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Eternal Feedback Vertex Sets: A New Graph Protection Model Using Guards

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Abstract Graph protection using mobile guards has received a lot of attention in the literature. It has been considered in different forms, including Eternal Dominating set, Eternal Independent set, and Eternal Vertex Cover set. In this paper, we introduce and study two new models of graph protection, namely *Eternal Feedback Vertex sets (EFVS)* and *m-Eternal Feedback Vertex sets (m-EFVS)*. Both models are based on an initial selection of a feedback vertex set (*FVS*), where a vertex in *FVS* can be replaced with a neighboring vertex such that the resulting set is a *FVS* too. We prove bounds for both the *eternal* and *m-eternal feedback vertex numbers* on, mainly, distance graphs, circulant graphs, and grids. Also, we deduce other inequalities for both parameters on cycles, complete graphs and complete bipartite graphs.

Keywords Graph protection, eternal security problem, bipartite graphs, distance graphs, grids

Mathematics Subject Classification (2020) Combinatorics

1 Introduction

The use of mobile guards to protect a graph is a problem that has already been initiated and studied in the literature [1]. The problem consists in defending the vertices, as well as edges, of a graph G against any attack using defense units, called guards, stationing at the vertices of G .

A protection problem can be modeled by a two-player game between a defender and an attacker. The defender chooses the set of vertices needed to hold the guards in the first turn, and must defend any attack on any vertex,

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by finding a new guard configuration at each turn. On the other hand, the attacker chooses at each turn the location of the attack. An attack is defended if a guard can be moved to the attacked vertex across one edge. The defender wins the game if each sequence of attacks is successfully defended, the attacker wins otherwise. We note that the sequence of attacks may be infinite in length.

Several variants of the problem have been studied in the literature, including eternal dominating set [1], eternal independent set [2], and eternal vertex cover set [3]. In this paper, we define two new variants of graph protection, namely *Eternal Feedback Vertex Sets (EFVS)* and *m-Eternal Feedback Vertex Sets (m-EFVS)*. The two variants are modeled in the same way using a two-player game, with the particular condition that the set of vertices chosen by the defender must be, at the same time, a feedback vertex set (*FVS*) and a dominating set (*DS*) in all turns. In other words, the *Eternal Feedback Vertex Set* on a graph can be seen as an infinite game between a defender and an attacker, where the defender chooses a set of guards F_i , such that for any $i \geq 1$, F_i is a feedback vertex set. At turn i , the attacker chooses a vertex r_i , called an attack, in $V \setminus F_{i-1}$ and the defender must defend the attack by moving to r_i a guard on a vertex v_j adjacent to r_i . The new guards configuration is $F_i = F_{i-1} \cup \{r_i\} \setminus \{v_j\}$. The sets F_i , for any $i \geq 1$, are also dominating sets. In the case where the defender protects vertices by moving more than one guard, we talk about *m-Eternal Feedback Vertex Sets*.

Protecting a graph with mobile guards is a problem that has already been studied in many recent papers with different situations, such as eternal dominating [1], eternal independent [4], and eternal vertex cover sets [3], etc. It involves the placement of mobile guards on the vertices of a graph in order to protect them against any attack. We are mainly interested in protecting graphs against infinite sequences of attacks executed one at a time. Good surveys of these topics can be found in [5,6].

The study of graph protection started with the domination theory (a thorough survey of this theory can be found in [7]). In 2004, Burger *et al.* [1] studied, for the first time, the eternal dominating problem, where it was called *infinite order domination*, and sometimes referred to as the *one-guard moves model*. The set of vertices D_i holding the guards, for each turn i , is required to be a dominating set, and D_{i+1} is obtained from D_i by moving one guard to the attacked vertex from an adjacent vertex in D_i . The authors proved that the size of a smallest eternal dominating set of a graph G , called *eternal dominating number*, lies between the independence and clique covering numbers, *i.e.* $\alpha(G) \leq \gamma^\infty(G) \leq \theta(G)$. Many other results have also been obtained [1].

Goddard *et al.* [2] computed the eternal domination number for some particular classes of graphs, and showed that for any graph G , if $\alpha(G) = 2$, then $\gamma^\infty(G) \leq 3$. They also presented a new variant of the eternal domination problem, known as *m-eternal domination set*, and referred to as *guards move model*. For this model, each D_i , $i > 1$, is required to be a dominating set, and D_{i+1} is obtained from D_i by allowing each guard to move to a neighboring vertex, *i.e.* each guard in D_i may move to an adjacent vertex. The size of a smallest m-eternal dominating set of G , known as the *m-eternal domination*

number and denoted γ_m^∞ , has been computed for grids and many other classes of graphs [5].

In [8], authors dealt with eternal domination on directed graphs. They studied both parameters, γ^∞ and γ_m^∞ , on this class of graphs, and proved that computing the oriented eternal dominating number is NP-hard. Also, they generalized known results on graphs to digraphs. Many other dominating models have been defined in the recent years, such as the eternal total domination and eternal connected domination [9].

Similarly, the eternal independent set problem was introduced by Hartnell *et al.* [4] and also studied in [10]. They defined it with the two variants, one guard move model and many guards move model, and the two parameters, the eternal and m-eternal independent number (α^∞ and α_m^∞) have been studied on different classes of graphs leading to several results [4, 10].

Our main focus in this work is to compute both parameters, eternal feedback vertex (EFV) and m-eternal feedback vertex (m-EFV) numbers, namely F^∞ and F_m^∞ , corresponding to the smallest eternal and m-eternal feedback vertex set, on some particular classes of graphs. We organized the paper as follows. In *Section 2*, we present different definitions and notation we use in the rest of the paper. In *Section 3*, we provide formal definitions for our new models. Then, we compare the two parameters, F^∞ and F_m^∞ , with known graph parameters in *Section 4*. In *Section 5*, we deduce some inequalities for F^∞ and F_m^∞ on cycles, complete graphs and complete bipartite graphs, and provide a detailed study for distance graphs, circulant graphs and grids. We close up the paper with a conclusion and some future problems to be considered.

2 Definitions and Notations

In this paper, all graphs are considered finite, without multiple edges and without loops. For a graph $G = (V, E)$, V is the vertex set and E is the edge set. For each vertex $v \in V$, let $N(v)$ denote the open neighborhood of v , *i.e.* $N(v) = \{u \in V \mid vu \in E\}$, and $N[v] = N(v) \cup \{v\}$. Given a subset of nodes $S \subseteq V$, S is an *independent set* of G if there is no edge uv for every $u, v \in S$. The *independence number* $\alpha(G)$ is the size of a largest independent set of G . A set S is a *dominating set* of G if $\bigcup_{v \in S} N[v] = V$. The *domination number* $\gamma(G)$ is the size of a smallest dominating set of G . A set $S \subseteq V$ is said to be a *feedback vertex set (FVS)* if its removal results in a forest. We denote by $F(G)$ the minimum cardinality over all FVS, *i.e.* $F(G) = \min\{|S|, S \text{ is a FVS}\}$. A *minimum feedback vertex set (MFVS)* is a FVS of cardinality $F(G)$.

