^1(D)}.$$

By Theorem 10 we have for any  $f \in C^b(\overline{D})$

$$P_t f = \int_D p_t^\gamma(y, x) f(y) dy$$

thus for any  $g \in L^\infty(D)$  by Fubini-Tonelli

$$\begin{aligned} \int_D P_t f g dx &= \int_D \int_D p_t(y, x) f(y) g(x) dy dx \\ &= \int_D f(y) \left( \int_D p_t(y, x) g(x) dx \right) dy \end{aligned}$$

Thus by Maximum principle applied to  $\int_D p_t(y, x) g(x) dx$  that is solution of (5) by Theorem 8, we have for any  $g \in L^\infty(D)$

$$\left| \int_D P_t f g \right| \leq \|f\|_{L^1(D)} \|g\|_{L^\infty(D)}$$

which implies by Hahn-Banach:

$$\|P_t f\|_{L^1(D)} \leq \|f\|_{L^1(D)}$$

which is the desired result.  $\square$

**Lemma 12.** *Let  $\lambda > 0$  and  $\varphi$  be defined as in Theorem 9, then*

$$P_t \varphi = e^{-\lambda t} \varphi, \forall t > 0.$$

We can now show that  $(P_t)_{t \geq 0}$  is a strongly continuous semigroup:

**Proposition 13.** *The operators  $(P_t)_{t \geq 0}$  with  $P_0 = I$  and  $(P_t)_{t > 0}$  defined above form a  $C_0$ -semigroup of  $L^1(D)$ . As a consequence, the semigroup  $(Q_t)_{t \geq 0}$  associated to the adjoint Cauchy problem is also strongly continuous on  $\mathbb{R}_+$  in  $L^1(D)$ .*

*Proof.* It is already known that  $(P_t)_{t > 0}$  is strongly continuous on  $\mathbb{R}_+^*$  by Theorem 8. It remains to show that it is strongly continuous at  $t = 0$ . By Proposition I.5.3 p. 38 in [7], it suffices to show that there exist  $\delta > 0$ ,  $M \geq 1$ , and a dense subset  $\mathcal{D} \subset L^1(D)$  such that

- (1)  $\|P_t\|_{L^1(D)} \leq M$  for all  $t \in [0, \delta]$ ,
- (2)  $\lim_{t \rightarrow 0^+} P_t g = g$  strongly in  $L^1(D)$  for all  $g \in \mathcal{D}$ .

By Lemma 11, we know that  $\|P_t\|_{L^1(D)} \leq 1$ . It remains to establish (2) for  $\mathcal{D} = C_c(D)$ . Let  $g \in C_c(D)$  that we assume nonnegative and denote  $h(t, x) := P_t g(x)$  for any  $x \in D$ , we know that  $h \in C(\mathbb{R}_+ \times D)$ , thus for any  $t \in [0, T]$ ,  $x \mapsto h(t, x)$  is measurable. In the same way, for every  $x \in D$ ,  $t \mapsto h(t, x)$  is continuous. Let  $\varphi$  defined in Theorem 9 and Lemma 12. As  $g \in C_c(D)$  and  $\varphi > 0$  everywhere, there exists  $C > 0$  such that

$$g \leq C\varphi \text{ a.e. on } D$$

and by positivity of  $P_t$  and Lemma 12, we have

$$h \leq C e^{-\lambda t} \varphi \leq C\varphi \text{ a.e. on } D, \forall t \geq 0$$

with  $\varphi \in L^1(D)$ . In others words,  $h$  is dominated on  $[0, T]$  by a integrable function on  $D$  which implies that:

$$t \rightarrow \int_D h(t, x) dx \text{ is continuous on } [0, T].$$

As  $h$  is nonnegative, Schaffé's lemma implies:

$$(10) \quad \|P_t g - g\|_{L^1(D)} \rightarrow 0 \text{ as } t \rightarrow 0^+ \forall \text{ nonnegative } g \in C_c(D)$$

Now consider  $g \in C_c(D)$  that can change sign and let  $g = g_+ - g_-$  with  $g_{\pm} \in C_c(D)$  and  $g_{\pm} \geq 0$ . We have

$$\|P_t g - g\|_{L^1(D)} \leq \|P_t g_+ - g_+\|_{L^1(D)} + \|P_t g_- - g_-\|_{L^1(D)}$$

thus by (10)

$$\|P_t g - g\|_{L^1(D)} \rightarrow 0 \text{ as } t \rightarrow 0^+ \forall g \in C_c(D),$$

Consequently,  $(P_t)_{t \geq 0}$  is a  $C_0$ -semigroup of  $L^1(D)$ .

