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ON THE EXISTENCE OF WEAK SOLUTION OF THE KINETIC FOKKER-PLANCK EQUATION IN A BOUNDED DOMAIN WITH ABSORBING BOUNDARY

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INTRODUCTION

In the present paper, we discuss the existence and unicity of weak solutions of the kinetic Fokker-Planck equation in a bounded domain with absorbing boundary. More precisely, let an integer $d \geq 2$ and let Ω be a bounded, open and smooth subset of \mathbb{R}^d . As its boundary is smooth, we can define for any $q \in \partial\Omega$ the outer normal vector n_q . Thus, we can decompose $\partial\Omega = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$ with:

$$\begin{aligned}\Gamma^+ &= \{(q, p) \in \partial\Omega \times \mathbb{R}^d \mid p \cdot n_q > 0\}, \\ \Gamma^- &= \{(q, p) \in \partial\Omega \times \mathbb{R}^d \mid p \cdot n_q < 0\}, \\ \Gamma^0 &= \{(q, p) \in \partial\Omega \times \mathbb{R}^d \mid p \cdot n_q = 0\}.\end{aligned}$$

Let $\sigma \in \mathbb{R}_+^*$, $\gamma \in \mathbb{R}$ and $F \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, we introduce the operator:

$$(1) \quad \mathcal{L} = -p \cdot \nabla_q - F(q) \cdot \nabla_p + \gamma \nabla_p \cdot (p \cdot) + \frac{\sigma^2}{2} \Delta_p,$$

and its formal adjoint in $L^2(dx)$

$$(2) \quad \mathcal{L}^* = p \cdot \nabla_q + F(q) \cdot \nabla_p - \gamma p \cdot \nabla_p + \frac{\sigma^2}{2} \Delta_p.$$

Following [8, 9], we denote $D := \Omega \times \mathbb{R}^d$, $\bar{D} = \bar{\Omega} \times \mathbb{R}^d$ and $\partial D := \partial\Omega \times \mathbb{R}^d$. The variable $x \in D$ represents the couple $(q, p) \in D$ and $dx = dqdp$. We will use preferably this notation except when the two variables q and p do not have symmetrical roles, such as the beginning of subsection 2.1.

The so-called kinetic Fokker-Planck equation is the following initial-boundary problem $f \equiv f(t, x)$:

$$(3) \quad \begin{cases} \partial_t f &= \mathcal{L}f \text{ on } (0, T) \times D, \quad T > 0, \\ f(t, x) &= 0 \text{ on } (0, T) \times \Gamma^-, \\ f(0, x) &= f^{in}(x) \text{ on } D. \end{cases}$$

It models the collective behavior of particles evolving inside Ω with velocity in \mathbb{R}^d , with initial distribution f^{in} , submitted to an exterior force F with a thermal bath encoded by the term of diffusion in velocity $\frac{\sigma^2}{2} \Delta_p$ and the friction term $\gamma \nabla_p \cdot (p \cdot)$. More precisely, the quantity $f(x)dx \equiv f(t, q, p)dtdqdp$ represents the number of particles at time interval $[t, t + dt]$, in the volume $B(q, dq) \subset \Omega$ and with velocities in $B(p, dp) \subset \mathbb{R}^d$. The homogeneous boundary condition $f(t, x) = 0$ on $(0, T) \times \Gamma^-$ means that whenever a particle coming from the interior of Ω hits the boundary, it exits and disappears forever.

In the case of $\Omega = \mathbb{R}^d$, the theory of the kinetic Fokker-Planck equation is well-established. The bounded case is another matter because of the trace issue in kinetic theory. For more details, we refer to the enlightening discussion in Stéphane Mischler's "habilitation à diriger les recherches" p. 29-32. Let us just recall here that a possible strategy is to bypass the trace issue by define a weaker notion of solution with a

smaller space of test functions canceling on the problematic part of the boundary. This strategy is followed in [5] that we recall now in more detail:

Let $T > 0$, define the cylinder domain $Q_T := [0, T[\times \overline{D}$ and the following space of test functions:

$$\mathcal{D}_{Q_T} := \{ \phi \in C_c^\infty(Q_T) \mid \phi(t, x) = 0 \text{ whenever } x \in \Gamma^+ \cup \Gamma^0 \}$$

We recall that $C_c^\infty(Q_T)$ denotes the space of smooth functions with compact support in Q_T . In [5], it is denoted by $C_0^\infty(Q_T)$ without being explicitly defined but the context implies the compact support in velocity. Indeed, the definition of a weak solution in [5] (see below) implicitly requires that $(\partial_t \phi + \mathcal{L}^* \phi)$ belongs to $L^\infty(Q_T)$ whereas a smooth function "vanishing at infinity" may not have its derivative essentially bounded.

In [5] a function $f \in L^\infty([0, T]; L^1(D))$ is called a weak solution of (3) if it verifies [5, formula (1.7)]:

$$\int_D f(\partial_t \phi + \mathcal{L}^* \phi) dt dx + \int_D f^{in}(x) \phi(0, x) dx = 0, \quad \forall \phi \in \mathcal{D}_{Q_T}.$$

It is easy to check that a classical solution of (3) would be also such a weak solution. However, we emphasize that, as the space of test functions is smaller, the notion of weak solution is therefore weaker than the usual. In [5], it is claimed to have established the existence of a weak solution provided that f^{in} satisfies a few conditions [5, Theorem 1.1]. But the proof of a crucial lemma [5, Lemma 2.3] has a gap which turns out difficult to fill (see the discussion in Appendix A in the first version of [2]). Carrillo's paper is sometimes cited decisively for other results such as [10] where the authors extend Carrillo's theory to the case of reflexive boundary with a test function space calibrated to bypass the trace issue in the same spirit as Carrillo's approach. To the best of my knowledge, I am not aware of any work that rigorously proves the existence and a fortiori the uniqueness of a weak solution. Meanwhile, in a series of articles, [8, 9], the existence of classical solutions with several interesting properties to the initial-boundary problem associated to \mathcal{L}^* is established with probabilistic tools. The aim of the present paper is to deduce by duality a theory of weak solution from these articles.

