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Pfaffian Interaction and BCD -quiver Matrix Models

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Abstract

We study matrix models involving Pfaffian interactions as generalizations of the standard $\beta = 1$ and $\beta = 4$ matrix models. We present the Pfaffian formulas for the partition function and the characteristic polynomial averages. We also explore the matrix chain with the Pfaffian interaction, which realizes the BCD -type quiver matrix models.

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1 Introduction

Pfaffian is an analog of the determinant, which is one of the central concepts in the linear algebra. In the context of statistical physics and quantum field theory, Pfaffian is frequently utilized to describe real (Majorana) fermions, while the determinant is used for complex (Dirac) fermions. It has been also known that the Pfaffian structure is often discussed for certain classes of random matrices, called the orthogonal and symplectic ensembles, which are classified into the $\beta = 1$ and $\beta = 4$ models due to the Wigner–Dyson classification. Similarly to the $\beta = 2$ model (unitary ensemble), which exhibits the determinantal structure, the correlation functions (including the partition function) are in general described using the Pfaffian.¹ See [Meh04, For10] for details.

In this paper, we explore formal eigenvalue integrals, involving various Pfaffian interactions, that we call the generalized $\beta = 1$ and $\beta = 4$ models. We obtain the Pfaffian formulas for the corresponding partition functions and also the average of characteristic polynomials. As an application of the Pfaffian interaction, we then construct the matrix chain model, involving several sets of eigenvalues. In order to classify such a matrix chain, we use the graphical description based on the Dynkin-quiver diagram. Such a multi-matrix model is often called the quiver matrix model, which has been originally introduced to study its connection with conformal field theory [MMM91, KMM⁺93]. For example, the standard linear matrix chain is classified into the type A quiver model,

$$A_n : \circ - \circ - \cdots - \circ - \circ \quad (1.1)$$

The remaining classical classes are known as the BCD -type,

$$B_n : \circ - \circ - \cdots - \circ \rightleftarrows \circ \quad C_n : \circ - \circ - \cdots - \circ \rightleftarrows \circ \quad D_n : \circ - \circ - \cdots - \circ \begin{array}{l} \nearrow \circ \\ \searrow \circ \end{array} \quad (1.2)$$

In particular, the B and C types involve the oriented arrows (non-simply-laced quivers), so that its realization is not straightforward as for the other simply-laced classes. We show that such a non-simply-laced quiver matrix model can be constructed using the Pfaffian interaction, and discuss its relation to the simply-laced cases through the folding process based on the outer automorphism of the Dynkin-quiver diagram.

The remaining part of this paper is organized as follows. In Section 2, we collect the notations and the fundamental properties and formulas regarding the determinant and the Pfaffian, that we use in this paper. In Section 3, we introduce the generalized $\beta = 1$ and $\beta = 4$ matrix models involving the Pfaffian interactions, and show the Pfaffian formulas for their partition functions. In Section 4, we consider the correlation functions of the characteristic polynomials with respect to the generalized $\beta = 1$ and $\beta = 4$ models, and again show the Pfaffian formulas for them. In Section 5, we explore the matrix chain model, and show that the quiver matrix models classified into the type BCD could be constructed using the Pfaffian interaction.

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2 Preliminary

In this Section, we present the notations and the preliminary results that we will use in the following parts.

¹One may also use the quaternion determinant, which is essentially equivalent to the Pfaffian.

2.1 Notations

For $n \in \mathbb{N}$, we define a set of integers,

$$[n] := (1, \dots, n). \quad (2.1)$$

For a subset $I \subset [n]$, we denote

$$\Sigma(I) = \sum_{i \in I} i. \quad (2.2)$$

Let X be an $n \times m$ matrix. For $I \subset [n]$, $J \subset [m]$, we denote by $X(I; J)$ the submatrix of X obtained by picking up rows indexed by I and columns indexed by J . If X is a skew-symmetric matrix, we write $X(I) = X(I; I)$ for simplicity.

Operator formalism

Let $d\mu(x)$ be an integral measure for the variable x . We denote a function (wave function) associated with a formal state $|f\rangle$ in the Hilbert space by

$$f(x) = \langle x | f \rangle. \quad (2.3)$$

For the integral operator K associated with the corresponding kernel $K(x, y)$, we write

$$\langle x | K | f \rangle = \int d\mu(y) K(x, y) f(y). \quad (2.4)$$

In this convention, the kernel of the integral operator K is given by

$$K(x, y) = \langle x | K | y \rangle. \quad (2.5)$$

Hence, the identity operator is given by the integral over the complete set,

$$\int d\mu(x) |x\rangle \langle x| = 1. \quad (2.6)$$

We denote the transposition of the kernel,

$$K^T(y, x) = \langle y | K^T | x \rangle. \quad (2.7)$$

We also write the inner product as follows,

$$\langle f | g \rangle = \int d\mu(x) f(x) g(x), \quad (2.8a)$$

$$\langle f | K | g \rangle = \int d\mu(x) d\mu(y) f(x) K(x, y) g(y). \quad (2.8b)$$

2.2 Determinantal formulas

2.2.1 Vandermonde determinant

We first show several determinantal formulas regarding the Vandermonde determinant, which is a main building block of the matrix model. See also the monographs on this subject [Meh04, For10, EKR15] for more details.

Proposition 2.1 (Vandermonde determinant). Let $N \in \mathbb{N}$. For a set of variables $X = (x_i)_{i \in [N]}$, the N -variable Vandermonde determinant is given by a rank N determinant,

$$\Delta_N(X) = \prod_{1 \leq i < j \leq N} (x_i - x_j) = \det_{i, j \in [N]} p_{N-j}(x_i), \quad (2.9)$$

where $p_k(x) = x^k + \dots$ is a degree- k monic polynomial.

Using this expression, we have the following Lemma, which is a crucial identity to discuss the $\beta = 4$ model.

Lemma 2.2. For the N -variable Vandermonde determinant, the following determinantal formula holds,

$$\Delta_N(X)^4 = (-1)^{N(N+1)/2} \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) p'_{2N-j}(x_i)]. \quad (2.10)$$

Proof. The $2N$ -variable Vandermonde determinant of $X = (x_i)_{i \in [N]}$ and $Y = (y_i)_{i \in [N]}$ may be written as follows,

$$\Delta_{2N}(X; Y) = \prod_{\substack{i, j \in [N] \\ i < j}} (x_i - x_j)(y_i - y_j)(x_i - y_j)(x_j - y_i) \prod_{i \in [N]} (x_i - y_i). \quad (2.11)$$

On the other hand, the $2N$ -variable Vandermonde determinant is given by

$$\Delta_{2N}(X; Y) = \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) p_{2N-j}(y_i)] = \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) p_{2N-j}(y_i) - p_{2N-j}(x_i)]. \quad (2.12)$$

Hence, we obtain

$$\begin{aligned} \Delta_N(X)^4 &= (-1)^{N(N+1)/2} \lim_{y_i \rightarrow x_i} \Delta_{2N}(X; Y) \prod_{i \in [N]} (y_i - x_i)^{-1} \\ &= (-1)^{N(N+1)/2} \lim_{y_i \rightarrow x_i} \det_{\substack{i \in [N] \\ j \in [2N]}} \left[p_{2N-j}(x_i) \frac{p_{2N-j}(y_i) - p_{2N-j}(x_i)}{y_i - x_i} \right] \\ &= (-1)^{N(N+1)/2} \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) p'_{2N-j}(x_i)]. \end{aligned} \quad (2.13)$$

This completes the proof. \square

2.2.2 Schur polynomials

In this paper, we shall use the Schur polynomial as the basis of the symmetric polynomial of the formal eigenvalues. See, e.g. [Mac15] for the details on this subject.

Definition 2.3 (Schur polynomial). Let λ be a partition, a non-increasing sequence of non-negative integers,

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > \lambda_{\ell+1} = \cdots = 0), \quad (2.14)$$

where $\ell = \ell(\lambda)$ is called the length of the partition. Denoting the transposed partition by λ^\top , we have $\ell(\lambda) = \lambda_1^\top$. We define the size of the partition by

$$|\lambda| = \sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\infty} \lambda_i^\top = |\lambda^\top|. \quad (2.15)$$

Then, the Schur polynomial of N variables, $X = (x_i)_{i \in [N]}$, is defined as follows,

$$s_\lambda(X) = \frac{1}{\Delta_N(X)} \det_{i, j \in [N]} x_i^{\lambda_j + N - j}. \quad (2.16)$$

If $\ell(\lambda) > N$, it always vanishes $s_\lambda(X) = 0$. We also remark $s_\emptyset(X) = 1$.

For the latter convenience, we also define a modified version of the Schur polynomial as follows.

Definition 2.4. Let $N \in \mathbb{N}$, and we denote $X = (x_i)_{i \in [N]}$, $Y = (y_i)_{i \in [N]}$. For a partition λ , we define

$$a_\lambda(X; Y) = \det_{i \in [N], j \in [2N]} \begin{bmatrix} x_i^{\lambda_j + 2N - j} & (y_i^{\lambda_j + 2N - j})' \end{bmatrix}. \quad (2.17)$$

Then, we define a modified Schur polynomial as follows,

$$s_\lambda(X; Y) = \frac{a_\lambda(X; Y)}{a_\emptyset(X; Y)}. \quad (2.18)$$

Regarding the Schur polynomials, we will use the following decomposition in the study of the characteristic polynomials.

Lemma 2.5 (Schur polynomial expansion). Let $X = (x_i)_{i \in [N]}$ and $Y = (y_i)_{i \in [M]}$. We have the following expansions with the Schur polynomial,

$$\prod_{i \in [N], j \in [M]} (x_i - y_j) = \sum_{\lambda \subseteq (M^N)} (-1)^{|\lambda|} s_{\lambda^\vee}(X) s_\lambda(Y), \quad (2.19a)$$

$$\begin{aligned} \prod_{i \in [N], j \in [M]} (x_i - y_j)^{-1} &= \det_N X^{-M} \sum_{\lambda | \ell(\lambda) \leq \min(M, N)} s_\lambda(X^{-1}) s_\lambda(Y), \\ &= \det_M Y^{-N} \sum_{\lambda | \ell(\lambda) \leq \min(M, N)} s_\lambda(X) s_\lambda(Y^{-1}), \end{aligned} \quad (2.19b)$$

where we define the dual partition

$$\lambda^\vee = (\lambda_1^\vee, \dots, \lambda_M^\vee) = (N - \lambda_M^\top, \dots, N - \lambda_1^\top), \quad (2.20)$$

and the length of the partition denoted by $\ell(\lambda) = \lambda_1$.

