



Uniqueness and tube property for the Swiss cheese large deviations

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UNIQUENESS AND TUBE PROPERTY FOR THE SWISS CHEESE LARGE DEVIATIONS

DIRK ERHARD AND JULIEN POISAT

ABSTRACT. We consider the simple random walk on the Euclidean lattice, in three dimensions and higher, conditioned to visit fewer sites than expected, when the deviation from the mean scales like the mean. The associated large deviation principle was first derived in 2001 by van den Berg, Bolthausen and den Hollander in the continuous setting, that is for the volume of a Wiener sausage, and later taken up by Phetpradap in the discrete setting. One of the key ideas in their work is to condition the range of the random walk to a certain skeleton, that is a sub-sequence of the random walk path taken along an appropriate mesoscopic scale. In this paper we prove that (i) the rate function obtained by van den Berg, Bolthausen and den Hollander has a unique minimizer over the set of probability measures modulo shifts, at least for deviations of the range well below the mean, and (ii) the empirical measure of the skeleton converges under the conditioned law, in a certain manner, to this minimizer. To this end we use an adaptation of the topology recently introduced by Mukherjee and Varadhan to compactify the space of probability measures.

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1. INTRODUCTION

Let $S = (S_n)_{n \in \mathbb{N}_0}$ be a discrete-time simple random walk on \mathbb{Z}^d , whose increments $(S_n - S_{n-1})_{n \in \mathbb{N}}$ are independent and uniformly distributed on the $2d$ unit vectors. We assume throughout the paper that $d \geq 3$ and we denote by P_x and E_x the probability and expectation with respect to the simple random walk starting from $x \in \mathbb{Z}^d$. We omit the subscript when $x = 0$. The *range* of the random walk up to time n is the set of all vertices visited by the process up to time n , which we denote by

$$(1.1) \quad S_{(0,n]} = \{S_1, \dots, S_n\},$$

and its cardinality (volume of the range) is denoted by

$$(1.2) \quad R_n = \# \{S_1, \dots, S_n\}.$$

The almost-sure asymptotic behavior for the volume of the range is given by the following Law of Large Numbers [8]

$$(1.3) \quad \frac{R_n}{n} \longrightarrow \kappa_d := P(S_k \neq 0, \forall k \geq 0), \quad n \rightarrow \infty.$$

The limit, sometimes called the *escape probability*, is positive as the random walk is transient in dimension $d \geq 3$. This probability coincides with the inverse of the Green function at the origin:

$$(1.4) \quad 1/\kappa_d = \sum_{n \geq 0} P(S_n = 0).$$

In this paper we consider the law of the simple random walk conditioned on the large deviation event that $R_n \leq bn$, in the limit of large n and when $b < \kappa_d$, that is for *lower deviations scaling like the mean*.

1.1. Moderate deviations of the range of simple random walk. The problem of evaluating the probability of the large deviation event above was first considered by van den Berg, Bolthausen and den Hollander [19] in the context of Wiener sausages instead of random walks. Let $B = (B_s)_{s \geq 0}$ be a standard Brownian motion on \mathbb{R}^d and $a > 0$.

The Wiener sausage with radius a and running up to time $t \geq 0$ is defined as the random subset

$$(1.5) \quad W^a(t) = \{x \in \mathbb{R}^d : \exists s \in [0, t] : |x - W_s| \leq a\},$$

where throughout the paper $|\cdot|$ denotes the Euclidean norm. The following almost-sure convergence [17, 20] is the analogue of (1.3) in the continuum:

$$(1.6) \quad \frac{\text{Leb}[W^a(t)]}{t} \longrightarrow \kappa_a := \text{cap}(\bar{\mathcal{B}}(0, a)), \quad t \rightarrow \infty,$$

where Leb is Lebesgue measure, $\bar{\mathcal{B}}(0, a)$ is the closed Euclidean ball with radius a centred at the origin, and $\text{cap}(\bar{\mathcal{B}}(0, a))$ denotes its Newtonian capacity.

For any $\kappa > 0$ and $b > 0$, define

$$(1.7) \quad I_\kappa(b) = \inf_{\phi \in \mathcal{D}_\kappa(b)} \left[\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 dx \right],$$

where

$$(1.8) \quad \mathcal{D}_\kappa(b) = \left\{ \phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x) dx = 1, \int_{\mathbb{R}^d} (1 - e^{-\kappa \phi^2(x)}) dx \leq b \right\}.$$

The main result in [19] reads:

Theorem 1.1 (Van den Berg, Bolthausen and den Hollander : moderate deviations for the volume of the Wiener sausage). *For all $b > 0$,*

$$(1.9) \quad \lim_{t \rightarrow \infty} t^{\frac{2}{d}-1} \log \mathbb{P}(\text{Leb}[W^a(t)] \leq bt) = -I_{\kappa_a}(b),$$

where κ_a is chosen as in (1.6).

Here, the term *moderate* refers to the exponent $1 - 2/d$ being smaller than one.

Remark 1.2. *It was shown in [19, Theorem 3] that $I_{\kappa_a}(b) > 0$ if and only if $b \in (0, \kappa_a)$.*

This result was later adapted to the random walk setting in Phetdradap's Ph.D thesis [15].

Theorem 1.3 (Phetdradap : deviations for the range of the simple random walk). *For all $b > 0$,*

$$(1.10) \quad \lim_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(R_n \leq bn) = -\frac{1}{d} I_{\kappa_d}(b),$$

where κ_d is chosen as in (1.3).

The function that governs the large deviation is the same in both cases, up to a multiplicative constant $1/d$. The lattice structure survives in the limit through the constant κ_d (instead of κ_a). It may be however pulled out from the variational formula by a simple scaling argument. In the sequel we shall write κ instead of κ_d and drop the dependence of $\mathcal{D}(b)$ and $I(b)$ on κ , in order to lighten notation. This should not lead to any confusion since we only deal with the simple random walk from now on.

While Theorems 1.1 and 1.3 settle the issue of the large deviation cost, the question about the law of the random walk *conditioned* on this large deviation event remains. In the continuous setting, van den Berg, Bolthausen and den Hollander [19] set forth a heuristic picture coined as the *Swiss cheese* strategy: the conditioned Brownian motion

should behave as if pushed by a drift field towards the origin, folding itself onto scale $t^{1/d}$ instead of the typical scale $t^{1/2}$. While doing so, the Wiener sausage *covers only part of the space and leaves random holes whose sizes are of order one and whose density varies on scale $t^{1/d}$* , to quote the authors. The function that drives the drift field is expected to be the minimizer of the rate function, provided there exists a unique minimizer, at least modulo spatial shifts. The uniqueness issue will be addressed in Section 1.2 below. Provided existence and uniqueness, we further show that the minimizer is the limiting profile for the occupation time measure of a certain subsequence of the random walk path (later called *skeleton*). The limit is for a certain topology, explained in Section 1.3, that disregards space shifts. This property, referred to as *tube property*, is exposed in Section 1.4. We believe that the two main results of this paper (Theorems 1.4 and 1.6) are a first step towards a rigorous description of that so-called *Swiss cheese* picture, conjectured to be linked to the model of random interlacements [18]. To the authors knowledge, the only available results on the conditioned random walk path were obtained by Asselah and Schapira [1, 2, 3].

1.2. Existence and uniqueness of minimizers modulo shifts. We first need to recall some definitions from [19]. Let $D^1(\mathbb{R}^d)$ be the set of locally integrable functions $\psi: \mathbb{R}^d \mapsto \mathbb{C}$ such that $\nabla \psi \in L^2(\mathbb{R}^d)$ (in the sense of distributions) and such that for all $a > 0$ the set $\{|f| > a\}$ has finite Lebesgue measure [13]. When $d \geq 5$, let us define

$$(1.11) \quad u_d^* := 1 - \left[\inf_{\psi \in \Sigma^*} \|\psi\|_2^2 \right]^{-1}$$

where $\|\cdot\|_2$ is the L^2 -norm and Σ^* is the set of local minimizers of $\|\nabla \psi\|_2$ among all $\psi \in D^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} (e^{-\psi^2} - 1 + \psi^2) = 1$. It is known that $2/d \leq u_d^* < 1$ [19, Theorem 5]. In [19, Theorems 4 and 5], the authors proved that, for all $b \in (0, \kappa)$ when $d \in \{3, 4\}$ or for all $b \in (0, u_d^* \kappa]$ when $d \geq 5$, the variational problem in (1.7) has a minimizer that is strictly positive, has a unique global maximum is radially symmetric (modulo shifts) and strictly decreasing in the radial component. Moreover, any other minimizer is of the same type. However, uniqueness was still open. In this paper we prove the following:

Theorem 1.4 (Uniqueness of minimizers). *The variational problem in (1.7) has at most one minimizer (modulo space shifts) for almost every $b \in (0, \kappa)$ when $d = 3$ and almost every $b \in (0, \frac{2}{d}\kappa)$ when $d \geq 4$.*

Remark 1.5. *If we were to know that $b \mapsto I(b)$ were differentiable, then our proof would allow us to remove the “almost every” part from our statement. Our method to prove uniqueness fails when b is close enough to κ and $d \geq 4$. It thus remains an open question to determine whether there is a unique minimizer modulo spatial shifts for all $b \in (0, \kappa)$ and $d = 4$. When b is close enough to κ and $d \geq 5$, it is known that there is no minimizer but rather a minimizing sequence of probability measure that loses mass [19, Theorem 5]. As we shall see below, one can embed the space of sub-probability measures modulo shifts into a larger space (its compactification with respect to a certain topology) on which we may also write a variational principle. The fundamental question then is if this new variational problem characterizes the Swiss cheese in the sense that one has equality in (1.10) when one replaces the right hand side in (1.10) by the new variational*

problem and whether there exists a unique minimizing sub-probability measure modulo spatial shifts.

From what precedes, we may assert that the variational problem in (1.7) has a unique minimizer (modulo space shifts) for almost all $b \in (0, \kappa)$ when $d = 3$, and for (at least) almost all $b \in (0, \frac{2}{d}\kappa)$ when $d \geq 4$. For such values of b , we shall then denote by ϕ_b the unique minimizer centered at the origin, and by

$$(1.12) \quad \mathbf{m}_b := \{\phi_b^2 * \delta_x : x \in \mathbb{R}^d\}$$

the set of minimizers, where $*$ is the convolution operation, δ_x is the Dirac mass at $x \in \mathbb{R}^d$ and, with a slight abuse of notation, ϕ_b^2 stands for the measure with density ϕ_b^2 with respect to Lebesgue measure. As we shall see, \mathbf{m}_b is the limit of the occupation time measure of a certain *skeleton* of the random walk conditioned on the large deviation event. In the next section we introduce the topology under consideration.

1.3. Compactification of the space of probability measures. The empirical and pair empirical measures of many Markov chains and processes such as simple random walk on \mathbb{Z}^d or Brownian motion on \mathbb{R}^d only satisfy a weak large deviation principle. This is due to the lack of exponential tightness. However, the fact that the large deviation upper bound only holds for compact sets is often a big obstacle. In our context this is not different. To circumvent that problem Mukherjee and Varadhan [14] introduced a new topology which takes the shift invariance of many models in statistical mechanics into account and allows to compactify the space of measures, see also [4, 7, 10, 6] for applications. In this section we summarize the construction of this topology.

Let $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ be the space of probability measures on \mathbb{R}^d and $\mathcal{M}_{\leq 1} = \mathcal{M}_{\leq 1}(\mathbb{R}^d)$ be the space of sub-probability measures on \mathbb{R}^d . We consider the action of the shifts θ_x , for $x \in \mathbb{R}^d$, defined by:

$$(1.13) \quad \int_{\mathbb{R}^d} f(u)(\theta_x \alpha)(du) = \int_{\mathbb{R}^d} f(u+x)\alpha(du)$$

for all continuous and bounded functions $f: \mathbb{R}^d \mapsto \mathbb{R}$ and $\alpha \in \mathcal{M}_{\leq 1}$. We shall denote by $\widetilde{\mathcal{M}}_1$ (resp. $\widetilde{\mathcal{M}}_{\leq 1}$) the space of equivalence classes of \mathcal{M}_1 (resp. $\mathcal{M}_{\leq 1}$) under the action of the shifts θ_x . For any $\alpha \in \mathcal{M}_{\leq 1}$ we denote by $\tilde{\alpha}$ its orbit, i.e., equivalence class. For $k \geq 2$, we define \mathcal{F}_k as the space of continuous functions $f: (\mathbb{R}^d)^k \mapsto \mathbb{R}$ that are *translation invariant*, i.e.

$$(1.14) \quad f(u_1+x, \dots, u_k+x) = f(u_1, \dots, u_k), \quad \forall x, u_1, \dots, u_k \in \mathbb{R}^d,$$

and *vanishing at infinity*, in the sense that

$$(1.15) \quad \lim_{\max_{i \neq j} |u_i - u_j| \rightarrow \infty} f(u_1, \dots, u_k) = 0.$$

For $k \geq 2$, $f \in \mathcal{F}_k$ and $\alpha \in \mathcal{M}_{\leq 1}$, we write

$$(1.16) \quad \Lambda(f, \alpha) := \int f(u_1, \dots, u_k) \prod_{1 \leq i \leq k} \alpha(du_i),$$

which actually only depends on the orbit $\tilde{\alpha}$. Let us define

$$(1.17) \quad \mathcal{F} := \bigcup_{k \geq 2} \mathcal{F}_k,$$

for which there exists a countable dense set (under the uniform metric) denoted by

$$(1.18) \quad \{f_r(u_1, \dots, u_{k_r}), r \in \mathbb{N}\}.$$

We then define the following set of empty, finite, or countable collections of sub-probability measure orbits:

$$(1.19) \quad \tilde{\mathcal{X}} := \left\{ \xi = \{\tilde{\alpha}_i\}_{i \in I} : \tilde{\alpha}_i \in \widetilde{\mathcal{M}}_{\leq 1}, \sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1 \right\}.$$

For every $\xi_1, \xi_2 \in \tilde{\mathcal{X}}$, define

$$(1.20) \quad \mathbf{D}(\xi_1, \xi_2) := \sum_{r \geq 1} \frac{1}{2^r} \frac{1}{1 + \|f_r\|_\infty} \left| \sum_{\tilde{\alpha} \in \xi_1} \Lambda(f_r, \alpha) - \sum_{\tilde{\alpha} \in \xi_2} \Lambda(f_r, \alpha) \right|.$$

It was then shown in [14] that the space $\tilde{\mathcal{X}}$ equipped with \mathbf{D} is a compact metric space and that $\widetilde{\mathcal{M}}_1$ is dense in $\tilde{\mathcal{X}}$. Moreover, the set of sub-probability measure orbits is naturally embedded into $\tilde{\mathcal{X}}$. We refer the interested reader to [14] for details. Let us however close this section with a simple (one-dimensional) example in order to better grasp the idea behind this topology. Consider a sequence of probability measure $(\mu_n)_{n \geq 1}$ defined by

$$(1.21) \quad \mu_n = \frac{1}{2} \mathcal{N}(n, 1) + \frac{1}{3} \mathcal{N}(-n, 2) + \frac{1}{6} \mathcal{N}(0, n),$$

where $\mathcal{N}(m, \sigma^2)$ is the normal distribution with mean m and variance σ^2 . This sequence does not converge in the weak topology but the sequence $(\tilde{\mu}_n)_{n \geq 1}$ does converge in the \mathbf{D} -topology to the limit $\xi = \{\frac{1}{2} \mathcal{N}(\cdot, 1), \frac{1}{3} \mathcal{N}(\cdot, 2)\} \in \tilde{\mathcal{X}}$, where $\mathcal{N}(\cdot, \sigma^2)$ denotes the normal distribution modulo space shifts. The reason behind that is that the two components $\frac{1}{2} \mathcal{N}(n, 1)$ and $\frac{1}{3} \mathcal{N}(-n, 2)$ coincide after a shift with $\frac{1}{2} \mathcal{N}(0, 1)$ and $\frac{1}{3} \mathcal{N}(0, 2)$, whereas the last component of μ_n simply goes to zero.