We first recall that a real valued function f defined on a local compact space E vanishing "at infinity" is defined by

$$\forall \varepsilon > 0, \exists \text{ compact } K_\varepsilon \subset E : \sup_{x \in K_\varepsilon} |f(x)| \leq \varepsilon.$$

We denote by $C_0^\infty(E)$ the set of smooth functions vanishing at infinity on E .

We introduce the following space:

$$\mathcal{D}_{\tilde{Q}_T} := \{ \phi \in C_0^\infty(\tilde{Q}_T) \mid \phi \in L^1(D; L^\infty((0, T))) \text{ and } (\partial_t + \mathcal{L}^*)\phi \in L^\infty(Q_T) \},$$

with

$$\tilde{Q}_T := [0, T[\times (\overline{\Omega} \setminus (\Gamma^+ \cup \Gamma^0)) \times \mathbb{R}^d.$$

Notice that by extension by continuity, we have $\phi \in C^b([0, T] \times \overline{D})$ with $\phi(T, \cdot) = 0$ and $\phi = 0$ on $\Gamma^+ \cup \Gamma^0$ for any $\phi \in \mathcal{D}_{\tilde{Q}_T}$ which implies $\mathcal{D}_{Q_T} \subset \mathcal{D}_{\tilde{Q}_T}$. We now give a new definition of weak solution.

Definition 1. *A function $f \in L^\infty([0, T]; L^1(D))$ is said to be a weak solution of (3) with initial condition f^{in} if it satisfies*

$$\int_D f(\partial_t \phi + \mathcal{L}^* \phi) dt dx + \int_D f^{in}(x) \phi(0, x) dx = 0, \quad \forall \phi \in \mathcal{D}_{\tilde{Q}_T}.$$

We can now state the first main result:

Theorem 2 (Existence and uniqueness). *For any $f^{in} \in L^1(D)$, there exists a unique weak solution $f \in L^\infty([0, T]; L^1(D))$ of (3) with initial condition f^{in} in the sense of Definition 1.*

The existence and the uniqueness of the weak solution together with the properties derived in [8, 9] implies the existence of a strongly continuous semigroup with nice properties:

Theorem 3. *Let $f^{in} \in L^1(D)$, then:*

- (1) (*Semigroup property*) *There exists a positive, irreducible and immediately compact C_0 semigroup $(P_t)_{t \geq 0}$ such that for any $f^{in} \in L^1(D)$, $t \mapsto P_t f^{in}$ is the unique weak solution of (3) with initial condition f^{in} in the sense of Definition 1.*
- (2) (*Regularity*) *For any $t > 0$, we have $P_t f^{in} \in C^\infty(D) \cap C^b(\overline{D})$*
- (3) (*Balanced exponential growth*), *let denotes the spectral bound of the operator \mathcal{L}*

$$s(\mathcal{L}) := \sup \{ \operatorname{Re} \lambda \mid \lambda \in \sigma(\mathcal{L}) \}$$

then $s(\mathcal{L}) \in \sigma(\mathcal{L})$ with $s(\mathcal{L}) < 0$ and up to a multiplicative constant, there exists unique nonnegative functions φ and $\psi \in C^b(\overline{D}) \cap L^1(D)$ such that

$$\mathcal{L}\varphi = s(\mathcal{L})\varphi, \quad \mathcal{L}^*\psi = s(\mathcal{L})\psi.$$

Moreover, normalizing ψ such that $\int_D \psi \varphi = 1$, we have

$$\lim_{t \rightarrow +\infty} \left\| \left\| e^{-s(\mathcal{L})t} P_t - \varphi \otimes \psi \right\| \right\|_{L^1(D)} = 0,$$

where

$$\forall f \in L^1(D) \quad (\varphi \otimes \psi)(f) = \left(\int_D \psi f \right) \varphi,$$

and $\|\cdot\|_{L^1(D)}$ denotes the norm operator in $L^1(D)$.

As $f(t, \cdot) \in C^b(\overline{D})$, we can define $\gamma f \in C^b(\partial D)$ and $\gamma_\pm f := \gamma f|_{\pm(p \cdot n_q) > 0}$ but we have more:

Theorem 4 (Trace). *Let γf the trace of f defined as above, then $\gamma f \in L^1(\partial\Omega \times \mathbb{R}^d; |p \cdot n_q| d\sigma_{\partial\Omega} dp)$ where $d\sigma_{\partial\Omega}$ denotes the Lebesgue measure on $\partial\Omega$. And for any $t > 0$ and for any $\phi \in C^\infty(D) \cap C(\overline{D})$ we have*

$$\int_D (\mathcal{L}f)\phi = \int_D f \mathcal{L}^*\phi + \int_{\partial D} \gamma f \phi(n_q \cdot p) dp d\sigma_{\partial\Omega}.$$

Remark 1. *The results on the existence and integrability of the traces are all the more remarkable that in general it is not true in mathematical kinetic theory that even for $f \in L^1$ such that $q \cdot \nabla_p f \in L^1$:*

$$\gamma f \in L^1(\Omega \times \mathcal{V}; |n_q \cdot p| dp dq)$$

see for instance [3] for a counter-example.

As the proofs of Theorem 2 and Theorem 3 rely heavily upon [8, 9] and upon the theory of Banach lattice, for sake of completeness, we will first recall the most important points of [8, 9] and of the theory of Banach lattice In Section 1. Then, we will establish Theorem 2 and Theorem 3 in Section 2. After that, the proof of will be given in Section 3.