Proof. This follows from the Cauchy sum formula. See, e.g., [Mac15] for details. \square

Lemma 2.6 (Cauchy determinant). We define the Cauchy determinant for $X = (x_i)_{i \in [N]}$ and $Y = (y_i)_{i \in [M]}$ as follows,

$$\Delta_{N|M}(X|Y) := \frac{\Delta_N(X) \Delta_M(Y)}{\prod_{j \in [M]} \prod_{i \in [N]} (x_i - y_j)}. \quad (2.21)$$

Then, the following formulas hold.

$$N \geq M: \quad \Delta_{N|M}(X|Y) = \sum_{\lambda | \ell(\lambda) \leq M} \det_{\substack{i \in [N], j \in [M] \\ k \in [N-M]}} \begin{bmatrix} x_i^{-\lambda_j - M + j - 1} & \\ & x_i^{k-1} \end{bmatrix} \det_{i, j \in [M]} y_i^{\lambda_j + M - j} = \det_{\substack{i \in [N], j \in [M] \\ k \in [N-M]}} \begin{bmatrix} 1 \\ x_i - y_j \\ x_i^{k-1} \end{bmatrix} \quad (2.22a)$$

$$N \leq M: \quad \Delta_{N|M}(X|Y) = \sum_{\lambda | \ell(\lambda) \leq N} \det_{i, j \in [N]} x_i^{\lambda_j + N - j} \det_{\substack{i \in [M], j \in [N] \\ k \in [M-N]}} \begin{bmatrix} y_i^{-\lambda_j - N + j - 1} & \\ & y_i^{k-1} \end{bmatrix} = \det_{\substack{i \in [N], j \in [M] \\ k \in [M-N]}} \begin{bmatrix} 1 \\ x_i - y_j \\ y_j^{k-1} \end{bmatrix} \quad (2.22b)$$

Proof. The expressions as the summation over the partitions are obtained from the Schur polynomial expansion (Lemma 2.5). In addition, for the case $N \geq M$, the summation over the partition is given by

$$\sum_{\ell(\lambda) \leq M} = \sum_{0 \leq \lambda_M \leq \dots \leq \lambda_1 \leq \infty} = \sum_{0 \leq r_M < \dots < r_1 \leq \infty} = \frac{1}{M!} \sum_{\substack{r_\alpha = 0, \dots, \infty \\ r_\alpha \neq r_\beta}} \quad (2.23)$$

where we write $r_\alpha = \lambda_\alpha + M - \alpha$. Then, we have

$$\Delta_{N|M}(X|Y) = \det_{\substack{i \in [N], j \in [M] \\ k \in [N-M]}} \left[\begin{array}{c} \sum_{r=0}^{\infty} x_i^{-r-1} y_j^r \\ x_i^{k-1} \end{array} \right] = \det_{\substack{i \in [N], j \in [M] \\ k \in [N-M]}} \left[\begin{array}{c} 1 \\ x_i - y_j \\ x_i^{k-1} \end{array} \right]. \quad (2.24)$$

The other case $N \leq M$ is similarly analyzed by exchanging the x and y variables. \square

2.3 Pfaffian formulas

We present several formulas involving Pfaffians used in this paper.

Proposition 2.7. For $n, m \in 2\mathbb{N}$, let A and \tilde{A} be invertible skew-symmetric matrices of size n and m , respectively. Let B be an $n \times m$ generic matrix. Then, the following factorization formula holds for the Pfaffian of the block matrix,

$$\text{Pf} \begin{bmatrix} A & B \\ -B^T & \tilde{A} \end{bmatrix} = \text{Pf } A \text{ Pf} \left[\tilde{A} + B^T A^{-1} B \right] = \text{Pf } \tilde{A} \text{ Pf} \left[A + B \tilde{A}^{-1} B^T \right]. \quad (2.25)$$

For $A = \tilde{A} = 0$, we obtain

$$\text{Pf} \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix} = \det B. \quad (2.26)$$

Remark 2.8. This formula is a Pfaffian analog of the block matrix determinant formula,

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B) = \det D \det(A - BD^{-1}C). \quad (2.27)$$

Lemma 2.9 (Laplace-type expansion [Oka19]). For an $m \times m$ skew-symmetric matrix Z , and an $m \times n$ matrix W with $m > n$, we have the following expansion of the Pfaffian,

$$\text{Pf} \begin{bmatrix} Z & W \\ -W^T & 0 \end{bmatrix} = \sum_{\substack{I \subset [n] \\ |I|=m-n}} (-1)^{\Sigma(I) + \binom{m}{2}} \text{Pf } Z(I) \det W([m] \setminus I; [n]) \quad (2.28)$$

where I runs over all $(m - n)$ -element subsets of $[n]$.

Lemma 2.10 (Cauchy–Binet-type formula [IW95, Oka19]). Let $l \in \mathbb{Z}_{\geq 0}$, $m, n \in \mathbb{N}$, such that $m - l \in 2\mathbb{N}$, and $m - l \leq n$. For an $n \times n$ skew-symmetric matrix Z , an $m \times n$ matrix W , and an $m \times l$ matrix X , we have the following expansion of the Pfaffian,

$$\text{Pf} \begin{bmatrix} WZW^T & X \\ -X^T & 0 \end{bmatrix} = \sum_{\substack{I \subset [n] \\ |I|=m-l}} \text{Pf } Z(I) \det[W([m]; I) \ X([m]; [l])] \quad (2.29)$$

where I runs over all $(m - l)$ -element subsets of $[n]$. If $l = 0$, there is no contribution of the matrix X ,

$$\text{Pf} [WZW^T] = \sum_{\substack{I \subset [n] \\ |I|=m}} \text{Pf } Z(I) \det W([m]; I). \quad (2.30)$$

If $m = n$ (hence $l = 0$), the only possibility is $I = \emptyset$, so that we have

$$\text{Pf} [WZW^T] = \text{Pf } Z \det W. \quad (2.31)$$

2.4 Integration formulas

We then introduce the integration formulas that will be crucial to perform the eigenvalue integrals discussed in this paper.

Proposition 2.11 (Andréief's formula). Let $(f_i(x))_{i \in \mathbb{N}}$ and $(g_i(x))_{i \in \mathbb{N}}$ be sequences of integrable functions. Denoting $d\mu(X) = \prod_{i \in [N]} d\mu(x_i)$, the following identity holds,

$$\frac{1}{N!} \int d\mu(X) \det_{1 \leq i, j \leq N} f_i(x_j) \det_{1 \leq i, j \leq N} g_i(x_j) = \det_{1 \leq i, j \leq N} \langle f_i | g_j \rangle. \quad (2.32)$$

Proposition 2.12 (de Bruijn's formula). Let $N, M \in \mathbb{N}$, such that $N + M \in 2\mathbb{N}$. Let $A(x, y), B(x, y)$ be skew-symmetric functions, and $S(x, y)$ generic integrable two-variable function. Let $(f_i(x))_{i \in \mathbb{N}}, (g_i(x))_{i \in \mathbb{N}}$ be sequences of integrable functions as before. Denoting $d\mu(X) = \prod_{i \in [N]} d\mu(x_i)$, etc, the following identities hold,

$$\frac{1}{(2N)!} \int d\mu(X) \det_{i, j \in [2N]} f_j(x_i) \text{Pf}_{i, j \in [2N]} A(x_i, x_j) = \text{Pf}_{i, j \in [2N]} \langle f_i | A | f_j \rangle, \quad (2.33a)$$

$$\begin{aligned} & \frac{1}{N!M!} \int d\mu(X) d\mu(Y) \det_{\substack{i \in [N], j \in [M] \\ k \in [N+M]}} [f_k(x_i) g_k(y_j)] \text{Pf}_{\substack{i, j \in [N] \\ k, l \in [M]}} \begin{bmatrix} A(x_i, x_j) & S(x_i, y_l) \\ -S^T(y_k, x_j) & B(y_k, y_l) \end{bmatrix} \\ &= \text{Pf}_{i, j \in [N+M]} [\langle f_i | A | f_j \rangle + \langle f_i | S | g_j \rangle - \langle g_i | S^T | f_j \rangle + \langle g_i | B | g_j \rangle]. \end{aligned} \quad (2.33b)$$

Remark 2.13. These formulas are due to C. Andréief [And86] and N. G. de Bruijn [dB55]. See [For19] for a historical survey on these formulas.

3 Matrix model with Pfaffian interaction

In this Section, we introduce the generalized $\beta = 1$ and $\beta = 4$ matrix models. We show that the corresponding partition functions are concisely described by the Pfaffian formula.

3.1 $\beta = 1$ model

We start to define the following model that we call the generalized $\beta = 1$ matrix model.

Definition 3.1 (Generalized $\beta = 1$ matrix model). Let $N \in 2\mathbb{N}$, and $A(x, y)$ be a skew-symmetric function, $A(y, x) = -A(x, y)$. Let $(f_{i-1}(x))_{i \in [N]}$ be a sequence of the integrable functions. We denote the integral measure by $d\mu(X) = \prod_{i \in [N]} d\mu(x_i)$. Then, we define the generalized $\beta = 1$ matrix model partition function as follows,

$$\mathcal{Z}_N^{(1)} = \frac{1}{N!} \int d\mu(X) \det_{i, j \in [N]} f_{j-1}(x_i) \text{Pf}_{i, j \in [N]} A(x_i, x_j). \quad (3.1)$$

In fact, this generalized $\beta = 1$ model is reduced to the standard $\beta = 1$ model as follows.

Proposition 3.2. Putting $A(x, y) = \text{sgn}(x - y)$ and $f_{i-1}(x) = p_{N-i}(x)$, the generalized $\beta = 1$ model reduces to the standard $\beta = 1$ matrix model (real symmetric matrix model; orthogonal ensemble),

$$\mathcal{Z}_N^{(1)} \xrightarrow[f_{i-1} \rightarrow p_{N-i}]{A \rightarrow \text{sgn}} \frac{1}{N!} \int d\mu(X) |\Delta_N(X)|. \quad (3.2)$$

Proof. Putting $A(x, y) = \text{sgn}(x - y)$, we have

$$\text{Pf}_{i, j \in [N]} \text{sgn}(x_i - x_j) = \prod_{1 \leq i < j \leq N} \text{sgn}(x_i - x_j). \quad (3.3)$$

Putting $f_{i-1}(x) = p_{N-i}(x)$, the determinant factor becomes the Vandermonde determinant (2.9). Hence, we have

$$\mathcal{Z}_N^{(1)} \longrightarrow \frac{1}{N!} \int d\mu(X) \Delta_N(X) \prod_{1 \leq i < j \leq N} \text{sgn}(x_i - x_j) = \frac{1}{N!} \int d\mu(X) |\Delta_N(X)|. \quad (3.4)$$

This completes the proof. \square

Then, the following Pfaffian formula holds for the generalized $\beta = 1$ model partition function.

