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# HIGGSING $qq$ -CHARACTER AND IRREDUCIBILITY

TARO KIMURA

*Dedicated to Professor Hiraku Nakajima on his 60th anniversary*

ABSTRACT. We show that the  $qq$ -character of the irreducible highest weight module for finite-type and affine quivers is obtained by Higgsing, specialization of the equivariant parameters of the associated framing space in the quiver variety.

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## 1. INTRODUCTION AND SUMMARY

The  $qq$ -character is a two parameter deformation of the ordinary character associated with a module constructed on a generic quiver, which has been originally introduced in the context of supersymmetric gauge theory through double quantization of Seiberg–Witten geometry by Nekrasov [Nek15]. The  $qq$ -character has a geometric realization using Nakajima’s quiver variety [Nak94, Nak98, Nak99], which could be interpreted as a natural generalization of the geometric construction of the  $q$ -character of Yangian and quantum affine algebra [Kni95, FR98]:<sup>1</sup> Let  $\Gamma = (\Gamma_0, \Gamma_1)$  be a quiver, where  $\Gamma_0$  is a set of the nodes and  $\Gamma_1$  is a set of the edges. We denote the quiver variety associated with quiver  $\Gamma$  by  $\mathfrak{M}_{v,w}^\Gamma$  with the dimension vectors,  $v = (v_i)_{i \in \Gamma_0}$  and  $w = (w_i)_{i \in \Gamma_0}$ , such that the vector spaces attached to the nodes and the corresponding framing spaces are given by  $\mathbf{V}_i = \mathbb{C}^{v_i}$  and  $\mathbf{W}_i = \mathbb{C}^{w_i}$  for  $i \in \Gamma_0$ . We also denote  $\mathbf{V} = (\mathbf{V}_i)_{i \in \Gamma_0}$  and  $\mathbf{W} = (\mathbf{W}_i)_{i \in \Gamma_0}$ , and the automorphism groups by  $G_v = \mathrm{GL}(\mathbf{V}) = \prod_{i \in \Gamma_0} \mathrm{GL}(\mathbf{V}_i)$  and  $G_w = \mathrm{GL}(\mathbf{W}) = \prod_{i \in \Gamma_0} \mathrm{GL}(\mathbf{W}_i)$ . Then, in addition to the quantum deformation parameters  $q_{1,2} \in \mathbb{C}^\times$ , the  $q$ - and  $qq$ -characters depend on the equivariant parameters of  $G_w$  denoted by  $\underline{x} = (x_{i,\alpha})_{i \in \Gamma_0, \alpha=1, \dots, w_i}$ , that we call the *weight parameters* for short. These weight parameters are identified with the zeros of the Drinfeld polynomials,<sup>2</sup> which play a central role in the construction of the finite dimensional representation of Yangian and quantum affine algebra [CP91, CP94]. It has been known that the finite dimensional module constructed from generic Drinfeld polynomials is not irreducible in general, and in order to obtain the irreducible highest weight module, zeros of the Drinfeld polynomial, namely the weight parameters, should obey the  *$q$ -segment condition*. The purpose of this paper is to elucidate the role of the weight parameters in the relation to the irreducibility of the  $qq$ -character.

**1.1. Irreducible reduction of  $qq$ -character.** Let us briefly demonstrate what happens for the  $q$ - and  $qq$ -characters via the specialization of the weight parameters. We denote the  $q$ - and  $qq$ -characters associated with the weight dimension vector  $w$  and the weight parameters  $\underline{x}$  by  $\mathbb{T}_{w,\underline{x}}^{(q)}$  and  $\mathbb{T}_{w,\underline{x}}^{(q_1,2)}$ . Both of them are constructed from the  $Y$ -function assigned to each node denoted by  $(Y_{i,x})_{i \in \Gamma_0, x \in \mathbb{C}^\times}$ , which is associated with the fundamental weight. Then, the simplest example is the weight one module of  $A_1$  quiver. Denoting  $Y_{1,x} = Y_x$ , we have the  $q$ - and  $qq$ -characters as follows:

$$\mathbb{T}_{1,x}^{(q)} = Y_x + Y_{xq}^{-1}, \quad \mathbb{T}_{1,x}^{(q_1,2)} = Y_x + Y_{xq_1q_2}^{-1}. \quad (1.1)$$

We provide the details on the construction of the  $qq$ -character in §2. Apparently, the  $qq$ -character is reduced to the  $q$ -character in the limits,  $\lim_{q_1 \rightarrow 1} \mathbb{T}_{1,x}^{(q_1,2)} = \mathbb{T}_{1,x}^{(q_2)}$  and  $\lim_{q_2 \rightarrow 1} \mathbb{T}_{1,x}^{(q_1,2)} = \mathbb{T}_{1,x}^{(q_1)}$ .

<sup>1</sup>There exists another two parameter deformation of the character, called the  $t$ -analog of  $q$ -character [Nak00, Nak01], which also has a natural geometric construction based on the (graded version of) quiver variety. The role of the  $t$ -parameter is different from the second deformation parameter of the  $qq$ -character, so that the relation between the  $qq$ -character and the  $t$ -analog of  $q$ -character is still not clear at this moment.

<sup>2</sup>Namely, the Drinfeld polynomial is a characteristic polynomial of  $G_w$ .

The next example is the weight two module. For generic weight parameters  $\underline{x} = (x_1, x_2)$ , the  $qq$ -character of weight two is given by

$$\mathbb{T}_{2,\underline{x}}^{(q_1,2)} = Y_{x_1} Y_{x_2} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \frac{Y_{x_2}}{Y_{x_1 q_1 q_2}} + \mathcal{S}\left(\frac{x_1}{x_2}\right) \frac{Y_{x_1}}{Y_{x_2 q_1 q_2}} + \frac{1}{Y_{x_1 q_1 q_2} Y_{x_2 q_1 q_2}}, \quad (1.2)$$

where we have a rational function, called the  $\mathcal{S}$ -function, in the coefficient. See §2.5 for the definition. This weight two character corresponds to the tensor product of the weight one characters, which is not further decomposed at this point.<sup>3</sup> Then, we specialize the weight parameters. Noticing that the  $\mathcal{S}$ -function has zeros  $\mathcal{S}(z) = 0$  at  $z = q_{1,2}$ , we put  $\underline{x} = (x, xq_1)$  and obtain the reduced  $qq$ -character containing three monomials,

$$\mathbb{T}_{2,\underline{x}}^{(q_1,2)} \xrightarrow{\underline{x} \rightarrow (x, xq_1)} \tilde{\mathbb{T}}_{2,x}^{(q_1,2)} = Y_x Y_{xq_1} + \mathcal{S}(q_1^{-1}) \frac{Y_x}{Y_{xq_1^2 q_2}} + \frac{1}{Y_{xq_1 q_2} Y_{xq_1^2 q_2}}. \quad (1.3)$$

We call this the irreducible  $qq$ -character of weight two of  $A_1$  quiver denoted by  $\tilde{\mathbb{T}}_{2,x}^{(q_1,2)}$ . This reduction mechanism to obtain the irreducible  $qq$ -character is the main subject of this paper.

Recalling  $\lim_{q_1 \rightarrow 1} \mathcal{S}(q_1^{-1}) = 2$  and  $\lim_{q_2 \rightarrow 1} \mathcal{S}(q_1^{-1}) = 1$ , the irreducible  $qq$ -character  $\tilde{\mathbb{T}}_{2,x}^{(q_1,2)}$  is further reduced to the  $q$ -characters as follows:

$$\lim_{q_1 \rightarrow 1} \tilde{\mathbb{T}}_{2,x}^{(q_1,2)} = Y_x^2 + 2 \frac{Y_x}{Y_{xq_2}} + Y_{xq_2}^{-2} = (Y_x + Y_{xq_2}^{-1})^2 = \left(\mathbb{T}_{1,x}^{(q_2)}\right)^2, \quad (1.4a)$$

$$\lim_{q_2 \rightarrow 1} \tilde{\mathbb{T}}_{2,x}^{(q_1,2)} = Y_x Y_{xq_1} + \frac{Y_x}{Y_{xq_1^2}} + \frac{1}{Y_{xq_1} Y_{xq_1^2}} = \mathbb{T}_{2,x}^{(q_1)}. \quad (1.4b)$$

This implies that we have the irreducible three-dimensional module in the limit  $q_2 \rightarrow 1$ , while we have the tensor product module, which is not irreducible, in the other limit  $q_1 \rightarrow 1$ . Such a reduction to the irreducible module is specific to the  $qq$ -character: The (tensor) product of the  $q$ -characters with the weight parameters  $(x, xq_1)$  is given by

$$\mathbb{T}_{1,x}^{(q)} \mathbb{T}_{1,xq}^{(q)} = Y_x Y_{xq} + \frac{Y_x}{Y_{xq^2}} + \frac{1}{Y_{xq} Y_{xq^2}} + 1 = \mathbb{T}_{2,x}^{(q)} + \mathbb{T}_{0,x}^{(q)}. \quad (1.5)$$

Although it is decomposed into the irreducible pieces,  $\mathbb{T}_{2,x}^{(q)}$  and  $\mathbb{T}_{0,x}^{(q)}$ , one cannot eliminate either one of them.<sup>4</sup> See §3 for more detailed analysis on  $A_1$  quiver.

We remark that since the  $qq$ -character is symmetric under exchanging  $q_1 \leftrightarrow q_2$  in this example, we may also consider another specialization  $\underline{x} = (x, xq_2)$  to obtain the irreducible  $qq$ -character. Such a symmetry between  $q_1$  and  $q_2$  is expected to hold for the Kirillov–Reshetikhin (KR) module of simply-laced quivers [KR90], while it would be violated in other cases. See §4.1.3 for  $A_2$  quiver (weight  $w = (1, 1)$ ) and §5 for  $BC_2$  quiver examples.

**1.2. Higgs mechanism.** Specialization of the weight parameters demonstrated above has a natural interpretation as the Higgs mechanism. Geometrically realizing the  $qq$ -character in the eight dimensional setup, called the gauge origami, the weight parameters are interpreted as the Coulomb moduli of the dual gauge theory defined in the transverse surface [Nek17]. From this point of view, the gauge symmetry is originally described by  $U(\mathbf{W}) = \prod_{i \in \Gamma_0} U(w_i)$ , which is a compact version of  $G_w = GL(\mathbf{W})$ . Specialization of the weight parameters implies freezing all the Coulomb moduli except for the center of mass factor. Therefore, we have the symmetry breaking,  $U(\mathbf{W}) \rightarrow U(1)$ . We remark that the Higgs mechanism freezes the non-Abelian part of the gauge group. The  $U(1)$  factors of each  $U(w_i)$  are fixed by the bifundamental mass parameters except for the overall center of mass factor in this context. It has been established that the  $qq$ -character has no singularity with respect to this center of mass factor, a.k.a., the compactness theorem [Nek16].

<sup>3</sup>From this point of view, the  $qq$ -character  $\mathbb{T}_{2,\underline{x}}^{(q_1,2)}$  is irreducible, but not corresponding to the irreducible representation of the associated (quantum) algebra. In this paper, the notion of the irreducibility is used in the later sense.

<sup>4</sup>This relation among the  $q$ -characters  $(\mathbb{T}_{w,x}^{(q)})_{w=0,1,2}$  is interpreted as the simplest example of the T-system [KNS10].

Such an interpretation of the parameter specialization as Higgsing has been discussed in the various contexts. It has been known that one can move on to the root of Higgs branch of the moduli space of supersymmetric vacua by tuning the Coulomb moduli [Dor98, DHT99]. In the context of topological string, the idea of Higgsing is used to describe the geometric transition [GV98]. For the primary example of the transition between the resolved and deformed conifolds, one may discuss the open string degrees of freedom on the Lagrangian submanifold on the deformed conifold side, which is given as the cotangent bundle  $T^*S^3$  and its Lagrangian  $S^3$ . From this point of view, the Higgsing condition on the Kähler parameter is interpreted as the flux quantization condition. Along this direction, we may apply a similar interpretation to the current situation. Geometrically, the irreducible part is interpreted as a subvariety of the whole quiver variety: The graded quiver variety is decomposed into direct product under the  $q$ -segment condition [Nak99, §4]. Hence, the Higgsed  $qq$ -character, which describes the irreducible module, is expected to correspond to this irreducible subvariety of the quiver variety. This is a geometric representation theoretical interpretation of the Higgsing process.

**1.3. Summary and organization.** In this paper, we have the following conjectures regarding the  $qq$ -character and its irreducibility through the Higgsing process.

**Conjecture 1.1.** Let  $\Gamma$  be a generic quiver and  $w = (w_i)_{i \in \Gamma_0} \in \mathbb{Z}_{\geq 0}^{\text{rk } \Gamma}$  be a generic weight dimension vector. Then, there exists a canonical specialization of the weight parameters  $\underline{x} = (x_{i,\alpha})$ , such that the  $qq$ -character is reduced to that for the irreducible highest weight module parametrized by  $w$ .

This Conjecture is for generic modules and for generic quivers. See Propositions 6.1 and 7.1 for the tensor product modules of the Fock modules of the quantum toroidal algebras associated with  $\widehat{A}_0$  and  $\widehat{A}_{r-1}$  quivers. We have a more specific Conjecture regarding the KR module of a finite-type quiver.

**Conjecture 1.2** (Theorems 3.2, 3.3 for  $A_1$  quiver). Let  $\Gamma$  be a finite-type quiver and fix  $i \in \Gamma_0$ . Considering the weight dimension vector having a single non-zero entry,  $w_i \neq 0$  and  $w_j = 0$  for  $j \neq i$ , the following holds:

- (1) The  $qq$ -character is reduced to that for the irreducible highest-weight module if the weight parameters  $\underline{x} = (x_{i,\alpha})$  obey the  $q_1$ -segment condition.
- (2) This irreducible  $qq$ -character is further reduced to the irreducible  $q_1$ -character in the limit  $q_2 \rightarrow 1$ .
- (3) The irreducible  $qq$ -character is factorized into the product of the  $q_2$ -characters in the other limit  $q_1 \rightarrow 1$ . Hence it is not irreducible in general.

The remaining part of this paper is organized as follows: In §2, as a preliminary, we provide an algorithm to construct the  $qq$ -character. In §3, we consider  $A_1$  quiver as a primary example and provide a proof of Conjecture 1.2 in this case. In §4 and §5, we consider  $A_2$  and  $BC_2$  quivers to show several examples, which support Conjecture 1.2 for the KR modules and Conjecture 1.1 for other cases. In §6 and §7, we consider affine quivers  $\widehat{A}_0$  and  $\widehat{A}_r$  and prove Conjecture 1.1 for the tensor product of the Fock modules of the corresponding quantum toroidal algebras.

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2. CONSTRUCTION OF  $qq$ -CHARACTER

In this Section, we provide an algorithm to calculate the  $qq$ -character. We apply the algebraic construction based on the iWeyl reflection. See [Nek15, KP22] for the alternative geometric construction based on the equivariant integral over the quiver variety.

**2.1. Quiver.** Let  $\Gamma = (\Gamma_0, \Gamma_1)$  be a quiver with the set of nodes  $\Gamma_0$  and the set of edges  $\Gamma_1$ . We call  $\text{rk } \Gamma = |\Gamma_0|$  the rank of quiver. We define a decorated quiver  $\Gamma_d = (\Gamma, d)$  with the set of positive integers  $d = (d_i)_{i \in \Gamma_0} \in \mathbb{Z}_{>1}^{\text{rk } \Gamma}$  assigned to each node, which would be identified with the relative root length of the corresponding root system. We denote  $d_{ij} = \gcd(d_i, d_j)$ .