1. OF BANACH LATTICES AND PROBABILISTIC INVESTIGATIONS OF THE KINETIC FOKKER-PLANCK EQUATION

First, we recall the main elements of the theory of Banach lattice and a few of useful theorems around positive operators and semigroups of operators in a Banach lattice. Then, we recall and synthesize the main results of [8, 9].

1.1. **Banach lattices.** First, we underline that $L^1(D)$ is a ordered set with the partial order defined by

$$0 \leq f \text{ if and only if } 0 \leq f(x) \text{ a.e. on } D.$$

This ordered space is a Banach lattice, meaning that the ordering is compatible with the vector structure and is such that

$$\forall f, g \in L^1(D), \text{ if } |g| \leq |f|, \text{ then } \|g\|_{L^1(D)} \leq \|f\|_{L^1(D)}.$$

A semigroup $\mathcal{S} \equiv (S_t)_{t \geq 0}$ is said to be positive if it preserves positivity, i.e.

$$f \geq 0 \Rightarrow S_t f \geq 0, \text{ for each } t \geq 0.$$

A closed subspace I of $L^1(D)$ is said to be an (order) ideal if

$$\forall (f, g) \in L^1(D) \times I, |f| \leq |g| \Rightarrow f \in I.$$

The null subspace $\{0\}$ and $L^1(D)$ are ideals. A semigroup \mathcal{S} is said to be irreducible if and only if the only ideals that are invariant for \mathcal{S} are $\{0\}$ and $L^1(D)$. That is equivalent to the following property (for instance [12, Proposition III.8.3 p.186]):

$$\forall 0 \leq f \in L^1(D), 0 \leq g \in L^\infty(D) \text{ with } f \neq 0, g \neq 0, \exists t > 0 : \int_D g S_t f > 0.$$

In others words, an irreducible semigroup spreads the support of any nonnegative function through the entire space. A nonnegative function $f \in L^1(D)$ is a quasi-interior point if it is a.e. strictly positive on D . In $L^\infty(D)$ that is also a Banach lattice, a non negative function g is a quasi-interior point if there exists $\delta > 0$ such that $\delta \mathbb{1}_D \leq g$ a.e. The ground bound of a C_0 -semigroup \mathcal{S} is

$$\omega_0(\mathcal{S}) := \inf_{\omega \in \mathbb{R}} \{ \exists M \geq 0 : \|S_t\| \leq M e^{\omega t} \}$$

It is known [1, Proposition 12.1 p. 181] that

$$\omega_0(\mathcal{S}) = \lim_{t \rightarrow \infty} \frac{\log \|S_t\|}{t} = \inf_{t \geq 0} \frac{\log \|S_t\|}{t},$$

and that for any $t > 0$, the spectral radius of S_t is equal to:

$$(4) \quad r(S_t) = e^{t\omega_0(\mathcal{S})}.$$

and let A an unbounded operator, we recall its spectral bound is

$$s(A) := \sup \{ \operatorname{Re} \mu \mid \mu \in \sigma(A) \}$$

where $\sigma(A)$ is the spectrum of A . If \mathcal{S} is a positive C_0 -semigroup on $L^1(D)$ then its growth bound coincides with the spectral bound of its generator (see for instance [1, Theorem 12.17 p.193]).

Lemma 5. *Let A be a positive, compact and irreducible operator on $L^1(D)$*

- (1) *then $r(A) > 0$ and there exists $u \in L^1(D)$ and $v \in L^\infty(D)$ strictly positive a.e., unique up to a multiplicative constant such that*

$$\begin{cases} Au &= r(A)u, \\ A'v &= r(A)v. \end{cases}$$

- (2) *Besides, let $\lambda > 0$ such there exists nonnegative $w \in L^1(D)$ with $Aw = \lambda w$ then $\lambda = r(A)$ and $w = u$ up to a multiplicative constant.*

Proof. The first point is classical in Banach lattice theory and is known as De Pagter's theorem [6]. The second point is almost immediate, but despite my best efforts, I have not found citable results in the literature. For the sake of completeness, I provide the evidence. As w is nonnegative and v strictly positive a.e., we have $\int_{\Omega \times \mathbb{R}^d} vw > 0$. Keeping that in mind, $Aw = \mu w$ implies

$$\int_D Awv = \mu \int_D wv$$

thus, as $\int_D wv > 0$ is strictly positive

$$r(A) = \mu,$$

and the equality of w with u is a direct consequence of the uniqueness of the eigenvector associated to $r(A)$. \square

The theory of the balanced growth is very rich. The following proposition is a special case, it is a synthesis of Propositions 2.3 and 2.4 and 2.5 and Remark 2.2 in [13]) and for the precise form of P , see Proposition III.8.5 p. 189 in [12].

Lemma 6. *Let $(S_t)_{t \geq 0}$ a irreducible, positive and compact semigroup on a Banach lattice E , then there exists a positive projection of rank one, $P = x_0 \otimes x'_0$ where x_0 is, up to a constant, the eigenvalue associated to $s(A)$ and $x'_0 \in E'$ such that $x'_0(x_0) = 1$, and constants $\delta > 0$ and $M \geq 1$ such that*

$$\left\| e^{-ts(A)} S_t - P \right\|_{L^1(D)} \leq M e^{-\delta t}.$$

1.2. A brief recalling of the results of Tony Lelièvre and alii. We recall here the theory developed by Tony Lelièvre, Mouad Ramil, and Julien Reygnier in a probabilistic framework for the kinetic Fokker-Planck in a bounded domain with absorbing [8, 9]. From here, we use there notations except the ones of the operators \mathcal{L} and \mathcal{L}^* . Indeed, in [8, 9], \mathcal{L} (respectively, \mathcal{L}^*) corresponds to \mathcal{L}^* (respectively \mathcal{L}) in the present paper, because the operator \mathcal{L}^* defined in (2).