Proposition 3.3. Let $N \in 2\mathbb{N}$. The following Pfaffian formula holds for the generalized $\beta = 1$ model partition function,

$$\mathcal{Z}_N^{(1)} = \text{Pf}_{i,j \in [N]} \mathbf{N}_{i-1,j-1}^{(1)}, \quad (3.5)$$

where we define a skew-symmetric matrix,

$$\mathbf{N}_{i,j}^{(1)} = \langle f_i | A | f_j \rangle. \quad (3.6)$$

Proof. We can directly obtain this expression from de Bruijn's integration formula (2.33a). \square

Remark 3.4 (Skew-orthogonal functions). Defining the skew-orthogonal functions $(F_i)_{i \in [N]}$ obeying the relation for $i, j \in [N/2]$,

$$\langle F_{2i-2} | A | F_{2j-1} \rangle = -\langle F_{2j-1} | A | F_{2i-2} \rangle = h_i^{(1)} \delta_{i,j}, \quad (3.7a)$$

$$\langle F_{2i-2} | A | F_{2j-2} \rangle = \langle F_{2i-1} | A | F_{2i-1} \rangle = 0, \quad (3.7b)$$

the partition function is given by

$$\mathcal{Z}_N^{(1)} = \prod_{i \in [N/2]} h_i^{(1)}. \quad (3.8)$$

Similarly to the $\beta = 2$ matrix model, we may introduce the Christoffel–Darboux kernel for the $\beta = 1$ model as well, which will be a building block of the probability distribution function of the formal eigenvalues.

Definition 3.5 (Christoffel–Darboux kernel). Let $N \in 2\mathbb{N}$. We denote the inverse matrix of $\mathbf{N}^{(1)}$ by $\tilde{\mathbf{N}}^{(1)}$. Then, we define the Christoffel–Darboux (CD) kernel for the $\beta = 1$ model,

$$\begin{aligned} K_N^{(1)}(x, y) &= \sum_{i,j=0}^{N-1} f_i(x) \tilde{\mathbf{N}}_{i,j}^{(1)} f_j(y) \\ &= \sum_{i=0}^{N/2-1} \frac{F_{2i-1}(x) F_{2i-2}(y) - F_{2i-2}(x) F_{2i-1}(y)}{h_i^{(1)}}, \end{aligned} \quad (3.9)$$

where $(F_i)_{i \in [N]}$ is a set of the skew-orthogonal functions defined in (3.7). The corresponding integral operator is given by

$$K_N^{(1)} = \sum_{i,j=0}^{N-1} |f_i\rangle \tilde{\mathbf{N}}_{i,j}^{(1)} \langle f_j|. \quad (3.10)$$

Remark 3.6. The CD kernel introduced here is slightly different from the quaternion kernel used in the standard analysis of the $\beta = 1$ model. See, e.g., [Meh04, For10] for details.

In fact, the CD kernel shows the property, called the self-reproducing property, which will be an important property to compute the correlation function.

Proposition 3.7. The CD kernel $K_N^{(1)}$ is self-reproducing,

$$K_N^{(1)} \cdot A \cdot K_N^{(1)} = K_N^{(1)}, \quad (3.11a)$$

$$\mathrm{Tr} \left[K_N^{(1)} \cdot A \right] = N. \quad (3.11b)$$

Proof. This property can be checked by direct calculation,

$$K_N^{(1)} \cdot A \cdot K_N^{(1)} = \sum_{i,j,k,l=0}^{N-1} |f_i\rangle \tilde{\mathbf{N}}_{i,j}^{(1)} \underbrace{\langle f_j | A | f_k \rangle}_{\mathbf{N}_{j,k}^{(1)}} \tilde{\mathbf{N}}_{k,l}^{(1)} \langle f_l| = \sum_{i,l=0}^{N-1} |f_i\rangle \tilde{\mathbf{N}}_{i,l}^{(1)} \langle f_l| = K_N^{(1)}. \quad (3.12)$$

The trace condition is shown as follows,

$$\mathrm{Tr} \left[K_N^{(1)} \cdot A \right] = \sum_{i,j=0}^{N-1} \int \int d\mu(x) d\mu(y) f_i(x) \tilde{\mathbf{N}}_{i,j}^{(1)} f_j(y) A(y, x) = \sum_{i,j=0}^{N-1} \tilde{\mathbf{N}}_{i,j}^{(1)} \mathbf{N}_{j,i}^{(1)} = N. \quad (3.13)$$

□

Remark 3.8. From the completeness condition, we obtain

$$K_\infty^{(1)}(x, y) = \lim_{N \rightarrow \infty} K_N^{(1)}(x, y) = \tilde{A}(x, y), \quad (3.14)$$

where $\tilde{A}(x, y)$ is the kernel of the inverse operator A^{-1} ,

$$\int d\mu(z) A(x, z) \tilde{A}(z, y) = \int d\mu(z) \tilde{A}(x, z) A(z, y) = \delta(x - y). \quad (3.15)$$

Using this CD kernel, we may concisely express the probability distribution function as follows.

Corollary 3.9. The probability distribution function of the formal eigenvalues associated with the $\beta = 1$ matrix model (3.1),

$$\mathbf{P}_N^{(1)}(X) d\mu(X) = \frac{1}{N! \mathcal{Z}_N^{(1)}} \det_{i,j \in [N]} f_{j-1}(x_i) \mathrm{Pf}_{i,j \in [N]} A(x_i, x_j) d\mu(X), \quad (3.16)$$

is written in terms of the CD kernel

$$\mathbf{P}_N^{(1)}(X) d\mu(X) = \frac{1}{N!} \mathrm{Pf}_{i,j \in [N]} K_N^{(1)}(x_i, x_j) \mathrm{Pf}_{i,j \in [N]} A(x_i, x_j) d\mu(X). \quad (3.17)$$

Remark 3.10. This structure of the probability distribution function (involving both the CD kernel and the interaction kernel) is similarly discussed in the context of the coupled matrix model [EM98, BK22].

3.2 $\beta = 4$ model

We then consider the generalized version of the $\beta = 4$ matrix model.

Definition 3.11 (Generalized $\beta = 4$ matrix model). Let $N \in \mathbb{N}$, and $(f_{i-1}(x))_{i \in [2N]}$, $(g_{i-1}(x))_{i \in [2N]}$ be sequences of the integrable functions. We define $2N$ variables $(z_i, z_{N+j}) = (x_i, y_j)$ for $i, j \in [N]$, and the corresponding integral measure by $d\mu(Z) = \prod_{i \in [2N]} d\mu(z_i)$. For skew-symmetric functions, $A(x, y)$, $B(x, y)$, and a generic integrable function $S(x, y)$, we define the generalized $\beta = 4$ model partition function as follows,

$$\mathcal{Z}_N^{(4)} = \frac{1}{N!^2} \int d\mu(Z) \det_{i \in [N], j \in [2N]} [f_{j-1}(x_i) g_{j-1}(y_i)] \mathrm{Pf}_{i,j \in [N]} A(x_i, x_j; y_i, y_j), \quad (3.18)$$

where we denote

$$\mathbf{A}(x, \tilde{x}; y, \tilde{y}) = \begin{bmatrix} A(x, \tilde{x}) & S(x, \tilde{y}) \\ -S^T(y, \tilde{x}) & B(y, \tilde{y}) \end{bmatrix}. \quad (3.19)$$

As is the case of the $\beta = 1$ model, the generalized $\beta = 4$ model is reduced to the standard $\beta = 4$ model via the specialization as follows.

Lemma 3.12. Putting

$$f_{i-1}(x) = p_{2N-i}(x), \quad g_{i-1}(x) = p'_{2N-i}(x), \quad A(x, y) = B(x, y) = 0, \quad S(x, y) = \delta(x - y), \quad (3.20)$$

the generalized $\beta = 4$ model reduces to the standard $\beta = 4$ matrix model (quaternion self-dual; symplectic ensemble),

$$\mathcal{Z}_{N,N}^{(4)} \longrightarrow \frac{1}{N!} \int d\mu(X) \Delta_N(X)^4. \quad (3.21)$$

Proof. Applying the formula (2.26) and Andréief's formula (2.32), we obtain

$$\begin{aligned} \mathcal{Z}_{N,N}^{(4)} &= \frac{1}{N!^2} \int d\mu(Z) \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) \ p'_{2N-j}(y_i)] \text{Pf}_{i,j \in [N]} \begin{bmatrix} 0 & \delta(x_i - y_j) \\ -\delta(y_i - x_j) & 0 \end{bmatrix} \\ &= \frac{1}{N!^2} \int d\mu(Z) \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) \ p'_{2N-j}(y_i)] \det_{i,j \in [N]} \delta(x_i - y_j) \\ &= \frac{1}{N!} \int d\mu(X) \det_{\substack{i \in [N] \\ j \in [2N]}} [p_{2N-j}(x_i) \ p'_{2N-j}(x_i)] \\ &= \frac{1}{N!} \int d\mu(X) \Delta_N(X)^4. \end{aligned} \quad (3.22)$$

This completes the proof. \square

Then, the following Pfaffian formula holds for the generalized $\beta = 4$ model partition function.

Proposition 3.13. The following Pfaffian formula holds for the generalized $\beta = 4$ model partition function,

$$\mathcal{Z}_N^{(4)} = \text{Pf}_{i,j \in [2N]} \mathbf{N}_{i-1, j-1}^{(4)}, \quad (3.23)$$

where we define a skew-symmetric matrix,

$$\begin{aligned} \mathbf{N}_{i,j}^{(4)} &= \langle f_i | A | f_j \rangle + \langle f_i | S | g_j \rangle - \langle g_i | S^T | f_j \rangle + \langle g_i | B | g_j \rangle \\ &= [\langle f_i | \ \langle g_i |] \begin{bmatrix} A & S \\ -S^T & B \end{bmatrix} \begin{bmatrix} |f_j\rangle \\ |g_j\rangle \end{bmatrix} \\ &=: \langle f_i, g_i | \mathbf{A} | f_j, g_j \rangle. \end{aligned} \quad (3.24)$$

Proof. We obtain this formula by using de Bruijn's integration formula (2.33b). \square

Remark 3.14 (Skew-biorthogonal functions). Defining the skew-biorthogonal functions $(F_i, G_i)_{i \in [N]}$ obeying the relation for $i, j \in [N]$,

$$\langle F_{2i-2}, G_{2i-2} | \mathbf{A} | F_{2j-1}, G_{2j-1} \rangle = -\langle F_{2j-1}, G_{2j-1} | \mathbf{A} | F_{2i-2}, G_{2i-2} \rangle = h_i^{(4)} \delta_{i,j}, \quad (3.25a)$$

$$\langle F_{2i-2}, G_{2i-2} | \mathbf{A} | F_{2j-2}, G_{2j-2} \rangle = \langle F_{2i-1}, G_{2i-1} | \mathbf{A} | F_{2j-1}, G_{2j-1} \rangle = 0, \quad (3.25b)$$

the partition function is given by

$$\mathcal{Z}_N^{(4)} = \prod_{i \in [N]} h_{i-1}^{(4)}. \quad (3.26)$$

Similarly to the $\beta = 1$ model, we consider the CD kernel for the $\beta = 4$ model, which is a matrix analog of the ordinary kernel.