**2.2. Index functor.** Let  $\mathbf{X}$  be the bundle for which we denote the set of Grothendieck roots (exponentials of the Chern roots) by  $X = \{x_\alpha\}_{\alpha=1, \dots, \text{rk } \mathbf{X}}$ . Then, the corresponding character is given by  $\text{ch } \mathbf{X} = \sum_{\alpha=1}^{\text{rk } \mathbf{X}} x_\alpha = \sum_{x \in X} x$ . We denote the wedge products of the bundle  $\mathbf{X}$  by

$$\wedge \mathbf{X} = \bigoplus_{k=0}^{\text{rk } \mathbf{X}} (-1)^k \wedge^k \mathbf{X}. \quad (2.1)$$

Then, we denote the index functor, which converts the additive class to the multiplicative class, by

$$\mathbb{I}[\mathbf{X}] = \prod_{\alpha=1}^{\text{rk } \mathbf{X}} (1 - x_\alpha) = \text{ch } \wedge \mathbf{X}. \quad (2.2)$$

We also use the twisted version. We denote the twisted sum of the wedge products by

$$\wedge_y \mathbf{X} = \bigoplus_{k=0}^{\text{rk } \mathbf{X}} (-y)^k \wedge^k \mathbf{X}. \quad (2.3)$$

For  $\mathbf{X}'$  a one-dimensional bundle with  $\text{ch } \mathbf{X}' = x'$ , we have

$$\mathbb{I}[\mathbf{X} \otimes \mathbf{X}'] = \prod_{\alpha=1}^{\text{rk } \mathbf{X}} (1 - x_\alpha x') = \text{ch } \wedge_{x'} \mathbf{X}, \quad (2.4)$$

which is the characteristic polynomial with respect to  $X$ .

**2.3. Y-function.** For a quiver  $\Gamma$ , we assign a formal bundle to each node of  $\Gamma$ ,  $\mathbf{Y} = \{\mathbf{Y}_i\}_{i \in \Gamma_0}$ . We then define the Y-functions for the quiver  $\Gamma$ ,

$$Y_{i,X} = \mathbb{I}[\mathbf{X} \otimes \mathbf{Y}_i] = \prod_{\alpha=1}^{\text{rk } \mathbf{X}} Y_{i,x_\alpha}, \quad i \in \Gamma_0, \quad x_\alpha \in \mathbb{C}^\times. \quad (2.5)$$

Hence, each Y-function is the characteristic polynomial associated with  $\mathbf{Y}_i$ ,

$$Y_{i,x} = \text{ch } \wedge_x \mathbf{Y}_i. \quad (2.6)$$

In the context of quiver gauge theory, this Y-function is defined from the observable sheaf  $\{\mathbf{Y}_i\}$ , the pullback of the inclusion map of the universal sheaf defined on the instanton moduli space [NP12, NPS13]. In particular, if  $\text{rk } \mathbf{X} = 1$ , (logarithm of) the corresponding Y-function (hence, the bundle  $\mathbf{Y}_i$ ) plays a role of the fundamental weight of the Lie algebra  $\mathfrak{g}_\Gamma$  associated with the quiver  $\Gamma$  denoted by  $(\varpi_i)_{i \in \Gamma_0}$ ,

$$Y_{i,x} = e^{(\varpi_i, x)}. \quad (2.7)$$

In this context, the parameter  $x$  is interpreted as the spectral parameter to define the corresponding evaluation module of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}}_\Gamma)$ . In the following, we will often use the notation

$$Y_{i,x;j,k} = Y_{i,xq_1^j q_2^k}. \quad (2.8)$$

**2.4. A-function.** We define a  $q_{1,2}$ -deformation of the Cartan matrix associated with the quiver  $\Gamma$ ,

$$c_{ji} = (1 + q_1^{d_i} q_2) \delta_{ij} - \sum_{e:i \rightarrow j} \sum_{r=0}^{d_i/d_{ij}-1} \mu_e q_1^{rd_{ij}} - \sum_{e:j \rightarrow i} \sum_{r=0}^{d_i/d_{ij}-1} \mu_e^{-1} q_1^{(r+1)d_{ij}} q_2 \quad (i, j \in \Gamma_0). \quad (2.9)$$

We may trivialize the parameters  $(\mu_e)_{e \in \Gamma_1}$  except for the cyclic quiver. Hence, we simply put  $\mu_e = 1$  except in §6 and §7 where we discuss cyclic quivers. Then, we define the A-function,

$$\begin{aligned} \mathbf{A}_{i,X} &= \mathbb{I} \left[ \sum_{j \in \Gamma_0} (\mathbf{X} \otimes \mathbf{Y}_j) c_{ji} \right] \\ &= \mathbf{Y}_{i,X} \mathbf{Y}_{i,X;d_i,1} \prod_{e:i \rightarrow j} \prod_{r=0}^{d_i/d_{ij}-1} \mathbf{Y}_{j,\mu_e X;rd_{ij},0}^{-1} \prod_{e:j \rightarrow i} \prod_{r=0}^{d_i/d_{ij}-1} \mathbf{Y}_{j,\mu_e^{-1} X;(r+1)d_{ij},1}^{-1}. \end{aligned} \quad (2.10)$$

Compared with the bundle  $\mathbf{Y}$ , the combination appearing in the A-function  $\mathbf{A}_i = \sum_{j \in \Gamma_0} \mathbf{Y}_j c_{ji}$  plays a role of the simple root of the corresponding quiver,

$$\mathbf{A}_{i,x} = e^{(\alpha_i, x)}. \quad (2.11)$$

**2.5. S-function.** We define a rational function, that we call the S-function,<sup>5</sup>

$$\mathcal{S}(z) = \frac{(1 - z/q_1)(1 - z/q_2)}{(1 - z)(1 - z/q)} = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} (1 - q_1^{-n})(1 - q_2^{-n}) \right) \quad (2.12)$$

where the latter expression is available for  $|z| < 1$ ,  $|q_{1,2}| > 1$ . The S-function obeys the inversion relation

$$\begin{aligned} \mathcal{S}(z) &= \mathcal{S}(z^{-1}q) \quad (z \neq 1, q) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{z^{-n}}{n} (1 - q_1^n)(1 - q_2^n) \right) \quad (|z| > 1, |q_{1,2}| < 1). \end{aligned} \quad (2.13)$$

We also define the higher-degree S-function,

$$\mathcal{S}_r(z) = \frac{(1 - z/q_1^r)(1 - z/q_2)}{(1 - z)(1 - z/q_1^r q_2)} = \prod_{s=0}^{r-1} \mathcal{S}(z q_1^{-s}), \quad (2.14)$$

which obeys the reflection relation

$$\mathcal{S}_r(z) = \mathcal{S}_r(z^{-1} q_1^r q_2), \quad z \neq 1, q_1^r q_2. \quad (2.15)$$

For generic  $z$ , the S-function becomes trivial in the classical limit,

$$\mathcal{S}_r(z) \xrightarrow{q_{1,2} \rightarrow 1} 1. \quad (2.16)$$

**2.6. iWeyl reflection.** We now describe an algorithm to calculate the  $qq$ -character:

- (1) Start with the highest-weight monomial associated with  $w = (w_i)_{i \in \Gamma_0} \in \mathbb{Z}_{\geq 0}^{\text{rk } \Gamma}$  with the weight parameters  $\underline{x} = (x_{i,\alpha})_{i \in \Gamma_0, \alpha=1, \dots, w_i}$ ,<sup>6</sup>

$$\mathbf{Y}_{w, \underline{x}} := \prod_{i \in \Gamma_0} \prod_{\alpha=1}^{w_i} \mathbf{Y}_{i, x_{i,\alpha}}. \quad (2.17)$$

<sup>5</sup>We change the convention  $(q_1, q_2) \rightarrow (q_1^{-1}, q_2^{-1})$  compared with our previous publications [KP15, KP16, KP17].

<sup>6</sup>For  $i \in \Gamma_0$ , denote by  $\mathbf{W}_i$  a vector space with the Grothendieck roots  $(x_{i,\alpha})_{\alpha=1, \dots, w_i}$ , which would be identified with the framing space of the quiver variety  $\mathfrak{M}_{v,w}^\Gamma$ . Then, the Drinfeld polynomials  $\{\mathbf{P}_{i,x}\}_{i \in \Gamma_0}$  associated with the finite-dimensional module of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}}_\Gamma)$  are given by the characteristic polynomials with respect to  $\mathbf{W}_i$ ,  $\mathbf{P}_{i,x} = \text{ch } \wedge_x \mathbf{W}_i$ .

(2) Apply the iWeyl reflection for the Y-function to generate the monomials,

$$\text{iWeyl} : Y_{i,x} \mapsto Y_{i,x} A_{i,x}^{-1} = Y_{i,x;d_i,1}^{-1} \prod_{e:i \rightarrow j} \prod_{r=0}^{d_i/d_{i_j}-1} Y_{j,x;r d_{i_j},0} \prod_{e:j \rightarrow i} \prod_{r=0}^{d_i/d_{i_j}-1} Y_{j,x;(r+1)d_{i_j},1}. \quad (2.18)$$

The reflection is applied to the Y-function in the numerator of the monomials.

(a) If the monomial contains several Y-functions which belong to the same node  $i \in \Gamma_0$ , multiply the S-functions:<sup>7</sup>

$$Y_{i,x} \frac{\prod_{\alpha=1}^n Y_{i,x_\alpha}}{\prod_{\beta=1}^m Y_{i,x_\beta}} \mapsto \left( \frac{\prod_{\alpha=1}^n \mathcal{S}_{d_i}(x_\alpha/x)}{\prod_{\beta=1}^m \mathcal{S}_{d_i}(x_\beta/x)} \right) Y_{i,x} A_{i,x}^{-1} \frac{\prod_{\alpha=1}^n Y_{i,x_\alpha}}{\prod_{\beta=1}^m Y_{i,x_\beta}}. \quad (2.19)$$

(b) If the monomial contains the factor  $(Y_{i,x})^n$ , it should be considered as  $\lim_{x_\alpha \rightarrow x} \prod_{\alpha=1}^n Y_{i,x_\alpha}$ , which gives rise to derivatives of the Y-function [Nek15, KP15].

(3) If there is no Y-function in the numerator, no further reflection is applied (lowest-weight monomial). For finite-type quivers ( $\det c^{[0]} > 0$ , where  $c^{[0]}$  is the classical limit of the Cartan matrix (2.9) obtained by replacing all the  $\mathbb{C}^\times$  variables with one), it is guaranteed that this process is terminated within finite time reflections, while the sequence of reflections are not terminated for generic quivers (affine and hyperbolic types;  $\det c^{[0]} \leq 0$ ).

This is an algorithm to generate  $qq$ -characters geometrically defined as an equivariant integral over the corresponding quiver variety [Nek15, KP22]. The  $qq$ -character of the tensor product does not exhibit further decomposition for generic weight parameters. In order to obtain the  $qq$ -character of the irreducible module, we should specialize the weight parameters  $\underline{x}$ . In the following, we study several examples to demonstrate how to obtain the irreducible module in the context of the  $qq$ -character.

### 3. $A_1$ QUIVER

In this case, we denote  $Y_{i,x} = Y_x$  for simplicity. The iWeyl reflection of  $A_1$  quiver is given by<sup>8</sup>

$$\text{iWeyl} : Y_x \mapsto Y_{xq}^{-1}. \quad (3.1)$$

The fundamental  $qq$ -character is then given by<sup>9</sup>

$$\mathbb{T}_{1,x} = Y_x + Y_{xq}^{-1}. \quad (3.2)$$

We obtain the following expression from the direct computation based on the algorithm shown in §2.6.

**Proposition 3.1.** For  $A_1$  quiver, the  $qq$ -character of weight  $w = (w)$  with the parameters  $\underline{x} = (x_1, \dots, x_w)$  is given by

$$\mathbb{T}_{w,\underline{x}} = Y_{x_1} \cdots Y_{x_w} + \cdots = \sum_{I \sqcup J = \{1, \dots, w\}} \prod_{i \in I, j \in J} \mathcal{S}\left(\frac{x_i}{x_j}\right) \prod_{i \in I} Y_{x_i} \prod_{j \in J} Y_{x_j}^{-1}, \quad (3.3)$$

which contains  $2^w$  monomials in total.

<sup>7</sup>This S-factor is interpreted as the OPE factor in the vertex operator formalism of Y- and A-functions [KP15].

<sup>8</sup>We use the notation  $q = q_1 q_2$ .

<sup>9</sup>We do not write the  $q_{1,2}$  dependence of the  $qq$ -character explicitly as far as no confusion.



**3.1. Weight two.** From the formula (3.3), the  $qq$ -character of weight two  $w = (2)$  with the weight parameters  $\underline{x} = (x_1, x_2)$  is given by

$$\mathsf{T}_{2,\underline{x}} = \mathsf{Y}_{x_1} \mathsf{Y}_{x_2} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \frac{\mathsf{Y}_{x_2}}{\mathsf{Y}_{x_1 q}} + \mathcal{S}\left(\frac{x_1}{x_2}\right) \frac{\mathsf{Y}_{x_1}}{\mathsf{Y}_{x_2 q}} + \frac{1}{\mathsf{Y}_{x_1 q} \mathsf{Y}_{x_2 q}}. \quad (3.4)$$

For generic weight parameters, this  $qq$ -character contains four monomials, corresponding to the tensor product of two two-dimensional modules of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  associated with  $A_1$  quiver. Recalling that the  $\mathcal{S}$ -function has zeros at  $z = q_1$  and  $q_2$ , we may obtain a three-dimensional module, which corresponds to the irreducible module of weight two, by specializing the weight parameters,

$$\mathsf{T}_{2,\underline{x}} \longrightarrow \widetilde{\mathsf{T}}_{2,x}^{(q_1,2)} = \begin{cases} \mathsf{Y}_x \mathsf{Y}_{x;1,0} + \mathcal{S}(q_1^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;2,1}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;2,1}^{-1} & (x_1 = x, x_2/x_1 = q_1) \\ \mathsf{Y}_x \mathsf{Y}_{x;0,1} + \mathcal{S}(q_2^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;1,2}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;1,2}^{-1} & (x_1 = x, x_2/x_1 = q_2) \end{cases} \quad (3.5)$$

The highest weight monomials  $(\mathsf{Y}_x \mathsf{Y}_{x q_m})_{m=1,2}$  are given by

$$\mathsf{Y}_x \mathsf{Y}_{x;1,0} = \mathsf{Y}_x \mathsf{Y}_{x q_1} = \mathbb{I}[\mathbf{W}_{2_1,x}^{(1)} \otimes \mathbf{Y}_1], \quad \mathsf{Y}_x \mathsf{Y}_{x;0,1} = \mathsf{Y}_x \mathsf{Y}_{x q_2} = \mathbb{I}[\mathbf{W}_{2_2,x}^{(1)} \otimes \mathbf{Y}_1], \quad (3.6)$$

where we define the vector spaces  $(\mathbf{W}_{k_m,x}^{(i)})_{i \in \Gamma_0}$  having the character

$$\text{ch } \mathbf{W}_{k_m,x}^{(i)} = x + x q_m + \cdots + x q_m^{k_m-1} \quad (m = 1, 2). \quad (3.7)$$

Namely, this specialization (3.5) corresponds to the KR module of weight two with the shift parameter  $q_{1,2}$ . We also denote the corresponding  $qq$ -character by  $\widetilde{\mathsf{T}}_{2,x}^{(q_1,2)} = \mathsf{T}[\mathbf{W}_{2_1,2,x}^{(1)}]$ .

We remark a geometric interpretation of the truncation of the  $qq$ -character. Before the truncation, there are four monomials in the expression (3.4). The first and the last ones have no specific geometric structure as the corresponding quiver variety is just a point. The second and the third ones correspond to the fixed point contributions of the cotangent bundle  $T^*\mathbb{P}^1$ , which is the quiver variety of  $(v, w) = (1, 2)$ . Tuning the weight parameters, we can extract either of two monomials as the subvariety contribution.