Let $(X_t^x = (q_t^x, p_t^x))_{t \geq 0}$ the Langevin process starting at $x \in D$ at $t = 0$ defined by the following SDE:

$$\begin{cases} dq_t^x &= p_t^x dt, \\ dp_t^x &= F(q_t^x) dt - \gamma p_t^x dt + \sigma dB_t, \\ (q_0^x, p_0^x) &= x, \end{cases}$$

where $(B_t)_{t \geq 0}$ is a d -dimensional Brownian motion. Define also the first exit time from D of the process $(X_t^x)_{t \geq 0}$:

$$\tau_D^x := \inf \{t > 0 : X_t^x \notin D\}.$$

We denote the following unbounded operator:

$$\mathcal{L}_\gamma^* \equiv p \cdot \nabla_q + F(q) \cdot \nabla_p - \gamma p \cdot \nabla_p + \frac{\sigma^2}{2} \Delta_p.$$

It is the operator defined in (2) with the subscript γ in order to emphasize its dependence on the parameter γ . It is the infinitesimal generator of the Langevin process whereas (3) describes the evolution of Langevin process absorbed at ∂D . That explains why in a probabilistic framework such as in [8, 9], the initial problem with \mathcal{L}_γ^* is most often studied while in mathematical physics, it is (3) that arouses the most interest.

That being said, the first main result [8, Theorem 2.10] is then

Theorem 7. *Let $\gamma \in \mathbb{R}$ and let $f \in C^b(D \cup \Gamma^-)$ and define*

$$u_\gamma : (t, x) \mapsto \mathbb{E} \left[\mathbb{1}_{\tau_D^x > t} f(X_t^x) \right]$$

then

(1) (Initial and boundary values) the function u satisfies :

$$u_\gamma(0, x) = \begin{cases} f(x) & \text{if } x \in D \cup \Gamma^-, \\ 0 & \text{if } x \in \Gamma^+ \cup \Gamma^0, \end{cases}$$

and

$$\forall t > 0, \forall x \in \Gamma^+ \cup \Gamma^0, u(t, x) = 0.$$

(2) (Continuity) we have :

$$u_\gamma \in C^b((\mathbb{R}_+ \times \overline{D}) \setminus (\{0\} \times (\Gamma^- \cup \Gamma^0)))$$

(3) (Interior regularity) $u \in C^\infty(\mathbb{R}_+^*)$ and, for all $t > 0$, $x \in D$

$$\partial_t u_\gamma = \mathcal{L}_\gamma^* u_\gamma$$

Moreover, $f \mapsto \left((t, x) \mapsto \mathbb{E} \left[\mathbb{1}_{\tau_\partial^x > t} f(X_t^x) \right] \right)$ is a semigroup with nice properties including the existence of a transition kernel. We sum up the main results (specially Theorem 2.7 and Theorem 2.9 in [9] and Theorem 2.14 in [8]).

Theorem 8. Let $\gamma \in \mathbb{R}$, there exists

$$(t, x, y) \mapsto p_t^\gamma(x, y) \in C^\infty(\mathbb{R}_+^* \times D \times D) \cap C(\mathbb{R}_+^* \times \overline{D} \times \overline{D})$$

which satisfies for all $t > 0$

- $p_t^\gamma(x, y) > 0$ for all $x \notin \Gamma^+ \cup \Gamma^0$ and $y \notin \Gamma^- \cup \Gamma^0$
- $p_t^\gamma(x, y) = 0$ if $x \in \Gamma^+ \cup \Gamma^0$ or $y \in \Gamma^- \cup \Gamma^0$

Moreover, for $f \in C^b(\overline{D})$, the functions u_γ, v_γ defined by

$$\forall t > 0, \forall x \in D, u_\gamma(t, x) := \int_D p_t^\gamma(x, y) f(y) dy, \quad v_\gamma(t, x) := \int_D p_t^\gamma(y, x) f(y) dy$$

are in $C^\infty(\mathbb{R}_+^* \times D)$ and satisfy

$$(5) \quad \begin{cases} \partial_t u_\gamma & = \mathcal{L}_\gamma^* u_\gamma, \\ u_\gamma & = 0 \text{ on } \Gamma^+, \\ u_\gamma(0, \cdot) & = f; \end{cases}$$

and

$$(6) \quad \begin{cases} \partial_t v_\gamma & = \mathcal{L}_\gamma v_\gamma, \\ v_\gamma & = 0 \text{ on } \Gamma^-, \\ v_\gamma(0, \cdot) & = f. \end{cases}$$

Besides, the solution of (5) satisfies Maximum principle. For any $t > 0$, the operator

$$Q_t^\gamma f : x \in \overline{D} \mapsto \mathbb{E} \left[\mathbb{1}_{\tau_\partial^x > t} f(X_t^x) \right] = \int p_t^\gamma(y, x) f(y) dy$$

is well-defined and forms a positive and irreducible semigroup of compact operators $(Q_t^\gamma)_{t>0}$ on $L^n(D)$ to $L^n(D)$ and $C^b(\overline{D})$ to $C^b(\overline{D})$. Moreover, it maps $L^n(D)$ into $C^b(\overline{D})$ continuously for any $n \in [1, \infty]$.