Definition 3.15 (Matrix Christoffel–Darboux kernel). Let $N \in \mathbb{N}$. We denote the inverse matrix of $\mathbf{N}^{(4)}$ by $\tilde{\mathbf{N}}^{(4)}$. Then, we define the matrix Christoffel–Darboux (CD) kernel for the $\beta = 4$ model,

$$\begin{aligned} & \mathbf{K}_N^{(4)}(x, \tilde{x}; y, \tilde{y}) \\ &= \sum_{i,j=0}^{2N-1} \begin{bmatrix} f_i(x) \\ g_i(y) \end{bmatrix} \tilde{\mathbf{N}}_{i,j}^{(4)} \begin{bmatrix} f_j(\tilde{x}) & g_j(\tilde{y}) \end{bmatrix} \\ &= \sum_{i=0}^{N-1} \frac{1}{h_i^{(4)}} \begin{bmatrix} F_{2i+1}(x)F_{2i}(\tilde{x}) - F_{2i}(x)F_{2i+1}(\tilde{x}) & F_{2i+1}(x)G_{2i}(\tilde{y}) - F_{2i}(x)G_{2i+1}(\tilde{y}) \\ G_{2i+1}(y)F_{2i}(\tilde{x}) - G_{2i}(y)F_{2i+1}(\tilde{x}) & G_{2i+1}(y)G_{2i}(\tilde{y}) - G_{2i}(y)G_{2i+1}(\tilde{y}) \end{bmatrix}, \end{aligned} \quad (3.27)$$

where $(F_{i-1}, G_{i-1})_{i \in [2N]}$ is a set of the skew-biorthogonal functions defined in (3.25). In the operator formalism, it is given by

$$\mathbf{K}_N^{(4)} = \sum_{i,j=0}^{2N-1} \begin{bmatrix} |f_i\rangle \\ |g_i\rangle \end{bmatrix} \tilde{\mathbf{N}}_{i,j}^{(4)} [\langle f_j| \quad \langle g_j|] = \sum_{i,j=0}^{2N-1} \begin{bmatrix} |f_i\rangle \tilde{\mathbf{N}}_{i,j}^{(4)} \langle f_j| & |f_i\rangle \tilde{\mathbf{N}}_{i,j}^{(4)} \langle g_j| \\ |g_i\rangle \tilde{\mathbf{N}}_{i,j}^{(4)} \langle f_j| & |g_i\rangle \tilde{\mathbf{N}}_{i,j}^{(4)} \langle g_j| \end{bmatrix}. \quad (3.28)$$

This matrix CD kernel similarly exhibits the self-reproducing property.

Proposition 3.16. The CD kernel $\mathbf{K}_N^{(4)}$ is self-reproducing,

$$\mathbf{K}_N^{(4)} \cdot \mathbf{A} \cdot \mathbf{K}_N^{(4)} = \mathbf{K}_N^{(4)}, \quad (3.29a)$$

$$\text{Tr} \left[\mathbf{K}_N^{(4)} \cdot \mathbf{A} \right] = 2N. \quad (3.29b)$$

Proof. This property can be checked by direct calculation as follows,

$$\begin{aligned} \mathbf{K}_N^{(4)} \cdot \mathbf{A} \cdot \mathbf{K}_N^{(4)} &= \sum_{i,j,k,l=0}^{2N-1} \begin{bmatrix} |f_i\rangle \\ |g_i\rangle \end{bmatrix} \tilde{\mathbf{N}}_{i,j}^{(4)} [\langle f_j| \quad \langle g_j|] \mathbf{A} \begin{bmatrix} |f_k\rangle \\ |g_k\rangle \end{bmatrix} \tilde{\mathbf{N}}_{k,l}^{(4)} [\langle f_l| \quad \langle g_l|] \\ &= \sum_{i,j,k,l=0}^{2N-1} \begin{bmatrix} |f_i\rangle \\ |g_i\rangle \end{bmatrix} \tilde{\mathbf{N}}_{i,j}^{(4)} \mathbf{N}_{j,k}^{(4)} \tilde{\mathbf{N}}_{k,l}^{(4)} [\langle f_l| \quad \langle g_l|] = \mathbf{K}_N^{(4)}. \end{aligned} \quad (3.30)$$

The trace condition is shown as follows,

$$\begin{aligned} \text{Tr} \left[\mathbf{K}_N^{(4)} \cdot \mathbf{A} \right] &= \sum_{i,j=0}^{2N-1} \int d\mu(x) d\mu(\tilde{x}) d\mu(y) d\mu(\tilde{y}) \text{Tr} \left(\begin{bmatrix} f_i(x) \\ g_i(y) \end{bmatrix} \tilde{\mathbf{N}}_{i,j}^{(4)} \begin{bmatrix} f_j(\tilde{x}) & g_j(\tilde{y}) \end{bmatrix} \begin{bmatrix} A(x, \tilde{x}) & S(x, \tilde{y}) \\ -S(y, \tilde{x}) & B(y, \tilde{y}) \end{bmatrix} \right) \\ &= \sum_{i,j=0}^{2N-1} \tilde{\mathbf{N}}_{i,j}^{(4)} \mathbf{N}_{j,i}^{(4)} = 2N. \end{aligned} \quad (3.31)$$

This completes the proof. \square

Remark 3.17. From the completeness condition, we obtain

$$\mathbf{K}_\infty^{(4)} = \lim_{N \rightarrow \infty} \mathbf{K}_N^{(4)} = \tilde{\mathbf{A}}, \quad (3.32)$$

which is the inverse of the integral operator \mathbf{A} ,

$$\mathbf{A} \cdot \tilde{\mathbf{A}} = \tilde{\mathbf{A}} \cdot \mathbf{A} = \mathbf{1}. \quad (3.33)$$

Using the matrix CD kernel, the probability distribution function of the formal eigenvalues may be written as follows.

Corollary 3.18. The probability distribution function of the formal eigenvalues associated with the $\beta = 4$ matrix model (3.18),

$$\begin{aligned} & \mathbf{P}_N^{(4)}(X; Y) d\mu(X; Y) \\ &= \frac{1}{N!^2 \mathcal{Z}_N^{(4)}} \det_{i \in [N], j \in [2N]} [f_{j-1}(x_i) g_{j-1}(y_i)] \text{Pf}_{i, j \in [N]} \begin{bmatrix} A(x_i, x_j) & S(x_i, y_j) \\ -S^T(y_i, x_j) & B(y_i, y_j) \end{bmatrix} d\mu(X; Y) \end{aligned} \quad (3.34)$$

is written in terms of the CD kernel,

$$\mathbf{P}_N^{(4)}(X; Y) d\mu(X; Y) = \frac{1}{N!^2} \text{Pf}_{i, j \in [N]} \mathbf{K}_N^{(4)}(x_i, x_j; y_i, y_j) \text{Pf}_{i, j \in [N]} \begin{bmatrix} A(x_i, x_j) & S(x_i, y_j) \\ -S^T(y_i, x_j) & B(y_i, y_j) \end{bmatrix} d\mu(X; Y). \quad (3.35)$$

4 Characteristic polynomials

In this Section, we consider the characteristic polynomial averages of the $\beta = 1$ and $\beta = 4$ matrix models. We first define the expectation value with respect to the corresponding probability distribution functions.

Definition 4.1 (Expectation value). We define the expectation value with respect to the probability distribution function $\mathbf{P}_N^{(\beta)}(X)$ for $\beta = 1, 4$ as follows,

$$\langle \mathcal{O}(X) \rangle_\beta = \int d\mu(X) \mathbf{P}_N^{(\beta)}(X) \mathcal{O}(X). \quad (4.1)$$

We remark that the number of variables are N and $2N$ for $\beta = 1$ and $\beta = 4$, respectively.

4.1 $\beta = 1$ model

We consider the following special case of the $\beta = 1$ model (3.1) for $N \in 2\mathbb{N}$ with a skew-symmetric function $A(x, y) = -A(y, x)$:

$$\mathcal{Z}_N^{(1)} = \frac{1}{N!} \int d\mu(X) \Delta_N(X) \text{Pf}_{i, j \in [N]} A(x_i, x_j). \quad (4.2)$$

Then, we study the Schur polynomial average with respect to the $\beta = 1$ model, which would be a building block of the correlation functions discussed below.

Lemma 4.2 (Schur polynomial average). Let $N \in 2\mathbb{N}$. The Schur polynomial average of the $\beta = 1$ model is given by a rank N Pfaffian as follows,

$$\langle s_\lambda(X) \rangle_1 = \frac{1}{\mathcal{Z}_N^{(1)}} \text{Pf}_{i, j \in [N]} \left[\langle x^{\lambda_i + N - i} \mid A \mid x^{\lambda_j + N - j} \rangle \right]. \quad (4.3)$$

Proof. From the definition of the Schur polynomial (2.16), we have

$$\langle s_\lambda(X) \rangle_1 = \frac{1}{N! \mathcal{Z}_N^{(1)}} \int d\mu(X) \det_{i, j \in [N]} x_i^{\lambda_j + N - j} \text{Pf}_{i, j \in [N]} A(x_i, x_j). \quad (4.4)$$

Then, applying the de Bruijn's formula (2.33a), we arrive at the result. \square

Applying this Schur polynomial average together with the Schur polynomial expansion of the characteristic polynomial, we obtain the Pfaffian formula for the characteristic polynomial average.

Proposition 4.3 (Characteristic polynomial). Let $N, M \in 2\mathbb{N}$. For $Z = (z_\alpha)_{\alpha \in [M]}$, the characteristic polynomial average of the $\beta = 1$ model is given by the rank M Pfaffian of the degree- $(N + M)$ CD kernel,

$$\left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X) \right\rangle_1 = \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N+M}^{(1)}}{\mathcal{Z}_N^{(1)}} \text{Pf}_{\alpha, \beta \in [M]} K_{N+M}^{(1)}(z_\alpha, z_\beta). \quad (4.5)$$

Proof. We apply the Schur polynomial expansion formula (Lemma 2.5) to obtain

$$\begin{aligned} & \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X) \right\rangle_1 \\ &= \sum_{\lambda \subseteq (M^N)} (-1)^{|\lambda|} s_{\lambda^\vee}(Z) \langle s_\lambda(X) \rangle_1 \\ &= \frac{1}{\mathcal{Z}_N^{(1)} \Delta_M(Z)} \sum_{\lambda \subseteq (M^N)} (-1)^{|\lambda|} \det_{\alpha, \beta \in [M]} z_\alpha^{\lambda_\beta^\vee + M - \beta} \text{Pf}_{i, j \in [N]} \left[\langle x^{\lambda_i + N - i} | A | x^{\lambda_j + N - j} \rangle \right]. \end{aligned} \quad (4.6)$$

Applying the Laplace-type expansion (Lemma 2.9) and the Pfaffian formula for the block matrix (2.25), we have

$$\begin{aligned} \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X) \right\rangle_1 &= \frac{1}{\mathcal{Z}_N^{(1)} \Delta_M(Z)} \text{Pf}_{i, j \in [N+M]} \begin{bmatrix} \langle p_{N+M-i} | A | p_{N+M-j} \rangle & p_{N+M-i}(z_\alpha) \\ -p_{N+M-i}(z_\beta) & 0 \end{bmatrix} \\ &= \frac{\mathcal{Z}_{N+M}^{(1)}}{\mathcal{Z}_N^{(1)} \Delta_M(Z)} \text{Pf}_{\alpha, \beta \in [M]} \left[\sum_{i, j=0}^{N+M-1} p_i(z_\alpha) \tilde{N}_{i, j}^{(1)} p_j(z_\beta) \right] \\ &= \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N+M}^{(1)}}{\mathcal{Z}_N^{(1)}} \text{Pf}_{\alpha, \beta \in [M]} K_{N+M}^{(1)}(z_\alpha, z_\beta). \end{aligned} \quad (4.7)$$

This completes the proof. \square

Next, we consider the average of the characteristic polynomial inverse. For this purpose, we define the dual CD kernel as follows.

