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DIFFERENTIATING L_∞ GROUPOIDS

PART I

DU LI, LEONID RYVKIN, ARNE WESSEL, AND CHENCHANG ZHU

ABSTRACT. Differentiating an n -groupoid via the differential-geometric fat point a priori only yields a presheaf of graded manifolds. In this article we prove that this presheaf is representable by the tangent complex of the n -groupoid. As an immediate consequence we obtain that the tangent complex carries the structure of a Lie n -algebroid.

CONTENTS

1. Introduction	1
2. Higher groupoids and graded manifolds	3
2.1. Simplicial manifolds and their horns	3
2.2. Graded manifolds and their tangent spaces	5
2.3. Iterated tangent bundles for manifolds	8
2.4. Limits for N-manifolds	9
2.5. Compatible connections for \mathbb{Z} -graded vector bundles	11
3. Representability of the tangent functor	14
3.1. Description of $\mathcal{T}(X_\bullet)$ as a limit	15
3.2. The central pullback diagram	16
3.3. Embedding $T[1]^k M/T[k]M$ into $(T[1]^{k-1}M)^{[k-1]}$	18
3.4. A monomorphism of $T[1]^k \delta^k X$ to $(T[1]^k X_{k-1})^{[k] \setminus 0}$	22
3.5. The induction step	23
4. Examples	26
References	31

1. INTRODUCTION

Lie n -groupoids ($n = 1, 2, \dots, \infty$) offer a clear and concrete way to represent geometric n -stacks in differential geometry. When $n = 1$, they reduce to the well-studied Lie groupoids, which play a fundamental role in differential geometry, topology and non-commutative geometry. The infinitesimal counterpart of a Lie n -groupoid is a Lie n -algebroid, which can be understood as a kind of differential graded (d.g.) manifold concentrated in positive degrees, referred to as an NQ manifold. D.g. manifolds are extensively used in mathematical physics, particularly in BV theory, BRST theory, AKSZ theory, and topological field theory (TFT).

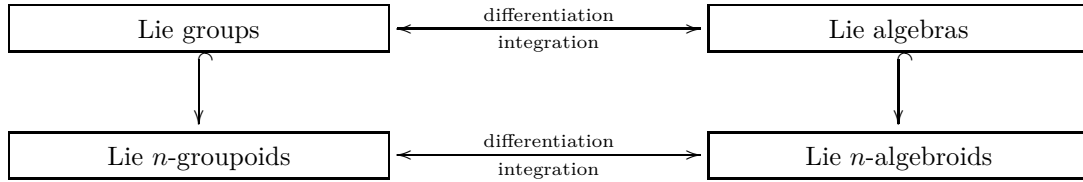
Therefore, how Lie n -groupoids and Lie n -algebroids **correspond** to each other, becomes an important and interesting problem. Inspired by the relation between Lie groups and Lie algebras, which are Lie

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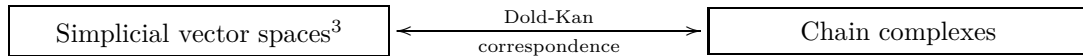
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1-groupoids over a point and Lie 1-algebroids over a point respectively, we expect the following relation



When all the Lie brackets vanish, and the base manifold is simply a point, a Lie n -algebroid is just a chain complex of vector spaces of length n . Thus the Dold-Kan correspondence may be viewed as the abelian case of the integration and differentiation correspondence.



In fact, even prior to its significant appearance in homological algebra, the problem of integration and differentiation problem is deeply rooted in differential geometry. This is evident from Lie’s three theorems as well as from Van Est’s theory. In recent times, there have been several significant milestones in this area. A few of these include: Integration of Lie algebroids [CF03], classifying the first complete integrability obstruction for Lie algebroids; integration of Poisson manifolds examined from the perspective of sigma models [CF01], creating a link to mathematical physics; integration achieved via stacky Lie groupoids that surpasses integrability obstruction from a higher point of view [TZ06], opening a link to higher structures; and integrability obstruction and surpassing construction via higher structures for infinite-dimensional Lie algebras [Nee02, WZ16]. As shown in these cases, differentiation is straightforward when $n = 1$, but integration poses a more complex and compelling challenge. This is because unlike finite-dimensional Lie algebras, many Lie-algebra-like structures cannot be integrated. However, in the higher case when $n \geq 2$, providing a universal differentiation is itself an unsolved problem. This is exactly the topic of our project.

Indeed, the higher aspect of integration and differentiation problems has become a hot topic recently, starting with Getzler’s integration of nilpotent L_∞ -algebras [Get09]. Henriques [Hen08] investigated general L_∞ -algebra integration, while Ševera and Siran [Šv20] focused on the local integration of L_∞ -algebroids. Using Lurie’s deformation functor [Lur11], Nuiten [Nui18] proposed a differentiation procedure in the derived geometry setting. Ševera [Šev] explored the differentiation of L_∞ -groupoids to presheaves in d.g. manifolds. This article also contains an idea of how to show the representability of the differentiation, i.e., how to pass from presheaves on graded manifolds to actual graded manifolds. This idea has been further pursued in [Li14] and [JSW16], effectively reducing the problem to a challenging combinatorial one. Note that there is a combinatorial gap in [JSW16, Section 4.3]⁴, as communicated to the authors. In [Li14], Lemma 8.34 exhibits a combinatorial gap.

In this article, we devote the entire Section 3.3 to solving the above combinatorial problem, and thus provide a complete proof for the conjecture from [Šev] that the tangent presheaf of an L_∞ -groupoid is representable. In addition, unlike previous proof attempts, our proof does not go through local coordinates and rather relies on connections and jet bundles. We develop a rigorous theory on graded manifolds involving their limits, splittings and compatible connections. This leads to a very natural appearance of the tangent complex as the differentiation, and illuminates the differential geometric nature of the differentiation problem.

As mentioned previously, we aim at providing a universal differentiation to all L_∞ -groupoids. In this article we use a graded version of a “fat point” D_\bullet in the world of higher groupoids and give an explicit formula of the resulting tangent object. “Fat point” is a well-known mathematical slang, coming probably from Weil⁵, appears mostly in the context of algebraic geometry. For example, $P_\epsilon := \text{Spec}(\mathbb{R}[x]/\langle x^2 \rangle)$ is

³Notice that a simplicial vector space always satisfies Kan conditions [Moo56].

⁴More precisely, it is around Eqs. (4.17) and (4.18).

⁵This should have come from André Weil’s local algebra and his idea of “points proches” [Wei53] at least this is the earliest literature that the authors can trace back to this idea.

the fat point in the world of \mathbb{R} -varieties: shooting with P_ϵ we obtain tangent spaces of a variety (see e.g. [Per08, Section V]). We prove in Theorem 3.3 that shooting with the simplicial “fat point” D_\bullet to a Lie n -groupoid X_\bullet , one obtains exactly the tangent complex of X_\bullet . That is,

Theorem (3.3). *Let D_\bullet be the simplicial nerve of the pair groupoid of D (cf. Examples 2.14 and 2.1), and X_\bullet a Lie n -groupoid. Then*

$$(1) \quad \text{Hom}(D_\bullet, X_\bullet) \cong \ker Tp_0^1|_{X_0}[1] \oplus \ker Tp_0^2|_{X_0} \oplus \cdots \oplus \ker Tp_0^n|_{X_n}[n],$$

where $p_0^k : X_k \rightarrow \mathfrak{h}^n(X)$ is the horn projection (see Eq. (5)).

The tangent complex of a Lie n -groupoid plays the role of the tangent object in the world of Lie n -groupoids, thus also differentiable n -stacks. Its first explicit formula in quotient form appears in [Get, MT18a]. Here, what we use in (1) is an isomorphic model given in [CZ23]. This formula is also observed by Du Li [Li14] in a local form and in [JSW16]. With this form, it is clear that the tangent complex of a Lie n -groupoid is a graded vector bundle of degree n . As defined in Def. 3.1, this procedure is functorial in X_\bullet . Thus D_\bullet serves well as a “fat point” in the world of higher Lie groupoids. After proving the theorem, we apply our method to various examples, including the Lie 2-groupoids of Courant algebroids [LBv15, MT11, MT18b, SZ], string Lie 2-groups, a form of higher groups called n -tower to explain higher topological orders in condensed matter physics [LZW19, ZLW19], and simplicial Lie groups [Jur12].

Since the D -action on $\text{Hom}(D_\bullet, X_\bullet)$ induces a natural homological vector field ([Šev]), an immediate corollary of our result is

Corollary. *The tangent complex of a Lie n -groupoid carries the structure of a Lie n -algebroid.*

In a follow-up article we will compute the higher brackets in an explicit form. This will require a very careful choice of the connections and splittings that are part of the isomorphism (1) and a better understanding of the Lie n -algebroid structure on an iterated shifted tangent bundle.

Acknowledgement We thank enormously Rui Fernandes who proposed to us a method using S_n symmetry based on the technique in [dHF18] to reduce dramatically the complicity of the combinatoric problem. This method would have almost solved the problem but we discovered that at a certain place it could not generalize to Lie n -groupoids when $n \geq 2$. We would like to thank very much Christian Blohmann and Lory Kadiyan, who shared with us their insights on tangent functors in general categories. We also give our warmest thanks to Miquel Cueva, Florian Dorsch, Madeline Jotz, Branislav Jurco, Joost Nuiten, Dimitry Roytenberg, Pavol Severa, Tilmann Wurzbacher. We are supported by DFG ZH 274/3-1, the Procope Project GraNum of the DAAD and Campus France, RTG 2491. Finally we thank very much the ESI Vienna program Higher Structures and Field Theory, Huazhong University of Science and Technology, and the IHP Paris Trimester Higher Structures in Geometry and Mathematical Physics, where the project was worked on.

2. HIGHER GROUPOIDS AND GRADED MANIFOLDS

2.1. Simplicial manifolds and their horns. A **simplicial manifold** X_\bullet is a contravariant functor from Δ the category of finite ordinals

$$(2) \quad [0] = \{0\}, \quad [1] = \{0, 1\}, \quad \dots, \quad [l] = \{0, 1, \dots, l\}, \quad \dots,$$

with order-preserving maps to the category of manifolds. More precisely, X_\bullet consists of a tower of manifolds X_l , face maps $d_k^l : X_l \rightarrow X_{l-1}$ for $k = 0, \dots, l$, and degeneracy maps $s_k^l : X_l \rightarrow X_{l+1}$. These maps satisfy the following simplicial identities

$$(3) \quad \begin{aligned} d_i^{l-1} d_j^l &= d_{j-1}^{l-1} d_i^l & \text{if } i < j, \\ s_i^l s_j^{l-1} &= s_{j+1}^l s_i^{l-1} & \text{if } i \leq j, \end{aligned} \quad d_i^l s_j^{l-1} = \begin{cases} s_{j-1}^{l-2} d_i^{l-1} & \text{if } i < j, \\ \text{id} & \text{if } i = j, j+1, \\ s_j^{l-2} d_{i-1}^{l-1} & \text{if } i > j+1. \end{cases}$$

We may drop the upper indices and just write d_i and s_i for simplicity when the context is clear later in our article.

Similarly, we may define other simplicial objects in other categories, such as simplicial sets, simplicial vector spaces, which we will meet in this article. For any category \mathcal{C} , we denote by \mathcal{C}_\bullet the category of its simplicial objects.

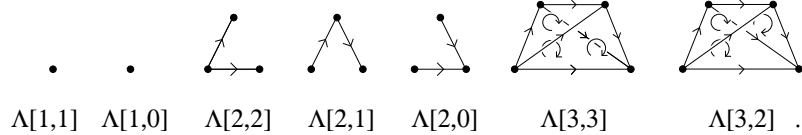
Example 2.1 (Pair groupoids in \mathcal{C}). *In any category \mathcal{C} , which admits finite products, for any object $X \in \mathcal{C}$, there exists a pair groupoid $X \times X \rightrightarrows X$, whose nerve we denote by X_\bullet . It can be seen as the functor $[n] \mapsto X^{[n]} \cong X \times \dots \times X$ ($n+1$ copies).*

Example 2.2 (Discrete groupoids in \mathcal{C}). *In any category \mathcal{C} , for any object $X \in \mathcal{C}$, the functor $[n] \mapsto X$ defines a simplicial manifold (with all faces and degeneracies being the identity). It is distinct from the nerve of the pair groupoid and somewhat abusively we will simply denote it $X \in \mathcal{C}_\bullet$.*

The following simplicial sets, which may be viewed as simplicial manifolds with discrete topology (e.g. in Def. 2.3), play an important role for us. They are the simplicial l -**simplex** $\Delta[l]$ and the **horn** $\Lambda[l, j]$,

$$(4) \quad \begin{aligned} (\Delta[l])_k &= \{f : [k] \rightarrow [l] \mid f(i) \leq f(j), \forall i \leq j\}, \\ (\Lambda[l, j])_k &= \{f \in (\Delta[l])_k \mid \{0, \dots, j-1, j+1, \dots, l\} \not\subseteq \{f(0), \dots, f(k)\}\}. \end{aligned}$$

In fact the horn $\Lambda[l, j]$ is a simplicial set obtained from the simplicial l -simplex $\Delta[l]$ by taking away its unique non-degenerate l -simplex as well as the j -th of its $l+1$ non-degenerate $(l-1)$ -simplices, as in the following picture (in this paper all the arrows are oriented from bigger numbers to smaller numbers):



Definition 2.3. A **Lie n -groupoid** [Get09, Hen08, Zhu09a] is a simplicial manifold X_\bullet where the natural projections

$$(5) \quad p_j^l : X_l = \text{Hom}(\Delta[l], X_\bullet) \rightarrow \text{Hom}(\Lambda[l, j], X_\bullet) =: X_{l,j}$$

are surjective submersions⁶ for all $1 \leq l \geq j \geq 0$ and diffeomorphisms for all $0 \leq j \leq l < n$. It is further a **Lie n -group** if $X_0 = \text{pt}$.

If we replace surjective submersions by submersions, and isomorphisms by injective étale maps, in the requirement for the natural projections p_j^l we say that X_\bullet is a **local Lie n -groupoid**, see [Zhu09b, Defi.1.1]. Similarly, it is further a **local Lie n -group** if $X_0 = \text{pt}$.

Remark 2.4. A simplicial map between two simplicial sets S and T is a family of maps $S_n \rightarrow T_n$ that intertwine the face and degeneracy maps on S and T . If we fix S , we may view the set of all simplicial maps $S \rightarrow T$ as the limit of a certain diagram; the diagram is indexed by the category \mathcal{I} with objects $\square S_n$ and arrows $(s, f) : s \mapsto f^*(s)$ for all $s \in S_n$, $f : [m] \rightarrow [n]$; the diagram maps $s \in S_n$ to the corresponding element in T_n and the arrow (s, f) to $f^* : T_n \rightarrow T_m$. If we replace T by a simplicial object in a category \mathcal{C} , then this defines a diagram in \mathcal{C} . We let $\text{Hom}(S, T)$ be its limit in \mathcal{C} , if it exists (in general, it may be defined only as a presheaf over \mathcal{C}). Or in the case of a **Lie n -groupoid**, we can roughly view simplicial sets $\Delta[m]$ and $\Lambda[m, j]$ as simplicial manifolds with their discrete topology so that $\text{Hom}(S, X)$ denotes the set of homomorphisms of simplicial manifolds, with its natural topology resp. smooth structure. Thus $\text{Hom}(\Delta[m], X)$ is just another name for X_m . However it is not obvious that $\text{Hom}(\Lambda[m, j], X)$ is again a manifold, and it is a result of [Hen08, Corollary 2.5].

Example 2.5 (Lie 1-groupoids). It is well known that any Lie 1-groupoid K_\bullet is always the simplicial nerve NK_\bullet of a usual Lie groupoid $K \rightrightarrows M$. Here we recall that $NK_0 = M$ and for $i \geq 0$ define

⁶Submersions for Banach manifolds is also introduced in [Lan95]. It is sometimes called a split submersion in other literature, namely we need local splitting in charts.

$NK_i = \{(k_1, \dots, k_i) \in K^{\times i} \mid s(k_j) = t(k_{j+1})\}$. The faces and degeneracies are given by

$$\begin{aligned} d_0(k) &= s(k), & d_1(k) &= t(k), \\ d_0(k_1, \dots, k_i) &= (k_2, \dots, k_i), & d_j(k_1, \dots, k_i) &= (k_1, \dots, k_j k_{j+1}, \dots, k_i), \\ d_i(k_1, \dots, k_i) &= (k_1, \dots, k_{i-1}), & s_j(k_1, \dots, k_i) &= (k_1, \dots, k_j, 1_{s(k_j)}, k_{j+1}, \dots, k_i). \end{aligned}$$

Example 2.6 (Lie 1-group). A particular case of the previous example is when we consider a Lie group G . Then its simplicial nerve NG_\bullet is given by $NG_i = G^{\times i}$ and the faces and degeneracies are just

$$\begin{aligned} d_0(g_1, \dots, g_i) &= (g_2, \dots, g_i), & d_j(g_1, \dots, g_i) &= (g_1, \dots, g_j g_{j+1}, \dots, g_i), \\ d_i(g_1, \dots, g_i) &= (g_1, \dots, g_{i-1}), & s_j(g_1, \dots, g_i) &= (g_1, \dots, g_j, e, g_{j+1}, \dots, g_i). \end{aligned}$$

2.2. Graded manifolds and their tangent spaces. We are interested in two sorts of graded manifolds the first of which are N-manifolds. Our objects of interest — Lie n -algebroids are N-manifolds equipped with a degree 1 differential Q , also known as A homological vector field. However, we will need our version of the fat point and the test object to be \mathbb{Z} -graded manifolds.

Various flavours of \mathbb{Z} -graded manifolds exist throughout the literature, cf. e.g. [Vor02] or [Vys22, KS21]. Here we give a short recall with a slightly modified definition of \mathbb{Z} -graded manifolds according to our needs in this article, while the one for N-manifolds is equivalent to the standard references (see e.g. [ZZ12, Section 1.1] and references therein).

Definition 2.7. A \mathbb{Z} -graded manifold \mathcal{M} is a \mathbb{Z} -graded graded-commutative algebra $C(\mathcal{M})$ (called the “function algebra”) such that there exist a \mathbb{Z} -graded vector bundle V_\bullet and an isomorphism

$$(6) \quad C(\mathcal{M}) \cong \Gamma(S^{\geq 0} V_\bullet^*).$$

Furthermore, when V_\bullet is concentrated on strictly negative degrees, we call \mathcal{M} a N-manifold. Through out the article, unless otherwise mentioned, we always assume that a graded vector bundle V_\bullet has finite rank in each component V_i .