As for  $(Q_t)_{t \geq 0}$ , it is a direct consequence of (9), i.e.

$$Q_t^\gamma = e^{d\gamma t} S P_t^{-\gamma} S.$$

□

Let us make also an important remark

**Remark 2.** For any  $t > 0$ , the kernel  $p_t$  satisfies a Gaussian upper-bound (see [8, Corollary 2.22] or [9, Theorem 2.7]) that is established by studying the Langevin process on the space  $\mathbb{R}^{2d}$  after extending the driving force over the whole space. The authors have deduced an inequality valid on the whole space which they then truncate on  $\Omega \times \mathbb{R}^d$ . As the two functions are continuous, the proof of the authors implies that the inequality is valid on  $\bar{\Omega} \times \mathbb{R}^d$  and thus on  $\partial\Omega \times \mathbb{R}^d$ .

The recall being done, for the sake of readability, we split the proof of Theorem 2 into several steps. First we show that for any  $f^{in} \in L^1(\Omega \times \mathbb{R}^d)$ ,  $t \mapsto P_t f^{in}$  belongs to the good function space.

**Lemma 14.** Let  $f^{in} \in L^1(D)$  and  $f := P_t f^{in}$  with  $(P_t)_{t > 0}$  defined in Theorems 8 and 7. Then  $f \in L^\infty([0, T]; L^1(D))$ .

*Proof.* It is simply a direct consequence that for any  $t > 0$ , we have  $\|P_t f^{in}\|_{L^1(D)} \leq \|f^{in}\|_{L^1(D)}$ . □

We now show that for any smooth initial condition, the solution in the sense of Theorems 8 and 7 is also a weak solution.

**Lemma 15.** Let  $f^{in} \in C^b(D \cup \Gamma^+)$  and  $u := P_t f^{in}$  with  $(P_t)_{t > 0}$  defined in Theorems 8 and 7, then for any  $\phi \in \mathcal{D}_{Q_T}$  we have

$$\int_{Q_T} u(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp + \int_D f^{in}(q, p) \phi(0, q, p) dq = 0.$$

In others words,  $u$  is a weak solution of (3)

*Proof.* We cannot apply integration by parts (Green Theorem) to the cylindric domain  $Q_T$  because  $f$  is not  $C^1$  on  $Q_T$ . Therefore, we introduce  $\varepsilon \in (0, T)$  and the subset:

$$Q_{\varepsilon, T} := [\varepsilon, T] \times D.$$

Let  $f^{in} \in C^b(D \cup \Gamma^+)$  and let  $u$  be defined in Theorem 7. We have  $u \in C^b(\bar{Q}_{\varepsilon, T}) \cap C^\infty(Q_{\varepsilon, T})$ , so that by Green Theorem applied to the cylindric domain (see Theorem 5.5 p. 47 in [11]), we have for any  $\phi \in \mathcal{D}_{\bar{Q}_T}$

$$\begin{aligned} 0 &= \iint_{Q_{\varepsilon, T}} (\partial_t u - \mathcal{L}u) \phi dt dx \\ &= \iint_{Q_{\varepsilon, T}} (-\partial_t \phi - \mathcal{L}^* \phi) u dt dx - \iint_D u(\varepsilon, x) \phi(\varepsilon, x) dx \end{aligned}$$

We know that, since  $u \in C^b((\mathbb{R}_+ \times \bar{D}) \setminus (\{0\} \times (\Gamma^- \cup \Gamma^0)))$ ,

$$u(\varepsilon, \cdot) \phi(\varepsilon, \cdot) \rightarrow f^{in} \phi(0, \cdot) \text{ as } \varepsilon \rightarrow 0^+ \text{ a.e. on } \Omega \times \mathbb{R}^d.$$