Eventually, we recall here a special case of Theorem 2.13 in [9], the last property being a consequence of the irreducibility of the semigroup (see Lemma 4.3 in [8])

Theorem 9. *Under assumptions above, there exists $\lambda_\gamma > 0$ such that $\lambda_\gamma + \gamma d > 0$ and positive functions $\varphi_\gamma, \psi_\gamma \in C^b(\overline{D}) \cap L^1(D)$, unique up to multiplicative constants such*

$$(7) \quad \begin{cases} \mathcal{L}\varphi_\gamma + \lambda_\gamma\varphi_\gamma & = 0 \\ \varphi_\gamma & = 0 \text{ on } \Gamma^- \cup \Gamma^0 \end{cases}$$

and

$$(8) \quad \begin{cases} \mathcal{L}^*\psi_\gamma + \lambda_\gamma\psi_\gamma & = 0 \\ \psi_\gamma & = 0 \text{ on } \Gamma^+ \cup \Gamma^0. \end{cases}$$

2. PROOF OF THEOREM 2 AND THEOREM 3

We establish here Theorem 2 and Theorem 3. The proof of Theorem 2 is split into several steps.

2.1. A semigroup for the kinetic Fokker-Planck equation. In order to use the results of [8, 9] recalled above, first notice that if $u_{-\gamma}$ is the solution associated to $\mathcal{L}_{-\gamma}^*$ as in Theorems 7 and 8 then it is $C^\infty(\mathbb{R}_+^* \times D)$ and thus we can derive it in classical sense. It is then easy to check that the function

$$v(t, q, p) := e^{d\gamma t} u_{-\gamma}(t, q, -p)$$

solves (3) in the classical sense. That remark motivates the introduction of the operator defined on the set of measurable functions $L^0(D)$

$$(Sf)(q, p) := f(q, -p) \quad \forall f \in L^0(D).$$

The operator S is obviously bounded on $L^n(D)$ and $C^b(\overline{D})$, moreover we have $S^2 = I$ which imply that the operators defined by

$$(9) \quad P_t^\gamma := e^{d\gamma t} S Q_t^{-\gamma} S,$$

with Q_t^γ defined in Theorem 8, form a semigroup, which, together with Theorem 2.8 in [9], implies the following theorem:

Theorem 10. *The operators $(P_t^\gamma)_{t \geq 0}$ form a positive and irreducible semigroup of compact operators on $L^n(D)$ for any $n \in [1, \infty]$ and $C^b(\overline{D})$. Moreover, it maps $L^n(D)$ into $C^b(\overline{D})$. And for any $f \in L^n(D)$, $v_\gamma := P_t^\gamma f$ is in $C^\infty(\mathbb{R}_+^* \times D)$ and satisfies*

$$\begin{cases} \partial_t v_\gamma & = \mathcal{L}_\gamma v_\gamma \text{ on } D, \\ v_\gamma & = 0 \text{ on } \Gamma^-, \\ v_\gamma(0, \cdot) & = f. \end{cases}$$

Moreover, we have the identity for any $f \in C^b(\overline{D})$ and $v_\gamma = P_t^\gamma f$

$$v_\gamma = \int_D p_t^\gamma(y, x) f(y) dy.$$

where p_t^γ is the transition kernel defined in Theorem 8.

From here on, we drop the subscript γ whenever it is obvious from the context what the value of γ is, P_t for P_t^γ and so on. We now show that P_t is a substochastic semigroup:

Lemma 11. *For any $t > 0$*

$$\|P_t\|_{L^1(D)} \leq 1.$$

As consequence, $\forall f \in L^1(D)$, $t \mapsto P_t f \in L^\infty(\mathbb{R}_+; L^1(D))$.

Proof. By density, it suffices to show that for any $t > 0$ and for any $f \in C^b(\overline{D})$, we have

$$\|P_t f\|_{L^1(D)} \leq \|f\|_{L^1(D)}.$$

By Theorem 10 we have for any $f \in C^b(\overline{D})$

$$P_t f = \int_D p_t^\gamma(y, x) f(y) dy$$

thus for any $g \in L^\infty(D)$ by Fubini-Tonelli

$$\begin{aligned} \int_D P_t f g dx &= \int_D \int_D p_t(y, x) f(y) g(x) dy dx \\ &= \int_D f(y) \left(\int_D p_t(y, x) g(x) dx \right) dy \end{aligned}$$

Thus by Maximum principle applied to $\int_D p_t(y, x) g(x) dx$ that is solution of (5) by Theorem 8, we have for any $g \in L^\infty(D)$

$$\left| \int_D P_t f g \right| \leq \|f\|_{L^1(D)} \|g\|_{L^\infty(D)}$$

which implies by Hahn-Banach:

$$\|P_t f\|_{L^1(D)} \leq \|f\|_{L^1(D)}$$

which is the desired result. \square

Lemma 12. *Let $\lambda > 0$ and φ be defined as in Theorem 9, then*

$$P_t \varphi = e^{-\lambda t} \varphi, \forall t > 0.$$

We can now show that $(P_t)_{t \geq 0}$ is a strongly continuous semigroup:

Proposition 13. *The operators $(P_t)_{t \geq 0}$ with $P_0 = I$ and $(P_t)_{t > 0}$ defined above form a C_0 -semigroup of $L^1(D)$. As a consequence, the semigroup $(Q_t)_{t \geq 0}$ associated to the adjoint Cauchy problem is also strongly continuous on \mathbb{R}_+ in $L^1(D)$.*

Proof. It is already known that $(P_t)_{t > 0}$ is strongly continuous on \mathbb{R}_+^* by Theorem 8. It remains to show that it is strongly continuous at $t = 0$. By Proposition I.5.3 p. 38 in [7], it suffices to show that there exist $\delta > 0$, $M \geq 1$, and a dense subset $\mathcal{D} \subset L^1(D)$ such that

- (1) $\|P_t\|_{L^1(D)} \leq M$ for all $t \in [0, \delta]$,
- (2) $\lim_{t \rightarrow 0^+} P_t g = g$ strongly in $L^1(D)$ for all $g \in \mathcal{D}$.