*Classical limit.* In the classical limit, the  $\mathcal{S}$ -factor behaves as follows,

$$\mathcal{S}(q_1^{-1}) = (1 + q_1^{-1}) \frac{1 - q_1^{-1} q_2^{-1}}{1 - q_1^{-2} q_2^{-1}} \longrightarrow \begin{cases} 2 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \quad (3.8a)$$

$$\mathcal{S}(q_2^{-1}) = (1 + q_2^{-1}) \frac{1 - q_1^{-1} q_2^{-1}}{1 - q_1^{-1} q_2^{-2}} \longrightarrow \begin{cases} 1 & (q_1 \rightarrow 1) \\ 2 & (q_2 \rightarrow 1) \end{cases} \quad (3.8b)$$

Hence, the  $qq$ -character of weight two (3.5) is reduced as

$$\mathsf{Y}_x \mathsf{Y}_{x;1,0} + \mathcal{S}(q_1^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;2,1}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;2,1}^{-1} \longrightarrow \begin{cases} \mathsf{Y}_x^2 + 2 \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;0,1}} + \mathsf{Y}_{x;0,1}^{-2} = (\mathsf{Y}_x + \mathsf{Y}_{x;0,1}^{-1})^2 & (q_1 \rightarrow 1) \\ \mathsf{Y}_x \mathsf{Y}_{x;1,0} + \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;2,0}} + \mathsf{Y}_{x;1,0}^{-1} \mathsf{Y}_{x;2,0}^{-1} & (q_2 \rightarrow 1) \end{cases} \quad (3.9a)$$

$$\mathsf{Y}_x \mathsf{Y}_{x;0,1} + \mathcal{S}(q_2^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;1,2}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;1,2}^{-1} \longrightarrow \begin{cases} \mathsf{Y}_x \mathsf{Y}_{x;0,1} + \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;0,2}} + \mathsf{Y}_{x;0,1}^{-1} \mathsf{Y}_{x;0,2}^{-1} & (q_1 \rightarrow 1) \\ \mathsf{Y}_x^2 + 2 \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;1,0}} + \mathsf{Y}_{x;1,0}^{-2} = (\mathsf{Y}_x + \mathsf{Y}_{x;1,0}^{-1})^2 & (q_2 \rightarrow 1) \end{cases} \quad (3.9b)$$

Therefore, we obtain the  $q$ -character associated with the KR module  $\mathbf{W}_{k_1,x}^{(1)}$  ( $\mathbf{W}_{k_2,x}^{(1)}$ ) in the limit  $q_2 \rightarrow 1$  ( $q_1 \rightarrow 1$ ), while the degenerated  $q$ -character, corresponding to the tensor product of the two-dimensional modules, is obtained in the limit  $q_1 \rightarrow 1$  ( $q_2 \rightarrow 1$ ). Such a degenerated situation (folded  $q$ -character) has been recently discussed in the literature [CK18, FHR21, KO22].

**3.2. Weight three.** The  $qq$ -character of weight three with generic weight parameters  $\underline{x} = (x_1, x_2, x_3)$  is given by

$$\begin{aligned} \mathbb{T}_{3, \underline{x}} &= Y_{x_1} Y_{x_2} Y_{x_3} \\ &+ \mathcal{S}\left(\frac{x_1}{x_3}\right) \mathcal{S}\left(\frac{x_2}{x_3}\right) \frac{Y_{x_1} Y_{x_2}}{Y_{x_3 q}} + \mathcal{S}\left(\frac{x_1}{x_2}\right) \mathcal{S}\left(\frac{x_3}{x_2}\right) \frac{Y_{x_1} Y_{x_3}}{Y_{x_2 q}} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \mathcal{S}\left(\frac{x_3}{x_1}\right) \frac{Y_{x_2} Y_{x_3}}{Y_{x_1 q}} \\ &+ \mathcal{S}\left(\frac{x_1}{x_2}\right) \mathcal{S}\left(\frac{x_1}{x_3}\right) \frac{Y_{x_1}}{Y_{x_2 q} Y_{x_3 q}} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \mathcal{S}\left(\frac{x_2}{x_3}\right) \frac{Y_{x_2}}{Y_{x_1 q} Y_{x_3 q}} + \mathcal{S}\left(\frac{x_3}{x_1}\right) \mathcal{S}\left(\frac{x_3}{x_2}\right) \frac{Y_{x_3}}{Y_{x_1} Y_{x_2 q}} \\ &+ \frac{1}{Y_{x_1 q} Y_{x_2 q} Y_{x_3 q}}. \end{aligned} \quad (3.10)$$

Specializing the parameters  $\underline{x} = (x, xq_1, xq_1^2)$ , which corresponds to the KR module of weight three  $\mathbf{W}_{3_1, x}^{(1)}$ , we obtain

$$\mathbb{T}_{3, \underline{x}} \longrightarrow \mathbb{T}[\mathbf{W}_{3_1, x}^{(1)}] = Y_x Y_{x;1,0} Y_{x;2,0} + \mathcal{S}_2(q_1^{-1}) \left( \frac{Y_x Y_{x;1,0}}{Y_{x;3,1}} + \frac{Y_x}{Y_{x;2,1} Y_{x;3,1}} \right) + \frac{1}{Y_{x;1,1} Y_{x;2,1} Y_{x;3,1}}. \quad (3.11)$$

This corresponds to the irreducible four-dimensional module of  $U_q(\widehat{\mathfrak{sl}}_2)$  associated with  $A_1$  quiver. We obtain a similar expression by another specialization  $\underline{x} = (x, xq_2, xq_2^2)$ .

*Classical limit.* In the classical limit, the  $\mathcal{S}$ -factor is given by

$$\mathcal{S}_2(q_1^{-1}) = (1 + q_1^{-1} + q_1^{-2}) \frac{1 - q_1^{-1} q_2^{-1}}{1 - q_1^{-3} q_2^{-1}} \longrightarrow \begin{cases} 3 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \quad (3.12)$$

Hence, the  $qq$ -character is reduced to the degenerated  $q$ -character ( $q_1 \rightarrow 1$ ) and the ordinary  $q$ -character of weight three ( $q_2 \rightarrow 1$ ) as follows,

$$\mathbb{T}[\mathbf{W}_{3_1, x}^{(1)}] \longrightarrow \begin{cases} Y_x^3 + \frac{3Y_x^2}{Y_{x;0,1}} + \frac{3Y_x}{Y_{x;0,1}^2} + \frac{1}{Y_{x;0,1}^3} = (Y_x + Y_{x;0,1}^{-1})^3 & (q_1 \rightarrow 1) \\ Y_x Y_{x;1,0} Y_{x;2,0} + \frac{Y_x Y_{x;1,0}}{Y_{x;3,0}} + \frac{Y_x}{Y_{x;2,0} Y_{x;3,0}} + \frac{1}{Y_{x;1,0} Y_{x;2,0} Y_{x;3,0}} & (q_2 \rightarrow 1) \end{cases} \quad (3.13)$$

**3.3. Generic weight.** We consider the specialization of the  $qq$ -character of generic weight.

**Theorem 3.2.** Under the specialization  $\underline{x} = (x, xq_1, \dots, xq_1^{w-1})$  corresponding to the degree- $w$  KR module  $\mathbf{W}_{w_1, x}^{(1)}$ , the  $qq$ -character is given by  $w + 1$  monomials,

$$\mathbb{T}[\mathbf{W}_{w_1, x}^{(1)}] = \sum_{v=0}^w \left( \prod_{\substack{i=1, \dots, w-v \\ j=1, \dots, v}} \mathcal{S}(q_1^{1-i-j}) \right) \prod_{i=1, \dots, w-v} Y_{x; i-1, 0} \prod_{j=w-v+1, \dots, w} Y_{x; j, 1}^{-1}. \quad (3.14)$$

*Proof.* Under the specialization  $\underline{x} = (x, xq_1, \dots, xq_1^{w-1})$ , due to the  $\mathcal{S}$ -factors, there are only the contributions of  $I = \{1, \dots, w-v\}$ ,  $J = \{w-v+1, \dots, w\}$  remaining in the  $qq$ -character formula (3.3), which gives rise to the expression above.  $\square$

This proves the part (1) of Conjecture 1.2 for  $A_1$  quiver. We remark that the  $\mathcal{S}$ -factor is identified with the fixed point contribution of the equivariant integral over the quiver variety of  $A_1$ , identified with the cotangent bundle of the Grassmannian  $T^*\text{Gr}(v, w)$ .

*Classical limit.* Let us consider the classical limit of the  $qq$ -character. We have the following results in the limits  $q_1 \rightarrow 1$  and  $q_2 \rightarrow 1$ , respectively.

**Theorem 3.3.** The classical limit of the  $qq$ -character gives rise to the degenerated  $q$ -character and the weight- $w$   $q$ -character,

$$\mathbb{T}[\mathbf{W}_{w_1, x}^{(1)}] \longrightarrow \begin{cases} \sum_{v=0}^w \binom{w}{v} (\mathbb{Y}_x)^{w-v} (\mathbb{Y}_{x;0,1})^{-v} = (\mathbb{Y}_x + \mathbb{Y}_{x;0,1}^{-1})^w & (q_1 \rightarrow 1) \\ \sum_{v=0}^w \prod_{i=1, \dots, w-v} \mathbb{Y}_{x; i-1, 0} \prod_{j=w-v+1, \dots, w} \mathbb{Y}_{x; j, 0}^{-1} & (q_2 \rightarrow 1) \end{cases}. \quad (3.15)$$

*Proof.* Applying the expression (2.12), we may rewrite the  $\mathcal{S}$ -factor in the  $qq$ -character (3.14) as follows,

$$\begin{aligned} \prod_{\substack{i=1, \dots, w-v \\ j=1, \dots, v}} \mathcal{S}(q_1^{1-i-j}) &= \frac{(q_1^{-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-w+v-1}; q_1^{-1})_\infty (q_1^{-v-1}; q_1^{-1})_\infty (q_1^{-w} q_2^{-1}; q_1^{-1})_\infty}{(q_1^{-1}; q_1^{-1})_\infty (q_1^{-w+v-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-v-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-w}; q_1^{-1})_\infty} \\ &= \frac{(q_1^{-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-w} q_2^{-1}; q_1^{-1})_\infty}{(q_1^{-w+v-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-v-1} q_2^{-1}; q_1^{-1})_\infty} \frac{\Gamma_{q_1^{-1}}(w) \Gamma_{q_1^{-1}}(1)}{\Gamma_{q_1^{-1}}(v) \Gamma_{q_1^{-1}}(w-v)} \longrightarrow \begin{cases} \binom{w}{v} & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases}, \end{aligned} \quad (3.16)$$

where we define the  $q$ -gamma function for  $|q| < 1$ ,

$$\Gamma_q(x) = (1-q)^x \frac{(q; q)_\infty}{(q^x; q)_\infty} \xrightarrow{q \rightarrow 1} \Gamma(x) \quad (3.17)$$

with the  $q$ -shifted factorial ( $q$ -Pochhammer symbol)

$$(z; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n). \quad (3.18)$$

Then, we obtain the expression above.  $\square$

This completes a proof of the parts (2) and (3) of Conjecture 1.2 for  $A_1$  quiver.

#### 4. $A_2$ QUIVER

In this case, there are two fundamental  $\mathbb{Y}$ -functions, and the corresponding iWeyl reflection is given as follows,

$$\text{iWeyl}: (\mathbb{Y}_{1,x}, \mathbb{Y}_{2,x}) \longmapsto \left( \frac{\mathbb{Y}_{2,x}}{\mathbb{Y}_{1,xq}}, \frac{\mathbb{Y}_{1,xq}}{\mathbb{Y}_{2,xq}} \right). \quad (4.1)$$

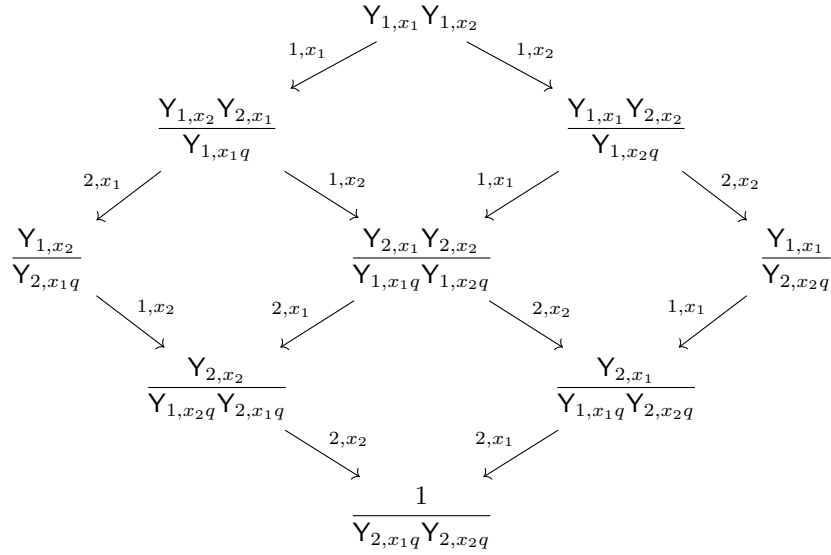
Then, the fundamental  $qq$ -characters are given by

$$\mathbb{T}_{1,x} = \mathbb{T}_{(1,0),x} = \mathbb{Y}_{1,x} + \frac{\mathbb{Y}_{2,x}}{\mathbb{Y}_{1,xq}} + \frac{1}{\mathbb{Y}_{2,xq}}, \quad (4.2a)$$

$$\mathbb{T}_{2,x} = \mathbb{T}_{(0,1),x} = \mathbb{Y}_{2,x} + \frac{\mathbb{Y}_{1,xq}}{\mathbb{Y}_{2,xq}} + \frac{1}{\mathbb{Y}_{1,xq^2}}, \quad (4.2b)$$

which correspond to two three-dimensional modules of  $U_q(\widehat{\mathfrak{sl}}_3)$  associated with  $A_2$  quiver.

**4.1. Weight two.** For  $A_2$  quiver, we have three possible weight two situations,  $w = (2, 0), (0, 2)$  and  $w = (1, 1)$ . The former two cases correspond to the KR modules,  $\left( \mathbf{W}_{2_m, x}^{(i)} \right)_{i=1,2, m=1,2}$ , while the last one is not.


 FIGURE 1. Hasse diagram for the iWeyl reflection flow of the weight  $w = (2, 0)$  for  $A_2$  quiver.

4.1.1.  $w = (2, 0)$ . We consider the  $qq$ -character associated with weight  $w = (2, 0)$  with generic parameters  $\underline{x} = (x_1, x_2)$ ,

$$\begin{aligned} \mathbb{T}_{(2,0); \underline{x}} &= Y_{1,x_1} Y_{1,x_2} + \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{1,x_1 q} Y_{1,x_2 q}} + \frac{1}{Y_{2,x_1 q} Y_{2,x_2 q}} \\ &+ \mathcal{S}\left(\frac{x_2}{x_1}\right) \left[ \frac{Y_{1,x_2} Y_{2,x_1}}{Y_{1,x_1 q}} + \frac{Y_{1,x_2}}{Y_{2,x_1 q}} + \frac{Y_{2,x_2}}{Y_{1,x_2 q} Y_{2,x_1 q}} \right] \\ &+ \mathcal{S}\left(\frac{x_1}{x_2}\right) \left[ \frac{Y_{1,x_1} Y_{2,x_2}}{Y_{1,x_2 q}} + \frac{Y_{1,x_1}}{Y_{2,x_2 q}} + \frac{Y_{2,x_1}}{Y_{1,x_1 q} Y_{2,x_2 q}} \right]. \end{aligned} \quad (4.3)$$

The flow of the iWeyl reflections is described by the Hasse diagram shown in Fig. 1. The  $qq$ -character (4.3) contains 9 monomials, corresponding to the tensor product of two three-dimensional fundamental modules of  $w = (1, 0)$ .

We specialize the weight parameters  $\underline{x} = (x, xq_1)$ , which corresponds to the weight-two KR module  $\mathbf{W}_{2_1, x}^{(1)}$ . Then, we have the following.

**Proposition 4.1.** The  $qq$ -character of six-dimensional module of weight two of the algebra  $U_q(\widehat{\mathfrak{sl}}_3)$  is given by

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{2_1, x}^{(1)}] &= Y_{1,x} Y_{1,x;1,0} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,1} Y_{1,x;2,1}} + \frac{1}{Y_{2,x;1,1} Y_{2,x;2,1}} \\ &+ \mathcal{S}(q_1^{-1}) \left[ \frac{Y_{1,x} Y_{2,x;1,0}}{Y_{1,x;2,1}} + \frac{Y_{1,x}}{Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;1,1} Y_{2,x;2,1}} \right]. \end{aligned} \quad (4.4)$$

On the other hand, the eliminated factors in the  $qq$ -character are interpreted as the contribution of the three-dimensional module,

$$\frac{Y_{1,x_2} Y_{2,x_1}}{Y_{1,x_1 q}} + \frac{Y_{1,x_2}}{Y_{2,x_1 q}} + \frac{Y_{2,x_2}}{Y_{1,x_2 q} Y_{2,x_1 q}} \xrightarrow{(x_1, x_2) \rightarrow (x, xq)} \mathbb{T}_{(0,1), x}. \quad (4.5)$$

We remark that this specialization of the parameter  $(x_1, x_2) \rightarrow (x, xq)$  corresponds to the pole of the  $\mathcal{S}$ -function.<sup>10</sup> This is consistent with the decomposition of the tensor product into the irreducible modules,  $3 \otimes 3 = 6 \oplus 3$ .