Moreover

$$|u(\varepsilon, \cdot)\phi(\varepsilon, \cdot)| \leq \|f^{in}\|_{L^\infty(D)} \left( \sup_{\varepsilon \geq 0} \phi(\varepsilon, \cdot) \right).$$

As  $(\sup_\varepsilon \phi(\varepsilon, \cdot)) \in L^1(\Omega \times \mathbb{R}^d)$  since it is in  $\mathcal{D}_{\tilde{Q}_T}$ , we can apply dominated convergence to get:

$$\iint_D u(\varepsilon, x)\phi(\varepsilon, x)dx \xrightarrow{\varepsilon \rightarrow 0^+} \iint_D f^{in}\phi dx$$

Hence  $u$  satisfies for all  $\phi \in \mathcal{D}_{\tilde{Q}_T}$

$$\iint_{Q_{\varepsilon, T}} (\partial_t \phi + \mathcal{L}^* \phi) u dt dx + \iint_D f^{in}(x)\phi(0, x)dx = 0$$

which is the desired result.  $\square$

Let now extend the result above to the case of  $f^{in} \in L^1(D)$ .

**Lemma 16.** *Let  $f^{in} \in L^1(D)$  and for any  $t > 0$ ,  $f(t, \cdot) := P_t f^{in}$  then  $f \in L^\infty([0, T]; L^1(D))$  and for any  $\phi \in \mathcal{D}_{Q_T}$  we have*

$$\int_{Q_T} f(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp + \int_D f^{in}(q, p)\phi(0, q, p) dq = 0.$$

In others words,  $f$  is a weak solution of (3)

*Proof.* Let  $f^{in} \in L^1(D)$ . We know that for any  $t > 0$ ,  $P_t \in \mathcal{L}(L^p(D))$ . Keeping it in mind, take  $f_n^{in} \in C_0^\infty(D)$  such that  $f_n^{in} \xrightarrow{n \rightarrow +\infty} f^{in}$  in  $L^1(D)$ . Then, denoting  $f_n := P_t f_n^{in}$  we have by Lemma 15 for any  $\phi \in \mathcal{D}_{\tilde{Q}_T}$  and for any  $n \in \mathbb{N}$ :

$$(11) \quad \int_{Q_T} f_n(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp + \int_D f_n^{in}(q, p)\phi(0, q, p) dq = 0.$$

By Lebesgue partial converse theorem (see Theorem 4.9 p. 94 in [4]), we can moreover assume that, up to a subsequence, there exists  $g \in L^1(D)$  such that:

$$|f_n^{in}(x)| \leq g \text{ a.e. in } x \in D,$$

and

$$f_n^{in} \rightarrow f^{in} \text{ a.e. in } x \in D.$$

That implies immediately:

$$(12) \quad \int_D f_n^{in}(q, p)\phi(0, x)dx \rightarrow \int_D f^{in}(x)\phi(0, x)dx \text{ as } n \rightarrow +\infty,$$

and

$$(13) \quad \forall (t, x) \in [0, T] \times D, \quad \int_D p_t(y, x) f_n^{in} dy \xrightarrow{n \rightarrow +\infty} \int_D p_t(y, x) f^{in} dy.$$

Besides, observe that

$$0 \leq f_n(t, x) \leq \int_D f_n^{in}(y) p_t(y, x) dy \leq \int_D g(y) p_t(y, x) dy$$

with  $\int_D g(y) p_t(y, x) dy \in L^1([0, T] \times D)$  and by (13)

$$\int_D f_n^{in}(y) p_t(y, x) dy \rightarrow \int_D f^{in}(y) p_t(y, x) dy \text{ a.e. in } (t, x) \in [0, T] \times D$$

as  $n \rightarrow +\infty$ . Therefore by dominated convergence we have

$$f_n \rightarrow f \text{ in } L^1([0, T] \times D) - \text{strong}$$

as  $n \rightarrow +\infty$ . That implies:

$$(14) \quad \int_{Q_T} f_n(-\partial_t \phi + \mathcal{L}^* \phi) dt dq dp \rightarrow \int_{Q_T} f(-\partial_t \phi + \mathcal{L}^* \phi) dt dq dp$$

as  $n \rightarrow +\infty$ . Applying (12) and (14) to (11) shows that  $f$  is a weak solution to the initial boundary problem (3) with  $f^{in}$  as initial condition.