By Lemma 11, we know that $\|P_t\|_{L^1(D)} \leq 1$. It remains to establish (2) for $\mathcal{D} = C_c(D)$. Let $g \in C_c(D)$ that we assume nonnegative and denote $h(t, x) := P_t g(x)$ for any $x \in D$, we know that $h \in C(\mathbb{R}_+ \times D)$, thus for any $t \in [0, T]$, $x \mapsto h(t, x)$ is measurable. In the same way, for every $x \in D$, $t \mapsto h(t, x)$ is continuous. Let φ defined in Theorem 9 and Lemma 12. As $g \in C_c(D)$ and $\varphi > 0$ everywhere, there exists $C > 0$ such that

$$g \leq C\varphi \text{ a.e. on } D$$

and by positivity of P_t and Lemma 12, we have

$$h \leq C e^{-\lambda t} \varphi \leq C\varphi \text{ a.e. on } D, \forall t \geq 0$$

with $\varphi \in L^1(D)$. In others words, h is dominated on $[0, T]$ by a integrable function on D which implies that:

$$t \rightarrow \int_D h(t, x) dx \text{ is continuous on } [0, T].$$

As h is nonnegative, Schaffé's lemma implies:

$$(10) \quad \|P_t g - g\|_{L^1(D)} \rightarrow 0 \text{ as } t \rightarrow 0^+ \forall \text{ nonnegative } g \in C_c(D)$$

Now consider $g \in C_c(D)$ that can change sign and let $g = g_+ - g_-$ with $g_{\pm} \in C_c(D)$ and $g_{\pm} \geq 0$. We have

$$\|P_t g - g\|_{L^1(D)} \leq \|P_t g_+ - g_+\|_{L^1(D)} + \|P_t g_- - g_-\|_{L^1(D)}$$

thus by (10)

$$\|P_t g - g\|_{L^1(D)} \rightarrow 0 \text{ as } t \rightarrow 0^+ \forall g \in C_c(D),$$

Consequently, $(P_t)_{t \geq 0}$ is a C_0 -semigroup of $L^1(D)$.

As for $(Q_t)_{t \geq 0}$, it is a direct consequence of (9), i.e.

$$Q_t^\gamma = e^{d\gamma t} S P_t^{-\gamma} S.$$

□

Let us make also an important remark

Remark 2. For any $t > 0$, the kernel p_t satisfies a Gaussian upper-bound (see [8, Corollary 2.22] or [9, Theorem 2.7]) that is established by studying the Langevin process on the space \mathbb{R}^{2d} after extending the driving force over the whole space. The authors have deduced an inequality valid on the whole space which they then truncate on $\Omega \times \mathbb{R}^d$. As the two functions are continuous, the proof of the authors implies that the inequality is valid on $\bar{\Omega} \times \mathbb{R}^d$ and thus on $\partial\Omega \times \mathbb{R}^d$.

The recall being done, for the sake of readability, we split the proof of Theorem 2 into several steps. First we show that for any $f^{in} \in L^1(\Omega \times \mathbb{R}^d)$, $t \mapsto P_t f^{in}$ belongs to the good function space.

Lemma 14. Let $f^{in} \in L^1(D)$ and $f := P_t f^{in}$ with $(P_t)_{t > 0}$ defined in Theorems 8 and 7. Then $f \in L^\infty([0, T]; L^1(D))$.

Proof. It is simply a direct consequence that for any $t > 0$, we have $\|P_t f^{in}\|_{L^1(D)} \leq \|f^{in}\|_{L^1(D)}$. □

We now show that for any smooth initial condition, the solution in the sense of Theorems 8 and 7 is also a weak solution.

Lemma 15. Let $f^{in} \in C^b(D \cup \Gamma^+)$ and $u := P_t f^{in}$ with $(P_t)_{t > 0}$ defined in Theorems 8 and 7, then for any $\phi \in \mathcal{D}_{Q_T}$ we have

$$\int_{Q_T} u(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp + \int_D f^{in}(q, p) \phi(0, q, p) dq = 0.$$

In others words, u is a weak solution of (3)

Proof. We cannot apply integration by parts (Green Theorem) to the cylindric domain Q_T because f is not C^1 on Q_T . Therefore, we introduce $\varepsilon \in (0, T)$ and the subset:

$$Q_{\varepsilon, T} := [\varepsilon, T] \times D.$$

Let $f^{in} \in C^b(D \cup \Gamma^+)$ and let u be defined in Theorem 7. We have $u \in C^b(\bar{Q}_{\varepsilon, T}) \cap C^\infty(Q_{\varepsilon, T})$, so that by Green Theorem applied to the cylindric domain (see Theorem 5.5 p. 47 in [11]), we have for any $\phi \in \mathcal{D}_{\bar{Q}_T}$

$$\begin{aligned} 0 &= \iint_{Q_{\varepsilon, T}} (\partial_t u - \mathcal{L}u) \phi dt dx \\ &= \iint_{Q_{\varepsilon, T}} (-\partial_t \phi - \mathcal{L}^* \phi) u dt dx - \iint_D u(\varepsilon, x) \phi(\varepsilon, x) dx \end{aligned}$$

We know that, since $u \in C^b((\mathbb{R}_+ \times \bar{D}) \setminus (\{0\} \times (\Gamma^- \cup \Gamma^0)))$,

$$u(\varepsilon, \cdot) \phi(\varepsilon, \cdot) \rightarrow f^{in} \phi(0, \cdot) \text{ as } \varepsilon \rightarrow 0^+ \text{ a.e. on } \Omega \times \mathbb{R}^d.$$

Moreover

$$|u(\varepsilon, \cdot)\phi(\varepsilon, \cdot)| \leq \|f^{in}\|_{L^\infty(D)} \left(\sup_{\varepsilon \geq 0} \phi(\varepsilon, \cdot) \right).$$

As $(\sup_\varepsilon \phi(\varepsilon, \cdot)) \in L^1(\Omega \times \mathbb{R}^d)$ since it is in $\mathcal{D}_{\tilde{Q}_T}$, we can apply dominated convergence to get:

$$\iint_D u(\varepsilon, x)\phi(\varepsilon, x)dx \xrightarrow{\varepsilon \rightarrow 0^+} \iint_D f^{in}\phi dx$$

Hence u satisfies for all $\phi \in \mathcal{D}_{\tilde{Q}_T}$

$$\iint_{Q_{\varepsilon, T}} (\partial_t \phi + \mathcal{L}^* \phi) u dt dx + \iint_D f^{in}(x)\phi(0, x)dx = 0$$

which is the desired result. \square

Let now extend the result above to the case of $f^{in} \in L^1(D)$.

