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HIGGSING qq -CHARACTER AND IRREDUCIBILITY

TARO KIMURA

Dedicated to Professor Hiraku Nakajima on his 60th anniversary

ABSTRACT. We show that the qq -character of the irreducible highest weight module for finite-type and affine quivers is obtained by Higgsing, specialization of the equivariant parameters of the associated framing space in the quiver variety.

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1. INTRODUCTION AND SUMMARY

The qq -character is a two parameter deformation of the ordinary character associated with a module constructed on a generic quiver, which has been originally introduced in the context of supersymmetric gauge theory through double quantization of Seiberg–Witten geometry by Nekrasov [Nek15]. The qq -character has a geometric realization using Nakajima’s quiver variety [Nak94, Nak98, Nak99], which could be interpreted as a natural generalization of the geometric construction of the q -character of Yangian and quantum affine algebra [Kni95, FR98]:¹ Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, where Γ_0 is a set of the nodes and Γ_1 is a set of the edges. We denote the quiver variety associated with quiver Γ by $\mathfrak{M}_{v,w}^\Gamma$ with the dimension vectors, $v = (v_i)_{i \in \Gamma_0}$ and $w = (w_i)_{i \in \Gamma_0}$, such that the vector spaces attached to the nodes and the corresponding framing spaces are given by $\mathbf{V}_i = \mathbb{C}^{v_i}$ and $\mathbf{W}_i = \mathbb{C}^{w_i}$ for $i \in \Gamma_0$. We also denote $\mathbf{V} = (\mathbf{V}_i)_{i \in \Gamma_0}$ and $\mathbf{W} = (\mathbf{W}_i)_{i \in \Gamma_0}$, and the automorphism groups by $G_v = \mathrm{GL}(\mathbf{V}) = \prod_{i \in \Gamma_0} \mathrm{GL}(\mathbf{V}_i)$ and $G_w = \mathrm{GL}(\mathbf{W}) = \prod_{i \in \Gamma_0} \mathrm{GL}(\mathbf{W}_i)$. Then, in addition to the quantum deformation parameters $q_{1,2} \in \mathbb{C}^\times$, the q - and qq -characters depend on the equivariant parameters of G_w denoted by $\underline{x} = (x_{i,\alpha})_{i \in \Gamma_0, \alpha=1, \dots, w_i}$, that we call the *weight parameters* for short. These weight parameters are identified with the zeros of the Drinfeld polynomials,² which play a central role in the construction of the finite dimensional representation of Yangian and quantum affine algebra [CP91, CP94]. It has been known that the finite dimensional module constructed from generic Drinfeld polynomials is not irreducible in general, and in order to obtain the irreducible highest weight module, zeros of the Drinfeld polynomial, namely the weight parameters, should obey the *q -segment condition*. The purpose of this paper is to elucidate the role of the weight parameters in the relation to the irreducibility of the qq -character.

1.1. Irreducible reduction of qq -character. Let us briefly demonstrate what happens for the q - and qq -characters via the specialization of the weight parameters. We denote the q - and qq -characters associated with the weight dimension vector w and the weight parameters \underline{x} by $\mathbb{T}_{w,\underline{x}}^{(q)}$ and $\mathbb{T}_{w,\underline{x}}^{(q_1,2)}$. Both of them are constructed from the Y -function assigned to each node denoted by $(Y_{i,x})_{i \in \Gamma_0, x \in \mathbb{C}^\times}$, which is associated with the fundamental weight. Then, the simplest example is the weight one module of A_1 quiver. Denoting $Y_{1,x} = Y_x$, we have the q - and qq -characters as follows:

$$\mathbb{T}_{1,x}^{(q)} = Y_x + Y_{xq}^{-1}, \quad \mathbb{T}_{1,x}^{(q_1,2)} = Y_x + Y_{xq_1q_2}^{-1}. \quad (1.1)$$

We provide the details on the construction of the qq -character in §2. Apparently, the qq -character is reduced to the q -character in the limits, $\lim_{q_1 \rightarrow 1} \mathbb{T}_{1,x}^{(q_1,2)} = \mathbb{T}_{1,x}^{(q_2)}$ and $\lim_{q_2 \rightarrow 1} \mathbb{T}_{1,x}^{(q_1,2)} = \mathbb{T}_{1,x}^{(q_1)}$.

¹There exists another two parameter deformation of the character, called the t -analog of q -character [Nak00, Nak01], which also has a natural geometric construction based on the (graded version of) quiver variety. The role of the t -parameter is different from the second deformation parameter of the qq -character, so that the relation between the qq -character and the t -analog of q -character is still not clear at this moment.

²Namely, the Drinfeld polynomial is a characteristic polynomial of G_w .

The next example is the weight two module. For generic weight parameters $\underline{x} = (x_1, x_2)$, the qq -character of weight two is given by

$$\mathbb{T}_{2,\underline{x}}^{(q_1,2)} = Y_{x_1} Y_{x_2} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \frac{Y_{x_2}}{Y_{x_1 q_1 q_2}} + \mathcal{S}\left(\frac{x_1}{x_2}\right) \frac{Y_{x_1}}{Y_{x_2 q_1 q_2}} + \frac{1}{Y_{x_1 q_1 q_2} Y_{x_2 q_1 q_2}}, \quad (1.2)$$

where we have a rational function, called the \mathcal{S} -function, in the coefficient. See §2.5 for the definition. This weight two character corresponds to the tensor product of the weight one characters, which is not further decomposed at this point.³ Then, we specialize the weight parameters. Noticing that the \mathcal{S} -function has zeros $\mathcal{S}(z) = 0$ at $z = q_{1,2}$, we put $\underline{x} = (x, xq_1)$ and obtain the reduced qq -character containing three monomials,

$$\mathbb{T}_{2,\underline{x}}^{(q_1,2)} \xrightarrow{\underline{x} \rightarrow (x, xq_1)} \tilde{\mathbb{T}}_{2,x}^{(q_1,2)} = Y_x Y_{xq_1} + \mathcal{S}(q_1^{-1}) \frac{Y_x}{Y_{xq_1^2 q_2}} + \frac{1}{Y_{xq_1 q_2} Y_{xq_1^2 q_2}}. \quad (1.3)$$

We call this the irreducible qq -character of weight two of A_1 quiver denoted by $\tilde{\mathbb{T}}_{2,x}^{(q_1,2)}$. This reduction mechanism to obtain the irreducible qq -character is the main subject of this paper.

Recalling $\lim_{q_1 \rightarrow 1} \mathcal{S}(q_1^{-1}) = 2$ and $\lim_{q_2 \rightarrow 1} \mathcal{S}(q_1^{-1}) = 1$, the irreducible qq -character $\tilde{\mathbb{T}}_{2,x}^{(q_1,2)}$ is further reduced to the q -characters as follows:

$$\lim_{q_1 \rightarrow 1} \tilde{\mathbb{T}}_{2,x}^{(q_1,2)} = Y_x^2 + 2 \frac{Y_x}{Y_{xq_2}} + Y_{xq_2}^{-2} = (Y_x + Y_{xq_2}^{-1})^2 = \left(\mathbb{T}_{1,x}^{(q_2)}\right)^2, \quad (1.4a)$$

$$\lim_{q_2 \rightarrow 1} \tilde{\mathbb{T}}_{2,x}^{(q_1,2)} = Y_x Y_{xq_1} + \frac{Y_x}{Y_{xq_1^2}} + \frac{1}{Y_{xq_1} Y_{xq_1^2}} = \mathbb{T}_{2,x}^{(q_1)}. \quad (1.4b)$$

This implies that we have the irreducible three-dimensional module in the limit $q_2 \rightarrow 1$, while we have the tensor product module, which is not irreducible, in the other limit $q_1 \rightarrow 1$. Such a reduction to the irreducible module is specific to the qq -character: The (tensor) product of the q -characters with the weight parameters (x, xq_1) is given by

$$\mathbb{T}_{1,x}^{(q)} \mathbb{T}_{1,xq}^{(q)} = Y_x Y_{xq} + \frac{Y_x}{Y_{xq^2}} + \frac{1}{Y_{xq} Y_{xq^2}} + 1 = \mathbb{T}_{2,x}^{(q)} + \mathbb{T}_{0,x}^{(q)}. \quad (1.5)$$

Although it is decomposed into the irreducible pieces, $\mathbb{T}_{2,x}^{(q)}$ and $\mathbb{T}_{0,x}^{(q)}$, one cannot eliminate either one of them.⁴ See §3 for more detailed analysis on A_1 quiver.

We remark that since the qq -character is symmetric under exchanging $q_1 \leftrightarrow q_2$ in this example, we may also consider another specialization $\underline{x} = (x, xq_2)$ to obtain the irreducible qq -character. Such a symmetry between q_1 and q_2 is expected to hold for the Kirillov–Reshetikhin (KR) module of simply-laced quivers [KR90], while it would be violated in other cases. See §4.1.3 for A_2 quiver (weight $w = (1, 1)$) and §5 for BC_2 quiver examples.

1.2. Higgs mechanism. Specialization of the weight parameters demonstrated above has a natural interpretation as the Higgs mechanism. Geometrically realizing the qq -character in the eight dimensional setup, called the gauge origami, the weight parameters are interpreted as the Coulomb moduli of the dual gauge theory defined in the transverse surface [Nek17]. From this point of view, the gauge symmetry is originally described by $U(\mathbf{W}) = \prod_{i \in \Gamma_0} U(w_i)$, which is a compact version of $G_w = GL(\mathbf{W})$. Specialization of the weight parameters implies freezing all the Coulomb moduli except for the center of mass factor. Therefore, we have the symmetry breaking, $U(\mathbf{W}) \rightarrow U(1)$. We remark that the Higgs mechanism freezes the non-Abelian part of the gauge group. The $U(1)$ factors of each $U(w_i)$ are fixed by the bifundamental mass parameters except for the overall center of mass factor in this context. It has been established that the qq -character has no singularity with respect to this center of mass factor, a.k.a., the compactness theorem [Nek16].

³From this point of view, the qq -character $\mathbb{T}_{2,\underline{x}}^{(q_1,2)}$ is irreducible, but not corresponding to the irreducible representation of the associated (quantum) algebra. In this paper, the notion of the irreducibility is used in the later sense.

⁴This relation among the q -characters $(\mathbb{T}_{w,x}^{(q)})_{w=0,1,2}$ is interpreted as the simplest example of the T-system [KNS10].

Such an interpretation of the parameter specialization as Higgsing has been discussed in the various contexts. It has been known that one can move on to the root of Higgs branch of the moduli space of supersymmetric vacua by tuning the Coulomb moduli [Dor98, DHT99]. In the context of topological string, the idea of Higgsing is used to describe the geometric transition [GV98]. For the primary example of the transition between the resolved and deformed conifolds, one may discuss the open string degrees of freedom on the Lagrangian submanifold on the deformed conifold side, which is given as the cotangent bundle T^*S^3 and its Lagrangian S^3 . From this point of view, the Higgsing condition on the Kähler parameter is interpreted as the flux quantization condition. Along this direction, we may apply a similar interpretation to the current situation. Geometrically, the irreducible part is interpreted as a subvariety of the whole quiver variety: The graded quiver variety is decomposed into direct product under the q -segment condition [Nak99, §4]. Hence, the Higgsed qq -character, which describes the irreducible module, is expected to correspond to this irreducible subvariety of the quiver variety. This is a geometric representation theoretical interpretation of the Higgsing process.

1.3. Summary and organization. In this paper, we have the following conjectures regarding the qq -character and its irreducibility through the Higgsing process.

Conjecture 1.1. Let Γ be a generic quiver and $w = (w_i)_{i \in \Gamma_0} \in \mathbb{Z}_{\geq 0}^{\text{rk } \Gamma}$ be a generic weight dimension vector. Then, there exists a canonical specialization of the weight parameters $\underline{x} = (x_{i,\alpha})$, such that the qq -character is reduced to that for the irreducible highest weight module parametrized by w .

This Conjecture is for generic modules and for generic quivers. See Propositions 6.1 and 7.1 for the tensor product modules of the Fock modules of the quantum toroidal algebras associated with \widehat{A}_0 and \widehat{A}_{r-1} quivers. We have a more specific Conjecture regarding the KR module of a finite-type quiver.

Conjecture 1.2 (Theorems 3.2, 3.3 for A_1 quiver). Let Γ be a finite-type quiver and fix $i \in \Gamma_0$. Considering the weight dimension vector having a single non-zero entry, $w_i \neq 0$ and $w_j = 0$ for $j \neq i$, the following holds:

- (1) The qq -character is reduced to that for the irreducible highest-weight module if the weight parameters $\underline{x} = (x_{i,\alpha})$ obey the q_1 -segment condition.
- (2) This irreducible qq -character is further reduced to the irreducible q_1 -character in the limit $q_2 \rightarrow 1$.
- (3) The irreducible qq -character is factorized into the product of the q_2 -characters in the other limit $q_1 \rightarrow 1$. Hence it is not irreducible in general.

The remaining part of this paper is organized as follows: In §2, as a preliminary, we provide an algorithm to construct the qq -character. In §3, we consider A_1 quiver as a primary example and provide a proof of Conjecture 1.2 in this case. In §4 and §5, we consider A_2 and BC_2 quivers to show several examples, which support Conjecture 1.2 for the KR modules and Conjecture 1.1 for other cases. In §6 and §7, we consider affine quivers \widehat{A}_0 and \widehat{A}_r and prove Conjecture 1.1 for the tensor product of the Fock modules of the corresponding quantum toroidal algebras.

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2. CONSTRUCTION OF qq -CHARACTER

In this Section, we provide an algorithm to calculate the qq -character. We apply the algebraic construction based on the iWeyl reflection. See [Nek15, KP22] for the alternative geometric construction based on the equivariant integral over the quiver variety.

2.1. Quiver. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with the set of nodes Γ_0 and the set of edges Γ_1 . We call $\text{rk } \Gamma = |\Gamma_0|$ the rank of quiver. We define a decorated quiver $\Gamma_d = (\Gamma, d)$ with the set of positive integers $d = (d_i)_{i \in \Gamma_0} \in \mathbb{Z}_{>1}^{\text{rk } \Gamma}$ assigned to each node, which would be identified with the relative root length of the corresponding root system. We denote $d_{ij} = \gcd(d_i, d_j)$.

2.2. Index functor. Let \mathbf{X} be the bundle for which we denote the set of Grothendieck roots (exponentials of the Chern roots) by $X = \{x_\alpha\}_{\alpha=1, \dots, \text{rk } \mathbf{X}}$. Then, the corresponding character is given by $\text{ch } \mathbf{X} = \sum_{\alpha=1}^{\text{rk } \mathbf{X}} x_\alpha = \sum_{x \in X} x$. We denote the wedge products of the bundle \mathbf{X} by

$$\wedge \mathbf{X} = \bigoplus_{k=0}^{\text{rk } \mathbf{X}} (-1)^k \wedge^k \mathbf{X}. \quad (2.1)$$

Then, we denote the index functor, which converts the additive class to the multiplicative class, by

$$\mathbb{I}[\mathbf{X}] = \prod_{\alpha=1}^{\text{rk } \mathbf{X}} (1 - x_\alpha) = \text{ch } \wedge \mathbf{X}. \quad (2.2)$$

We also use the twisted version. We denote the twisted sum of the wedge products by

$$\wedge_y \mathbf{X} = \bigoplus_{k=0}^{\text{rk } \mathbf{X}} (-y)^k \wedge^k \mathbf{X}. \quad (2.3)$$

For \mathbf{X}' a one-dimensional bundle with $\text{ch } \mathbf{X}' = x'$, we have

$$\mathbb{I}[\mathbf{X} \otimes \mathbf{X}'] = \prod_{\alpha=1}^{\text{rk } \mathbf{X}} (1 - x_\alpha x') = \text{ch } \wedge_{x'} \mathbf{X}, \quad (2.4)$$

which is the characteristic polynomial with respect to X .