1.4. Tube property. In this section we state the second and last main theorem of this paper. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We cut the random walk trajectory in blocks of length

$$(1.22) \quad \ell := \ell(n, \varepsilon) = \lfloor \varepsilon n^{2/d} \rfloor.$$

The number of blocks is denoted by (we assume that $n \in \ell \mathbb{N}$ for simplicity)

$$(1.23) \quad M = \frac{n}{\ell} \sim \frac{1}{\varepsilon} n^{1-2/d},$$

as $n \rightarrow \infty$. The (renormalized) *skeleton* process is defined as

$$(1.24) \quad \hat{S}_i^{(\varepsilon)} := \frac{S_{i\ell}}{n^{1/d}} \in \frac{\mathbb{Z}^d}{n^{1/d}}, \quad 0 \leq i \leq M,$$

and its pair empirical measure, which is a random measure on $(\frac{\mathbb{Z}^d}{n^{1/d}})^2$, is denoted by

$$(1.25) \quad L_{M, \varepsilon}^{(2)} := \frac{1}{M} \sum_{0 \leq i \leq M} \delta_{(\hat{S}_{i-1}^{(\varepsilon)}, \hat{S}_i^{(\varepsilon)})},$$

where δ is the Dirac measure. We might sometimes omit the subscript ε to lighten notations. Although the pair empirical measure is central in the proof, only its first marginal is necessary to state our second main result. We denote it by

$$(1.26) \quad L_{M,\varepsilon} := \frac{1}{M} \sum_{0 \leq i < M} \delta_{\hat{S}_i^{(\varepsilon)}}.$$

Theorem 1.6 (Tube property). *Let $b \in (0, \kappa)$ if $d = 3$ or $b \in (0, \frac{2}{d}\kappa)$ if $d \geq 4$ be such that the variational problem in (1.7) has a unique minimizer modulo space shifts. Let $\mathcal{U}(\mathbf{m}_b)$ be an open neighbourhood of \mathbf{m}_b w.r.t. the \mathbf{D} -topology. There exists ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$,*

$$(1.27) \quad \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(\tilde{L}_{M,\varepsilon} \notin \mathcal{U}(\mathbf{m}_b) | R_n \leq bn) < 0.$$

Theorem 1.6 relies on a strengthening of the large deviation upper bound in Theorem 1.3 to the \mathbf{D} -topology introduced in Section 1.3. To ease notation we define for $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$

$$(1.28) \quad J(\alpha) := \begin{cases} J(\phi) = \frac{1}{2} \|\nabla \phi\|_2^2 & \text{if } \phi := \sqrt{\frac{d\alpha}{dx}} \text{ exists and is in } H^1(\mathbb{R}^d), \\ +\infty & \text{else,} \end{cases}$$

$$(1.29) \quad \Gamma(\alpha) := \begin{cases} \Gamma(\phi) = \int (1 - e^{-\kappa \phi^2(x)}) dx & \text{if } \phi := \sqrt{\frac{d\alpha}{dx}} \text{ exists,} \\ +\infty & \text{else.} \end{cases}$$

Both functions are translation invariant and may be extended to $\tilde{\mathcal{X}}$ by setting:

$$(1.30) \quad \tilde{J}(\xi) := \sum_{i \in I} J(\alpha_i), \quad \tilde{\Gamma}(\xi) := \sum_{i \in I} \Gamma(\alpha_i) + \kappa \left(1 - \sum_{i \in I} \alpha_i(\mathbb{R}^d)\right), \quad \xi = \{\tilde{\alpha}_i\}_{i \in I}.$$

We draw the reader's attention to the fact that even though $\tilde{\mathcal{M}}_{\leq 1}$ is embedded into $\tilde{\mathcal{X}}$, $\tilde{\Gamma}(\{\tilde{\alpha}\}) = \Gamma(\alpha)$ if and only if α is a *probability* measure. We will comment on the presence of the second term in the definition of $\tilde{\Gamma}$ further in the paper, see Lemma 3.1 and the comment just below. Theorem 1.6 then relies on the following extension of Theorem 1.3.

Proposition 1.7 (Large Deviations Upper Bound at the level of orbits). *For any set $F \subseteq \tilde{\mathcal{X}}$ closed in the \mathbf{D} -topology,*

$$(1.31) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(R_n \leq bn, \tilde{L}_{M,\varepsilon} \in F) \leq -\frac{1}{d} \inf_{\xi \in F \cap \tilde{\mathcal{D}}(b)} \tilde{J}(\xi),$$

where

$$(1.32) \quad \tilde{\mathcal{D}}(b) = \left\{ \xi \in \tilde{\mathcal{X}} : \tilde{\Gamma}(\xi) \leq b \right\}.$$

Note that only the elements of $\tilde{\mathcal{X}}$ with finite entropy contribute to the infimum in (1.31) so the value assigned to Γ for sub-probability measures with no density w.r.t. Lebesgue measure is actually irrelevant.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.4, while Section 3 contains the proof of Theorem 1.6 given the validity of

Proposition 1.7. The remaining part, that is Section 4, is dedicated to the proof of that proposition.

2. PROOF OF THEOREM 1.4: UNIQUENESS

Let us first collect some known facts. It was shown in [19] that any minimizer of (1.7) satisfies the Euler-Lagrange equation

$$(2.1) \quad \Delta\phi + f_{\lambda,\mu}(\phi) = 0,$$

where the nonlinearity $f_{\lambda,\mu}$ is given by

$$(2.2) \quad f_{\lambda,\mu}(x) = \lambda x + \mu x \kappa e^{-\kappa x^2}, \quad x \in \mathbb{R},$$

λ and μ being the Lagrange multipliers. We further know from [19] that (i) the set of minimizers is stable under space shifts and (ii) any minimizer of this variational problem is (up to spatial shifts) radially symmetric (strictly) decreasing with $\lim_{|x| \rightarrow \infty} \phi(x) = 0$. Hence, we may and will assume from now on that ϕ is maximized at the origin. This implies that we can assume that (2.1) is equipped with the boundary conditions

$$(2.3) \quad \begin{cases} \phi \text{ radially symmetric decreasing with } \lim_{|x| \rightarrow \infty} \phi(x) = 0, \\ \nabla\phi(0) = 0. \end{cases}$$

Moreover, letting $y(r) := \phi(r, 0, \dots, 0)$ for $r \geq 0$, the function y satisfies the (one-dimensional) Euler-Lagrange equation (see [19, Proof of Lemma 11])

$$(2.4) \quad \begin{cases} y''(r) + \frac{d-1}{r}y'(r) + f_{\lambda,\mu}(y(r)) = 0, \\ \lim_{r \rightarrow \infty} y(r) = 0, \\ y'(0) = 0. \end{cases}$$

The proof of Theorem 1.4 consists of two steps:

- (Step 1) We prove that the pair of Lagrange multipliers (λ, μ) is uniquely determined by b .
- (Step 2) We prove that (2.4) has a unique solution.

Step 1. Determination of the Lagrange multipliers. In this step we prove the following

Proposition 2.1. *Assume that I is differentiable at $b \in (0, \kappa)$. Let ϕ be a minimizer of the variational problem in (1.7). Then ϕ is a solution of the Euler-Lagrange equation in (2.1) and the Lagrange multipliers satisfy:*

$$(2.5) \quad \begin{aligned} \text{(i)} \quad & \lambda + \mu b = 2(1 - \frac{2}{d})I(b), \\ \text{(ii)} \quad & \mu = 2I'(b). \end{aligned}$$

Note that Proposition 2.1 uniquely determines the Lagrange multipliers $(\mu(b), \lambda(b))$.

Proof of Proposition 2.1. For the proof we fix a solution ϕ to (2.1). Let us first prove (i). By Pohozaev's identity [5, Proposition 1, Chapter 2] ϕ satisfies

$$(2.6) \quad \|\nabla\phi\|_2^2 = \frac{2d}{d-2} \int F_{\lambda,\mu}(\phi(x)) dx,$$

where

$$(2.7) \quad F_{\lambda,\mu}(z) = \int_0^z f_{\lambda,\mu}(v) dv = \frac{1}{2}\lambda z^2 + \frac{1}{2}\mu(1 - e^{-\kappa z^2}).$$

To apply this identity, one may actually check that

- $f_{\lambda,\mu}: \mathbb{R} \rightarrow \mathbb{R}$ is continuous;
- $\phi \in L_{\text{loc}}^\infty$;
- $\nabla \phi \in L^2$;
- $F_{\lambda,\mu}(\phi) \in L^1$.

Thus, for any function ϕ that is a minimizer of (1.7) and a solution to (2.1), we can write

$$(2.8) \quad I(b) = \frac{1}{2}\|\nabla \phi\|_2^2 = \frac{d}{2(d-2)} \left(\lambda \|\phi\|_2^2 + \mu \int_{\mathbb{R}^d} (1 - e^{-\kappa \phi^2(x)}) dx \right) = \frac{d}{2(d-2)} (\lambda + \mu b).$$

Here, we used the fact that the minimizer is in $\mathcal{D}(b)$ and that the second constraint is saturated as a consequence of [19, Lemma 12] and its proof. We can deduce therefore that

$$(2.9) \quad \lambda + \mu b = \frac{2(d-2)}{d} I(b).$$

Equation (2.9) shows that λ is uniquely determined by μ .

Let us now prove (ii). Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space. We define the functional $\mathcal{H}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{R}^2$ by

$$(2.10) \quad \mathcal{H}(h) = \left(\int h(x)\phi(x) dx, \int h(x)\kappa\phi(x)e^{-\kappa\phi^2(x)} dx \right)$$

We split the proof in two parts.

Step (a) Let us first prove that \mathcal{H} is surjective. To see why, assume that \mathcal{H} is not surjective. As \mathcal{H} is linear, this implies that the dimension of the range of \mathcal{H} is one. In particular, there is a vector $(\alpha, \beta) \in \mathbb{R}^2$ such that $\mathcal{H}(h)$ is orthogonal to (α, β) for all $h \in \mathcal{S}(\mathbb{R}^d)$. Thus, for all $h \in \mathcal{S}(\mathbb{R}^d)$

$$(2.11) \quad \langle h, \alpha\phi + \beta\kappa\phi e^{-\kappa\phi^2} \rangle = 0,$$

which implies that

$$(2.12) \quad \alpha\phi(x) + \beta\kappa\phi(x)e^{-\kappa\phi(x)^2} = 0 \quad \text{for almost all } x \in \mathbb{R}^d.$$

We conclude that

$$(2.13) \quad \phi(\mathbb{R}^d) \subseteq \left\{ 0, \sqrt{-\frac{1}{\kappa} \log(-\frac{\alpha}{\kappa\beta})} \right\}.$$

Moreover, we know that ϕ as a minimizer is radially symmetric and strictly decreasing. This however is only possible if $\phi(x) > 0$ for all $x \in \mathbb{R}^d$, thus $\phi(\mathbb{R}^d) = \sqrt{-\frac{1}{\kappa} \log(-\frac{\alpha}{\kappa\beta})}$ which contradicts the fact that $\|\phi\|_2 = 1$. Hence, \mathcal{H} is surjective.

Step (b) Since \mathcal{H} is surjective we may pick $h \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{H}(h) = (0, 1)$. We now define three functionals:

$$(2.14) \quad J_1(\phi) = \frac{1}{2} \|\nabla \phi\|_2^2, \quad J_2(\phi) = \|\phi\|_2^2, \quad J_3(\phi) = \int (1 - e^{-\kappa \phi(x)^2}) dx.$$

With this choice of h , as $\varepsilon \rightarrow 0$, a direct calculation shows that

$$(2.15) \quad J_3\left(\frac{\phi + \varepsilon h}{\|\phi + \varepsilon h\|_2}\right) = b + 2\varepsilon + o(\varepsilon).$$

Note furthermore that since ϕ and h are orthogonal in L^2 we have that

$$(2.16) \quad \|\phi + \varepsilon h\|_2 = \sqrt{1 + \varepsilon^2 \|h\|_2^2} > 1.$$

Hence,

$$(2.17) \quad I(b + 2\varepsilon + o(\varepsilon)) \leq J_1\left(\frac{\phi + \varepsilon h}{\|\phi + \varepsilon h\|_2}\right) \leq J_1(\phi + \varepsilon h).$$

Thus, expanding I and J_1 around b and ϕ respectively, we see that

$$(2.18) \quad I(b + 2\varepsilon + o(\varepsilon)) = I(b) + 2\varepsilon I'(b) + o(\varepsilon) \leq J_1(\phi) - \varepsilon \int \Delta \phi(x) h(x) dx + o(\varepsilon).$$

By subtracting $I(b) = J_1(\phi)$ and dividing by ε in the previous inequality, and since ε can be positive or negative, we conclude that $I'(b) = -\frac{1}{2} \int \Delta \phi(x) h(x) dx$. We may conclude the proof by using that ϕ solves the Euler-Lagrange equation and that $\mathcal{H}(h) = (0, 1)$. ■

Step 2. *Uniqueness of solution to Equation (2.4):* Let us write (for simplicity we omit the dependence on b):

$$(2.19) \quad f(r) = r\psi(r), \quad \psi(r) := \lambda + \mu\kappa e^{-\kappa r^2}.$$

We shall use Theorem 1 in Serrin and Tang [16]. Let us first check that Hypothesis (H1) therein is satisfied in our case. By Proposition 2.1(ii), we have that $\mu < 0$, so that by Proposition 2.1(i), $\psi(+\infty) = \lambda > 0$. Next we will show that $\psi(0) = \lambda + \kappa\mu < 0$. First, assume $d \geq 5$ and define $h(u) = (1 - u)^{\frac{2}{d}-1} \chi(u)$, where $\chi(u) := 2\kappa^{2/d} I(\kappa u)$ for $u \in (0, 1)$, as in [19, Eq. (1.10)]. By [19, Theorem 5(iii)], $h'(u) \leq 0$, from which we obtain

$$(2.20) \quad (1 - \frac{2}{d})\chi(u) + (1 - u)\chi'(u) \leq 0, \quad \text{for a.e. } u \in (0, u_d^*).$$

Combining with (i) and (ii) in Proposition 2.1, we get

$$(2.21) \quad \lambda + \kappa\mu = (1 - \frac{2}{d})I(b) + (\kappa - b)I'(b) \leq 0, \quad \text{for a.e. } b \in (0, \kappa u_d^*).$$

Let us now discard the possibility of equality in the line above. By [19, Theorem 3(iii)], we also have $\frac{2}{du}\chi(u) + \chi'(u) \leq 0$ for a.e. $u \in (0, 1)$, so that equality in (2.20) and the fact that $\chi(u) > 0$ (see [19, Theorem 3]) yields $u \geq 2/d$. This would contradict our assumption that $b = \kappa u < 2\kappa/d$. If $d = 4$, the same argument goes through by applying [19, Theorem 4(ii)] instead and noticing that the exponents $1 - \frac{2}{d}$ and $\frac{2}{d}$ coincide. If $d = 3$, we apply [19, Theorem 4(ii)] to the function $h(u) = (1 - u)^{-2/d} \chi(u)$ instead and get

$$(2.22) \quad 0 \geq \frac{2}{d}I(b) + (\kappa - b)I'(b) > (1 - \frac{2}{d})I(b) + (\kappa - b)I'(b), \quad b \in (0, \kappa).$$

This settles our claim that $\psi(0) < 0$. Then, the equation $\psi(r) = 0$ has a unique positive solution which we denote by α . One can readily check that f is continuous on $(0, \infty)$ with $f(r) \leq 0$ for $r \in (0, \alpha]$ and $f(r) > 0$ for $r > \alpha$.

Let us now check Hypothesis (H2), according to which the function

$$(2.23) \quad g(r) := \frac{rf'(r)}{f(r)}$$

should be non-increasing on (α, ∞) . By a direct computation,

$$(2.24) \quad g(r) = 1 + r \frac{\psi'(r)}{\psi(r)} = 1 + 2\kappa r^2 \left[\frac{1}{1 + \frac{\kappa\mu}{\lambda} e^{-\kappa r^2}} - 1 \right].$$

By definition, $\psi(\alpha) = 0$, from which we get

$$(2.25) \quad \frac{\kappa\mu}{\lambda} = -e^{\kappa\alpha^2},$$

so that

$$(2.26) \quad g(r) - 1 = 2\kappa r^2 \left[\frac{1}{1 - e^{\kappa(\alpha^2 - r^2)}} - 1 \right].$$

Let us re-parametrize the problem by setting $u = u(r) := e^{-\kappa r^2} \in (0, e^{-\kappa\alpha^2})$, as $r \in (\alpha, \infty)$. Since $u(r)$ is decreasing in r , we must now check that the right-hand side in (2.26) is non-decreasing in u . We get

$$(2.27) \quad g(r) - 1 = 2(-\log u) \left[\frac{1}{1 - e^{\kappa\alpha^2} u} - 1 \right],$$

and

$$(2.28) \quad \frac{d}{du} g(r) = \frac{2e^{\kappa\alpha^2} \Xi(u)}{(1 - e^{\kappa\alpha^2} u)^2}, \quad \Xi(u) := -\log u - (1 - e^{\kappa\alpha^2} u).$$

It is now straightforward to show that $\Xi(u) \geq 0$ for $0 < u < e^{-\kappa\alpha^2}$, which completes the proof.

We may now conclude the proof of Theorem 1.4, noting that:

Lemma 2.2. *The function $b \mapsto I(b)$ is almost-everywhere differentiable on $[0, \infty)$.*

Proof of Lemma 2.2. Note that $b \mapsto I(b)$ is a monotone function. Thus, by Lebesgue's theorem on the differentiability of monotone functions we can conclude that I is almost everywhere differentiable. ■

3. PROOF OF THEOREM 1.6: TUBE PROPERTY

In this section we prove Theorem 1.6 assuming the validity of Proposition 1.7 and the first statement in Proposition 4.11.

Lemma 3.1 (Scaling properties). *Let $\phi \in H^1(\mathbb{R}^d)$. For $a \in (0, \infty)$, define $\phi_{[a]}(x) := a^{d/2} \phi(ax)$. Then,*

$$(3.1) \quad \|\phi_{[a]}\|_2 = \|\phi\|_2, \quad \|\nabla \phi_{[a]}\|_2 = a \|\nabla \phi\|_2.$$

Moreover, the function

$$(3.2) \quad a \in (0, \infty) \mapsto \int_{\mathbb{R}^d} \left(1 - e^{-\kappa \phi_{[a]}^2(x)}\right) dx$$

is continuous, non-increasing and converges to 0 as $a \rightarrow \infty$, and to $\kappa \|\phi\|_2^2$ as $a \rightarrow 0$.