Remark 2.8. In this definition and through out in this article, $S^{\geq 0}$ denotes the (augmented) symmetric algebra, that is, it includes the arity 0 part $\Gamma(S^0 V_\bullet^*) := C^\infty(M)$, which has degree 0. Otherwise, the degree of an element in $\Gamma(S^{\geq 0} V_\bullet^*)$ is the sum of the degrees of each term in the symmetric product.

For convenience, we define graded manifolds using just the global function algebra and not using a sheaf of algebras. This way, for us the Batchelor-Gawedzki Theorem, which has certain subtleties in the \mathbb{Z} -graded case (cf. e.g. [KS21]) is part of the definition for us.

Definition 2.9. A morphism of \mathbb{Z} -graded manifolds $\mathcal{M} \rightarrow \mathcal{N}$ is a \mathbb{Z} -graded algebra morphism $\Phi : C(\mathcal{N}) \rightarrow C(\mathcal{M})$.

Remark 2.10. We define our maps purely as algebra homomorphisms, without an underlying base map, because for our later differentiation construction we will need that morphisms can map smooth coordinates (i.e. those coming from $C^\infty(M)$) to formal ones (i.e. degree 0 elements in $\Gamma(S^{\geq 1} V_\bullet^*)$).

We denote the category of \mathbb{Z} -graded manifolds by $\mathbb{Z}\text{Mfd}$ and the category of N-manifolds by NMfd the subcategory of graded manifolds whose vector bundles have non-trivial components only in strictly positive degree. Note that the function algebras of N-Manifolds are not strictly positively graded: In degree 0 they have the algebra of smooth functions of the underlying manifold. Since morphisms preserve degrees, NMfd is a fully faithful subcategory of $\mathbb{Z}\text{Mfd}$.

Remark 2.11.

(1) Clearly, a \mathbb{Z} -graded vector bundle V_\bullet gives us a \mathbb{Z} -graded manifold $\mathfrak{S}(V_\bullet)$, and

$$(7) \quad \mathfrak{S} : \mathbb{Z}\text{Vectbd} \rightarrow \mathbb{Z}\text{Mfd}$$

is a functor from the category of \mathbb{Z} -graded vector bundles to that of \mathbb{Z} -graded manifolds. Given a \mathbb{Z} -graded manifold \mathcal{M} , a choice of a vector bundle V_\bullet together with a choice i_{V_\bullet} of the isomorphism (6), that is, $(V_\bullet, i_{V_\bullet})$, is called a **splitting** of \mathcal{M} . However, later in the article, when there is no confusion of choice of isomorphisms in the context, we also refer V_\bullet alone as a splitting of \mathcal{M} .

- (2) We call a morphism $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ between \mathbb{Z} -graded manifolds **strict** with respect to some splitting of \mathcal{M} and \mathcal{N} (or simply *strict* for simplicity when the context is clear) if it comes from a morphism of the corresponding \mathbb{Z} -graded vector bundles. More precisely, there are a splitting V_\bullet of \mathcal{M} and W_\bullet of \mathcal{N} and a \mathbb{Z} -graded vector bundle morphism $f_\bullet : V_\bullet \rightarrow W_\bullet$ such that $S^{\geq 0}(f_\bullet)$ is the composed map

$$\Gamma(S^{\geq 0}W_\bullet^*) \xrightarrow{\cong} C(\mathcal{N}) \xrightarrow{\Phi} C(\mathcal{M}) \xrightarrow{\cong} \Gamma(S^{\geq 0}V_\bullet^*).$$

- (3) Moreover, \mathfrak{S} clearly restricts to $\mathbb{Z}^{< 0}$ -graded vector bundles, gives rise to a functor

$$\mathfrak{S} : \mathbb{Z}^{< 0}\text{Vectbd} \rightarrow \text{NMfd}$$

from the category of $\mathbb{Z}^{< 0}$ -graded vector bundles to that of N -manifolds. We use the same terminology of “splitting” and “strict” as above.

- (4) The concept of a strict morphism involves splittings of the graded manifolds. Thus the composition of strict maps is **not** necessarily strict. We need the splittings of the graded manifold in the middle to be the same, because an automorphism of a graded manifold does **not** necessarily come from a vector bundle morphism between two different splittings.
- (5) There exist morphisms of graded manifolds that are not strict no matter which splittings we choose. Here is an example: We consider the graded manifold $\mathcal{M} = \mathbb{R}^2[1]$, whose function algebra is generated by two elements ϵ_1, ϵ_2 in degree 1 and the graded manifold $\mathcal{N} = \mathbb{R}[2]$, whose function algebra is generated by one element δ in degree 2. The map $\delta \mapsto \epsilon_1\epsilon_2$ defines a non-trivial morphism $\mathcal{M} \rightarrow \mathcal{N}$, however any strict morphism from \mathcal{M} to \mathcal{N} would be trivial for degree reasons.
- (6) Strict embeddings are clearly monomorphisms in either $\mathbb{Z}\text{Mfd}$ or NMfd . Recall that a **monomorphism** in a category is a left-cancellative morphism. That is, $X \xrightarrow{f} Y$ is a monomorphism if for all objects Z and all morphisms $g_1, g_2 : Z \rightarrow X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

Given a graded manifold \mathcal{M} , the manifold M and the graded vector bundle $V \rightarrow M$ such that $C(\mathcal{M}) \cong \Gamma(S^{\geq 0}V^*)$ are unique up to isomorphism. We call M the **body** of \mathcal{M} and we say that \mathcal{M} is over M . When one of the graded manifolds \mathcal{M}, \mathcal{N} is over a point, we define their **product** by the usual tensor product $C(\mathcal{M} \times \mathcal{N}) = C(\mathcal{M}) \otimes C(\mathcal{N})$.

Remark 2.12. Notice the isomorphism (6) is highly non-canonical. Thus the inclusion⁷ from $C^\infty(M)$ to $C(\mathcal{M})$ is not canonical, unless M is a point. This makes defining products of graded manifolds \mathcal{M}, \mathcal{N} intrinsically in general a little subtle⁸. Since we only need products of graded manifolds in the situation when one of them is over a point, we give the above definition and refer the general definition to a future work.

Example 2.13. Let M be a manifold. We denote by $T[1]M$ the \mathbb{Z} -graded manifold whose function algebra is $\Omega^\bullet(M)$. It has a canonical splitting, given by vector bundle $V_\bullet = V_{-1} = TM$ in degree -1 .

More generally, for a \mathbb{Z} -graded or $\mathbb{Z}^{< 0}$ -graded vector bundle V_\bullet , we denote $V[k]_\bullet$ (or $V_\bullet[k]$ by abuse of notation sometimes) its k -shift (to the left), that is, $V[k]_i = V_{k+i}$. In particular, if V is a vector bundle, $V[k]$ is a \mathbb{Z} -graded vector bundle which concentrates in degree $-k$.

Thus with this notation, $T[k]M$ then has a double meaning: it is the \mathbb{Z} -graded manifold corresponding to $\Omega^\bullet(M)$ with a suitable shift of degree, and it is also a \mathbb{Z} -graded vector bundle with TM on degree $-k$. That is, $T[k]M = \mathfrak{S}(T[k]M)$. We hope that this little confusion is harmless for readers.

Example 2.14. We will denote the split graded manifold $\mathbb{R}[-1]$ by D , where \mathbb{R} is the 1-dimensional trivial vector bundle over a point. We write its function algebra as $C(D) = \mathbb{R}[\epsilon]$ with $\deg(\epsilon) = -1$ and hence $\epsilon^2 = 0$. Careful: \mathbb{Z} -graded manifold $\mathbb{R}[-1]$ is very different from $\mathbb{R}[1]$, even though both of them look like an odd line in super geometry, where objects are $\mathbb{Z}/2\mathbb{Z}$ -graded manifolds.

⁷In fact neither the inclusion from $C^\infty(M)$ to $C(\mathcal{M})$ nor the projection in the other direction in general is canonical.

⁸One can of course pick splittings and construct the graded manifold corresponding to the product of the graded vector bundles. But then one will need to pay attention to choices of splittings.

In algebraic geometry, the concept of “fat point” is used to detect tangent objects [Per08, Wei53]. We will see that D plays the role of graded and 1-shifted version of “fat point” for differential geometry. In order to make sense of it, we require the notion of presheaf.

Let \mathcal{C} be a category. We denote by $\mathbf{pSh}(\mathcal{C})$ be the category of **presheaves** (Set-valued contravariant functors) on \mathcal{C} . There is a functor $X \mapsto \mathbf{hom}(\cdot, X) : \mathcal{C} \rightarrow \mathbf{pSh}(\mathcal{C})$ and we call a presheaf **representable**, if it is isomorphic to something in the image of this functor. Furthermore, for $A, B \in \mathcal{C}$, we define a presheaf on \mathcal{C} , when the product of A with any object in \mathcal{C} exists,

$$(8) \quad \mathbf{Hom}(A, B) : T \mapsto \mathbf{hom}(A \times T, B),$$

where $T \in \mathcal{C}$ is an arbitrary test object and the uncapitalized \mathbf{hom} denotes the set of morphisms in \mathcal{C} . Using this notion, we can define the tangent object of a graded manifold. We have the following theorem:

Theorem 2.15. *Let \mathcal{M} be \mathbb{Z} -graded manifold with a splitting $V_\bullet \rightarrow M$. Then the presheaf $\mathbf{Hom}(D, \mathcal{M})$ is representable and a connection on V_\bullet^* induces a splitting of $\mathbf{Hom}(D, \mathcal{M})$ by $V_\bullet \oplus T[1]M \oplus V[1]_\bullet$. Thus we denote $\mathbf{Hom}(D, \mathcal{M})$ by $T[1]\mathcal{M}$, the 1-shifted tangent bundle of \mathcal{M} .*

This theorem seems folklorically known to the graded geometry community, but we failed to find a precise reference, so we provide a proof here. Moreover, we will need to keep precise track of the spot where we use the connection in order to investigate the compatibility between splittings and morphisms later.

Proof. Consider a morphism in $\mathbf{Hom}(D, \mathcal{M})(\mathcal{T}) = \mathbf{hom}(D \times \mathcal{T}, \mathcal{M})$, where \mathcal{T} is a \mathbb{Z} -graded manifold with body T . Since the body of D is simply a point, such a morphism is given by an algebra morphism $\Psi : C(\mathcal{M}) \rightarrow \mathbb{R}[\epsilon] \otimes C(\mathcal{T})$. For $\alpha \in C(\mathcal{M})$ we can decompose Ψ as

$$\Psi(\alpha) = \Psi_0(\alpha) + \epsilon \cdot \Psi_1(\alpha),$$

where Ψ_0 and Ψ_1 map to $C(\mathcal{T})$. The algebra $C(\mathcal{M})$ is generated by elements $f \in C^\infty(M)$ and $\sigma \in \Gamma(V_k^*), k \in \mathbb{Z}$. We can translate the fact that Ψ is an algebra homomorphism to the following equations:

$$(9) \quad \Psi_0(fg) = \Psi_0(f)\Psi_0(g)$$

$$(10) \quad \Psi_0(f\sigma) = \Psi_0(f)\Psi_0(\sigma)$$

$$(11) \quad \Psi_1(fg) = \Psi_0(f)\Psi_1(g) + \Psi_1(f)\Psi_0(g)$$

$$(12) \quad \Psi_1(f\sigma) = \Psi_0(f)\Psi_1(\sigma) + \Psi_1(f)\Psi_0(\sigma),$$

for $f, g \in C^\infty(M)$ and $\sigma \in \Gamma(V_k^*), k \in \mathbb{Z}$. The first two equations imply that Ψ_0 gives rise to a morphism of graded manifold from \mathcal{T} to \mathcal{M} . The last two equations show that Ψ_1 satisfies a Leibniz rule over Ψ_0 .

In the following, we will need some knowledge on jet bundles and differential operators and we give explicit references in [Nes03]. The fact that Ψ is an algebra homomorphism implies that Ψ_1 satisfies

$$(13) \quad [f, [g, \Psi_1]] = 0, \quad \forall f, g \in C^\infty(M),$$

where $[f, P]$ denotes the operator $[f, P](s) = P(fs) - fP(s)$ for all $s \in C(\mathcal{M})$. Eq. (13) implies that Ψ_1 is a differential operator of order 1 in $\mathbf{Diff}_1(C^\infty(M) \oplus \Gamma(V_\bullet^*), C(\mathcal{T}))$ between the two $C^\infty(M)$ -modules. Here $C^\infty(M)$ acts on $C(\mathcal{T})$ via Ψ_0 . We want to apply Theorem 14.25 from [Nes03], stating that differential operators of order l from a projective module into a geometrical one correspond to module homomorphisms from the corresponding l -jet bundle. Clearly the module $\Gamma(V_k^*)$ (resp. $C^\infty(M)$) is projective, so we only need to verify that $C(\mathcal{T})$ is geometrical as a $C^\infty(M)$ -module. For an \mathbb{R} -algebra A , an A -module is called geometrical (section 12.11 in [Nes03]), if the space of invisible elements is trivial:

$$\mathit{Inv}(P) := \bigcap_{h \in \mathbf{hom}_{\mathbf{alg}}(A, \mathbb{R})} \ker(h) \cdot P \stackrel{!}{=} 0.$$

Now we show that $C(\mathcal{T})$ is geometric as a $C^\infty(M)$ -module. By the naturality of geometrization (15.29 in [Nes03]), we only need to show that that $C(\mathcal{T})$ is a geometric $C_0(\mathcal{T})$ -module, since $C^\infty(M)$ acts on $C(\mathcal{T})$ via Ψ_0 . Since a projective module is always geometric [Nes03, Section12], using a splitting of \mathcal{T} , we can

see that $C(\mathcal{T})$ is a geometric $C^\infty(T)$ -module. Moreover, we get morphisms $C^\infty(T) \xrightarrow{i} C_0(\mathcal{T}) \xrightarrow{p} C^\infty(T)$ composing to the identity. This means that the map $i^* : \text{hom}_{\text{alg}}(C_0(\mathcal{T}), \mathbb{R}) \rightarrow \text{hom}_{\text{alg}}(C^\infty(T), \mathbb{R})$ is surjective. This implies (by going through the proof of 15.29 in [Nes03] backwards) that $C(\mathcal{T})$ is a geometric $C_0(\mathcal{T})$ -module.

Therefore Theorem 14.25 from [Nes03] is applicable and Ψ_1 is uniquely determined by the $C^\infty(M)$ -module map $J^1\Phi_1 : \Gamma(T^*M \oplus J^1V_\bullet^*) \rightarrow C(\mathcal{T})$, with components,

$$J_{T^*}^1\Psi_1 : \Gamma(T^*M) \rightarrow C_1(\mathcal{T}), \quad J_k^1\Psi_1 : \Gamma(J^1V_k^*) \rightarrow C_{k+1}(\mathcal{T}), \quad \forall k \in \mathbb{Z}.$$

Now we would like to see how much and how redundantly the information of (Ψ_0, Ψ_1) can be encoded using $(\Psi_0, J^1\Psi_1)$. Notice that the space $J^1V_k^*$ contains $T^*M \otimes V_k^*$ as a subspace (see (14)), on which the values of $J_k^1\Psi_1$ are already determined by Ψ_0 and $J_{T^*}^1\Psi_1$. Hence, to get rid of the redundancy, we have to pick a complement of $T^*M \otimes V_k^*$ in $J^1V_k^*$, i.e. a splitting $s_k : V_k^* \rightarrow J^1V_k^*$ of the following sequence of vector bundles over M :

$$(14) \quad 0 \rightarrow T^*M \otimes V_k^* \rightarrow J^1V_k^* \rightarrow V_k^* \rightarrow 0,$$

or, in other words, a TM connection on V_k^* for each $k \in \mathbb{Z}$. We now fix a series of such connections s_\bullet .

By property of 1-jet, $\Phi_1 = J^1\Phi_1 \circ j^1 : \Gamma(V_k^*) \rightarrow C(\mathcal{T})$. Let $\sigma \in \Gamma(V_k^*)$ and $f \in C^\infty(M)$. By definition $j^1(f\sigma) = df \otimes \sigma + fj^1\sigma$. Since $J_k^1\Psi_1$ is $C^\infty(M)$ -linear with $C^\infty(M)$ acting on $C(\mathcal{T})$ via Ψ_0 , (12) implies that

$$J_k^1\Psi_1(df \otimes \sigma) = \Psi_1(f)\Psi_0(\sigma).$$

So it suffices to know the values of $J_k^1\Psi_1$ on a complement of $T^*M \otimes V_k^*$, i.e. on the image of s_\bullet . This means that the information of Ψ is uniquely encoded by:

$$\Psi_0 : \Gamma(V_\bullet)^* \rightarrow C(\mathcal{T}), \quad J_{T^*}^1\Psi_1 : \Gamma(T^*M) \rightarrow C_1(\mathcal{T}), \quad J_k^1\Psi_1 \circ s_k : \Gamma(V_k^*) \rightarrow C_{k+1}(\mathcal{T}).$$

Thus $\text{Hom}(D, \mathcal{M})$ is representable and a connection on V_\bullet^* gives a splitting of it by $V_\bullet \oplus T[1]M \oplus V[1]_\bullet$. \square

For ordinary manifolds, the notation introduced in this theorem is consistent with the existing one.

Example 2.16. *Let $\mathcal{M} = M$ be an ordinary manifold (with $C(\mathcal{M}) = C^\infty(M)$). Then $\text{Hom}(D, \mathcal{M})$ is representable by $T[1]M$. This statement can be seen as a special case of the previous theorem.*

Inspired by the above theorem, for a \mathbb{Z} -graded vector bundle V_\bullet over M , we define its 1-shifted tangent to be

$$T[1]V_\bullet := V_\bullet \oplus T[1]M \oplus V[1]_\bullet.$$

Thus Theorem 2.15 tells us that a connection on V_\bullet^* gives us an isomorphism

$$(15) \quad T[1]\mathfrak{S}(V_\bullet) \cong \mathfrak{S}(T[1]V_\bullet).$$

2.3. Iterated tangent bundles for manifolds. The object of study for us will be the iterated tangent bundles of an ordinary manifold:

$$\text{Hom}(D^k, M) = T[1]\text{Hom}(D^{k-1}, M) = T[1]T[1]\text{Hom}(D^{k-2}, M) = \dots =: T[1]^k M$$

Now the simplicial and graded worlds start merging: Since the nerve of the pair groupoid

$$(16) \quad D_\bullet = \{D_k = D^{[k]}\}_{k \in \mathbb{N}}$$

carries a simplicial structure, the iterated tangent bundles will carry a cosimplicial one, which—in much larger generality - is the object of study of [BK].