2.3. Y-function. For a quiver Γ , we assign a formal bundle to each node of Γ , $\mathbf{Y} = \{\mathbf{Y}_i\}_{i \in \Gamma_0}$. We then define the Y-functions for the quiver Γ ,

$$Y_{i,X} = \mathbb{I}[\mathbf{X} \otimes \mathbf{Y}_i] = \prod_{\alpha=1}^{\text{rk } \mathbf{X}} Y_{i,x_\alpha}, \quad i \in \Gamma_0, \quad x_\alpha \in \mathbb{C}^\times. \quad (2.5)$$

Hence, each Y-function is the characteristic polynomial associated with \mathbf{Y}_i ,

$$Y_{i,x} = \text{ch } \wedge_x \mathbf{Y}_i. \quad (2.6)$$

In the context of quiver gauge theory, this Y-function is defined from the observable sheaf $\{\mathbf{Y}_i\}$, the pullback of the inclusion map of the universal sheaf defined on the instanton moduli space [NP12, NPS13]. In particular, if $\text{rk } \mathbf{X} = 1$, (logarithm of) the corresponding Y-function (hence, the bundle \mathbf{Y}_i) plays a role of the fundamental weight of the Lie algebra \mathfrak{g}_Γ associated with the quiver Γ denoted by $(\varpi_i)_{i \in \Gamma_0}$,

$$Y_{i,x} = e^{(\varpi_i, x)}. \quad (2.7)$$

In this context, the parameter x is interpreted as the spectral parameter to define the corresponding evaluation module of the quantum affine algebra $U_q(\widehat{\mathfrak{g}}_\Gamma)$. In the following, we will often use the notation

$$Y_{i,x;j,k} = Y_{i,xq_1^j q_2^k}. \quad (2.8)$$

2.4. A-function. We define a $q_{1,2}$ -deformation of the Cartan matrix associated with the quiver Γ ,

$$c_{ji} = (1 + q_1^{d_i} q_2) \delta_{ij} - \sum_{e:i \rightarrow j} \sum_{r=0}^{d_i/d_{ij}-1} \mu_e q_1^{rd_{ij}} - \sum_{e:j \rightarrow i} \sum_{r=0}^{d_i/d_{ij}-1} \mu_e^{-1} q_1^{(r+1)d_{ij}} q_2 \quad (i, j \in \Gamma_0). \quad (2.9)$$

We may trivialize the parameters $(\mu_e)_{e \in \Gamma_1}$ except for the cyclic quiver. Hence, we simply put $\mu_e = 1$ except in §6 and §7 where we discuss cyclic quivers. Then, we define the A-function,

$$\begin{aligned} \mathbf{A}_{i,X} &= \mathbb{I} \left[\sum_{j \in \Gamma_0} (\mathbf{X} \otimes \mathbf{Y}_j) c_{ji} \right] \\ &= \mathbf{Y}_{i,X} \mathbf{Y}_{i,X;d_i,1} \prod_{e:i \rightarrow j} \prod_{r=0}^{d_i/d_{ij}-1} \mathbf{Y}_{j,\mu_e X;rd_{ij},0}^{-1} \prod_{e:j \rightarrow i} \prod_{r=0}^{d_i/d_{ij}-1} \mathbf{Y}_{j,\mu_e^{-1} X;(r+1)d_{ij},1}^{-1}. \end{aligned} \quad (2.10)$$

Compared with the bundle \mathbf{Y} , the combination appearing in the A-function $\mathbf{A}_i = \sum_{j \in \Gamma_0} \mathbf{Y}_j c_{ji}$ plays a role of the simple root of the corresponding quiver,

$$\mathbf{A}_{i,x} = e^{(\alpha_i, x)}. \quad (2.11)$$

2.5. S-function. We define a rational function, that we call the S-function,⁵

$$\mathcal{S}(z) = \frac{(1 - z/q_1)(1 - z/q_2)}{(1 - z)(1 - z/q)} = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} (1 - q_1^{-n})(1 - q_2^{-n}) \right) \quad (2.12)$$

where the latter expression is available for $|z| < 1$, $|q_{1,2}| > 1$. The S-function obeys the inversion relation

$$\begin{aligned} \mathcal{S}(z) &= \mathcal{S}(z^{-1}q) \quad (z \neq 1, q) \\ &= \exp \left(\sum_{n=1}^{\infty} \frac{z^{-n}}{n} (1 - q_1^n)(1 - q_2^n) \right) \quad (|z| > 1, |q_{1,2}| < 1). \end{aligned} \quad (2.13)$$

We also define the higher-degree S-function,

$$\mathcal{S}_r(z) = \frac{(1 - z/q_1^r)(1 - z/q_2)}{(1 - z)(1 - z/q_1^r q_2)} = \prod_{s=0}^{r-1} \mathcal{S}(z q_1^{-s}), \quad (2.14)$$

which obeys the reflection relation

$$\mathcal{S}_r(z) = \mathcal{S}_r(z^{-1} q_1^r q_2), \quad z \neq 1, q_1^r q_2. \quad (2.15)$$

For generic z , the S-function becomes trivial in the classical limit,

$$\mathcal{S}_r(z) \xrightarrow{q_{1,2} \rightarrow 1} 1. \quad (2.16)$$

2.6. iWeyl reflection. We now describe an algorithm to calculate the qq -character:

- (1) Start with the highest-weight monomial associated with $w = (w_i)_{i \in \Gamma_0} \in \mathbb{Z}_{\geq 0}^{\text{rk } \Gamma}$ with the weight parameters $\underline{x} = (x_{i,\alpha})_{i \in \Gamma_0, \alpha=1, \dots, w_i}$,⁶

$$\mathbf{Y}_{w, \underline{x}} := \prod_{i \in \Gamma_0} \prod_{\alpha=1}^{w_i} \mathbf{Y}_{i, x_{i,\alpha}}. \quad (2.17)$$

⁵We change the convention $(q_1, q_2) \rightarrow (q_1^{-1}, q_2^{-1})$ compared with our previous publications [KP15, KP16, KP17].

⁶For $i \in \Gamma_0$, denote by \mathbf{W}_i a vector space with the Grothendieck roots $(x_{i,\alpha})_{\alpha=1, \dots, w_i}$, which would be identified with the framing space of the quiver variety $\mathfrak{M}_{v,w}^\Gamma$. Then, the Drinfeld polynomials $\{\mathbf{P}_{i,x}\}_{i \in \Gamma_0}$ associated with the finite-dimensional module of the quantum affine algebra $U_q(\widehat{\mathfrak{g}}_\Gamma)$ are given by the characteristic polynomials with respect to \mathbf{W}_i , $\mathbf{P}_{i,x} = \text{ch } \wedge_x \mathbf{W}_i$.

(2) Apply the iWeyl reflection for the Y-function to generate the monomials,

$$\text{iWeyl} : Y_{i,x} \mapsto Y_{i,x} A_{i,x}^{-1} = Y_{i,x;d_i,1}^{-1} \prod_{e:i \rightarrow j} \prod_{r=0}^{d_i/d_{i_j}-1} Y_{j,x;rd_{i_j},0} \prod_{e:j \rightarrow i} \prod_{r=0}^{d_i/d_{i_j}-1} Y_{j,x;(r+1)d_{i_j},1}. \quad (2.18)$$

The reflection is applied to the Y-function in the numerator of the monomials.

(a) If the monomial contains several Y-functions which belong to the same node $i \in \Gamma_0$, multiply the S-functions:⁷

$$Y_{i,x} \frac{\prod_{\alpha=1}^n Y_{i,x_\alpha}}{\prod_{\beta=1}^m Y_{i,x_\beta}} \mapsto \left(\frac{\prod_{\alpha=1}^n \mathcal{S}_{d_i}(x_\alpha/x)}{\prod_{\beta=1}^m \mathcal{S}_{d_i}(x_\beta/x)} \right) Y_{i,x} A_{i,x}^{-1} \frac{\prod_{\alpha=1}^n Y_{i,x_\alpha}}{\prod_{\beta=1}^m Y_{i,x_\beta}}. \quad (2.19)$$

(b) If the monomial contains the factor $(Y_{i,x})^n$, it should be considered as $\lim_{x_\alpha \rightarrow x} \prod_{\alpha=1}^n Y_{i,x_\alpha}$, which gives rise to derivatives of the Y-function [Nek15, KP15].

(3) If there is no Y-function in the numerator, no further reflection is applied (lowest-weight monomial). For finite-type quivers ($\det c^{[0]} > 0$, where $c^{[0]}$ is the classical limit of the Cartan matrix (2.9) obtained by replacing all the \mathbb{C}^\times variables with one), it is guaranteed that this process is terminated within finite time reflections, while the sequence of reflections are not terminated for generic quivers (affine and hyperbolic types; $\det c^{[0]} \leq 0$).

This is an algorithm to generate qq -characters geometrically defined as an equivariant integral over the corresponding quiver variety [Nek15, KP22]. The qq -character of the tensor product does not exhibit further decomposition for generic weight parameters. In order to obtain the qq -character of the irreducible module, we should specialize the weight parameters \underline{x} . In the following, we study several examples to demonstrate how to obtain the irreducible module in the context of the qq -character.

3. A_1 QUIVER

In this case, we denote $Y_{i,x} = Y_x$ for simplicity. The iWeyl reflection of A_1 quiver is given by⁸

$$\text{iWeyl} : Y_x \mapsto Y_{xq}^{-1}. \quad (3.1)$$

The fundamental qq -character is then given by⁹

$$\mathbb{T}_{1,x} = Y_x + Y_{xq}^{-1}. \quad (3.2)$$

We obtain the following expression from the direct computation based on the algorithm shown in §2.6.

Proposition 3.1. For A_1 quiver, the qq -character of weight $w = (w)$ with the parameters $\underline{x} = (x_1, \dots, x_w)$ is given by

$$\mathbb{T}_{w,\underline{x}} = Y_{x_1} \cdots Y_{x_w} + \cdots = \sum_{I \sqcup J = \{1, \dots, w\}} \prod_{i \in I, j \in J} \mathcal{S}\left(\frac{x_i}{x_j}\right) \prod_{i \in I} Y_{x_i} \prod_{j \in J} Y_{x_j}^{-1}, \quad (3.3)$$

which contains 2^w monomials in total.

⁷This S-factor is interpreted as the OPE factor in the vertex operator formalism of Y- and A-functions [KP15].

⁸We use the notation $q = q_1 q_2$.

⁹We do not write the $q_{1,2}$ dependence of the qq -character explicitly as far as no confusion.

3.1. Weight two. From the formula (3.3), the qq -character of weight two $w = (2)$ with the weight parameters $\underline{x} = (x_1, x_2)$ is given by

$$\mathsf{T}_{2,\underline{x}} = \mathsf{Y}_{x_1} \mathsf{Y}_{x_2} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \frac{\mathsf{Y}_{x_2}}{\mathsf{Y}_{x_1 q}} + \mathcal{S}\left(\frac{x_1}{x_2}\right) \frac{\mathsf{Y}_{x_1}}{\mathsf{Y}_{x_2 q}} + \frac{1}{\mathsf{Y}_{x_1 q} \mathsf{Y}_{x_2 q}}. \quad (3.4)$$

For generic weight parameters, this qq -character contains four monomials, corresponding to the tensor product of two two-dimensional modules of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ associated with A_1 quiver. Recalling that the \mathcal{S} -function has zeros at $z = q_1$ and q_2 , we may obtain a three-dimensional module, which corresponds to the irreducible module of weight two, by specializing the weight parameters,

$$\mathsf{T}_{2,\underline{x}} \longrightarrow \widetilde{\mathsf{T}}_{2,x}^{(q_1,2)} = \begin{cases} \mathsf{Y}_x \mathsf{Y}_{x;1,0} + \mathcal{S}(q_1^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;2,1}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;2,1}^{-1} & (x_1 = x, x_2/x_1 = q_1) \\ \mathsf{Y}_x \mathsf{Y}_{x;0,1} + \mathcal{S}(q_2^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;1,2}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;1,2}^{-1} & (x_1 = x, x_2/x_1 = q_2) \end{cases} \quad (3.5)$$

The highest weight monomials $(\mathsf{Y}_x \mathsf{Y}_{x q_m})_{m=1,2}$ are given by

$$\mathsf{Y}_x \mathsf{Y}_{x;1,0} = \mathsf{Y}_x \mathsf{Y}_{x q_1} = \mathbb{I}[\mathbf{W}_{2_1,x}^{(1)} \otimes \mathbf{Y}_1], \quad \mathsf{Y}_x \mathsf{Y}_{x;0,1} = \mathsf{Y}_x \mathsf{Y}_{x q_2} = \mathbb{I}[\mathbf{W}_{2_2,x}^{(1)} \otimes \mathbf{Y}_1], \quad (3.6)$$

where we define the vector spaces $(\mathbf{W}_{k_m,x}^{(i)})_{i \in \Gamma_0}$ having the character

$$\text{ch } \mathbf{W}_{k_m,x}^{(i)} = x + x q_m + \cdots + x q_m^{k_m-1} \quad (m = 1, 2). \quad (3.7)$$

Namely, this specialization (3.5) corresponds to the KR module of weight two with the shift parameter $q_{1,2}$. We also denote the corresponding qq -character by $\widetilde{\mathsf{T}}_{2,x}^{(q_1,2)} = \mathsf{T}[\mathbf{W}_{2_1,2,x}^{(1)}]$.

We remark a geometric interpretation of the truncation of the qq -character. Before the truncation, there are four monomials in the expression (3.4). The first and the last ones have no specific geometric structure as the corresponding quiver variety is just a point. The second and the third ones correspond to the fixed point contributions of the cotangent bundle $T^*\mathbb{P}^1$, which is the quiver variety of $(v, w) = (1, 2)$. Tuning the weight parameters, we can extract either of two monomials as the subvariety contribution.

Classical limit. In the classical limit, the \mathcal{S} -factor behaves as follows,

$$\mathcal{S}(q_1^{-1}) = (1 + q_1^{-1}) \frac{1 - q_1^{-1} q_2^{-1}}{1 - q_1^{-2} q_2^{-1}} \longrightarrow \begin{cases} 2 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \quad (3.8a)$$

$$\mathcal{S}(q_2^{-1}) = (1 + q_2^{-1}) \frac{1 - q_1^{-1} q_2^{-1}}{1 - q_1^{-1} q_2^{-2}} \longrightarrow \begin{cases} 1 & (q_1 \rightarrow 1) \\ 2 & (q_2 \rightarrow 1) \end{cases} \quad (3.8b)$$

Hence, the qq -character of weight two (3.5) is reduced as

$$\mathsf{Y}_x \mathsf{Y}_{x;1,0} + \mathcal{S}(q_1^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;2,1}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;2,1}^{-1} \longrightarrow \begin{cases} \mathsf{Y}_x^2 + 2 \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;0,1}} + \mathsf{Y}_{x;0,1}^{-2} = (\mathsf{Y}_x + \mathsf{Y}_{x;0,1}^{-1})^2 & (q_1 \rightarrow 1) \\ \mathsf{Y}_x \mathsf{Y}_{x;1,0} + \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;2,0}} + \mathsf{Y}_{x;1,0}^{-1} \mathsf{Y}_{x;2,0}^{-1} & (q_2 \rightarrow 1) \end{cases} \quad (3.9a)$$

$$\mathsf{Y}_x \mathsf{Y}_{x;0,1} + \mathcal{S}(q_2^{-1}) \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;1,2}} + \mathsf{Y}_{x;1,1}^{-1} \mathsf{Y}_{x;1,2}^{-1} \longrightarrow \begin{cases} \mathsf{Y}_x \mathsf{Y}_{x;0,1} + \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;0,2}} + \mathsf{Y}_{x;0,1}^{-1} \mathsf{Y}_{x;0,2}^{-1} & (q_1 \rightarrow 1) \\ \mathsf{Y}_x^2 + 2 \frac{\mathsf{Y}_x}{\mathsf{Y}_{x;1,0}} + \mathsf{Y}_{x;1,0}^{-2} = (\mathsf{Y}_x + \mathsf{Y}_{x;1,0}^{-1})^2 & (q_2 \rightarrow 1) \end{cases} \quad (3.9b)$$

Therefore, we obtain the q -character associated with the KR module $\mathbf{W}_{k_1,x}^{(1)}$ ($\mathbf{W}_{k_2,x}^{(1)}$) in the limit $q_2 \rightarrow 1$ ($q_1 \rightarrow 1$), while the degenerated q -character, corresponding to the tensor product of the two-dimensional modules, is obtained in the limit $q_1 \rightarrow 1$ ($q_2 \rightarrow 1$). Such a degenerated situation (folded q -character) has been recently discussed in the literature [CK18, FHR21, KO22].