We denote the *path graph*, *cycle graph* and *complete graph*, of order n , by P_n , C_n and K_n , respectively, and the *complete bipartite graph* with partite sets of cardinalities n and m by $K_{n,m}$.

A graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$, is a *distance graph*, denoted $P_n(D)$, if it is defined on a set of distances $D = \{d_1, \dots, d_k\}$, where d_i is a positive integer, and two distinct vertices $v_i, v_j \in V$ being adjacent if and only

if $|i - j| \in D$. We sometimes omit the brackets and we write $P_n(d_1, \dots, d_k)$ instead of $P_n(\{d_1, \dots, d_k\})$. A graph G with $V = \{v_1, v_2, \dots, v_n\}$ is a *circulant graph*, denoted $C_n(D)$, if it is defined on $D = \{d_1, d_2, \dots, d_k\}$, where the set of neighbors of a given vertex v_i is $\{v_k | k = (i \pm d_j) \bmod n, j = 1, 2, \dots, k\}$, such that $1 \leq d_1 < d_2 < \dots < d_k \leq \frac{n}{2}$. We sometimes omit the brackets and we write $C_n(d_1, \dots, d_k)$ instead of $C_n(\{d_1, \dots, d_k\})$.

Given two graphs G and H , we denote the *cartesian product* of G and H by $G \times H$.

3 Eternal and m-Eternal Feedback Vertex Sets

In the same way as the eternal dominating problem [1,2], we introduce the *Eternal Feedback Vertex Set Problem (EFVS)* as the problem of protecting a graph using mobile guards, where the subset of vertices holding guards must be, at each turn, both a feedback and a dominating set.

In our variant, we consider the case where the attack is defended if a guard moves to the attacked vertex across only one edge. Using the two-players game model, first, the defender chooses a set F_0 of k vertices which hold the guards. At turn i , the attacker attacks by choosing a vertex $r_i \in V \setminus F_{i-1}$, and the defender must defend the attack by moving to r_i a guard from an adjacent vertex v_i . The new guards configuration is $F_i = F_{i-1} \cup \{r_i\} \setminus \{v_i\}$. The defender wins the game if any infinite sequence of attacks is defended. The *eternal feedback vertex number*, denoted $F^\infty(G)$, is the minimum number of guards necessary for the defender to win, and the *eternal feedback vertex set* is a set that can initially be chosen by the defender in a winning strategy.

Let $G = (V, E)$ be a graph. The set $EFVS(G)$, of eternal feedback vertex sets of G , is the greatest set of subsets of V such that for every subset $S \in EFVS(G)$ and every $r \in V \setminus S$, there is a vertex $v \in S$ such that $vr \in E$ and $S \cup \{r\} \setminus \{v\} \in EFVS(G)$. The size of a smallest eternal feedback set of G is the *eternal feedback vertex number* $F^\infty(G)$, i.e $F^\infty(G) = \min\{|S|, S \in EFVS\}$.

For the *m-Eternal Feedback Vertex Problem*¹, it is the variant where the defender is authorized to move several guards at a time. It is defined in the same way as the *EFVS*, except that when the attack occurs, each guard is allowed to move to a neighboring vertex through only one hop. Several guards movement can be considered either to defend the attacked vertex, or to make a better position of guards for the coming turns.

Let $G = (V, E)$ be a graph and let $S_1, S_2 \subset V$ be two subsets. A multi-move f from S_1 to S_2 is a one-to-one mapping from S_1 to S_2 such that for every $x \in S_1$, we have $f(x) = x$ or $(x, f(x)) \in E$. The set $MEFVS(G)$ of m-eternal feedback vertex sets of G is the greatest set of subsets of V such that for every $S \in MEFVS(G)$ and every $r \in V(G) \setminus S$, there is a multimove f such that $r \in f(S)$ and $f(S) \in MEFVS(G)$. The size of a smallest m-eternal

¹ Notice that the m in m-eternal does not represent a value.

feedback vertex set of G is the *m-eternal feedback vertex number* $F_m^\infty(G)$; *i.e.* $F_m^\infty(G) = \min\{|S|, S \in MEFVS\}$.

Considering the cycle graph C_6 where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ (see *Figure 1*). Removing any vertex from C_6 results in a path, *i.e.* a minimum feedback vertex set contains only one vertex. The smallest dominating set of C_6 must contain at least two vertices. Thus, we must start the game with at least two guards, and as a defender, we choose two vertices to hold the guards. Without loss of generality, let $F_1 = \{v_2, v_5\}$ be the initial set of guards. Assume that the attacker choose vertex v_3 , so the defender must move the guard from v_2 to v_3 . The new configuration $F_2 = \{v_3, v_5\}$ is not a dominating set (see *Figure 1a*), and so using only two guards the defender will lose the game. If we start the game with three guards, and without loss of generality, let $F_1 = \{v_2, v_4, v_6\}$ (see *Figure 1b*). A guard on a shaded vertex can move to an unshaded neighbor, and the resulting guard configuration induces a dominating feedback vertex set. *Figure 1b* illustrates the case where v_3 is attacked and the guard moves from v_2 to v_3 .

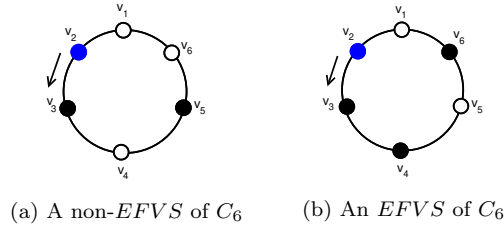


Fig. 1: C_6 , the guards are placed on the black shaded vertices

One can notice that $F^\infty(P_3) = 2$ and $F^\infty(C_3) = 1$. *Figure 2* illustrates possible configurations on the house graph G , where $F^\infty(G) = 2$.

4 EFVS and m-EFVS on general graphs

On general graphs, the following observation, comparing the two numbers F^∞ and F_m^∞ with some known graph parameters, comes directly from the definitions of the *EFVS* and *m-EFVS*.

Observation 1 *Let G be a general graph, then*

$$F^\infty(G) \geq F_m^\infty(G) \geq F(G). \quad (1)$$

Given the result comparing both the eternal domination and m-eternal domination numbers of a graph G with the domination and independent numbers of G :

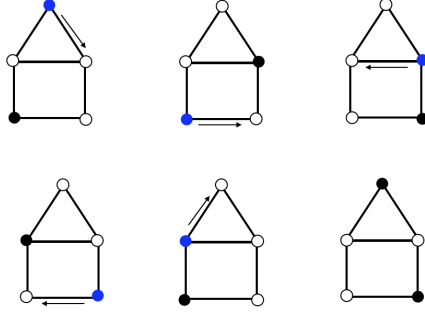


Fig. 2: Eternal feedback vertex set on house graph (shaded vertices are guards).