Proof of Lemma 3.1. We refer the reader to [19, Section 5, proof of Lemma 12]. \blacksquare

The limit $a \rightarrow 0$ in Lemma 3.1 corresponds to *evanescent mass* and better enlightens our definition of $\tilde{\Gamma}$ on $\tilde{\mathcal{X}}$ in (1.30).

Lemma 3.2. *Let $b \in (0, \kappa)$ if $d = 3$ or $b \in (0, \frac{2}{d}\kappa)$ if $d \geq 4$ be such that the variational problem in (1.7) has a unique minimizer modulo space shift. Let $\mathcal{U}(\mathfrak{m}_b)$ be an open neighbourhood of \mathfrak{m}_b w.r.t. the \mathbf{D} -topology. Then,*

$$(3.3) \quad \inf_{\substack{\xi \in \tilde{\mathcal{D}}(b) \\ \xi \notin \mathcal{U}(\mathfrak{m}_b)}} \tilde{J}(\xi) > \inf_{\xi \in \tilde{\mathcal{D}}(b)} \tilde{J}(\xi).$$

Proof of Lemma 3.2. Assume that the left-hand side in (3.3) is finite, otherwise there is nothing to prove. Then, we may safely restrict the infimum on the left-hand side to $\tilde{\mathcal{D}}_K(b) := \tilde{\mathcal{D}}(b) \cap \{\xi \in \tilde{\mathcal{X}} : \tilde{J}(\xi) \leq K\}$ for some $K > 0$. By Proposition 4.11, $\tilde{\mathcal{D}}_K(b)$ is a closed set. Moreover, $(\tilde{\mathcal{X}}, \mathbf{D})$ is a compact metric space, therefore there exists a minimizer, further denoted by ξ , for the function \tilde{J} on the compact set $\tilde{\mathcal{D}}_K(b) \cap \mathcal{U}(\mathfrak{m}_b)^c$. We distinguish between cases according to the number of elements in ξ .

Case 0. Assume that $\xi = \emptyset$. Then, $\tilde{\Gamma}(\xi) = \kappa > b$, hence $\xi \notin \tilde{\mathcal{D}}(b)$, which contradicts our assumption.

Case 1. Assume that ξ has a single element, i.e. $\xi = \{\tilde{\alpha}\}$ for some sub-probability measure α . Necessarily, $\alpha(dx) = \phi^2(x)dx$ for some $\phi \in H^1(\mathbb{R}^d)$, otherwise $\tilde{J}(\xi) = +\infty$. There are then two further subcases:

Case 1a. If $\int \phi^2 = 1$, then by Theorem 1.4 (uniqueness of the minimizer among the set of probability measures modulo space shifts), $\tilde{J}(\xi) > \tilde{J}(\mathfrak{m}_b)$, which closes this case.

Case 1b. Assume now that $\int \phi^2 \in (0, 1)$. Using Lemma 3.1 and arguing as in [19, Proof of Lemma 12], one may check that

$$(3.4) \quad \int (1 - e^{-\kappa \phi^2}) + \kappa \left(1 - \int \phi^2\right) = b,$$

which we may rewrite as

$$(3.5) \quad \int (e^{-\kappa \phi^2} - 1 + \kappa \phi^2) = \kappa - b.$$

If there exists another element $\xi' \in \tilde{\mathcal{D}}(b)$ such that $\tilde{J}(\xi') < \tilde{J}(\xi)$ then obviously the infimum of \tilde{J} over $\tilde{\mathcal{D}}(b)$ is strictly smaller than $\tilde{J}(\xi)$ and there is nothing more to prove. Therefore, we may assume from now on that ϕ is a local minimizer of $\|\nabla \phi\|_2$ under the constraint in (3.5), and write the associated Euler-Lagrange equation.

- In the case $d \in \{3, 4\}$, we may use [19, Lemma 14] to get that $\int \phi^2 = +\infty$, which contradicts our assumption.

- In the case $d \geq 5$, let us define $u := b/\kappa \in (0, 1)$ and $\psi^2(x) := \kappa\phi^2(\kappa(1-u)^{1/d}x)$. Then, by a straightforward change of variable, $\int \psi^2 = (1-u)^{-1} \int \phi^2 < (1-u)^{-1}$ and ψ is a minimizer of $(1-u)^{1-2/d} \|\nabla \psi\|_2^2$ under the constraint $\int e^{-\psi^2} - 1 + \psi^2 = 1$. Arguing as in [19, Proof of Theorem 5(ii), Item 1], we obtain that $\|\psi\|_2^2 \geq (1-u_d^*)^{-1}$, which leads to a contradiction when $u \leq u_d^*$, i.e. $b \leq u_d^* \kappa$.

Case 2. Finally, let us assume that ξ contains at least two elements, i.e. (i) at least two distinct elements, or (ii) at least one element with multiplicity at least two. Let us denote them by $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. In the sequel we pick $\alpha_i(dx) = \phi_i^2(x)dx$, with $\phi_i \in H^1(\mathbb{R}^d)$ for $i \in \{1, 2\}$, two elements of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ such that $\min(\phi_1^2, \phi_2^2) > 0$ on a set of positive measure. Consider $\alpha_3 := \alpha_1 + \alpha_2$, and

$$(3.6) \quad \xi' := \{\tilde{\alpha}_3, \xi \setminus \{\tilde{\alpha}_1, \tilde{\alpha}_2\}\}.$$

By the convexity inequality for J , see e.g. [13, Theorem 7.8], $J(\alpha_3) \leq J(\alpha_1) + J(\alpha_2)$, hence $\tilde{J}(\xi') \leq \tilde{J}(\xi)$. To complete the argument, let us first notice that for every $u, v \in [0, 1]$, $1 - uv \leq (1-u) + (1-v)$, with the inequality being strict as soon as $\max(u, v) < 1$. By our choice of α_1 and α_2 , this yields

$$(3.7) \quad \int 1 - e^{-\kappa(\phi_1^2 + \phi_2^2)} < \int (1 - e^{-\kappa\phi_1^2}) + \int (1 - e^{-\kappa\phi_2^2}),$$

hence $\tilde{\Gamma}(\xi') < \tilde{\Gamma}(\xi) \leq b$. Let us now define $\alpha_4(dx) := a^d \alpha_3(a dx)$ and $\xi'' := \{\alpha_4, \xi' \setminus \{\alpha_3\}\}$. By using Lemma 3.1 and choosing $a \in (0, 1)$ close enough to one, we obtain $\tilde{\Gamma}(\xi'') \leq b$ and $\tilde{J}(\xi'') < \tilde{J}(\xi') \leq \tilde{J}(\xi)$, which completes the proof. \blacksquare

The reader may check that the above proof actually yields the following:

Corollary 3.3. *Under the same assumptions as in Theorem 1.6, \mathbf{m}_b is the unique minimizer of the rate function \tilde{J} in $\tilde{\mathcal{D}}(b)$.*

We may now prove Theorem 1.6. Let $\mathcal{U}(\mathbf{m}_b)$ be an open neighbourhood of \mathbf{m}_b with respect to the \mathbf{D} -topology. By Lemma 3.2, the quantity

$$(3.8) \quad \delta := \inf_{\substack{\xi \in \tilde{\mathcal{D}}(b) \\ \xi \notin \mathcal{U}(\mathbf{m}_b)}} \tilde{J}(\xi) - \inf_{\xi \in \tilde{\mathcal{D}}(b)} \tilde{J}(\xi)$$

is (strictly) positive. By Proposition 1.7, there exists ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$(3.9) \quad \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(\tilde{L}_{M,\varepsilon} \notin \mathcal{U}(\mathbf{m}_b), R_n \leq bn) \leq -\frac{1}{d} \inf_{\substack{\xi \in \tilde{\mathcal{D}}(b) \\ \xi \notin \mathcal{U}(\mathbf{m}_b)}} \tilde{J}(\xi) + \frac{\delta}{2d}.$$

Using Theorem 1.3 and the fact that the infimum of J over $\mathcal{D}(b)$ coincides with the infimum of \tilde{J} over $\tilde{\mathcal{D}}(b)$ (Corollary 3.3) we obtain

$$(3.10) \quad \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(\tilde{L}_{M,\varepsilon} \notin \mathcal{U}(\mathbf{m}_b) | R_n \leq bn) \leq -\frac{1}{d} \inf_{\substack{\xi \in \tilde{\mathcal{D}}(b) \\ \xi \notin \mathcal{U}(\mathbf{m}_b)}} \tilde{J}(\xi) + \frac{1}{d} \inf_{\xi \in \tilde{\mathcal{D}}(b)} \tilde{J}(\xi) + \frac{\delta}{2d},$$

which is less than $-\delta/(2d) < 0$. This completes the proof of Theorem 1.6.

4. PROOF OF PROPOSITION 1.7: LARGE DEVIATION UPPER BOUND

This section is devoted to the proof of the large deviation upper bound in the \mathbf{D} -topology, which is key to obtain Theorem 1.6. In Section 4.1 we first reduce the deviations for the volume of the range of the random walk to the deviations of its expectation conditioned to the skeleton and show that the conditional expectation may be expressed as a certain functional of the skeleton pair empirical measure. That first step closely follows [15, 19], the only difference being that we do not fold the random walk on a torus. Our compactification method rather relies on the use of the compact metric space $(\tilde{\mathcal{X}}, \mathbf{D})$ instead of the more standard weak topology on the space of probability measures. To be more precise, an adaptation of the \mathbf{D} -topology is needed to obtain a large deviation principle for the *pair* empirical measure. This will be provided in Section 4.2, using previous work of the authors [9]. The continuity properties that are needed to apply the standard contraction principle to the relevant functionals are given in Section 4.3. The main differences with [15, 19] will be discussed at the beginning of that section. All these ingredients will be combined in Section 4.4 to finally prove Proposition 1.7. Remaining sections contain the deferred proofs of more technical and lengthy lemmas.

4.1. Approximation of the conditional range. The very first step in our way to Proposition 1.7 is the use of the following concentration inequality, adapted from [15, Proposition 2.2.2].

Proposition 4.1 (Concentration inequality). *For all $\delta > 0$,*

$$(4.1) \quad \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(|R_n - \mathbb{E}(R_n | \hat{S})| \geq \delta n) = -\infty.$$

As a consequence, we may restrict our attention to the conditional expectation of the volume of the range. The only difference with [15, Proposition 2.2.2] is that the random walk has not been folded on the torus. Hence only mild modifications are needed to get Proposition 4.1. Those are deferred to Appendix A.2.

The next step is to express the conditional expectation as a function of the skeleton empirical measure. Although this step follows exactly [15, 19], we shortly reproduce the computation here, for the reader's convenience and to make the paper more self-contained. More notation is needed beforehand: we introduce the auxiliary functions

$$(4.2) \quad q_{\ell, \Lambda}(x, y) := \mathbb{P}_x(S_{(0, \ell]} \cap \Lambda \neq \emptyset | S_\ell = y), \quad x, y \in \mathbb{Z}^d, \quad \Lambda \subset \mathbb{Z}^d,$$

$$(4.3) \quad \bar{q}_{\ell, \Lambda}(x, y) := q_{\ell, \Lambda}(\lfloor xn^{1/d} \rfloor, \lfloor yn^{1/d} \rfloor), \quad x, y \in \mathbb{R}^d, \quad \Lambda \subset \mathbb{Z}^d,$$

and when $\Lambda = \Lambda(n, z)$ is the singleton $\{\lfloor zn^{1/d} \rfloor\}$ ($z \in \mathbb{R}^d$) we define

$$(4.4) \quad \bar{q}_\ell(x, y; z) := \bar{q}_{\ell, \Lambda(n, z)}(x, y), \quad x, y \in \mathbb{R}^d.$$

Let us now define

$$(4.5) \quad \mathfrak{q}_{\varepsilon, n, \Lambda}(x, y) := -M \log[1 - \bar{q}_{\ell, \Lambda}(x, y)], \quad x, y \in \mathbb{R}^d,$$

where we remind the reader of the definition of M in (1.23). Let $z \in \mathbb{R}^d$. When $\Lambda = \Lambda(n, z)$ is the singleton $\{\lfloor zn^{1/d} \rfloor\}$, we define

$$(4.6) \quad \mathfrak{q}_{\varepsilon, n}(x, y; z) := \mathfrak{q}_{\varepsilon, n, \Lambda(n, z)}(x, y) \quad x, y \in \mathbb{R}^d.$$

With all this notation in hand, we may finally write:

$$\begin{aligned}
(4.7) \quad \frac{1}{n} \mathbb{E}(R_n | \hat{S}) &= \frac{1}{n} \sum_{z \in \mathbb{Z}^d} [1 - \mathbb{P}(z \notin S_{(0,n]} | \hat{S})] \\
&= \frac{1}{n} \sum_{z \in \mathbb{Z}^d} \left[1 - \prod_{1 \leq i \leq M} \mathbb{P}(z \notin S_{((i-1)\ell, i\ell]} | \hat{S}) \right] \\
&= \frac{1}{n} \sum_{z \in \mathbb{Z}^d} \left[1 - \prod_{1 \leq i \leq M} [1 - q_{\ell, \{z\}}(\hat{S}_{i-1}^{(\varepsilon)} n^{1/d}, \hat{S}_i^{(\varepsilon)} n^{1/d})] \right] \\
&= \int_{\mathbb{R}^d} dz \left[1 - \prod_{1 \leq i \leq M} [1 - \bar{q}_{\ell}(\hat{S}_{i-1}^{(\varepsilon)}, \hat{S}_i^{(\varepsilon)}; z)] \right] \\
&= \int_{\mathbb{R}^d} dz \left[1 - \exp \left(- \int \mathbf{q}_{\varepsilon, n}(x, y; z) L_{M, \varepsilon}^{(2)}(dx, dy) \right) \right].
\end{aligned}$$

In what follows, we replace the function $\mathbf{q}_{\varepsilon, n}$ by a function that does not depend on n anymore, see Proposition 4.3 below. First, let us define for $x, y \in \mathbb{R}^d$,

$$(4.8) \quad \varphi_{\varepsilon}(x, y) := \int_0^{\varepsilon} ds \frac{p_{s/d}(-x) p_{(\varepsilon-s)/d}(y)}{p_{\varepsilon/d}(y-x)}.$$

where p_s denotes the Brownian heat kernel, i.e.

$$(4.9) \quad p_s(x) = \frac{1}{(2\pi s)^{d/2}} \exp \left(-\frac{|x|^2}{2s} \right), \quad x \in \mathbb{R}^d,$$

and for all $\mu \in \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d)$,

$$(4.10) \quad \phi_{\infty, \varepsilon}(\mu) := \int dz \left(1 - \exp \left\{ -\frac{\kappa_d}{\varepsilon} \int \mu(dx, dy) \varphi_{\varepsilon}(x-z, y-z) \right\} \right).$$

Furthermore we shall restrict the pair empirical measure to a certain subset of $\mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d)$ on which we control the maximal distance run by the random walk along each “edge” of the skeleton. Namely, we define

$$(4.11) \quad \mathcal{M}(A, \varepsilon_0) := \{ \mu \in \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d) : \mu(\{(x, y) : |x - y| \geq A\}) \leq \varepsilon_0 \}, \quad A, \varepsilon_0 > 0.$$

This restriction is harmless due to the following:

Proposition 4.2. *Let $\varepsilon, \varepsilon_0 > 0$ and $b \in (0, \kappa)$ be fixed. There exists $A > 0$ such that*

$$(4.12) \quad \lim_{n \rightarrow \infty} \mathbb{P}(L_{M, \varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0) | R_n \leq bn) = 0.$$

Proof of Proposition 4.2. We use the rough upper bound

$$(4.13) \quad \mathbb{P}(L_{M, \varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0) | R_n \leq bn) \leq \frac{\mathbb{P}(L_{M, \varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0))}{\mathbb{P}(R_n \leq bn)}.$$

Note that

$$(4.14) \quad \mathbb{P}(L_{M, \varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0)) \leq \mathbb{P}(\text{card}\{0 < i \leq M : |S_{i\ell} - S_{(i-1)\ell}| \geq A\sqrt{\frac{\ell}{\varepsilon}}\} > \varepsilon_0 M).$$

Notice that for each $C > 0$ (to be determined later) there exists A such that

$$(4.15) \quad \sup_{\ell \geq 1} \mathbb{P}\left(S_\ell \geq A\sqrt{\frac{\ell}{\varepsilon}}\right) \leq \frac{\varepsilon_0}{C}.$$

For such value of A we get the binomial estimate:

$$(4.16) \quad \mathbb{P}(L_{M,\varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0)) \leq \mathbb{P}(\text{Bin}(M, \varepsilon_0/C) \geq \varepsilon_0 M).$$

By a standard Large Deviation estimate (see Lemma A.1), we get (recall that ε_0 is fixed)

$$(4.17) \quad \mathbb{P}(L_{M,\varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0)) \leq \exp(-[1 + o(1)]\varepsilon_0 \Xi(C) \frac{1}{\varepsilon} n^{1-2/d}),$$

where $\lim_{C \rightarrow \infty} \Xi(C) = +\infty$ and the $o(1)$ holds as $n \rightarrow \infty$. In view of (4.13) and Theorem 1.3, it is enough to choose C large enough such that $\varepsilon_0 \Xi(C) > (\varepsilon/d)I(b)$, and then pick A accordingly, in order to conclude. \blacksquare

We may finally state the main result of this subsection. The proof is deferred to Appendix B, due to its length.