3.2. Weight three. The qq -character of weight three with generic weight parameters $\underline{x} = (x_1, x_2, x_3)$ is given by

$$\begin{aligned} \mathbb{T}_{3, \underline{x}} &= Y_{x_1} Y_{x_2} Y_{x_3} \\ &+ \mathcal{S}\left(\frac{x_1}{x_3}\right) \mathcal{S}\left(\frac{x_2}{x_3}\right) \frac{Y_{x_1} Y_{x_2}}{Y_{x_3 q}} + \mathcal{S}\left(\frac{x_1}{x_2}\right) \mathcal{S}\left(\frac{x_3}{x_2}\right) \frac{Y_{x_1} Y_{x_3}}{Y_{x_2 q}} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \mathcal{S}\left(\frac{x_3}{x_1}\right) \frac{Y_{x_2} Y_{x_3}}{Y_{x_1 q}} \\ &+ \mathcal{S}\left(\frac{x_1}{x_2}\right) \mathcal{S}\left(\frac{x_1}{x_3}\right) \frac{Y_{x_1}}{Y_{x_2 q} Y_{x_3 q}} + \mathcal{S}\left(\frac{x_2}{x_1}\right) \mathcal{S}\left(\frac{x_2}{x_3}\right) \frac{Y_{x_2}}{Y_{x_1 q} Y_{x_3 q}} + \mathcal{S}\left(\frac{x_3}{x_1}\right) \mathcal{S}\left(\frac{x_3}{x_2}\right) \frac{Y_{x_3}}{Y_{x_1} Y_{x_2 q}} \\ &+ \frac{1}{Y_{x_1 q} Y_{x_2 q} Y_{x_3 q}}. \end{aligned} \quad (3.10)$$

Specializing the parameters $\underline{x} = (x, xq_1, xq_1^2)$, which corresponds to the KR module of weight three $\mathbf{W}_{3_1, x}^{(1)}$, we obtain

$$\mathbb{T}_{3, \underline{x}} \longrightarrow \mathbb{T}[\mathbf{W}_{3_1, x}^{(1)}] = Y_x Y_{x;1,0} Y_{x;2,0} + \mathcal{S}_2(q_1^{-1}) \left(\frac{Y_x Y_{x;1,0}}{Y_{x;3,1}} + \frac{Y_x}{Y_{x;2,1} Y_{x;3,1}} \right) + \frac{1}{Y_{x;1,1} Y_{x;2,1} Y_{x;3,1}}. \quad (3.11)$$

This corresponds to the irreducible four-dimensional module of $U_q(\widehat{\mathfrak{sl}}_2)$ associated with A_1 quiver. We obtain a similar expression by another specialization $\underline{x} = (x, xq_2, xq_2^2)$.

Classical limit. In the classical limit, the \mathcal{S} -factor is given by

$$\mathcal{S}_2(q_1^{-1}) = (1 + q_1^{-1} + q_1^{-2}) \frac{1 - q_1^{-1} q_2^{-1}}{1 - q_1^{-3} q_2^{-1}} \longrightarrow \begin{cases} 3 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \quad (3.12)$$

Hence, the qq -character is reduced to the degenerated q -character ($q_1 \rightarrow 1$) and the ordinary q -character of weight three ($q_2 \rightarrow 1$) as follows,

$$\mathbb{T}[\mathbf{W}_{3_1, x}^{(1)}] \longrightarrow \begin{cases} Y_x^3 + \frac{3Y_x^2}{Y_{x;0,1}} + \frac{3Y_x}{Y_{x;0,1}^2} + \frac{1}{Y_{x;0,1}^3} = (Y_x + Y_{x;0,1}^{-1})^3 & (q_1 \rightarrow 1) \\ Y_x Y_{x;1,0} Y_{x;2,0} + \frac{Y_x Y_{x;1,0}}{Y_{x;3,0}} + \frac{Y_x}{Y_{x;2,0} Y_{x;3,0}} + \frac{1}{Y_{x;1,0} Y_{x;2,0} Y_{x;3,0}} & (q_2 \rightarrow 1) \end{cases} \quad (3.13)$$

3.3. Generic weight. We consider the specialization of the qq -character of generic weight.

Theorem 3.2. Under the specialization $\underline{x} = (x, xq_1, \dots, xq_1^{w-1})$ corresponding to the degree- w KR module $\mathbf{W}_{w_1, x}^{(1)}$, the qq -character is given by $w + 1$ monomials,

$$\mathbb{T}[\mathbf{W}_{w_1, x}^{(1)}] = \sum_{v=0}^w \left(\prod_{\substack{i=1, \dots, w-v \\ j=1, \dots, v}} \mathcal{S}(q_1^{1-i-j}) \right) \prod_{i=1, \dots, w-v} Y_{x; i-1, 0} \prod_{j=w-v+1, \dots, w} Y_{x; j, 1}^{-1}. \quad (3.14)$$

Proof. Under the specialization $\underline{x} = (x, xq_1, \dots, xq_1^{w-1})$, due to the \mathcal{S} -factors, there are only the contributions of $I = \{1, \dots, w-v\}$, $J = \{w-v+1, \dots, w\}$ remaining in the qq -character formula (3.3), which gives rise to the expression above. \square

This proves the part (1) of Conjecture 1.2 for A_1 quiver. We remark that the \mathcal{S} -factor is identified with the fixed point contribution of the equivariant integral over the quiver variety of A_1 , identified with the cotangent bundle of the Grassmannian $T^*\text{Gr}(v, w)$.

Classical limit. Let us consider the classical limit of the qq -character. We have the following results in the limits $q_1 \rightarrow 1$ and $q_2 \rightarrow 1$, respectively.

Theorem 3.3. The classical limit of the qq -character gives rise to the degenerated q -character and the weight- w q -character,

$$\mathbb{T}[\mathbf{W}_{w_1, x}^{(1)}] \longrightarrow \begin{cases} \sum_{v=0}^w \binom{w}{v} (\mathbb{Y}_x)^{w-v} (\mathbb{Y}_{x;0,1})^{-v} = (\mathbb{Y}_x + \mathbb{Y}_{x;0,1}^{-1})^w & (q_1 \rightarrow 1) \\ \sum_{v=0}^w \prod_{i=1, \dots, w-v} \mathbb{Y}_{x; i-1, 0} \prod_{j=w-v+1, \dots, w} \mathbb{Y}_{x; j, 0}^{-1} & (q_2 \rightarrow 1) \end{cases}. \quad (3.15)$$

Proof. Applying the expression (2.12), we may rewrite the \mathcal{S} -factor in the qq -character (3.14) as follows,

$$\begin{aligned} \prod_{\substack{i=1, \dots, w-v \\ j=1, \dots, v}} \mathcal{S}(q_1^{1-i-j}) &= \frac{(q_1^{-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-w+v-1}; q_1^{-1})_\infty (q_1^{-v-1}; q_1^{-1})_\infty (q_1^{-w} q_2^{-1}; q_1^{-1})_\infty}{(q_1^{-1}; q_1^{-1})_\infty (q_1^{-w+v-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-v-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-w}; q_1^{-1})_\infty} \\ &= \frac{(q_1^{-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-w} q_2^{-1}; q_1^{-1})_\infty}{(q_1^{-w+v-1} q_2^{-1}; q_1^{-1})_\infty (q_1^{-v-1} q_2^{-1}; q_1^{-1})_\infty} \frac{\Gamma_{q_1^{-1}}(w) \Gamma_{q_1^{-1}}(1)}{\Gamma_{q_1^{-1}}(v) \Gamma_{q_1^{-1}}(w-v)} \longrightarrow \begin{cases} \binom{w}{v} & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases}, \end{aligned} \quad (3.16)$$

where we define the q -gamma function for $|q| < 1$,

$$\Gamma_q(x) = (1-q)^x \frac{(q; q)_\infty}{(q^x; q)_\infty} \xrightarrow{q \rightarrow 1} \Gamma(x) \quad (3.17)$$

with the q -shifted factorial (q -Pochhammer symbol)

$$(z; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n). \quad (3.18)$$

Then, we obtain the expression above. \square

This completes a proof of the parts (2) and (3) of Conjecture 1.2 for A_1 quiver.

4. A_2 QUIVER

In this case, there are two fundamental \mathbb{Y} -functions, and the corresponding iWeyl reflection is given as follows,

$$\text{iWeyl}: (\mathbb{Y}_{1,x}, \mathbb{Y}_{2,x}) \longmapsto \left(\frac{\mathbb{Y}_{2,x}}{\mathbb{Y}_{1,xq}}, \frac{\mathbb{Y}_{1,xq}}{\mathbb{Y}_{2,xq}} \right). \quad (4.1)$$

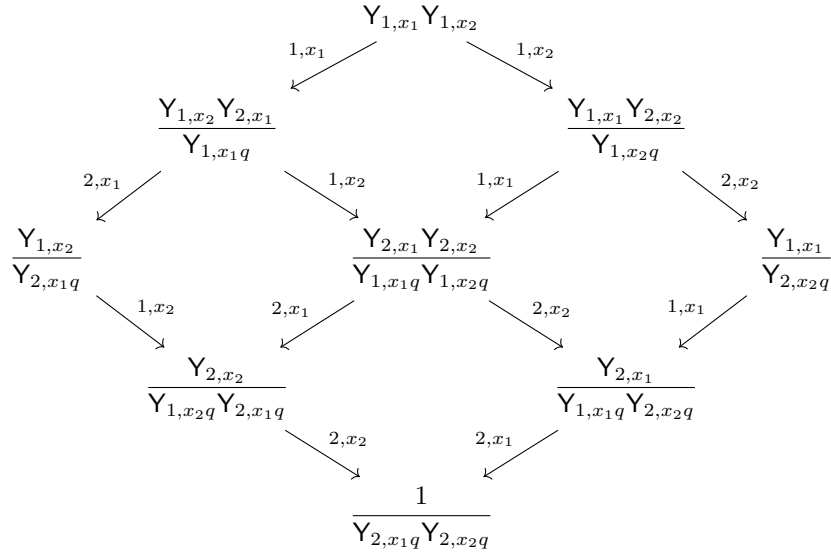
Then, the fundamental qq -characters are given by

$$\mathbb{T}_{1,x} = \mathbb{T}_{(1,0),x} = \mathbb{Y}_{1,x} + \frac{\mathbb{Y}_{2,x}}{\mathbb{Y}_{1,xq}} + \frac{1}{\mathbb{Y}_{2,xq}}, \quad (4.2a)$$

$$\mathbb{T}_{2,x} = \mathbb{T}_{(0,1),x} = \mathbb{Y}_{2,x} + \frac{\mathbb{Y}_{1,xq}}{\mathbb{Y}_{2,xq}} + \frac{1}{\mathbb{Y}_{1,xq^2}}, \quad (4.2b)$$

which correspond to two three-dimensional modules of $U_q(\widehat{\mathfrak{sl}}_3)$ associated with A_2 quiver.

4.1. Weight two. For A_2 quiver, we have three possible weight two situations, $w = (2, 0), (0, 2)$ and $w = (1, 1)$. The former two cases correspond to the KR modules, $\left(\mathbf{W}_{2_m, x}^{(i)} \right)_{i=1,2, m=1,2}$, while the last one is not.


 FIGURE 1. Hasse diagram for the iWeyl reflection flow of the weight $w = (2, 0)$ for A_2 quiver.

4.1.1. $w = (2, 0)$. We consider the qq -character associated with weight $w = (2, 0)$ with generic parameters $\underline{x} = (x_1, x_2)$,

$$\begin{aligned} \mathbb{T}_{(2,0); \underline{x}} &= Y_{1,x_1} Y_{1,x_2} + \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{1,x_1 q} Y_{1,x_2 q}} + \frac{1}{Y_{2,x_1 q} Y_{2,x_2 q}} \\ &+ \mathcal{S}\left(\frac{x_2}{x_1}\right) \left[\frac{Y_{1,x_2} Y_{2,x_1}}{Y_{1,x_1 q}} + \frac{Y_{1,x_2}}{Y_{2,x_1 q}} + \frac{Y_{2,x_2}}{Y_{1,x_2 q} Y_{2,x_1 q}} \right] \\ &+ \mathcal{S}\left(\frac{x_1}{x_2}\right) \left[\frac{Y_{1,x_1} Y_{2,x_2}}{Y_{1,x_2 q}} + \frac{Y_{1,x_1}}{Y_{2,x_2 q}} + \frac{Y_{2,x_1}}{Y_{1,x_1 q} Y_{2,x_2 q}} \right]. \end{aligned} \quad (4.3)$$

The flow of the iWeyl reflections is described by the Hasse diagram shown in Fig. 1. The qq -character (4.3) contains 9 monomials, corresponding to the tensor product of two three-dimensional fundamental modules of $w = (1, 0)$.

We specialize the weight parameters $\underline{x} = (x, xq_1)$, which corresponds to the weight-two KR module $\mathbf{W}_{2_1, x}^{(1)}$. Then, we have the following.

Proposition 4.1. The qq -character of six-dimensional module of weight two of the algebra $U_q(\widehat{\mathfrak{sl}}_3)$ is given by

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{2_1, x}^{(1)}] &= Y_{1,x} Y_{1,x;1,0} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,1} Y_{1,x;2,1}} + \frac{1}{Y_{2,x;1,1} Y_{2,x;2,1}} \\ &+ \mathcal{S}(q_1^{-1}) \left[\frac{Y_{1,x} Y_{2,x;1,0}}{Y_{1,x;2,1}} + \frac{Y_{1,x}}{Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;1,1} Y_{2,x;2,1}} \right]. \end{aligned} \quad (4.4)$$

On the other hand, the eliminated factors in the qq -character are interpreted as the contribution of the three-dimensional module,

$$\frac{Y_{1,x_2} Y_{2,x_1}}{Y_{1,x_1 q}} + \frac{Y_{1,x_2}}{Y_{2,x_1 q}} + \frac{Y_{2,x_2}}{Y_{1,x_2 q} Y_{2,x_1 q}} \xrightarrow{(x_1, x_2) \rightarrow (x, xq)} \mathbb{T}_{(0,1), x}. \quad (4.5)$$

We remark that this specialization of the parameter $(x_1, x_2) \rightarrow (x, xq)$ corresponds to the pole of the \mathcal{S} -function.¹⁰ This is consistent with the decomposition of the tensor product into the irreducible modules, $3 \otimes 3 = 6 \oplus 3$.

¹⁰As shown in [KP15], the qq -character provides the generating current of the q -deformation of W-algebra associated with quiver (quiver W-algebra). In this context, the pole term plays a crucial role in the quadratic relations for the generating

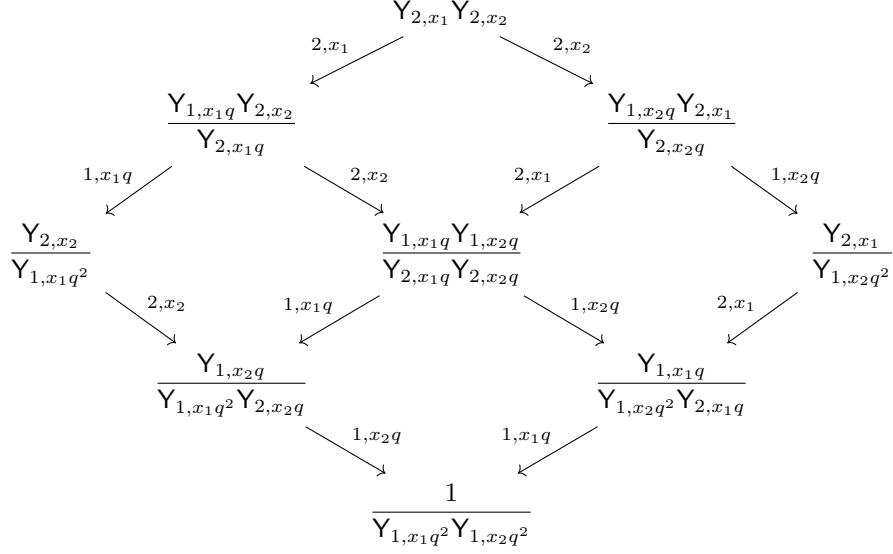


FIGURE 2. Hasse diagram for the iWeyl reflection flow of the weight $w = (0, 2)$ for A_2 quiver.