Definition 4.4 (Dual Christoffel–Darboux kernel). Let $N \in 2\mathbb{N}$. We define the integral operator as follows,

$$\bar{A}_N = A - \sum_{k, l=0}^{N-1} A | p_k \rangle \tilde{N}_{k, l}^{(1)} \langle p_l | A = A \left(\tilde{A} - K_N^{(1)} \right) A. \quad (4.8)$$

Recalling the relation (3.14), we may write

$$\bar{A}_N = \sum_{i=N/2}^{\infty} A (|\psi_{2i-1}\rangle \langle \psi_{2i-2}| - |\psi_{2i-2}\rangle \langle \psi_{2i-1}|) A, \quad (4.9)$$

where $(\psi_k)_{k \in \mathbb{N}}$ is a set of degree- k skew-orthonormal polynomials, $\langle \psi_{2i-2} | A | \psi_{2j-1} \rangle = \delta_{i, j}$, $\langle \psi_{2i} | A | \psi_{2j} \rangle = \langle \psi_{2i-1} | A | \psi_{2j-1} \rangle = 0$. Then, we define the dual Christoffel–Darboux kernel through the double Hilbert transform,

$$\begin{aligned} \tilde{K}_N^{(1)}(z, w) &= \int d\mu(x) d\mu(\tilde{x}) \frac{\bar{A}_N(x, \tilde{x})}{(z-x)(w-\tilde{x})} \\ &= \int d\mu(x) d\mu(\tilde{x}) \sum_{i=N/2}^{\infty} \frac{\langle x | A (|\psi_{2i-1}\rangle \langle \psi_{2i-2}| - |\psi_{2i-2}\rangle \langle \psi_{2i-1}|) | A | \tilde{x} \rangle}{(z-x)(w-\tilde{x})}. \end{aligned} \quad (4.10)$$

Using this dual CD kernel, we obtain the following Pfaffian formula for the characteristic polynomial inverse.

Proposition 4.5 (Characteristic polynomial inverse). Let $N, M \in 2\mathbb{N}$. For $Z = (z_\alpha)_{\alpha \in [M]}$, the characteristic polynomial inverse average of the $\beta = 1$ model is given by a Pfaffian of the dual CD kernel,

$$N \geq M : \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-1} \right\rangle_1 = (-1)^{NM} \frac{\mathcal{Z}_{N-M}^{(1)}/\mathcal{Z}_N^{(1)}}{\Delta_M(Z)} \text{Pf}_{\alpha, \beta \in [M]} \tilde{K}_{N-M}^{(1)}(z_\alpha, z_\beta), \quad (4.11a)$$

$$N \leq M : \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-1} \right\rangle_1 = \frac{1/\mathcal{Z}_N^{(1)}}{\Delta_M(Z)} \text{Pf}_{\alpha, \beta \in [M]} \tilde{K}_0^{(1)}(z_\alpha, z_\beta) \\ \times \text{Pf}_{k, l \in [M-N]} \left[\sum_{\alpha, \beta=1}^M \langle p_{M-N-k} | z_\alpha \rangle \langle z_\alpha | \frac{1}{\tilde{K}_0^{(1)}} | z_\beta \rangle \langle z_\beta | p_{M-N-l} \rangle \right]. \quad (4.11b)$$

Proof. We first consider the case $N \geq M$. In this case, we apply the Schur polynomial expansion formula (Lemma 2.5) to obtain

$$\left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-1} \right\rangle_1 = (-1)^{NM} \sum_{\ell(\lambda) \leq M} s_\lambda(Z) \left\langle \frac{s_\lambda(X^{-1})}{\det X^M} \right\rangle_1 \\ = \frac{(-1)^{NM}}{\mathcal{Z}_N^{(1)} \Delta_M(Z)} \sum_{\ell(\lambda) \leq M} \det z_\alpha^{\lambda_\beta + M - \beta} \\ \times \text{Pf}_{\substack{\alpha, \beta \in [M] \\ k, l \in [N-M]}} \left[\begin{array}{cc} \langle x^{-\lambda_\alpha - M + \alpha - 1} | A | x^{-\lambda_\beta - M + \beta - 1} \rangle & \langle x^{-\lambda_\alpha - M + \alpha - 1} | A | x^{N-M-l} \rangle \\ \langle x^{N-M-k} | A | x^{-\lambda_\beta - M + \beta - 1} \rangle & \langle x^{N-M-k} | A | x^{N-M-l} \rangle \end{array} \right] \\ = \frac{(-1)^{NM}}{\Delta_M(Z)} \frac{\mathcal{Z}_{N-M}^{(1)}}{\mathcal{Z}_N^{(1)}} \sum_{\ell(\lambda) \leq M} \det z_\alpha^{\lambda_\beta + M - \beta} \text{Pf}_{\alpha, \beta \in [M]} \left[\langle x^{-\lambda_\alpha - M + \alpha - 1} | \bar{A}_{N-M} | x^{-\lambda_\beta - M + \beta - 1} \rangle \right]. \quad (4.12)$$

Applying the Cauchy–Binet-type expansion (Lemma 2.10), we have

$$\sum_{\ell(\lambda) \leq M} \det z_\alpha^{\lambda_\beta + M - \beta} \text{Pf}_{\alpha, \beta \in [M]} \left[\langle x^{-\lambda_\alpha - M + \alpha - 1} | \bar{A}_{N-M} | x^{-\lambda_\beta - M + \beta - 1} \rangle \right] \\ = \frac{1}{M!} \sum_{\substack{r_\alpha=0, \dots, \infty \\ r_\alpha \neq r_\beta}} \det z_\alpha^{r_\beta} \text{Pf}_{\alpha, \beta \in [M]} \left[\langle x^{-r_\alpha - 1} | \bar{A}_{N-M} | x^{-r_\beta - 1} \rangle \right] \\ = \text{Pf}_{\alpha, \beta \in [M]} \left[\sum_{r, s=0}^{\infty} z_\alpha^r z_\beta^s \langle x^{-r-1} | \bar{A}_{N-M} | x^{-s-1} \rangle \right] = \text{Pf}_{\alpha, \beta \in [M]} \left[\tilde{K}_{N-M}^{(1)}(z_\alpha, z_\beta) \right]. \quad (4.13)$$

The last expression is obtained as follows,

$$\sum_{r, s=0}^{\infty} z^r w^s \langle x^{-r-1} | \bar{A}_{N-M} | x^{-s-1} \rangle = \int d\mu(x) d\mu(\tilde{x}) \frac{\bar{A}_{N-M}(x, \tilde{x})}{(z-x)(w-\tilde{x})} \\ = \tilde{K}_{N-M}^{(1)}(z, w). \quad (4.14)$$

This completes the derivation for the case $N \geq M$. For the case $N \leq M$, on the other hand,

the Schur polynomial expansion yields

$$\begin{aligned} \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-1} \right\rangle_1 &= \sum_{\ell(\lambda) \leq N} \frac{s_\lambda(Z^{-1})}{\det Z^N} \langle s_\lambda(X) \rangle_1 \\ &= \frac{1}{\mathcal{Z}_N^{(1)} \Delta_M(Z)} \sum_{\ell(\lambda) \leq N} \det_{\substack{\alpha \in [M], \beta \in [N] \\ k \in [M-N]}} \begin{bmatrix} z_\alpha^{-\lambda_\beta - N + \beta - 1} \\ p_{M-N-k}(z_\alpha) \end{bmatrix} \text{Pf}_{\alpha, \beta \in [N]} \langle x^{\lambda_\alpha + N - \alpha} \mid A \mid x^{\lambda_\beta + N - \beta} \rangle. \end{aligned} \quad (4.15)$$

Applying the Cauchy–Binet-type expansion (2.29), we obtain

$$\begin{aligned} &\left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-1} \right\rangle_1 \\ &= \frac{1/\mathcal{Z}_N^{(1)}}{\Delta_M(Z)} \text{Pf}_{\substack{\alpha, \beta \in [M] \\ k, l \in [M-N]}} \begin{bmatrix} \tilde{K}_0^{(1)}(z_\alpha, z_\beta) & p_{M-N-l}(z_\alpha) \\ -p_{M-N-k}(z_\beta) & 0 \end{bmatrix} \\ &= \frac{1/\mathcal{Z}_N^{(1)}}{\Delta_M(Z)} \text{Pf}_{\alpha, \beta \in [M]} \tilde{K}_0^{(1)}(z_\alpha, z_\beta) \text{Pf}_{k, l \in [M-N]} \left[\sum_{\alpha, \beta=1}^M \langle p_{M-N-k} \mid z_\alpha \rangle \langle z_\alpha \mid \frac{1}{\tilde{K}_0^{(1)}} \mid z_\beta \rangle \langle z_\beta \mid p_{M-N-l} \rangle \right]. \end{aligned} \quad (4.16)$$

This completes the proof. \square

Remark 4.6. Applying the characteristic polynomial formulas for the two-point function ($M = 2$), we obtain an alternative form of the CD kernels,

$$\frac{\mathcal{Z}_{N+2}^{(1)}}{\mathcal{Z}_N^{(1)}} K_{N+2}^{(1)}(z, w) = (z - w) \langle \det(z - X) \det(w - X) \rangle_1, \quad (4.17a)$$

$$\frac{\mathcal{Z}_{N-2}^{(1)}}{\mathcal{Z}_N^{(1)}} \tilde{K}_{N-2}^{(1)}(z, w) = (z - w) \left\langle \det(z - X)^{-1} \det(w - X)^{-1} \right\rangle_1, \quad (4.17b)$$

which was originally observed in [BS06]. Such a relation between the characteristic polynomial average and the CD kernel is also available for the $\beta = 2$ model (a.k.a., generalized Heine formula). See, e.g. [EKR15] for details.