Proposition 4.3. *Let $\varepsilon > 0$. For all $\varepsilon_0, A > 0$, there exists $n_0(\varepsilon, \varepsilon_0, A)$ such that for all $n \geq n_0(\varepsilon, \varepsilon_0, A)$,*

$$(4.18) \quad \frac{1}{n} \mathbb{E}(R_n | \hat{S}) \geq \phi_{\infty, \varepsilon}(L_{M,\varepsilon}^{(2)}) - \varepsilon_0,$$

on the event $L_{M,\varepsilon}^{(2)} \in \mathcal{M}(A, \varepsilon_0)$.

Remark 4.4. *Proposition 4.3 partially translates [15, Lemma 2.2.10 (a)] to the infinite volume case. However, our proof will deviate from the one in [15] exactly to avoid issues coming from the fact that the random walk considered here is on \mathbb{R}^d and not on a torus.*

Let us end this section with an observation that shall become useful later on

Lemma 4.5. *For all $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $z \in \mathbb{R}^d \mapsto \frac{1}{\varepsilon} \varphi_\varepsilon(x - z, y - z)$ is the density of a probability measure on \mathbb{R}^d and $z \in \mathbb{R}^d \mapsto \frac{1}{\varepsilon} n^{1-2/d} \bar{q}_\ell(x, y; z)$ is a sub-probability measure.*

Proof of Lemma 4.5. The first part of the statement is straightforward since for all $s \in (0, \varepsilon)$,

$$(4.19) \quad \int dz p_{s/d}(z - x) p_{(\varepsilon-s)/d}(y - z) = p_{\varepsilon/d}(y - x),$$

by the Chapman-Kolmogorov equation. For the second part of the statement, we write

$$\begin{aligned} n^{1-2/d} \int dz \bar{q}_\ell(x, y; z) &= n^{1-2/d} \int dz \mathbb{P}_{\lfloor xn^{1/d} \rfloor}(S_{(0,\ell]} \cap \{\lfloor zn^{1/d} \rfloor\} \neq \emptyset | S_\ell = \lfloor yn^{1/d} \rfloor) \\ &= n^{-2/d} \sum_{z \in \mathbb{Z}^d} \mathbb{P}_{\lfloor xn^{1/d} \rfloor}(S_{(0,\ell]} \cap \{z\} \neq \emptyset | S_\ell = \lfloor yn^{1/d} \rfloor) \\ &\leq n^{-2/d} \sum_{z \in \mathbb{Z}^d} \sum_{1 \leq i \leq \ell} \mathbb{P}_{\lfloor xn^{1/d} \rfloor}(S_i = z | S_\ell = \lfloor yn^{1/d} \rfloor) = n^{-2/d} \ell = \varepsilon. \end{aligned}$$

\blacksquare

4.2. Large deviation principle for pair empirical measures modulo shifts. With Proposition 4.3 in hand it is only natural to apply a large deviation principle for the skeleton *pair* empirical measure. The Mukherjee-Varadhan topology introduced in Section 1.3 was adapted in [9] to allow strong large deviation principles for *pair* empirical measures. In this section we explain the modifications needed to obtain this topology from the one in Section 1.3. Let $\mathcal{M}_1^{(2)} := \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ and $\mathcal{M}_{\leq 1}^{(2)} := \mathcal{M}_{\leq 1}(\mathbb{R}^d \times \mathbb{R}^d)$ be the space of sub-probability measures on $\mathbb{R}^d \times \mathbb{R}^d$. We consider the action of the shifts $\theta_{x,x}$ for $x \in \mathbb{R}^d$, defined by:

$$(4.21) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u, v)(\theta_{x,x}\nu)(du, dv) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(u+x, v+x)\nu(du, dv)$$

for all continuous bounded functions $f: \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ and $\nu \in \mathcal{M}_{\leq 1}^{(2)}$. We shall denote by $\widetilde{\mathcal{M}}_1^{(2)}$ (resp. $\widetilde{\mathcal{M}}_{\leq 1}^{(2)}$) the space of equivalence classes of $\mathcal{M}_1^{(2)}$ (resp. $\mathcal{M}_{\leq 1}^{(2)}$) under the collection of shifts $\theta_{x,x}$. For any $\alpha \in \mathcal{M}_1^{(2)}$ we denote by $\tilde{\alpha}$ its orbit, i.e., equivalence class. Recall the definition of \mathcal{F}_k in Section 1.3. For $k \geq 1$, $f \in \mathcal{F}_{2k}$ and $\alpha \in \mathcal{M}_{\leq 1}^{(2)}$, we write

$$(4.22) \quad \Lambda(f, \alpha) := \int f(u_1, v_1, \dots, u_k, v_k) \prod_{1 \leq i \leq k} \alpha(du_i, dv_i),$$

which only depends on the orbit $\tilde{\alpha}$. This time, we define

$$(4.23) \quad \mathcal{F}^{(2)} := \bigcup_{k \geq 2} \mathcal{F}_{2k},$$

for which there exists a countable dense set (under the uniform metric) denoted by

$$(4.24) \quad \{f_r^{(2)}(u_1, v_1, \dots, u_{k_r}, v_{k_r}) : r \in \mathbb{N}\},$$

see [14, Section 2.2]. We define

$$(4.25) \quad \tilde{\mathcal{X}}^{(2)} := \left\{ \xi = \{\tilde{\alpha}_i\}_{i \in I} : \alpha_i \in \mathcal{M}_{\leq 1}^{(2)}, \sum_{i \in I} \alpha_i(\mathbb{R}^d \times \mathbb{R}^d) \leq 1 \right\},$$

where I may be empty, finite or countable. For any $\xi_1, \xi_2 \in \tilde{\mathcal{X}}^{(2)}$, define

$$(4.26) \quad \mathbf{D}_2(\xi_1, \xi_2) := \sum_{r \geq 1} \frac{1}{2^r} \frac{1}{1 + \|f_r^{(2)}\|_\infty} \left| \sum_{\tilde{\alpha} \in \xi_1} \Lambda(f_r^{(2)}, \alpha) - \sum_{\tilde{\alpha} \in \xi_2} \Lambda(f_r^{(2)}, \alpha) \right|.$$

It was then shown in [9] that $\tilde{\mathcal{X}}^{(2)}$ equipped with \mathbf{D}_2 is a compact metric space, and that $\widetilde{\mathcal{M}}^{(2)}$ is dense in $\tilde{\mathcal{X}}^{(2)}$.

In what follows, (π_ε) denotes the Brownian semigroup, i.e. for $\varepsilon > 0$ and $x \in \mathbb{R}^d$,

$$(4.27) \quad \pi_\varepsilon(x, dy) := p_\varepsilon(x, y)dy, \quad p_\varepsilon(x, y) := p_\varepsilon(y - x),$$

(recall (4.9)) and for all $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$ we write

$$(4.28) \quad (\alpha \otimes \pi_\varepsilon)(dx, dy) := \alpha(dx)\pi_\varepsilon(x, y) \in \mathcal{M}_{\leq 1}(\mathbb{R}^d \times \mathbb{R}^d).$$

We denote by h the relative entropy, defined by

$$(4.29) \quad h(\mu|\nu) = \begin{cases} \int \log\left(\frac{d\mu}{d\nu}\right) d\mu & \text{if } \mu \ll \nu; \\ +\infty & \text{else;} \end{cases} \quad \mu, \nu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d \times \mathbb{R}^d).$$

If $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}^{(2)}$, we denote by $\alpha_{i,1}$ and $\alpha_{i,2}$ the projections of α_i (that is an arbitrary element of the orbit $\tilde{\alpha}_i$) onto the first and last d coordinates respectively. Then, we define, for every $\varepsilon > 0$:

$$(4.30) \quad \tilde{J}_\varepsilon^{(2)}(\xi) := \sum_{i \in I} h(\alpha_i | \alpha_{i,1} \otimes \pi_\varepsilon)$$

if $\alpha_{i,1} = \alpha_{i,2}$ for all $i \in I$, and $\tilde{J}_\varepsilon^{(2)}(\xi) := +\infty$ otherwise. The following result is key to our analysis:

Proposition 4.6 (Pair empirical LDP upper bound, Theorem 7.2 in [9]). *For any closed set F in $\tilde{\mathcal{X}}^{(2)}$,*

$$(4.31) \quad \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log P(\tilde{L}_{M,\varepsilon}^{(2)} \in F) \leq - \inf_{\xi \in F} \frac{1}{\varepsilon} \tilde{J}_\varepsilon^{(2)}(\xi).$$

For the rest of the paper, let us give ourselves a slightly more convenient notation for projections. For all $\mu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d \times \mathbb{R}^d)$, we define

$$(4.32) \quad \widetilde{\text{Pr}}(\mu) = \{\text{Pr}(\mu) * \delta_x : x \in \mathbb{R}^d\},$$

where Pr denotes the usual projection onto the first d coordinates of a sub-probability measure in $\mathbb{R}^d \times \mathbb{R}^d$. Given $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}^{(2)}$, we define

$$(4.33) \quad \widetilde{\text{Pr}}(\xi) = \{\widetilde{\text{Pr}}(\tilde{\alpha}_i)\}_{i \in I}.$$

Lemma 4.7. *The mapping $\widetilde{\text{Pr}}$ is continuous from $(\tilde{\mathcal{X}}^{(2)}, \mathbf{D}_2)$ to $(\tilde{\mathcal{X}}, \mathbf{D})$.*

Proof of Lemma 4.7. Suppose that $\{\xi_n\}_{n \in \mathbb{N}}$ is a sequence of elements in $\tilde{\mathcal{X}}^{(2)}$ that converges to $\xi \in \tilde{\mathcal{X}}^{(2)}$ for the \mathbf{D}_2 metric. Consider $f \in \mathcal{F}_k$. Then we can write $f(u_1, \dots, u_k) = \hat{f}(u_1, v_1, \dots, u_k, v_k)$ where $\hat{f} \in \mathcal{F}_{2k}$ is constant along the v -variables. Thus,

$$(4.34) \quad \begin{aligned} \sum_{\tilde{\alpha} \in \xi} \int f(u_1, \dots, u_k) \prod_{1 \leq i \leq k} (\text{Pr } \alpha)(du_i) &= \sum_{\tilde{\alpha} \in \xi} \int \hat{f}(u_1, v_1, \dots, u_k, v_k) \prod_{1 \leq i \leq k} \alpha(du_i, dv_i) \\ &= \lim_{n \rightarrow \infty} \sum_{\tilde{\alpha} \in \xi_n} \int \hat{f}(u_1, v_1, \dots, u_k, v_k) \prod_{1 \leq i \leq k} \alpha(du_i, dv_i) \\ &= \lim_{n \rightarrow \infty} \sum_{\tilde{\alpha} \in \xi_n} \int f(u_1, \dots, u_k) \prod_{1 \leq i \leq k} (\text{Pr } \alpha)(du_i). \end{aligned}$$

The definition of the respective metrics, together with the fact that $\widetilde{\text{Pr}}(\xi) = \{\widetilde{\text{Pr}}(\tilde{\alpha}_i)\}_{i \in I}$, yield the claim. \blacksquare

4.3. Lower semi-continuity. The goal of this section is two-fold. Firstly, we provide the minimal continuity properties later required to apply the contraction principle to the relevant functional of the skeleton empirical measure. Here, continuity is meant for the \mathbf{D} and \mathbf{D}_2 topologies, hence extra work is needed compared to [15, 19]. Secondly, we provide a series of approximations that bridge the gap between the functional appearing in Proposition 4.3 (that is $\phi_{\infty,\varepsilon}$) and the one appearing in Theorem 1.6 (that is $\tilde{\Gamma}$). Adjustments from [15, 19] are required.

Let us first extend $\phi_{\infty,\varepsilon}$ to $\tilde{\mathcal{X}}^{(2)}$. Since $\phi_{\infty,\varepsilon}$ is well-defined for sub-probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ and is invariant by the shifts $(\theta_{x,x})_{x \in \mathbb{R}^d}$ (recall the definition from Section 4.2), we may define for every $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}^{(2)}$,

$$(4.35) \quad \tilde{\phi}_{\infty,\varepsilon}(\xi) := \sum_{i \in I} \phi_{\infty,\varepsilon}(\alpha_i) + \kappa \left(1 - \sum_{i \in I} \alpha_i(\mathbb{R}^d \times \mathbb{R}^d)\right).$$

Lemma 4.8. *The mapping $\tilde{\phi}_{\infty,\varepsilon}$ is lower-semi continuous with respect to the metric \mathbf{D}_2 .*

The proof of Lemma 4.8, which is quite long, is deferred to Section 4.5. Let us now define (see (2.89) in [19])

$$(4.36) \quad \Psi_\varepsilon(\alpha) := \int dx \left(1 - e^{-\frac{\kappa}{\varepsilon} \int_0^\varepsilon ds \int p_{s/d}(x-y) \alpha(dy)}\right),$$

which is defined for $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$ and is translation invariant, while for $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}$, we define

$$(4.37) \quad \tilde{\Psi}_\varepsilon(\xi) = \sum_{i \in I} \Psi_\varepsilon(\alpha_i) + \kappa \left(1 - \sum_{i \in I} \alpha_i(\mathbb{R}^d)\right).$$

Recall the definition of $\widetilde{\text{Pr}}$ from Section 4.2. The following result generalizes [19, Lemma 6].

Lemma 4.9. *For all $K > 0$,*

$$(4.38) \quad \lim_{\varepsilon \rightarrow 0} \sup_{\xi \in \tilde{\mathcal{X}}^{(2)} : \tilde{J}_{\varepsilon/d}^{(2)}(\xi) \leq K\varepsilon} |\tilde{\phi}_{\infty,\varepsilon}(\xi) - \tilde{\Psi}_\varepsilon(\widetilde{\text{Pr}}(\xi))| = 0.$$

Proof of Lemma 4.9. Let $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}^{(2)}$. We write $\sigma_i := \alpha_i(\mathbb{R}^d \times \mathbb{R}^d) = (\text{Pr}(\alpha_i) \otimes \pi_{\varepsilon/d})(\mathbb{R}^d \times \mathbb{R}^d)$ and $\bar{\alpha}_i = \alpha_i/\sigma_i$ for all $i \in I$ and $\varepsilon > 0$. Using that $\text{Pr}(\alpha_i)(\mathbb{R}^d) = \alpha_i(\mathbb{R}^d \times \mathbb{R}^d)$ and adapting the proof of [19, Lemma 6], we get

$$(4.39) \quad |\tilde{\phi}_{\infty,\varepsilon}(\xi) - \tilde{\Psi}_\varepsilon(\widetilde{\text{Pr}}(\xi))| \leq \kappa \sum_{i \in I} \sigma_i \|\bar{\alpha}_i - \text{Pr}(\bar{\alpha}_i) \otimes \pi_{\varepsilon/d}\|_{\text{tv}}.$$

Moreover, by the standard Pinsker inequality [19, Lemma 5(e)]

$$(4.40) \quad \|\bar{\alpha}_i - \text{Pr}(\bar{\alpha}_i) \otimes \pi_{\varepsilon/d}\|_{\text{tv}} \leq 8 \sqrt{J_{\varepsilon/d}^{(2)}(\bar{\alpha}_i)} = 8 \sqrt{\frac{J_{\varepsilon/d}^{(2)}(\alpha_i)}{\sigma_i}}.$$

Recall that $\sum_{i \in I} \sigma_i \leq 1$. By the Cauchy-Schwarz inequality, we obtain

$$(4.41) \quad |\tilde{\phi}_{\infty,\varepsilon}(\xi) - \tilde{\Psi}_\varepsilon(\widetilde{\text{Pr}}(\xi))| \leq 8\kappa \sqrt{\sum_{i \in I} J_{\varepsilon/d}^{(2)}(\alpha_i)} = 8\kappa \sqrt{\tilde{J}_{\varepsilon/d}^{(2)}(\mu)},$$

which completes the proof. ■

Recall the definition of $\tilde{\Gamma}$ in (1.30). We define for any $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}$ and $\delta > 0$,

$$(4.42) \quad \tilde{\Gamma}_\delta(\xi) := \sum_{i \in I} \int dx \left(1 - e^{-\kappa(p_{\delta/d} * \alpha_i)(x)} \right) + \kappa \left(1 - \sum_{i \in I} \alpha_i(\mathbb{R}^d) \right).$$

Lemma 4.10. *For all $K > 0$,*

$$(4.43) \quad \lim_{\varepsilon \rightarrow 0} \sup_{\xi \in \tilde{\mathcal{X}}: \tilde{J}_{\varepsilon/d}(\xi) \leq K\varepsilon} \left| \tilde{\Gamma}(\xi) - \tilde{\Psi}_\varepsilon(\xi) \right| = 0.$$

Moreover, there exists $C \in (0, \infty)$ such that for any $\delta \geq \varepsilon$,

$$(4.44) \quad \sup_{\xi \in \tilde{\mathcal{X}}: \tilde{J}_{\varepsilon/d}(\xi) \leq K\varepsilon} \left| \tilde{\Gamma}(\xi) - \tilde{\Gamma}_\delta(\xi) \right| \leq C\sqrt{\delta}.$$

The proof is deferred to Section 4.6. We conclude this section with the following:

Proposition 4.11. *For all $K > 0$, the restriction of $\tilde{\Gamma}$ to $\mathcal{A}(K) := \{\xi \in \tilde{\mathcal{X}}: \tilde{J}(\xi) \leq K\}$ is lower semi-continuous. Moreover, for all $\delta > 0$, the map $\tilde{\Gamma}_\delta$ is lower semi-continuous on $\tilde{\mathcal{X}}$.*

The proof is deferred to Section 4.7.