Classical limit. Recalling the behavior of the \mathcal{S} -factor in the classical limit (3.8), this qq -character is further reduced in the classical limit as follows,

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(1)}] &\xrightarrow{q_1 \rightarrow 1} Y_{1,x}^2 + \left(\frac{Y_{2,x}}{Y_{1,x;0,1}} \right)^2 + Y_{2,x;0,1}^{-2} + 2 \left(\frac{Y_{1,x} Y_{2,x}}{Y_{1,x;0,1}} + \frac{Y_{1,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x}}{Y_{1,x;0,1} Y_{2,x;0,1}} \right) \\ &= \left(Y_{1,x} + \frac{Y_{2,x}}{Y_{1,x;0,1}} + \frac{1}{Y_{2,x;0,1}} \right)^2, \end{aligned} \quad (4.6a)$$

$$\xrightarrow{q_2 \rightarrow 1} Y_{1,x} Y_{1,x;1,0} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,0} Y_{1,x;2,0}} + \frac{1}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{Y_{1,x} Y_{2,x;1,0}}{Y_{1,x;2,0}} + \frac{Y_{1,x}}{Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;1,0} Y_{2,x;2,0}}. \quad (4.6b)$$

In the limit $q_1 \rightarrow 1$, we obtain the degenerated tensor product of $w = (1, 0)$, while we obtain the q -character associated with the KR module $\mathbf{W}_{21,x}^{(1)}$ in the limit $q_2 \rightarrow 1$.

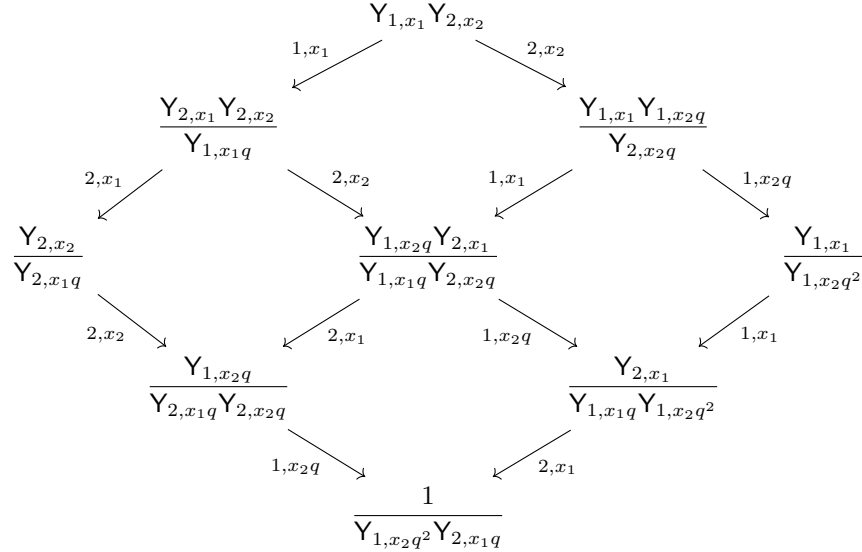
4.1.2. $w = (0, 2)$. The qq -character of weight $w = (0, 2)$ is similarly calculated as before. For the weight parameters $\underline{x} = (x_1, x_2)$, we obtain

$$\begin{aligned} \mathbb{T}_{(0,2),\underline{x}} &= Y_{2,x_1} Y_{2,x_2} + \frac{Y_{1,x_1 q} Y_{1,x_2 q}}{Y_{2,x_1 q} Y_{2,x_2 q}} + \frac{1}{Y_{1,x_1 q^2} Y_{1,x_2 q^2}} \\ &\quad + \mathcal{S} \left(\frac{x_2}{x_1} \right) \left[\frac{Y_{1,x_1 q} Y_{2,x_2}}{Y_{2,x_1 q}} + \frac{Y_{2,x_2}}{Y_{1,x_1 q^2}} + \frac{Y_{1,x_2 q}}{Y_{1,x_1 q^2} Y_{2,x_2 q}} \right] \\ &\quad + \mathcal{S} \left(\frac{x_1}{x_2} \right) \left[\frac{Y_{1,x_2 q} Y_{2,x_1}}{Y_{2,x_2 q}} + \frac{Y_{2,x_1}}{Y_{1,x_2 q^2}} + \frac{Y_{1,x_1 q}}{Y_{1,x_2 q^2} Y_{2,x_1 q}} \right]. \end{aligned} \quad (4.7)$$

The iWeyl reflection flow is described by the Hasse diagram in Fig. 2.

Specializing the parameters as $\underline{x} = (x, xq_1)$, we obtain the qq -character of the KR module $\mathbf{W}_{21,x}^{(2)}$ as follows.

currents. For the A -type quiver, in particular, the quadratic relations [Oda01, Koj21] are interpreted as a double quantization of Pieri's formula for the fundamental representations.


 FIGURE 3. Hasse diagram for the iWeyl reflection flow of the weight $w = (1, 1)$ for A_2 quiver.

Proposition 4.2.

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(2)}] &= Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,1} Y_{1,x;2,1}}{Y_{2,x;1,1} Y_{2,x;2,1}} + \frac{1}{Y_{1,x;2,2} Y_{1,x;3,2}} \\ &\quad + \mathcal{S}(q_1^{-1}) \left[\frac{Y_{1,x;2,1} Y_{2,x}}{Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;3,2}} + \frac{Y_{1,x;1,1}}{Y_{1,x;3,2} Y_{2,x;1,1}} \right]. \end{aligned} \quad (4.8)$$

We may obtain a similar expression from another specializations $\underline{x} = (x, xq_2)$.

Classical limit. The classical limit of this qq -character is given by

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(2)}] &\xrightarrow{q_1 \rightarrow 1} Y_{2,x}^2 + \left(\frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} \right)^2 + \frac{1}{Y_{1,x;0,2}^2} + 2 \left(\frac{Y_{1,x;0,1} Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x}}{Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,2} Y_{2,x;0,1}} \right) \\ &= \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right)^2, \end{aligned} \quad (4.9a)$$

$$\xrightarrow{q_2 \rightarrow 1} Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,0} Y_{1,x;2,0}}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{1}{Y_{1,x;2,0} Y_{1,x;3,0}} + \frac{Y_{1,x;2,0} Y_{2,x}}{Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;3,0}} + \frac{Y_{1,x;1,0}}{Y_{1,x;3,0} Y_{2,x;1,0}}. \quad (4.9b)$$

We obtain the degenerated tensor product of $w = (0, 1)$ in the limit $q_1 \rightarrow 1$, while we obtain the q -character associated with the KR module $\mathbf{W}_{21,x}^{(2)}$ in the limit $q_2 \rightarrow 1$.

4.1.3. $w = (1, 1)$. We consider the qq -character of weight $w = (1, 1)$,

$$\begin{aligned} \mathbb{T}_{(1,1); \underline{x}} &= Y_{1,x_1} Y_{2,x_2} + \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{1,x_1 q}} + \frac{Y_{1,x_1} Y_{1,x_2 q}}{Y_{2,x_2 q}} + \frac{Y_{1,x_2 q}}{Y_{2,x_1 q} Y_{2,x_2 q}} + \frac{Y_{2,x_1}}{Y_{1,x_1 q} Y_{1,x_2 q^2}} + \frac{1}{Y_{1,x_2 q^2} Y_{2,x_1 q}} \\ &\quad + \mathcal{S} \left(\frac{x_2}{x_1} \right) \frac{Y_{2,x_2}}{Y_{2,x_1 q}} + \mathcal{S} \left(\frac{x_1}{x_2} \right) \frac{Y_{1,x_2 q} Y_{2,x_1}}{Y_{1,x_1 q} Y_{2,x_2 q}} + \mathcal{S} \left(\frac{x_1}{x_2 q} \right) \frac{Y_{1,x_1}}{Y_{1,x_2 q^2}}, \end{aligned} \quad (4.10)$$

which is described by the Hasse diagram in Fig. 3. The qq -character (4.10) contains 9 monomials, interpreted as the tensor product of two three-dimensional fundamental modules, $w = (1, 0)$ and $w = (0, 1)$. In this case, we shall have the decomposition, $3 \otimes 3 = 8 \oplus 1$.

In order to obtain the irreducible module of the weight $w = (1, 1)$, we have three possible specializations: (a) $\underline{x} = (x, xq_1)$, (b) $\underline{x} = (xq_1, x)$, and (c) $\underline{x} = (xq_1^2q_2, x)$.

	(x_1, x_2)	$\mathcal{S}(x_2/x_1)$	$\mathcal{S}(x_1/x_2)$	$\mathcal{S}(x_1/x_2q)$	
(a)	(x, xq_1)	0	$\mathcal{S}(q_1^{-1})$	$\mathcal{S}(q_1^{-2}q_2^{-1})$	(4.11)
(b)	(xq_1, x)	$\mathcal{S}(q_1^{-1})$	0	$\mathcal{S}(q_2^{-1})$	
(c)	$(xq_1^2q_2, xq_1)$	$\mathcal{S}(q_1^{-2}q_2^{-1})$	$\mathcal{S}(q_1^{-1})$	0	

Proposition 4.3. We have the following specializations of the qq -character, corresponding to the 8-dimensional module of A_2 quiver,

$$\begin{aligned} T_{(1,1),\underline{x}} \xrightarrow{(a)} & Y_{1,x}Y_{2,x;1,0} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;1,1}} + \frac{Y_{1,x}Y_{1,x;2,1}}{Y_{2,x;2,1}} + \frac{Y_{1,x;2,1}}{Y_{2,x;1,1}Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;1,1}Y_{1,x;3,2}} + \frac{1}{Y_{1,x;3,2}Y_{2,x;1,1}} \\ & + \mathcal{S}(q_1^{-1})\frac{Y_{1,x;2,1}Y_{2,x}}{Y_{1,x;1,1}Y_{2,x;2,1}} + \mathcal{S}(q_1^{-2}q_2^{-1})\frac{Y_{1,x}}{Y_{1,x;3,2}}, \end{aligned} \quad (4.12a)$$

$$\begin{aligned} \xrightarrow{(b)} & Y_{1,x;1,0}Y_{2,x} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,1}} + \frac{Y_{1,x;1,0}Y_{1,x;1,1}}{Y_{2,x;1,1}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}Y_{2,x;2,1}} + \frac{Y_{2,x;1,0}}{Y_{1,x;2,1}Y_{1,x;2,2}} + \frac{1}{Y_{1,x;2,2}Y_{2,x;2,1}} \\ & + \mathcal{S}(q_1^{-1})\frac{Y_{2,x}}{Y_{2,x;2,1}} + \mathcal{S}(q_2^{-1})\frac{Y_{1,x;1,0}}{Y_{1,x;2,2}}, \end{aligned} \quad (4.12b)$$

$$\begin{aligned} \xrightarrow{(c)} & Y_{1,x;2,1}Y_{2,x} + \frac{Y_{2,x;2,1}Y_{2,x}}{Y_{1,x;3,2}} + \frac{Y_{1,x;1,1}Y_{1,x;2,1}}{Y_{2,x;1,1}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}Y_{2,x;3,2}} + \frac{Y_{2,x;2,1}}{Y_{1,x;2,2}Y_{1,x;3,2}} + \frac{1}{Y_{1,x;2,2}Y_{2,x;3,2}} \\ & + \mathcal{S}(q_1^{-2}q_2^{-1})\frac{Y_{2,x}}{Y_{2,x;3,2}} + \mathcal{S}(q_1^{-1})\frac{Y_{1,x;1,1}Y_{2,x;2,1}}{Y_{1,x;3,2}Y_{2,x;1,1}}. \end{aligned} \quad (4.12c)$$

Classical limit. In this case, we can discuss several classical limits of the qq -character. From the classical limit of the \mathcal{S} -factor (3.8) together with $\mathcal{S}(q_1^{-2}q_2^{-1}) \xrightarrow{q_1,2 \rightarrow 1} 1$, we have the following expressions:

$$(a) \quad \underline{x} = (x, xq_1)$$

$$\begin{aligned} (q_1 \rightarrow 1) : & Y_{1,x}Y_{2,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + \frac{Y_{1,x}Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}}{Y_{1,x;0,1}Y_{1,x;0,2}} + \frac{1}{Y_{1,x;0,2}Y_{2,x;0,1}} \\ & + 2\frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x}}{Y_{1,x;0,2}} = \left(Y_{1,x} + \frac{Y_{2,x}}{Y_{1,x;0,1}} + \frac{1}{Y_{2,x;0,1}} \right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right), \end{aligned} \quad (4.13a)$$

$$\begin{aligned} (q_2 \rightarrow 1) : & Y_{1,x}Y_{2,x;1,0} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;1,0}} + \frac{Y_{1,x}Y_{1,x;2,0}}{Y_{2,x;2,0}} + \frac{Y_{1,x;2,0}}{Y_{2,x;1,0}Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;1,0}Y_{1,x;3,0}} + \frac{1}{Y_{1,x;3,0}Y_{2,x;1,0}} \\ & + \frac{Y_{1,x;2,0}Y_{2,x}}{Y_{1,x;1,0}Y_{2,x;2,0}} + \frac{Y_{1,x}}{Y_{1,x;3,0}}. \end{aligned} \quad (4.13b)$$

(b) $\underline{x} = (xq_1, x)$

$$(q_1 \rightarrow 1) : \quad Y_{1,x}Y_{2,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + \frac{Y_{1,x}Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}}{Y_{1,x;0,1}Y_{1,x;0,2}} + \frac{1}{Y_{1,x;0,2}Y_{2,x;0,1}} \\ + 2\frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x}}{Y_{1,x;0,2}} = \left(Y_{1,x} + \frac{Y_{2,x}}{Y_{1,x;0,1}} + \frac{1}{Y_{2,x;0,1}} \right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right), \quad (4.14a)$$

$$(q_2 \rightarrow 1) : \quad Y_{1,x;1,0}Y_{2,x} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,0}} + \frac{Y_{1,x;1,0}^2}{Y_{2,x;1,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0}Y_{2,x;2,0}} + \frac{Y_{2,x;1,0}}{Y_{1,x;2,0}^2} + \frac{1}{Y_{1,x;2,0}Y_{2,x;2,0}} \\ + \frac{Y_{2,x}}{Y_{2,x;2,0}} + 2\frac{Y_{1,x;1,0}}{Y_{1,x;2,0}} = \left(Y_{1,x;1,0} + \frac{Y_{2,x;1,0}}{Y_{1,x;2,0}} + \frac{1}{Y_{2,x;2,0}} \right) \left(Y_{2,x} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0}} + \frac{1}{Y_{1,x;2,0}} \right). \quad (4.14b)$$

(c) $\underline{x} = (xq_1^2q_2, x)$

$$(q_1 \rightarrow 1) : \quad Y_{1,x;0,1}Y_{2,x} + \frac{Y_{2,x;0,1}Y_{2,x}}{Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}Y_{2,x;0,2}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}^2} + \frac{1}{Y_{1,x;0,2}Y_{2,x;0,2}} \\ + \frac{Y_{2,x}}{Y_{2,x;0,2}} + 2\frac{Y_{1,x;0,1}}{Y_{1,x;0,2}} = \left(Y_{1,x;0,1} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}} \right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{1}{Y_{1,x;0,2}} \right), \quad (4.15a)$$

$$(q_2 \rightarrow 1) : \quad Y_{1,x;2,0}Y_{2,x} + \frac{Y_{2,x;2,0}Y_{2,x}}{Y_{1,x;3,0}} + \frac{Y_{1,x;1,0}Y_{1,x;2,0}}{Y_{2,x;1,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0}Y_{2,x;3,0}} + \frac{Y_{2,x;2,0}}{Y_{1,x;2,0}Y_{1,x;3,0}} + \frac{1}{Y_{1,x;2,0}Y_{2,x;3,0}} \\ + \frac{Y_{2,x}}{Y_{2,x;3,0}} + \frac{Y_{1,x;1,0}Y_{2,x;2,0}}{Y_{1,x;3,0}Y_{2,x;1,0}}. \quad (4.15b)$$

We observe that (4.13b) and (4.15b) provide the q -character of the adjoint representation of the algebra $U_q(\widehat{\mathfrak{sl}}_3)$ (See, for example, [NN08, §3.1]). The remaining cases are the degenerate limits, where we obtain the product of two three-dimensional fundamental q -characters.