Theorem 2 [1, 2, 11] For any graph G , we have

$$\gamma(G) \leq \gamma_m^\infty(G) \leq \alpha(G) \leq \gamma^\infty(G) \leq \binom{\alpha(G) + 1}{2}.$$

and based on the fact that each eternal (*resp.* m-eternal) feedback vertex set is an eternal (*resp.* a m-eternal) dominating set, the following inequalities hold:

Theorem 3 Let G be any graph, then $\gamma_m^\infty(G) \leq F_m^\infty(G)$ and $\gamma^\infty(G) \leq F^\infty(G)$.

Consequently, we have the following corollary:

Corollary 1 For any graph G , $\alpha(G) \leq \gamma^\infty(G) \leq F^\infty(G)$.

The relation between the eternal and m-eternal feedback vertex numbers of a graph and its induced subgraphs is given by the following:

Proposition 1 For any graph G , and any induced subgraph H of G , we have

$$F^\infty(G) \geq F^\infty(H) \text{ and } F_m^\infty(G) \geq F_m^\infty(H).$$

Proof For the first inequality, we suppose that $F^\infty(G) < F^\infty(H)$. Thus, for any EFVS F of G , F is not an eternal FVS of H . On the other hand, the sequence of attacks in G restricted to the induced subgraph H requires only $F^\infty(G)$ guards, which is a contradiction. The same proof is used for the second inequality. ■

It was proved in [8] that the eternal (*resp.* m-eternal) dominating number of a graph is the sum of the eternal (*resp.* m-eternal) dominating number of its components. This is trivially true also for the eternal (*resp.* m-eternal) feedback number for any graph.

Corollary 2 Given a graph G consisted of the connected components G_1, G_2, \dots, G_i , $i > 0$, we have:

$$F^\infty(G) = \sum_{i=1} F^\infty(G_i), \quad \text{and} \quad F_m^\infty(G) = \sum_{i=1} F_m^\infty(G_i).$$

5 EFVS and m-EFVS on particular classes of Graphs

In this section, we compute F^∞ and F_m^∞ on some particular classes of graphs. First, we note that for any acyclic graph G , it is trivial to see that the feedback vertex number is equal to zero, hence finding an eternal feedback vertex set for G is equivalent to find an eternal dominating set.

For cycles, complete graphs, and complete bipartite graphs, we have the following observations:

Observation 4 Given a cycle C_n and a complete graph K_n , for any $n \geq 3$, and given a complete bipartite graph $K_{n,m}$, where $V = A \cup B$, $|A| = n$ and $|B| = m$, then:

1. $\lfloor \frac{n}{2} \rfloor \leq F^\infty(C_n) \leq \lceil \frac{n}{2} \rceil$.
2. $F_m^\infty(C_n) = \lceil \frac{n}{3} \rceil$.
3. $F^\infty(K_n) = F(K_n) = n - 2$.
4. $F^\infty(K_{n,m}) = n + m - 3$.
5. $F_m^\infty(K_{n,m}) = \max\{n, m\}$.

For cycles, removing any vertex produces a path graph, thus finding the eternal feedback vertex number is equivalent to finding the eternal domination number. For m-Eternal FVS, an example on C_{12} , *Figure3*, illustrates that $F_m^\infty = \lceil \frac{n}{3} \rceil$.

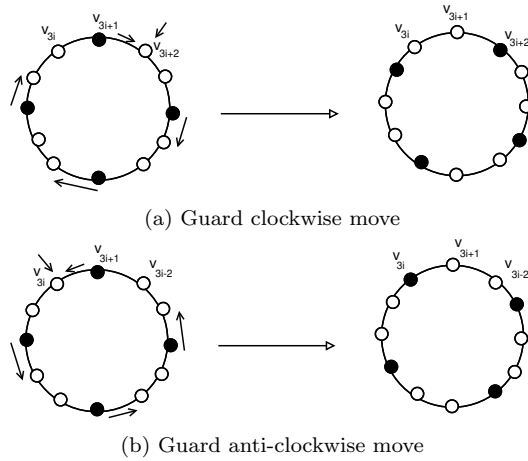


Fig. 3: Example of m-Eternal Feedback Vertex Set on C_{12} . The guards are placed on the black shaded vertices.

In a complete graph K_n , any three vertices form a cycle, so any feedback vertex set must contain at least $n - 2$ vertices and thus both eternal and m-eternal feedback vertex numbers of a complete graph are equal to its feedback vertex number.

Now, for complete bipartite graphs where $A = \{v_1, \dots, v_n\}$ and $B = \{u_1, \dots, u_m\}$, a cycle $v_i v_j u_k u_l$ is obtained for all $i, j, k, l \geq 1$. So we can see that, at all turns, we must place the guards in such a way that if A holds less than $n - 2$ guards, then B must contain at least $m - 1$ guards. Therefore, the only way to get this is to place $n + m - 3$ guards. *Figure 4b* shows that if we use less than $n + m - 3$ guards to protect a complete graph, $K_{n,m}$, a cycle will be obtained.

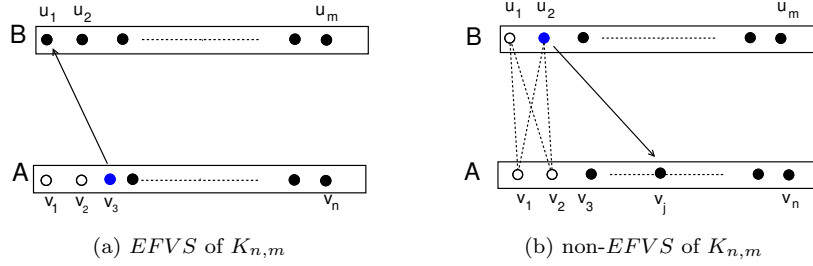


Fig. 4: $K_{n,m}$, the guards are placed on the black shaded vertices, and the blue nodes holding the guard before moving.

In the case of m -Eternal FVS, all vertices of A (or B ; we select the maximum) must hold guards (see *Figure 5*). Note that each partition in $K_{n,m}$ is an independent set, and so no cycle is obtained by removing a set partition.