Corollary 2.17. *(of Theorem 2.15) Let M be a manifold with a fixed connection ∇ on T^*M . The presheaf $T[1]^k M := \text{Hom}(D^k, M)$ is representable,*

$$(17) \quad T[1]^k M := \text{Hom}(D^{\times k}, M) \cong \mathfrak{S}(T[1]M^{\binom{k}{1}} \oplus T[2]M^{\binom{k}{2}} \oplus \dots \oplus T[k]M^{\binom{k}{k}}),$$

where $T[i]M^{\binom{k}{i}} := T[i]M \oplus \dots \oplus T[i]M$ denotes $\binom{k}{i}$ copies of direct sum of $T[i]M$ over M .

Proof. This follows from Example 2.13 and a repeated application of Theorem 2.15 with the observation made above Eq. (15). The equation $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$ will be repeatedly applied in the induction process. \square

In the sequel, we will have to do some combinatorics on (17). For simplicity, let us denote the graded vector bundle on the right hand side by T_{\bullet}^k , that is,

$$(18) \quad T_{-i}^k = \bigoplus_{\{I \subset [k-1]; |I|=i\}} TM_I,$$

where I as an i -element subsets of $[k-1] = \{0, \dots, k-1\}$ can be also understood as a strictly increasing i -multi-index with values in $[k-1]$, and TM_I is a copy of TM indexed by I .

Notation 2.18. *We introduce some notation. Let $I \subset [k-1]$ be a multi-index.*

- For v in TM , we write v^I for an element in T_{\bullet}^k , which is v in the I -th component TM_I and 0 in all other components.
- For $i \in \mathbb{Z}$, we write $\overset{i \rightarrow}{I}$ for the multi-index with all indices in I after i pushed to the right by one. For example $I = (2, 3, 5, 7)$, then $\overset{3 \rightarrow}{I} = (2, 3, 6, 8)$, $\overset{4 \rightarrow}{I} = (2, 3, 6, 8)$, $\overset{-1 \rightarrow}{I} = (3, 4, 6, 8)$, and $\overset{10 \rightarrow}{I} = (2, 3, 5, 7)$
- Similarly, $\overset{i \leftarrow}{I}$ denotes the index where everything in I after i is pulled by one to the left, e.g. $J = (2, 3, 6, 8)$, then $\overset{2 \leftarrow}{J} = (2, 2, 5, 7)$, $\overset{3 \leftarrow}{J} = (2, 3, 5, 7)$ and $\overset{4 \leftarrow}{J} = (2, 3, 5, 7)$. We notice that while $\overset{i \rightarrow}{I}$ is still a strictly increasing $|I|$ -multi-index, $\overset{i \leftarrow}{I}$ might be not such a multi-index anymore because a number can appear twice as the example shows. In such cases, when $\overset{i \leftarrow}{I}$ is not anymore a strictly increasing $|I|$ -multi-index, we take $v^{\overset{i \leftarrow}{I}} = 0$ as there is no such a component in the sum of (18).

This immediately allows us to translate the simplicial identities on D_{\bullet} , to co-simplicial identities on $T[1]_{\bullet}^* M$ with splitting (18).

Lemma 2.19. *The face map $d_j : D^{[k]} \rightarrow D^{[k-1]}$, for $j \in [k]$ corresponds to the coface map $d_j^* : T_{\bullet}^{k-1} \rightarrow T_{\bullet}^k$, and the degeneracy maps $s_i : D^{[k-1]} \rightarrow D^{[k]}$, for $i \in [k-1]$ corresponds to the codegeneracy map $s_i^* : T_{\bullet}^k \rightarrow T_{\bullet}^{k-1}$. They re-arrange the indices as following:*

$$d_j^* : \begin{array}{cccccccc} 0, & 1, & \dots, & j-1, & j, & \dots, & k-1 & , \\ \downarrow & \downarrow & & \downarrow & \searrow & & \searrow & \\ 0, & 1, & \dots & j-1, & j, & j+1, & \dots, & k \end{array} \quad , \quad s_i^* : \begin{array}{cccccccc} 0, & 1, & \dots, & i-1, & i, & \dots, & k \\ \downarrow & \downarrow & & \downarrow & \swarrow & & \swarrow & \\ 0, & 1, & \dots & i, & \dots, & k-1 & \end{array} .$$

More precisely, for $j \in [k]$ and $i \in [k-1]$

$$(19) \quad d_j^*(v^I) = v^{\overset{j-1 \rightarrow}{I}}, \quad s_i^*(v^I) = v^{\overset{i \leftarrow}{I}},$$

where by convention $v^{\overset{i \leftarrow}{I}} = 0$ when $i, i+1 \in I$.

2.4. Limits for N-manifolds. It is relatively complicated to describe limits explicitly in the category of graded manifolds. Notice that there are morphisms (see Remark 2.11) between graded manifolds which do not come from morphisms between graded vector bundles, but some sort of non-strict L_∞ -style maps. It is not clear how to describe a fiber product, for example, if one leg is such a non-strict map. However, pullbacks of graded vector bundles can be easily described if one leg is a submersion on the base. In fact, the underlying sets for limits of graded vector bundles can be easily explicitly described as every graded vector bundle has an underlying set. Luckily, in the case of N-manifolds, which is the case we need in this article, the functor $\mathfrak{S} : \mathbb{Z}^{<0}\text{Vectbd} \rightarrow \text{NMfd}$ preserves limits. Therefore, when we have strict morphisms, it is easy to describe a limit of N-manifolds via the explicit formula in the world of $\mathbb{Z}^{<0}$ -graded vector bundles.

We now set off to show this fact. For this we introduce the notion of vector bundle comorphisms ([HM93]).

Definition 2.20. Let $E \rightarrow M$ and $F \rightarrow N$ be vector bundles. A vector bundle comorphism $E \rightarrow F$ is a couple (ϕ, Φ) , where $\phi : M \rightarrow N$ is smooth and $\Phi : \phi^*F \rightarrow E$ is a vector bundle morphism. We will denote vector bundle comorphisms by diagrams:

$$\begin{array}{ccc} E & \leftarrow & F \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

Example 2.21. A morphism of vector bundles $E \rightarrow F$ over $M \rightarrow N$, induces a vector bundle comorphism $E^* \leftarrow F^*$. In fact any vector bundle comorphism corresponds to a vector bundle morphism of the dual vector bundles.

Observation 2.22. Let $E \rightarrow M$ and $F \rightarrow N$ be two vector bundles, and $\phi : M \rightarrow N$ a smooth map. Then $\Gamma(F)$ is naturally a $C^\infty(N)$ -module structure with point-wise multiplication. Moreover, $\Gamma(E)$ is also a $C^\infty(N)$ -module, with the product induced by the point-wise multiplication:

$$g \cdot \xi := (\phi^*g)\xi, \quad \text{for } g \in C^\infty(N), \quad \xi \in \Gamma(E).$$

Then we have the following lemma:

Lemma 2.23. Let $\Phi : \Gamma(E) \rightarrow \Gamma(F)$ be a $C^\infty(N)$ -module morphism. Given $\sigma \in \Gamma(E)$ and $x \in M$, we claim that if $\sigma(\phi(x)) = 0$, then $\Phi(\sigma)(x) = 0$.

Proof. As the statement is point-wise, we may take a local frame F^i of F and we set $\sigma = \sum_i f_i F^i$ in a neighborhood of x . Then

$$(20) \quad \Phi(\sigma) = \sum_i \phi^* f_i \Phi(F^i).$$

If $\sigma(\phi(x)) = 0$, then $f_i(\phi(x)) = 0$. Thus by (20), $\Phi(\sigma)(x) = 0$. \square

Corollary 2.24. We use the same notation as the last lemma. Then the set of $C^\infty(N)$ -module morphisms $\text{hom}_{C^\infty(N)}(\Gamma(F), \Gamma(E))$ is exactly the set of comorphisms $E \leftarrow F$.

Proof. Given $\Phi \in \text{hom}_{C^\infty(N)}(\Gamma(F), \Gamma(E))$, we define $\varphi : E \leftarrow F$ by

$$\varphi(w, x) := \Phi(\sigma)(x), \quad \text{for } x \in M, w \in F_{\phi(x)}$$

where $\sigma \in \Gamma(F)$ is a section such that $\sigma(\phi(x)) = w$. By Lemma 2.23, this is well defined. It is not hard to verify that such Φ and φ one-to-one correspond to each other. \square

Proposition 2.25. The functor $\mathfrak{S} : \mathbb{Z}^{<0}\text{Vectbd} \rightarrow \text{NMfd}$ preserves limits.

Proof. This follows from the fact that \mathfrak{S} is the right adjoint to a functor $\mathfrak{F} : \text{NMfd} \rightarrow \mathbb{Z}^{<0}\text{Vectbd}$. We define \mathfrak{F} on the subcategory of split N-manifolds, which by the N-graded version of the Batchelor-Gawedzki theorem [BP13, Eq. (2) Theorem 1] is equivalent to the category of all N-manifolds. Let \mathcal{M} be a split N-manifold which corresponds to a negatively graded vector bundle $V_\bullet \rightarrow M$, that is $C(\mathcal{M}) \cong \Gamma(S^{\geq 0}(V_\bullet^*))$. We set $\mathfrak{F}(\mathcal{M}) = S^{\geq 0}(V_\bullet)$, with its total grading. We take another negatively graded vector bundle $W_\bullet \rightarrow N$, then by definition,

$$(21) \quad \text{hom}(\mathcal{M}, \mathfrak{S}(W_\bullet)) = \text{hom}_{C^\infty(N)\text{alg}}(\Gamma(S^{\geq 0}(W_\bullet^*)), \Gamma(S^{\geq 0}(V_\bullet^*))).$$

Here we notice that similar to Observation 2.22, $\Gamma(S^{\geq 0}(W_\bullet^*))$ and $\Gamma(S^{\geq 0}(V_\bullet^*))$ are both C^∞ -algebras with the symmetric product. Similarly, (co)morphisms of vector bundles can also extend to (co)morphisms of vector bundles with additional \mathbb{R} -algebra structure fiber-wise provided the (co)morphisms preserve the

algebra structures. We denote the set of comorphisms by $\text{cohom}(-, -)$, and the one preserving additional algebra structures by $\text{cohom}_{\text{alg}}(-, -)$. Then Corollary 2.24 and Example 2.21 implies that

$$(22) \quad \begin{aligned} & \text{hom}_{C^\infty(N)\text{alg}}(\Gamma(S^{\geq 0}(W_\bullet^*)), \Gamma(S^{\geq 0}(V_\bullet^*))) = \text{cohom}_{\text{alg}}(S^{\geq 0}(W_\bullet^*), S^{\geq 0}(V_\bullet^*)) \\ & = \text{cohom}(W_\bullet^*, S^{\geq 0}(V_\bullet^*)) = \text{hom}(S^{\geq 0}(V_\bullet), W_\bullet) = \text{hom}(\mathfrak{F}(\mathcal{M}), W_\bullet). \end{aligned}$$

We thus have proven the adjunction formula $\text{hom}(\mathcal{M}, \mathfrak{S}(W_\bullet)) = \text{hom}(\mathfrak{F}(\mathcal{M}), W_\bullet)$. \square

We would like to be able to separate out the highest degree generators from an iterated tangent bundle. I.e. we would like to have diagrams of graded manifolds of the sort:

$$T[k]M \rightarrow T[1]^k M \rightarrow \frac{T[1]^k M}{T[k]M}$$

After choosing splittings this statement becomes evident: we can just include the highest degree ($=k$) TM into the larger vector bundle and also realize the quotient as a strict morphism. In fact, the sequence is canonical and does not depend on the choice of a splitting, as follows from the following Lemma:

Lemma 2.26. *Let \mathcal{M} be a N -manifold with corresponding $\mathbb{Z}^{<0}$ -graded vector bundle V_\bullet . Then the following algebras define graded manifolds:*

- *The smallest subalgebra of $C(\mathcal{M})$ containing $C_0(\mathcal{M}) \oplus C_1(\mathcal{M}) \oplus \dots \oplus C_{k-1}(\mathcal{M})$ —we denote the corresponding N -manifold by $\mathcal{M}_{<k}$.*
- *The algebra $C(\mathcal{M})/I$, where I is the ideal of $C(\mathcal{M})$ generated by $C_1(\mathcal{M}) \oplus \dots \oplus C_{k-1}(\mathcal{M})$ —we denote the corresponding N -manifold by $\mathcal{M}_{\geq k}$.*

There are natural maps $\mathcal{M}_{\geq k} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{<k}$. A splitting of \mathcal{M} induces a splitting of $\mathcal{M}_{\geq k}$ and $\mathcal{M}_{<k}$, such that this sequence of maps becomes $V_{\geq k} \rightarrow V_\bullet \rightarrow V_{<k}$.

Proof. The proof is a direct computation, which is implicitly carried out when proving the N -graded version of the Batchelor-Gawedzki theorem. [BP13, Theorem 1] \square

Our object of interest $T[k]M$ is naturally split, since it has generators in only one degree. So any distribution $E \subset TM$ induces a graded submanifold $E[k] \rightarrow T[k]M$ and the above sequence can be changed to:

$$(23) \quad E[k] \rightarrow T[1]^k M \rightarrow \frac{T[1]^k M}{E[k]}$$

Moreover, the associated sequence of graded vector bundles over M splits (non-canonically), so $T[1]^k M \cong \frac{T[1]^k M}{E[k]} \times_M E[k]$.

2.5. Compatible connections for \mathbb{Z} -graded vector bundles. Given a morphism of \mathbb{Z} -graded vector bundles $\Phi : E_\bullet \rightarrow F_\bullet$ over manifolds M and N , we would like the induced morphism $T[1]\mathfrak{S}(E_\bullet) \rightarrow T[1]\mathfrak{S}(F_\bullet)$ to also be realized as a morphism of \mathbb{Z} -graded vector bundles. I.e. we want to pick connections on E_\bullet and F_\bullet such that $\mathfrak{T}[1]\mathfrak{S}(\Phi)$ preserves the corresponding splittings of $T[1]\mathfrak{S}(E_\bullet)$ and $T[1]\mathfrak{S}(F_\bullet)$. For this we introduce the notion of compatible connections of vector bundle comorphisms.

Vector bundle comorphisms have one important property: They induce morphisms on the level of sections: Let $\Phi : E \leftarrow - - F$ be a comorphism over $\phi : M \rightarrow N$. Then any section $\sigma \in \Gamma(F)$ induces a section $\Phi \circ \phi^* \sigma \in \Gamma(E)$. Recall that that jet bundles are quotients of sections. Then a comorphism $E \leftarrow - - F$ induces the following comorphism

$$(24) \quad J^1 \Phi : J^1 E \leftarrow - - J^1 F, \quad \text{by } j^1 \sigma \mapsto j^1(\Phi \circ \phi^* \sigma),$$

The following lemma uses standard techniques (cf. e.g. [Sau89], Chapter 4.2, where the statement is proven for invertible morphisms rather than general comorphisms).

Lemma 2.27. *We use the notion as in the above text. Then the following diagram commutes:*

$$\begin{array}{ccccc} T^*M \otimes E & \longrightarrow & J^1E & \longrightarrow & E \\ \uparrow & & \uparrow & & \uparrow \\ T^*N \otimes F & \longrightarrow & J^1F & \longrightarrow & F \end{array}$$

Proof. By definition of a comorphism, we need to use ϕ^* to pull back the sequence of vector bundles over N to a sequence of vector bundles over M , i.e. the above lemma actually means, that the following diagram of vector bundles over M commutes:

$$\begin{array}{ccccc} T^*M \otimes E & \longrightarrow & J^1E & \longrightarrow & E \\ \uparrow & & \uparrow & & \uparrow \\ \phi^*(T^*N \otimes F) & \longrightarrow & \phi^*(J^1F) & \longrightarrow & \phi^*F \end{array}$$

Recall that the natural map $J^1E \rightarrow E$ is defined as $(j^1\sigma)_x \mapsto \sigma_x$ for a section $\sigma \in \Gamma(E)$ and point $x \in M$. The right square of the diagram commutes by the definition of $J^1\Phi$ (see Eq. (24)).

For the left square, let us describe the proof with the help of local coordinates y^1, \dots, y^n on N and a local frame F^j on F . Then at point $\bar{y} \in N$, the vector bundle morphism

$$(T^*N \otimes F) \rightarrow J^1(F), \quad \text{is defined by } \sum_{i,j} a_{i,j} dy^i|_{\bar{y}} \otimes F^j|_{\bar{y}} \mapsto (j^1\sigma)_{\bar{y}},$$

with section $\sigma \in \Gamma(F)$ given by $\sigma(y) = \sum_{i,j} a_{i,j} (y^i - \bar{y}^i) F^j$.