5. BC_2 QUIVER

We consider BC_2 quiver, which is the minimal rank non-simply-laced example. In this case, we have the decoration (root) parameters $d = (d_1, d_2) = (2, 1)$, and the iWeyl reflections for the Y -functions are given by

$$\text{iWeyl} : (Y_{1,x}, Y_{2,x}) \longmapsto \left(\frac{Y_{2,x}Y_{2,xq_1}}{Y_{1,xq_1^2q_2}}, \frac{Y_{1,xq_1q_2}}{Y_{2,xq_1q_2}} \right). \quad (5.1)$$

Then, the fundamental qq -characters are given as follows,

$$\mathbb{T}_{(1,0),x} = Y_{1,x} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,1}} + \mathcal{S}(q_1^{-1})\frac{Y_{2,x}}{Y_{2,x;2,1}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}Y_{2,x;2,1}} + \frac{1}{Y_{1,x;3,2}}, \quad (5.2a)$$

$$\mathbb{T}_{(0,1),x} = Y_{2,x} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1}} + \frac{Y_{2,x;2,1}}{Y_{1,x;3,2}} + \frac{1}{Y_{2,x;3,2}}. \quad (5.2b)$$

They correspond to the five and four dimensional modules associated with the quantum affine algebra $U_q(\widehat{\mathfrak{so}}_5) = U_q(\widehat{\mathfrak{sp}}_2)$ corresponding to BC_2 quiver.

In the following, we construct the weight two qq -characters based on the iWeyl reflections presented here, and examine the irreducibility of the corresponding modules.

5.1. $w = (2, 0)$. We consider the qq -character of weight $w = (2, 0)$,

$$\begin{aligned}
\mathbb{T}_{(2,0),\underline{x}} &= Y_{1,x_1} Y_{1,x_2} + \frac{Y_{2,x_1} Y_{2,x_1;1,0} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_1;2,1} Y_{1,x_2;2,1}} + \frac{Y_{1,x_1;1,1} Y_{1,x_2;1,1}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;1,1} Y_{2,x_2;2,1}} + \frac{1}{Y_{1,x_1;3,2} Y_{1,x_2;3,2}} \\
&+ \mathcal{S}_2\left(\frac{x_2}{x_1}\right) \left[\frac{Y_{1,x_2} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1}} + \frac{Y_{1,x_1;1,1} Y_{1,x_2}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1}} + \frac{Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_1;3,2} Y_{1,x_2;2,1}} + \frac{Y_{1,x_2;1,1}}{Y_{1,x_2;3,2} Y_{2,x_2;1,1} Y_{2,x_2;2,1}} \right. \\
&\quad \left. + \mathcal{S}(q_1^{-1}) \left(\frac{Y_{1,x_2} Y_{2,x_1}}{Y_{2,x_1;2,1}} + \frac{Y_{2,x_1} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_2;2,1} Y_{2,x_1;2,1}} + \frac{Y_{1,x_1;1,1} Y_{2,x_2}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;2,1}} + \frac{Y_{2,x_2}}{Y_{1,x_1;3,2} Y_{2,x_2;2,1}} \right) \right. \\
&\quad \left. + \mathcal{S}_2\left(\frac{x_2}{x_1} q_1^{-1} q_2^{-1}\right) \frac{Y_{1,x_2}}{Y_{1,x_1;3,2}} + \mathcal{S}_2\left(\frac{x_2}{x_1} q_1\right) \frac{Y_{1,x_1;1,1} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_2;1,1} Y_{2,x_1;1,1} Y_{2,x_1;2,1}} \right] \\
&+ \mathcal{S}_2\left(\frac{x_1}{x_2}\right) \left[\frac{Y_{1,x_1} Y_{2,x_2} Y_{2,x_2;1,0}}{Y_{1,x_2;2,1}} + \frac{Y_{1,x_2;1,1} Y_{1,x_1}}{Y_{2,x_2;1,1} Y_{2,x_2;2,1}} + \frac{Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_2;3,2} Y_{1,x_1;2,1}} + \frac{Y_{1,x_1;1,1}}{Y_{1,x_1;3,2} Y_{2,x_1;1,1} Y_{2,x_1;2,1}} \right. \\
&\quad \left. + \mathcal{S}(q_1^{-1}) \left(\frac{Y_{1,x_1} Y_{2,x_2}}{Y_{2,x_2;2,1}} + \frac{Y_{2,x_2} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1} Y_{2,x_2;2,1}} + \frac{Y_{1,x_2;1,1} Y_{2,x_1}}{Y_{2,x_2;1,1} Y_{2,x_2;2,1} Y_{2,x_1;2,1}} + \frac{Y_{2,x_1}}{Y_{1,x_2;3,2} Y_{2,x_1;2,1}} \right) \right. \\
&\quad \left. + \mathcal{S}_2\left(\frac{x_1}{x_2} q_1^{-1} q_2^{-1}\right) \frac{Y_{1,x_1}}{Y_{1,x_2;3,2}} + \mathcal{S}_2\left(\frac{x_1}{x_2} q_1\right) \frac{Y_{1,x_2;1,1} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;1,1} Y_{2,x_2;1,1} Y_{2,x_2;2,1}} \right] \\
&+ \mathcal{S}(q_1^{-1})^2 \mathcal{S}\left(\frac{x_2}{x_1} q_1^{-1}\right) \mathcal{S}\left(\frac{x_1}{x_2} q_1^{-1}\right) \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{2,x_1;2,1} Y_{2,x_2;2,1}}. \tag{5.3}
\end{aligned}$$

The corresponding Hasse diagram is shown in Fig. 4. This qq -character contains 25 monomials, corresponding to the tensor product of two modules of $w = (1, 0)$. We shall see the decomposition of this module into the irreducible ones, $5 \otimes 5 = 14 \oplus 10 \oplus 1$.

Recalling that $\mathcal{S}_2(z) = 0$ at $z = q_1^2, q_2$, we specialize the weight parameters as $\underline{x} = (x, xq_1^2)$. This is consistent with that the KR module $\mathbf{W}_{2_1,x}^{(1)}$ should be considered with q_1^2 -shift for the node $d_1 = 2$.

Proposition 5.1. The qq -character associated with the KR module $\mathbf{W}_{2_1,x}^{(1)}$ is given as follows,

$$\begin{aligned}
\mathbb{T}[\mathbf{W}_{2_1,x}^{(1)}] &= Y_{1,x} Y_{1,x;2,0} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;2,1} Y_{1,x;4,1}} + \frac{Y_{1,x;1,1} Y_{1,x;3,1}}{Y_{2,x;1,1} Y_{2,x;2,1} Y_{2,x;3,1} Y_{2,x;4,1}} + \frac{1}{Y_{1,x;3,2} Y_{1,x;5,2}} \\
&+ \mathcal{S}_2(q_1^{-2}) \left[\frac{Y_{1,x} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;4,1}} + \frac{Y_{1,x;3,1} Y_{1,x}}{Y_{2,x;3,1} Y_{2,x;4,1}} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;5,2} Y_{1,x;2,1}} + \frac{Y_{1,x;1,1}}{Y_{1,x;3,2} Y_{2,x;1,1} Y_{2,x;2,1}} \right. \\
&\quad \left. + \mathcal{S}(q_1^{-1}) \left(\frac{Y_{1,x} Y_{2,x;2,0}}{Y_{2,x;4,1}} + \frac{Y_{2,x;2,0} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1} Y_{2,x;4,1}} + \frac{Y_{1,x;3,1} Y_{2,x}}{Y_{2,x;3,1} Y_{2,x;4,1} Y_{2,x;2,1}} + \frac{Y_{2,x}}{Y_{1,x;5,2} Y_{2,x;2,1}} \right) \right. \\
&\quad \left. + \mathcal{S}_2(q_1^{-3} q_2^{-1}) \frac{Y_{1,x}}{Y_{1,x;5,2}} + \mathcal{S}_2(q_1^{-1}) \frac{Y_{1,x;3,1} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,1} Y_{2,x;3,1} Y_{2,x;4,1}} \right], \tag{5.4}
\end{aligned}$$

which corresponds to the 14-dimensional module of BC_2 quiver for the weight $w = (2, 0)$.

We remark that the eliminated terms due to the factor $\mathcal{S}_2(x_2/x_1)$ correspond to the 10-dimensional module of weight $w = (0, 2)$ discussed in §5.2.

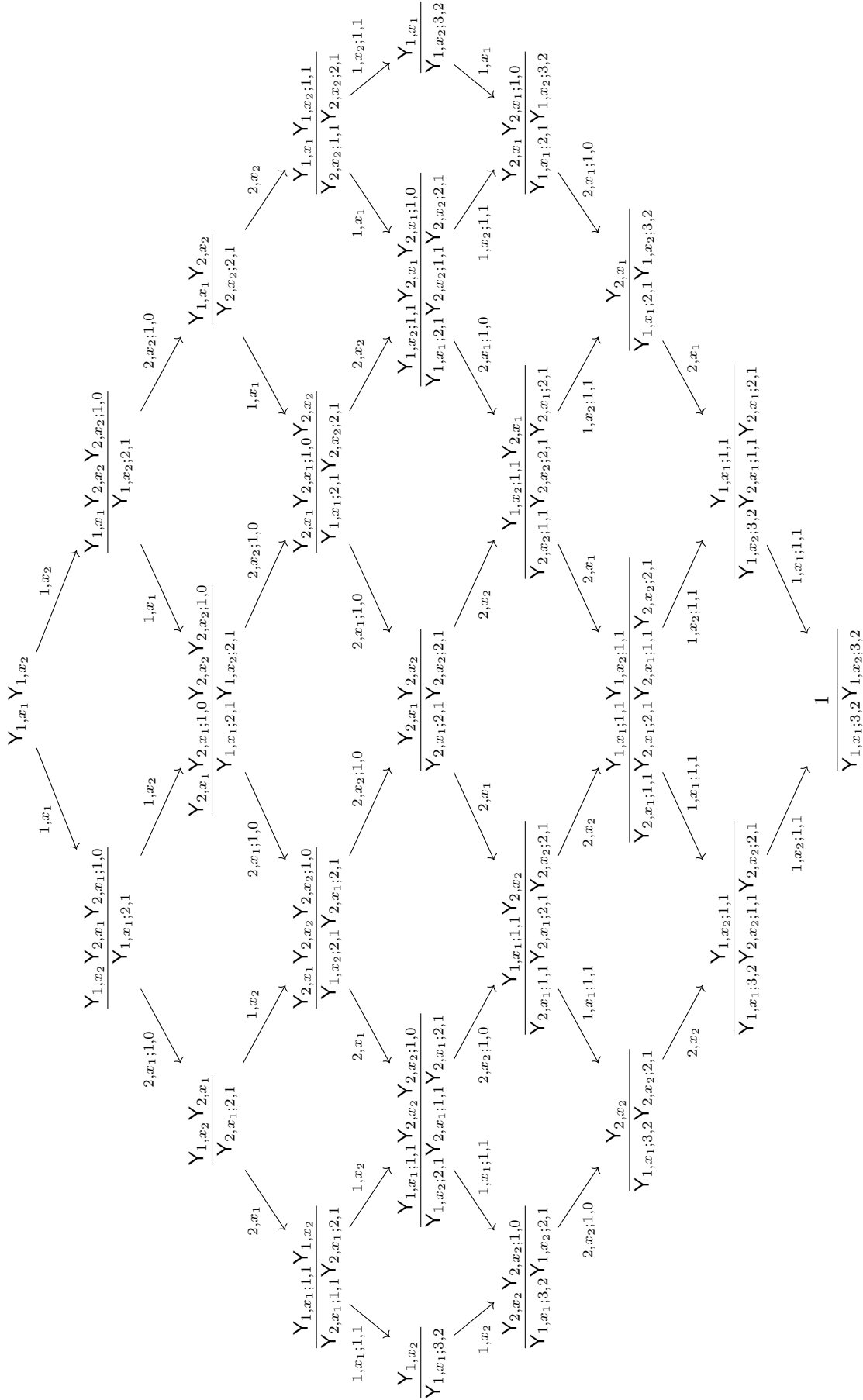


FIGURE 4. Hasse diagram for the iWeyl reflection flow of the weight $w = (2, 0)$ for BC_2 quiver

Classical limit. In addition to the classical limit of the \mathcal{S} -factors (3.8) and (3.12), we also have $\mathcal{S}(q_1^{-3}q_2^{-1}) \xrightarrow{q_{1,2} \rightarrow 1} 1$ and

$$\mathcal{S}_2(q_1^{-2}) = (1 + q_1^{-2}) \frac{1 - q_1^{-2}q_2^{-1}}{1 - q_1^{-4}q_2^{-1}} \longrightarrow \begin{cases} 2 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \quad (5.5)$$

Hence, the qq -character is reduced in the classical limit as follows,

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{2,1,x}^{(1)}] &\xrightarrow{q_1 \rightarrow 1} Y_{1,x}^2 + \frac{Y_{2,x}^4}{Y_{1,x;0,1}^2} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}^4} + \frac{1}{Y_{1,x;0,2}^2} \\ &+ 2 \left(\frac{Y_{1,x} Y_{2,x}^2}{Y_{1,x;0,1}} + \frac{Y_{1,x} Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}^2}{Y_{1,x;0,1} Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,2} Y_{2,x;0,1}} + \frac{Y_{1,x}}{Y_{1,x;0,2}} \right) \\ &+ 4 \left(\frac{Y_{1,x} Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x}^3}{Y_{1,x;0,1} Y_{2,x;0,1}} + \frac{Y_{1,x;0,1} Y_{2,x}}{Y_{2,x;0,1}^3} + \frac{Y_{2,x}}{Y_{1,x;0,2} Y_{2,x;0,1}} \right) + 6 \frac{Y_{2,x}^2}{Y_{2,x;0,1}^3} \\ &= \left(Y_{1,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + 2 \frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{1}{Y_{1,x;0,2}} \right)^2, \end{aligned} \quad (5.6a)$$