4.2 $\beta = 4$ model

We consider the following special case of the $\beta = 4$ model (3.18) for $N \in \mathbb{N}$ and skew-symmetric functions $A(x, y) = -A(y, x)$ and $B(x, y) = -B(y, x)$, and a generic integrable function $S(x, y)$:

$$\begin{aligned} \mathcal{Z}_N^{(4)} &= \frac{1}{N!^2} \int d\mu(X; Y) \det_{i \in [N], j \in [2N]} \begin{bmatrix} p_{2N-j}(x_i) & p'_{2N-j}(y_i) \end{bmatrix} \text{Pf}_{i, j \in [N]} \begin{bmatrix} A(x_i, x_j) & S(x_i, y_j) \\ -S^T(y_i, x_j) & B(y_i, y_j) \end{bmatrix} \\ &= \frac{1}{N!^2} \int d\mu(X; Y) \det_{i \in [N], j \in [2N]} \begin{bmatrix} p_{2N-j}(x_i) & p'_{2N-j}(y_i) \end{bmatrix} \text{Pf}_{i, j \in [N]} \mathbf{A}(x_i, x_j; y_i, y_j). \end{aligned} \quad (4.18)$$

In order to consider the characteristic polynomial average for the $\beta = 4$ model, we compute the average of the modified Schur polynomial as follows.

Lemma 4.7 (Modified Schur polynomial average). Let $N \in \mathbb{N}$. Then, the modified Schur polynomial average of the $\beta = 4$ model is given by a rank $2N$ Pfaffian as follows,

$$\langle s_\lambda(X; Y) \rangle_4 = \frac{1}{\mathcal{Z}_N^{(4)}} \text{Pf}_{i, j \in [2N]} \langle x^{\lambda_i + 2N - i} \mid \mathbf{A} \mid y^{\lambda_j + 2N - j} \rangle, \quad (4.19)$$

where we here denote the norm by

$$\langle f \mid \mathbf{A} \mid g \rangle := \langle f, g' \mid \mathbf{A} \mid f, g' \rangle. \quad (4.20)$$

Proof. From the definition of the modified Schur polynomial (2.18), we have

$$\begin{aligned} & \langle \mathfrak{s}_\lambda(X; Y) \rangle_4 \\ &= \frac{1}{N!^2 \mathcal{Z}_N^{(4)}} \int d\mu(X; Y) \det_{i \in [N], j \in [2N]} \left[x_i^{\lambda_j + 2N - j} \quad (y_i^{\lambda_j + 2N - j})' \right] \text{Pf}_{i, j \in [N]} \begin{bmatrix} A(x_i, x_j) & S(x_i, y_j) \\ -S^T(y_i, x_j) & B(y_i, y_j) \end{bmatrix}. \end{aligned} \quad (4.21)$$

Then, applying the de Bruijn's formula (2.33b), we arrive at the formula. \square

Remark 4.8. Putting

$$A(x, y) = B(x, y) = 0, \quad S(x, y) = \delta(x - y), \quad (4.22)$$

as in Lemma 3.12, the average of the modified Schur polynomial is reduced to that for the ordinary $2N$ -variable Schur polynomial with respect to the symplectic ensemble,

$$\langle \mathfrak{s}_\lambda(X; Y) \rangle_4 \xrightarrow{(4.22)} \langle s_\lambda(X; X) \rangle_4. \quad (4.23)$$

In order to describe the characteristic polynomial average, we define the auxiliary kernel, which is the $(1, 1)$ -component of the matrix kernel $K_N^{(4)}$, by

$$K_N^{(4)}(x, y) = \sum_{i, j=0}^{2N-1} p_i(x) \tilde{N}_{i, j}^{(4)} p_j(y). \quad (4.24)$$

Then, we may write the matrix kernel in terms of the auxiliary kernel as follows,

$$K_N^{(4)}(x, \tilde{x}; y, \tilde{y}) = \begin{bmatrix} K_N^{(4)}(x, \tilde{x}) & \partial_{\tilde{y}} K_N^{(4)}(x, \tilde{y}) \\ \partial_y K_N^{(4)}(y, \tilde{x}) & \partial_{y, \tilde{y}}^2 K_N^{(4)}(y, \tilde{y}) \end{bmatrix}. \quad (4.25)$$

With this auxiliary kernel, we obtain the following Pfaffian formula for the characteristic polynomial average.

Proposition 4.9. Let $N \in \mathbb{N}$, $M \in 2\mathbb{N}$, and we denote $Z = (z_\alpha)_{\alpha \in [M]}$. Then, the following Pfaffian formula holds,

$$\sum_{\lambda \subseteq (M^{2N})} (-1)^{|\lambda|} s_\lambda(Z) \langle \mathfrak{s}_\lambda(X; Y) \rangle_4 = \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N+M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} K_{N+M/2}^{(4)}(z_\alpha, z_\beta). \quad (4.26)$$

Proof. Applying the Laplace-type expansion (Lemma 2.9) and the formula for the block matrix (2.25), we have

$$\begin{aligned} & \sum_{\lambda \subseteq (M^{2N})} (-1)^{|\lambda|} s_\lambda(Z) \langle \mathfrak{s}_\lambda(X; Y) \rangle_4 \\ &= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \sum_{\lambda \subseteq (M^{2N})} (-1)^{|\lambda|} \det_{\alpha, \beta \in [M]} z_\alpha^{\lambda_\beta + M - \beta} \text{Pf}_{i, j \in [2N]} \langle x^{\lambda_i + 2N - i} \mid \mathbf{A} \mid y^{\lambda_j + 2N - j} \rangle \\ &= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \begin{bmatrix} \langle x^{2N+M-i} \mid \mathbf{A} \mid y^{2N+M-j} \rangle & z_\alpha^{2N+M-i} \\ -z_\beta^{2N+M-j} & 0 \end{bmatrix} \\ &= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \begin{bmatrix} \langle p_{2N+M-i} \mid \mathbf{A} \mid p_{2N+M-j} \rangle & p_{2N+M-i}(z_\alpha) \\ -p_{2N+M-j}(z_\beta) & 0 \end{bmatrix} \\ &= \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N+M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \left[\sum_{i, j=0}^{2N+M-1} p_i(z_\alpha) \tilde{N}_{i, j}^{(4)} p_j(z_\beta) \right] = \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N+M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} K_{N+M/2}^{(4)}(z_\alpha, z_\beta). \end{aligned} \quad (4.27)$$

This completes the proof. \square

Corollary 4.10 (Characteristic polynomial). The characteristic polynomial average with respect to the symplectic ensemble is given by a Pfaffian,

$$\left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^2 \right\rangle_4 = \frac{\mathcal{Z}_{N+M/2}^{(4)} / \mathcal{Z}_N^{(4)}}{\Delta_M(Z)} \text{Pf}_{\alpha, \beta \in [M]} K_{N+M/2}^{(4)}(z_\alpha, z_\beta) \Big|_{(4.22)} \quad (4.28)$$

We then consider the characteristic polynomial inverse for the $\beta = 4$ model. For this purpose, we introduce the dual CD kernel as follows.

Definition 4.11 (Dual Christoffel–Darboux kernel). Let $N \in \mathbb{N}$. We define the integral operator as follows,

$$\bar{A}_N = A \left(\tilde{A} - \mathbf{K}_N^{(4)} \right) A. \quad (4.29)$$

Recalling the relation (3.32), we may write

$$\bar{A}_N = \sum_{i=N}^{\infty} A \begin{bmatrix} |\phi_{2i-1}\rangle \langle \phi_{2i-2}| - |\phi_{2i-2}\rangle \langle \phi_{2i-1}| & |\phi_{2i-1}\rangle \langle \phi'_{2i-2}| - |\phi_{2i-2}\rangle \langle \phi'_{2i-1}| \\ \langle \phi'_{2i-1}| \langle \phi_{2i-2}| - \langle \phi'_{2i-2}| \langle \phi_{2i-1}| & \langle \phi'_{2i-1}| \langle \phi'_{2i-2}| - \langle \phi'_{2i-2}| \langle \phi'_{2i-1}| \end{bmatrix} A, \quad (4.30)$$

where $(\phi_k)_{k \in \mathbb{N}}$ is a set of degree- k skew-orthonormal polynomials, $\langle \phi_{2i-2} | A | \phi_{2j-1} \rangle = \delta_{i,j}$, $\langle \phi_{2i} | A | \phi_{2j} \rangle = \langle \phi_{2i-1} | A | \phi_{2j-1} \rangle = 0$. Then, we define the dual matrix Christoffel–Darboux kernel through the double Hilbert transform,

$$\tilde{\mathbf{K}}_N^{(4)}(z, w) = \int d\mu(x) d\mu(\tilde{x}) \frac{\bar{A}_N(x, \tilde{x})}{(z-x)(w-\tilde{x})}. \quad (4.31)$$

We may also define an auxiliary kernel (the $(1, 1)$ -component of the matrix kernel $\tilde{\mathbf{K}}_N^{(4)}$),

$$\tilde{K}_N^{(4)}(z, w) = \int d\mu(x) d\mu(\tilde{x}) \sum_{i=N}^{\infty} \frac{\langle x | A | (|\phi_{2i-1}\rangle \langle \phi_{2i-2}| - |\phi_{2i-2}\rangle \langle \phi_{2i-1}|) | A | \tilde{x} \rangle}{(z-x)(w-\tilde{x})}. \quad (4.32)$$

Using this dual CD kernel, we obtain the following Pfaffian formula.

Proposition 4.12. Let $N \in \mathbb{N}$, $M \in 2\mathbb{N}$, and we denote $Z = (z_\alpha)_{\alpha \in [M]}$. Then, the following Pfaffian formulas hold.

$$2N \geq M : \sum_{\ell(\lambda) \leq M} \frac{s_\lambda(Z^{-1})}{\det Z^{2N}} \langle s_\lambda(X; Y) \rangle_4 = \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N-M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \tilde{\mathbf{K}}_{N-M/2}^{(4)}(z_\alpha, z_\beta), \quad (4.33a)$$

$$\begin{aligned} 2N \leq M : & \sum_{\ell(\lambda) \leq 2N} \frac{s_\lambda(Z^{-1})}{\det Z^{2N}} \langle s_\lambda(X; Y) \rangle_4 \\ &= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \tilde{\mathbf{K}}_0^{(4)}(z_\alpha, z_\beta) \text{Pf}_{k, l \in [M-2N]} \left[\sum_{\alpha, \beta=1}^M \langle p_{M-2N-k} | z_\alpha \rangle \langle z_\alpha | \frac{1}{\tilde{\mathbf{K}}_0^{(4)}} | z_\beta \rangle \langle z_\beta | p_{M-2N-l} \rangle \right]. \end{aligned} \quad (4.33b)$$