The map $\phi^*(J^1F) \rightarrow J^1E$ is also best understood in local coordinates. Let us take x^1, \dots, x^m local coordinates on M and E^1, \dots, E^k a local frame of E . Then a section ξ of E is locally given by $x \mapsto \xi(x) = (x, \sum_j \beta_j(x) E^j)$. The first jet also contains the first derivatives:

$$x \mapsto j^1\xi(x) = (x, \sum_j \beta_j(x) E^j, \sum_{i,j} \partial_{x^i} \beta_j(x) dx^i \otimes E^j).$$

Analogously a section σ of F is $y \mapsto \sigma(y) = (y, \sum_j \alpha_j(y) F^j)$ and its jet is

$$y \mapsto j^1\sigma(y) = (y, \sum_j \alpha_j(y) F^j, \sum_{i,j} \partial_{y^i} \alpha_j(y) dy^i \otimes F^j),$$

where y^1, \dots, y^n are local coordinates on N and F^1, \dots, F^l are a local frame of F . Then a pullback section $\phi^*\sigma \in \Gamma(\phi^*F)$ is $x \mapsto \phi^*\sigma(x) = (x, \sum_j \alpha_j(\phi(x)) F^j)$ and its jet is

$$x \mapsto j^1\phi^*\sigma(x) = (x, \sum_j \alpha_j(\phi(x)) F^j, \sum_{i,j} \partial_{x^i} \alpha_j(\phi(x)) dx^i \otimes F^j).$$

The expression $\Phi \circ \phi^*\sigma \in \Gamma(E)$ takes the form,

$$x \mapsto \Phi \circ \phi^*\sigma(x) = (x, \sum_{i,j} \Phi_j^i(x) \alpha_i(\phi(x)) E^j),$$

where in local coordinates, $\phi = (\phi^1, \dots, \phi^n)$ is a collections of maps depending on x^1, \dots, x^m , and $\Phi = \sum_{i,j} \Phi_j^i(x) F_i^* \otimes E^j$ is a matrix depending on a point in M , with F_i^* 's the dual frame of F^i 's. We now can calculate $j^1(\Phi \circ \phi^*\sigma)$:

$$j^1(\Phi \circ \phi^*\sigma)(x) = \left(x, \sum_{i,j} \Phi_j^i(x) \alpha_i(\phi(x)) E^j, \sum_{i,j,p} \left(\partial_p \Phi_j^i(x) \alpha_i(\phi(x)) + \sum_q \Phi_j^i(x) (\partial_{y^q} \alpha_i)(\partial_{x^p} \phi^q(x)) \right) dx^p \otimes E^j \right).$$

In particular we know that if we denote the derivative coordinates in $J^1 F$ by $u_{i,q}$, that is, $u_{i,q} = \partial_{y^q} \alpha_i(y)$, the map $\phi^*(J^1 F) \rightarrow J^1 E$ becomes:

$$\left(x, \sum_i \alpha_i F^i, \sum_{i,q} u_{i,q} dy^q \otimes F^i \right) \mapsto \left(x, \sum_{i,j} \Phi_j^i(x) \alpha_i E^j, \sum_{i,j,p} \left(\partial_p \Phi_j^i(x) \alpha_i + \sum_q \Phi_j^i(x) u_{i,q} \partial_{x^p} \phi^q(x) \right) dx^p \otimes E^j \right).$$

In particular for elements of the form $(x, 0, dy^q \otimes u_{i,q} F^i) \in \phi^*(T^*N \otimes F) \subset \phi^* J^1 F$, since $\alpha_i(x) = 0$ for these elements, we have

$$\begin{aligned} & \sum_{i,j,p} \left(\partial_{x^p} \Phi_j^i(x) \alpha_i + \sum_q \Phi_j^i(x) u_{i,q} \partial_{x^p} \phi^q(x) \right) dx^p \otimes E^j \\ &= \sum_{i,j,p} \sum_q \Phi_j^i(x) u_{i,q} \partial_p \phi^q(x) dx^p \otimes E^j = (\phi^* \otimes \Phi) \left(\sum_{i,q} dy^q \otimes u_{i,q} F^i \right), \end{aligned}$$

which is exactly in the image of $\phi^*(T^*N \otimes F) \rightarrow T^*M \otimes E$. Thus the left square commutes. \square

Definition 2.28. Let $E \dashrightarrow F$ be a vector bundle comorphism over $M \rightarrow N$ and $s : E \rightarrow J^1 E$ and $s' : F \rightarrow J^1 F$ be connections. We say that s, s' are compatible if the following diagram commutes

$$\begin{array}{ccc} J^1 E & \xleftarrow{s} & E \\ \uparrow & & \uparrow \\ J^1 F & \xleftarrow{s'} & F \end{array}$$

This definition generalizes easily to the graded case by taking levelwise compatible connections.

Example 2.29. Let $\phi : M \rightarrow N$ be smooth and $F \rightarrow N$ a vector bundle. Then we have the following diagram of vector bundle morphisms:

$$\begin{array}{ccccc} \phi^*(T^*N \otimes F) & \longrightarrow & \phi^* J^1 F & \longrightarrow & \phi^* F \\ \downarrow & & \downarrow & & \parallel \\ T^*M \otimes \phi^* F & \longrightarrow & J^1(\phi^* F) & \longrightarrow & \phi^* F \end{array}$$

A section $s : F \rightarrow J^1 F$ induces a section $\phi^* s : \phi^* F \rightarrow \phi^* J^1 F$, which in turn induces a unique connection $s' : \phi^* F \rightarrow J^1 \phi^* F$, which is compatible with s .abusively we denote this pullback connection s' by $\phi^* s$ and note that this coincides with the classical definition of pullback connection.

Example 2.30. Given a connection s on $F \rightarrow N$, there is a unique connection s^* on $F^* \rightarrow N$ such that the induced connection on $F \otimes F^*$ is compatible with the trivial connection on $N \times \mathbb{R}$ via the trace comorphism $N \times \mathbb{R} \leftarrow F \otimes F^*$. One can verify that the dualization is compatible with pullbacks, i.e. $\phi^*(s^*) = (\phi^* s)^*$ as connections on $\phi^* F^*$ (cf. e.g. [Tu17]).

The reason why we are interested in compatible connections is the following result, which may be viewed as the functorial version of Theorem 2.15.

Theorem 2.31. Let $E_\bullet \xrightarrow{L} F_\bullet$ be a morphism of \mathbb{Z} -graded vector bundles over $M \xrightarrow{\phi} N$ and s, s' compatible connections on the corresponding comorphism between E_\bullet, F_\bullet . Then the morphism of graded manifolds $T[1]\mathfrak{S}(E_\bullet) \xrightarrow{T[1]\mathfrak{S}(L)} T[1]\mathfrak{S}(F_\bullet)$ respects the splittings induced by s, s' given in Theorem 2.15.

Proof. The theorem amounts to prove the following commutative diagram:

$$\begin{array}{ccc} T[1]\mathfrak{S}(E_\bullet)\mathfrak{S}(L) & \xrightarrow{T[1]\mathfrak{S}(L)} & T[1]\mathfrak{S}(F_\bullet) \\ \downarrow sp_{s'} & & \downarrow sp_{s'} \\ E_\bullet \oplus T[1]M \oplus E[\mathbb{1}]_\bullet & \xrightarrow{L \oplus T[1]\phi \oplus L[\mathbb{1}]} & F_\bullet \oplus T[1]M \oplus F[\mathbb{1}]_\bullet \end{array},$$

where $sp_s, sp_{s'}$ denotes the splittings induced by s, s' respectively in Theorem 2.15. Then the proof amounts to check what happens with a \mathbb{Z} -graded vector bundle morphism when following the construction of Theorem 2.15. We look at $T[1]\mathfrak{S}(L)$ on \mathcal{T} -points for a test \mathbb{Z} -graded manifold \mathcal{T} , i.e. the map,

$$\mathrm{hom}(\Gamma(S^\bullet(E_\bullet^*)), C(\mathcal{T}) \otimes \mathbb{R}[\epsilon]) \rightarrow \mathrm{hom}(\Gamma(S^\bullet(F_\bullet^*)), C(\mathcal{T}) \otimes \mathbb{R}[\epsilon]).$$

Following the notation in Theorem 2.15, an element on the left can be described as $\Psi_0 + \epsilon\Psi_1$. The image of such an element is given by $\Psi_0 \circ L^* + \epsilon \cdot \Psi_1 \circ L^* =: \Phi_0 + \epsilon \cdot \Phi_1$, where L^* denotes the map $L^* : \Gamma(S^\bullet(F_\bullet^*)) \rightarrow \Gamma(S^\bullet(E_\bullet^*))$ induced by (the symmetric powers of the dual of) L .

The fact that $\Phi_0 = \Psi_0 \circ L^*$ immediately implies that the component of $T[1]\mathfrak{S}(L)$ on the $E_\bullet \rightarrow F_\bullet$ part is just precomposition with L^* , therefore given by L in the graded vector bundle picture. Now we have to apply the jet construction of Theorem 2.15 to $\Phi_1 = \Psi_1 \circ L^*$. The restriction of Φ_1 to $\Gamma(F_\bullet^*)$ can be described as follows:

$$\Gamma(F_\bullet^*) \xrightarrow{L^*} \Gamma(E_\bullet^*) \xrightarrow{\Psi_1} C(\mathcal{T})$$

This diagram induces a diagram of jet bundles:

$$\Gamma(J^1 F_\bullet^*) \xrightarrow{J^1 L^*} \Gamma(J^1 E_\bullet^*) \xrightarrow{J^1 \Psi_1} C(\mathcal{T}).$$

We are now interested in the composition $J^1 \Phi_1 \circ s' = J^1 \Psi_1 \circ J^1 L^* \circ s' = J^1 \Psi_1 \circ s \circ L^*$, where the last equality follows from the compatibility of the connections. In particular, the component of $T[1]\mathfrak{S}(L)$ on the $E[1]_\bullet \rightarrow F[1]_\bullet$ part is just composition with L^* , thus is given by $L[1]$ in the graded vector bundle picture. With an analogous computation, the component of $T[1]\mathfrak{S}(L)$ on the $T[1]M \rightarrow T[1]N$ part coincides with the tangent map $T[1]\phi : T[1]M \rightarrow T[1]N$ (without a choice of connection). \square

A particular case, where compatible connections can always be found is the case of fiberwise surjective vector bundle morphisms.

Lemma 2.32. *Let $E \rightarrow F$ be a fiberwise surjective vector bundle morphism and s a connection on F^* . Then there exists a compatible connection on E^* .*

Proof. The map $\phi^* F^* \rightarrow E^*$ is injective, i.e. $E^* \cong \phi^* F^* \oplus K$ for some vector bundle K . Pick as the connection on E^* the direct sum $s \oplus s^K$, where s^K is any connection on K . \square

Corollary 2.33. *Let $M \rightarrow N$ be a submersion. Then for any connection on T^*N , there exists a compatible connection on T^*M . In particular the morphisms $T[1]^k M \rightarrow T[1]^k N$ can all be realized by strict morphisms (i.e. coming from morphisms of graded vector bundles).*

3. REPRESENTABILITY OF THE TANGENT FUNCTOR

Definition 3.1. [Šev] *The tangent functor $\mathcal{T} : \mathrm{Mfd}_\bullet \rightarrow \mathrm{pSh}(\mathbb{Z}\mathrm{Mfd})$ from the category of simplicial manifolds Mfd_\bullet to that of presheaves of \mathbb{Z} -graded manifolds $\mathrm{pSh}(\mathbb{Z}\mathrm{Mfd})$ is given by,*

$$\mathcal{T}(X_\bullet) := \mathrm{Hom}(D_\bullet, X_\bullet) : T \mapsto \mathrm{hom}(T \times D_\bullet, X_\bullet), \quad \forall X_\bullet \in \mathrm{Mfd}_\bullet, \forall T \in \mathbb{Z}\mathrm{Mfd},$$

on the level of objects, and by post-composition on the level of morphisms. Here T denotes also the constant simplicial manifold with T on each level.

Remark 3.2. *The space $\mathrm{Hom}(D_\bullet, X_\bullet)(T)$ consists of maps in $\mathrm{hom}(T \times D_I, X_I)$ which are compatible with each other with respect to face and degeneracy morphisms. Therefore $\mathrm{Hom}(D_\bullet, X_\bullet)$ is the limit of the following diagrams*

$$\begin{array}{ccc} \mathrm{Hom}(D_k, X_k) & \xrightarrow{(d_J^D)^*} & \mathrm{Hom}(D_{k+1}, X_k)^{\times [k+1]} \\ (s_I^X)_* \downarrow & & \uparrow (d_J^X)_* \\ \mathrm{Hom}(D_k, X_{k+1})^{\times [k]} & \xleftarrow{(s_I^D)^*} & \mathrm{Hom}(D_{k+1}, X_{k+1}) \end{array}$$

where $I = \{0, \dots, k\}$ and $J = \{0, \dots, k+1\}$.

The goal of this section is proving the following central theorem.

Theorem 3.3. *Let X_\bullet be a Lie n -groupoid. The presheaf $\mathcal{T}(X_\bullet)$ is a N -manifold, that is, it is representable in NMfd . Moreover, it is (non-canonically) isomorphic to the tangent object,*

$$(25) \quad \ker Tp_0^1|_{X_0}[1] \oplus \ker Tp_0^2|_{X_0}[2] \oplus \cdots \oplus \ker Tp_0^n|_{X_0}[n],$$

where $p_0^k : X_k \rightarrow \mathfrak{h}^k(X)$ is the horn projection.

The theorem is a consequence of the results in the below sections. We start by proving that $\mathcal{T}(X_\bullet)$ is the limit of certain presheaves H^k (Lemma 3.5). Then we show the representability of these presheaves inductively (Lemma 3.8).

Remark 3.4. *The non-canonical isomorphism in Theorem 3.3 of the infinitesimal data of a Lie n -groupoid X_\bullet to the tangent object (25) depends on a choice of compactible connections on TX_k and $T\mathfrak{h}^k(X)$, and the isomorphism (46), for $k = 2, \dots, n$. The connections on different levels can be independent. This is the case even when X_0 is a point. Notice that choosing a splitting of an N -manifold depends on some choices of splitting of a certain short exact sequence on each ≥ 2 -level (see [BP13, Eq. (2) Theorem 1]). But here provide explicitly and geometrically what these choices are in our case.*

3.1. Description of $\mathcal{T}(X_\bullet)$ as a limit. We introduce a sequence of presheaves $H^k : \mathbb{Z}\text{Mfd} \rightarrow \text{Set}$:

$$(26) \quad H^k(T) = \left\{ (f_0, \dots, f_k) \mid f_l \in \text{hom}(T \times D_l, X_l) \text{ for } l < k, \right. \\ \left. f_k \in \text{hom}(T \times d_{1, \dots, k}^{-1}(\star), X_k) \text{ such that conditions H1. - H4. hold} \right\}$$

Here $d_{1, \dots, k} : D_k \rightarrow D_0$ is the face map corresponding to the inclusion $[0] = \{0\} \rightarrow [k] = \{0, \dots, k\}$ and \star is the basepoint of $D_0 = D$, and $\star \rightarrow D$ corresponds to $\mathbb{R}[\epsilon] \rightarrow \mathbb{R}$. This gives us,

$$(27) \quad d_{1, \dots, k}^{-1}(\star) = \star \times D^{\times k} \xrightarrow{\iota_k} D^{\times k+1} = D_k,$$

which is in general not an ‘‘inclusion’’ in the usual sense.

- (H1) $s_i^X f_l = f_{l+1} s_i^D$ for $l \leq k-2$ and $i \in \{0, \dots, l\}$.
- (H2) $d_i^X f_{l+1} = f_l d_i^D$ for $l \leq k-2$ and $i \in \{0, \dots, l+1\}$.
- (H3) $s_i^X f_{k-1}|_{d_{1, \dots, k-1}^{-1}(\star)} = f_k s_i^D|_{d_{1, \dots, k-1}^{-1}(\star)}$ for $i \in 0, \dots, k-1$.
- (H4) $d_i^X f_k = f_{k-1} d_i^D|_{d_{1, \dots, k}^{-1}(\star)}$ for $i \in 0, \dots, k$.

Here s_i^X, d_i^X denote the simplicial morphisms for X_\bullet , and s_i^D, d_i^D denote those for D_\bullet . Moreover, $|_{d_{1, \dots, k-1}^{-1}(\star)}$ and $|_{d_{1, \dots, k}^{-1}(\star)}$ denote the natural precomposition with ι_{k-1} and ι_k respectively.

Lemma 3.5. *Let X_\bullet be any simplicial manifold and H^k the presheaves defined above. Then $\mathcal{T}(X_\bullet) = \lim H^k$.*

Proof. There are natural maps

$$H \rightarrow \dots \rightarrow H^{k+1} \rightarrow H^k \rightarrow \dots \rightarrow H^0$$

On T -points it is just given by restricting to a sub-collection of the $\{f_i\}$. So we have a canonical map $H \rightarrow \lim H^k$. In order to prove the equality, we have to show that there is an inverse map $\lim H^k \rightarrow H$. We define this map on T -points.

An element in $\lim H^k(T)$ is represented by a series of elements $(f_0^{(0)}) \in H^0(T), (f_0^{(0)}, f_1^{(0)}) \in H^1(T), \dots, (f_0^{(i)}, \dots, f_i^{(i)}) \in H^i(T), \dots$. By definition, all these maps have to be compatible with restrictions and simplicial identities whenever defined. In particular, $f_a^{(i)} = f_a^{(j)}$ whenever $i, j \geq a+1$. We can hence define the simplicial morphism $F \in \text{hom}(T \times D_\bullet, X_\bullet)$ by $F_a = f_a^{(a+1)} \in \text{hom}(T \times D_a, X_a)$. The simplicial identities for F between levels a and $a+1$ then follow from $F_a = f_a^{(a+1)} = f_a^{(a+2)}, F_{a+1} = f_{a+1}^{(a+2)}$ and the fact that $f_a^{(a+2)}, f_{a+1}^{(a+2)}$ satisfy all simplicial identities. \square

For H^0 all conditions are void and $d_{1,\dots,k}$ should be understood as $d_\emptyset : D \rightarrow D$, and is the identity. Therefore $d_\emptyset^{-1}(\star) = \star$. This means that:

Lemma 3.6. $H^0(T) = \text{hom}(T, X_0)$. Thus $H^0 = X_0$ is representable.

3.2. The central pullback diagram. In this subsection we will build H^k from H^{k-1} via a pullback diagram.

Lemma 3.7. $H^k(T)$ can be realized as the following pullback:

$$(28) \quad \begin{array}{ccc} H^k(T) & \xrightarrow{\mathcal{F}_k} & \text{hom}(T \times \star \times D^k, X_k) \\ \downarrow & & \downarrow \\ H^{k-1}(T) & \xrightarrow{((s_0^X)_*, \dots, (s_{k-1}^X)_*, (d_1^D)^*, \dots, (d_k^D)^*) \circ \mathcal{F}_{k-1}} & \text{hom}(T \times \star \times D^{k-1}, X_k)^{[k-1]} \times \text{hom}(T \times \star \times D^k, X_{k-1})^{[k] \setminus 0} \end{array}$$

Here, for $l \geq 1$, \mathcal{F}_l is the forgetful map that remembers only the information on the l -th level restricted on $\star \times D^l$, that is

$$(29) \quad H^l \xrightarrow{\mathcal{F}_l} \text{Hom}(\star \times D^l, X_l).$$

Proof. We start by observing that $(f_0, \dots, f_k) \in H^k(T)$ is uniquely determined by f_k and then determine the conditions on f_k explicitly.

(Step1) **An element in $(f_0, \dots, f_k) \in H^k(T)$ is uniquely determined by f_k .**

For $l < k$ consider the iterated inclusion $I_{l,k}$ from $D_l = D^{[l]} = D^{l+1}$ to D_k by augmenting ‘‘points’’ from the left, i.e. $D^{l+1} \rightarrow \star^{k-l} \times D^{l+1} \hookrightarrow D^{k+1}$. We recall that $d^{0,\dots,k-l-1} : [l] \rightarrow [k]$ is a map forgetting $0, \dots, k-l-1$, that is, $i \mapsto k-l+i$ for $i = 0, \dots, l$. Thus $d_{0,\dots,k-l-1}^D \circ I_{l,k} = \text{id}$. Abusively, we write $I_{l,k}$ for $(\text{id}_T \times I_{l,k})$. The image of $I_{l,k}$ lies in $\star \times D^k = d_{1,\dots,k}^{-1}(\star)$, so $f_k \circ I_{l,k}$ is well-defined and we can deduce

$$(30) \quad f_l = f_l \circ d_{0,\dots,k-l-1}^D \circ I_{l,k} = d_{0,\dots,k-l-1}^X \circ f_k \circ I_{l,k}.$$

Here we have used the fact that f_i 's are compatible with degeneracies, and one can verify that this equation holds even for the case $l = k-1$. Hence, f_l is uniquely determined by f_k for $l = 0, \dots, k-1$.