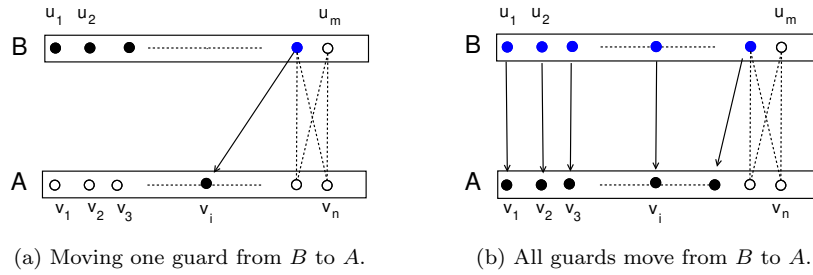


Fig. 5: $K_{n,m}$, the guards are placed on the black shaded vertices, and blue nodes are guard positions before moving.

5.1 Distance graphs and Circulant graphs

Along this section, we use the following bonds of the minimum feedback vertex set number, F , of different families of distance and circulant graphs, summarized in *Table 1* [13].

D	$F(P_n(D))$		$F(C_n(D))$	
	Lower bound	Upper bound	Lower bound	Upper bound
$\{1, 2, 4\}$	$4\lfloor \frac{n}{8} \rfloor$	$\lceil \frac{n}{2} \rceil$	$4\lfloor \frac{n}{8} \rfloor$	$\lceil \frac{n+4}{2} \rceil$
$\{1, t\}$	$\lceil \frac{n-t}{3} \rceil$	$\lceil \frac{n-2}{3} \rceil$	$\lceil \frac{n+1}{3} \rceil$	$\lceil \frac{n+t}{3} \rceil + 1$
$\{1, s, t\}$	$\lceil \frac{2n-s-t}{5} \rceil$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	$\lceil \frac{n+t}{2} \rceil$
$\{1, 2, 3, \dots, t\}$	$\begin{cases} \lfloor \frac{n}{t+1} \rfloor (t-1) & \text{if } n = 0, 1 \pmod{t+1} \\ n - 2\lfloor \frac{n}{t+1} \rfloor - 2 & \text{otherwise} \end{cases}$		$n - 2\lfloor \frac{n}{t+1} \rfloor - 1$	$n - 2\lfloor \frac{n}{t+1} \rfloor$

 Table 1: Bounds of $MFVS$ on distance graphs and circulant graphs.

We note that hereafter, for figures, we illustrate only edges needed to explain the proofs, and so the other edges can be easily deduced from the definition of the graph class.

5.1.1 Distance Graphs

Depending on the values of D , we first consider the particular case where $D = \{1, 2, 4\}$, and then the more general cases where $D = \{1, t\}$, and $D = \{1, 2, 3, \dots, t\}$, where t is a non-negative integer.

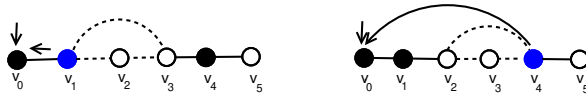
Case 1. $D = \{1, 2, 4\}$.

Given a distance graphs $P_n(D)$ of n vertices, where two distinct vertices v_i, v_j are adjacent in P_n if and only if $|i - j| \in \{1, 2, 4\}$.

Theorem 5 For any integer n , we have $F^\infty(P_n(1, 2, 4)) = 5\lfloor \frac{n}{8} \rfloor$.

Proof Let F_1 be any $EFVS$ of a distance graph $G = P_n(1, 2, 4)$. According to Proposition 5 in [13], there are two possible cases for F_1 :

- (i) Only two vertices over the first six are in F_1 . In this case, it is obligatory to choose vertices v_1 and v_4 , as removing them is the only possibility to get an acyclic graph. Suppose that any other vertex is attacked, without loss of generality let be v_0 . To defend v_0 , we must move the guard either from v_1 to v_0 , or from v_4 to v_0 . In both cases, at least one cycle will be generated, such as $v_1v_2v_3$ for the first move and $v_2v_3v_4$ for the second move (see Figure 6). For illustration, if we assume that the guard moves from v_1 to v_0 , then the new guard configuration is $\{v_0, v_1\}$. According to the construction of $P_n(1, 2, 4)$, v_1 is adjacent to v_2 , v_2 is adjacent to v_3 and v_1 is adjacent to v_3 . Hence, a cycle is generated by $\{v_1, v_2, v_3\}$.


 Fig. 6: $P_6(1, 2, 4)$, guards are on black shaded vertices.

- (ii) At least three vertices over the first six are in F_1 . We arbitrary choose three non-consecutive vertices to hold guards in the first turn, considered as a minimum eternal FVS for $P_6(1, 2, 4)$. Consider now the first 12 vertices of G , and suppose that we apply the same strategy on each 6 vertices. A counter example can be occurred as follows. Without loss of generality and as we consider a game with infinite turns, let's suppose that at turn i , we have $F_i = \{v_0, v_3, v_5, v_6, v_8, v_{11}\}$, and vertex v_7 is attacked. All possible guard moves will generate at least one cycle. In fact, if the guard is moved from v_3 to v_7 , v_5 to v_7 , v_6 to v_7 , v_8 to v_7 , or v_{11} to v_7 , we get respectively cycle $v_2v_3v_4$, $v_1v_2v_4v_5$, $v_2v_4v_6$, $v_8v_9v_{10}$, or $v_9v_{10}v_{11}$ (see *Figure 7*).

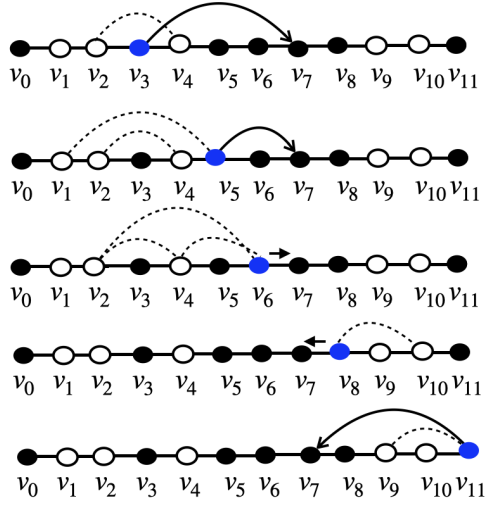


Fig. 7: The different possible cycles generated while defending an attack on vertex v_7 of $P_{12}(1, 2, 4)$.

So, for this case, we must fix the guards on vertices v_6 and v_7 to protect them. Accordingly, we can see that in general at least 5 vertices must be selected to hold guards for each 8 vertices. Consider F_1 is obtained by any FVS of three vertices selected from the six first vertices (for each block of 8 vertices), with the addition of the two vertices v_6 and v_7 , we can see easily that the defender can avoid any attack (see *Figure 8*).

Knowing that we are in the eternal variant of the feedback vertex problem, we can see that this example can be generalized for a large n , and thus the defender can win the game starting with 5 guards on each consecutive 8 vertices, and hence $F^\infty(P_n(D)) = 5 \lfloor \frac{n}{8} \rfloor$.