As for the uniqueness of the weak solution, as the problem is linear, it suffices to show that  $f \in L^1(Q_T)$  such that

$$\int_{Q_T} f(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp = 0, \quad \forall \phi \in \mathcal{D}_{\tilde{Q}_T}$$

implies

$$f = 0.$$

If for any  $\psi \in C_c((0, T) \times D)$ , there exists  $\phi \in \mathcal{D}_{\tilde{Q}_T}$  solution of the backward problem

$$(15) \quad \begin{cases} \partial_t \phi + \mathcal{L}^* \phi & = \psi \\ \phi|_{\Gamma^0 \cup \Gamma^-} & = 0 \\ \phi(T, \cdot) & = 0 \end{cases}$$

that would imply that  $f = 0$  in distributional sense and thus  $f = 0$  in  $L^1(Q_T)$ . Consider the forward problem:

$$(16) \quad \begin{cases} \partial_t \tilde{\phi} - \mathcal{L}^* \tilde{\phi} & = \psi \\ \tilde{\phi}|_{\Gamma^0 \cup \Gamma^-} & = 0 \\ \tilde{\phi}(0, \cdot) & = 0 \end{cases}$$

It is obvious that  $t \mapsto \tilde{\phi}(T - t, \cdot)$  is solution of (15). As  $(Q_t)_{t \geq 0}$  is strongly continuous by Proposition 13, we have by Duhamel formula [7, Corollary III.1.7 p. 161]

$$\tilde{\phi} = \int_0^t Q_{t-s} \psi(s) ds$$

thus

$$\begin{aligned} \phi &= \int_0^{T-t} Q_{T-t-s} \psi(s) ds \\ &= \int_t^T Q_{T-s} \psi(s-t) ds \end{aligned}$$

It remains to show that  $\phi \in \mathcal{D}_{\tilde{Q}_T}$  to conclude. By linearity, we can assume that  $\psi$  is nonnegative. That being said, first, by the properties of the function  $p_t$  (Theorems 8 and 7 above), we have  $\phi \in C_0^\infty(\tilde{Q}_T)$ . Besides, we have obviously  $(\partial_t + \mathcal{L}^*)\phi \in L^\infty(Q_T)$ . Eventually, let  $\psi_\lambda$  being the positive eigenvector associated to  $\mathcal{L}^*$ , see (8) in Theorem 9, with eigenvalue  $\lambda$  and  $\int_D \psi_\lambda = 1$ . Let  $K$  compact of  $(0, T) \times D$  such that  $\text{supp } \psi \subset K$ . As  $\psi_\lambda$  is positive everywhere on  $D$  and constant in time, there exists  $c > 0$  such that  $\psi_\lambda \geq c$  for any  $(t, x) \in K$ . Therefore, there exists  $C > 0$  such that

$$\psi \leq C \psi_\lambda \text{ everywhere on } (0, T) \times D.$$

As  $(Q_t)_{t > 0}$  are positive operators, the properties of  $\psi_\lambda$  and the inequality above imply

$$Q_{t-s} \psi(s) \leq C e^{-\lambda(t-s)} \psi_\lambda.$$

Integrating the inequality above in time  $s \in [0, t]$  leads to:

$$\tilde{\phi} \leq \frac{C}{\lambda} (1 - e^{-\lambda t}) \psi_\lambda,$$

thus

$$0 \leq \phi \leq \frac{C}{\lambda} \left(1 - e^{-\lambda(T-t)}\right) \psi_\lambda$$

which implies that  $\phi \in L^1(D; L^\infty((0, T)))$ . Consequently,  $\phi \in \mathcal{D}_{\tilde{Q}_T}$  which is the desired conclusion.  $\square$