4.4. Conclusion: Proof of Proposition 1.7. Along this section, we shall say that a real-valued sequence (u_n) is *negligible* if for some $c > \frac{1}{d}I(b)$ (see Theorem 1.3) we have $|u_n| \leq \exp(-cn^{1-2/d})$ for all n large enough. Let $\varepsilon_0 > 0$. By Proposition 4.1,

$$(4.45) \quad \mathbb{P}(R_n \leq bn, \tilde{L}_{M,\varepsilon} \in F) \leq \mathbb{P}(\mathbb{E}(R_n|\hat{S}) \leq (b + \varepsilon_0)n, \tilde{L}_{M,\varepsilon} \in F) + O(e^{-C(\varepsilon, \varepsilon_0)n^{1-2/d}}),$$

where $C(\varepsilon, \varepsilon_0)$ goes to infinity as ε goes to zero. Hence the second term is *negligible* provided ε is chosen small enough. By Proposition 4.3, we obtain for all $A > 0$ and $n \geq n_0(\varepsilon, \varepsilon_0, A)$,

$$(4.46) \quad \mathbb{P}(\mathbb{E}(R_n|\hat{S}) \leq (b + \varepsilon_0)n, \tilde{L}_{M,\varepsilon} \in F) \leq (a) + (b),$$

where

$$(4.47) \quad \begin{aligned} (a) &= \mathbb{P}(\phi_{\infty, \varepsilon}(L_{M,\varepsilon}^{(2)}) \leq b + 2\varepsilon_0, \tilde{L}_{M,\varepsilon} \in F), \\ (b) &= \mathbb{P}(L_{M,\varepsilon}^{(2)} \notin \mathcal{M}(A, \varepsilon_0)). \end{aligned}$$

By Proposition 4.2 the term (b) is *negligible* upon choosing A suitable. Let us now focus on (a), which is the main term and which we may write as

$$(4.48) \quad \begin{aligned} (a) &= \mathbb{P}(\tilde{\phi}_{\infty, \varepsilon}(\tilde{L}_{M,\varepsilon}^{(2)}) \leq b + 2\varepsilon_0, \tilde{L}_{M,\varepsilon} \in F) \\ &= \mathbb{P}(\tilde{\phi}_{\infty, \varepsilon}(\tilde{L}_{M,\varepsilon}^{(2)}) \leq b + 2\varepsilon_0, \tilde{\text{Pr}}(\tilde{L}_{M,\varepsilon}^{(2)} \in F)). \end{aligned}$$

By Lemmas 4.7 and 4.8, the set $\widetilde{\text{Pr}}^{-1}(F) \cap \widetilde{\phi}_{\infty, \varepsilon}^{-1}(-\infty, b+2\varepsilon_0]$ is closed for the \mathbf{D}_2 -topology. Thus, combining this with Proposition 4.6, we get

$$(4.49) \quad \limsup_{n \rightarrow \infty} n^{2/d-1} \log \mathbb{P}(\widetilde{\phi}_{\infty, \varepsilon}(\widetilde{L}_{M, \varepsilon}^{(2)}) \leq b + 2\varepsilon_0, \widetilde{\text{Pr}}(\widetilde{L}_{M, \varepsilon}^{(2)}) \in F) \\ \leq - \inf_{\xi \in \widetilde{\text{Pr}}^{-1}(F) \cap \widetilde{\phi}_{\infty, \varepsilon}^{-1}(-\infty, b+2\varepsilon_0]} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}^{(2)}(\xi).$$

Define

$$(4.50) \quad \mathcal{G}_\varepsilon(b) := \{\xi \in \widetilde{\mathcal{X}} : \widetilde{\Psi}_\varepsilon(\xi) \leq b\}.$$

Now, we argue that for ε sufficiently small

$$(4.51) \quad \inf_{\xi \in \widetilde{\text{Pr}}^{-1}(F) \cap \widetilde{\phi}_{\infty, \varepsilon}^{-1}(-\infty, b+2\varepsilon_0]} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}^{(2)}(\xi) \geq \inf_{\xi \in \widetilde{\text{Pr}}^{-1}(F \cap \mathcal{G}_\varepsilon(b+2\varepsilon_0))} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}^{(2)}(\xi) = \inf_{\xi \in F \cap \mathcal{G}_\varepsilon(b+2\varepsilon_0)} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}(\xi).$$

The lower bound is obtained by Lemma 4.9, while the last equality follows from Lemma 4.7 and the contraction principle. Summing up, we have proven so far that for all ε sufficiently small

$$(4.52) \quad \limsup_{n \rightarrow \infty} n^{\frac{2}{d}-1} \log \mathbb{P}(R_n \leq bn, \widetilde{L}_{M, \varepsilon} \in F) \leq - \inf_{\xi \in F \cap \mathcal{G}_\varepsilon(b+2\varepsilon_0)} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}(\xi).$$

We now investigate the limit of the right-hand side as $\varepsilon \rightarrow 0$. To that end we first note that if

$$(4.53) \quad \liminf_{\varepsilon \rightarrow 0} \inf_{\xi \in F \cap \mathcal{G}_\varepsilon(b+2\varepsilon_0)} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}(\xi) = \infty,$$

then we can immediately conclude the result. Hence, we can assume that the above limit is finite, and therefore that at least a long a sub-sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ there is a constant $K > 0$ such that for all $n \in \mathbb{N}$

$$(4.54) \quad \inf_{\xi \in F \cap \mathcal{G}_{\varepsilon_n}(b+2\varepsilon_0)} \frac{1}{\varepsilon_n} \widetilde{J}_{\varepsilon_n/d}(\xi) \leq K.$$

In the sequel we will suppress the sub-sequence from the notation. To continue we will need the following result, whose proof is deferred to Section 4.8:

Proposition 4.12. *For any compact set $\mathcal{K} \subseteq \widetilde{\mathcal{X}}$ (in the \mathbf{D} -topology),*

$$(4.55) \quad \liminf_{\varepsilon \rightarrow 0} \inf_{\xi \in \mathcal{K}} \frac{1}{\varepsilon} \widetilde{J}_{\varepsilon/d}(\xi) \geq \frac{1}{d} \inf_{\xi \in \mathcal{K}} \widetilde{J}(\xi).$$

We now finish the proof of Proposition 1.7. Define

$$(4.56) \quad \widetilde{\mathcal{D}}_\delta(b) := \{\xi \in \widetilde{\mathcal{X}} : \widetilde{\Gamma}_\delta(\xi) \leq b\},$$

and recall the definition of $\widetilde{\mathcal{D}}(b)$ in the formulation of Proposition 1.7. By Lemma 4.10, and the observation made around Equation (4.54) we can write for any ε sufficiently

small, $\delta \geq \varepsilon_0$, and some fixed constant $C > 0$,

$$\begin{aligned}
(4.57) \quad & - \inf_{\xi \in F \cap \mathcal{G}_\varepsilon(b+2\varepsilon_0)} \frac{1}{\varepsilon} \tilde{J}_{\varepsilon/d}(\xi) = - \inf_{\substack{\xi \in F \cap \mathcal{G}_\varepsilon(b+2\varepsilon_0), \\ \tilde{J}_{\varepsilon/d}(\xi) \leq K\varepsilon}} \frac{1}{\varepsilon} \tilde{J}_{\varepsilon/d}(\xi) \\
& \leq - \inf_{\substack{\xi \in F \cap \tilde{\mathcal{D}}(b+3\varepsilon_0), \\ \tilde{J}_{\varepsilon/d}(\xi) \leq K\varepsilon}} \frac{1}{\varepsilon} \tilde{J}_{\varepsilon/d}(\xi) \\
& \leq - \inf_{\substack{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta}), \\ \tilde{J}_{\varepsilon/d}(\xi) \leq K\varepsilon}} \frac{1}{\varepsilon} \tilde{J}_{\varepsilon/d}(\xi) \\
& \leq - \inf_{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta})} \frac{1}{\varepsilon} \tilde{J}_{\varepsilon/d}(\xi).
\end{aligned}$$

Note that $F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta}) \subseteq \tilde{\mathcal{X}}$ is closed by the second part of Proposition 4.11, thus compact. Hence, Proposition 4.12 implies that

$$(4.58) \quad - \liminf_{\varepsilon \rightarrow 0} \inf_{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta})} \frac{1}{\varepsilon} \tilde{J}_{\varepsilon/d}(\xi) \leq -\frac{1}{d} \inf_{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta})} \tilde{J}(\xi)$$

It remains to send $\delta \rightarrow 0$. To that end, we note that using the same arguments as in Equation (4.54) we can again assume that there is some $K > 0$ such that at least along a subsequence of δ 's converging to zero we have that

$$(4.59) \quad \inf_{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta})} \tilde{J}(\xi) \leq K$$

for all such δ . We will again suppress the choice of subsequence from the notation. Thus, recalling that $\mathcal{A}(K) := \{\xi \in \tilde{\mathcal{X}} : \tilde{J}(\xi) \leq K\}$, we can write

$$(4.60) \quad - \inf_{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta})} \tilde{J}(\xi) = - \inf_{\xi \in F \cap \tilde{\mathcal{D}}_\delta(b+C\sqrt{\delta}) \cap \mathcal{A}(K)} \tilde{J}(\xi) \leq - \inf_{\xi \in F \cap \tilde{\mathcal{D}}(b+2C\sqrt{\delta}) \cap \mathcal{A}(K)} \tilde{J}(\xi),$$

where we used Lemma 4.10 to obtain the last inequality. Letting $\delta \rightarrow 0$ with the help of Lemma 4.13 below and using that

$$(4.61) \quad \inf_{\xi \in F \cap \tilde{\mathcal{D}}(b) \cap \mathcal{A}(K)} \tilde{J}(\xi) \geq \inf_{\xi \in F \cap \tilde{\mathcal{D}}(b)} \tilde{J}(\xi),$$

we can conclude the proof of Proposition 1.7.

Lemma 4.13. *Let $F \subseteq \tilde{\mathcal{X}}$ be a closed set. Then,*

$$(4.62) \quad \lim_{\delta \rightarrow 0} \inf_{\xi \in F \cap \tilde{\mathcal{D}}(b+\delta) \cap \mathcal{A}(K)} \tilde{J}(\xi) = \inf_{\xi \in F \cap \tilde{\mathcal{D}}(b) \cap \mathcal{A}(K)} \tilde{J}(\xi).$$

Proof of Lemma 4.13. Since $\tilde{\mathcal{D}}(b) \subseteq \tilde{\mathcal{D}}(b+\delta)$, we directly have

$$(4.63) \quad \limsup_{\delta \rightarrow 0} \inf_{F \cap \tilde{\mathcal{D}}(b+\delta) \cap \mathcal{A}(K)} \tilde{J} \leq \inf_{F \cap \tilde{\mathcal{D}}(b) \cap \mathcal{A}(K)} \tilde{J},$$

From now on, we focus on the reversed inequality. By Proposition 4.11 (first part of the statement) and the compactness of $\tilde{\mathcal{X}}$, the set $F \cap \tilde{\mathcal{D}}(b+\delta) \cap \mathcal{A}(K)$ is actually compact.

Hence, there exists a sequence $(\xi_\delta)_{\delta>0}$ of minimizers of the left hand side in (4.62). By the compactness of $\tilde{\mathcal{X}}$ we may extract a subsequence converging to some $\xi_0 \in \tilde{\mathcal{X}}$, which for ease of notation we again denote by $(\xi_\delta)_{\delta>0}$. By the lower semi-continuity of \tilde{J} , we get that $\xi_0 \in \mathcal{A}(K)$ and by the lower semi-continuity of $\tilde{\Gamma}$ restricted to $\mathcal{A}(K)$, we get that $\tilde{\Gamma}(\xi_0) \leq \liminf_{\delta \rightarrow 0} \tilde{\Gamma}(\xi_\delta) = b$, hence $\xi_0 \in \tilde{\mathcal{D}}(b)$. In conclusion,

$$(4.64) \quad \liminf_{\delta \rightarrow 0} \inf_{F \cap \tilde{\mathcal{D}}(b+\delta) \cap \mathcal{A}(K)} \tilde{J} = \liminf_{\delta \rightarrow 0} \tilde{J}(\xi_\delta) \geq \tilde{J}(\xi_0) \geq \inf_{F \cap \tilde{\mathcal{D}}(b) \cap \mathcal{A}(K)} \tilde{J}.$$

■

4.5. Proof of Lemma 4.8. We proceed in several steps.

(i) *Truncation procedure.* For $\eta \in (0, \varepsilon/2)$ and $\mu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d \times \mathbb{R}^d)$, let us define

$$(4.65) \quad \phi_{\infty, \varepsilon, \eta}(\mu) := \int_{\mathbb{R}^d} dz \left(1 - e^{-\frac{\kappa}{\varepsilon} \int \varphi_{\varepsilon, \eta}(z-x, z-y) \mu(dx, dy)} \right)$$

where

$$(4.66) \quad \varphi_{\varepsilon, \eta}(x, y) := \int_{\eta}^{\varepsilon-\eta} \frac{p_{s/d}(-x) p_{(\varepsilon-s)/d}(y)}{p_{\varepsilon/d}(x-y)} ds,$$

and for $\xi = \{\tilde{\alpha}_i\}_{i \in I}$,

$$(4.67) \quad \tilde{\phi}_{\infty, \varepsilon, \eta}(\xi) := \sum_{i \in I} \phi_{\infty, \varepsilon, \eta}(\alpha_i) + \kappa \left(1 - \frac{2\eta}{\varepsilon} \right) \left(1 - \sum_{i \in I} \alpha_i(\mathbb{R}^d \times \mathbb{R}^d) \right).$$

It is enough to show that $\tilde{\phi}_{\infty, \varepsilon, \eta}$ is lower semi-continuous with respect to the metric \mathbf{D}_2 , since $\tilde{\phi}_{\infty, \varepsilon}$ is the supremum of $\tilde{\phi}_{\infty, \varepsilon, \eta}$ with respect to η .