Lemma 16. *Let $f^{in} \in L^1(D)$ and for any $t > 0$, $f(t, \cdot) := P_t f^{in}$ then $f \in L^\infty([0, T]; L^1(D))$ and for any $\phi \in \mathcal{D}_{Q_T}$ we have*

$$\int_{Q_T} f(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp + \int_D f^{in}(q, p)\phi(0, q, p) dq = 0.$$

In others words, f is a weak solution of (3)

Proof. Let $f^{in} \in L^1(D)$. We know that for any $t > 0$, $P_t \in \mathcal{L}(L^p(D))$. Keeping it in mind, take $f_n^{in} \in C_0^\infty(D)$ such that $f_n^{in} \xrightarrow{n \rightarrow +\infty} f^{in}$ in $L^1(D)$. Then, denoting $f_n := P_t f_n^{in}$ we have by Lemma 15 for any $\phi \in \mathcal{D}_{\tilde{Q}_T}$ and for any $n \in \mathbb{N}$:

$$(11) \quad \int_{Q_T} f_n(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp + \int_D f_n^{in}(q, p)\phi(0, q, p) dq = 0.$$

By Lebesgue partial converse theorem (see Theorem 4.9 p. 94 in [4]), we can moreover assume that, up to a subsequence, there exists $g \in L^1(D)$ such that:

$$|f_n^{in}(x)| \leq g \text{ a.e. in } x \in D,$$

and

$$f_n^{in} \rightarrow f^{in} \text{ a.e. in } x \in D.$$

That implies immediately:

$$(12) \quad \int_D f_n^{in}(q, p)\phi(0, x)dx \rightarrow \int_D f^{in}(x)\phi(0, x)dx \text{ as } n \rightarrow +\infty,$$

and

$$(13) \quad \forall (t, x) \in [0, T] \times D, \quad \int_D p_t(y, x) f_n^{in} dy \xrightarrow{n \rightarrow +\infty} \int_D p_t(y, x) f^{in} dy.$$

Besides, observe that

$$0 \leq f_n(t, x) \leq \int_D f_n^{in}(y) p_t(y, x) dy \leq \int_D g(y) p_t(y, x) dy$$

with $\int_D g(y) p_t(y, x) dy \in L^1([0, T] \times D)$ and by (13)

$$\int_D f_n^{in}(y) p_t(y, x) dy \rightarrow \int_D f^{in}(y) p_t(y, x) dy \text{ a.e. in } (t, x) \in [0, T] \times D$$

as $n \rightarrow +\infty$. Therefore by dominated convergence we have

$$f_n \rightarrow f \text{ in } L^1([0, T] \times D) - \text{strong}$$

as $n \rightarrow +\infty$. That implies:

$$(14) \quad \int_{Q_T} f_n(-\partial_t \phi + \mathcal{L}^* \phi) dt dq dp \rightarrow \int_{Q_T} f(-\partial_t \phi + \mathcal{L}^* \phi) dt dq dp$$

as $n \rightarrow +\infty$. Applying (12) and (14) to (11) shows that f is a weak solution to the initial boundary problem (3) with f^{in} as initial condition.

As for the uniqueness of the weak solution, as the problem is linear, it suffices to show that $f \in L^1(Q_T)$ such that

$$\int_{Q_T} f(\partial_t \phi + \mathcal{L}^* \phi) dt dq dp = 0, \quad \forall \phi \in \mathcal{D}_{\tilde{Q}_T}$$

implies

$$f = 0.$$

If for any $\psi \in C_c((0, T) \times D)$, there exists $\phi \in \mathcal{D}_{\tilde{Q}_T}$ solution of the backward problem

$$(15) \quad \begin{cases} \partial_t \phi + \mathcal{L}^* \phi & = \psi \\ \phi|_{\Gamma^0 \cup \Gamma^-} & = 0 \\ \phi(T, \cdot) & = 0 \end{cases}$$

that would imply that $f = 0$ in distributional sense and thus $f = 0$ in $L^1(Q_T)$. Consider the forward problem:

$$(16) \quad \begin{cases} \partial_t \tilde{\phi} - \mathcal{L}^* \tilde{\phi} & = \psi \\ \tilde{\phi}|_{\Gamma^0 \cup \Gamma^-} & = 0 \\ \tilde{\phi}(0, \cdot) & = 0 \end{cases}$$

It is obvious that $t \mapsto \tilde{\phi}(T-t, \cdot)$ is solution of (15). As $(Q_t)_{t \geq 0}$ is strongly continuous by Proposition 13, we have by Duhamel formula [7, Corollary III.1.7 p. 161]

$$\tilde{\phi} = \int_0^t Q_{t-s} \psi(s) ds$$

thus

$$\begin{aligned} \phi &= \int_0^{T-t} Q_{T-t-s} \psi(s) ds \\ &= \int_t^T Q_{T-s} \psi(s-t) ds \end{aligned}$$