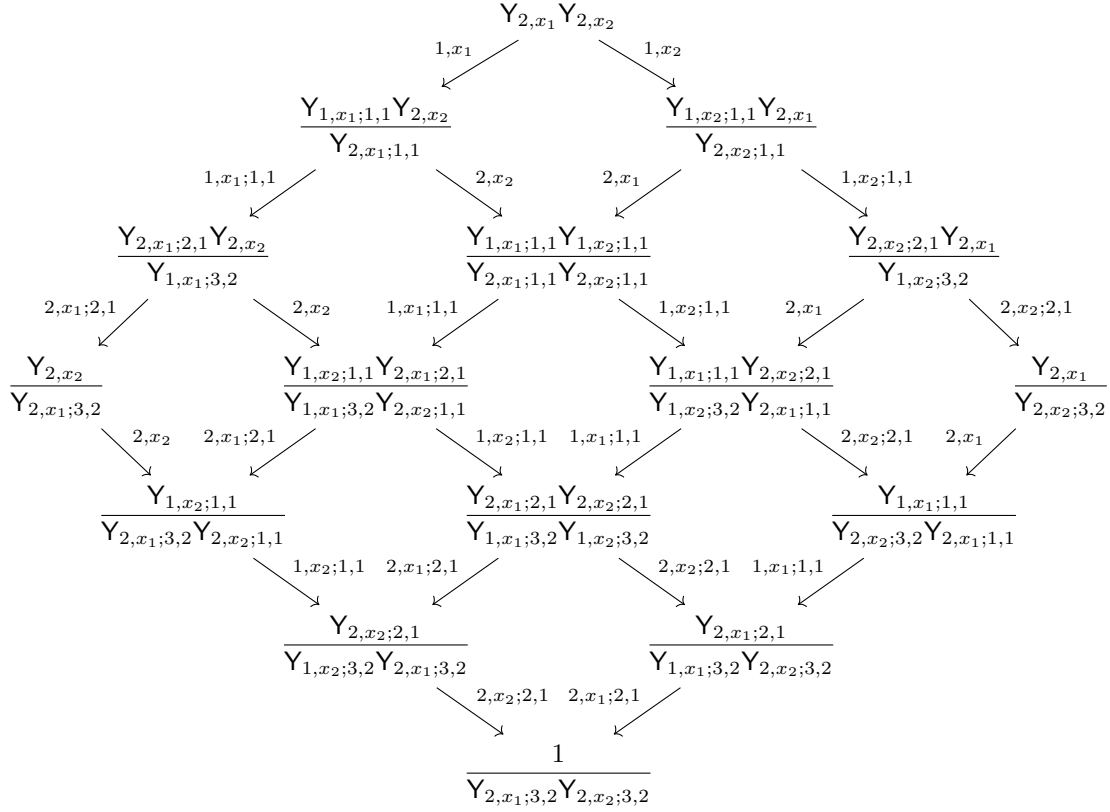
$$\begin{aligned} &\xrightarrow{q_2 \rightarrow 1} Y_{1,x} Y_{1,x;2,0} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;2,0} Y_{1,x;4,0}} + \frac{Y_{1,x;1,0} Y_{1,x;3,0}}{Y_{2,x;1,0} Y_{2,x;2,0} Y_{2,x;3,0} Y_{2,x;4,0}} + \frac{1}{Y_{1,x;3,0} Y_{1,x;5,0}} \\ &+ \frac{Y_{1,x} Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;4,0}} + \frac{Y_{1,x;3,0} Y_{1,x}}{Y_{2,x;3,0} Y_{2,x;4,0}} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;5,0} Y_{1,x;2,0}} + \frac{Y_{1,x;1,0}}{Y_{1,x;3,0} Y_{2,x;1,0} Y_{2,x;2,0}} \\ &+ \frac{Y_{1,x} Y_{2,x;2,0}}{Y_{2,x;4,0}} + \frac{Y_{2,x;2,0} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,0} Y_{2,x;4,0}} + \frac{Y_{1,x;3,0} Y_{2,x}}{Y_{2,x;3,0} Y_{2,x;4,0} Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{1,x;5,0} Y_{2,x;2,0}} \\ &+ \frac{Y_{1,x}}{Y_{1,x;5,0}} + \frac{Y_{1,x;3,0} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;1,0} Y_{2,x;3,0} Y_{2,x;4,0}}. \end{aligned} \quad (5.6b)$$

We obtain the q -character of the 14-dimensional module in the limit $q_2 \rightarrow 1$. On the other hand, the limit $q_1 \rightarrow 1$ provides the tensor product of the fundamental module $w = (1, 0)$. We remark that, in the limit $q_1 \rightarrow 1$, the fundamental qq -character (5.2a) is already degenerated due to the \mathcal{S} -factor. The resulting module is associated with the six-dimensional module of AD_3 quiver, which is the fractionalization of BC_2 quiver [KP22].

5.2. $w = (0, 2)$. The qq -character of weight $w = (0, 2)$ is given as follows,

$$\begin{aligned} \mathbb{T}_{(0,2),\underline{x}} &= Y_{2,x_1} Y_{2,x_2} + \frac{Y_{1,x_1;1,1} Y_{1,x_2;1,1}}{Y_{2,x_1;1,1} Y_{2,x_2;1,1}} + \frac{Y_{2,x_1;2,1} Y_{2,x_2;2,1}}{Y_{1,x_1;3,2} Y_{1,x_2;3,2}} + \frac{1}{Y_{2,x_1;3,2} Y_{2,x_2;3,2}} \\ &+ \mathcal{S} \left(\frac{x_2}{x_1} \right) \left[\frac{Y_{1,x_1;1,1} Y_{2,x_2}}{Y_{2,x_1;1,1}} + \frac{Y_{2,x_1;2,1} Y_{2,x_2}}{Y_{1,x_1;3,2}} + \mathcal{S} \left(\frac{x_2}{x_1} q_1^{-2} q_2^{-1} \right) \frac{Y_{2,x_2}}{Y_{2,x_1;3,2}} + \frac{Y_{1,x_2;1,1}}{Y_{2,x_1;3,2} Y_{2,x_2;1,1}} + \frac{Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;3,2}} \right] \\ &+ \mathcal{S} \left(\frac{x_1}{x_2} \right) \left[\frac{Y_{1,x_2;1,1} Y_{2,x_1}}{Y_{2,x_2;1,1}} + \frac{Y_{2,x_2;2,1} Y_{2,x_1}}{Y_{1,x_2;3,2}} + \mathcal{S} \left(\frac{x_1}{x_2} q_1^{-2} q_2^{-1} \right) \frac{Y_{2,x_1}}{Y_{2,x_2;3,2}} + \frac{Y_{1,x_1;1,1}}{Y_{2,x_2;3,2} Y_{2,x_1;1,1}} + \frac{Y_{2,x_1;2,1}}{Y_{1,x_1;3,2} Y_{2,x_2;3,2}} \right] \\ &+ \mathcal{S}_2 \left(\frac{x_2}{x_1} \right) \frac{Y_{1,x_2;1,1} Y_{2,x_1;2,1}}{Y_{1,x_1;3,2} Y_{2,x_2;1,1}} + \mathcal{S}_2 \left(\frac{x_1}{x_2} \right) \frac{Y_{1,x_1;1,1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;1,1}}, \end{aligned} \quad (5.7)$$

where the i Weyl reflection flow is described by the Hasse diagram in Fig. 5. There exist 16 monomials in the qq -character. We shall show the decomposition, $4 \otimes 4 = 10 \oplus 5 \oplus 1$, as follows.


 FIGURE 5. Hasse diagram for the iWeyl reflection flow of the weight $w = (0, 2)$ for BC_2 quiver.

Proposition 5.2. Specializing the weight parameters $\underline{x} = (x, xq_1)$, we obtain the qq -character for the KR module $\mathbf{W}_{2_1, \underline{x}}^{(2)}$ as follows,

$$\begin{aligned}
 \mathbb{T}[\mathbf{W}_{2_1, \underline{x}}^{(2)}] &= Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,1} Y_{1,x;2,1}}{Y_{2,x;1,1} Y_{2,x;2,1}} + \frac{Y_{2,x;2,1} Y_{2,x;3,1}}{Y_{1,x;3,2} Y_{1,x;4,2}} + \frac{1}{Y_{2,x;3,2} Y_{2,x;4,2}} \\
 &+ \mathcal{S}(q_1^{-1}) \left[\frac{Y_{1,x;2,1} Y_{2,x}}{Y_{2,x;2,1}} + \frac{Y_{2,x;3,1} Y_{2,x}}{Y_{1,x;4,2}} + \mathcal{S}(q_1^{-3} q_2^{-1}) \frac{Y_{2,x}}{Y_{2,x;4,2}} + \frac{Y_{1,x;1,1}}{Y_{2,x;4,2} Y_{2,x;1,1}} + \frac{Y_{2,x;2,1}}{Y_{1,x;3,2} Y_{2,x;4,2}} \right] \\
 &+ \mathcal{S}_2(q_1) \frac{Y_{1,x;2,1}}{Y_{1,x;3,2}} + \mathcal{S}_2(q_1^{-1}) \frac{Y_{1,x;1,1} Y_{2,x;3,1}}{Y_{1,x;4,2} Y_{2,x;1,1}}, \tag{5.8}
 \end{aligned}$$

which corresponds to the 10-dimensional module (adjoint representation of $\mathrm{SO}(5) = \mathrm{Sp}(2)$) with the affine extension.

Classical limit. In the classical limit, the \mathcal{S} -factors appearing in the qq -character $\mathbb{T}[\mathbf{W}_{2_1, \underline{x}}^{(2)}]$ are given by $\mathcal{S}(q_1^{-3} q_2^{-1}) \xrightarrow{q_{1,2} \rightarrow 1} 1$ and

$$\mathcal{S}_2(q_1) = -q_1^{-1} \frac{1 - q_1 q_2^{-1}}{1 - q_1^{-1} q_2^{-1}} \longrightarrow \begin{cases} -1 & (q_1 \rightarrow 1) \\ 1 & (q_2 \rightarrow 1) \end{cases} \tag{5.9}$$

Therefore, we obtain the following reductions,

$$\begin{aligned} \mathbb{T}[\mathbf{W}_{21,x}^{(2)}] &\xrightarrow{q_1 \rightarrow 1} Y_{2,x}^2 + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}^2} + \frac{Y_{2,x;0,1}^2}{Y_{1,x;0,2}^2} + \frac{1}{Y_{2,x;0,2}^2} \\ &\quad + 2 \left(\frac{Y_{1,x;0,1} Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1} Y_{2,x}}{Y_{1,x;0,2}} + \frac{Y_{2,x}}{Y_{2,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,2} Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2} Y_{2,x;0,2}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,2}} \right) \\ &= \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}} \right)^2, \end{aligned} \quad (5.10a)$$

$$\begin{aligned} &\xrightarrow{q_2 \rightarrow 1} Y_{2,x} Y_{2,x;1,0} + \frac{Y_{1,x;1,0} Y_{1,x;2,0}}{Y_{2,x;1,0} Y_{2,x;2,0}} + \frac{Y_{2,x;2,0} Y_{2,x;3,0}}{Y_{1,x;3,0} Y_{1,x;4,0}} + \frac{1}{Y_{2,x;3,0} Y_{2,x;4,0}} + \frac{Y_{1,x;2,0} Y_{2,x}}{Y_{2,x;2,0}} \\ &\quad + \frac{Y_{2,x;3,0} Y_{2,x}}{Y_{1,x;4,0}} + \frac{Y_{2,x}}{Y_{2,x;4,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;4,0} Y_{2,x;1,0}} + \frac{Y_{2,x;2,0}}{Y_{1,x;3,0} Y_{2,x;4,0}} + \frac{Y_{1,x;2,0}}{Y_{1,x;3,0}} + \frac{Y_{1,x;1,0} Y_{2,x;3,0}}{Y_{1,x;4,0} Y_{2,x;1,0}}. \end{aligned} \quad (5.10b)$$

The limit $q_1 \rightarrow 1$ is the degenerate limit, which provides the tensor product of two four-dimensional modules, while we obtain the q -character of the 10-dimensional module with the affine extension in the limit $q_2 \rightarrow 1$.

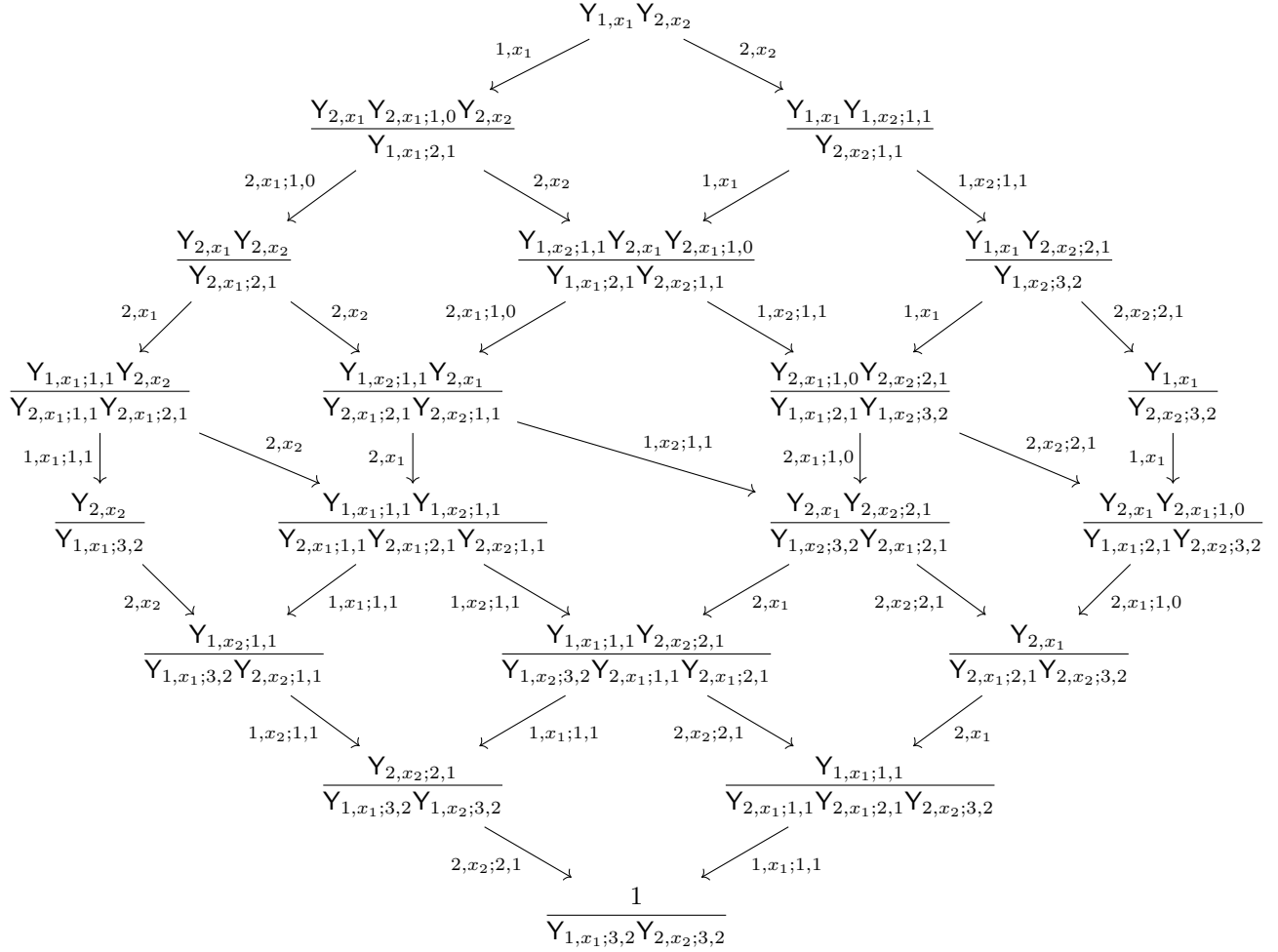
5.3. $w = (1, 1)$. We consider the qq -character of weight $w = (1, 1)$,

$$\begin{aligned} \mathbb{T}_{(1,1),\underline{x}} &= Y_{1,x_1} Y_{2,x_2} + \frac{Y_{2,x_1} Y_{2,x_1;1,0} Y_{2,x_2}}{Y_{1,x_1;2,1}} + \frac{Y_{1,x_1} Y_{1,x_2;1,1}}{Y_{2,x_2;1,1}} + \frac{Y_{2,x_1} Y_{2,x_1;1,0} Y_{2,x_2;2,1}}{Y_{1,x_1;2,1} Y_{1,x_2;3,2}} \\ &\quad + \frac{Y_{1,x_1;1,1} Y_{1,x_2;1,1}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;1,1}} + \frac{Y_{2,x_2;2,1}}{Y_{1,x_1;3,2} Y_{1,x_2;3,2}} + \frac{Y_{1,x_1;1,1}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1} Y_{2,x_2;3,2}} + \frac{1}{Y_{1,x_1;3,2} Y_{2,x_2;3,2}} \\ &\quad + \mathcal{S}_2 \left(\frac{x_1}{x_2} q_1^{-1} q_2^{-1} \right) \left[\frac{Y_{1,x_1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2}} + \frac{Y_{1,x_1}}{Y_{2,x_2;3,2}} + \frac{Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1} Y_{2,x_2;3,2}} \right] + \mathcal{S}_2 \left(\frac{x_2}{x_1} q_1 q_2 \right) \frac{Y_{1,x_2;1,1} Y_{2,x_1} Y_{2,x_1;1,0}}{Y_{1,x_1;2,1} Y_{2,x_2;1,1}} \\ &\quad + \mathcal{S}_2 \left(\frac{x_2}{x_1} \right) \left[\frac{Y_{1,x_1;1,1} Y_{2,x_2}}{Y_{2,x_1;1,1} Y_{2,x_1;2,1}} + \frac{Y_{2,x_2}}{Y_{1,x_1;3,2}} + \frac{Y_{1,x_2;1,1}}{Y_{1,x_1;3,2} Y_{2,x_2;1,1}} \right] + \mathcal{S}_2 \left(\frac{x_1}{x_2} \right) \frac{Y_{1,x_1;1,1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;1,1} Y_{2,x_1;2,1}} \\ &\quad + \mathcal{S}(q_1^{-1}) \left[\mathcal{S} \left(\frac{x_2}{x_1} q_1^{-1} \right) \frac{Y_{2,x_1} Y_{2,x_2}}{Y_{2,x_1;2,1}} + \mathcal{S} \left(\frac{x_1}{x_2} \right) \left(\frac{Y_{1,x_2;1,1} Y_{2,x_1}}{Y_{2,x_1;2,1} Y_{2,x_2;1,1}} + \frac{Y_{2,x_1} Y_{2,x_2;2,1}}{Y_{1,x_2;3,2} Y_{2,x_1;2,1}} \right) + \mathcal{S} \left(\frac{x_1}{x_2} q_1^{-2} q_2^{-1} \right) \frac{Y_{2,x_1}}{Y_{2,x_1;2,1} Y_{2,x_2;3,2}} \right], \end{aligned} \quad (5.11)$$

where the iWeyl reflection flow is described in Fig. 6. This qq -character shows the decomposition of the tensor product, $4 \otimes 5 = 16 \oplus 4$.