(Step2) **Let $f^{(k-1)} = (f_0^{(k-1)}, \dots, f_{k-1}^{(k-1)}) \in H^{k-1}(T)$. An element $f_k \in \text{hom}(T \times d_{1,\dots,k}^{-1}(\star), X_k)$ generates an element of $H^k(T)$, which restricts to $f^{(k-1)}$ if and only if**

$$(sIk) \quad s_i^X f_{k-1}^{(k-1)} = f_k s_i^D |_{d_{1,\dots,k-1}^{-1}(\star)} \quad \text{for } i \in I = \{0, \dots, k-1\}.$$

$$(dJk) \quad d_i^X f_k = f_{k-1}^{(k-1)} d_i^D |_{d_{1,\dots,k}^{-1}(\star)} \quad \text{for } i \in J = \{1, \dots, k\}.$$

That is, given an element $f_k \in \text{hom}(T \times d_{1,\dots,k}^{-1}(\star), X_k)$, it extends to an element $f = (f_0, \dots, f_k) \in H^k$ with property $f_i = f_i^{(k-1)}$ ($i = 0, \dots, k-2$) and $f_{k-1} \iota_{k-1} = f_{k-1}^{(k-1)}$ for an $f^{(k-1)} = (f_0^{(k-1)}, \dots, f_{k-1}^{(k-1)}) \in H^{k-1}(T)$ if and only if (sIk) and (dJk) hold.

The necessity of the conditions is obvious because conditions H3 and H4 for f imply (sIk), (dJk) respectively. Notice that it is not a typo that $i \neq 0$ in (dJk).

For the sufficiency, we define $\{f_i\}_{i < k}$ using formula (30) and obtain $f = (f_0, \dots, f_k)$. We need to show that f satisfies H1, H2, H3, H4, and $f_{k-1} \iota_{k-1} = f_{k-1}^{(k-1)}$ for $f^{(k-1)} = (f_0^{(k-1)}, \dots, f_{k-1}^{(k-1)}) \in H^{k-1}(T)$.

The $I_{l,k}$ we have used in the previous step can be rewritten as $I_{l,k} = \iota_k \circ \iota_{k-1} \circ \dots \circ \iota_{l+1}$. Notice that $\iota_k = s_0^D$, by (30) and (s0k), we calculate:

$$f_{k-1} \iota_{k-1} = d_0^X f_k \iota_{k-1} = d_0^X f_k s_0^D \iota_{k-1} = d_0^X s_0^X f_{k-1}^{(k-1)} \iota_{k-1} = f_{k-1}^{(k-1)} \iota_{k-1}.$$

As $\iota_{k-1} : D^{k-1} \rightarrow d_{1,\dots,k-1}^{-1}(\star)$ is an isomorphism, it follows that $f_{k-1}|_{d_{1,\dots,k-1}^{-1}(\star)} = f_{k-1}^{(k-1)}$, i.e. $f_{k-1}\iota_{k-1} = f_{k-1}^{(k-1)}$.

Let us look at the analogue for $k-1$ of formula (30), stating that $f_{k-1}^{(k-1)}$ completely determines $f_i^{(k-1)}$ for $i \leq k-1$:

$$f_l^{(k-1)} = d_{0,\dots,(k-1)-l-1}^X \circ f_{k-1}^{(k-1)} \circ I_{l,k-1}$$

Since the image of $I_{l,k-1}$ lies in the image of ι_{k-1} , this is equal to:

$$= d_{0,\dots,(k-1)-l-1}^X \circ f_{k-1} \circ I_{l,k-1} = d_{0,\dots,(k-1)-l-1}^X \circ d_0^X \circ f_k \circ I_{1,k} \circ I_{l,k-1} = f_l$$

In particular this implies, that all simplicial identities for $\{f_0, \dots, f_{k-2}\}$ hold. Next we check the simplicial identities involving f_{k-1} and f_{k-2} :

To see the relation $s_a^X f_{k-2} = f_{k-1} s_a^D$, for $a = 0, \dots, k-2$, we can apply formula (30) to the relation between f_{k-1} and f_{k-2}

$$s_a^X f_{k-2} = s_a^X d_0^X f_{k-1} \iota_{k-1} = d_0^X s_{a+1}^X f_{k-1} \iota_{k-1} = d_0^X f_k s_{a+1}^D \iota_{k-1} = d_0^X f_k \iota_k s_a^D = f_{k-1} s_a^D.$$

Similarly $d_a^X f_{k-1} = f_{k-2} d_a^D$ for $a \in \{0, \dots, k-1\}$ can be verified by:

$$d_a^X f_{k-1} = d_a^X d_0 f_k I_k = d_0^X d_{a+1}^X f_k I_k = d_0^X f_{k-1} d_{a+1}^D I_k = d_0^X f_{k-1} I_{k-1} d_a^D = f_{k-2} d_a^D.$$

Finally, almost all the identities between f_k and f_{k-1} follow directly from (sIk) and (dJk), for instance for H4 ($i \neq 0$) we have

$$d_i^X f_k \iota_k = f_{k-1}^{(k-1)} d_i^D \iota_k = f_{k-1}^{(k-1)} \iota_k d_{i-1}^D = f_{k-1} \iota_k d_{i-1}^D = f_{k-1} d_i^D \iota_k.$$

The only special identity is $d_0^X f_k = f_{k-1} d_0^D|_{d_{1,\dots,k}^{-1}(\star)}$, which we can verify by applying from the right d_0^D to the equation defining f_{k-1} .

$$f_{k-1} \circ d_0^D = d_0^X \circ f_k \circ \iota_k \circ d_0^D$$

The statement now follows from $\iota_k \circ d_0^D|_{\star \times D^k} = id_{\star \times D^k}$. □

Since the presheaves on the right hand side are representable, we could rewrite the diagram as:

$$(31) \quad \begin{array}{ccc} H^k & \xrightarrow{\mathcal{F}_k} & T[1]^k X_k \\ \downarrow & & \downarrow (s_I^D)^*, (d_J^X)_* \\ H^{k-1} & \xrightarrow{((s_I^X)_*, (d_J^D)^*) \circ \mathcal{F}_{k-1}} & (T[1]^{k-1} X_k)^{[k-1]} \times (T[1]^k X_{k-1})^{[k] \setminus 0} \end{array},$$

where $J = (1, \dots, k)$ and $I = (0, \dots, k-1)$, and we furthermore abbreviate the tuple $((d_1^X)_*, \dots, (d_k^X)_*)$ to $(d_J^X)_*$, and similarly for any other such tuple of maps.

If H^{k-1} is representable, showing the representability of H^k is equivalent to showing that there is an N-manifold fitting into the pullback diagram (31). More precisely, we want to inductively prove

Lemma 3.8. *For all $k \geq 0$, H^k is representable by $\oplus_{i=1}^k \ker T p_0^i|_{X_0}$.*

In order to do so, we first have to get a better understanding of the image of $T[1]^k X_k$ in $(T[1]^k X_{k-1})^{[k] \setminus 0} \times (T[1]^{k-1} X_k)^{[k-1]}$ (for notations see Section 2.3). We will first investigate the image of $(s_I^D)^*$, then $(d_J^X)_*$ and then combine them. But before this, let us verify the initial cases for the induction. The case when $k = 0$ is verified in Lemma 3.6. We now verify that $H^1 = \ker T[1] p_0^1|_{X_0}$, and \mathcal{F}_1 is an embedding.

3.2.1. *The case when $k = 1$.* When $k = 1$, conditions (H1) (H2) are empty, and (H3) and (H4) are

$$\begin{aligned} \text{(H3)} \quad & s_0^X f_0|_{\star} = f_1 s_0^D|_{\star}; \\ \text{(H4)} \quad & d_j^X f_1 = f_0 d_j^D|_{\star \times D}, \quad j = 0, 1. \end{aligned}$$

We have

$$H^1(T) = \{(f_0, f_1) | f_0 \in \text{hom}(T \times D, X_0), f_1 \in \text{hom}(T \times \star \times D, X_1), \text{ such that H3 and H4 hold.}\}$$

By (Step1), (f_0, f_1) is uniquely determined by f_1 . By (Step2), given a T -point $f_0^{(0)} \in H^0(T)$, $f_1 \in \text{hom}(T \times \star \times D, X_1)$ is anything satisfying

$$(32) \quad s_0^X f_0^{(0)} = f_1 s_0^D|_{\star}, \quad d_1^X f_1 = f_0^{(0)} d_1^D|_{\star \times D} = f_0^{(0)}(\star) = x_0.$$

As $H^0 = X_0$ and $\text{hom}(T \times \star \times D, X_1) = T[1]X_1(T)$ are both representable, the above equations are also satisfied by the points they correspond to. That is, $f_0^{(0)}$ corresponds to a point $x_0 \in H^0 = X_0$, and f_1 corresponds to a vector $v \in T[1]X_1$. The first equation in (32) implies that $f_1|_{(\star, \star)} = s_0^X(x_0)$. This shows that $v \in T_{x_0}[1]X_1$. The second equation in (32) shows that $Td_1^X v = 0_{x_0}$. Thus $H^1 = \ker T[1]p_0^1|_{X_0}$.

In this case, \mathcal{F}_1 is an embedding. Noticing that $H^0 = X_0 \xrightarrow{\mathcal{F}_0=id} X_0$, the fact $H^0 = X_0 \xrightarrow{(\delta_1, (s_0^X)_\star)} T[1]X_0 \times X_1$ is an embedding implies that \mathcal{F}_1 is an embedding via the pullback diagram (31). However we should not expect that \mathcal{F}_k in general to be a strict embedding. This will be addressed in a later work.

3.3. Embedding $T[1]^k M/T[k]M$ into $(T[1]^{k-1} M)^{[k-1]}$. In this section we show that for any manifold M , the N-manifold $T[1]^k M/T[k]M$ can be seen as an N-submanifold of $T[1]^{k-1} M^{[k-1]}$. More precisely, for a fixed k , consider following diagrams for all $0 \leq i \leq j \leq k-2$:

$$(33) \quad \begin{array}{ccc} T[1]^k M/T[k]M & \xrightarrow{\sigma_i} & T[1]^{k-1} M \\ \downarrow \sigma_{j+1} & & \downarrow \sigma_j \\ T[1]^{k-1} M & \xrightarrow{\sigma_i} & T^{k-2}[1]M. \end{array}$$

Here $T[1]^l M$ is understood as $\text{Hom}(\star \times D^l, M)$ for $l = k, k-1, k-2$, and σ_i corresponds to degeneracies of D^l restricted to $\star \times D^{l-1}$ for $l = k, k-1$. Let us explain a bit: Indicated by diagram (28), we will later consider $\text{Hom}(\star \times D^l, M)$ with $\star \times D^l \subset D^{l+1}$, instead of directly $\text{Hom}(D^l, M)$. The inclusion $D^l \cong \star \times D^l \subset D^{l+1}$ corresponds to the map

$$\mathbb{R}[\epsilon_0, \dots, \epsilon_l] = C(D^{l+1}) \rightarrow C(\star \times D^l) = \mathbb{R}[\epsilon_1, \dots, \epsilon_l], \quad \epsilon_0 \mapsto 0, \epsilon_j \mapsto \epsilon_j, \text{ for } j = 1, \dots, l.$$

Here ϵ_i 's all have degree -1, thus they anti-commute. The map σ_i corresponds to degeneracies of D^l restricted to $\star \times D^{l-1}$ for $l = k, k-1$. As a consequence, we have one more $\sigma_i : T[1]^k M/T[k]M \rightarrow T[1]^{k-1} M$ than we would have simply took $T[1]^k M = \text{Hom}(D^k, M)$ and $T[1]^{k-1} M = \text{Hom}(D^{k-1}, M)$ and understood σ_i comes from $(s_i^D)^*$ on the level $k-2$ in (16). In fact, the maps $\sigma_1, \dots, \sigma_{k-1}$ correspond to the usual degeneracies $(s_0^D)^*, \dots, (s_{k-2}^D)^*$ if we take $T[1]^k M = \text{Hom}(D^k, M)$ and $T[1]^{k-1} M = \text{Hom}(D^{k-1}, M)$, and the map σ_0 , corresponds to the inclusion $D^{k-1} \rightarrow \star \times D^{k-1} \rightarrow D^k$.

Now we prove the key Lemma of this subsection. It is also the key step repairing the mistake in Li's Lemma 8.23. The proof is a long and fine combinatorial demonstration. Thus we break it down into three observations to make the complex proof easier to understand and follow.

Lemma 3.9. *The N-manifold $T[1]^k M/T[k]M$ is the limit of the above family of diagrams (33) in the category NMfd. Moreover, the natural map $(\sigma_0, \dots, \sigma_{k-1}) : T[1]^k M/T[k]M \rightarrow (T[1]^{k-1} M)^{[k-1]}$ is a strict embedding with respect to splittings determined by a chosen connection on TM .*

Proof. Upon fixing a connection on TM , we can view these diagrams as diagrams in the category of $\mathbb{Z}^{<0}$ -graded vector bundles and show that there is a limit there. We have a similar splitting of $T[1]^l M$

by T^l as in (18), but as here we are taking $\text{Hom}(\star \times D^l, M)$ instead of $\text{Hom}(D^l, M)$ directly as therein, the indices are shifted by 1. That is, here $T[1]^l M$ has a splitting by $\bigoplus_{i=1}^l \tilde{T}_{-i}^l$ with

$$(34) \quad \tilde{T}_{-i}^l = \bigoplus_{I \subset \{1, \dots, l\}: |I|=i} TM_I.$$

Since all morphisms involved are ($\mathbb{Z}^{<0}$ -graded) vector bundle morphisms over the identity of M , the lemma boils down to show the following observation:

Observation 3.10. *For any family of elements $\{v_0, \dots, v_{k-1}\}$ in \tilde{T}_\bullet^{k-1} —the graded vector bundle corresponding to $T[1]^{k-1}M = \text{Hom}(\star \times D^{k-1}, M)$, satisfying compatibility equations*

$$(35) \quad \sigma_i v_{j+1} = \sigma_j v_i, \quad \forall 0 \leq i \leq j \leq k-2,$$

there is a unique element $w \in T[1]M^{(1)} \oplus T[2]M^{(2)} \oplus \dots \oplus T[k]M^{(k-1)} \subset \tilde{T}_\bullet^k$ (which is the graded vector bundle corresponding to $T[1]^k M / T[k]M$) mapping to them.

An element v_i in \tilde{T}_\bullet^{k-1} is given by its components $\{v_i^I \in TM_I\}$, where I runs through all subsets of the set $\{1, \dots, k-1\}$. The desired element w will have components $\{w^I \in TM_I\}$ for all non-full subsets $I \subset \{1, \dots, k\}$.

Let us try to understand the equation $\sigma_i w = v_i$ for a fixed multiindex $I \subset \{1, \dots, k-1\}$. We use notation in Notation 2.18. Notice that even we have a shift, the notations therein stay with the same meaning.

There are two options:

- $i \notin I$: Then the only contribution to $(\sigma_i w)^I$ comes from w^J for $J = \overset{i \rightarrow}{I}$.
- $i \in I$. In this case I can still come from $\overset{i \rightarrow}{I}$, but also from $\overset{i-1 \rightarrow}{I}$.

Hence $\sigma_i w = v_i$ is equivalent to,

$$(36) \quad v_i^I = \begin{cases} w^{\overset{i \rightarrow}{I}} & i \notin I \\ w^{\overset{i \rightarrow}{I}} + w^{\overset{i-1 \rightarrow}{I}} & i \in I \end{cases}$$

for all multi-indices $I \subset \{1, 2, \dots, k-1\}$ and all $i \in \{0, \dots, k-1\}$. Thus our proof boils down to show that linear equations in Eq. (36) provide a unique solution w with fixed v_i satisfying Eq. (35) for $i \in \{0, \dots, k-1\}$. We prove this statement in two steps, which are summarized in Observation 3.11 and Observation 3.12.

We first notice that

- Components with different degree, namely index-length, in \tilde{T}_\bullet^l do not interfere with each other. Especially the component of w of index-length j only depends on the component of v_i 's with index-length j .
- There is no equation containing the highest degree w^J with $J = (1, 2, 3, 4, \dots, k)$. This is the reason why the pullback will be $T[1]^k M / T[k]M$ and not $T[1]^k M$.

Now we try to revert (36) to get all equations to a form in which a fixed w^J (i.e. a fixed multi-index J) is involved on the right hand side. We do this by case-scanning. Let us fix J and i , we have the following case distinction:

- (1) $i \in J$ and $i+1 \in J$: Then J can not be achieved as some $\overset{i \rightarrow}{I}$ or $\overset{i-1 \rightarrow}{I}$. I.e. there is no equation involving v_i and w^J .

- (2) $i \in J$ and $i+1 \notin J$: Then J can not be of the form $\overset{i-1 \rightarrow}{I}$, but it can be of the form $J = \overset{i \rightarrow}{I}$, and

in this case $I = \overset{i \leftarrow}{J}$. This $v_i^{\overset{i \leftarrow}{J}}$ appears in the equation $v_i^{\overset{i \leftarrow}{J}} = w^J + w^{\overset{i-1 \rightarrow}{\overset{i \leftarrow}{J}}}$. The second summand (pulling everything after i first, then pushing everything after $i-1$) turns out to change i into $i+1$

and leave the rest as it is. We denote such operation by $\overset{i \curvearrowright i+1}{J}$. That is, we have $\overset{i-1 \rightarrow}{\overset{i \leftarrow}{J}} = \overset{i \curvearrowright i+1}{J}$.