Proof. We first consider the case $2N \geq M$. The summation over the partition is given as follows,

$$\begin{aligned}
& \sum_{\ell(\lambda) \leq M} \frac{s_\lambda(Z^{-1})}{\det Z^{2N}} \langle s_\lambda(X; Y) \rangle_4 \\
&= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \sum_{\ell(\lambda) \leq M} \det z_\alpha^{-\lambda_\beta - 2N + \beta - 1} \\
&\quad \times \text{Pf}_{\substack{\alpha, \beta \in [M] \\ k, l \in [2N - M]}} \left[\begin{array}{cc} \langle x^{\lambda_\alpha + 2N - \alpha} | \mathbf{A} | y^{\lambda_\beta + 2N - \beta} \rangle & \langle x^{\lambda_\alpha + 2N - \alpha} | \mathbf{A} | y^{2N - M - l} \rangle \\ \langle x^{2N - M - k} | \mathbf{A} | y^{\lambda_\beta + 2N - \beta} \rangle & \langle x^{2N - M - k} | \mathbf{A} | y^{2N - M - l} \rangle \end{array} \right] \\
&= \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N-M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \frac{1}{M!} \sum_{\substack{r_\alpha = 0, \dots, \infty \\ r_\alpha \neq r_\beta}} \det z_\alpha^{-r_\beta + M - 2N - 1} \text{Pf}_{\alpha, \beta \in [M]} \langle x^{r_\alpha - M + 2N} | \bar{\mathbf{A}}_{N-M/2} | y^{r_\beta - M + 2N} \rangle \\
&= \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N-M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \left[\sum_{r, s=0}^{\infty} z_\alpha^{-r + M - 2N - 1} z_\beta^{-s + M - 2N - 1} \langle x^{r - M + 2N} | \bar{\mathbf{A}}_{N-M/2} | y^{s - M + 2N} \rangle \right] \\
&= \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N-M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \tilde{\mathbf{K}}_{N-M/2}^{(4)}(z_\alpha, z_\beta). \tag{4.34}
\end{aligned}$$

In order to obtain the last expression, we may apply the same argument to the $\beta = 1$ case (4.14). The case $2N \leq M$ can be obtained in a similar way. Indeed,

$$\begin{aligned}
& \sum_{\ell(\lambda) \leq 2N} \frac{s_\lambda(Z^{-1})}{\det Z^{2N}} \langle s_\lambda(X; Y) \rangle_4 \\
&= \frac{1}{\mathcal{Z}_N^{(4)} \Delta_M(Z)} \sum_{\ell(\lambda) \leq 2N} \det \begin{array}{c} z_\alpha^{-\lambda_\beta - 2N + \beta - 1} \\ p_{M-2N-k}(z_\alpha) \end{array} \text{Pf}_{\alpha, \beta \in [2N]} \langle x^{\lambda_\alpha + 2N - \alpha} | \mathbf{A} | y^{\lambda_\beta + 2N - \beta} \rangle \\
&= \frac{1}{\mathcal{Z}_N^{(4)} \Delta_M(Z)} \text{Pf}_{\substack{\alpha, \beta \in [M] \\ k, l \in [M-2N]}} \left[\begin{array}{cc} \tilde{\mathbf{K}}_0^{(4)}(z_\alpha, z_\beta) & p_{M-2N-l}(z_\alpha) \\ -p_{M-2N-k}(z_\beta) & 0 \end{array} \right] \\
&= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \tilde{\mathbf{K}}_0^{(4)}(z_\alpha, z_\beta) \text{Pf}_{k, l \in [M-2N]} \left[\sum_{\alpha, \beta=1}^M \langle p_{M-2N-k} | z_\alpha \rangle \langle z_\alpha | \frac{1}{\tilde{\mathbf{K}}_0^{(4)}} | z_\beta \rangle \langle z_\beta | p_{M-2N-l} \rangle \right] \\
&\tag{4.35}
\end{aligned}$$

where first the Cauchy-Binet-type formula (2.29) and then the Pfaffian of a block matrix (2.25) have been used. This completes the proof. \square

Corollary 4.13 (Characteristic polynomial inverse). The characteristic polynomial inverse with respect to the symplectic ensemble is therefore given by the Pfaffians

$$2N \geq M : \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-2} \right\rangle_4 = \frac{1}{\Delta_M(Z)} \frac{\mathcal{Z}_{N-M/2}^{(4)}}{\mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \tilde{\mathbf{K}}_{N-M/2}^{(4)}(z_\alpha, z_\beta) \Big|_{(4.22)}, \tag{4.36a}$$

$$\begin{aligned}
& 2N \leq M : \left\langle \prod_{\alpha \in [M]} \det(z_\alpha - X)^{-2} \right\rangle_4 \\
&= \frac{1}{\Delta_M(Z) \mathcal{Z}_N^{(4)}} \text{Pf}_{\alpha, \beta \in [M]} \tilde{\mathbf{K}}_0^{(4)}(z_\alpha, z_\beta) \text{Pf}_{k, l \in [M-2N]} \left[\sum_{\alpha, \beta=1}^M \langle p_{M-2N-k} | z_\alpha \rangle \langle z_\alpha | \frac{1}{\tilde{\mathbf{K}}_0^{(4)}} | z_\beta \rangle \langle z_\beta | p_{M-2N-l} \rangle \right]. \\
&\tag{4.36b}
\end{aligned}$$

5 BCD -quiver matrix models

We consider more generalized situations, that we call quiver matrix models, involving several sets of formal eigenvalues with interactions. We in particular explore the BCD -type quivers.

5.1 D -type quiver models

We first consider the matrix model that we call the D -type quiver model as follows.

Definition 5.1. Let $L, N \in \mathbb{N}$ and $(X_k, \tilde{X}_k)_{k \in [L]} = (x_{k,i}, \tilde{x}_{k,i})_{i \in [N]}^{k \in [L]}$ a set of formal eigenvalues. Denoting $Z_k = (z_{k,i}, z_{k,N+i})_{i \in [N]} = (x_{k,i}, \tilde{x}_{k,i})_{i \in [N]}$ and $\underline{X} = (X_k, \tilde{X}_k)_{k \in [L]}$, we define the partition function of the D_{L+1} -type quiver matrix model,

$$\begin{aligned} \mathcal{Z}[D_{L+1}] &= \int \frac{d\mu(\underline{X})}{(2N)!^{L-1} N!^2} \frac{\Delta_{2N}(Z_1)^2 \cdots \Delta_{2N}(Z_{L-1})^2 \Delta_N(X_L)^2 \Delta_N(\tilde{X}_L)^2}{\prod_{i,j \in [2N]} (z_{1,i} - z_{2,j}) \cdots (z_{L-2,i} - z_{L-1,j}) \prod_{i \in [2N], j \in [N]} (z_{L-1,i} - x_{L,j})(z_{L-1,i} - \tilde{x}_{L,j})}. \end{aligned} \quad (5.1)$$

This model is graphically described using the quiver diagram of type D_{L+1} as shown in (1.2). For this D -type model, the following Pfaffian formula holds.

Proposition 5.2. We denote $\omega(x, y) = \frac{1}{x-y}$. Then, the following Pfaffian formula holds for the partition function of the D -type matrix model,

$$\begin{aligned} \mathcal{Z}[D_{L+1}] &= \text{Pf}_{i,j \in [2N]} \left[\int \prod_{k=1}^L d\mu(x_k; \tilde{x}_k) \left(\frac{p_{2N-i}(x_1)}{x_1 - x_2} \cdots \frac{1}{x_{L-1} - x_L} \frac{1}{x_L - \tilde{x}_L} \frac{1}{\tilde{x}_L - \tilde{x}_{L-1}} \cdots \frac{p_{2N-j}(\tilde{x}_1)}{\tilde{x}_2 - \tilde{x}_1} \right. \right. \\ &\quad \left. \left. - \frac{p_{2N-j}(\tilde{x}_1)}{\tilde{x}_1 - \tilde{x}_2} \cdots \frac{1}{\tilde{x}_{L-1} - \tilde{x}_L} \frac{1}{\tilde{x}_L - x_L} \frac{1}{x_L - x_{L-1}} \cdots \frac{p_{2N-i}(x_1)}{x_2 - x_1} \right) \right] \\ &=: \text{Pf}_{i,j \in [2N]} [\langle p_{2N-i} | \omega^{2L-1} | p_{2N-j} \rangle - \langle p_{2N-j} | \omega^{2L-1} | p_{2N-i} \rangle]. \end{aligned} \quad (5.2)$$

Proof. Recalling the relation

$$\Delta_N(X_L) \Delta_N(\tilde{X}_L) = \frac{\Delta_{2N}(Z_L)}{\prod_{i,j \in [N]} (x_{L,i} - \tilde{x}_{L,j})}, \quad (5.3)$$

we may rewrite the partition function as follows,

$$\begin{aligned} \mathcal{Z}[D_{L+1}] &= \int \frac{d\mu(\underline{X})}{(2N)!^{L-1} N!^2} \frac{\Delta_{2N}(Z_1)^2 \cdots \Delta_{2N}(Z_{L-1})^2 \Delta_{2N}(Z_L)^2}{\prod_{i,j \in [2N]} (z_{1,i} - z_{2,j}) \cdots (z_{L-1,i} - z_{L,j}) \prod_{i,j \in [N]} (x_{L,i} - \tilde{x}_{L,j})} \\ &= \int \frac{d\mu(\underline{X})}{(2N)!^{L-1} N!^2} \det_{i,j \in [2N]} p_{2N-i}(z_{1,j}) \prod_{k=1}^{L-1} \left[\det_{i,j \in [2N]} \left(\frac{1}{z_{k,i} - z_{k+1,j}} \right) \right] \det_{i,j \in [N]} \left(\frac{1}{x_{L,i} - \tilde{x}_{L,j}} \right). \end{aligned} \quad (5.4)$$

Applying Andréief formula (2.32) and de Bruijn's formula (2.33) recursively, we obtain

$$\begin{aligned} \mathcal{Z}[D_{L+1}] &= \int \frac{d\mu(Z_L)}{N!^2} \det_{\substack{i \in [2N] \\ j \in [N]}} \left[\langle p_{2N-i} | \omega^{L-1} | x_{L,j} \rangle \right] \det_{i,j \in [N]} \langle x_{L,i} | \omega | \tilde{x}_{L,j} \rangle \\ &= \int \frac{d\mu(Z_L)}{N!^2} \det_{\substack{i \in [2N] \\ j \in [N]}} \left[\langle p_{2N-i} | \omega^{L-1} | x_{L,j} \rangle \right] \text{Pf}_{i,j \in [N]} \begin{bmatrix} 0 & \langle x_{L,i} | \omega | \tilde{x}_{L,j} \rangle \\ -\langle \tilde{x}_{L,i} | \omega | x_{L,j} \rangle & 0 \end{bmatrix} \\ &= \text{Pf}_{i,j \in [2N]} [\langle p_{2N-i} | \omega^{2L-1} | p_{2N-j} \rangle - \langle p_{2N-j} | \omega^{2L-1} | p_{2N-i} \rangle]. \end{aligned} \quad (5.5)$$

This completes the proof. \square

Remark 5.3. We may apply the same approach to consider the average of the characteristic polynomial coupled with the left-most matrix X_1 , and obtain the Pfaffian formula in terms of the corresponding CD kernel. See also [BK22] for the A -type quiver model calculation.

5.2 Matrix chain with the Pfaffian interaction

We then consider the matrix chain generalization of the $\beta = 1$ and $\beta = 4$ models.