<sup>10</sup>As shown in [KP15], the  $qq$ -character provides the generating current of the  $q$ -deformation of W-algebra associated with quiver (quiver W-algebra). In this context, the pole term plays a crucial role in the quadratic relations for the generating

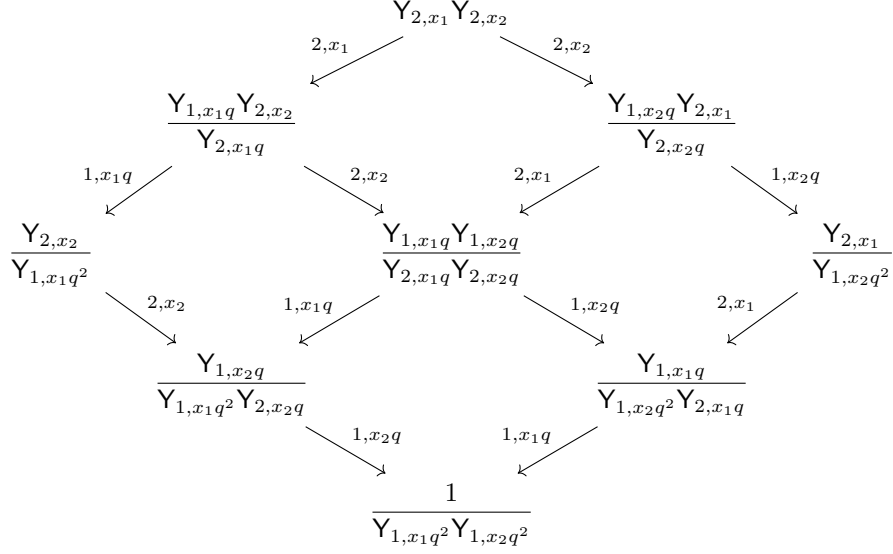


FIGURE 2. Hasse diagram for the iWeyl reflection flow of the weight  $w = (0, 2)$  for  $A_2$  quiver.

*Classical limit.* Recalling the behavior of the  $\mathcal{S}$ -factor in the classical limit (3.8), this  $qq$ -character is further reduced in the classical limit as follows,

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(1)}] &\xrightarrow{q_1 \rightarrow 1} Y_{1,x}^2 + \left( \frac{Y_{2,x}}{Y_{1,x;0,1}} \right)^2 + Y_{2,x;0,1}^{-2} + 2 \left( \frac{Y_{1,x} Y_{2,x}}{Y_{1,x;0,1}} + \frac{Y_{1,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x}}{Y_{1,x;0,1} Y_{2,x;0,1}} \right) \\ &= \left( Y_{1,x} + \frac{Y_{2,x}}{Y_{1,x;0,1}} + \frac{1}{Y_{2,x;0,1}} \right)^2, \end{aligned} \quad (4.6a)$$

$$\xrightarrow{q_2 \rightarrow 1} Y_{1,x} Y_{1,x;1,0} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,0} Y_{1,x;2,0}} + \frac{1}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{Y_{1,x} Y_{2,x;1,0}}{Y_{1,x;2,0}} + \frac{Y_{1,x}}{Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;1,0} Y_{2,x;2,0}}. \quad (4.6b)$$

In the limit  $q_1 \rightarrow 1$ , we obtain the degenerated tensor product of  $w = (1, 0)$ , while we obtain the  $q$ -character associated with the KR module  $\mathbf{W}_{21,x}^{(1)}$  in the limit  $q_2 \rightarrow 1$ .

4.1.2.  $w = (0, 2)$ . The  $qq$ -character of weight  $w = (0, 2)$  is similarly calculated as before. For the weight parameters  $\underline{x} = (x_1, x_2)$ , we obtain

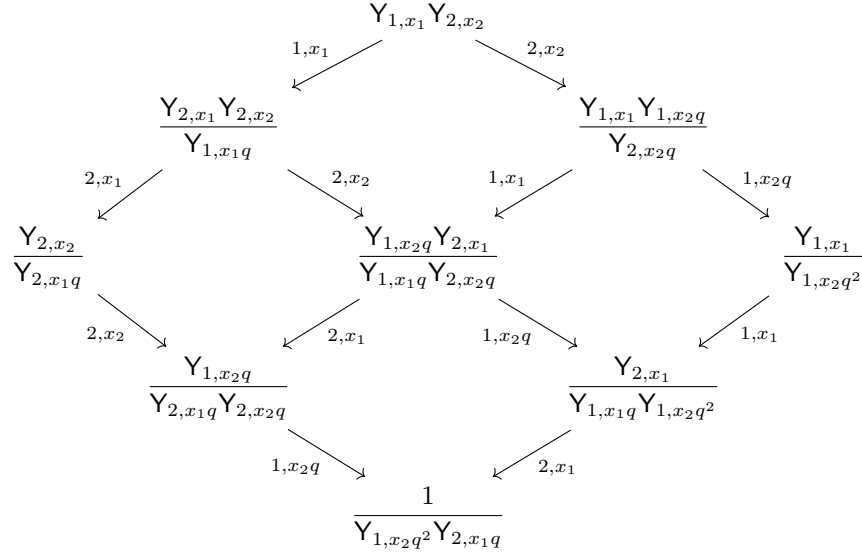
$$\begin{aligned} \mathbb{T}_{(0,2),\underline{x}} &= Y_{2,x_1} Y_{2,x_2} + \frac{Y_{1,x_1 q} Y_{1,x_2 q}}{Y_{2,x_1 q} Y_{2,x_2 q}} + \frac{1}{Y_{1,x_1 q^2} Y_{1,x_2 q^2}} \\ &\quad + \mathcal{S}\left(\frac{x_2}{x_1}\right) \left[ \frac{Y_{1,x_1 q} Y_{2,x_2}}{Y_{2,x_1 q}} + \frac{Y_{2,x_2}}{Y_{1,x_1 q^2}} + \frac{Y_{1,x_2 q}}{Y_{1,x_1 q^2} Y_{2,x_2 q}} \right] \\ &\quad + \mathcal{S}\left(\frac{x_1}{x_2}\right) \left[ \frac{Y_{1,x_2 q} Y_{2,x_1}}{Y_{2,x_2 q}} + \frac{Y_{2,x_1}}{Y_{1,x_2 q^2}} + \frac{Y_{1,x_1 q}}{Y_{1,x_2 q^2} Y_{2,x_1 q}} \right]. \end{aligned} \quad (4.7)$$

The iWeyl reflection flow is described by the Hasse diagram in Fig. 2.

Specializing the parameters as  $\underline{x} = (x, xq_1)$ , we obtain the  $qq$ -character of the KR module  $\mathbf{W}_{21,x}^{(2)}$  as follows.

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currents. For the  $A$ -type quiver, in particular, the quadratic relations [Oda01, Koj21] are interpreted as a double quantization of Pieri's formula for the fundamental representations.

FIGURE 3. Hasse diagram for the iWeyl reflection flow of the weight  $w = (1, 1)$  for  $A_2$  quiver.

**Proposition 4.2.**

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(2)}] &= Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,1} Y_{1,x;2,1}}{Y_{2,x;1,1} Y_{2,x;2,1}} + \frac{1}{Y_{1,x;2,2} Y_{1,x;3,2}} \\ &\quad + \mathcal{S}(q_1^{-1}) \left[ \frac{Y_{1,x;2,1} Y_{2,x}}{Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;3,2}} + \frac{Y_{1,x;1,1}}{Y_{1,x;3,2} Y_{2,x;1,1}} \right]. \end{aligned} \quad (4.8)$$

We may obtain a similar expression from another specializations  $\underline{x} = (x, xq_2)$ .

*Classical limit.* The classical limit of this  $qq$ -character is given by

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(2)}] &\xrightarrow{q_1 \rightarrow 1} Y_{2,x}^2 + \left( \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} \right)^2 + \frac{1}{Y_{1,x;0,2}^2} + 2 \left( \frac{Y_{1,x;0,1} Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x}}{Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,2} Y_{2,x;0,1}} \right) \\ &= \left( Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right)^2, \end{aligned} \quad (4.9a)$$

$$\xrightarrow{q_2 \rightarrow 1} Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,0} Y_{1,x;2,0}}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{1}{Y_{1,x;2,0} Y_{1,x;3,0}} + \frac{Y_{1,x;2,0} Y_{2,x}}{Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;3,0}} + \frac{Y_{1,x;1,0}}{Y_{1,x;3,0} Y_{2,x;1,0}}. \quad (4.9b)$$

We obtain the degenerated tensor product of  $w = (0, 1)$  in the limit  $q_1 \rightarrow 1$ , while we obtain the  $q$ -character associated with the KR module  $\mathbf{W}_{21,x}^{(2)}$  in the limit  $q_2 \rightarrow 1$ .

4.1.3.  $w = (1, 1)$ . We consider the  $qq$ -character of weight  $w = (1, 1)$ ,

$$\begin{aligned} \mathbb{T}_{(1,1); \underline{x}} &= Y_{1,x_1} Y_{2,x_2} + \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{1,x_1 q}} + \frac{Y_{1,x_1} Y_{1,x_2 q}}{Y_{2,x_2 q}} + \frac{Y_{1,x_2 q}}{Y_{2,x_1 q} Y_{2,x_2 q}} + \frac{Y_{2,x_1}}{Y_{1,x_1 q} Y_{1,x_2 q^2}} + \frac{1}{Y_{1,x_2 q^2} Y_{2,x_1 q}} \\ &\quad + \mathcal{S} \left( \frac{x_2}{x_1} \right) \frac{Y_{2,x_2}}{Y_{2,x_1 q}} + \mathcal{S} \left( \frac{x_1}{x_2} \right) \frac{Y_{1,x_2 q} Y_{2,x_1}}{Y_{1,x_1 q} Y_{2,x_2 q}} + \mathcal{S} \left( \frac{x_1}{x_2 q} \right) \frac{Y_{1,x_1}}{Y_{1,x_2 q^2}}, \end{aligned} \quad (4.10)$$

which is described by the Hasse diagram in Fig. 3. The  $qq$ -character (4.10) contains 9 monomials, interpreted as the tensor product of two three-dimensional fundamental modules,  $w = (1, 0)$  and  $w = (0, 1)$ . In this case, we shall have the decomposition,  $3 \otimes 3 = 8 \oplus 1$ .

In order to obtain the irreducible module of the weight  $w = (1, 1)$ , we have three possible specializations: (a)  $\underline{x} = (x, xq_1)$ , (b)  $\underline{x} = (xq_1, x)$ , and (c)  $\underline{x} = (xq_1^2q_2, x)$ .

	$(x_1, x_2)$	$\mathcal{S}(x_2/x_1)$	$\mathcal{S}(x_1/x_2)$	$\mathcal{S}(x_1/x_2q)$	
(a)	$(x, xq_1)$	0	$\mathcal{S}(q_1^{-1})$	$\mathcal{S}(q_1^{-2}q_2^{-1})$	(4.11)
(b)	$(xq_1, x)$	$\mathcal{S}(q_1^{-1})$	0	$\mathcal{S}(q_2^{-1})$	
(c)	$(xq_1^2q_2, xq_1)$	$\mathcal{S}(q_1^{-2}q_2^{-1})$	$\mathcal{S}(q_1^{-1})$	0	

**Proposition 4.3.** We have the following specializations of the  $qq$ -character, corresponding to the 8-dimensional module of  $A_2$  quiver,

$$\begin{aligned} T_{(1,1),\underline{x}} \xrightarrow{(a)} & Y_{1,x}Y_{2,x;1,0} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;1,1}} + \frac{Y_{1,x}Y_{1,x;2,1}}{Y_{2,x;2,1}} + \frac{Y_{1,x;2,1}}{Y_{2,x;1,1}Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;1,1}Y_{1,x;3,2}} + \frac{1}{Y_{1,x;3,2}Y_{2,x;1,1}} \\ & + \mathcal{S}(q_1^{-1})\frac{Y_{1,x;2,1}Y_{2,x}}{Y_{1,x;1,1}Y_{2,x;2,1}} + \mathcal{S}(q_1^{-2}q_2^{-1})\frac{Y_{1,x}}{Y_{1,x;3,2}}, \end{aligned} \quad (4.12a)$$

$$\begin{aligned} \xrightarrow{(b)} & Y_{1,x;1,0}Y_{2,x} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,1}} + \frac{Y_{1,x;1,0}Y_{1,x;1,1}}{Y_{2,x;1,1}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}Y_{2,x;2,1}} + \frac{Y_{2,x;1,0}}{Y_{1,x;2,1}Y_{1,x;2,2}} + \frac{1}{Y_{1,x;2,2}Y_{2,x;2,1}} \\ & + \mathcal{S}(q_1^{-1})\frac{Y_{2,x}}{Y_{2,x;2,1}} + \mathcal{S}(q_2^{-1})\frac{Y_{1,x;1,0}}{Y_{1,x;2,2}}, \end{aligned} \quad (4.12b)$$

$$\begin{aligned} \xrightarrow{(c)} & Y_{1,x;2,1}Y_{2,x} + \frac{Y_{2,x;2,1}Y_{2,x}}{Y_{1,x;3,2}} + \frac{Y_{1,x;1,1}Y_{1,x;2,1}}{Y_{2,x;1,1}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}Y_{2,x;3,2}} + \frac{Y_{2,x;2,1}}{Y_{1,x;2,2}Y_{1,x;3,2}} + \frac{1}{Y_{1,x;2,2}Y_{2,x;3,2}} \\ & + \mathcal{S}(q_1^{-2}q_2^{-1})\frac{Y_{2,x}}{Y_{2,x;3,2}} + \mathcal{S}(q_1^{-1})\frac{Y_{1,x;1,1}Y_{2,x;2,1}}{Y_{1,x;3,2}Y_{2,x;1,1}}. \end{aligned} \quad (4.12c)$$

*Classical limit.* In this case, we can discuss several classical limits of the  $qq$ -character. From the classical limit of the  $\mathcal{S}$ -factor (3.8) together with  $\mathcal{S}(q_1^{-2}q_2^{-1}) \xrightarrow{q_1,2 \rightarrow 1} 1$ , we have the following expressions:

$$(a) \quad \underline{x} = (x, xq_1)$$

$$\begin{aligned} (q_1 \rightarrow 1) : & Y_{1,x}Y_{2,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + \frac{Y_{1,x}Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}}{Y_{1,x;0,1}Y_{1,x;0,2}} + \frac{1}{Y_{1,x;0,2}Y_{2,x;0,1}} \\ & + 2\frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x}}{Y_{1,x;0,2}} = \left( Y_{1,x} + \frac{Y_{2,x}}{Y_{1,x;0,1}} + \frac{1}{Y_{2,x;0,1}} \right) \left( Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right), \end{aligned} \quad (4.13a)$$

$$\begin{aligned} (q_2 \rightarrow 1) : & Y_{1,x}Y_{2,x;1,0} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;1,0}} + \frac{Y_{1,x}Y_{1,x;2,0}}{Y_{2,x;2,0}} + \frac{Y_{1,x;2,0}}{Y_{2,x;1,0}Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;1,0}Y_{1,x;3,0}} + \frac{1}{Y_{1,x;3,0}Y_{2,x;1,0}} \\ & + \frac{Y_{1,x;2,0}Y_{2,x}}{Y_{1,x;1,0}Y_{2,x;2,0}} + \frac{Y_{1,x}}{Y_{1,x;3,0}}. \end{aligned} \quad (4.13b)$$