■

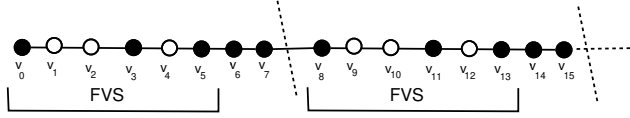


Fig. 8: A winning configuration for $P_n(1, 2, 4)$, guards are on black shaded vertices.

The following corollary follows directly from *Table 1*, *Equation (1)* and *Theorem 5*.

Corollary 3 *For any integer n , $4\lfloor \frac{n}{8} \rfloor \leq F_m^\infty(P_n(1, 2, 4)) \leq 5\lfloor \frac{n}{8} \rfloor$.*

Case 2. $D = \{1, t\}$.

This is the case where edges of the distance graph are constructed for each distance t , besides the unit distance.

In [14], the authors proved the following lemma.

Lemma 1 [14] *Let G be a path graph with n vertices, then $\gamma^\infty(P_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 6 *For any integers n and t , such that $1 \leq t < n$, we have :*

$$F^\infty(P_n(1, t)) = \lceil \frac{n}{2} \rceil + 1.$$

Proof If $t \geq n$, then $P_n(1, t)$ is a path (an acyclic graph), so we are in the case where $F(P_n(1, t)) = 0$, and $F^\infty(P_n(1, t)) = \gamma^\infty(P_n(1, t))$. For the case where $t \leq n$ and $n \geq 2t$, we have two possible cases.

(i) $n \bmod t = 0$. Depending on the parity of $\frac{n}{t}$, we have two cases.

(a) $\frac{n}{t}$ is even. We partition the graph into $\frac{n}{t}$ blocks, namely $T_1, T_2, \dots, T_{\frac{n}{t}}$, of t vertices each. Let $V(T_i) = \{v_1^i, v_2^i, \dots, v_t^i\}$ for all $1 \leq i \leq \frac{n}{t}$. According to *Lemma 1*, we have $\gamma_m^\infty(P_n) = \lceil \frac{n}{2} \rceil$. Thus, $F^\infty(P_n) \geq F_m^\infty(P_n) \geq \lceil \frac{n}{2} \rceil$ (*Equation 1*). On the other hand, and as paths are a subclass of distance graphs (a distance graph $P_n(1, t)$ is a path with some extra edges), so we have $F^\infty(P_n(1, t)) \geq F^\infty(P_n) \geq \lceil \frac{n}{2} \rceil$.

First, we prove that $F^\infty(P_n(1, t)) > \lceil \frac{n}{2} \rceil$. For that, we show that any subset F of G of $\lceil \frac{n}{2} \rceil$ vertices cannot be an *EFVS*. Without loss of generality, let $F = \{v_l^{2k+1}, 0 \leq k \leq \frac{n}{2t}, 1 \leq l \leq t\}$ since starting with any other subset of vertices of cardinality $\lceil \frac{n}{2} \rceil$ as a guard configuration, we inevitably end up with configuration F after some number of turns (note that we consider an infinite game). In fact, for instance where $G = P_{16}(1, 4)$, we let $F_1 = \{v_1^1, v_2^1, v_3^2, v_4^2, v_1^3, v_2^3, v_3^4, v_4^4\}$. After a sequence of attacks on vertices v_3^1, v_4^1, v_3^3 , and v_4^3 found in blocks T_1 and T_3 respectively, the guards should be arranged on T_1 and T_3 (see *Figure 9*). We denote this configuration F , and we have just to prove that F ,

which is clearly of cardinality $\lceil \frac{n}{2} \rceil$, is not an *EFVS*. Assume that the defender starts the game with F , and the attacker chooses to attack vertex $v_1^{\frac{n}{t}}$. The defender has only two choices to avoid the attack, either by moving the guard from vertex $v_1^{\frac{n}{t}-1}$ to $v_1^{\frac{n}{t}}$, which results in cycle $v_1^{(n/t)-2}v_2^{(n/t)-2}\dots v_t^{(n/t)-2}v_1^{(n/t)-1}$ (see *Figure 10*); or by moving the guard from $v_t^{\frac{n}{t}-1}$ to $v_1^{\frac{n}{t}}$, which results in a non-dominating guard configuration $F' = F \setminus \{v_t^{\frac{n}{t}-1}\} \cup \{v_1^{\frac{n}{t}}\}$. Hence F is not an *EFVS*, and so $F^\infty(P_n(1, t)) \neq \lceil \frac{n}{2} \rceil$.

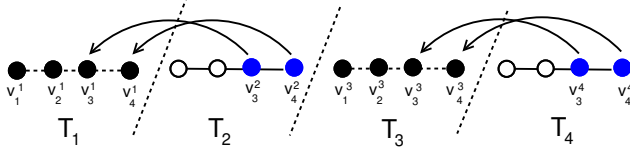


Fig. 9: $P_{16}(1, 4)$, the guards are moved from blue vertices after an attack on the two last vertices of blocks T_1 and T_3 . Black shaded vertices are guard positions after attacks (which results in a F configuration).

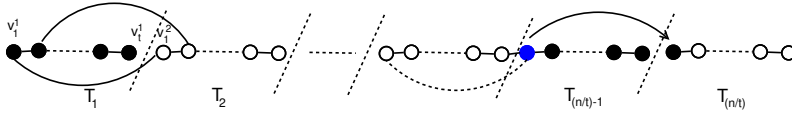


Fig. 10: $P_n(1, t)$, the guards are on black shaded vertices.

Second, we show that $F^\infty(P_n(1, t)) = \lceil \frac{n}{2} \rceil + 1$. For that, let $F' = F \cup \{v_t^{\frac{n}{t}}\}$ be the set that holds the guards in the initial turn. We have $|F'| = |F| + 1 = \lceil \frac{n}{2} \rceil + 1$, and F' is a dominating *FVS* of G (since F is a dominating *FVS*). Starting the game with F' , the defender will win the game by applying the following strategy. Depending on the block of the attacked vertex, we have two defense strategies: (a') where $k = 1$, *i.e.* the attack occurs on the two first consecutive blocks, the defender can move the guard from v_l^1 to v_l^2 when v_l^2 is attacked, for all $1 \leq l \leq t$. (b') where $k \geq 2$, we deal with each attacked vertex according to its position in the block. If vertices v_l^{2k} , for $2 \leq l < t$, are attacked then the defender can move the guard from v_l^{2k-1} to v_l^{2k} . If vertices v_t^{2k} are attacked, the guard can be moved from v_1^{2k+1} to v_t^{2k} . Finally, vertices v_1^{2k} are protected by moving the guard from v_t^{2k} to v_1^{2k} . Note that all the obtained guard configurations F' are dominating *FVS* as $G \setminus F'$ has no chord of length t joining two consecutive blocks. Thus, fixing a

guard on $v_t^{\frac{n}{t}}$ allows to deal with the non-domination encountered above (where $F^\infty(P_n(1, t)) = \lceil \frac{n}{2} \rceil$).