Therefore the unique equation involving v_i and w^J is:

$$v_i^{\overset{i\leftarrow}{J}} = w^J + w^{\overset{i\curvearrowright i+1}{J}}.$$

- (3) $i \notin J$ and $i+1 \in J$: This case is the opposite of the previous one, J can not be $\overset{i\rightarrow}{I}$, but it can be realized as $\overset{i-1\rightarrow}{I}$ with $I = \overset{i-1\leftarrow}{J}$. We get the equation:

$$v_i^{\overset{i-1\leftarrow}{J}} = w^{\overset{i\curvearrowright i+1}{J}} + w^J$$

- (4) $i \notin J$ and $i+1 \notin J$: This is the case, where (36) has only one summand on the right hand side, and we get

$$v_i^{\overset{i\leftarrow}{J}} = w^J$$

In summary for a multi-index J ranging from 1 to k and an index $0 \leq i \leq k-1$, we have the following observation:

Observation 3.11. *A rewriting of Eq. (36) is given in the following cases:*

	$i \in J$	$i \notin J$
$i+1 \in J$	no equation	$v_i^{\overset{i-1\leftarrow}{J}} = w^{\overset{i\curvearrowright i+1}{J}} + w^J$, if $i \geq 1$
$i+1 \notin J$	$v_i^{\overset{i\leftarrow}{J}} = w^J + w^{\overset{i\curvearrowright i+1}{J}}$	$v_i^{\overset{i\leftarrow}{J}} = w^J$.

Notice that the equation including $\overset{0\curvearrowright 1}{J}$ vanishes by definition. That is the reason why in the right top field, there is the additional assumption $i \geq 1$.

We now are going to show

Observation 3.12. *When $k \geq 2$, for any multi-index J ranging from 1 to k with length $l := |J| < k$, the formulas in (36) uniquely determine $\{w^J \mid |J| = l\}$.*

We pick the lexicographical order on the multi-indices of length l and do an induction-proof backwards.

- A) Initial case: For this, we only need to show that for the last multi-index $J = (k-l+1, \dots, k)$, we have

- (a) **We can define** $w^J = v_0^{\overset{0\leftarrow}{J}}$, as $0, 1 \notin J$.
(b) **All equations containing only w^J in Observation 3.11 are satisfied if we define w^J by (a).**

Proof for A). The formulas containing only w^J are those of the type $w^J = v_i^{\overset{i\leftarrow}{J}}$, i.e. those $i > 0$ where $i, i+1 \notin J$. For those we have to show that

$$v_i^{\overset{i\leftarrow}{J}} = v_0^{\overset{0\leftarrow}{J}},$$

so that there is no contradiction among equations involving w^J if w^J is defined by (a).

We consider the $\overset{0\leftarrow}{I}$ component of equation $\sigma_0 v_i = \sigma_{i-1} v_0$, where $I = \overset{i\leftarrow}{J} = \overset{0\leftarrow}{J} = (k-l, \dots, k-1)$. Since $0, i-1 \notin \overset{0\leftarrow}{I}$ and $\overset{0\leftarrow}{I} = \overset{i-1\leftarrow}{I}$ we get

$$v_i^{\overset{i\leftarrow}{J}} = v_i^{\overset{0\leftarrow}{I}} = (\sigma_0 v_i)^{\overset{0\leftarrow}{I}} = (\sigma_{i-1} v_0)^{\overset{i-1\leftarrow}{I}} = v_0^{\overset{0\leftarrow}{I}} = v_0^{\overset{0\leftarrow}{J}}.$$

□

- B) Induction Step: Fix a multi-index J and assume that $w^{\tilde{J}}$ are defined and uniquely defined for all $\tilde{J} > J$ with respect to the lexicographical order, and that all equations involving only $\{w^{\tilde{J}} \mid \tilde{J} > J\}$ in Observation 3.11 are satisfied. Then

(a) **We can define w^J as follows:** Let $j = \max\{l \in \{1, \dots, k\} | l \notin J\} - 1$, then $j + 1 \notin J$. We set w^J by:

$$w^J = \begin{cases} v_j^{j\leftarrow J} & \text{if } j \notin J \\ v_j^{j\leftarrow J} - w^{j\curvearrowright j+1} & \text{if } j \in J \end{cases}$$

(b) **All equations involving only $\{w^{\tilde{J}} | \tilde{J} \geq J\}$ in Observation 3.11 are satisfied.**

Proof for B). In order to avoid the case distinction in the calculation, we define

$$(37) \quad w^{l\curvearrowright l+1} := 0, \quad \text{when } l \notin J \text{ or } l + 1 \in J.$$

With this notation, the right bottom equation is a special case of the left bottom equation, thus we save the discussion of one case. The formulas involving w^J and not involving any multi-index of w which is smaller than J are all of the type $v_i^{i\leftarrow J} - w^{i\curvearrowright i+1}$, i.e. formulas with v_i where $i + 1 \notin J$, keeping in mind the notation (37). By our construction of j , it follows that $i < j$. We need to verify

$$(38) \quad v_i^{i\leftarrow J} - w^{i\curvearrowright i+1} = v_j^{j\leftarrow J} - w^{j\curvearrowright j+1},$$

so that w^J is well defined by (a).

We first notice that we can rewrite the equation,

$$(\sigma_l v_k)^I = \begin{cases} v_k^{l\rightarrow I} & l \notin I \\ v_k^{l\rightarrow I} + v_k^{l-1\rightarrow I} & l \in I \end{cases}$$

to $(\sigma_l v_k)^I = v_k^{l\rightarrow I} + v_k^{l\curvearrowright l+1}$, using the notation with the curved arrows with convention (37).

Then for $i < j$, we have $\sigma_i v_j = \sigma_{j-1} v_i$. On the component $I = \begin{matrix} j-1\leftarrow \\ i\leftarrow \\ j\leftarrow \end{matrix} J = \begin{matrix} i\leftarrow \\ j\leftarrow \end{matrix} J$, $\sigma_i v_j = \sigma_{j-1} v_i$ reads:

$$(39) \quad \begin{aligned} (\sigma_i v_j)^I &= v_j^{i\rightarrow I} + v_j^{i\curvearrowright i+1} = v_j^{j\leftarrow I} + v_j^{j\curvearrowright j+1} \\ &= (\sigma_{j-1} v_i)^I = v_i^{j-1\rightarrow I} + v_i^{j-1\curvearrowright j} = v_i^{i\leftarrow I} + v_i^{i\curvearrowright i+1}. \end{aligned}$$

Here $\begin{matrix} i\rightarrow \\ j\leftarrow \end{matrix} I = \begin{matrix} j\leftarrow \end{matrix} I$ by definition of I .

We will now prove Equation (38), the proof works differently for $i + 1 = j$ and $i + 1 < j$:

When $i + 1 < j$, by Eq. (39), Eq. (38) reduces to

$$v_i^{j\curvearrowright j+1} - w^{j\curvearrowright j+1} = v_j^{j\leftarrow I} - w^{i\curvearrowright i+1}$$

By induction hypothesis, $w^{i\curvearrowright i+1} = v_j^{j\leftarrow I} - w^{j\curvearrowright j+1}$ and $w^{j\curvearrowright j+1} = v_i^{i\leftarrow I} - w^{j\curvearrowright j+1}$. As $i + 1 < j$, $w^{j\curvearrowright j+1} = w^{i\curvearrowright i+1}$. This concludes the proof for the case $i + 1 < j$.

When $i + 1 = j$, then we know $j = i + 1, j + 1 = i + 2 \notin J$. Equation (38) then reads:

$$v_i^{i\leftarrow I} - w^{i\curvearrowright i+1} = v_j^{j\leftarrow I}$$

By induction hypothesis, we can now replace $w \overset{i \curvearrowright i+1}{J} \overset{j \leftarrow}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1}$ by $v_j \overset{j \leftarrow}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1}$ and subsequently $w \overset{j \curvearrowright j+1}{i \curvearrowright i+1}$ by $v_i \overset{i \leftarrow}{j \curvearrowright j+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1}$ making the following equation the aim of proof:

$$v_i \overset{i \leftarrow}{j \curvearrowright j+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1} \overset{j \leftarrow}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} = v_j \overset{j \leftarrow}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1}.$$

Upon observing that $\overset{j \leftarrow}{i \curvearrowright i+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1} \overset{j \leftarrow}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} = \overset{j \leftarrow}{i \curvearrowright i+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1} \overset{i \curvearrowright i+1}{j \curvearrowright j+1} \overset{j \leftarrow}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1} \overset{j \curvearrowright j+1}{i \curvearrowright i+1}$, this equation reduces exactly to Eq. (39). Thus Eq. (38) is satisfied also in the $j = i + 1$ case, completing the proof of the induction step, hence the proof of Observation 3.12. \square

Observation 3.12 implies Observation 3.10 and thus concludes the proof of this Lemma. \square

Remark 3.13. We believe that [HJ23, Theorem 2.7] is proving some similar result in another language and may be used to prove this Lemma if sufficient translation is provided. However, straightforwardly, the space P therein is completely different from our limit $T[1]^k M/T[k]M$ as the maps $p_j^{n \setminus \{i\}}$ to form the limit are different from ours, namely the σ_i 's. Translating to our simplicial language, $p_j^{n \setminus \{i\}}$ are degeneracy maps $D^{n-1} \rightarrow D^n$ of the nerve of the group $D \Rightarrow \star$ but not those of the nerve of the pair groupoid $D \times D \Rightarrow D$ as in our case. We leave the usage of this Theorem in the proof of our result to a future, possibly joint work.

3.4. A monomorphism of $T[1]^k \mathfrak{d}^k X$ to $(T[1]^k X_{k-1})^{[k] \setminus 0}$. The map $d_J^X : X_k \rightarrow X_{k-1}^{[k] \setminus 0}$, for $J = (1, \dots, k)$, are easier to take care of and one does not need a choice of connection. Given a Lie ∞ -groupoid X_\bullet , the maps (d_1^X, \dots, d_k^X) factor through the horn $\mathfrak{d}^k X$, which can be seen as a subset of $X_{k-1}^{[k] \setminus 0}$. Moreover, it is exactly the limit of the following diagrams for $1 \leq i < j \leq k$:

$$\begin{array}{ccc} \mathfrak{d}^k(X) & \xrightarrow{d_i^X} & X_{k-1} \\ \downarrow d_j^X & & \downarrow d_{j-1}^X \\ X_{k-1} & \xrightarrow{d_i^X} & X_{k-2} \end{array}$$

Lemma 3.14. Let X_\bullet be a Lie ∞ -groupoid. Then $T[1]^k \mathfrak{d}^k(X)$ is the limit of the following diagrams in the category of \mathbb{Z} -graded manifolds:

$$\begin{array}{ccc} T[1]^k \mathfrak{d}^k(X) & \xrightarrow{T[1]^k d_i^X} & T[1]^k X_{k-1} \\ \downarrow T[1]^k d_j^X & & \downarrow T[1]^k d_{j-1}^X \\ T[1]^k X_{k-1} & \xrightarrow{T[1]^k d_i^X} & T[1]^k X_{k-2}. \end{array}$$

Proof. Here the proof follows directly from the fact that the functor $T[1]$ preserves limits, since it is the right-adjoint to the functor $D \times$, i.e. $\text{hom}(D \times A, B) = \text{hom}(A, T[1]B)$. \square

Remark 3.15. This Lemma gives us a map $T[1]^k \mathfrak{d}^k(X) \xrightarrow{T[1]^k i_{k,0}} (T[1]^k X_{k-1})^{[k] \setminus 0}$ of \mathbb{Z} -graded manifolds over the embedding $\mathfrak{d}^k(X) \xrightarrow{i_{k,0}} X_{k-1}^{[k] \setminus 0}$. Since $T[1]$ is the right-adjoint to the functor $D \times$ and right-adjoints preserves monomorphism, $T[1]^k i_{k,0}$ is a monomorphism in $\mathbb{Z}\text{Mfd}$. Recall that monomorphism means left-cancellation (see Remark 2.11). To have left-cancellation in $\mathbb{Z}\text{Mfd}$ implies to have left-cancellation in its subcategory NMfd . Thus $T[1]^k i_{k,0}$ is also a monomorphism in NMfd .

3.5. The induction step. We are now prepared to characterize the image of $T[1]^k X_k$ inside $(T[1]^k X_{k-1})^{[k] \setminus 0} \times (T[1]^{k-1} X_k)^{[k-1]}$. The tangent maps of horn projection fit into the following commutative diagrams of N-manifolds

$$\begin{array}{ccc} T[k]X_k & \longrightarrow & T[1]^k X_k \\ \downarrow T[k]p_0^k & & \downarrow T[1]^k p_0^k \\ T[k]\mathfrak{h}^k(X) & \longrightarrow & T[1]^k \mathfrak{h}^k(X) \end{array}$$

Since p_0^k is a surjective submersion, according to Cor.2.33, we may choose compatible connections on T^*X_k and $T^*\mathfrak{h}^k(X)$ such that $T[1]^k p_0^k$ is a strict morphism, namely it comes from a morphism of graded vector bundles

$$(40) \quad \bigoplus_{i=1}^k \bigoplus_{I \subset \{1, \dots, k\}; |I|=i} T(X_k)_I \xrightarrow{\oplus_i \oplus_I (Tp_0^k)_I} \bigoplus_{i=1}^k \bigoplus_{I \subset \{1, \dots, k\}; |I|=i} T\mathfrak{h}^k(X)_I.$$

It is clear that the above map $\oplus_i \oplus_I (Tp_0^k)_I$ is a surjective submersion. Notice that $T[k]X_k$ and $T[k]\mathfrak{h}^k(X)$ are components when $i = k$ of the above direct sums on the left and right respectively, and the map $T[k]p_0^k$ is a component of $\oplus (Tp_0^k)_I$. After mod out these components,

$$(41) \quad \bigoplus_{i=1}^{k-1} \bigoplus_{I \subset \{1, \dots, k\}; |I|=i} T(X_k)_I \xrightarrow{\oplus_i \oplus_I (Tp_0^k)_I} \bigoplus_{i=1}^{k-1} \bigoplus_{I \subset \{1, \dots, k\}; |I|=i} T\mathfrak{h}^k(X)_I,$$

is again a surjective submersion. This shows that

$$T[1]^k X_k / T[k]X_k \xrightarrow{(p_0^k)_*} T[1]^k \mathfrak{h}^k(X) / T[k]\mathfrak{h}^k(X),$$

is again a strict morphism and it comes from a surjective submersion of graded vector bundles thus a strict surjective submersion. It is also clear that the map $T[1]^k \mathfrak{h}^k(X) \xrightarrow{q} T[1]^k \mathfrak{h}^k(X) / T[k]\mathfrak{h}^k(X)$ is also a strict surjective submersion. Let

$$A := T[1]^k X_k / T[k]X_k, \quad B := T[1]^k \mathfrak{h}^k(X), \quad C := T[1]^k \mathfrak{h}^k(X) / T[k]\mathfrak{h}^k(X).$$

Thus with the above maps $p := (p_0^k)_*$ and q , the limit $A \times_C B$ is again an N-manifold which comes with a splitting given by a pair of compatible connections on T^*X_k and $T^*\mathfrak{h}^k(X)$ according to Section 2.4. We further show that:

Lemma 3.16. *The limit $A \times_C B \cong T[1]^k X_k / \ker Tp_0^k[k]$. Thus the natural map $T[1]^k X_k \rightarrow A \times_C B$ is a strict surjective submersion with respect to splittings determined by compatible connections on T^*X_k and $T^*\mathfrak{h}^k(X)$.*

Proof. We apply compatible connections to descend the argument to the level of graded vector bundles. Once it is descended to the level of graded vector bundles, things are very clear: C contains all the information of B except for the highest degree $T[k]\mathfrak{h}^k(X)$. Notice that $T[k]X_k / \ker Tp_0^k[k] = (p_0^k)^* T[k]\mathfrak{h}^k(X)$, a linear algebra calculation in the form of $(V'/W') \times_{V/(W'/W)} V = V'/W$ gives us the desired result. \square

Using the results in Section 3.3 and Section 3.4, we want to obtain a map $i_A \times i_B : A \times_C B \rightarrow (T[1]^{k-1} X_k)^{[k-1]} \times (T[1]^k X_{k-1})^{[k] \setminus 0}$ such that $H^{k-1} \xrightarrow{(\circ) \circ \mathcal{F}_{k-1}} (T[1]^{k-1} X_k)^{[k-1]} \times (T[1]^k X_{k-1})^{[k] \setminus 0}$ lifts to $A \times_C B$. For this, we need to understand the nature of $A \times_C B$ as the limit of a diagram system.

Lemma 3.17. *The fiber product $A \times_C B$ is the limit of the following diagrams in the category of N-manifolds:*

$$\begin{array}{ccccc} A \times_C B & \xrightarrow{(s_i^D)^*} & T[1]^{k-1} X_k & , & A \times_C B & \xrightarrow{(d_{j'}^X)^*} & T[1]^k X_{k-1} \\ \downarrow (s_{i'+1}^D)^* & & \downarrow (s_{i'}^D)^* & & \downarrow (d_j^X)^* & & \downarrow (d_{j-1}^X)^* \\ T[1]^{k-1} X_k & \xrightarrow{(s_i^D)^*} & T[1]^{k-2} X_k & & T[1]^k X_{k-1} & \xrightarrow{(d_{j'}^X)^*} & T[1]^k X_{k-2} \end{array}$$

$$\begin{array}{ccc}
A \times_C B & \xrightarrow{(d_i^X)_*} & T[1]^k X_{k-1} \\
\downarrow (s_j^D)_* & & \downarrow (s_j^D)_* \\
T[1]^{k-1} X_k & \xrightarrow{(d_i^X)_*} & T[1]^{k-1} X_{k-1}.
\end{array}$$

More precisely, for a test object $T \in \text{NMfd}$, a collection of maps $a_I = (a_0, \dots, a_{k-1}) : T \rightarrow T[1]^{k-1} X_k$, $b_J = (b_1, \dots, b_k) : T \rightarrow T[1]^k X_{k-1}$ satisfying the following equations lift to a unique map $T \rightarrow A \times_C B$:

- (a) $(s_{i'}^D)_* a_i = (s_i^D)_* a_{i'+1}$, for $0 \leq i \leq i' \leq k-2$,
- (b) $(d_{j'}^X)_* b_j = (d_{j-1}^X)_* b_{j'}$, for $1 \leq j' < j \leq k$,
- (c) $(d_j^X)_* a_i = (s_i^D)_* b_j$, for $j \in J = \{1, \dots, k\}$, $i \in I = \{0, \dots, k-1\}$.

Condition (a), (b), (c) correspond to the first, second, and third type of diagrams respectively, and the range of the indices in these diagrams are indicated in the corresponding conditions.