(b)  $\underline{x} = (xq_1, x)$ 

$$(q_1 \rightarrow 1) : \quad Y_{1,x} Y_{2,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + \frac{Y_{1,x} Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}}{Y_{1,x;0,1} Y_{1,x;0,2}} + \frac{1}{Y_{1,x;0,2} Y_{2,x;0,1}} \\ + 2 \frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x}}{Y_{1,x;0,2}} = \left( Y_{1,x} + \frac{Y_{2,x}}{Y_{1,x;0,1}} + \frac{1}{Y_{2,x;0,1}} \right) \left( Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right), \quad (4.14a)$$

$$(q_2 \rightarrow 1) : \quad Y_{1,x;1,0} Y_{2,x} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,0}} + \frac{Y_{1,x;1,0}^2}{Y_{2,x;1,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{Y_{2,x;1,0}}{Y_{1,x;2,0}^2} + \frac{1}{Y_{1,x;2,0} Y_{2,x;2,0}} \\ + \frac{Y_{2,x}}{Y_{2,x;2,0}} + 2 \frac{Y_{1,x;1,0}}{Y_{1,x;2,0}} = \left( Y_{1,x;1,0} + \frac{Y_{2,x;1,0}}{Y_{1,x;2,0}} + \frac{1}{Y_{2,x;2,0}} \right) \left( Y_{2,x} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0}} + \frac{1}{Y_{1,x;2,0}} \right). \quad (4.14b)$$

(c)  $\underline{x} = (xq_1^2 q_2, x)$ 

$$(q_1 \rightarrow 1) : \quad Y_{1,x;0,1} Y_{2,x} + \frac{Y_{2,x;0,1} Y_{2,x}}{Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1} Y_{2,x;0,2}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}^2} + \frac{1}{Y_{1,x;0,2} Y_{2,x;0,2}} \\ + \frac{Y_{2,x}}{Y_{2,x;0,2}} + 2 \frac{Y_{1,x;0,1}}{Y_{1,x;0,2}} = \left( Y_{1,x;0,1} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}} \right) \left( Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right), \quad (4.15a)$$

$$(q_2 \rightarrow 1) : \quad Y_{1,x;2,0} Y_{2,x} + \frac{Y_{2,x;2,0} Y_{2,x}}{Y_{1,x;3,0}} + \frac{Y_{1,x;1,0} Y_{1,x;2,0}}{Y_{2,x;1,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0} Y_{2,x;3,0}} + \frac{Y_{2,x;2,0}}{Y_{1,x;2,0} Y_{1,x;3,0}} + \frac{1}{Y_{1,x;2,0} Y_{2,x;3,0}} \\ + \frac{Y_{2,x}}{Y_{2,x;3,0}} + \frac{Y_{1,x;1,0} Y_{2,x;2,0}}{Y_{1,x;3,0} Y_{2,x;1,0}}. \quad (4.15b)$$

We observe that (4.13b) and (4.15b) provide the  $q$ -character of the adjoint representation of the algebra  $U_q(\widehat{\mathfrak{sl}}_3)$  (See, for example, [NN08, §3.1]). The remaining cases are the degenerate limits, where we obtain the product of two three-dimensional fundamental  $q$ -characters.

## 5. $BC_2$ QUIVER

We consider  $BC_2$  quiver, which is the minimal rank non-simply-laced example. In this case, we have the decoration (root) parameters  $d = (d_1, d_2) = (2, 1)$ , and the iWeyl reflections for the  $Y$ -functions are given by

$$\text{iWeyl} : (Y_{1,x}, Y_{2,x}) \longmapsto \left( \frac{Y_{2,x} Y_{2,xq_1}}{Y_{1,xq_1^2 q_2}}, \frac{Y_{1,xq_1 q_2}}{Y_{2,xq_1 q_2}} \right). \quad (5.1)$$

Then, the fundamental  $qq$ -characters are given as follows,

$$\mathbb{T}_{(1,0),x} = Y_{1,x} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1}} + \mathcal{S}(q_1^{-1}) \frac{Y_{2,x}}{Y_{2,x;2,1}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1} Y_{2,x;2,1}} + \frac{1}{Y_{1,x;3,2}}, \quad (5.2a)$$

$$\mathbb{T}_{(0,1),x} = Y_{2,x} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}} + \frac{Y_{2,x;2,1}}{Y_{1,x;3,2}} + \frac{1}{Y_{2,x;3,2}}. \quad (5.2b)$$

They correspond to the five and four dimensional modules associated with the quantum affine algebra  $U_q(\widehat{\mathfrak{so}}_5) = U_q(\widehat{\mathfrak{sp}}_2)$  corresponding to  $BC_2$  quiver.

In the following, we construct the weight two  $qq$ -characters based on the iWeyl reflections presented here, and examine the irreducibility of the corresponding modules.



5.1.  $w = (2, 0)$ . We consider the  $qq$ -character of weight  $w = (2, 0)$ ,

$$\begin{aligned}
\mathbb{T}_{(2,0),\underline{x}} &= Y_{1,x_1} Y_{1,x_2} + \frac{Y_{2,x_1} Y_{2,x_1;1,0} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_1;2,1} Y_{1,x_2;2,1}} + \frac{Y_{1,x_1;1,1} Y_{1,x_2;1,1}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;1,1} Y_{2,x_2;2,1}} + \frac{1}{Y_{1,x_1;3,2} Y_{1,x_2;3,2}} \\
&+ \mathcal{S}_2\left(\frac{x_2}{x_1}\right) \left[ \frac{Y_{1,x_2} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1}} + \frac{Y_{1,x_1;1,1} Y_{1,x_2}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1}} + \frac{Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_1;3,2} Y_{1,x_2;2,1}} + \frac{Y_{1,x_2;1,1}}{Y_{1,x_2;3,2} Y_{2,x_2;1,1} Y_{2,x_2;2,1}} \right. \\
&\quad \left. + \mathcal{S}(q_1^{-1}) \left( \frac{Y_{1,x_2} Y_{2,x_1}}{Y_{2,x_1;2,1}} + \frac{Y_{2,x_1} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_2;2,1} Y_{2,x_1;2,1}} + \frac{Y_{1,x_1;1,1} Y_{2,x_2}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;2,1}} + \frac{Y_{2,x_2}}{Y_{1,x_1;3,2} Y_{2,x_2;2,1}} \right) \right. \\
&\quad \left. + \mathcal{S}_2\left(\frac{x_2}{x_1} q_1^{-1} q_2^{-1}\right) \frac{Y_{1,x_2}}{Y_{1,x_1;3,2}} + \mathcal{S}_2\left(\frac{x_2}{x_1} q_1\right) \frac{Y_{1,x_1;1,1} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_2;1,1} Y_{2,x_1;1,1} Y_{2,x_1;2,1}} \right] \\
&+ \mathcal{S}_2\left(\frac{x_1}{x_2}\right) \left[ \frac{Y_{1,x_1} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_2;2,1}} + \frac{Y_{1,x_2;1,1} Y_{1,x_1}}{Y_{2,x_2;1,1} Y_{2,x_2;2,1}} + \frac{Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_2;3,2} Y_{1,x_1;2,1}} + \frac{Y_{1,x_1;1,1}}{Y_{1,x_1;3,2} Y_{2,x_1;1,1} Y_{2,x_1;2,1}} \right. \\
&\quad \left. + \mathcal{S}(q_1^{-1}) \left( \frac{Y_{1,x_1} Y_{2,x_2}}{Y_{2,x_2;2,1}} + \frac{Y_{2,x_2} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1} Y_{2,x_2;2,1}} + \frac{Y_{1,x_2;1,1} Y_{2,x_1}}{Y_{2,x_2;1,1} Y_{2,x_2;2,1} Y_{2,x_1;2,1}} + \frac{Y_{2,x_1}}{Y_{1,x_2;3,2} Y_{2,x_1;2,1}} \right) \right. \\
&\quad \left. + \mathcal{S}_2\left(\frac{x_1}{x_2} q_1^{-1} q_2^{-1}\right) \frac{Y_{1,x_1}}{Y_{1,x_2;3,2}} + \mathcal{S}_2\left(\frac{x_1}{x_2} q_1\right) \frac{Y_{1,x_2;1,1} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;1,1} Y_{2,x_2;1,1} Y_{2,x_2;2,1}} \right] \\
&+ \mathcal{S}(q_1^{-1})^2 \mathcal{S}\left(\frac{x_2}{x_1} q_1^{-1}\right) \mathcal{S}\left(\frac{x_1}{x_2} q_1^{-1}\right) \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{2,x_1;2,1} Y_{2,x_2;2,1}}. \tag{5.3}
\end{aligned}$$

The corresponding Hasse diagram is shown in Fig. 4. This  $qq$ -character contains 25 monomials, corresponding to the tensor product of two modules of  $w = (1, 0)$ . We shall see the decomposition of this module into the irreducible ones,  $5 \otimes 5 = 14 \oplus 10 \oplus 1$ .

Recalling that  $\mathcal{S}_2(z) = 0$  at  $z = q_1^2, q_2$ , we specialize the weight parameters as  $\underline{x} = (x, xq_1^2)$ . This is consistent with that the KR module  $\mathbf{W}_{2_1,x}^{(1)}$  should be considered with  $q_1^2$ -shift for the node  $d_1 = 2$ .

**Proposition 5.1.** The  $qq$ -character associated with the KR module  $\mathbf{W}_{2_1,x}^{(1)}$  is given as follows,

$$\begin{aligned}
\mathbb{T}[\mathbf{W}_{2_1,x}^{(1)}] &= Y_{1,x} Y_{1,x;2,0} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;2,1} Y_{1,x;4,1}} + \frac{Y_{1,x;1,1} Y_{1,x;3,1}}{Y_{2,x;1,1} Y_{2,x;2,1} Y_{2,x;3,1} Y_{2,x;4,1}} + \frac{1}{Y_{1,x;3,2} Y_{1,x;5,2}} \\
&+ \mathcal{S}_2(q_1^{-2}) \left[ \frac{Y_{1,x} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;4,1}} + \frac{Y_{1,x;3,1} Y_{1,x}}{Y_{2,x;3,1} Y_{2,x;4,1}} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;5,2} Y_{1,x;2,1}} + \frac{Y_{1,x;1,1}}{Y_{1,x;3,2} Y_{2,x;1,1} Y_{2,x;2,1}} \right. \\
&\quad \left. + \mathcal{S}(q_1^{-1}) \left( \frac{Y_{1,x} Y_{2,x;2,0}}{Y_{2,x;4,1}} + \frac{Y_{2,x;2,0} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1} Y_{2,x;4,1}} + \frac{Y_{1,x;3,1} Y_{2,x}}{Y_{2,x;3,1} Y_{2,x;4,1} Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;5,2} Y_{2,x;2,1}} \right) \right. \\
&\quad \left. + \mathcal{S}_2(q_1^{-3} q_2^{-1}) \frac{Y_{1,x}}{Y_{1,x;5,2}} + \mathcal{S}_2(q_1^{-1}) \frac{Y_{1,x;3,1} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,1} Y_{2,x;3,1} Y_{2,x;4,1}} \right], \tag{5.4}
\end{aligned}$$

which corresponds to the 14-dimensional module of  $BC_2$  quiver for the weight  $w = (2, 0)$ .

We remark that the eliminated terms due to the factor  $\mathcal{S}_2(x_2/x_1)$  correspond to the 10-dimensional module of weight  $w = (0, 2)$  discussed in §5.2.

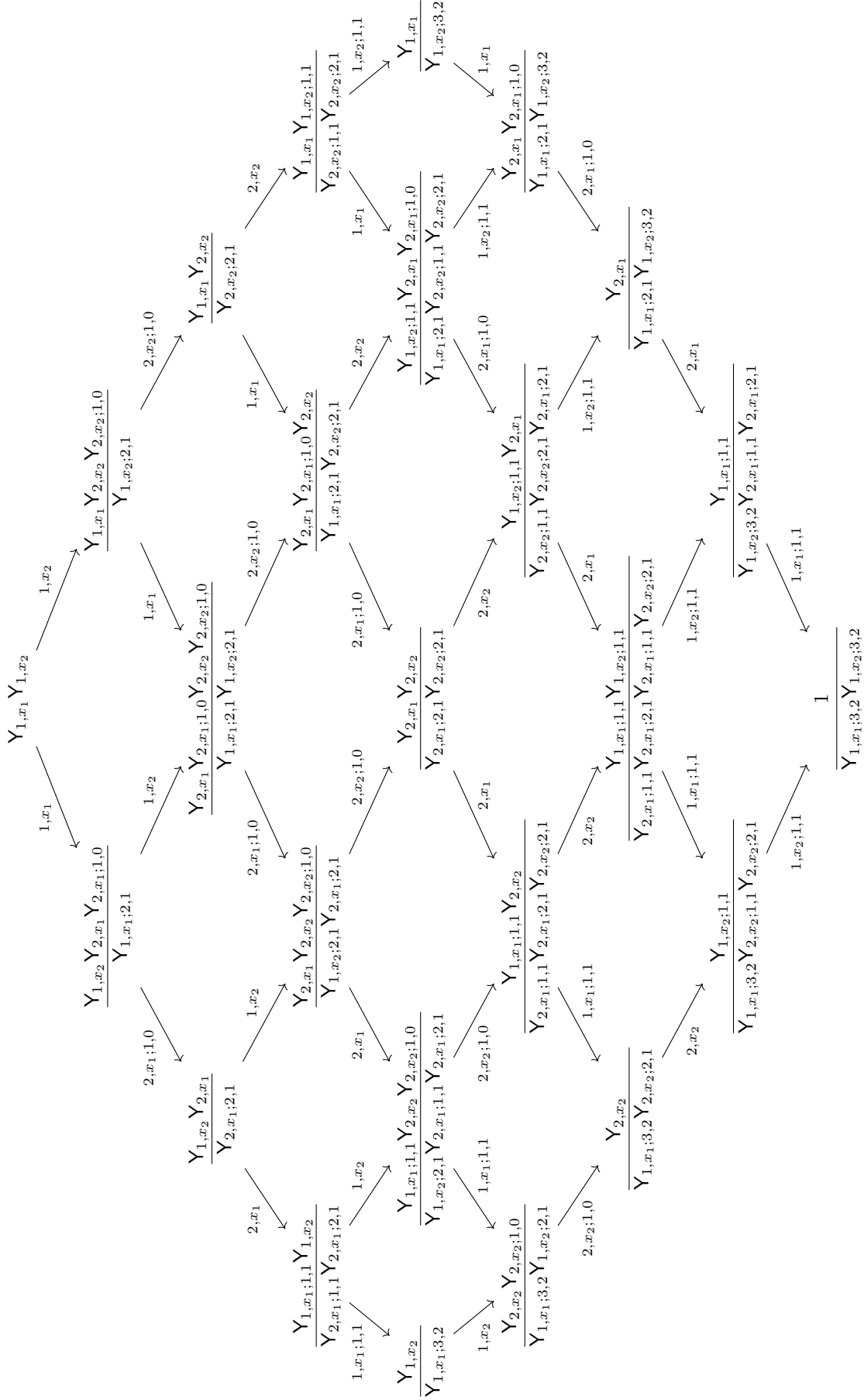


FIGURE 4. Hasse diagram for the iWeyl reflection flow of the weight  $w = (2, 0)$  for  $BC_2$  quiver

*Classical limit.* In addition to the classical limit of the  $\mathcal{S}$ -factors (3.8) and (3.12), we also have  $\mathcal{S}(q_1^{-3}q_2^{-1}) \xrightarrow{q_{1,2} \rightarrow 1} 1$  and

$$\mathcal{S}_2(q_1^{-2}) = (1 + q_1^{-2}) \frac{1 - q_1^{-2}q_2^{-1}}{1 - q_1^{-4}q_2^{-1}} \longrightarrow \begin{cases} 2 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \quad (5.5)$$

Hence, the  $qq$ -character is reduced in the classical limit as follows,

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{2,1,x}^{(1)}] &\xrightarrow{q_1 \rightarrow 1} Y_{1,x}^2 + \frac{Y_{2,x}^4}{Y_{1,x;0,1}^2} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}^4} + \frac{1}{Y_{1,x;0,2}^2} \\ &+ 2 \left( \frac{Y_{1,x} Y_{2,x}^2}{Y_{1,x;0,1}} + \frac{Y_{1,x} Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}^2}{Y_{1,x;0,1} Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,2} Y_{2,x;0,1}} + \frac{Y_{1,x}}{Y_{1,x;0,2}} \right) \\ &+ 4 \left( \frac{Y_{1,x} Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x}^3}{Y_{1,x;0,1} Y_{2,x;0,1}} + \frac{Y_{1,x;0,1} Y_{2,x}}{Y_{2,x;0,1}^3} + \frac{Y_{2,x}}{Y_{1,x;0,2} Y_{2,x;0,1}} \right) + 6 \frac{Y_{2,x}^2}{Y_{2,x;0,1}^3} \\ &= \left( Y_{1,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + 2 \frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{1}{Y_{1,x;0,2}} \right)^2, \end{aligned} \quad (5.6a)$$