- (b) $\frac{n}{t}$ is odd. As above, we partition the graph into $\frac{n}{t}$ blocks, say $T_1, T_2, \dots, T_{\frac{n}{t}}$, of t vertices each. Let G' be the graph obtained from G by removing vertices of the last block $T_{\frac{n}{t}}$, where $n' = n - t$ is the number of vertices of G' . We deal with G' as follows. We have $\frac{n'}{t}$ is even, and so $F^\infty(G') = \lceil \frac{n'}{2} \rceil + 1$. For a winning strategy, we have to just choose any dominating set of cardinality $\frac{t}{2}$ for block $T_{\frac{n}{t}}$. We note that block $T_{\frac{n}{t}}$ is a path graph with t vertices, and as $F(T_{\frac{n}{t}}) = 0$ so any eternal dominating set is an *EFVS*. Accordingly, we have:

$$F^\infty(G) \leq F^\infty(G') + F^\infty(T_{\frac{n}{t}}) \leq \lceil \frac{n'}{2} \rceil + 1 + \frac{t}{2} \leq \lceil \frac{n}{2} \rceil + 1.$$

Now, we assume that there exists an *EFVS* of G with cardinal less than $\lceil \frac{n}{2} \rceil + 1$. There is at least $\lceil \frac{n'}{2} \rceil + 1$ guards for G' , so the path graph $G[T_{\frac{n}{t}}]$ will be protected by less than $\lceil \frac{t}{2} \rceil$, which leads to a contradiction, *i.e.* after some moves of the guards, we necessarily get three consecutive vertices in $T_{\frac{n}{t}}$ without guards. And hence $F^\infty(G) = \lceil \frac{n}{2} \rceil + 1$.

- (ii) $n \bmod t \neq 0$. We partition the graph into $\lceil \frac{n}{t} \rceil$ blocks, $T_1, \dots, T_{\lceil \frac{n}{t} \rceil}$, each of t vertices except the last block, $T_{\lceil \frac{n}{t} \rceil}$, contain $r < t$ vertices. Similar to case (i), we deal with two cases according to the parity of $\lceil \frac{n}{t} \rceil$.
- (a) $\lceil \frac{n}{t} \rceil$ is even. Let $G' = G \setminus T_{\lceil \frac{n}{t} \rceil}$ be the graph obtained from G by deleting the last block, and let n' be the number of vertices of G' . We use the same strategy as in case (i)(b) on graph G' , and we have $F^\infty(G') = \lceil \frac{n'}{2} \rceil + 1$. In fact, $T_{\lceil \frac{n}{t} \rceil}$ is a path graph with $r < t$ vertices, thus the last block can be defended by $\lceil \frac{r}{2} \rceil$ guards. Hence,

$$F^\infty(G) \leq F^\infty(G') + F^\infty(T_{\lceil \frac{n}{t} \rceil}) = \lceil \frac{n'}{2} \rceil + 1 + \lceil \frac{r}{2} \rceil = \lceil \frac{n}{2} \rceil + 1.$$

Now, we assume that the defender can win the game with less than $\lceil \frac{n}{2} \rceil + 1$ guards. We know that for a winning strategy, it is obligatory to have at least $\lceil \frac{n'}{2} \rceil + 1$ to protect G' in each turn of the game. And so the path graph $T_{\lceil \frac{n}{t} \rceil}$ is supposed to be protected by less than $\lceil \frac{r}{2} \rceil$ guards, which leads to a contradiction as this results in a non-dominating guard configuration. And hence, $F^\infty(G) = \lceil \frac{n}{2} \rceil + 1$.

- (b) $\lceil \frac{n}{t} \rceil$ is odd. Similarly, we let G' be the graph obtained from G by removing the last r vertices. We use exactly the same proof as in (ii)(a), except that in this case we deal with G' according to (i)(a).

■

We note that the proof of *Theorem 6* present as well a linear solving algorithm of the *EFVS* problem on $P_n(1, t)$.

Directly from [13], *Equation (1)* and *Theorem 6*, we get the following corollary.

Corollary 4 For any n and t , and $G = P_n(1, t)$ we have:

$$\begin{aligned} \lfloor \frac{n-t}{2} \rfloor \leq F_m^\infty(G) \leq \lceil \frac{n}{2} \rceil + 1, & \quad \text{if } \frac{n}{2} \leq t \leq n. \\ \lfloor \frac{n-t}{4} \rfloor \leq F_m^\infty(G) \leq \lceil \frac{n}{2} \rceil + 1, & \quad \text{if } \frac{n}{3} \leq t \leq \frac{n}{2}. \\ \lfloor \frac{n-t}{3} \rfloor \leq F_m^\infty(G) \leq \lceil \frac{n}{2} \rceil + 1, & \quad \text{if } t \geq 2 \text{ and } n \geq t. \end{aligned}$$

Case 3. $D = \{1, 2, 3, \dots, t\}$.

In this case, the graph is more dense and contains all edges established between each pair of vertices considering all possible distances less or equal than t .

Theorem 7 Let $G = P_n(1, 2, 3, \dots, t)$. For any $n \geq 5$ and $t \leq \frac{n-2}{2}$, we have :

$$F^\infty(P_n(D)) = \begin{cases} \lfloor \frac{n}{t+1} \rfloor (t-1) & n \bmod (t+1) = 0, 1, \\ (n-2) \lfloor \frac{n}{t+1} \rfloor - 2 & \text{otherwise.} \end{cases}$$

Proof Directly from Table 1 and Observation 1, we have:

$$F^\infty(P_n(D)) \geq F(P_n(D)) \geq \begin{cases} \lfloor \frac{n}{t+1} \rfloor (t-1) & n \bmod (t+1) = 0, 1 \\ (n-2) \lfloor \frac{n}{t+1} \rfloor - 2 & \text{otherwise.} \end{cases}$$

To prove the equality it is sufficient to prove that the *FVS* constructed in [13] can be an initial *EFVS* for a wining strategy. Let $G = P_n(1, 2, 3, \dots, t)$, we decompose G into q blocks of $(t+1)$ vertices each, and one block with r vertices, where $n = q(t+1) + r$, $q \geq 1$ and $0 \leq r \leq t+1$, and let v_i^s denote the i^{th} vertex in block s , for $i \geq 0$ and $s \geq 1$.