Proof. Given a test object T with a collection of maps a_I, b_J with $I = \{0, \dots, k-1\}$ and $J = \{1, \dots, k\}$, condition (a) and (b) are the conditions for $a_I : T \rightarrow (T[1]^{k-1} X_k)^{[k] \setminus 0}$ and $b_J : T \rightarrow (T[1]^k X_{k-1})^{[k-1]}$ to admit lifts to $a : T \rightarrow A$ and $b : T \rightarrow B$ respectively. We observe the following diagram:

$$(42) \quad \begin{array}{ccccc}
A \times_C B & \xrightarrow{pr_A} & B & \xrightarrow{i_B} & (T[1]^k X_{k-1})^{[k] \setminus 0} \\
\downarrow pr_B & & \downarrow q & & \downarrow \\
A & \xrightarrow{p} & C & \xrightarrow{} & (T[1]^k X_{k-1} / T[k] X_{k-1})^{[k] \setminus 0} \\
\downarrow i_A & & \downarrow i_C & & \downarrow \\
(T[1]^{k-1} X_k)^{[k-1]} & \xrightarrow{} & (T[1]^{k-1} \mathfrak{H}^k(X))^{[k-1]} & \xrightarrow{(T[1]^{k-1} i_{k,0})^{[k-1]}} & (T[1]^{k-1} X_{k-1})^{[k-1] \times [k] \setminus 0}
\end{array}$$

Here, i_A and i_C are the strict embeddings described in Section 3.3, taking M being X_k and $\mathfrak{H}^k(X)$ respectively⁹; $T[1]^{k-1} i_{k,0}$ is the monomorphism described in Section 3.4, the upper-left square is a pullback diagram¹⁰; and the remaining three squares are commutative diagrams. The composition of two bottom maps is $[k-1]$ -tuples of $(d_j^X)_*$ and the composition of the two vertical maps on the most right is $[k] \setminus 0$ -tuples of $(s_I^D)_*$. Thus we have

$$(s_I^D)_*^{[k] \setminus 0} \circ i_B \circ b = (s_I^D)_*^{[k] \setminus 0} \circ b_J = (d_J^X)_*^{[k-1]} \circ a_I = (d_J^X)_*^{[k-1]} \circ i_A \circ a,$$

implied by condition (b), (c), (a) sequentially. Chasing through the above diagram, we have

$$(43) \quad (T[1]^{k-1} i_{k,0})^{[k-1]} \circ i_C \circ q \circ b = (T[1]^{k-1} i_{k,0})^{[k-1]} \circ i_C \circ p \circ a.$$

To show that $A \times_C B$ is the limit, we only need to show that $q \circ b = p \circ a$. Then the starting data of a_I, b_J give us unique a, b , which further satisfy $q \circ b = p \circ a$, thus we have a unique lift $f : T \rightarrow A \times_C B$.

Since we have (43), to show that $q \circ b = p \circ a$, it is sufficient to see that $(T[1]^{k-1} i_{k,0})^{[k-1]} \circ i_C$ is a monomorphism. The strict embedding i_C is certainly a monomorphism in NMfd and $T[1]^{k-1} i_{k,0}$ is also a monomorphism in NMfd by Section 3.4, thus $(T[1]^{k-1} i_{k,0})^{[k-1]} \circ i_C$ is a monomorphism. \square

Now we show the induction step in the following lemma:

Lemma 3.18. *Assume that H^{k-1} is representable as a graded manifold \mathcal{M} over X_0 . Then H^k is representable as a graded manifold by $\mathcal{M} \times_{X_0} \ker T(p_0^k)[k]$.*

⁹Notice that $(s_i^D)_*$ is exactly σ_i therein.

¹⁰To simplify the notation, various $T[1]^k d_j^X$ or $T[1]^{k-1} d_j^X$ therein are simply denoted by $(d_j^X)_*$.

Proof. Our starting point is the pullback diagram (31). The right vertical map factors through $A \times_C B$ by Lemma 3.17 and Lemma 3.16,

$$(44) \quad \begin{array}{ccc} H^k & \xrightarrow{\mathcal{F}_k} & T[1]^k X_k \\ \downarrow & & \downarrow \\ H^{k-1} & \xrightarrow{((s_I^X)_*, (d_J^D)^*) \circ \mathcal{F}_{k-1}} & (T[1]^{k-1} X_k)^{[k-1]} \times (T[1]^k X_{k-1})^{[k] \setminus 0} \end{array} \quad \begin{array}{c} \downarrow \\ A \times_C B \\ \downarrow \iota \end{array} \quad \begin{array}{c} \downarrow \\ (s_I^D)^*, (d_J^X)_* \\ \downarrow \end{array}$$

We first show that the bottom map $H^{k-1} \rightarrow (T[1]^k X_{k-1})^{[k] \setminus 0} \times (T[1]^{k-1} X_k)^{[k-1]}$ lifts to a unique map $H^{k-1} \rightarrow T[1]^k X_k / \ker(Tp_0^k)[k]$. For this, we only need to verify that $a_i = (s_i^X)_* \circ \mathcal{F}_{k-1}$ and $b_j = (d_j^D)^* \circ \mathcal{F}_{k-1}$ satisfy condition (a), (b), (c).

For condition (a), we look at the following diagrams, where the commutativity of the left square follows from the diagram (31) one level below, and that of the right square follows from simplicial identities.

$$\begin{array}{ccccc} H^{k-1} & \xrightarrow{\mathcal{F}_{k-1}} & T[1]^{k-1} X_{k-1} & \xrightarrow{(s_i^X)_*} & T[1]^{k-1} X_k \\ \downarrow \pi & & \downarrow (s_{i'}^D)^* & & \downarrow (s_{i'}^D)^* \\ H^{k-2} & \xrightarrow{(s_{i'}^X)_* \circ \mathcal{F}_{k-2}} & T[1]^{k-2} X_{k-1} & \xrightarrow{(s_i^X)_*} & T[1]^{k-2} X_k \end{array}$$

This means that $(s_{i'}^D)^* a_i = (s_{i'}^D)^* (s_i^X)_* \mathcal{F}_{k-1} = (s_i^X)_* (s_{i'}^X)_* \pi$. In particular for $i \leq i'$:

$$(s_{i'}^D)^* a_i = (s_i^X)_* (s_{i'}^X)_* \pi = (s_{i'+1}^X)_* (s_i^X)_* \pi = (s_i^D)^* a_{i'+1}.$$

Conditions (b), and (c) follow with a similar argument by simplicial identities and induction hypothesis. Secondly, we look at that the composed map

$$(45) \quad \iota : A \times_C B \xrightarrow{pr_A \times pr_B} A \times B \xrightarrow{i_A \times i_B} (T[1]^{k-1} X_k)^{[k-1]} \times (T[1]^k X_{k-1})^{[k] \setminus 0}.$$

By Section 3.4, $i_B := T[k]i_{k,0}$ is a monomorphism, and by Section 3.3, i_A is a strict embedding thus a also monomorphism. Moreover, it is a general fact on fiber product that $A \times_C B \xrightarrow{pr_A \times pr_B} A \times B$ is a monomorphism. We shortly recall the proof here: Let $g_1, g_2 : T \rightarrow A \times_C B$ such that $pr_A \times pr_B \circ g_1 = pr_A \times pr_B \circ g_2$. Then in particular $pr_A \circ g_1 = pr_A \circ g_2$ and the same for pr_B . Thus $q \circ pr_A \circ g_1 = q \circ pr_A \circ g_2 = p \circ pr_B \circ g_1 = p \circ pr_B \circ g_2$, thus there exists a unique map $g_1 = g_2 : T \rightarrow A \times_C B$ by the definition of fiber product. Thus ι is a monomorphism. Following a diagram chasing argument, one can see that H^k is also the fiber product of the smaller angled diagram, that is,

$$(46) \quad H^k \cong H^{k-1} \times_{A \times_C B} T[1]^k X_k.$$

Notice that

$$(47) \quad T[1]^k X_k \cong T[1]^k X_k / \ker Tp_0^k[k] \times_{X_k} \ker Tp_0^k[k] \cong A \times_C B \times_{X_k} \ker Tp_0^k[k].$$

Moreover fiber products in NMfd still satisfies base change via a typical diagram chasing argument. Thus we have,

$$(48) \quad \begin{aligned} H_k &\cong H_{k-1} \times_{A \times_C B} T[1]^k X_k \cong H_{k-1} \times_{A \times_C B} A \times_C B \times_{X_k} \ker Tp_0^k[k] \\ &\cong H_{k-1} \times_{X_k} \ker Tp_0^k[k] \cong H_{k-1} \times_{X_0} X_0 \times_{X_k} \ker Tp_0^k[k] = H_{k-1} \times_{X_0} \ker Tp_0^k[k] |_{X_0} \end{aligned}$$

□

Now Lemma 3.8 follows:

Proof for Lemma 3.8. The case $k = 0, 1$ are verified in Lemma 3.6 and Section 3.2.1. Now by induction hypothesis, we have already $H_{k-1} \cong \mathfrak{S}(\oplus_{i=1}^{k-1} Tp_0^i[i]|_{X_0})$, it is clear that $H_k \cong \mathfrak{S}(\oplus_{i=1}^k Tp_0^i[i]|_{X_0})$ by Lemma 3.18. Notice that in the calculation, for simplicity, we sometimes omit \mathfrak{S} , namely a graded vector bundle also means its represented graded manifolds. \square

This completes the proof of Theorem 3.3.

Remark 3.19. *It is clear that for each step when $k \geq 2$, we need to choose a pair of compatible connections on TX_k and $T\hat{\mathfrak{d}}^k(X)$ (see Lemma 3.16). But we do not need any connection for $k = 0, 1$. Thus the splitting of $\mathcal{T}(X_\bullet)$ depends on a choice of compatible connections on TX_k and $T\hat{\mathfrak{d}}^k(X)$, and isomorphisms (46) for $k = 2, \dots, n$.*

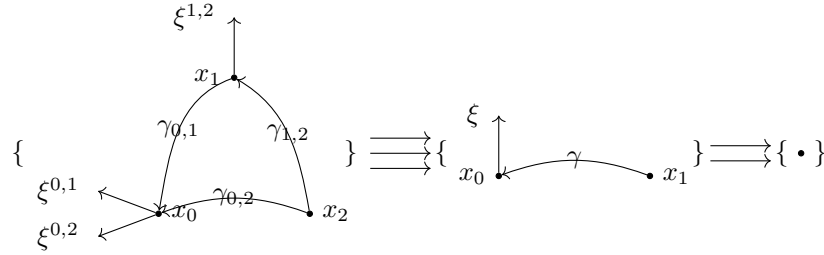
4. EXAMPLES

Example 4.1 (Lie groupoids). *In the case when X_\bullet is the nerve of a Lie groupoid, the horn projection $p_0^1 : X_1 \rightarrow X_0$ is simply $d_1 = \mathfrak{s}$ the source of the Lie groupoid. Thus the tangent complex of a Lie groupoid $X_1 \rightrightarrows X_0$ is simply $\ker T\mathfrak{s}|_{X_0}$ the vector bundle of the Lie algebroid. This especially tells us the tangent complex of a Lie group G is the Lie algebra \mathfrak{g} .*

In a forthcoming paper, we will demonstrate how to obtain (higher) Lie brackets on the tangent complexes. Now we focus on examples of tangent object of various higher Lie groupoids.

Example 4.2 (Courant Lie 2-groupoid). *The integration of Courant algebroids is a hot topic. Various integration models for special cases are given in [LBv15, MT11, MT18b, SZ]. As the nature of a Courant algebroid is a 2-shifted symplectic Lie 2-algebroid, one expects a 2-shifted symplectic Lie 2-groupoid as the integration object. As the topic of this article is differentiation rather than integration, we do not look for a 2-shifted symplectic Lie 2-groupoid structure. However, we take the finite dimensional model of a Lie 2-groupoid integrating a standard Courant algebroid, which we call a **Courant Lie 2-groupoid**, and calculate its tangent object. Notice that the finite dimensional models in the above literature are locally the same. The one we use here is from [SZ]. Given a manifold M , the Courant Lie 2-groupoid is*

$$(49) \quad X_2 : (\Pi_1(M) \times_M \Pi_1(M)) \times_{M \times 3} (T^*M)^{\times 3} \rightrightarrows X_1 := \Pi_1(M) \times_M T^*M \rightrightarrows X_0 := M,$$



where $\Pi_1(M) = \tilde{M} \times \tilde{M} / \pi_1(M)$ is the fundamental groupoid of M with \tilde{M} the simply connected cover of M .

We notice that $p_0^1 : \Pi_1(M) \times_M T^*M \rightarrow M$ is given by $(\gamma, \xi) \mapsto \gamma(0)$. We have $\ker Tp_0^1|_M = T^*M \oplus TM$. Similarly, $p_0^2 : X_2 \rightarrow \hat{\mathfrak{d}}^2(X)$ is given by

$$(\gamma_{1,2}, \gamma_{0,1}, \xi^{1,2}, \xi^{0,1}, \xi^{0,2}) \mapsto (\gamma_{0,1}, \gamma_{0,1} \cdot \gamma_{1,2}, \xi^{0,1}, \xi^{0,2}).$$

Thus $\ker Tp_0^2|_M = T^*M$. Thus we obtain $T[1]M \oplus T^*[1]M \oplus T^*[2]M$, which is the underlying graded vector bundle of a standard Courant algebroid, as the tangent of the Courant Lie 2-groupoid.

Example 4.3 (Semi-strict Lie 2-groupoids). *The above example of a Courant Lie 2-groupoid is a special case of a semi-strict Lie 2-groupoid. A Lie 2-groupoid can be viewed as [Zhu09a] a groupoid object in the 2-category **GpdBibd** where the space of objects is only a manifold (but not a general Lie groupoid). Here **GpdBibd** is the 2-category with Lie groupoids as objects, Hilsum-Skandalis (HS) bimodules as morphisms, and isomorphisms of HS bimodules as 2-morphisms. A semistrict Lie 2-groupoid is a particular Lie 2-groupoid, which is a groupoid object in the 2-category of **Gpd** with the space of objects a manifold,*

where \mathbf{Gpd} is a sub-2-category of $\mathbf{GpdBibd}$ containing only strict groupoid morphisms as morphisms. In particular, this means that a semistrict Lie 2-groupoid consists of a Lie groupoid $\mathcal{G} := G_1 \xrightarrow[\mathfrak{t}_G]{\mathfrak{s}_G} G_0$ over a manifold M with $\mathfrak{s}, \mathfrak{t} : G_0 \Rightarrow M$, such that there is an additional horizontal multiplication morphism $m : \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}$ of Lie groupoids, which is associative up to a 2-morphism called an associator. For the complete definition we refer to [SZ, Definition 5.2].

The Lie 2-groupoid X_\bullet described using the simplicial language corresponding to the semistrict Lie 2-groupoid $\mathcal{G} \Rightarrow M$ is given by

$$X_0 = M, \quad X_1 = G_0, \quad X_2 = (X_1 \times_{s, X_0, t} X_1) \times_{m_0, X_1, \mathfrak{t}_G} G_1,$$

where m_0 is the part of m on the level of objects. We refer to [Zhu09a, Eq. (19) Section 4.1] for face and degeneracy maps. Then $p_0^1 = \mathfrak{s} : X_1 \rightarrow X_0$. Thus $\ker Tp_0^1|_{X_0} = \ker T\mathfrak{s}|_M$. Moreover,

$$p_0^2 : X_2 = (X_1 \times_{s, X_0, t} X_1) \times_{m_0, X_1, \mathfrak{t}_G} G_1 \rightarrow X_1 \times_{X_0} X_1, \quad (\gamma_{12}, \gamma_{01}, \xi) \mapsto (\gamma_{01}, \mathfrak{s}_G(\xi)).$$

Here we use some notation inspired by the calculation in the last example. Since $\mathfrak{t}_G(\xi) = m_0(\gamma_{01}, \gamma_{12})$, when we compare the information on the left hand side with that on the right hand side, we obtain that $\ker Tp_0^2|_M = \ker T\mathfrak{s}_G|_M$. Here we use the embedding $M \xrightarrow{s_0} G_0 = X_1 \xrightarrow{s_0} X_2$. Thus we obtain

$$\ker T\mathfrak{s}[1]|_M \oplus \ker T\mathfrak{s}_G[2]|_M,$$

as the tangent object of a semistrict Lie 2-groupoid.

Example 4.4 (Strict Lie 2-groups and crossed modules of Lie groups). A crossed module $H \xrightarrow{\partial} G$ gives rise to a strict Lie 2-group whose underlying Lie groupoid is $G \times H \Rightarrow G$, which is a special example of semistrict Lie 2-groupoid with $M = pt$. Thus the previous example specially implies that its tangent is

$$\mathfrak{g}[1] \oplus \mathfrak{h}[2].$$

Example 4.5 (general central extensions for Lie 2-groups). Let G be a Lie group, $\mu : A \rightarrow B$ a homomorphism of abelian Lie groups. We consider the simplicial manifold BG , which is the nerve of the groupoid $G \rightrightarrows *$. Let $(U_i^{(n)})_{i \in I^{(n)}}$ be a simplicial covering of BG in the sense of [WZ16] and $\phi \in Z_U^3(BG, A \rightarrow B)$. We will follow the construction of [WZ16] of a stacky Lie groupoid central extension corresponding to cocycle ϕ and then [Zhu09a] to turn the stacky construction into a Lie 2-group \mathcal{G} in the simplicial formalism as in Def. 2.3.

$$(50) \quad 1 \rightarrow (A \rightarrow B) \rightarrow \mathcal{G} \rightarrow G \rightarrow 1$$

In fact, given a stacky 2-group $(\Gamma_1 \rightrightarrows_{\mathfrak{s}}^{\mathfrak{t}} \Gamma_0) \rightrightarrows *$, the corresponding simplicial 2-group is given by $X_0 = *$, $X_1 = \Gamma_0$ and $X_2 = E_m$, the bibundle representing the multiplication of Γ . In our case $\Gamma = \Gamma[\phi]$ is given by $\Gamma_0 = U_{[0]}^{(1)} \times B$ and $\Gamma_1 = U_{[1]}^{(1)} \times B \times A$, where $U_{[k]}^{(1)}$ denotes all possible k -fold intersections of $U^{(1)}$

$$\bigcap_{j=0}^{k-1} U_{i_j}^{(1)} = U_{i_0}^{(1)} \times_G \dots \times_G U_{i_{k-1}}^{(1)}.$$

We have source and target maps $\mathfrak{s}(u, b, a) = (\pi_0 u, b)$ and $\mathfrak{t}(u, b, a) = (\pi_1 u, b)$ of Γ . The zigzag giving rise to the multiplication is $\Gamma \times \Gamma \leftarrow \Gamma^2 \rightarrow \Gamma$, where Γ^2 is the groupoid

$$\Gamma_1^2 = U_{[1]}^{(2)} \times B^2 \times A^2 \rightrightarrows \Gamma_0^2 = U_{[0]}^{(2)} \times B^2$$

with analogous source and target maps. The bibundle E_m then is given by

$$((\Gamma_1 \times \Gamma_1) \times_{\mathfrak{t}, \Gamma_0 \times \Gamma_0} \Gamma_0^2 \times_{\Gamma_0, \mathfrak{t}} \times \Gamma_1) / \Gamma_1^2.$$

We need to specify the groupoid morphisms from Γ^2 . As a map from Γ_1^2 to $(\Gamma \times \Gamma)_1 = \Gamma_1 \times \Gamma_1$ we have:

$$(v_0, v_1, b_0, b_1, a_0, a_1) \mapsto (d_0 v_0, d_0 v_1, b_0, a_0) \times (d_2 v_0, d_2 v_1, b_1, a_1).$$

The map $\Gamma_1^2 \rightarrow \Gamma_1$ is given by

$$(v_0, v_1, b_0, b_1, a_0, a_1) \mapsto (d_1 v_0, d_1 v_1, b_0 + b_1 + \phi^{2,0,0}(v_0), a_0 + a_1 + \phi^{2,1,-1}(v_0, v_1)).$$

On the base Γ_0^2 the maps are $(v_0, b_0, b_1) \mapsto (d_0 v_0, b_0) \times (d_2 v_0, b_1)$ to and $(v_0, b_0, b_1) \mapsto (d_1 v_0, b_0 + b_1 + \phi^{2,0,0}(v_0))$ respectively. Hence

$$E_m = [((U_{[1]}^{(1)})^3 \times_{(U_{[0]}^{(1)})^3} U_1^{(2)})/U_{[1]}^{(2)}] \times B^2 \times A.$$

Since the first factor is just a cover of a small region in $G \times G$, this means that $\ker(Tp_0^2)|_{X_0} = T_e A$. Since $X_0 = *$, we also have $\ker(Tp_0^1)|_{X_0} = T_e G \oplus T_e B$. Thus the tangent object for the central extended Lie 2-group \mathcal{G} is

$$\mathfrak{g}[1] \oplus \mathfrak{b}[1] \oplus \mathfrak{a}[2],$$

where \mathfrak{g} , \mathfrak{b} and \mathfrak{a} are the Lie algebra of G , B and A correspondingly.