It remains to show that $\phi \in \mathcal{D}_{\tilde{Q}_T}$ to conclude. By linearity, we can assume that ψ is nonnegative. That being said, first, by the properties of the function p_t (Theorems 8 and 7 above), we have $\phi \in C_0^\infty(\tilde{Q}_T)$. Besides, we have obviously $(\partial_t + \mathcal{L}^*)\phi \in L^\infty(Q_T)$. Eventually, let ψ_λ being the positive eigenvector associated to \mathcal{L}^* , see (8) in Theorem 9, with eigenvalue λ and $\int_D \psi_\lambda = 1$. Let K compact of $(0, T) \times D$ such that $\text{supp } \psi \subset K$. As ψ_λ is positive everywhere on D and constant in time, there exists $c > 0$ such that $\psi_\lambda \geq c$ for any $(t, x) \in K$. Therefore, there exists $C > 0$ such that

$$\psi \leq C \psi_\lambda \text{ everywhere on } (0, T) \times D.$$

As $(Q_t)_{t > 0}$ are positive operators, the properties of ψ_λ and the inequality above imply

$$Q_{t-s} \psi(s) \leq C e^{-\lambda(t-s)} \psi_\lambda.$$

Integrating the inequality above in time $s \in [0, t]$ leads to:

$$\tilde{\phi} \leq \frac{C}{\lambda} (1 - e^{-\lambda t}) \psi_\lambda,$$

thus

$$0 \leq \phi \leq \frac{C}{\lambda} \left(1 - e^{-\lambda(T-t)}\right) \psi_\lambda$$

which implies that $\phi \in L^1(D; L^\infty((0, T)))$. Consequently, $\phi \in \mathcal{D}_{\tilde{Q}_T}$ which is the desired conclusion. \square

2.2. Proof of Theorem 3. Now, we establish here Theorem 3

- (1) By Theorem 2, for any $f^{in} \in L^1(D)$, there exists a unique weak solution f in sense of Definition 1, thus we have $f(t, \cdot) = P_t f^{in}$ with $(P_t)_{t \geq 0}$ defined in (9). By Theorem 10, Lemma 11 and Proposition 13, it is a positive, irreducible, contractive and compact C_0 - semigroup.
- (2) By Theorem 7 with the property of semigroup, we have for any $f^{in} \in L^1(D)$, $\forall t > 0$, $P_t f^{in} \in C^\infty(D) \cap C^b(\bar{D})$.
- (3) With the properties of the semigroup established above, the balanced growth law of $(P_t)_{t \geq 0}$ is a direct consequence of Lemma 6 with the eigenvectors identified in Theorem 9.

3. A TRACE THEORY FOR THE KINETIC FOKKER-PLANCK EQUATION

We now establish a trace theory for the kinetic Fokker-Planck equation. First, using the regularity of the weak solution, we can easily define its trace:

Lemma 17. *Let f be the weak solution with initial condition $f^{in} \in L^1(D)$. The trace of f on $\mathbb{R}_+^* \times \partial D$ is well defined with $\gamma f := f|_{\partial D}$ and $\gamma_\pm := \gamma f|_{\pm(p \cdot n_q) > 0}$. Moreover we have*

$$\gamma_- f = 0 \text{ and } \gamma_+ \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| dq dp).$$

Proof. By Theorem 8, we know that for any $t > 0$, P_t maps continuously $L^1(D)$ into $C^b(\bar{D})$. Thus we can define $\gamma f := f|_{\partial\Omega \times \mathbb{R}^d}$ and $\gamma_\pm := \gamma f|_{\pm(p \cdot n_q) > 0}$.

Besides, by semigroup property, we have for any $t > 0$, $P_t f^{in} = P_{t/2}(P_{t/2} f^{in})$ with $P_{t/2} f^{in} \in C^b(\bar{D})$ which implies by Theorem 7 that $\gamma_- f = 0$.

Now show that $\gamma_+ f \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| dq dp)$. As for any $t > 0$, $\gamma_+ f \in C^b(\partial D)$, it is measurable. Now we have by the semigroup property and by continuity:

$$\gamma_+ f(t, x) = \int_D p_{t/2}(x, y) f\left(\frac{t}{2}, y\right) dy$$

with $y \mapsto f\left(\frac{t}{2}, y\right) \in C^b(\partial D)$, thus

$$|\gamma_+ f(t, x)| \leq \left\| f\left(\frac{t}{2}, \cdot\right) \right\|_{C^b(\partial D)} \int_D p_{t/2}(x, y) dy.$$

The Gaussian estimates on $p_{t/2}$ (see [8, Corollary 2.22] and Remark 2) implies that

$$x \in \partial D \mapsto \int_D p_{t/2}(x, y) dy \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot p| d\sigma_{\partial\Omega} dp),$$

hence the desired result. \square

The existence of the trace and the regularity of the weak solution implies that the integration by parts, also known as Green's formula, is satisfied:

Lemma 18. *Let f be the weak solution with initial condition $f^{in} \in L^1(D)$ and let γf be its trace. Then, it satisfies Green's formula, meaning that for any $t > 0$ and for any $\phi \in C^\infty(D) \cap C(\bar{D})$ we have*

$$\int_D (\mathcal{L}f)\phi = \int_D f \mathcal{L}^* \phi + \int_{\partial D} \gamma f \phi (n_q \cdot p) dp d\sigma_{\partial\Omega}.$$

Proof. By unicity of the weak solution, we know that for any $t > 0$, $f = P_t f^{in}$ and thus $f \in C^\infty(D) \cap C(\overline{D})$ with $\gamma f \in L^1(\partial\Omega \times \mathbb{R}^d; |n_q \cdot |d\sigma_{\partial\Omega} dp)$ by Lemma 17. Hence, the classical theorem of Green's formula (Integration by parts) [11, Th 5.5 p. 47] applies here and leads to the desired result. \square

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