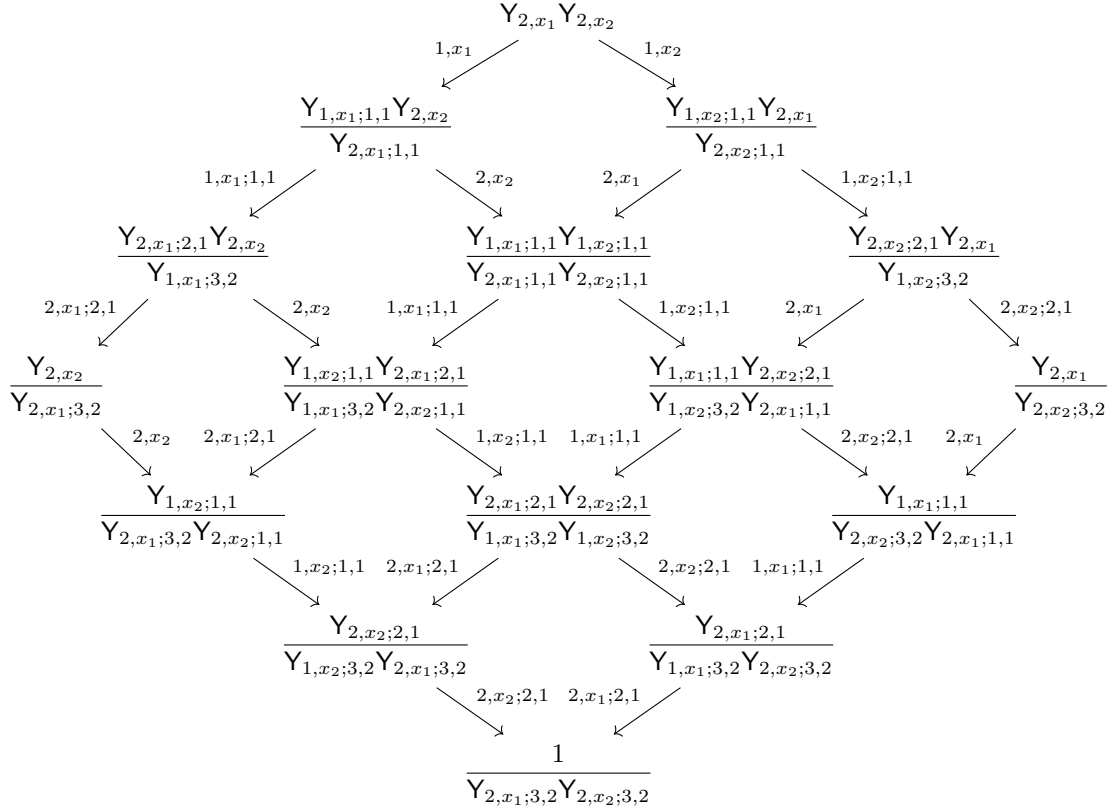
$$\begin{aligned} &\xrightarrow{q_2 \rightarrow 1} Y_{1,x} Y_{1,x;2,0} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;2,0} Y_{1,x;4,0}} + \frac{Y_{1,x;1,0} Y_{1,x;3,0}}{Y_{2,x;1,0} Y_{2,x;2,0} Y_{2,x;3,0} Y_{2,x;4,0}} + \frac{1}{Y_{1,x;3,0} Y_{1,x;5,0}} \\ &+ \frac{Y_{1,x} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;4,0}} + \frac{Y_{1,x;3,0} Y_{1,x}}{Y_{2,x;3,0} Y_{2,x;4,0}} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;5,0} Y_{1,x;2,0}} + \frac{Y_{1,x;1,0}}{Y_{1,x;3,0} Y_{2,x;1,0} Y_{2,x;2,0}} \\ &+ \frac{Y_{1,x} Y_{2,x;2,0}}{Y_{2,x;4,0}} + \frac{Y_{2,x;2,0} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,0} Y_{2,x;4,0}} + \frac{Y_{1,x;3,0} Y_{2,x}}{Y_{2,x;3,0} Y_{2,x;4,0} Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;5,0} Y_{2,x;2,0}} \\ &+ \frac{Y_{1,x}}{Y_{1,x;5,0}} + \frac{Y_{1,x;3,0} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,0} Y_{2,x;3,0} Y_{2,x;4,0}}. \end{aligned} \quad (5.6b)$$

We obtain the  $q$ -character of the 14-dimensional module in the limit  $q_2 \rightarrow 1$ . On the other hand, the limit  $q_1 \rightarrow 1$  provides the tensor product of the fundamental module  $w = (1, 0)$ . We remark that, in the limit  $q_1 \rightarrow 1$ , the fundamental  $qq$ -character (5.2a) is already degenerated due to the  $\mathcal{S}$ -factor. The resulting module is associated with the six-dimensional module of  $AD_3$  quiver, which is the fractionalization of  $BC_2$  quiver [KP22].

5.2.  $w = (0, 2)$ . The  $qq$ -character of weight  $w = (0, 2)$  is given as follows,

$$\begin{aligned} \mathbb{T}_{(0,2),\underline{x}} &= Y_{2,x_1} Y_{2,x_2} + \frac{Y_{1,x_1;1,1} Y_{1,x_2;1,1}}{Y_{2,x_1;1,1} Y_{2,x_2;1,1}} + \frac{Y_{2,x_1;2,1} Y_{2,x_2;2,1}}{Y_{1,x_1;3,2} Y_{1,x_2;3,2}} + \frac{1}{Y_{2,x_1;3,2} Y_{2,x_2;3,2}} \\ &+ \mathcal{S} \left( \frac{x_2}{x_1} \right) \left[ \frac{Y_{1,x_1;1,1} Y_{2,x_2}}{Y_{2,x_1;1,1}} + \frac{Y_{2,x_1;2,1} Y_{2,x_2}}{Y_{1,x_1;3,2}} + \mathcal{S} \left( \frac{x_2}{x_1} q_1^{-2} q_2^{-1} \right) \frac{Y_{2,x_2}}{Y_{2,x_1;3,2}} + \frac{Y_{1,x_2;1,1}}{Y_{2,x_1;3,2} Y_{2,x_2;1,1}} + \frac{Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;3,2}} \right] \\ &+ \mathcal{S} \left( \frac{x_1}{x_2} \right) \left[ \frac{Y_{1,x_2;1,1} Y_{2,x_1}}{Y_{2,x_2;1,1}} + \frac{Y_{2,x_2;2,1} Y_{2,x_1}}{Y_{1,x_2;3,2}} + \mathcal{S} \left( \frac{x_1}{x_2} q_1^{-2} q_2^{-1} \right) \frac{Y_{2,x_1}}{Y_{2,x_2;3,2}} + \frac{Y_{1,x_1;1,1}}{Y_{2,x_2;3,2} Y_{2,x_1;1,1}} + \frac{Y_{2,x_1;2,1}}{Y_{1,x_1;3,2} Y_{2,x_2;3,2}} \right] \\ &+ \mathcal{S}_2 \left( \frac{x_2}{x_1} \right) \frac{Y_{1,x_2;1,1} Y_{2,x_1;2,1}}{Y_{1,x_1;3,2} Y_{2,x_2;1,1}} + \mathcal{S}_2 \left( \frac{x_1}{x_2} \right) \frac{Y_{1,x_1;1,1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;1,1}}, \end{aligned} \quad (5.7)$$

where the  $i$ Weyl reflection flow is described by the Hasse diagram in Fig. 5. There exist 16 monomials in the  $qq$ -character. We shall show the decomposition,  $4 \otimes 4 = 10 \oplus 5 \oplus 1$ , as follows.


 FIGURE 5. Hasse diagram for the iWeyl reflection flow of the weight  $w = (0, 2)$  for  $BC_2$  quiver.

**Proposition 5.2.** Specializing the weight parameters  $\underline{x} = (x, xq_1)$ , we obtain the  $qq$ -character for the KR module  $\mathbf{W}_{2_1, \underline{x}}^{(2)}$  as follows,

$$\begin{aligned}
 \mathbb{T}[\mathbf{W}_{2_1, \underline{x}}^{(2)}] &= Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,1} Y_{1,x;2,1}}{Y_{2,x;1,1} Y_{2,x;2,1}} + \frac{Y_{2,x;2,1} Y_{2,x;3,1}}{Y_{1,x;3,2} Y_{1,x;4,2}} + \frac{1}{Y_{2,x;3,2} Y_{2,x;4,2}} \\
 &+ \mathcal{S}(q_1^{-1}) \left[ \frac{Y_{1,x;2,1} Y_{2,x}}{Y_{2,x;2,1}} + \frac{Y_{2,x;3,1} Y_{2,x}}{Y_{1,x;4,2}} + \mathcal{S}(q_1^{-3} q_2^{-1}) \frac{Y_{2,x}}{Y_{2,x;4,2}} + \frac{Y_{1,x;1,1}}{Y_{2,x;4,2} Y_{2,x;1,1}} + \frac{Y_{2,x;2,1}}{Y_{1,x;3,2} Y_{2,x;4,2}} \right] \\
 &+ \mathcal{S}_2(q_1) \frac{Y_{1,x;2,1}}{Y_{1,x;3,2}} + \mathcal{S}_2(q_1^{-1}) \frac{Y_{1,x;1,1} Y_{2,x;3,1}}{Y_{1,x;4,2} Y_{2,x;1,1}}, \tag{5.8}
 \end{aligned}$$

which corresponds to the 10-dimensional module (adjoint representation of  $\mathrm{SO}(5) = \mathrm{Sp}(2)$ ) with the affine extension.

*Classical limit.* In the classical limit, the  $\mathcal{S}$ -factors appearing in the  $qq$ -character  $\mathbb{T}[\mathbf{W}_{2_1, \underline{x}}^{(2)}]$  are given by  $\mathcal{S}(q_1^{-3} q_2^{-1}) \xrightarrow{q_{1,2} \rightarrow 1} 1$  and

$$\mathcal{S}_2(q_1) = -q_1^{-1} \frac{1 - q_1 q_2^{-1}}{1 - q_1^{-1} q_2^{-1}} \longrightarrow \begin{cases} -1 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \tag{5.9}$$

Therefore, we obtain the following reductions,

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(2)}] &\xrightarrow{q_1 \rightarrow 1} Y_{2,x}^2 + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}^2} + \frac{Y_{2,x;0,1}^2}{Y_{1,x;0,2}^2} + \frac{1}{Y_{2,x;0,2}^2} \\ &\quad + 2 \left( \frac{Y_{1,x;0,1} Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1} Y_{2,x}}{Y_{1,x;0,2}} + \frac{Y_{2,x}}{Y_{2,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,2} Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2} Y_{2,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,2}} \right) \\ &= \left( Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}} \right)^2, \end{aligned} \quad (5.10a)$$

$$\begin{aligned} &\xrightarrow{q_2 \rightarrow 1} Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,0} Y_{1,x;2,0}}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;3,0} Y_{1,x;4,0}} + \frac{1}{Y_{2,x;3,0} Y_{2,x;4,0}} + \frac{Y_{1,x;2,0} Y_{2,x}}{Y_{2,x;2,0}} \\ &\quad + \frac{Y_{2,x;3,0} Y_{2,x}}{Y_{1,x;4,0}} + \frac{Y_{2,x}}{Y_{2,x;4,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;4,0} Y_{2,x;1,0}} + \frac{Y_{2,x;2,0}}{Y_{1,x;3,0} Y_{2,x;4,0}} + \frac{Y_{1,x;2,0}}{Y_{1,x;3,0}} + \frac{Y_{1,x;1,0} Y_{2,x;3,0}}{Y_{1,x;4,0} Y_{2,x;1,0}}. \end{aligned} \quad (5.10b)$$

The limit  $q_1 \rightarrow 1$  is the degenerate limit, which provides the tensor product of two four-dimensional modules, while we obtain the  $q$ -character of the 10-dimensional module with the affine extension in the limit  $q_2 \rightarrow 1$ .

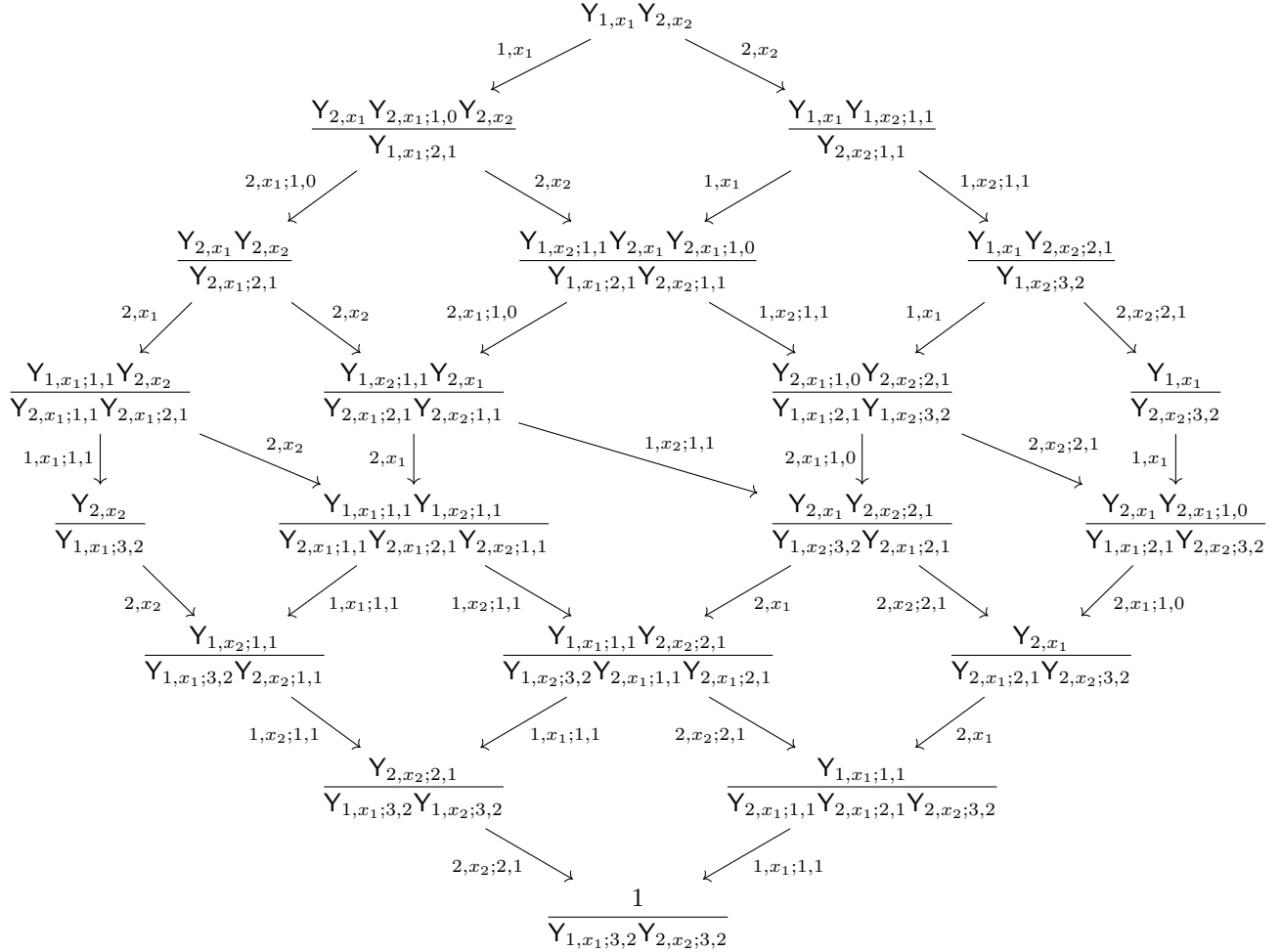
5.3.  $w = (1, 1)$ . We consider the  $qq$ -character of weight  $w = (1, 1)$ ,