Let $F = \{v_i^s, 2 \leq i \leq t, 1 \leq s \leq q\} \cup \{v_i^s, 2 \leq i \leq r-1, 1 \leq s \leq q\}$. F is a *FVS* of G . In fact, the graph obtained from G after removing the vertices of F is an acyclic graph. All the chords that join the vertices of the two consecutive blocks are removed except the chord that join the vertices v_1^s and v_0^{s+1} , for all $s \geq 1$. Thus, $G[V \setminus F]$ is a path graph (see Figure 11).



Fig. 11: The vertices of F are shaded

Assume that each vertex of F holds a guard, we can see that the first two vertices of each block are not in F , i.e. $v_i^s \in F$ for all $i \geq 2$ and $v_i^s \notin F$ for $i = 0, 1$ (see Figure 11). Thus, F is a dominating set of G , and all vertices are protected by at least one guard. For instance, vertex v_1^s is protected by v_2^s and v_0^s . We suppose that the first vertex on the block is attacked, without

loss of generality, let be v_0^1 , so the defender can avoid this attack with many ways, such as through moving a guard from v_t^1 to v_0^1 (no cycle is obtained) (see *Figure 12*).

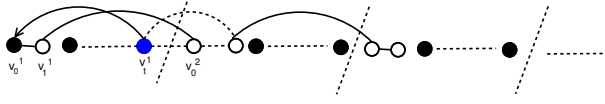


Fig. 12: $P_n(1, 2, \dots, t)$, $EFVS$ is shaded

Generalizing this model of protection to all the other blocks ensures that the new guard configurations, after each turn, is a dominating FVS , and the defender can avoid any attack. Therefore, starting with F , where $|F| = q(t-1)$ if $n \bmod (t+1) = 0, 1$, and $|F| = q(t-1) + r - 2$ otherwise, the defender wins the game, and the theorem is proved. ■

By *Equation (1)* and *Theorem 7* the following corollary follows.

Corollary 5 *Let $n \geq 5$ and $t \leq \frac{n-2}{2}$, then*

$$F_m^\infty(P_n(D)) = \begin{cases} \lfloor \frac{n}{t+1} \rfloor (t-1) & n = 0, 1 \bmod (t+1) \\ (n-2) \lfloor \frac{n}{t+1} \rfloor - 2 & \text{otherwise.} \end{cases}$$

5.1.2 Circulant Graphs

In this section, we consider the class of circulant graphs, and we deal with two cases, depending on the set of distance D , namely the case when $D = \{1, s, t\}$ and when $D = \{1, t\}$, where s and t are assumed satisfies $1 < s < t \leq \frac{n}{2}$.

We first establish the relationship between both parameters, F^∞ and F_m^∞ , on distance and circulant graphs. We have, by definition, distance graphs are a subclass of circulant graphs, *i.e* circulant graph is a distance graph with some extra edges (chords), so the following corollary holds.

Corollary 6 *For any $n \geq 1$ and any set D ,*

$$F^\infty(P_n(D)) \leq F^\infty(C_n(D)), \text{ and } F_m^\infty(P_n(D)) \leq F_m^\infty(C_n(D)).$$

For the general case, where $D = \{1, s, t\}$, the set of neighbors of a vertex v_i is $\{v_k | k = (i \pm w) \bmod n, w = 1, s, t, \text{ and } j = 1, 2, \dots, n\}$, where n is the number of vertices in the graph. The following bounds immediately follows from *Corollary 6*, *Equation (1)* and *Table 1*.

Corollary 7 *For any integers n, s and t , with $t \leq \frac{n-1}{2}$ and $2 \leq s < t$, $\lceil \frac{2n+1}{5} \rceil \leq F_m^\infty(C_n(1, s, t)) \leq F^\infty(C_n(1, s, t))$.*

For the case where $D = \{1, t\}$, the set of neighbors of a vertex v_i is $\{(i \pm w) \bmod n, w = 1, t, \text{ and } j = 1, 2, \dots, n\}$.

Theorem 8 For any $n \geq 5$ and $t < \frac{n}{2}$, we have:

$$F^\infty(C_n(1, t)) = \lceil \frac{n}{2} \rceil + 1.$$

Proof From Corollary 6 and Theorem 6, we have $F^\infty(C_n(1, t)) \geq \lceil \frac{n}{2} \rceil + 1$. To prove the equality, let $G = C_n(1, t)$, and let $H = P_{n-t}(1, t)$ be the subgraph of G induced by $\{t, t+1, \dots, n-1\}$. Directly from Theorem 6, we get $F^\infty(H) = \lceil \frac{n-t}{2} \rceil + 1$. On the other hand, for the cycle graph induced by the first t vertices $\{v_0, v_1, \dots, v_{t-1}\}$ of G , we can find an EFVS of size at most $\lceil \frac{t}{2} \rceil$. Therefore, there is an eternal feedback vertex set for $C_n(1, t)$ of size at most $(\lceil \frac{n-t}{2} \rceil + 1) + (\lceil \frac{t}{2} \rceil) = \lceil \frac{n}{2} \rceil + 1$. Hence, $F^\infty(C_n(1, t)) = \lceil \frac{n}{2} \rceil + 1$. ■

Using Equation (1) and Theorem 8, we obtain the following corollary.

Corollary 8 For any $n \geq 5$ and $t < \frac{n}{2}$, $F_m^\infty(C_n(1, t)) = \lceil \frac{n}{2} \rceil + 1$.

5.2 Grids

We compute F^∞ on a grid graph $G = P_n \times P_m$ obtained by the product of two paths P_n and P_m . Let assume that $n \leq m$ for all $n, m > 1$.

Theorem 9 Let G be a grid, such that $G = P_n \times P_m$, and let $n \leq m$. $F^\infty(G)$ is computed as follows:

$$F^\infty(G) = \begin{cases} \frac{m \times n}{2}, & \text{if } n \text{ or } m \text{ is even,} \\ \lceil \frac{m}{2} \rceil \times n, & \text{else.} \end{cases}$$

Proof Let $G = P_n \times P_m$, with n rows and m columns. We label the vertices of G according to their positions, such that $v_{i,j}$ is the vertex obtained on row i and column j , where $i \leq n$ and $j \leq m$. There are two possible cases depending on the parity of n and m .