Definition 5.4. Let $N \in 2\mathbb{N}$ ($\beta = 1$), \mathbb{N} ($\beta = 4$), $L \in \mathbb{N}$ and $\underline{X} = (X_k)_{k \in [L]} = (x_{k,i})_{i \in [N]}^{k \in [L]}$ a set of formal eigenvalues. Let $A(x, y)$, $B(x, y)$ and $S(x, y)$ be skew-symmetric and generic integrable functions as before. We denote by $(\omega_k(x, y))_{k=1, \dots, L-1}$ generic integrable two-variable functions, and by $(f_i(x), g_i(x))_{i \in \mathbb{Z}_{>0}}$ a set of integrable functions. Then, we define the partition function of the $\beta = 1$ and $\beta = 4$ matrix chain models as follows,

$$\mathcal{Z}_{NL}^{(1)} = \frac{1}{N!^L} \int d\mu(\underline{X}) \det_{i,j \in [N]} f_{j-1}(x_{1,i}) \prod_{k=1}^{L-1} \left[\det_{i,j \in [N]} \omega_k(x_{k,i}, x_{k+1,j}) \right] \text{Pf}_{i,j \in [N]} A(x_{L,i}, x_{L,j}), \quad (5.6a)$$

$$\begin{aligned} \mathcal{Z}_{NL}^{(4)} &= \frac{1}{N!^{2L}} \int d\mu(\underline{X}) \det_{i \in [N], j \in [2N]} [f_{j-1}(x_{1,i}) \ g_{j-1}(\tilde{x}_{1,i})] \\ &\quad \times \prod_{k=1}^{L-1} \left[\det_{i,j \in [N]} \Omega_k(x_{k,i}, \tilde{x}_{k,i}; x_{k+1,j}, \tilde{x}_{k+1,j}) \right] \text{Pf}_{i,j \in [N]} A(x_{L,i}, \tilde{x}_{L,i}; x_{L,j}, \tilde{x}_{L,j}) \end{aligned} \quad (5.6b)$$

where we write

$$\Omega_k(x, \tilde{x}; y, \tilde{y}) = \begin{bmatrix} \omega_k^{11}(x, y) & \omega_k^{12}(x, \tilde{y}) \\ \omega_k^{21}(\tilde{x}, y) & \omega_k^{22}(\tilde{x}, \tilde{y}) \end{bmatrix}, \quad A(x, \tilde{x}; y, \tilde{y}) = \begin{bmatrix} A(x, y) & S(x, \tilde{y}) \\ -S^T(\tilde{x}, y) & B(\tilde{x}, \tilde{y}) \end{bmatrix}. \quad (5.6c)$$

For these partition functions, we obtain the Pfaffian formulas as follows.

Proposition 5.5. We define the dressed integral operators,

$$A_{L-1} = \omega_1 \cdots \omega_{L-1} \cdot A \cdot \omega_{L-1}^T \cdots \omega_1^T, \quad (5.7a)$$

$$A_{L-1} = \Omega_1 \cdots \Omega_{L-1} \cdot A \cdot \Omega_{L-1}^T \cdots \Omega_1^T. \quad (5.7b)$$

Then, the partition functions of the coupled $\beta = 1$ and $\beta = 4$ models are given by Pfaffians,

$$\mathcal{Z}_{NL}^{(1)} = \text{Pf}_{i,j \in [N]} \langle f_{i-1} \mid A_{L-1} \mid f_{j-1} \rangle, \quad (5.8a)$$

$$\mathcal{Z}_{NL}^{(4)} = \text{Pf}_{i,j \in [N]} \langle f_{i-1}, g_{i-1} \mid A_{L-1} \mid f_{j-1}, g_{j-1} \rangle. \quad (5.8b)$$

Proof. We first consider the case $\beta = 1$. Applying Andréief's formula (2.32), we integrate the eigenvalues $(x_{k,i})_{i \in [N]}^{k=1, \dots, L-1}$ as follows,

$$\begin{aligned} &\frac{1}{N!^L} \int d\mu(\underline{X}) \det_{i,j \in [N]} f_{j-1}(x_{1,i}) \prod_{k=1}^{L-1} \left[\det_{i,j \in [N]} \omega_k(x_{k,i}, x_{k+1,j}) \right] \\ &= \frac{1}{N!} \int d\mu(X_L) \det_{i,j \in [N]} \langle f_{j-1} \mid \omega_1 \cdots \omega_{L-1} \mid x_{L,i} \rangle. \end{aligned} \quad (5.9)$$

We have a similar relation for the case $\beta = 4$,

$$\begin{aligned} &\frac{1}{N!^{2L}} \int d\mu(\underline{X}) \det_{i \in [N], j \in [2N]} [f_{j-1}(x_{1,i}) \ g_{j-1}(\tilde{x}_{1,i})] \prod_{k=1}^{L-1} \left[\det_{i,j \in [N]} \Omega_k(x_{k,i}, \tilde{x}_{k,i}; x_{k+1,j}, \tilde{x}_{k+1,j}) \right] \\ &= \frac{1}{N!^2} \int d\mu(X_L) \det_{i \in [N], j \in [2N]} \left[\langle f_{j-1} \mid \langle g_{j-1} \mid \Omega_1 \cdots \Omega_{L-1} \begin{bmatrix} x_{L,i} \\ \tilde{x}_{L,i} \end{bmatrix} \right] \right]. \end{aligned} \quad (5.10)$$

Then, we apply de Bruijn's formula (2.33) to integrate the remaining variables $(x_{L,i})_{i \in [N]}$, and obtain the result. \square

These generic models yield the following special cases that would have quiver interpretations.

5.2.1 B -type quiver models

We have the following examples for the matrix chain models. For the $\beta = 4$ model, we consider

$$f_{i-1}(x) = g_{i-1}(x) = p_{2N-i}(x), \quad A(x, y) = B(x, y) = 0, \quad S(x, y) = \delta(x - y),$$

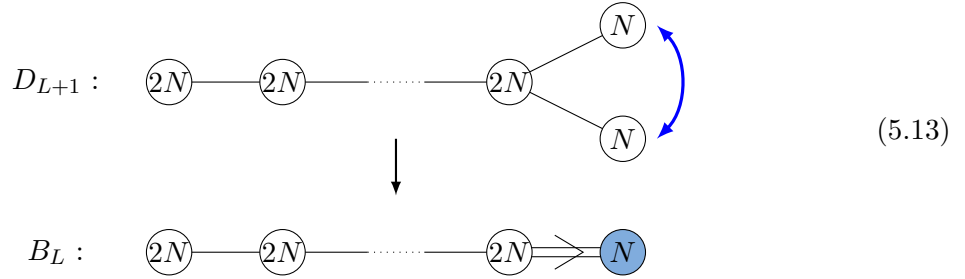
$$\omega_k^{l,m}(x, y) = \frac{1}{x - y} \quad (k = 1, \dots, L - 2). \quad \Omega_{L-1}(x, \tilde{x}; y, \tilde{y}) = \begin{bmatrix} \frac{1}{x-y} & \frac{1}{(x-\tilde{y})^2} \\ \frac{1}{\tilde{x}-y} & \frac{1}{(\tilde{x}-\tilde{y})^2} \end{bmatrix}, \quad (5.11)$$

and denote $Z_k = (z_{k,i}, z_{k,N+i})_{i \in [N]} = (x_{k,i}, \tilde{x}_{k,i})_{i \in [N]}$ and $\underline{X} = (X_k)_{k \in [L]}$. Then, we obtain

$$\mathcal{Z}_{N^L}^{(4)} = \int \frac{d\mu(\underline{X})}{(2N)!^{L-1} N!} \frac{\Delta_{2N}(Z_1)^2 \cdots \Delta_{2N}(Z_{L-1})^2 \Delta_N(X_L)^4}{\prod_{i,j \in [2N]} (z_{1,i} - z_{2,j}) \cdots (z_{L-2,i} - z_{L-1,j}) \prod_{i \in [2N], j \in [N]} (z_{L-1,i} - x_{L,j})^2}, \quad (5.12)$$

where we change the normalization of the partition function.

In fact, this model may have an interpretation as the B -type quiver matrix model, which is constructed from the D -type model (5.1): Under the process called the folding $X_L = \tilde{X}_L$, we obtain the partition function (5.12). This situation is graphically explained using the quiver diagrams:



In these diagrams, we denote the β -node $\Delta_N(X)^\beta$ by $\textcircled{\beta}$ ($\beta = 1$),² $\textcircled{2}$ ($\beta = 2$), $\textcircled{4}$ ($\beta = 4$), and the line connecting the nodes indicates the Cauchy-type interaction $\prod_{i,j \in [N]} (x_{k,i} - x_{k+1,j})^{-1}$ [BGS09].

5.2.2 C -type quiver models

For the $\beta = 1$ model, putting

$$f_{i-1}(x) = p_{N-i}(x), \quad \omega_k(x, y) = \frac{1}{x - y}, \quad A(x, y) = \text{sgn}(x - y), \quad (5.14)$$

we obtain

$$\mathcal{Z}_{N^L}^{(1)} = \int \frac{d\mu(X^L)}{N!^L} \frac{\Delta_N(X_1)^2 \cdots \Delta_N(X_{L-1})^2 |\Delta_N(X_L)|}{\prod_{i,j \in [N]} (x_{1,i} - x_{2,j}) \cdots (x_{L-1,i} - x_{L,j})}. \quad (5.15)$$

²We will consider the $\beta = 1$ node in the next part.

We also define the factored version of the symmetrized Cartan matrix,

$$\bar{b}_{ab} = \frac{1}{\gcd(b_{ab})} b_{ab}. \quad (5.21)$$

Then, for a Dynkin-quiver diagram Γ , we may write the partition function of the quiver matrix model as follows,

$$Z[\Gamma] = \int d\mu(\underline{X}) \prod_{a,b \in \Gamma} \prod_{i \in [N_a], j \in [N_b]} (x_{i,a} - x_{j,b})^{\bar{b}_{ab}(\Gamma^\vee)}, \quad (5.22)$$

where $\bar{b}_{ab}(\Gamma^\vee)$ is the factored symmetrized Cartan matrix for the Langlands dual quiver Γ^\vee . For the simply-laced quivers $\Gamma^\vee = \Gamma$, this reproduces the well-known construction [MMM91, KMM⁺93] as $\bar{b}_{ab} = c_{ab}$. From this point of view, the duality between $\beta = 1$ and $\beta = 4$ could be interpreted as the Feigin–Frenkel duality [FF92]. See [Fre07] for more details along this direction.

For example, for the B and C types, the symmetrization of the Cartan matrix is given as follows,

$$B_r : b = \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & & & \\ -2 & 4 & -2 & & \\ & & \ddots & \ddots & \\ & & & -2 & 4 & -2 \\ & & & & -2 & 2 \end{bmatrix} \quad (5.23a)$$

$$C_r : b = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 2 & -2 \\ & & & & -2 & 4 \end{bmatrix} \quad (5.23b)$$

Hence, the factored matrix is given by

$$\bar{b} = \begin{cases} b/2 & (\Gamma = B_r) \\ b & (\Gamma = C_r) \end{cases}. \quad (5.24)$$

Recalling $B_r^\vee = C_r$ and vice versa, the partition function defined in (5.22) reproduces the expressions (5.12) and (5.15).

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