Proposition 5.3. We specialize the weight parameters as (a) $\underline{x} = (xq_1^3 q_2, x)$ and (b) $\underline{x} = (x, xq_1^2)$ to obtain the qq -character for the 14-dimensional module:

$$\begin{aligned} \mathbb{T}_{(1,1),\underline{x}} &\xrightarrow{(a)} Y_{1,x;3,1} Y_{2,x} + \frac{Y_{2,x} Y_{2,x;3,1} Y_{2,x;4,1}}{Y_{1,x;5,2}} + \frac{Y_{1,x;1,1} Y_{1,x;3,1}}{Y_{2,x;1,1}} + \frac{Y_{2,x;2,1} Y_{2,x;3,1} Y_{2,x;4,1}}{Y_{1,x;3,2} Y_{1,x;5,2}} \\ &\quad + \frac{Y_{1,x;1,1} Y_{1,x;4,2}}{Y_{2,x;1,1} Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{Y_{2,x;2,1}}{Y_{1,x;3,2} Y_{1,x;6,3}} + \frac{Y_{1,x;4,2}}{Y_{2,x;3,2} Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{1}{Y_{1,x;6,3} Y_{2,x;3,2}} \\ &\quad + \mathcal{S}_2(q_1^{-3} q_2^{-1}) \left(\frac{Y_{1,x;4,2} Y_{2,x}}{Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{Y_{2,x}}{Y_{1,x;6,3}} + \frac{Y_{1,x;1,1}}{Y_{1,x;6,3} Y_{2,x;1,1}} \right) + \mathcal{S}_2(q_1^{-2}) \frac{Y_{1,x;1,1} Y_{2,x;3,1} Y_{2,x;4,1}}{Y_{1,x;5,2} Y_{2,x;1,1}} \\ &\quad + \mathcal{S}_2(q_1^{-1}) \left(\frac{Y_{1,x;4,2} Y_{2,x;2,1}}{Y_{1,x;3,2} Y_{2,x;4,2} Y_{2,x;5,2}} + \frac{Y_{1,x;1,1} Y_{2,x;3,1}}{Y_{2,x;1,1} Y_{2,x;5,2}} + \frac{Y_{2,x;2,1} Y_{2,x;3,1}}{Y_{1,x;3,2} Y_{2,x;5,2}} \right) + \mathcal{S}(q_1^{-1}) \mathcal{S}(q_1^{-4} q_2^{-1}) \frac{Y_{2,x} Y_{2,x;3,1}}{Y_{2,x;5,2}}, \end{aligned} \quad (5.12a)$$


 FIGURE 6. Hasse diagram for the iWeyl reflection flow of the weight $w = (1, 1)$ for BC_2 quiver.

$$\begin{aligned}
 T_{(1,1),x} \stackrel{(b)}{\rightarrow} & Y_{1,x} Y_{2,x;2,0} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;2,0}}{Y_{1,x;2,1}} + \frac{Y_{1,x} Y_{1,x;3,1}}{Y_{2,x;3,1}} + \frac{Y_{2,x} Y_{2,x;1,0} Y_{2,x;4,1}}{Y_{1,x;2,1} Y_{1,x;5,2}} \\
 & + \frac{Y_{1,x;1,1} Y_{1,x;3,1}}{Y_{2,x;1,1} Y_{2,x;2,1} Y_{2,x;3,1}} + \frac{Y_{2,x;4,1}}{Y_{1,x;3,2} Y_{1,x;5,2}} + \frac{Y_{1,x;1,1}}{Y_{2,x;1,1} Y_{2,x;2,1} Y_{2,x;5,2}} + \frac{1}{Y_{1,x;3,2} Y_{2,x;5,2}} \\
 & + S_2(q_1^{-3} q_2^{-1}) \left(\frac{Y_{1,x} Y_{2,x;4,1}}{Y_{1,x;5,2}} + \frac{Y_{1,x}}{Y_{2,x;5,2}} + \frac{Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1} Y_{2,x;5,2}} \right) + S_2(q_1^{-2}) \frac{Y_{1,x;1,1} Y_{2,x;4,1}}{Y_{1,x;5,2} Y_{2,x;1,1} Y_{2,x;2,1}} \\
 & + S_2(q_1^{-1}) \left(\frac{Y_{1,x;3,1} Y_{2,x} Y_{2,x;1,0}}{Y_{1,x;2,1} Y_{2,x;3,1}} + \frac{Y_{1,x;3,1} Y_{2,x}}{Y_{2,x;2,1} Y_{2,x;3,1}} + \frac{Y_{2,x} Y_{2,x;4,1}}{Y_{1,x;5,2} Y_{2,x;2,1}} \right) + S(q_1^{-1}) S(q_1^{-4} q_2^{-1}) \frac{Y_{2,x}}{Y_{2,x;2,1} Y_{2,x;5,2}}.
 \end{aligned} \tag{5.12b}$$

Classical limit. We then study the classical limit of the qq -character for the 14-dimensional module. In the limit $q_2 \rightarrow 1$, we obtain the corresponding q -character, while we observe the tensor product structure in the degenerate limit $q_1 \rightarrow 1$.

(a) $\underline{x} = (xq_1^3q_2, x)$

$$\begin{aligned}
(q_1 \rightarrow 1) : & \quad Y_{1,x;0,1}Y_{2,x} + \frac{Y_{2,x}Y_{2,x;0,1}^2}{Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}^3}{Y_{1,x;0,2}Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}Y_{1,x;0,2}}{Y_{2,x;0,1}Y_{2,x;0,2}^2} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}Y_{1,x;0,3}} \\
& + \frac{Y_{1,x;0,2}}{Y_{2,x;0,2}^3} + \frac{1}{Y_{1,x;0,3}Y_{2,x;0,2}} + \frac{Y_{1,x;0,2}Y_{2,x}}{Y_{2,x;0,2}^2} + \frac{Y_{2,x}}{Y_{1,x;0,3}} + \frac{Y_{1,x;0,1}}{Y_{1,x;0,3}Y_{2,x;0,1}} \\
& + 2\left(\frac{Y_{1,x;0,1}Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{Y_{2,x}Y_{2,x;0,1}}{Y_{2,x;0,2}}\right) + 3\left(\frac{Y_{1,x;0,2}Y_{2,x;0,1}}{Y_{1,x;0,2}Y_{2,x;0,2}^2} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,2}} + \frac{Y_{2,x;0,1}^2}{Y_{1,x;0,2}Y_{2,x;0,2}}\right) \\
& = \left(Y_{1,x;0,1} + \frac{Y_{2,x;0,1}^2}{Y_{1,x;0,2}} + 2\frac{Y_{2,x;0,1}}{Y_{2,x;0,2}} + \frac{Y_{1,x;0,2}}{Y_{2,x;0,2}^2} + \frac{1}{Y_{1,x;0,3}}\right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}}\right),
\end{aligned} \tag{5.13a}$$

$$\begin{aligned}
(q_2 \rightarrow 1) : & \quad Y_{1,x;3,0}Y_{2,x} + \frac{Y_{2,x;3,0}Y_{2,x;4,0}Y_{2,x}}{Y_{1,x;5,0}} + \frac{Y_{1,x;3,0}Y_{1,x;1,0}}{Y_{2,x;1,0}} + \frac{Y_{2,x;3,0}Y_{2,x;4,0}Y_{2,x;2,0}}{Y_{1,x;5,0}Y_{1,x;3,0}} \\
& + \frac{Y_{1,x;4,0}Y_{1,x;1,0}}{Y_{2,x;4,0}Y_{2,x;5,0}Y_{2,x;1,0}} + \frac{Y_{2,x;2,0}}{Y_{1,x;6,0}Y_{1,x;3,0}} + \frac{Y_{1,x;4,0}}{Y_{2,x;4,0}Y_{2,x;5,0}Y_{2,x;3,0}} + \frac{1}{Y_{1,x;6,0}Y_{2,x;3,0}} \\
& + \frac{Y_{1,x;4,0}Y_{2,x}}{Y_{2,x;4,0}Y_{2,x;5,0}} + \frac{Y_{2,x}}{Y_{1,x;5,0}} + \frac{Y_{1,x;1,0}}{Y_{1,x;6,0}Y_{2,x;1,0}} + \frac{Y_{1,x;1,0}Y_{2,x;3,0}Y_{2,x;4,0}}{Y_{1,x;5,0}Y_{2,x;1,0}} \\
& + \frac{Y_{1,x;4,0}Y_{2,x;2,0}}{Y_{1,x;3,0}Y_{2,x;4,0}Y_{2,x;5,0}} + \frac{Y_{1,x;1,0}Y_{2,x;3,0}}{Y_{2,x;5,0}Y_{2,x;1,0}} + \frac{Y_{2,x;3,0}Y_{2,x;2,0}}{Y_{1,x;3,0}Y_{2,x;5,0}} + \frac{Y_{2,x;3,0}Y_{2,x}}{Y_{2,x;5,0}}
\end{aligned} \tag{5.13b}$$

(b) $\underline{x} = (x, xq_1^2)$

$$\begin{aligned}
(q_1 \rightarrow 1) : & \quad Y_{1,x}Y_{2,x} + \frac{Y_{2,x}^3}{Y_{1,x;0,1}} + \frac{Y_{1,x}Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x}^2Y_{2,x;0,1}}{Y_{1,x;0,1}Y_{1,x;0,2}} + \frac{Y_{1,x;0,1}^2}{Y_{2,x;0,1}^3} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}^2} \\
& + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{1}{Y_{1,x;0,2}^2} + \frac{Y_{1,x}Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{Y_{1,x}}{Y_{2,x;0,2}} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}Y_{2,x;0,2}} \\
& + 2\left(\frac{Y_{1,x;0,1}}{Y_{1,x;0,2}Y_{2,x;0,1}} + \frac{Y_{2,x}}{Y_{2,x;0,1}Y_{2,x;0,2}}\right) + 3\left(\frac{Y_{2,x}^2}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}Y_{2,x}}{Y_{2,x;0,1}^2} + \frac{Y_{2,x}}{Y_{1,x;0,2}}\right) \\
& = \left(Y_{1,x} + \frac{Y_{2,x}^2}{Y_{1,x;0,1}} + 2\frac{Y_{2,x}}{Y_{2,x;0,1}} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}^2} + \frac{1}{Y_{1,x;0,2}}\right) \left(Y_{2,x} + \frac{Y_{1,x;0,1}}{Y_{2,x;0,1}} + \frac{Y_{2,x;0,1}}{Y_{1,x;0,2}} + \frac{1}{Y_{2,x;0,2}}\right)
\end{aligned} \tag{5.14a}$$

$$\begin{aligned}
(q_2 \rightarrow 1) & \quad Y_{1,x}Y_{2,x;2,0} + \frac{Y_{2,x}Y_{2,x;1,0}Y_{2,x;2,0}}{Y_{1,x;2,0}} + \frac{Y_{1,x}Y_{1,x;3,0}}{Y_{2,x;3,0}} + \frac{Y_{2,x}Y_{2,x;1,0}Y_{2,x;4,0}}{Y_{1,x;2,0}Y_{1,x;5,0}} \\
& + \frac{Y_{1,x;1,0}Y_{1,x;3,0}}{Y_{2,x;1,0}Y_{2,x;2,0}Y_{2,x;3,0}} + \frac{Y_{2,x;4,0}}{Y_{1,x;3,0}Y_{1,x;5,0}} + \frac{Y_{1,x;1,0}}{Y_{2,x;1,0}Y_{2,x;2,0}Y_{2,x;5,0}} + \frac{1}{Y_{1,x;3,0}Y_{2,x;5,0}} \\
& + \frac{Y_{1,x}Y_{2,x;4,0}}{Y_{1,x;5,0}} + \frac{Y_{1,x}}{Y_{2,x;5,0}} + \frac{Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,0}Y_{2,x;5,0}} + \frac{Y_{1,x;1,0}Y_{2,x;4,0}}{Y_{1,x;5,0}Y_{2,x;1,0}Y_{2,x;2,0}} \\
& + \frac{Y_{1,x;3,0}Y_{2,x}Y_{2,x;1,0}}{Y_{1,x;2,0}Y_{2,x;3,0}} + \frac{Y_{1,x;3,0}Y_{2,x}}{Y_{2,x;2,0}Y_{2,x;3,0}} + \frac{Y_{2,x}Y_{2,x;4,0}}{Y_{1,x;5,0}Y_{2,x;2,0}} + \frac{Y_{2,x}}{Y_{2,x;2,0}Y_{2,x;5,0}}
\end{aligned} \tag{5.14b}$$

6. \widehat{A}_0 QUIVER

We consider \widehat{A}_0 quiver, which is the simplest affine quiver. In this case, there exists a single node with the loop edge. The Cartan matrix is given by

$$c = 1 + q - \mu - \mu^{-1}q = (1 - \mu)(1 - \mu^{-1}q) = (1 - q_3)(1 - q_4), \tag{6.1}$$

where we define

$$(q_3, q_4) = (\mu, \mu^{-1}q) \quad (6.2)$$

and

$$\underline{q} = (q_1, q_2, q_3, q_4). \quad (6.3)$$

We assume $q_{3,4} \neq 1$,¹¹ so that the Cartan matrix is invertible. We remark the relation

$$q_1 q_2 = q_3 q_4 = q. \quad (6.4)$$

Then, we have the iWeyl reflection with a formal counting parameter $\mathfrak{q} \in \mathbb{C}^\times$,

$$\text{iWeyl}: \quad Y_x \longmapsto \mathfrak{q} \mathcal{S}(q_3) \frac{Y_{xq_3} Y_{xq_4}}{Y_{xq}}. \quad (6.5)$$

We remark that the extra factor $\mathcal{S}(q_3) = \mathcal{S}(q_4)$ is specific to the case with the loop edge.