- (i) n or m is even. First, let suppose that n and m are even. We assume that $F = \{v_{i,2k-1} | i, k \geq 1\}$ is the set of vertices chosen by the defender to hold the guards in the initial turn. F is a dominating feedback vertex set (we choose the vertices of each odd column), and $|F| = \frac{n \times m}{2}$. For an attack on any vertex $v_{i,j} \notin F$, say $v_{i,2k}$ where $k \geq 1$, the defender can remove a guard from the vertex $v_{i,2k-1}$ to $v_{i,2k}$, and the new guard configuration is FVS and dominating. Also, if at turn i , vertex $v_{i,2k-1} \notin F_i$ is attacked, then moving a guard from $v_{i,2k}$ to $v_{i,2k-1}$ ensures defending the attack. Thus, by applying this strategy, the defender will win the game. Now, suppose that the defender starts the game with a number of guards less than $\frac{n \times m}{2}$, so using the same strategy, there is a turn k where the

new configuration will no longer be a domination. In fact, without loss of generality let $F' = F \setminus \{v_{3,1}\}$ be the set obtained from F by deleting a vertex, say $v_{3,1}$. If $v_{2,4}$ is attacked, then if the guard moves from $v_{2,3}$ to $v_{2,4}$, so the new guard configuration is non-domination ($v_{1,3}$ is not protected by any guard), see *Figure 13a* for an example when $n = m = 4$. Otherwise, for $n > 4$, if the guard moves from $v_{2,5}$ to $v_{2,4}$, then as n is even, after a series of attacks, all possible moves lead to a non-dominating configuration. And hence we get the result that the minimal number of guards is $\frac{n \times m}{2}$.

In what follows, we generalize this result for any subset making an *EFVS* for G . In doing so, we first show that this is true for the particular case of grids, namely $P_2 \times P_2$, and then we generalize it for $P_m \times P_2$ and $P_n \times P_m$, for any n, m .

For $P_2 \times P_2$, which is a cycle graph of length four, and using *Observation 4,1*, we have $F^\infty(P_2 \times P_2) = \frac{n}{2} = \frac{4}{2} = \frac{n \times m}{2}$. To prove that $F^\infty(P_2 \times P_m) = \frac{n \times m}{2} = m$ is true for all m , we proceed by induction. We assume that it is true for all m , and prove it for $m + 2$. Let $G = P_2 \times P_{m+2}$ be a grid graph, and assume that $F^\infty(G) < \frac{2 \times (m+2)}{2} = m + 2$. Let $G' = P_2 \times P_m$ be the subgraph induced by the first m columns of G , and G'' be the subgraph induced by the last two columns. We have $F^\infty(G'') = F^\infty(P_2 \times P_2) = 2$, and from the hypothesis of induction we have $F^\infty(G') = \frac{2 \times m}{2} = m$. Thus, we contradict our assumption, $F^\infty(G) < m + 2$ (note that there is a turn where the subgraph G'' will hold only one guard which is a non-dominating configuration). Therefore, we need at least $m + 2$ guards, and so $F^\infty(P_2 \times P_m) = m$.

Now, we similarly generalize the result for all n and m . By induction, we suppose that it is true for any n and m , and prove that it remains for $n' = n + 2$ and any m . Let $G = P_{n'} \times P_m$ be a grid graph, where $n' = n + 2$, and n and m are even. And let G_1 be the subgraph induced by the first n rows of G . We have $G' = P_n \times P_m$, and so $F^\infty(G') = \frac{n \times m}{2}$ (induction hypothesis). On the other hand, we have $G'' = G[V(G) \setminus V(G')]$ is $P_2 \times P_m$, and so $F^\infty(G'') = m$ (which we have already proved). So assuming that $F^\infty(P_{n'} \times P_m)$ is less than $\frac{n' \times m}{2}$ leads to a contradiction. In fact, in this case, the defender must move a guard from G' to G'' to avoid an attack which results in a non dominating guard configuration. *Figure 13* illustrates the case of $P_4 \times P_4$. As we can see such a graph must be defended by $\frac{4 \times 4}{2} = 8$. The same prove can be used for the case where n is odd and m is even. For the case when n is even and m is odd, we deal with this case in the same manner as the previous one, by switching between the rows and columns. We let $F = \{v_{2k-1,j} | k, j \geq 1\}$ be the set of vertices chosen by the defender in the initial turn. F is a dominating feedback vertex set, and $|F| = \frac{n \times m}{2}$ (we choose the vertices of each odd row). For an attack on any vertex $v_{2k,j}$, where $k \geq 1$, the defender can move the guard from $v_{2k-1,j}$ to $v_{2k,j}$ and the new guard configuration is dominating *FVS* after any turn of the game. And thus, F is an *EFVS*.



Fig. 13: $P_4 \times P_4$, the guards are placed on the black shaded vertices, and blue nodes are guard positions before moving.

(ii) n and m are odd. We assume that the defender begin the game with $F' = \{v_{i,2k+1}, i \geq 1, k \geq 0\}$ (we choose all vertices of odd columns). F' is a dominating feedback vertex set and $|F'| = \lceil \frac{m}{2} \rceil \times n$. After any attack on vertices $v_{i,2k+2}$, the defender can move guards from $v_{i,2k+1}$ to $v_{i,2k+2}$, and the new guard configuration is dominating FVS after any turn, which is a winning strategy starting with F' . Thus, $F^\infty(G) \leq \lceil \frac{m}{2} \rceil \times n$. To prove the equality, we decompose graph G into two subgraphs, G_1 and G_2 , such that $G_1 = P_{n-1} \times P_m$ is the grid subgraph induced by the $n-1$ rows and m columns, and G_2 is the path graph induced by the last row and column of G .

From case (i), we have $F^\infty(G_1) = \frac{(n-1)(m)}{2}$, (note that $(n-1)$ is even and (m) is odd), and G_2 is a path graph with m vertices, so $F^\infty(G_2) = \lceil \frac{m}{2} \rceil$. Figure 14 illustrate the case where $n = m = 5$, $G = P_5 \times P_5$.

This complete the proof.

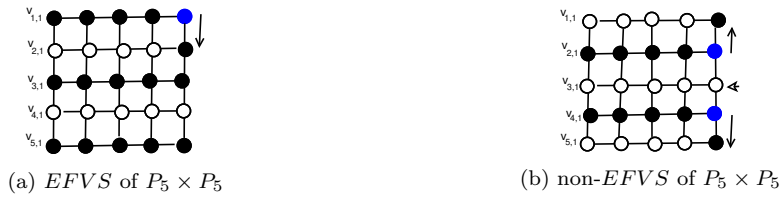


Fig. 14: $P_5 \times P_5$, the guards are placed on the black shaded vertices, and blue nodes are guard positions before moving.

■

6 Conclusion

In this paper, we proposed and studied new models of graph protection, namely *Eternal Feedback Vertex Set* and *m-Eternal Feedback Vertex Set*. Because of their difficulty, we considered the two problems in this first work, on relatively

simple classes of graphs. We deduced some inequalities for F^∞ and F_m^∞ on general graphs, cycles, complete graphs and complete bipartite graphs. Also, we computed both parameters on distance graphs, circulant graphs and grids. For future work, It is worthy to consider the general case of the problems where guards can be moved for more than one hop, as well as, to compute the two parameters, F^∞ and F_m^∞ , for these new variants on other classes of graphs, in particular those usually used for modeling games, such as k-dimensional graphs.

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Statements and Declarations

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