**2.2. Proof of Theorem 3.** Now, we establish here Theorem 3

- (1) By Theorem 2, for any  $f^{in} \in L^1(D)$ , there exists a unique weak solution  $f$  in sense of Definition 1, thus we have  $f(t, \cdot) = P_t f^{in}$  with  $(P_t)_{t \geq 0}$  defined in (9). By Theorem 10, Lemma 11 and Proposition 13, it is a positive, irreducible, contractive and compact  $C_0$ - semigroup.
- (2) By Theorem 7 with the property of semigroup, we have for any  $f^{in} \in L^1(D)$ ,  $\forall t > 0$ ,  $P_t f^{in} \in C^\infty(D) \cap C^b(\overline{D})$ .
- (3) With the properties of the semigroup established above, the balanced growth law of  $(P_t)_{t \geq 0}$  is a direct consequence of Lemma 6 with the eigenvectors identified in Theorem 9.

### 3. A TRACE THEORY FOR THE KINETIC FOKKER-PLANCK EQUATION

We now establish a trace theory for the kinetic Fokker-Planck equation. First, using the regularity of the weak solution, we can easily define its trace:

**Lemma 17.** *Let  $f$  be the weak solution with initial condition  $f^{in} \in L^1(D)$ . The trace of  $f$  on  $\mathbb{R}_+^* \times \partial D$  is well defined with  $\gamma f := f|_{\partial D}$  and  $\gamma_\pm := \gamma f|_{\pm(p \cdot n_q) > 0}$ . Moreover we have*

$$\gamma_- f = 0 \text{ and } \gamma_+ \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| dq dp).$$

*Proof.* By Theorem 8, we know that for any  $t > 0$ ,  $P_t$  maps continuously  $L^1(D)$  into  $C^b(\overline{D})$ . Thus we can define  $\gamma f := f|_{\partial\Omega \times \mathbb{R}^d}$  and  $\gamma_\pm := \gamma f|_{\pm(p \cdot n_q) > 0}$ .

Besides, by semigroup property, we have for any  $t > 0$ ,  $P_t f^{in} = P_{t/2}(P_{t/2} f^{in})$  with  $P_{t/2} f^{in} \in C^b(\overline{D})$  which implies by Theorem 7 that  $\gamma_- f = 0$ .

Now show that  $\gamma_+ f \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| dq dp)$ . As for any  $t > 0$ ,  $\gamma_+ f \in C^b(\partial D)$ , it is measurable. Now we have by the semigroup property and by continuity:

$$\gamma_+ f(t, x) = \int_D p_{t/2}(x, y) f\left(\frac{t}{2}, y\right) dy$$

with  $y \mapsto f\left(\frac{t}{2}, y\right) \in C^b(\partial D)$ , thus

$$|\gamma_+ f(t, x)| \leq \left\| f\left(\frac{t}{2}, \cdot\right) \right\|_{C^b(\partial D)} \int_D p_{t/2}(x, y) dy.$$

The Gaussian estimates on  $p_{t/2}$  (see [8, Corollary 2.22] and Remark 2) implies that

$$x \in \partial D \mapsto \int_D p_{t/2}(x, y) dy \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| d\sigma_{\partial\Omega} dp),$$

hence the desired result.  $\square$

The existence of the trace and the regularity of the weak solution implies that the integration by parts, also known as Green's formula, is satisfied:

**Lemma 18.** *Let  $f$  be the weak solution with initial condition  $f^{in} \in L^1(D)$  and let  $\gamma f$  be its trace. Then, it satisfies Green's formula, meaning that for any  $t > 0$  and for any  $\phi \in C^\infty(D) \cap C(\overline{D})$  we have*

$$\int_D (\mathcal{L}f)\phi = \int_D f \mathcal{L}^* \phi + \int_{\partial D} \gamma f \phi (n_q \cdot p) dp d\sigma_{\partial\Omega}.$$

*Proof.* By unicity of the weak solution, we know that for any  $t > 0$ ,  $f = P_t f^{in}$  and thus  $f \in C^\infty(D) \cap C(\overline{D})$  with  $\gamma f \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot |d\sigma_{\partial\Omega} dp)$  by Lemma 17. Hence, the classical theorem of Green's formula (Integration by parts) [11, Th 5.5 p. 47] applies here and leads to the desired result.  $\square$

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