(ii) *Rewriting the truncated function.* By expanding the exponential in (4.65), and using Lemma 4.5 we get for all $\mu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$(4.68) \quad \begin{aligned} & \tilde{\phi}_{\infty, \varepsilon, \eta}(\{\mu\}) - \kappa \left(1 - \frac{2\eta}{\varepsilon} \right) \\ &= \int_{\mathbb{R}^d} dz \sum_{n \geq 2} \frac{(-1)^{n+1}}{n!} \left(\frac{\kappa}{\varepsilon} \right)^n \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} \prod_{i=1}^n \varphi_{\varepsilon, \eta}(z - x_i, z - y_i) \mu(dx_i, dy_i). \end{aligned}$$

By Fubini-Tonelli's theorem, we may interchange the sum over n and the integral over z . Indeed,

$$(4.69) \quad \begin{aligned} & \int_{\mathbb{R}^d} dz \sum_{n \geq 2} \frac{1}{n!} \left(\frac{\kappa}{\varepsilon} \right)^n \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} \prod_{i=1}^n \varphi_{\varepsilon, \eta}(z - x_i, z - y_i) \mu(dx_i, dy_i) \\ &= \int_{\mathbb{R}^d} dz \sum_{n \geq 2} \frac{1}{n!} \left(\frac{\kappa}{\varepsilon} \right)^n \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_{\varepsilon, \eta}(z - x, z - y) \mu(dx, dy) \right)^n \\ &\leq \int_{\mathbb{R}^d} dz \left(e^{\frac{\kappa}{\varepsilon} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_{\varepsilon, \eta}(z-x, z-y) \mu(dx, dy)} - 1 \right) \\ &\leq e^{\frac{\kappa}{\varepsilon} \|\varphi_{\varepsilon, \eta}\|_{\infty}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\kappa}{\varepsilon} \varphi_{\varepsilon, \eta}(z - x, z - y) \mu(dx, dy) dz \\ &\leq \kappa e^{\frac{\kappa}{\varepsilon} \|\varphi_{\varepsilon, \eta}\|_{\infty}} \mu(\mathbb{R}^d \times \mathbb{R}^d) < \infty. \end{aligned}$$

We have used Lemma 4.5 and the comment slightly below Lemma 4.14 on the boundedness of the function $\varphi_{\varepsilon,\eta}$. We obtain thereby:

$$(4.70) \quad \begin{aligned} & \tilde{\phi}_{\infty,\varepsilon,\eta}(\{\mu\}) - \kappa(1 - \frac{2\eta}{\varepsilon}) \\ &= \sum_{n \geq 2} \frac{(-1)^{n+1}}{n!} \left(\frac{\kappa}{\varepsilon}\right)^n \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} V_n^{(\eta)}(x_1, y_1, \dots, x_n, y_n) \prod_{i=1}^n \mu(dx_i, dy_i), \end{aligned}$$

where

$$(4.71) \quad V_n^{(\eta)}(x_1, y_1, \dots, x_n, y_n) := \int_{\mathbb{R}^d} dz \prod_{i=1}^n \varphi_{\varepsilon,\eta}(z - x_i, z - y_i).$$

(iii) *Lower-semi continuity of the truncated function.* The V_n 's defined above (we omit η) are clearly *translation invariant*, and we will also prove that they are *continuous* (Step 1 below) and *vanishing at infinity* (Step 2 below). It then follows by the definition of the metric \mathbf{D}_2 that

$$(4.72) \quad \xi \mapsto \sum_{i \in I} \int V_n^{(\eta)}(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n \alpha_i(dx_j, dy_j)$$

is continuous. Moreover, by (4.69),

$$(4.73) \quad \begin{aligned} & \sum_{i \in I} \sum_{n \geq 2} \frac{1}{n!} \left(\frac{\kappa}{\varepsilon}\right)^n \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} V_n^{(\eta)}(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n \alpha_i(dx_j, dy_j) \\ & \leq \sum_{i \in I} \kappa e^{\frac{\kappa}{\varepsilon} \|\varphi_{\varepsilon,\eta}\|_\infty} \alpha_i(\mathbb{R}^d \times \mathbb{R}^d) < \infty, \end{aligned}$$

This allows us to use Fubini-Tonelli's theorem and obtain

$$(4.74) \quad \begin{aligned} & \sum_{i \in I} \sum_{n \geq 2} \frac{(-1)^{n+1}}{n!} \left(\frac{\kappa}{\varepsilon}\right)^n \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} V_n^{(\eta)}(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n \alpha_i(dx_j, dy_j) \\ &= \sum_{n \geq 2} \frac{(-1)^{n+1}}{n!} \left(\frac{\kappa}{\varepsilon}\right)^n \sum_{i \in I} \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} V_n^{(\eta)}(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n \alpha_i(dx_j, dy_j), \end{aligned}$$

with the sum over n on the right hand side converging absolutely. Therefore, the right-hand side is continuous with respect to the metric \mathbf{D}_2 .

(iv) *Conclusion.* As announced in Step (iii) above, it remains to prove the two required properties of the function $V_n^{(\eta)}$ in order to complete this section.

Step 1. *Continuity of $V_n^{(\eta)}$.* First, one can prove that for all z , the function

$$(4.75) \quad (x_1, y_1, \dots, x_n, y_n) \mapsto \prod_{i=1}^n \varphi_{\varepsilon,\eta}(z - x_i, z - y_i)$$

is continuous (by dominated convergence for instance). Now, observe that for any (x, y) in a compact set $K \subseteq \mathbb{R}^d \times \mathbb{R}^d$ (say $|x|, |y| \leq M$) and $\eta > 0$

$$(4.76) \quad \varphi_{\varepsilon, \eta}(z - x, z - y) \leq C(\varepsilon, \eta, M) \exp \left(-\frac{d|z|^2}{\varepsilon} + \frac{\varepsilon d M |z|}{\eta(\varepsilon - \eta)} \right),$$

which is integrable in z , hence the continuity of $V_n^{(\eta)}$, by dominated convergence. To get the inequality above, use that

$$(4.77) \quad \begin{aligned} \frac{|x - y|^2}{2\varepsilon} - \frac{|z - x|^2}{2s} - \frac{|z - y|^2}{2(\varepsilon - s)} &= \frac{|x - y|^2}{2\varepsilon} - \frac{|x|^2}{2s} - \frac{|y|^2}{2(\varepsilon - s)} - \frac{\varepsilon|z|^2}{2s(\varepsilon - s)} \\ &\quad + \frac{\langle z, x \rangle}{s} + \frac{\langle z, y \rangle}{(\varepsilon - s)} \\ &\leq -\frac{\varepsilon|z|^2}{2s(\varepsilon - s)} + \frac{|x||y|}{\varepsilon} + \frac{|z||x|}{s} + \frac{|z||y|}{(\varepsilon - s)}. \end{aligned}$$

Step 2. *Evanesence of $V_n^{(\eta)}$ at infinity.* Let us start with an estimate:

Lemma 4.14. *For all $x, y \in \mathbb{R}^d$, $\eta \in (0, \varepsilon/2)$,*

$$(4.78) \quad \varphi_{\varepsilon, \eta}(x, y) \leq C(\eta, \varepsilon) \min \left(1, (\sqrt{|x|} + \sqrt{|y|})^{-2} \right) \exp \left[\frac{d}{2\varepsilon} (|x - y|^2 - (|x| + |y|)^2) \right].$$

Note that the term in the exponential is nonpositive, and equals zero if and only if x and y are colinear with opposite directions (i.e. $\langle x, y \rangle = -|x||y|$). In particular, the function $\varphi_{\varepsilon, \eta}$ is bounded.

Proof of Lemma 4.14. For all $s \in (0, \varepsilon)$, $\alpha, \beta > 0$, define

$$(4.79) \quad f_{\alpha, \beta}(s) = \frac{\alpha}{s} + \frac{\beta}{\varepsilon - s}.$$

By computing the first four derivatives, we see that the function achieves its minimum $\frac{1}{\varepsilon}(\alpha^{1/2} + \beta^{1/2})^2$ at $\frac{\varepsilon\alpha^{1/2}}{\alpha^{1/2} + \beta^{1/2}}$ and that its second derivative achieves its minimum $\frac{2}{\varepsilon^3}(\alpha^{1/4} + \beta^{1/4})^4$ at $\frac{\varepsilon\alpha^{1/4}}{\alpha^{1/4} + \beta^{1/4}}$. Therefore, for all $s \in (0, \varepsilon)$, we get by using a second-order Taylor expansion:

$$(4.80) \quad f_{\alpha, \beta}(s) - \frac{1}{\varepsilon}(\alpha^{1/2} + \beta^{1/2})^2 \geq \frac{1}{\varepsilon^3}(\alpha^{1/4} + \beta^{1/4})^4 \left(s - \frac{\varepsilon\alpha^{1/2}}{\alpha^{1/2} + \beta^{1/2}} \right)^2.$$

Applying this inequality to $\alpha = |x|^2$ and $\beta = |y|^2$, we obtain

$$(4.81) \quad \begin{aligned} \varphi_{\varepsilon, \eta}(x, y) &\leq \frac{\varepsilon^{d/2}}{[2\pi\eta(\varepsilon - \eta)]^{-d/2}} \int_0^\varepsilon \exp \left[-\frac{d}{2\varepsilon^3} \left(s - \frac{\varepsilon|x|}{|x| + |y|} \right)^2 (\sqrt{|x|} + \sqrt{|y|})^4 \right] ds \\ &\quad \times \exp \left[\frac{d}{2\varepsilon} (|x - y|^2 - (|x| + |y|)^2) \right]. \end{aligned}$$

Using that $\int_{\mathbb{R}} e^{-cs^2} ds = \sqrt{\pi/c}$, we get the result. ■

Let us come back to Step 2. Let $n \geq 2$ and $\eta \in (0, \varepsilon/2)$. We now show that $V_n^{(\eta)}$ is vanishing. To this end, consider a sequence

$$(4.82) \quad (x_{1,N}, y_{1,N}, x_{2,N}, y_{2,N}, \dots, x_{n,N}, y_{n,N})$$

such that

$$(4.83) \quad \max_{1 \leq i \leq j \leq n} \{|x_{j,N} - x_{i,N}| \vee |y_{j,N} - y_{i,N}| \vee |x_{j,N} - y_{i,N}|\} \xrightarrow{N \rightarrow \infty} \infty.$$

In the sequel we suppress the N from the notation and assume w.l.o.g. that maximizing indexes i and j can be found in $\{1, 2\}$. Pick $M > 0$. For N large enough, we then have $\max(|y_1 - y_2|, |x_1 - x_2|, |x_1 - y_1|, |x_2 - y_2|) \geq M$. If $|y_1 - y_2| \geq M$, we may write

$$(4.84) \quad \mathbb{R}^d = A_1 \cup A_2, \quad \text{where } A_i := \{z \in \mathbb{R}^d : |z - y_i| \geq M/2\}, \quad i \in \{1, 2\}.$$

Using the boundedness of $\varphi_{\varepsilon, \eta}$ (Lemma 4.14) we see that it is sufficient to focus on the product of $\varphi_{\varepsilon, \eta}(\cdot - x_i, \cdot - y_i)$ for $i = 1, 2$. Using Lemma 4.14 and the fact that $z \mapsto \varphi_{\varepsilon, \eta}(z - x_2, z - y_2)$ is a sub-probability density, we get

$$(4.85) \quad \int_{A_1} dz \prod_{i=1}^2 \varphi_{\varepsilon, \eta}(z - x_i, z - y_i) \leq (\text{cst}) \int_{A_1} dz \frac{1}{|z - y_1|} \varphi_{\varepsilon, \eta}(z - x_2, z - y_2) \leq \frac{(\text{cst})}{M},$$

and similarly for the integral over A_2 . The other cases, namely $|x_1 - x_2| \geq M$, $|x_1 - y_1| \geq M$ and $|x_2 - y_2| \geq M$, can be dealt with in the same way.

4.6. Proof of Lemma 4.10. In this section, we prove Lemma 4.10 and take the occasion to correct a glitch in [19], see Remark 4.15 below. We proceed in two steps.

Step 1. Proof of (4.43). We further split this step in two parts. We first treat the case of ξ consisting of a single orbit and then treat the general case.

(i) *Single orbit case.* Let $\nu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$ and f be its density with respect to the Lebesgue measure. Assume $\xi = \{\tilde{\nu}\}$. Following (2.94) and (2.95) in the proof of [19, Lemma 7], we obtain:

$$(4.86) \quad |\tilde{\Gamma}(\xi) - \tilde{\Psi}_{d\varepsilon}(\xi)| \leq \frac{\kappa}{\varepsilon} \int_0^\varepsilon ds \|\nu\pi_s - \nu\|_{\text{tv}},$$

where

$$(4.87) \quad \|\nu\pi_s - \nu\|_{\text{tv}} := \int_{\mathbb{R}^d} |f(x) - \pi_s f(x)| dx.$$

The integrand is split and bounded as follows:

$$(4.88) \quad \|\nu\pi_s - \nu\|_{\text{tv}} \leq \|\nu\pi_s - \nu\pi_{s+\varepsilon}\|_{\text{tv}} + \|\nu\pi_{s+\varepsilon} - \nu\|_{\text{tv}}.$$

As in [19, Lemma 7], the first term in the sum is bounded by $8\sqrt{J_\varepsilon(\nu)}$. We now bound the second term. From this point on, the proof differs from [19, Lemma 7], see Remark 4.15 below. Writing $f - \pi_{s+\varepsilon}f = (\sqrt{f} - \sqrt{\pi_{s+\varepsilon}f})(\sqrt{f} + \sqrt{\pi_{s+\varepsilon}f})$ and using the Cauchy-Schwartz inequality, we obtain

$$(4.89) \quad \|\nu\pi_{s+\varepsilon} - \nu\|_{\text{tv}} = \int_{\mathbb{R}^d} |f(x) - \pi_{s+\varepsilon}f(x)| dx \leq 2H(\nu, \nu\pi_{s+\varepsilon}),$$

where

$$(4.90) \quad H(\nu, \nu\pi_{s+\varepsilon}) := \left(\int_{\mathbb{R}^d} (\sqrt{f(x)} - \sqrt{\pi_{s+\varepsilon}f(x)})^2 dx \right)^{1/2}$$

is the Hellinger distance between ν and $\nu\pi_{s+\varepsilon}$. Note that $\sqrt{\pi_{s+\varepsilon}f} \geq \pi_{s+\varepsilon}\sqrt{f}$, by Jensen's inequality. Therefore, using that ν and $\nu\pi_{s+\varepsilon}$ have the same mass,

$$(4.91) \quad H(\nu, \nu\pi_{s+\varepsilon})^2 = 2 \left(\int f(x) - \int \sqrt{f(x)} \sqrt{\pi_{s+\varepsilon}f(x)} \right) \leq 2L_{s+\varepsilon}(\sqrt{f}),$$

where we have defined

$$(4.92) \quad L_t(g) := \langle g, g \rangle - \langle g, \pi_t g \rangle, \quad g \in L^2(\mathbb{R}^d), \quad t \geq 0,$$

$\langle \cdot, \cdot \rangle$ being the usual inner product on the space of square integrable functions. By [13, Theorem 7.10], we know that for all $g \in L^2(\mathbb{R}^d)$, the map $t > 0 \mapsto L_t(g)/t$ is non-increasing (monotonicity is actually hidden in the proof of that theorem). Finally, note that for any $t > 0$

$$(4.93) \quad J_t(\nu) = \sup_{u>0} - \int \log \left(\frac{\pi_t u(x)}{u(x)} \right) f(x) dx \geq - \int \log \left(\frac{\pi_t \sqrt{f(x)}}{\sqrt{f(x)}} \right) f(x) dx \geq L_t(\sqrt{f}),$$

using $\log(1+z) \leq z$ for the last inequality. Summing up, we obtain for $0 \leq s \leq \varepsilon$,

$$(4.94) \quad \begin{aligned} \|\nu\pi_{s+\varepsilon} - \nu\|_{\text{tv}}^2 &\leq 4H(\nu, \nu\pi_{s+\varepsilon})^2 \leq 8L_{s+\varepsilon}(\sqrt{f}) = 8(s+\varepsilon) \frac{L_{s+\varepsilon}(\sqrt{f})}{s+\varepsilon} \\ &\leq 8(s+\varepsilon) \frac{L_\varepsilon(\sqrt{f})}{\varepsilon} \\ &\leq 16L_\varepsilon(\sqrt{f}) \\ &\leq 16J_\varepsilon(\nu), \end{aligned}$$

and finally,

$$(4.95) \quad |\Gamma(\nu) - \Psi_{d\varepsilon}(\nu)| \leq 12\kappa\sqrt{J_\varepsilon(\nu)}.$$

(ii) *General case.* Assume $\xi = \{\tilde{\alpha}_i, i \in I\}$. Similarly as in Step (i) and using the triangular inequality, we get:

$$(4.96) \quad |\tilde{\Gamma}(\xi) - \tilde{\Psi}_{d\varepsilon}(\xi)| \leq \sum_{i \in I} \frac{\kappa}{\varepsilon} \int_0^\varepsilon ds \|\alpha_i \pi_s - \alpha_i\|_{\text{tv}}.$$

Define $\sigma_i := \alpha_i(\mathbb{R}^d)$ and $\bar{\alpha}_i := \alpha_i/\sigma_i \in \mathcal{M}_1(\mathbb{R}^d)$ for all $i \in I$. From what precedes in (i),

$$(4.97) \quad \|\alpha_i \pi_s - \alpha_i\|_{\text{tv}} = \sigma_i \|\bar{\alpha}_i \pi_s - \bar{\alpha}_i\|_{\text{tv}} \leq 12\sigma_i \sqrt{J_\varepsilon(\bar{\alpha}_i)} = 12\sqrt{\sigma_i} \sqrt{J_\varepsilon(\alpha_i)}.$$

We may now conclude with the Cauchy-Schwarz inequality, since $\sum \sigma_i \leq 1$, that

$$(4.98) \quad |\tilde{\Gamma}(\xi) - \tilde{\Psi}_{d\varepsilon}(\xi)| \leq 12\kappa\sqrt{J_\varepsilon(\xi)},$$

which completes the proof.

Step 2. Proof of (4.44) We re-use the arguments from the previous step. For simplicity, let us stick to the case of a single orbit, i.e. $\xi = \{\tilde{\nu}\}$, where $\nu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$ with a density f w.r.t. Lebesgue measure. Then,

$$(4.99) \quad |\tilde{\Gamma}(\xi) - \tilde{\Gamma}_{d\delta}(\xi)| \leq \kappa \|\nu\pi_\delta - \nu\|_{\text{tv}} \leq 4\kappa\sqrt{L_\delta(\sqrt{f})}.$$

Since $\delta \geq \varepsilon$,

$$(4.100) \quad \frac{L_\delta(\sqrt{f})}{\delta} \leq \frac{L_\varepsilon(\sqrt{f})}{\varepsilon} \leq \frac{J_\varepsilon(\xi)}{\varepsilon},$$

which concludes the proof.