$$\begin{aligned} \mathbb{T}_{(1,1),\underline{x}} &= Y_{1,x_1} Y_{2,x_2} + \frac{Y_{2,x_1} Y_{2,x_1;1,0} Y_{2,x_2}}{Y_{1,x_1;2,1}} + \frac{Y_{1,x_1} Y_{1,x_2;1,1}}{Y_{2,x_2;1,1}} + \frac{Y_{2,x_1} Y_{2,x_1;1,0} Y_{2,x_2;2,1}}{Y_{1,x_1;2,1} Y_{1,x_2;3,2}} \\ &\quad + \frac{Y_{1,x_1;1,1} Y_{1,x_2;1,1}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;1,1}} + \frac{Y_{2,x_2;2,1}}{Y_{1,x_1;3,2} Y_{1,x_2;3,2}} + \frac{Y_{1,x_1;1,1}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;3,2}} + \frac{1}{Y_{1,x_1;3,2} Y_{2,x_2;3,2}} \\ &\quad + \mathcal{S}_2 \left( \frac{x_1}{x_2} q_1^{-1} q_2^{-1} \right) \left[ \frac{Y_{1,x_1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2}} + \frac{Y_{1,x_1}}{Y_{2,x_2;3,2}} + \frac{Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1} Y_{2,x_2;3,2}} \right] + \mathcal{S}_2 \left( \frac{x_2}{x_1} q_1 q_2 \right) \frac{Y_{1,x_2;1,1} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1} Y_{2,x_2;1,1}} \\ &\quad + \mathcal{S}_2 \left( \frac{x_2}{x_1} \right) \left[ \frac{Y_{1,x_1;1,1} Y_{2,x_2}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1}} + \frac{Y_{2,x_2}}{Y_{1,x_1;3,2}} + \frac{Y_{1,x_2;1,1}}{Y_{1,x_1;3,2} Y_{2,x_2;1,1}} \right] + \mathcal{S}_2 \left( \frac{x_1}{x_2} \right) \frac{Y_{1,x_1;1,1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;1,1} Y_{2,x_1;2,1}} \\ &\quad + \mathcal{S}(q_1^{-1}) \left[ \mathcal{S} \left( \frac{x_2}{x_1} q_1^{-1} \right) \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{2,x_1;2,1}} + \mathcal{S} \left( \frac{x_1}{x_2} \right) \left( \frac{Y_{1,x_2;1,1} Y_{2,x_1}}{Y_{2,x_1;2,1} Y_{2,x_2;1,1}} + \frac{Y_{2,x_1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;2,1}} \right) + \mathcal{S} \left( \frac{x_1}{x_2} q_1^{-2} q_2^{-1} \right) \frac{Y_{2,x_1}}{Y_{2,x_1;2,1} Y_{2,x_2;3,2}} \right], \end{aligned} \quad (5.11)$$

where the iWeyl reflection flow is described in Fig. 6. This  $qq$ -character shows the decomposition of the tensor product,  $4 \otimes 5 = 16 \oplus 4$ .

**Proposition 5.3.** We specialize the weight parameters as (a)  $\underline{x} = (xq_1^3 q_2, x)$  and (b)  $\underline{x} = (x, xq_1^2)$  to obtain the  $qq$ -character for the 14-dimensional module:

$$\begin{aligned} \mathbb{T}_{(1,1),\underline{x}} &\xrightarrow{(a)} Y_{1,x;3,1} Y_{2,x} + \frac{Y_{2,x} Y_{2,x;3,1} Y_{2,x;4,1}}{Y_{1,x;5,2}} + \frac{Y_{1,x;1,1} Y_{1,x;3,1}}{Y_{2,x;1,1}} + \frac{Y_{2,x;2,1} Y_{2,x;3,1} Y_{2,x;4,1}}{Y_{1,x;3,2} Y_{1,x;5,2}} \\ &\quad + \frac{Y_{1,x;1,1} Y_{1,x;4,2}}{Y_{2,x;1,1} Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{Y_{2,x;2,1}}{Y_{1,x;3,2} Y_{1,x;6,3}} + \frac{Y_{1,x;4,2}}{Y_{2,x;3,2} Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{1}{Y_{1,x;6,3} Y_{2,x;3,2}} \\ &\quad + \mathcal{S}_2(q_1^{-3} q_2^{-1}) \left( \frac{Y_{1,x;4,2} Y_{2,x}}{Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{Y_{2,x}}{Y_{1,x;6,3}} + \frac{Y_{1,x;1,1}}{Y_{1,x;6,3} Y_{2,x;1,1}} \right) + \mathcal{S}_2(q_1^{-2}) \frac{Y_{1,x;1,1} Y_{2,x;3,1} Y_{2,x;4,1}}{Y_{1,x;5,2} Y_{2,x;1,1}} \\ &\quad + \mathcal{S}_2(q_1^{-1}) \left( \frac{Y_{1,x;4,2} Y_{2,x;2,1}}{Y_{1,x;3,2} Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{Y_{1,x;1,1} Y_{2,x;3,1}}{Y_{2,x;1,1} Y_{2,x;5,2}} + \frac{Y_{2,x;2,1} Y_{2,x;3,1}}{Y_{1,x;3,2} Y_{2,x;5,2}} \right) + \mathcal{S}(q_1^{-1}) \mathcal{S}(q_1^{-4} q_2^{-1}) \frac{Y_{2,x} Y_{2,x;3,1}}{Y_{2,x;5,2}}, \end{aligned} \quad (5.12a)$$


 FIGURE 6. Hasse diagram for the iWeyl reflection flow of the weight  $w = (1, 1)$  for  $BC_2$  quiver.

$$\begin{aligned}
 T_{(1,1),x} \stackrel{(b)}{\rightarrow} & Y_{1,x} Y_{2,x;2,0} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;2,0}}{Y_{1,x;2,1}} + \frac{Y_{1,x} Y_{1,x;3,1}}{Y_{2,x;3,1}} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;4,1}}{Y_{1,x;2,1} Y_{1,x;5,2}} \\
 & + \frac{Y_{1,x;1,1} Y_{1,x;3,1}}{Y_{2,x;1,1} Y_{2,x;2,1} Y_{2,x;3,1}} + \frac{Y_{2,x;4,1}}{Y_{1,x;3,2} Y_{1,x;5,2}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1} Y_{2,x;2,1} Y_{2,x;5,2}} + \frac{1}{Y_{1,x;3,2} Y_{2,x;5,2}} \\
 & + S_2(q_1^{-3} q_2^{-1}) \left( \frac{Y_{1,x} Y_{2,x;4,1}}{Y_{1,x;5,2}} + \frac{Y_{1,x}}{Y_{2,x;5,2}} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1} Y_{2,x;5,2}} \right) + S_2(q_1^{-2}) \frac{Y_{1,x;1,1} Y_{2,x;4,1}}{Y_{1,x;5,2} Y_{2,x;1,1} Y_{2,x;2,1}} \\
 & + S_2(q_1^{-1}) \left( \frac{Y_{1,x;3,1} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1} Y_{2,x;3,1}} + \frac{Y_{1,x;3,1} Y_{2,x}}{Y_{2,x;2,1} Y_{2,x;3,1}} + \frac{Y_{2,x} Y_{2,x;4,1}}{Y_{1,x;5,2} Y_{2,x;2,1}} \right) + S(q_1^{-1}) S(q_1^{-4} q_2^{-1}) \frac{Y_{2,x}}{Y_{2,x;2,1} Y_{2,x;5,2}}.
 \end{aligned} \tag{5.12b}$$

*Classical limit.* We then study the classical limit of the  $qq$ -character for the 14-dimensional module. In the limit  $q_2 \rightarrow 1$ , we obtain the corresponding  $q$ -character, while we observe the tensor product structure in the degenerate limit  $q_1 \rightarrow 1$ .

(a)  $\underline{x} = (xq_1^3q_2, x)$ 

$$\begin{aligned}
(q_1 \rightarrow 1) : & \quad Y_{1,x;0,1}Y_{2,x} + \frac{Y_{2,x}Y_{2,x;0,1}^2}{Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}^3}{Y_{1,x;0,2}Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}Y_{1,x;0,2}}{Y_{2,x;0,1}Y_{2,x;0,2}^2} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}Y_{1,x;0,3}} \\
& + \frac{Y_{1,x;0,2}}{Y_{2,x;0,2}^3} + \frac{1}{Y_{1,x;0,3}Y_{2,x;0,2}} + \frac{Y_{1,x;0,2}Y_{2,x}}{Y_{2,x;0,2}^2} + \frac{Y_{2,x}}{Y_{1,x;0,3}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,3}Y_{2,x;0,1}} \\
& + 2\left(\frac{Y_{1,x;0,1}Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{Y_{2,x}Y_{2,x;0,1}}{Y_{2,x;0,2}}\right) + 3\left(\frac{Y_{1,x;0,2}Y_{2,x;0,1}}{Y_{1,x;0,2}Y_{2,x;0,2}^2} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,2}} + \frac{Y_{2,x;0,1}^2}{Y_{1,x;0,2}Y_{2,x;0,2}}\right) \\
& = \left(Y_{1,x;0,1} + \frac{Y_{2,x;0,1}^2}{Y_{1,x;0,2}} + 2\frac{Y_{2,x;0,1}}{Y_{2,x;0,2}} + \frac{Y_{1,x;0,2}}{Y_{2,x;0,2}^2} + \frac{1}{Y_{1,x;0,3}}\right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}}\right),
\end{aligned} \tag{5.13a}$$

$$\begin{aligned}
(q_2 \rightarrow 1) : & \quad Y_{1,x;3,0}Y_{2,x} + \frac{Y_{2,x;3,0}Y_{2,x;4,0}Y_{2,x}}{Y_{1,x;5,0}} + \frac{Y_{1,x;3,0}Y_{1,x;1,0}}{Y_{2,x;1,0}} + \frac{Y_{2,x;3,0}Y_{2,x;4,0}Y_{2,x;2,0}}{Y_{1,x;5,0}Y_{1,x;3,0}} \\
& + \frac{Y_{1,x;4,0}Y_{1,x;1,0}}{Y_{2,x;4,0}Y_{2,x;5,0}Y_{2,x;1,0}} + \frac{Y_{2,x;2,0}}{Y_{1,x;6,0}Y_{1,x;3,0}} + \frac{Y_{1,x;4,0}}{Y_{2,x;4,0}Y_{2,x;5,0}Y_{2,x;3,0}} + \frac{1}{Y_{1,x;6,0}Y_{2,x;3,0}} \\
& + \frac{Y_{1,x;4,0}Y_{2,x}}{Y_{2,x;4,0}Y_{2,x;5,0}} + \frac{Y_{2,x}}{Y_{1,x;5,0}} + \frac{Y_{1,x;1,0}}{Y_{1,x;6,0}Y_{2,x;1,0}} + \frac{Y_{1,x;1,0}Y_{2,x;3,0}Y_{2,x;4,0}}{Y_{1,x;5,0}Y_{2,x;1,0}} \\
& + \frac{Y_{1,x;4,0}Y_{2,x;2,0}}{Y_{1,x;3,0}Y_{2,x;4,0}Y_{2,x;5,0}} + \frac{Y_{1,x;1,0}Y_{2,x;3,0}}{Y_{2,x;5,0}Y_{2,x;1,0}} + \frac{Y_{2,x;3,0}Y_{2,x;2,0}}{Y_{1,x;3,0}Y_{2,x;5,0}} + \frac{Y_{2,x;3,0}Y_{2,x}}{Y_{2,x;5,0}}
\end{aligned} \tag{5.13b}$$

(b)  $\underline{x} = (x, xq_1^2)$ 

$$\begin{aligned}
(q_1 \rightarrow 1) : & \quad Y_{1,x}Y_{2,x} + \frac{Y_{2,x}^3}{Y_{1,x;0,1}} + \frac{Y_{1,x}Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x}^2Y_{2,x;0,1}}{Y_{1,x;0,1}Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}^3} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}^2} \\
& + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{1}{Y_{1,x;0,2}^2} + \frac{Y_{1,x}Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{Y_{1,x}}{Y_{2,x;0,2}} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}Y_{2,x;0,2}} \\
& + 2\left(\frac{Y_{1,x;0,1}}{Y_{1,x;0,2}Y_{2,x;0,1}} + \frac{Y_{2,x}}{Y_{2,x;0,1}Y_{2,x;0,2}}\right) + 3\left(\frac{Y_{2,x}^2}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}Y_{2,x}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}}{Y_{1,x;0,2}}\right) \\
& = \left(Y_{1,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + 2\frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{1}{Y_{1,x;0,2}}\right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}}\right)
\end{aligned} \tag{5.14a}$$

$$\begin{aligned}
(q_2 \rightarrow 1) & \quad Y_{1,x}Y_{2,x;2,0} + \frac{Y_{2,x}Y_{2,x;1,0}Y_{2,x;2,0}}{Y_{1,x;2,0}} + \frac{Y_{1,x}Y_{1,x;3,0}}{Y_{2,x;3,0}} + \frac{Y_{2,x}Y_{2,x;1,0}Y_{2,x;4,0}}{Y_{1,x;2,0}Y_{1,x;5,0}} \\
& + \frac{Y_{1,x;1,0}Y_{1,x;3,0}}{Y_{2,x;1,0}Y_{2,x;2,0}Y_{2,x;3,0}} + \frac{Y_{2,x;4,0}}{Y_{1,x;3,0}Y_{1,x;5,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0}Y_{2,x;2,0}Y_{2,x;5,0}} + \frac{1}{Y_{1,x;3,0}Y_{2,x;5,0}} \\
& + \frac{Y_{1,x}Y_{2,x;4,0}}{Y_{1,x;5,0}} + \frac{Y_{1,x}}{Y_{2,x;5,0}} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,0}Y_{2,x;5,0}} + \frac{Y_{1,x;1,0}Y_{2,x;4,0}}{Y_{1,x;5,0}Y_{2,x;1,0}Y_{2,x;2,0}} \\
& + \frac{Y_{1,x;3,0}Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,0}Y_{2,x;3,0}} + \frac{Y_{1,x;3,0}Y_{2,x}}{Y_{2,x;2,0}Y_{2,x;3,0}} + \frac{Y_{2,x}Y_{2,x;4,0}}{Y_{1,x;5,0}Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{2,x;2,0}Y_{2,x;5,0}}
\end{aligned} \tag{5.14b}$$

6.  $\widehat{A}_0$  QUIVER

We consider  $\widehat{A}_0$  quiver, which is the simplest affine quiver. In this case, there exists a single node with the loop edge. The Cartan matrix is given by

$$c = 1 + q - \mu - \mu^{-1}q = (1 - \mu)(1 - \mu^{-1}q) = (1 - q_3)(1 - q_4), \tag{6.1}$$

where we define

$$(q_3, q_4) = (\mu, \mu^{-1}q) \quad (6.2)$$

and

$$\underline{q} = (q_1, q_2, q_3, q_4). \quad (6.3)$$

We assume  $q_{3,4} \neq 1$ ,<sup>11</sup> so that the Cartan matrix is invertible. We remark the relation

$$q_1 q_2 = q_3 q_4 = q. \quad (6.4)$$

Then, we have the iWeyl reflection with a formal counting parameter  $\mathfrak{q} \in \mathbb{C}^\times$ ,

$$\text{iWeyl}: \quad Y_x \longmapsto \mathfrak{q} \mathcal{S}(q_3) \frac{Y_{xq_3} Y_{xq_4}}{Y_{xq}}. \quad (6.5)$$

We remark that the extra factor  $\mathcal{S}(q_3) = \mathcal{S}(q_4)$  is specific to the case with the loop edge.