Example 4.6 (String Lie 2-group). A famous sub-example of the above is the case of String Lie 2-group $\mathbf{String}(G)$. Given $G = Spin(n)$, or in general (see e.g. [FOT08, Theorem 1.47, 1.81]) a compact, connected and simple Lie group G , one can form its associated String Lie 2-group $\mathbf{String}(G)$, which is a central extension

$$1 \rightarrow \mathcal{B}S^1 \rightarrow \mathbf{String}(G) \rightarrow G \rightarrow 1,$$

of Lie 2-groups in the sense of [SP11], with extension class the generator of $H^2(\mathcal{B}G, \mathcal{B}S^1) = H^3(\mathcal{B}G, S^1) = H^4(\mathcal{B}G, \mathbb{Z}) = \mathbb{Z}$. It fits into the last example with $A = S^1$ and $B = \{1\}$. Thus the tangent object for $\mathbf{String}(G)$ is

$$\mathfrak{g}[1] \oplus \mathbb{R}[2].$$

Example 4.7 (n -tower). Given a complex of Lie groups

$$(51) \quad G \xleftarrow{q_2} \Pi_2 \xleftarrow{q_3} \Pi_3 \xleftarrow{q_4} \dots \xleftarrow{q_n} \Pi_n,$$

together with G -actions $\alpha_i : G \rightarrow \text{Aut}(\Pi_i)$ such that the q_i 's are G -equivariant with respect to the conjugation action of G on G itself, we now construct a Lie n -group inductively with the help of $n-1$ cocycles c_i 's which will be defined inductively too. We call these data an n -tower, which shows up in the construction of a sigma model for topological orders [LZW19].

The first cocycle $c_3 \in Z^3(G, (\Pi_2^0)^{\alpha_2})$ is a smooth normalized 3-cocycle of G with coefficients in $\Pi_2^0 := \ker q_2$ with action α_2 , that is, $c_3 : G^{\times 3} \rightarrow \Pi_2^0$ is a smooth map, and

$$\delta_{\alpha_2} c_3(g_1, g_2, g_3, g_4) = \alpha_2(g_1)c_3(g_2, g_3, g_4) - c_3(g_1 g_2, g_3, g_4) + c_3(g_1, g_2 g_3, g_4) - c_3(g_1, g_2, g_3 g_4) + c_3(g_1, g_2, g_3) = 0.$$

With c_3 , we can build up a Lie 2-group $\mathcal{G}_{c_3}(G, \Pi_2)$ as in [BL04, Section 8]. This Lie 2-group written in the simplicial Kan complex $X_{\bullet}^{(2)}$ has the form: $X_0^{(2)} = pt, X_1^{(2)} = G$,

$$X_2^{(2)} := \{(x_{01}^1, x_{02}^1, x_{12}^1; x_{012}^2) | x_{\bullet}^1 \in G, x_{012}^2 \in \Pi_2, x_{01}^1 x_{12}^1 (x_{02}^1)^{-1} = q_2(x_{012}^2)\} \cong G \times G \times \Pi_2,$$

$$X_3^{(2)} := \{(x_{01}^1, x_{02}^1, \dots, x_{23}^1; x_{012}^2, \dots, x_{123}^2) | x_{\bullet}^1 \in G, x_{\bullet}^2 \in \Pi_2,$$

$$dx^1 = q_2(x^2), d_{\alpha_2} x^2 = c_3(x^1)\} \cong G^{\times \binom{3}{1}} \times \Pi_2^{\times \binom{3}{2}},$$

and higher levels $X_m^{(2)} \cong G^{\times m} \times \Pi_2^{\times \binom{m}{2}}$ are determined by taking the coskeleton functor [Zhu09a, Section 2.3]. Here $(dx^1)_{abc} := x_{ab}^1 x_{bc}^1 (x_{ac}^1)^{-1}$ and

$$(52) \quad (d_{\alpha_2} x^2)_{abcd} := \alpha_2(x_{ab}^1) x_{bcd}^2 - x_{acd}^2 + x_{abd}^2 - x_{bcd}^2$$

Inductively, the second cocycle $c_4 \in Z^4(\mathcal{G}_{c_3}(G, \Pi_2), (\Pi_3^0)^{\alpha_3})$ is a smooth normalized¹¹ 4-cocycle of the Lie 2-group $\mathcal{G}_{c_3}(G, \Pi_2)$ with coefficients in $\Pi_3^0 := \ker d_2$ with action α_3 , where the smooth Lie 2-group cochain complex is the one of its simplicial nerve. More precisely, $c_4 : X_4^{(2)} \rightarrow \Pi_3^0$ is a smooth map, and

$$\delta_{\alpha_3} c_4(x) = \alpha_3(g)c_4(d_0(x)) + \sum_{j=1}^4 (-1)^j c_4(d_j(x)),$$

¹¹in the sense that its evaluation on degenerate simplices is 0.

where $g = d_2 \circ d_3 \circ d_4(x)$. With c_4 , we build a Lie 3-group $\mathcal{G}_{c_4}(G, \Pi_2, \Pi_3)$ whose simplicial Kan complex $X_\bullet^{(3)}$ has the form: $X_0^{(3)} = X_0^{(2)}$, $X_1^{(3)} = X_1^{(2)}$, $X_2^{(3)} = X_2^{(2)}$, and

$$X_3^{(3)} := \{(x_{01}^1, x_{02}^1, \dots, x_{23}^1; x_{012}^2, \dots, x_{123}^2; x_{0123}^3) : x_{\cdot}^1 \in G, x_{\cdot}^j \in \Pi_j, j = 2, 3, \\ dx^1 = q_2(x^2), d_{\alpha_2}x^2 = q_3(x^3) + c_3(x^1)\} \cong G^{\times(3)} \times \Pi_2^{\times(3)} \times \Pi_3^{\times(3)},$$

$$X_4^{(3)} := \{(x_{01}^1, x_{02}^1, \dots, x_{34}^1; x_{012}^2, \dots, x_{234}^2; x_{0123}^3, \dots, x_{1234}^3) : x_{\cdot}^1 \in G, x_{\cdot}^j \in \Pi_j, j = 2, 3, \\ dx^1 = q_2(x^2), d_{\alpha_2}x^2 = q_3(x^3) + c_3(x^1), d_{\alpha_3}x^3 = c_4(x^1; x^2)\} \cong G^{\times(4)} \times \Pi_2^{\times(4)} \times \Pi_3^{\times(4)},$$

where $d_{\alpha_3}x^3$ is a similar alternating sum in Π_3 as (52). This construction continues and it gives us inductively a Lie 2-group $X_\bullet^{(2)}$, a Lie 3-group $X_\bullet^{(3)}$, ..., and a Lie n -group $X_\bullet^{(n)}$ (see [LZW19, Appendix L] and [Mis] for details). At the end of the day, we have

$$(53) \quad \begin{aligned} X_m^{(n)} &= G^{\times(m)} \times \Pi_2^{\times(m)} \times \dots \times \Pi_m^{\times(m)}, \quad \text{when } m \leq n, \\ X_m^{(n)} &= G^{\times(m)} \times \Pi_2^{\times(m)} \times \dots \times \Pi_n^{\times(m)}, \quad \text{when } m \geq n. \end{aligned}$$

For $m \leq n$, after removing the only m -simplex in $X_m^{(n)}$, the horn space is

$$(54) \quad \mathfrak{h}^m(X_\bullet^{(n)}) = G^{\times(m)} \times \Pi_2^{\times(m)} \times \dots \times \Pi_{m-1}^{\times(m-1)},$$

containing $m(m-1)$ -simplices and p_0^m is simply the projection using the last equation in $X_m^{(n)}$. Thus $\ker Tp_0^m = \mathfrak{a}_m$, where \mathfrak{a}_m is the Lie algebra of Π_m when $m \geq 2$ and $\ker Tp_0^1 = \mathfrak{g}$. Therefore, the tangent of $X_\bullet^{(n)}$ is $\mathfrak{g}[1] \oplus \mathfrak{a}_2[2] \oplus \dots \oplus \mathfrak{a}_m[m]$.

Example 4.8 (strict extensions for Lie n -groups). As pointed out in [Šev], similar to [BL04, Section8] where a 3-cocycle is used to build a 2-group, if $n \geq 3$ and we are given a Lie group G , an abelian Lie group Π , a smooth $n+1$ -cocycle $c : G^{n+1} \rightarrow \Pi$, and an action $\alpha : G \rightarrow \text{Aut}(\Pi)$, we can build a Lie n -group X_\bullet , which may be viewed as an extension $G \rtimes_c \Pi$ of G by Π , with

$$X_m = G^{\times m}, \quad \text{when } m \leq n-1, \quad X_m = G^{\times m} \times \Pi^{\times(m)} \quad \text{when } m \geq n.$$

In fact, this is a special example of the above one when Π_2, \dots, Π_{n-1} are 0 and $\Pi_n = \Pi$. Thus its tangent is $\mathfrak{g}[1] \oplus \mathfrak{a}[n]$ where \mathfrak{a} is the Lie algebra of Π .

Example 4.9 (simplicial Lie groups and their simplicial classifying complex [Jur12]). Apart from n -tower, there is a much more classical generalization of crossed modules, namely simplicial groups. For the purpose of differentiation, we are interested in a simplicial Lie group G_\bullet , which is a group object in the category of simplicial manifolds. That is, each level G_i is a Lie group and all the face and degeneracy maps are Lie group morphisms.

A simplicial group G_\bullet always satisfies the Kan conditions. This is proven in the set-theoretic setting in some classical literature [Moo56, Theorem 3] and we refer to [Sch10] for a more complete history on the literature. The proof is to construct degenerate horn fillings explicitly, that is, for a horn $\lambda \in \mathfrak{h}^k(G_\bullet)$, one constructs a simplex $g_k \in G_k$ such that $p_i^k(g_k) = \lambda$. The construction basically uses only a combination of face and degeneracy maps and multiplication in G_\bullet , thus it survives differential geometry. Namely, if G_\bullet is further a simplicial Lie group, for a smooth curve $\lambda(t) \in \mathfrak{h}^k(G_\bullet)$, we have a smooth curve $g_k(t) \in G_k$ lifting it. Thus for any preimage $g'_k \in G_k$ of λ under p_i^k , $g'_k g_k^{-1} g_k(t)$ is a smooth curve lifting $\lambda(t)$. Thus a simplicial Lie group G_\bullet satisfies Kan conditions with surjective submersions as in Def. 2.3, namely G_\bullet is an L_∞ -groupoid. Notice that G_\bullet is usually not an L_∞ -group unless $G_0 = \{1\}$.

For a simplicial Lie group G_\bullet , its associated Moore complex (or normalized chain complex) is a $\mathbb{Z}^{\geq 0}$ -graded chain complex $((NG)_\bullet, \partial_\bullet)$ of (possibly non-abelian) groups with

$$(NG)_k = \cap_{i=1}^k \ker d_i^k, \quad \partial_k = d_0^k : (NG)_k \rightarrow (NG)_{k-1}.$$

Let \mathfrak{n}_k be the Lie algebra of $(NG)_k$. One can calculate that $\ker Tp_0^k|_{G_0} = \mathfrak{n}_k$, which is a vector bundle over G_0 with fiber \mathfrak{n}_k . Thus the tangent object for G_\bullet is

$$\underline{\mathfrak{n}}_1[1] \oplus \underline{\mathfrak{n}}_2[2] \oplus \dots \oplus \underline{\mathfrak{n}}_k[k] \oplus \dots$$

This result is also calculated in [JSW16, Remark 4.11].

Associated to a simplicial group G_\bullet , its simplicial classifying complex $\bar{W}G$ is given by

$$\bar{W}G_k = G_{k-1} \times \cdots \times G_0.$$

We should imagine an element of $\bar{W}G_k$ as a sequence of arrows — which are elements in G_i between x_j 's, which are in turn all $1 \in G_0$:

$$x_0 \xleftarrow{g_{k-1}} x_1 \xleftarrow{g_{k-2}} x_2 \quad \cdots \quad x_{k-2} \xleftarrow{g_1} x_{k-1} \xleftarrow{g_0} x_k.$$

We denote the face maps of $\bar{W}G_\bullet$ by \bar{d}_i^k , and they correspond to deleting x_i in the above picture, that is

$$\bar{d}_i^k(g_{k-1}, g_{k-2}, \dots, g_1, g_0) = \begin{cases} (g_{k-2}, \dots, g_0), & i = 0 \\ (d_0(g_{k-1})g_{k-2}, g_{k-3}, \dots, g_0), & i = 1 \\ \dots \\ (d_{i-1}(g_{k-1}), \dots, d_1(g_{k-i+1}), d_0(g_{k-i})g_{k-i-1}, g_{k-i-2}, \dots, g_0) \\ \dots \\ (d_{k-1}(g_{k-1}), \dots, d_1(g_1)), & i = k. \end{cases}$$

Thus we have

(55)

$$\ker T\bar{d}_1^k = \{(\delta g_{k-1}, \delta g_{k-2}, \dots, \delta g_0) | d_0(g_{k-1})\delta g_{k-2} + Td_0(\delta g_{k-1})g_{k-2} = 0, \delta g_{k-3} = \cdots = \delta g_0 = 0\}$$

$$\ker T\bar{d}_2^k = \{(\delta g_{k-1}, \delta g_{k-2}, \dots, \delta g_0) | Td_1(g_{k-1}) = 0, Td_0(\delta g_{k-2})g_{k-3} + d_0(g_{k-2})\delta g_{k-3} = 0, \delta g_{k-4} = \cdots = \delta g_0 = 0\}$$

...

$$\ker T\bar{d}_i^k = \{(\delta g_{k-1}, \delta g_{k-2}, \dots, \delta g_0) | Td_{i-1}(\delta g_{k-1}) = 0, Td_{i-2}(\delta g_{k-2}) = 0, \dots, Td_0(\delta g_{k-i})g_{k-i-1} + d_0(g_{k-i})\delta g_{k-i-1} = 0, \delta g_{k-i-2} = \cdots = \delta g_0 = 0\}$$

...

$$\ker T\bar{d}_k^k = \{(\delta g_{k-1}, \delta g_{k-2}, \dots, \delta g_0) | Td_{k-1}(\delta g_{k-1}) = 0, Td_{k-2}(\delta g_{k-2}) = 0, \dots, Td_1(\delta g_1) = 0\},$$

where δg_l denotes a tangent vector in $T_{g_l}G_l$ for all l . We denote the horn projection of $\bar{W}G_\bullet$ by \bar{p}_j^k . Then to calculate $\ker T\bar{p}_0^k = \cap_{i=1}^k \ker T\bar{d}_i^k$, we basically need to solve all the equations in the description of sets in (55). It can be seen that $\delta g_{k-1} \in \cap_{j=1}^{k-1} \ker T\bar{d}_j^{k-1} = \mathfrak{n}_{k-1}$, and $\delta g_{k-3} = \cdots = \delta g_0 = 0$. Moreover, $\delta g_{k-3} = \cdots = \delta g_0 = 0$ makes all the equations proposed on $\delta g_{k-3}, \dots, \delta g_0$ trivially satisfied. From $\ker T\bar{d}_1^k$, we observe that δg_{k-2} is determined by δg_{k-1} in the following way,

$$(56) \quad \delta g_{k-2} = d_0(g_{k-1})^{-1}Td_0(\delta g_{k-1})g_{k-2}.$$

Now we verify that (56) implies all the equations involving δg_{k-2} in (55). Notice that $\delta g_{k-3} = \cdots = \delta g_0 = 0$, thus the equations involving δg_{k-2} reduce to

$$(57) \quad Td_i(\delta g_{k-2}) = 0, \quad 0 \leq i \leq k-2.$$

Notice that d_i 's are group morphisms, thus

$$Td_i(d_0(g_{k-1})^{-1} \cdot (Td_0(\delta g_{k-1})) \cdot g_{k-2}) = d_i(d_0(g_{k-1})^{-1}) \cdot Td_i(Td_0(\delta g_{k-1})) \cdot d_i(g_{k-2}) = 0.$$

The last step uses the fact that $Td_i(Td_0(\delta g_{k-1})) = Td_0Td_{i+1}(\delta g_{k-1}) = 0$ because $\delta g_{k-1} \in T\ker d_j$ for $j \geq 1$. Thus $\ker T\bar{p}_0^k = \mathfrak{n}_{k-1}$, which is simply a vector space. Therefore the tangent object of $\bar{W}G$ is

$$\mathfrak{n}_0[1] \oplus \mathfrak{n}_1[2] \oplus \cdots \oplus \mathfrak{n}_k[k+1] \oplus \dots$$

Thus with Theorem 2.15, in the case of $\bar{W}G_\bullet$, we reproduce the result calculated in [Jur12].

For more examples, we also refer to [Li14, Section8.4] and [Šev, Section13].

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