Remark 4.15. Equation (4.43) in Lemma 4.10 extends [19, Lemma 7] to the space $\tilde{\mathcal{X}}$. Our proof actually corrects a flaw in the proof of [19, Lemma 7]. The latter proof uses [19, Lemma 5(b)], which deduces monotonicity of the map $t \mapsto J_t(\nu)/t$ from the sub-additivity of $t \mapsto J_t(\nu)$. However, such monotonicity cannot be derived from sub-additivity in general (though the reverse holds true). As a counter example, consider $f(t) = 1 + |\sin(1/t)|$ for $t > 0$. This function is sub-additive but $f(t)/t$ fails to be non-increasing on any right-neighborhood of 0.

4.7. Proof of Proposition 4.11. The proof contains two parts.

(i) Let us begin with the first part of the statement. Let $K > 0$. It follows from Lemma 4.10 that $\tilde{\Psi}_\varepsilon$ converges uniformly to Γ on $\mathcal{A}(K) = \{\xi \in \tilde{\mathcal{X}} : \tilde{J}(\xi) \leq K\}$, as $\varepsilon \rightarrow 0$. If $\tilde{\Psi}_\varepsilon$ restricted to $\mathcal{A}(K)$ is lower semi-continuous for all $\varepsilon > 0$ we deduce therefore that $\tilde{\Gamma}$ restricted to $\mathcal{A}(K)$ is also lower semi-continuous. It remains to prove that $\tilde{\Psi}_\varepsilon$ restricted to $\mathcal{A}(K)$ is lower semi-continuous. This follows from the three following facts:

- $\Psi_\varepsilon(\mu) = \phi_{\infty, \varepsilon}(\mu \otimes \pi_{\varepsilon/d})$ for all $\mu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$: see [19, Proof of Lemma 6] and use the fact that $(\mu \otimes \pi_{\varepsilon/d})(\mathbb{R}^d \times \mathbb{R}^d) = \mu(\mathbb{R}^d)$;
- $\tilde{\phi}_{\infty, \varepsilon}$ is lower semi-continuous for the \mathbf{D}_2 -topology on $\tilde{\mathcal{X}}^{(2)}$ (Lemma 4.8);
- $\xi \in \tilde{\mathcal{X}} \mapsto \xi \otimes \pi_{\varepsilon/d} \in \tilde{\mathcal{X}}^{(2)}$ is continuous for the \mathbf{D} -topology on $\tilde{\mathcal{X}}$. Here, when $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}$ we mean $\xi \otimes \pi_{\varepsilon/d} := \{\widetilde{\alpha_i \otimes \pi_{\varepsilon/d}}\}_{i \in I}$.

To prove the last point, pick $f: (\mathbb{R}^d \times \mathbb{R}^d)^n \rightarrow \mathbb{R} \in \mathcal{F}_{2n}$ and consider $\xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}$. Define, for $x_1, \dots, x_n \in \mathbb{R}^d$,

$$(4.101) \quad g_i(x_1, \dots, x_n) := \int_{(\mathbb{R}^d)^n} f(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n p_{\varepsilon/d}(x_j, y_j) dy_j,$$

so that

$$(4.102) \quad \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} f(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n (\alpha_i \otimes \pi_{\varepsilon/d})(dx_j, dy_j) =: \int_{(\mathbb{R}^d)^n} g(x_1, \dots, x_n) \prod_{j=1}^n \alpha_i(dx_j).$$

Let us prove that $g \in \mathcal{F}_n$. It is immediate that g is translation invariant and that it vanishes at infinity. Regarding continuity, let $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ such that $|x_j| \leq M$ for all $j \in \{1, \dots, n\}$ and some $M > 0$. Then, for all $j \in \{1, \dots, n\}$,

$$(4.103) \quad p_{\varepsilon/d}(x_j, y_j) \leq \frac{(\text{cst})}{\varepsilon^{d/2}} \exp\left(-\frac{d|y_j|^2}{2\varepsilon} + \frac{dM|y_j|}{2\varepsilon}\right),$$

which is integrable. The continuity of g now follows from the boundedness and continuity of f , and dominated convergence. From (4.102) and the fact that $g \in \mathcal{F}_n$, we may now conclude that

$$(4.104) \quad \xi \mapsto \sum_{i \in I} \int_{(\mathbb{R}^d \times \mathbb{R}^d)^n} f(x_1, y_1, \dots, x_n, y_n) \prod_{j=1}^n (\alpha_i \otimes \pi_{\varepsilon/d})(dx_j, dy_j)$$

is continuous, which completes this part of the proof.

(ii) Let us now prove the second part of the statement, namely that $\tilde{\Gamma}_\delta$ is lower semi-continuous. The proof of this part follows closely the proof of Lemma 4.8, so we will not provide all the details. First of all we note that in the same way as in the proof of Lemma 4.8 we can write for any sub-probability measure $\mu \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$,

$$(4.105) \quad \tilde{\Gamma}_{d\delta}(\{\tilde{\mu}\}) = \kappa - \sum_{n \geq 2} \frac{(-\kappa)^n}{n!} \int_{(\mathbb{R}^d)^n} W_n^\delta(x_1, x_2, \dots, x_n) \prod_{i=1}^n \mu(dx_i),$$

with the translation invariant function:

$$(4.106) \quad W_n^\delta(x_1, x_2, \dots, x_n) := \int_{\mathbb{R}^d} dz \prod_{i=1}^n p_\delta(z - x_i).$$

In view of the proof of Lemma 4.8 and of the boundedness of the heat kernel p_δ it is enough to show that W_n^δ is continuous and vanishes at infinity. Concerning the continuity, assume that for some $M > 0$ we have that $|x_i| \leq M$ for all $i \in \{1, 2, \dots, n\}$. Then for all i we can estimate

$$(4.107) \quad p_\delta(z - x_i) \leq \frac{(\text{cst})}{\delta^{d/2}} \exp\left(-\frac{|z|^2}{2\delta} + \frac{M|z|}{2\delta}\right),$$

which is integrable in z . Since p_δ is continuous, the continuity of W_n^δ follows by dominated convergence. It remains to show that W_n^δ vanishes at infinity, in the sense that

$$(4.108) \quad \lim_{\max_{i \neq j} |x_i - x_j| \rightarrow \infty} W_n^\delta(x_1, x_2, \dots, x_n) = 0.$$

To that end, and without any loss of generality, we can assume that $|x_1 - x_2| \rightarrow \infty$. Let $M = |x_1 - x_2|$ and define

$$(4.109) \quad \begin{aligned} A_1 &= \{z \in \mathbb{R}^d : |z - x_1| < \frac{M}{2}, |z - x_2| \geq \frac{M}{2}\}, \\ A_2 &= \{z \in \mathbb{R}^d : |z - x_1| \geq \frac{M}{2}, |z - x_2| < \frac{M}{2}\}, \\ A_3 &= \{z \in \mathbb{R}^d : |z - x_1| \geq \frac{M}{2}, |z - x_2| \geq \frac{M}{2}\}. \end{aligned}$$

By the choice of M , it follows that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^d$. On A_1 we can estimate

$$(4.110) \quad p_\delta(z - x_2) \leq \frac{(\text{cst})}{\delta^{d/2}} e^{-\frac{M^2}{8\delta}}.$$

Thus, by the boundedness of p_δ it follows that

$$(4.111) \quad \int_{A_1} dz \prod_{i=1}^n p_\delta(z - x_i) \leq \frac{(\text{cst})}{\delta^{d/2}} e^{-\frac{M^2}{8\delta}} \int dz p_\delta(z - x_1) = \frac{(\text{cst})}{\delta^{d/2}} e^{-\frac{M^2}{8\delta}},$$

which tends to zero as $M \rightarrow \infty$. The integrals over A_2 and A_3 can be dealt with in the same manner. This concludes the proof.

4.8. **Proof of Proposition 4.12.** The proof relies on the following lemma.

Lemma 4.16. *Suppose $(\xi_n) \rightarrow \xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}$. Then, for every sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ tending to zero, and for every $\delta > 0$, there exists $k = k(\delta)$ such that for every collection of positive constants $(c_i)_{1 \leq i \leq k}$ and every collection $(v_i)_{1 \leq i \leq k}$ of smooth non-negative functions with compact support,*

$$(4.112) \quad \liminf_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n}(\xi_n) \geq - \sum_{1 \leq i \leq k} \int \frac{\frac{1}{2} \Delta v_i(x)}{c_i + v_i(x)} \alpha_i(dx) - \delta.$$

Moreover, $k = k(\delta)$ can be chosen such that $\delta \mapsto k(\delta)$ is non-increasing with $\lim_{\delta \rightarrow 0} k(\delta) = |I|$, where the cardinality $|I|$ of the index set I may be either finite or infinite.

Let us recall the following standard fact about the entropy rate function:

$$(4.113) \quad J(\alpha) = \sup_{u, c} \int -\frac{1}{2} \frac{\Delta u}{c + u} d\alpha, \quad \alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d),$$

where the supremum runs over $c \in (0, \infty)$ and non-negative, smooth, and compactly supported functions u , see [14, Proof of Lemma 4.7]. By optimizing over the c_i 's and v_i 's in (4.112) and letting $\delta \rightarrow 0$, we thus obtain as an immediate corollary of Lemma 4.16:

Corollary 4.17. *Suppose $(\xi_n) \rightarrow \xi = \{\tilde{\alpha}_i\}_{i \in I} \in \tilde{\mathcal{X}}$. Then, for every sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ tending to zero,*

$$(4.114) \quad \liminf_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n/d}(\xi_n) \geq \frac{1}{d} \tilde{J}(\xi).$$

Proof of Lemma 4.16. The proof strategy is inspired by Lemma 4.2 in [14]. Let us first consider the simpler case when, for all n , ξ_n consists of a single orbit denoted by $\tilde{\mu}_n$. Then, as in [14] we may write at least along some subsequence, which we will suppress from the notation

$$(4.115) \quad \mu_n = \sum_{1 \leq i \leq k} \alpha_n^{(i)} + \beta_n.$$

with

$$(4.116) \quad \alpha_n^{(i)} * \delta_{a_n^{(i)}} \xrightarrow{\text{(weakly)}} \alpha_i, \quad n \rightarrow \infty,$$

and

$$(4.117) \quad \liminf_{n \rightarrow \infty} \inf_{i \neq j} |a_n^{(i)} - a_n^{(j)}| = +\infty$$

for some sequences $(a_n^{(i)})$ in \mathbb{R}^d . Moreover, for all $V \in \mathcal{F}_2$,

$$(4.118) \quad \int V(x, y) \alpha_n^{(i)}(dx) \beta_n(dy) \xrightarrow{n \rightarrow \infty} 0,$$

$$\limsup_{n \rightarrow \infty} \int V(x, y) \beta_n(dx) \beta_n(dy) \leq \delta.$$

Moreover, k depends monotonously on δ and tends to $|I|$ as δ tends to zero. For any function u which is smooth and positive,

$$(4.119) \quad \frac{1}{\varepsilon} \log \left(\frac{\pi_\varepsilon u}{u} \right)(x) \leq \left(\frac{\frac{1}{2} \Delta u}{u} \right)(x) + \left| \frac{\pi_\varepsilon u - u}{\varepsilon u}(x) - \frac{\frac{1}{2} \Delta u}{u}(x) \right|.$$

By integrating this inequality w.r.t. μ_n , we obtain

$$(4.120) \quad \int \frac{1}{\varepsilon} \log \left(\frac{\pi_\varepsilon u}{u} \right) \mu_n(dx) \leq \sum_{1 \leq i \leq k} \int \left(\frac{\frac{1}{2} \Delta u}{u} \right)(x) \alpha_n^{(i)}(dx) + \int \left(\frac{\frac{1}{2} \Delta u}{u} \right)(x) \beta_n(dx) \\ + \int \left| \frac{\pi_\varepsilon u - u}{\varepsilon u}(x) - \frac{\frac{1}{2} \Delta u}{u}(x) \right| \mu_n(dx).$$

We now pick u of the form

$$(4.121) \quad u(x) = \sum_{1 \leq i \leq k} (c_i + v_i(x + a_n^{(i)})) \varphi \left(\frac{x + a_n^{(i)}}{R} \right),$$

where

- $R > 0$,
- $(c_i)_{1 \leq i \leq k}$ are of positive numbers,
- $(v_i)_{1 \leq i \leq k}$ are non-negative, smooth and compactly supported functions,
- φ is smooth and satisfies $\varphi(x) = 0$ if $\|x\| \geq 2$ and $\varphi(x) = 1$ if $\|x\| \leq 1$.

We abbreviate $v_{i,R}(\cdot) = v_i(\cdot) \varphi(\frac{\cdot}{R})$, so that

$$(4.122) \quad u_n(x) = \sum_{1 \leq i \leq k} (c_i + v_{i,R}(x + a_n^{(i)})), \\ \Delta u_n(x) = \sum_{1 \leq i \leq k} \Delta v_{i,R}(x + a_n^{(i)}).$$

Then,

$$(4.123) \quad \limsup_{n \rightarrow \infty} \int \left(\frac{\frac{1}{2} \Delta u_n}{u_n} \right)(x) \alpha_n^{(i)}(dx) = \int \left(\frac{\frac{1}{2} \Delta v_{i,R}}{c_i + v_{i,R}} \right)(x) \alpha_i(dx), \\ \limsup_{n \rightarrow \infty} \int \left(\frac{\frac{1}{2} \Delta u_n}{u_n} \right)(x) \beta_n(dx) \leq \delta.$$

Moreover,

$$(4.124) \quad \left| \frac{\pi_\varepsilon u_n - u_n}{\varepsilon u_n}(x) - \frac{\frac{1}{2} \Delta u_n}{u_n}(x) \right| \leq \sum_{1 \leq i \leq k} \frac{1}{c_i} \left\| \frac{\pi_\varepsilon v_{i,R} - v_{i,R}}{\varepsilon} - \frac{1}{2} \Delta v_{i,R} \right\|_\infty,$$

which converges to 0 as $\varepsilon \rightarrow 0$. Note that the convergence is uniform because the $v_{i,R}$'s are in the domain of the generator $\frac{1}{2} \Delta$ (see e.g. [12, Chapter 6.2]). Finally, we obtain

$$(4.125) \quad \limsup_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \int \log \frac{\pi_{\varepsilon_n} u_n}{u_n}(x) \mu_n(dx) \leq \int \sum_{1 \leq i \leq k} \left(\frac{\frac{1}{2} \Delta v_{i,R}}{c_i + v_{i,R}} \right)(x) \alpha_i(dx) + \delta.$$

This completes the proof in the single-orbit case.

If for all $n \geq 1$, ξ_n consists of several orbits denoted by $\xi_n^{(j)}$, we can choose a subsequence such that, for all j , $\xi_n^{(j)}$ converges to $\{\tilde{\alpha}_i^{(j)}, i \in I\}$. From the single-orbit case, we get, for all $m \geq 1$,

$$(4.126) \quad \liminf_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n}(\xi_n) \geq \liminf_{n \rightarrow \infty} \sum_{j=1}^m \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n}(\xi_n^{(j)}) \geq \sum_{j=1}^m \sum_{i \in I} J(\alpha_i^{(j)}).$$

Letting $m \rightarrow \infty$, we get the result. ■

Proof of Proposition 4.12. Let (ε_n) be a sequence of positive real numbers converging to zero. From Corollary 4.17 we obtain that

$$(4.127) \quad \liminf_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n/d}(\xi_n) \geq \frac{1}{d} \tilde{J}(\xi)$$

as soon as ξ_n converges to ξ in $\tilde{\mathcal{X}}$. By compactness of \mathcal{K} and lower semi-continuity of the function \tilde{J}_ε , there exists $\xi_n \in \mathcal{K}$ such that

$$(4.128) \quad \inf_{\tilde{\mu} \in \mathcal{K}} \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n/d}(\tilde{\mu}) = \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n/d}(\xi_n).$$

and any subsequence of (ξ_n) has a limit (along a further subsequence) in \mathcal{K} , which we will denote by ξ . From what precedes,

$$(4.129) \quad \liminf_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \tilde{J}_{\varepsilon_n/d}(\xi_n) \geq \frac{1}{d} \tilde{J}(\xi) \geq \frac{1}{d} \inf_{\mathcal{K}} \tilde{J}.$$
■

APPENDIX A. TECHNICAL ESTIMATES

A.1. Large Deviation estimate for the binomial distribution.

Lemma A.1. *For all $M \in \mathbb{N}$ and $C > \varepsilon_0 > 0$,*

$$(A.1) \quad \mathbb{P}(\text{Bin}(M, \varepsilon_0/C) \geq \varepsilon_0 M) \leq \exp(-M \varepsilon_0 \Xi(C)),$$

with $\Xi(C) := \log C + \frac{1}{C} - 1$.

Proof of Lemma A.1. By Chernov's bound we may write for all $\lambda > 0$,

$$(A.2) \quad \begin{aligned} \mathbb{P}(\text{Bin}(M, \varepsilon_0/C) \geq \varepsilon_0 M) &\leq \exp \left(-M \left[\lambda \varepsilon_0 - \log \left(1 + \frac{\varepsilon_0}{C} (e^\lambda - 1) \right) \right] \right) \\ &\leq \exp \left(-\varepsilon_0 M \left[\lambda - \frac{1}{C} (e^\lambda - 1) \right] \right), \end{aligned}$$

and we conclude by picking $\lambda = \log C$. ■

A.2. Concentration inequality.