6.1. Weight one. The fundamental qq -character of the weight $w = (1)$ is given by an infinite sum over all the possible partitions [Nek15, KP15],

$$\mathbb{T}_{1,x} = \sum_{\lambda} \mathfrak{q}^{|\lambda|} Z_{\lambda}(\underline{q}) \prod_{s \in \partial_+ \lambda} Y_{x(s)} \prod_{s \in \partial_- \lambda} Y_{x(s)q}^{-1}, \quad (6.6)$$

where $|\lambda| = \sum_{k=1}^{\infty} \lambda_k$, and we denote by $\partial_+ \lambda$ and $\partial_- \lambda$ the outer and inner boundaries of the partition λ , where one can add/remove a box. For $s = (s_1, s_2)$ and the transpose of partition denoted by λ^T , we define

$$x(s) = xq_3^{s_1-1} q_4^{s_2-1}, \quad a(s) = \lambda_{s_2} - s_1, \quad \ell(s) = \lambda_{s_1}^T - s_2. \quad (6.7)$$

Then, we have

$$Z_{\lambda}(\underline{q}) = \prod_{s \in \lambda} \mathcal{S}(q_3^{\ell(s)+1} q_4^{-a(s)}) = \prod_{s \in \lambda} \mathcal{S}(q_3^{-\ell(s)} q_4^{a(s)+1}). \quad (6.8)$$

We remark that this factor is identified with the fixed point contribution of the Nekrasov partition function of 5d $\mathcal{N} = 1^* \text{U}(1)$ gauge theory with the Ω -background parameters (q_3, q_4) (not for (q_1, q_2)) with the adjoint mass q_1 (or q_2), which is geometrically interpreted as the $\chi_{q_1,2}$ -genus of the instanton moduli space [Nek02, NO03].

The qq -character (6.6) is interpreted to be associated with the infinite dimensional Fock module of the quantum toroidal algebra of \mathfrak{gl}_1 denoted by $U_q(\widehat{\mathfrak{gl}}_1)$ [NPS13, FJMM15, FJMM16] and its elliptic uplift [KO21].¹² See also [Liu22] for a related geometric representation theoretical perspective of the qq -character of \widehat{A}_0 quiver. Similarly to finite-type quivers, we can reduce the module by tuning the parameters. In this case, imposing the *resonance* condition [FFJ+10]¹³

$$q_3^{\mathbf{i}} q_4^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad q_3^{-\mathbf{i}+1} q_4^{\mathbf{i}} = q_{1,2}, \quad (6.9)$$

the rational function (6.8) becomes zero if $(\mathbf{i}, \mathbf{j}) \in \lambda$ for $(\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_{>0}^2$. Hence, the infinite sum (6.6) excludes the configuration such that $(\mathbf{i}, \mathbf{j}) \in \lambda$. This is called the *pit* condition at $(\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_{>0}^2$ [BFM15].

¹¹From the gauge theory point of view, supersymmetry is enhanced from 8 to 16 supercharges in the limit $q_{3,4} \rightarrow 1$.

¹²The qq -character of the MacMahon module of $U_q(\widehat{\mathfrak{gl}}_1)$ has been also constructed recently [KN23].

¹³We use a slightly different convention compared with the original paper [FFJ+10].

6.2. Generic weight. We then consider the qq -character of \widehat{A}_0 quiver for generic weight w . In this case, the qq -character is given by a summation over w -tuple partitions $\lambda = (\lambda_\alpha)_{\alpha=1,\dots,w}$ with the weight parameters $\underline{x} = (x_\alpha)_{\alpha=1,\dots,w}$,

$$\tau_{w,\underline{x}} = \sum_{\lambda} q^{|\lambda|} Z_{\lambda}(q; \underline{x}) \prod_{\alpha=1}^w \left[\prod_{s \in \partial_+ \lambda_\alpha} Y_{x_\alpha(s)} \prod_{s \in \partial_- \lambda_\alpha} Y_{x_\alpha(s)}^{-1} \right] \quad (6.10)$$

where $|\lambda| = \sum_{\alpha=1}^w |\lambda_\alpha|$, and we define

$$Z_{\lambda}(q; \underline{x}) = \prod_{\alpha=1}^w Z_{\lambda_\alpha}(q) \prod_{1 \leq \alpha < \beta \leq w} \left[\prod_{s \in \lambda_\alpha} \mathfrak{S} \left(\frac{x_\beta}{x_\alpha} q_3^{\ell_\beta(s)+1} q_4^{-a_\alpha(s)} \right) \prod_{s \in \lambda_\beta} \mathfrak{S} \left(\frac{x_\beta}{x_\alpha} q_3^{-\ell_\alpha(s)} q_4^{a_\beta(s)+1} \right) \right] \quad (6.11)$$

with

$$a_\alpha(s) = \lambda_{\alpha, s_2} - s_1, \quad \ell_\alpha(s) = \lambda_{\alpha, s_1}^T - s_2. \quad (6.12)$$

We remark

$$\mathfrak{S} \left(\frac{x_\beta}{x_\alpha} q_3^{\ell_\beta(s)+1} q_4^{-a_\alpha(s)} \right) = \mathfrak{S} \left(\frac{x_\alpha}{x_\beta} q_3^{-\ell_\beta(s)} q_4^{a_\alpha(s)+1} \right), \quad \mathfrak{S} \left(\frac{x_\beta}{x_\alpha} q_3^{-\ell_\alpha(s)} q_4^{a_\beta(s)+1} \right) = \mathfrak{S} \left(\frac{x_\alpha}{x_\beta} q_3^{\ell_\alpha(s)+1} q_4^{-a_\beta(s)} \right). \quad (6.13)$$

The factor $Z_{\lambda}(q; \underline{x})$ coincides with the fixed point contribution of the Nekrasov partition function of 5d $\mathcal{N} = 1^* \text{U}(w)$ gauge theory under identification of the weight parameters with the Coulomb moduli. From the representation theoretical point of view, the weight- w character is associated with the tensor product of the Fock modules. In this context, the weight parameters $(x_\alpha)_{\alpha=1,\dots,w}$ are identified with the evaluation parameters.

We may consider the same condition as the case of weight one (6.9), which imposes the same pit condition for all the partitions $(\lambda_\alpha)_{\alpha=1,\dots,w}$. For the tensor product of the Fock modules, there exists a family of the irreducible representations obtained by specialization of the weight parameters $\underline{x} = (x_\alpha)_{\alpha=1,\dots,w}$ [FFJ⁺10]. We consider the resonance condition

$$\frac{x_\beta}{x_\alpha} q_3^{\mathbf{i}} q_4^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad \frac{x_\alpha}{x_\beta} q_3^{-\mathbf{i}+1} q_4^{\mathbf{j}} = q_{1,2}. \quad (6.14)$$

In this case, we obtain the following restriction on the set of partitions,

$$\lambda_{\beta, k}^T - \lambda_{\alpha, k+\mathbf{j}-1}^T \geq \mathbf{i} \quad (\mathbf{i} \in \mathbb{Z}_{\leq 0}, \mathbf{j} \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 1}). \quad (6.15)$$

As the resulting modules are irreducible as shown in [FFJ⁺10], we have the following.

Proposition 6.1. Conjecture 1.1 holds for the tensor product module of the Fock modules associated with \widehat{A}_0 quiver.

Such a restriction, known as the Burge condition [Bur93], has been discussed also in the relation to the plane partitions [GK97], the Higgsing of 5d $\mathcal{N} = 1^*$ theory [CHZ12], and the minimal models of W-algebras [BF14, AB14, BFS15].

7. \widehat{A}_{r-1} QUIVER

We study the higher rank cyclic quiver \widehat{A}_{r-1} , which consists of r nodes. In this case, we consider the Cartan matrix given by

$$c = \begin{pmatrix} 1+q & -\mu_1 & & & -\mu_0^{-1}q \\ -\mu_1^{-1}q & 1+q & -\mu_2 & & \\ & -\mu_2^{-1}q & 1+q & \ddots & \\ & & \ddots & \ddots & -\mu_{r-1} \\ -\mu_0 & & & -\mu_{r-1}^{-1}q & 1+q \end{pmatrix} \quad (7.1)$$

and the determinant given by

$$\det c = (1 - \mu_{\text{tot}})(1 - \mu_{\text{tot}}^{-1}q^r), \quad \mu_{\text{tot}} := \prod_{i=0}^{r-1} \mu_i. \quad (7.2)$$

We assume $\mu_{\text{tot}} \neq 1$, $\mu_{\text{tot}}^{-1}q^r \neq 1$ in order that the Cartan matrix is invertible. We specialize the mass parameters to $\mu_i = \mu$ for $i = 0, \dots, r-1$ without loss of generality, so that $\mu_{\text{tot}} = \mu^r$. We use the same notation $(q_3, q_4) = (\mu, \mu^{-1}q)$ as before (6.2). Then, the iWeyl reflection is given as follows,

$$\text{iWeyl} : Y_{i,x} \mapsto \mathfrak{q}_i \frac{Y_{i+1,xq_3} Y_{i-1,xq_4}}{Y_{i,xq}}, \quad i \in \mathbb{Z}/r\mathbb{Z}, \quad (7.3)$$

where we interpret the node index periodic $i \equiv i + r \pmod{r}$. We also denote $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$.

7.1. Weight one. The fundamental qq -character associated with the i -th node, corresponding to the weight $w = (w_j)_{j=0, \dots, r-1}$ with $w_j = \delta_{i,j}$, is again given by summation over the partition,

$$\mathbb{T}_{i,x} = \sum_{\lambda} \mathfrak{q}_i^\lambda Z_\lambda^{(r)}(q) \prod_{s \in \partial_+ \lambda} Y_{i(s)+i,x(s)} \prod_{s \in \partial_- \lambda} Y_{i(s)+i,x(s)q}^{-1}, \quad (7.4)$$

where we define the coloring index $i(s) \equiv s_1 - s_2 \pmod{r}$ for $s = (s_1, s_2) \in \mathbb{Z}_{>0}^2$, and

$$\mathfrak{q}_i^\lambda = \prod_{s \in \lambda} \mathfrak{q}_{i(s)+i} = \prod_{j=0}^{r-1} \mathfrak{q}_{j+i}^{|\lambda|_j}, \quad |\lambda|_i = \#\{s \in \lambda \mid i(s) = i\}. \quad (7.5)$$

We also define

$$Z_\lambda^{(r)}(q) = \prod_{s \in \lambda \mid h(s) \in r\mathbb{Z}} \mathcal{S}(q_3^{\ell(s)+1} q_4^{-a(s)}) = \prod_{s \in \lambda \mid h(s) \in r\mathbb{Z}} \mathcal{S}(q_3^{-\ell(s)} q_4^{a(s)+1}), \quad (7.6)$$

with the hook length

$$h(s) = a(s) + \ell(s) + 1. \quad (7.7)$$

In this case, the rational factor (7.6) is identified with the fixed point contribution of the Nekrasov partition function of 5d $\mathcal{N} = 1^* \text{U}(1)$ theory on $\mathbb{C}^2/\mathbb{Z}_r$ with the Ω -background parameters (q_3, q_4) [FMP04]. The qq -character (7.4) is then associated with the colored Fock module of the higher rank quantum toroidal algebra $U_q(\widehat{\mathfrak{gl}}_r)$ [JM83, TU98, FJMM12].

Let us discuss truncation of the configuration. We tune the parameters as before (6.9). In this case, the rational function (7.6) becomes zero if $(\mathbf{i}, \mathbf{j}) \in \lambda$ and also $\mathbf{i} + \mathbf{j} - 1 \in r\mathbb{Z}$. The condition (6.9) is rewritten in terms of the total mass μ_{tot} instead of μ ,

$$\mu_{\text{tot}}^{(\mathbf{i}+\mathbf{j}-1)/r} q^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad \mu_{\text{tot}}^{-(\mathbf{i}+\mathbf{j}-1)/r} q^{\mathbf{j}} = q_{1,2}. \quad (7.8)$$

Hence, we can impose the pit condition at $(\mathbf{i}, \mathbf{j}) \in \mathbb{Z}_{>0}^2$ only if $\mathbf{i} + \mathbf{j} - 1 \in r\mathbb{Z}$ for \widehat{A}_{r-1} quiver.

7.2. Generic weight. We study the case with generic weight $w = (w_i)_{i=0,\dots,r-1} \in \mathbb{Z}_{\geq 0}^r$. We define the index set $I_w = \{(i, \alpha) \mid i = 0, \dots, r-1, \alpha = 1, \dots, w_i\}$ with $\mathbf{w} := |I_w| = \sum_{i=0}^{r-1} w_i$. Then, the qq -character is given as a summation over the set of partitions $\lambda = (\lambda_{i,\alpha})_{(i,\alpha) \in I_w}$,

$$\mathbb{T}_{w, \underline{x}} = \sum_{\lambda} \mathfrak{q}_w^{\lambda} Z_{\lambda}^{(r)}(\underline{q}; \underline{x}) \prod_{(i,\alpha) \in I_w} \left[\prod_{s \in \partial_+ \lambda_{i,\alpha}} Y_{i(s)+i, x_{i,\alpha}(s)} \prod_{s \in \partial_- \lambda_{i,\alpha}} Y_{i(s)+i, x_{i,\alpha}(s)}^{-1} q \right] \quad (7.9)$$

where we define

$$\mathfrak{q}_w^{\lambda} = \prod_{(i,\alpha) \in I_w} \prod_{s \in \lambda_{i,\alpha}} \mathfrak{q}_{i(s)+i}, \quad (7.10)$$

and the rational factor

$$Z_{\lambda}^{(r)}(\underline{q}; \underline{x}) = \prod_{(i,\alpha) \in I_w} Z_{\lambda_{i,\alpha}}^{(r)}(\underline{q}) \prod_{(i,\alpha) < (j,\beta)} \left[\prod_{\substack{s \in \lambda_{i,\alpha} \\ h_{j,\beta}^{i,\alpha}(s) \in r\mathbb{Z}}} \mathfrak{S} \left(\frac{x_{j,\beta}}{x_{i,\alpha}} q_3^{\ell_{j,\beta}(s)+1} q_4^{-a_{i,\alpha}(s)} \right) \prod_{\substack{s \in \lambda_{j,\beta} \\ h_{i,\alpha}^{j,\beta}(s) \in r\mathbb{Z}}} \mathfrak{S} \left(\frac{x_{j,\beta}}{x_{i,\alpha}} q_3^{-\ell_{i,\alpha}(s)} q_4^{a_{j,\beta}(s)+1} \right) \right] \quad (7.11)$$

with

$$h_{j,\beta}^{i,\alpha}(s) = a_{i,\alpha}(s) + \ell_{j,\beta}(s) + 1 - (i - j). \quad (7.12)$$

The arm and leg are similarly defined as in (6.12). In this case, this factor is identified with the fixed point contribution of the Nekrasov partition function of 5d $\mathcal{N} = 1^* \text{U}(\mathbf{w})$ theory on $\mathbb{C}^2/\mathbb{Z}_r$.

As in the previous case (§6.2), we have two possibilities to impose the restriction on the set of partitions. The first is the pit condition (7.8) obtained from the diagonal factor $Z_{\lambda_{i,\alpha}}^{(r)}(\underline{q})$ in (7.11). The second is from the pair contribution associated with (i, α) and (j, β) , which yields the irreducible modules constructed from the tensor product of the Fock modules for the quantum toroidal algebra $U_q(\widehat{\mathfrak{gl}}_r)$ [FJMM12]. In this case, we consider the following resonance condition

$$\frac{x_{j,\beta}}{x_{i,\alpha}} q_3^{\mathbf{i}} q_4^{-\mathbf{j}+1} = q_{1,2} \quad \text{or} \quad \frac{x_{i,\alpha}}{x_{j,\beta}} q_3^{-\mathbf{i}+1} q_4^{\mathbf{j}} = q_{1,2}. \quad (7.13)$$

If $\mathbf{i} + \mathbf{j} - 1 - (i - j) \in r\mathbb{Z}$, we have the restriction on the set of partitions,

$$\lambda_{j,\beta,k}^{\mathbf{T}} - \lambda_{i,\alpha,k+\mathbf{j}-1}^{\mathbf{T}} \geq \mathbf{i} \quad (\mathbf{i} \in \mathbb{Z}_{\leq 0}, \mathbf{j} \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 1}). \quad (7.14)$$

As the resulting modules are irreducible [FJMM12], we have the following.

Proposition 7.1. Conjecture 1.1 holds for the tensor product module of the Fock modules associated with \widehat{A}_{r-1} quiver.

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