**6.1. Weight one.** The fundamental  $qq$ -character of the weight  $w = (1)$  is given by an infinite sum over all the possible partitions [Nek15, KP15],

$$\mathbb{T}_{1,x} = \sum_{\lambda} \mathfrak{q}^{|\lambda|} Z_{\lambda}(\underline{q}) \prod_{s \in \partial_+ \lambda} Y_{x(s)} \prod_{s \in \partial_- \lambda} Y_{x(s)q}^{-1}, \quad (6.6)$$

where  $|\lambda| = \sum_{k=1}^{\infty} \lambda_k$ , and we denote by  $\partial_+ \lambda$  and  $\partial_- \lambda$  the outer and inner boundaries of the partition  $\lambda$ , where one can add/remove a box. For  $s = (s_1, s_2)$  and the transpose of partition denoted by  $\lambda^T$ , we define

$$x(s) = xq_3^{s_1-1} q_4^{s_2-1}, \quad a(s) = \lambda_{s_2} - s_1, \quad \ell(s) = \lambda_{s_1}^T - s_2. \quad (6.7)$$

Then, we have

$$Z_{\lambda}(\underline{q}) = \prod_{s \in \lambda} \mathcal{S}(q_3^{\ell(s)+1} q_4^{-a(s)}) = \prod_{s \in \lambda} \mathcal{S}(q_3^{-\ell(s)} q_4^{a(s)+1}). \quad (6.8)$$

We remark that this factor is identified with the fixed point contribution of the Nekrasov partition function of 5d  $\mathcal{N} = 1^* \text{U}(1)$  gauge theory with the  $\Omega$ -background parameters  $(q_3, q_4)$  (not for  $(q_1, q_2)$ ) with the adjoint mass  $q_1$  (or  $q_2$ ), which is geometrically interpreted as the  $\chi_{q_1,2}$ -genus of the instanton moduli space [Nek02, NO03].

The  $qq$ -character (6.6) is interpreted to be associated with the infinite dimensional Fock module of the quantum toroidal algebra of  $\mathfrak{gl}_1$  denoted by  $U_q(\widehat{\mathfrak{gl}}_1)$  [NPS13, FJMM15, FJMM16] and its elliptic uplift [KO21].<sup>12</sup> See also [Liu22] for a related geometric representation theoretical perspective of the  $qq$ -character of  $\widehat{A}_0$  quiver. Similarly to finite-type quivers, we can reduce the module by tuning the parameters. In this case, imposing the *resonance* condition [FFJ+10]<sup>13</sup>

$$q_3^{\mathbf{i}} q_4^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad q_3^{-\mathbf{i}+1} q_4^{\mathbf{i}} = q_{1,2}, \quad (6.9)$$

the rational function (6.8) becomes zero if  $(\mathbf{i}, \mathbf{j}) \in \lambda$  for  $(\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_{>0}^2$ . Hence, the infinite sum (6.6) excludes the configuration such that  $(\mathbf{i}, \mathbf{j}) \in \lambda$ . This is called the *pit* condition at  $(\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_{>0}^2$  [BFM15].

<sup>11</sup>From the gauge theory point of view, supersymmetry is enhanced from 8 to 16 supercharges in the limit  $q_{3,4} \rightarrow 1$ .

<sup>12</sup>The  $qq$ -character of the MacMahon module of  $U_q(\widehat{\mathfrak{gl}}_1)$  has been also constructed recently [KN23].

<sup>13</sup>We use a slightly different convention compared with the original paper [FFJ+10].



**6.2. Generic weight.** We then consider the  $qq$ -character of  $\widehat{A}_0$  quiver for generic weight  $w$ . In this case, the  $qq$ -character is given by a summation over  $w$ -tuple partitions  $\lambda = (\lambda_\alpha)_{\alpha=1,\dots,w}$  with the weight parameters  $\underline{x} = (x_\alpha)_{\alpha=1,\dots,w}$ ,

$$\tau_{w,\underline{x}} = \sum_{\lambda} q^{|\lambda|} Z_{\lambda}(q; \underline{x}) \prod_{\alpha=1}^w \left[ \prod_{s \in \partial_+ \lambda_\alpha} Y_{x_\alpha(s)} \prod_{s \in \partial_- \lambda_\alpha} Y_{x_\alpha(s)}^{-1} \right] \quad (6.10)$$

where  $|\lambda| = \sum_{\alpha=1}^w |\lambda_\alpha|$ , and we define

$$Z_{\lambda}(q; \underline{x}) = \prod_{\alpha=1}^w Z_{\lambda_\alpha}(q) \prod_{1 \leq \alpha < \beta \leq w} \left[ \prod_{s \in \lambda_\alpha} \mathfrak{S} \left( \frac{x_\beta}{x_\alpha} q_3^{\ell_\beta(s)+1} q_4^{-a_\alpha(s)} \right) \prod_{s \in \lambda_\beta} \mathfrak{S} \left( \frac{x_\beta}{x_\alpha} q_3^{-\ell_\alpha(s)} q_4^{a_\beta(s)+1} \right) \right] \quad (6.11)$$

with

$$a_\alpha(s) = \lambda_{\alpha, s_2} - s_1, \quad \ell_\alpha(s) = \lambda_{\alpha, s_1}^T - s_2. \quad (6.12)$$

We remark

$$\mathfrak{S} \left( \frac{x_\beta}{x_\alpha} q_3^{\ell_\beta(s)+1} q_4^{-a_\alpha(s)} \right) = \mathfrak{S} \left( \frac{x_\alpha}{x_\beta} q_3^{-\ell_\beta(s)} q_4^{a_\alpha(s)+1} \right), \quad \mathfrak{S} \left( \frac{x_\beta}{x_\alpha} q_3^{-\ell_\alpha(s)} q_4^{a_\beta(s)+1} \right) = \mathfrak{S} \left( \frac{x_\alpha}{x_\beta} q_3^{\ell_\alpha(s)+1} q_4^{-a_\beta(s)} \right). \quad (6.13)$$

The factor  $Z_{\lambda}(q; \underline{x})$  coincides with the fixed point contribution of the Nekrasov partition function of 5d  $\mathcal{N} = 1^* \text{U}(w)$  gauge theory under identification of the weight parameters with the Coulomb moduli. From the representation theoretical point of view, the weight- $w$  character is associated with the tensor product of the Fock modules. In this context, the weight parameters  $(x_\alpha)_{\alpha=1,\dots,w}$  are identified with the evaluation parameters.

We may consider the same condition as the case of weight one (6.9), which imposes the same pit condition for all the partitions  $(\lambda_\alpha)_{\alpha=1,\dots,w}$ . For the tensor product of the Fock modules, there exists a family of the irreducible representations obtained by specialization of the weight parameters  $\underline{x} = (x_\alpha)_{\alpha=1,\dots,w}$  [FFJ<sup>+</sup>10]. We consider the resonance condition

$$\frac{x_\beta}{x_\alpha} q_3^{\mathbf{i}} q_4^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad \frac{x_\alpha}{x_\beta} q_3^{-\mathbf{i}+1} q_4^{\mathbf{j}} = q_{1,2}. \quad (6.14)$$

In this case, we obtain the following restriction on the set of partitions,

$$\lambda_{\beta, k}^T - \lambda_{\alpha, k+\mathbf{j}-1}^T \geq \mathbf{i} \quad (\mathbf{i} \in \mathbb{Z}_{\leq 0}, \mathbf{j} \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 1}). \quad (6.15)$$

As the resulting modules are irreducible as shown in [FFJ<sup>+</sup>10], we have the following.

**Proposition 6.1.** Conjecture 1.1 holds for the tensor product module of the Fock modules associated with  $\widehat{A}_0$  quiver.

Such a restriction, known as the Burge condition [Bur93], has been discussed also in the relation to the plane partitions [GK97], the Higgsing of 5d  $\mathcal{N} = 1^*$  theory [CHZ12], and the minimal models of W-algebras [BF14, AB14, BFS15].

7.  $\widehat{A}_{r-1}$  QUIVER

We study the higher rank cyclic quiver  $\widehat{A}_{r-1}$ , which consists of  $r$  nodes. In this case, we consider the Cartan matrix given by

$$c = \begin{pmatrix} 1+q & -\mu_1 & & & -\mu_0^{-1}q \\ -\mu_1^{-1}q & 1+q & -\mu_2 & & \\ & -\mu_2^{-1}q & 1+q & \ddots & \\ & & \ddots & \ddots & -\mu_{r-1} \\ -\mu_0 & & & -\mu_{r-1}^{-1}q & 1+q \end{pmatrix} \quad (7.1)$$

and the determinant given by

$$\det c = (1 - \mu_{\text{tot}})(1 - \mu_{\text{tot}}^{-1}q^r), \quad \mu_{\text{tot}} := \prod_{i=0}^{r-1} \mu_i. \quad (7.2)$$

We assume  $\mu_{\text{tot}} \neq 1$ ,  $\mu_{\text{tot}}^{-1}q^r \neq 1$  in order that the Cartan matrix is invertible. We specialize the mass parameters to  $\mu_i = \mu$  for  $i = 0, \dots, r-1$  without loss of generality, so that  $\mu_{\text{tot}} = \mu^r$ . We use the same notation  $(q_3, q_4) = (\mu, \mu^{-1}q)$  as before (6.2). Then, the iWeyl reflection is given as follows,

$$\text{iWeyl} : Y_{i,x} \mapsto \mathfrak{q}_i \frac{Y_{i+1,xq_3} Y_{i-1,xq_4}}{Y_{i,xq}}, \quad i \in \mathbb{Z}/r\mathbb{Z}, \quad (7.3)$$

where we interpret the node index periodic  $i \equiv i + r \pmod{r}$ . We also denote  $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$ .

**7.1. Weight one.** The fundamental  $qq$ -character associated with the  $i$ -th node, corresponding to the weight  $w = (w_j)_{j=0,\dots,r-1}$  with  $w_j = \delta_{i,j}$ , is again given by summation over the partition,

$$\mathbb{T}_{i,x} = \sum_{\lambda} \mathfrak{q}_i^\lambda Z_{\lambda}^{(r)}(q) \prod_{s \in \partial_+ \lambda} Y_{i(s)+i,x(s)} \prod_{s \in \partial_- \lambda} Y_{i(s)+i,x(s)q}^{-1}, \quad (7.4)$$

where we define the coloring index  $i(s) \equiv s_1 - s_2 \pmod{r}$  for  $s = (s_1, s_2) \in \mathbb{Z}_{>0}^2$ , and

$$\mathfrak{q}_i^\lambda = \prod_{s \in \lambda} \mathfrak{q}_{i(s)+i} = \prod_{j=0}^{r-1} \mathfrak{q}_{j+i}^{|\lambda|_j}, \quad |\lambda|_i = \#\{s \in \lambda \mid i(s) = i\}. \quad (7.5)$$

We also define

$$Z_{\lambda}^{(r)}(q) = \prod_{s \in \lambda \mid h(s) \in r\mathbb{Z}} \mathcal{S}(q_3^{\ell(s)+1} q_4^{-a(s)}) = \prod_{s \in \lambda \mid h(s) \in r\mathbb{Z}} \mathcal{S}(q_3^{-\ell(s)} q_4^{a(s)+1}), \quad (7.6)$$

with the hook length

$$h(s) = a(s) + \ell(s) + 1. \quad (7.7)$$

In this case, the rational factor (7.6) is identified with the fixed point contribution of the Nekrasov partition function of 5d  $\mathcal{N} = 1^* \text{U}(1)$  theory on  $\mathbb{C}^2/\mathbb{Z}_r$  with the  $\Omega$ -background parameters  $(q_3, q_4)$  [FMP04]. The  $qq$ -character (7.4) is then associated with the colored Fock module of the higher rank quantum toroidal algebra  $U_q(\widehat{\mathfrak{gl}}_r)$  [JM83, TU98, FJMM12].

Let us discuss truncation of the configuration. We tune the parameters as before (6.9). In this case, the rational function (7.6) becomes zero if  $(\mathbf{i}, \mathbf{j}) \in \lambda$  and also  $\mathbf{i} + \mathbf{j} - 1 \in r\mathbb{Z}$ . The condition (6.9) is rewritten in terms of the total mass  $\mu_{\text{tot}}$  instead of  $\mu$ ,

$$\mu_{\text{tot}}^{(\mathbf{i}+\mathbf{j}-1)/r} q^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad \mu_{\text{tot}}^{-(\mathbf{i}+\mathbf{j}-1)/r} q^{\mathbf{j}} = q_{1,2}. \quad (7.8)$$

Hence, we can impose the pit condition at  $(\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_{>0}^2$  only if  $\mathbf{i} + \mathbf{j} - 1 \in r\mathbb{Z}$  for  $\widehat{A}_{r-1}$  quiver.

**7.2. Generic weight.** We study the case with generic weight  $w = (w_i)_{i=0,\dots,r-1} \in \mathbb{Z}_{\geq 0}^r$ . We define the index set  $I_w = \{(i, \alpha) \mid i = 0, \dots, r-1, \alpha = 1, \dots, w_i\}$  with  $\mathbf{w} := |I_w| = \sum_{i=0}^{r-1} w_i$ . Then, the  $qq$ -character is given as a summation over the set of partitions  $\lambda = (\lambda_{i,\alpha})_{(i,\alpha) \in I_w}$ ,

$$\mathbb{T}_{w,\underline{x}} = \sum_{\lambda} \mathbf{q}_w^{\lambda} Z_{\lambda}^{(r)}(\underline{q}; \underline{x}) \prod_{(i,\alpha) \in I_w} \left[ \prod_{s \in \partial_+ \lambda_{i,\alpha}} Y_{i(s)+i, x_{i,\alpha}(s)} \prod_{s \in \partial_- \lambda_{i,\alpha}} Y_{i(s)+i, x_{i,\alpha}(s)}^{-1} q \right] \quad (7.9)$$

where we define

$$\mathbf{q}_w^{\lambda} = \prod_{(i,\alpha) \in I_w} \prod_{s \in \lambda_{i,\alpha}} \mathbf{q}_{i(s)+i}, \quad (7.10)$$

and the rational factor

$$Z_{\lambda}^{(r)}(\underline{q}; \underline{x}) = \prod_{(i,\alpha) \in I_w} Z_{\lambda_{i,\alpha}}^{(r)}(\underline{q}) \prod_{(i,\alpha) < (j,\beta)} \left[ \prod_{\substack{s \in \lambda_{i,\alpha} \\ h_{j,\beta}^{i,\alpha}(s) \in r\mathbb{Z}}} \mathfrak{S} \left( \frac{x_{j,\beta}}{x_{i,\alpha}} q_3^{\ell_{j,\beta}(s)+1} q_4^{-a_{i,\alpha}(s)} \right) \prod_{\substack{s \in \lambda_{j,\beta} \\ h_{i,\alpha}^{j,\beta}(s) \in r\mathbb{Z}}} \mathfrak{S} \left( \frac{x_{j,\beta}}{x_{i,\alpha}} q_3^{-\ell_{i,\alpha}(s)} q_4^{a_{j,\beta}(s)+1} \right) \right] \quad (7.11)$$

with

$$h_{j,\beta}^{i,\alpha}(s) = a_{i,\alpha}(s) + \ell_{j,\beta}(s) + 1 - (i - j). \quad (7.12)$$

The arm and leg are similarly defined as in (6.12). In this case, this factor is identified with the fixed point contribution of the Nekrasov partition function of 5d  $\mathcal{N} = 1^* \text{U}(\mathbf{w})$  theory on  $\mathbb{C}^2/\mathbb{Z}_r$ .

As in the previous case (§6.2), we have two possibilities to impose the restriction on the set of partitions. The first is the pit condition (7.8) obtained from the diagonal factor  $Z_{\lambda_{i,\alpha}}^{(r)}(\underline{q})$  in (7.11). The second is from the pair contribution associated with  $(i, \alpha)$  and  $(j, \beta)$ , which yields the irreducible modules constructed from the tensor product of the Fock modules for the quantum toroidal algebra  $U_q(\widehat{\mathfrak{gl}}_r)$  [FJMM12]. In this case, we consider the following resonance condition

$$\frac{x_{j,\beta}}{x_{i,\alpha}} q_3^{\mathbf{i}} q_4^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad \frac{x_{i,\alpha}}{x_{j,\beta}} q_3^{-\mathbf{i}+1} q_4^{\mathbf{j}} = q_{1,2}. \quad (7.13)$$

If  $\mathbf{i} + \mathbf{j} - 1 - (i - j) \in r\mathbb{Z}$ , we have the restriction on the set of partitions,

$$\lambda_{j,\beta,k}^{\mathbf{T}} - \lambda_{i,\alpha,k+\mathbf{j}-1}^{\mathbf{T}} \geq \mathbf{i} \quad (\mathbf{i} \in \mathbb{Z}_{\leq 0}, \mathbf{j} \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 1}). \quad (7.14)$$

As the resulting modules are irreducible [FJMM12], we have the following.

**Proposition 7.1.** Conjecture 1.1 holds for the tensor product module of the Fock modules associated with  $\widehat{A}_{r-1}$  quiver.

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