Proof of Proposition 4.1. The proof follows the same lines of argument as in [15, Proposition 2.2.2], where the same result was proven for simple random walk on the torus. Thus, instead of giving a complete proof, we only point out the differences. In Equation (2.2.12) in [15] the estimate

$$(A.3) \quad \frac{1}{n} \hat{\mathcal{R}}_{n,\varepsilon}^K \leq N^d,$$

should be replaced by

$$(A.4) \quad \frac{1}{n} \hat{\mathcal{R}}_{n,\varepsilon}^K \leq 1,$$

which follows immediately from the definition of $\hat{\mathcal{R}}_{n,\varepsilon}^K$. Moreover, at the beginning of Step (4) the crude estimate

$$(A.5) \quad \frac{1}{n} \left| \mathcal{R}_{n,\varepsilon}^K - m_{n,\varepsilon}^K \right| \leq 1$$

should be used which implies exactly as in [15] that actually

$$(A.6) \quad \frac{1}{n} \left| \mathbb{E}_{n,\varepsilon} \mathcal{R}_{n,\varepsilon}^K - m_{n,\varepsilon}^K \right| \leq \frac{\delta}{3} + \mathbb{P}_{n,\varepsilon} \left(\frac{1}{n} \left| \mathcal{R}_{n,\varepsilon}^K - m_{n,\varepsilon}^K \right| \geq \frac{\delta}{3} \right).$$

From that point on the proof follows exactly [15]. ■

APPENDIX B. PROOF OF PROPOSITION 4.3

For the proof of Proposition 4.3 we will need to make use of the following local central limit theorem.

Proposition B.1 (Local limit theorem, see Theorem 1.2.1 in [11]). *Recall that $p_s(x) = (2\pi s)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{2s})$ for $s > 0$ and $x \in \mathbb{R}^d$. Define $E(n, x) = \mathbb{P}(S_n = x) - 2p_{n/d}(x)$ if n and x have the same parity, and zero otherwise. Then*

$$(B.1) \quad |E(n, x)| \leq O(n^{-\frac{d+2}{2}}) \wedge |x|^{-2} O(n^{-\frac{d}{2}}),$$

where $u_n = O(v_n)$ means $u_n \leq C v_n$ for some finite positive constant C and all n .

We also need to introduce truncated versions of some of the functions defined in Section 4.1, namely:

$$(B.2) \quad \mathfrak{q}_{\varepsilon,n,\Lambda}^\rho(x, y) := -M \log[1 - \bar{q}_{\ell,\Lambda}^\rho(x, y)],$$

where for $\rho > 0$,

$$(B.3) \quad \bar{q}_{\ell,\Lambda}^\rho(x, y) = \begin{cases} \bar{q}_{\ell,\Lambda}(x, y), & \text{if } d(\Lambda, \{\lfloor xn^{1/d} \rfloor, \lfloor yn^{1/d} \rfloor\}) > \rho n^{1/d}, \\ 0, & \text{otherwise,} \end{cases}$$

and with the convention $\bar{q}_{\ell,\Lambda}^0 = \bar{q}_{\ell,\Lambda}$. When $\Lambda = \Lambda(n, z)$ is the singleton $\{\lfloor zn^{1/d} \rfloor\}$ ($z \in \mathbb{R}^d$) we define

$$(B.4) \quad \bar{q}_\ell^\rho(x, y; z) := \bar{q}_{\ell,\Lambda(n,z)}^\rho(x, y), \quad x, y \in \mathbb{R}^d.$$

We also define

$$(B.5) \quad \varphi_\varepsilon^\rho(x, y) = \begin{cases} \varphi_\varepsilon(x, y) & \text{if } x, y \notin \bar{\mathcal{B}}(0, \rho) \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{\mathcal{B}}(0, \rho)$ is the (closed) Euclidean ball in \mathbb{R}^d with radius ρ and centered at the origin. Finally, we define for all $\mu \in \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d)$, $n \in \mathbb{N}$ and $\rho \geq 0$:

$$(B.6) \quad \phi_{n,\varepsilon,\rho}(\mu) = \int dz \left(1 - \exp \left\{ -\frac{1}{\varepsilon} \int \mu(dx, dy) n^{1-\frac{2}{d}} \bar{q}_\ell^\rho(x, y; z) \right\} \right)$$

and

$$(B.7) \quad \phi_{\infty,\varepsilon,\rho}(\mu) = \int dz \left(1 - \exp \left\{ -\frac{\kappa_d}{\varepsilon} \int \mu(dx, dy) \varphi_\varepsilon^\rho(x - z, y - z) \right\} \right),$$

with the conventions that $\phi_{\infty,\varepsilon,0} = \phi_{\infty,\varepsilon}$.

We now come to the proof of Proposition 4.3. We proceed in several steps.

Step 1. Removing the logarithm. By (4.7), we get

$$(B.8) \quad \frac{1}{n} \mathbb{E}(R_n | \hat{S}) = \Phi_{n,\varepsilon,0}(L_{M,\varepsilon}^{(2)}),$$

where

$$(B.9) \quad \Phi_{n,\varepsilon,0}(\mu) := \int_{\mathbb{R}^d} dz \left[1 - \exp \left(- \int \mathbf{q}_{\varepsilon,n}(x, y; z) \mu(dx, dy) \right) \right].$$

Recall the definition in (B.6). Since $\log(1+x) \leq x$ for all $x > -1$,

$$(B.10) \quad \Phi_{n,\varepsilon,0}(L_{M,\varepsilon}^{(2)}) \geq \phi_{n,\varepsilon,0}(L_{M,\varepsilon}^{(2)}).$$

Finally, we get that

$$(B.11) \quad \frac{1}{n} \mathbb{E}(R_n | \hat{S}) \geq \phi_{n,\varepsilon,0}(L_{M,\varepsilon}^{(2)}).$$

In the following steps we approximate $\phi_{n,\varepsilon,0}(L_{M,\varepsilon}^{(2)})$ by $\phi_{\infty,\varepsilon,0}(L_{M,\varepsilon}^{(2)})$.

Step 2. Truncation of the range and decomposition of the error term. For any $K > 0$ (whose precise value will be determined later), we have the trivial bound

$$(B.12) \quad \phi_{n,\varepsilon,0}(\mu) \geq \phi_{n,\varepsilon,Kn^{-1/d}}(\mu).$$

Therefore, we may write on the event $L_{M,\varepsilon}^{(2)} \in \mathcal{M}(A, \varepsilon_0)$:

$$(B.13) \quad \phi_{n,\varepsilon,0}(L_{M,\varepsilon}^{(2)}) \geq \phi_{\infty,\varepsilon,0}(L_{M,\varepsilon}^{(2)}) - (\mathcal{E}_1 + \mathcal{E}_2)(n, K),$$

where

$$(B.14) \quad \begin{aligned} \mathcal{E}_1(n, K) &:= \sup_{\mu \in \mathcal{M}(A, \varepsilon_0)} [\phi_{\infty,\varepsilon,0}(\mu) - \phi_{\infty,\varepsilon,Kn^{-1/d}}(\mu)], \\ \mathcal{E}_2(n, K) &:= \sup_{\mu \in \mathcal{M}(A, \varepsilon_0)} [\phi_{\infty,\varepsilon,Kn^{-1/d}}(\mu) - \phi_{n,\varepsilon,Kn^{-1/d}}(\mu)]. \end{aligned}$$

Note that there is no need for absolute values inside the supremum since we only aim at a lower bound. Recall that $\varepsilon, \varepsilon_0$ and A are fixed. We deal with the error terms $\mathcal{E}_1(n, K)$ and $\mathcal{E}_2(n, K)$ in Steps 3 and 4, respectively.

Step 3. Control on $\mathcal{E}_1(n, K)$. We will prove that for any $K > 0$,

$$(B.15) \quad \limsup_{n \rightarrow \infty} \mathcal{E}_1(n, K) \leq 2\kappa\varepsilon_0.$$

Therefore, throughout this proof step, the value of K is fixed (the precise value will be decided in Step 4 below). For all $\mu \in \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$(B.16) \quad \phi_{\infty,\varepsilon,0}(\mu) - \phi_{\infty,\varepsilon,Kn^{-1/d}}(\mu) \leq \int \mu(dx, dy) \frac{\kappa}{\varepsilon} \int_{\mathcal{B}(x, Kn^{-1/d}) \cup \mathcal{B}(y, Kn^{-1/d})} \varphi_\varepsilon(x - z, y - z) dz.$$

Recalling Lemma 4.5, we obtain that for all $\mu \in \mathcal{M}(A, \varepsilon_0)$,

$$(B.17) \quad \phi_{\infty,\varepsilon,0}(\mu) - \phi_{\infty,\varepsilon,Kn^{-1/d}}(\mu) \leq \kappa\varepsilon_0 + \mathcal{E}_{1,1}(n, K, \mu) + \mathcal{E}_{1,2}(n, K, \mu),$$

where

$$(B.18) \quad \mathcal{E}_{1,1}(n, K, \mu) := \int_{|x-y| \leq A} \mu(dx, dy) \frac{\kappa}{\varepsilon} \int_{\mathcal{B}(x, Kn^{-\frac{1}{d}})} \varphi_\varepsilon(x-z, y-z) dz$$

and $\mathcal{E}_{1,2}(n, K, \mu)$ is defined in the same way, with $\mathcal{B}(x, Kn^{-\frac{1}{d}})$ replaced by $\mathcal{B}(y, Kn^{-\frac{1}{d}})$. Since x and y play symmetric roles, it is enough to deal with $\mathcal{E}_{1,1}(n, K, \mu)$. Recalling (4.8), we see that

$$(B.19) \quad \begin{aligned} \mathcal{E}_{1,1}(n, K, \mu) &= \int_{|x-y| \leq A} \mu(dx, dy) \frac{\kappa}{\varepsilon} \int_{\mathcal{B}(x, Kn^{-\frac{1}{d}})} dz \int_0^\varepsilon ds \frac{p_{s/d}(x-z) p_{(\varepsilon-s)/d}(z-y)}{p_{\varepsilon/d}(x-y)} \\ &= \kappa \int_{|x-y| \leq A} \mu(dx, dy) \int_{\mathcal{B}(x, Kn^{-\frac{1}{d}})} dz \int_0^1 ds \frac{p_{\varepsilon s/d}(x-z) p_{\varepsilon(1-s)/d}(z-y)}{p_{\varepsilon/d}(x-y)} \end{aligned}$$

By using (4.19), we get that for all $\mu \in \mathcal{M}(A, \varepsilon_0)$,

$$(B.20) \quad \mathcal{E}_{1,1}(n, K, \mu) \leq \kappa(\varepsilon_0 + \mathcal{E}_{1,1}(n, K, \mu, \varepsilon_0)),$$

where

$$(B.21) \quad \mathcal{E}_{1,1}(n, K, \mu, \varepsilon_0) := \int_{|x-y| \leq A} \mu(dx, dy) \int_{\mathcal{B}(x, Kn^{-\frac{1}{d}})} dz \int_0^{1-\varepsilon_0} ds \frac{p_{\varepsilon s/d}(x-z) p_{\varepsilon(1-s)/d}(z-y)}{p_{\varepsilon/d}(x-y)}.$$

Using the expression of the Brownian kernel, there exists $C = C(A, \varepsilon_0, \varepsilon)$ such that

$$(B.22) \quad \sup_{\substack{s \leq 1-\varepsilon_0, \\ |x-y| \leq A}} \frac{p_{\varepsilon(1-s)/d}(z-y)}{p_{\varepsilon/d}(x-y)} \leq C.$$

Therefore,

$$(B.23) \quad \begin{aligned} \mathcal{E}_{1,1}(n, K, \mu, \varepsilon_0) &\leq C \int_{|x-y| \leq A} \mu(dx, dy) \int_0^{1-\varepsilon_0} P(|B_{\varepsilon s/d}| \leq Kn^{-\frac{1}{d}}) ds \\ &\leq C \int_0^1 P(|B_{\varepsilon s/d}| \leq Kn^{-\frac{1}{d}}) ds, \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$, by dominated convergence. This completes the proof of Step 3.

Step 4. Control on $\mathcal{E}_2(n, K)$. We will prove that there exists $K > 0$ (large enough) such that

$$(B.24) \quad \limsup_{n \rightarrow \infty} \mathcal{E}_2(n, K) \leq \varepsilon_0.$$

For convenience, we define

$$(B.25) \quad \mathcal{E}_2(n, K, \mu) := \phi_{\infty, \varepsilon, Kn^{-1/d}}(\mu) - \phi_{n, \varepsilon, Kn^{-1/d}}(\mu)$$

so that

$$(B.26) \quad \mathcal{E}_2(n, K) = \sup_{\mu \in \mathcal{M}(A, \varepsilon_0)} \mathcal{E}_2(n, K, \mu).$$

This step of the proof is more involved, so we divide it into several smaller steps.

Step 4a. Approximation of the escape probability. Let us further estimate $\mathcal{E}_2(n, K, \mu)$ as follows:

$$(B.27) \quad \mathcal{E}_2(n, K, \mu) \leq \mathcal{E}_{2,1}(n, K, \mu) + \mathcal{E}_{2,2}(n, K, \mu),$$

where

$$(B.28) \quad \begin{aligned} \mathcal{E}_{2,1}(n, K, \mu) &:= \phi_{\infty, \varepsilon, Kn^{-1/d}}^{(K)}(\mu) - \phi_{\infty, \varepsilon, Kn^{-1/d}}^{(K)}(\mu), \\ \mathcal{E}_{2,2}(n, K, \mu) &:= \phi_{\infty, \varepsilon, Kn^{-1/d}}^{(K)}(\mu) - \phi_{n, \varepsilon, Kn^{-1/d}}^{(K)}(\mu), \end{aligned}$$

and

$$(B.29) \quad \phi_{\infty, \varepsilon, \rho}^{(K)}(\mu) = \int dz \left(1 - \exp \left\{ - \frac{P(H_0 > H_{\partial B_K})}{\varepsilon} \int \mu(dx, dy) \varphi_\varepsilon^\rho(x - z, y - z) \right\} \right)$$

(compare with (B.7)). We used the following notations for hitting times in the formula above:

$$(B.30) \quad H_\Lambda := \inf\{n \geq 1 : S_n \in \Lambda\}, \quad \Lambda \subset \mathbb{Z}^d, \quad H_a = H_{\{a\}}, \quad a \in \mathbb{Z}.$$

By Lemma 4.5, we obtain

$$(B.31) \quad \sup_{\substack{\mu \in \mathcal{M}(A, \varepsilon_0) \\ n \geq 1}} \mathcal{E}_{2,1}(n, K, \mu) \leq |\kappa - P(H_0 > H_{\partial B_K})| = |P(H_0 = \infty) - P(H_0 > H_{\partial B_K})| \xrightarrow{K \rightarrow \infty} 0.$$

Therefore, there exists $K_1(\varepsilon_0)$ such that for all $K \geq K_1(\varepsilon_0)$,

$$(B.32) \quad \sup_{\substack{\mu \in \mathcal{M}(A, \varepsilon_0) \\ n \geq 1}} \mathcal{E}_{2,1}(n, K, \mu) \leq \varepsilon_0.$$

It now remains to control $\mathcal{E}_{2,2}(n, K, \mu)$ uniformly in $\mu \in \mathcal{M}(A, \varepsilon_0)$, as $n \rightarrow \infty$, which we will do in the following steps.

Step 4b. Truncation of the time interval. We now proceed to a truncation of the time interval $[0, \varepsilon]$ appearing in the definition of φ_ε . This truncation allows us to obtain uniform bounds later in the proof. We split $\mathcal{E}_{2,2}(n, K, \mu)$ as follows:

$$(B.33) \quad \mathcal{E}_{2,2}(n, K, \mu) \leq \mathcal{E}_{2,3}(n, K, \mu) + \mathcal{E}_{2,4}(n, K, \mu),$$

where

$$(B.34) \quad \begin{aligned} \mathcal{E}_{2,3}(n, K, \mu) &:= \phi_{\infty, \varepsilon, Kn^{-1/d}}^{(K)}(\mu) - \phi_{\infty, \varepsilon, Kn^{-1/d}}^{(K, \varepsilon_0)}(\mu), \\ \mathcal{E}_{2,4}(n, K, \mu) &:= \phi_{\infty, \varepsilon, Kn^{-1/d}}^{(K, \varepsilon_0)}(\mu) - \phi_{n, \varepsilon, Kn^{-1/d}}^{(\varepsilon_0)}(\mu), \end{aligned}$$

with

$$(B.35) \quad \begin{aligned} \phi_{n, \varepsilon, \rho}^{(\varepsilon_0)}(\mu) &:= \int dz \left(1 - \exp \left\{ - \frac{1}{\varepsilon} \int \mu(dx, dy) n^{1-\frac{2}{d}} \bar{q}_{\ell, \varepsilon_0}^\rho(x, y; z) \right\} \right) \\ \phi_{\infty, \varepsilon, \rho}^{(K, \varepsilon_0)}(\mu) &:= \int dz \left(1 - \exp \left\{ - \frac{P(H_0 > H_{\partial B_K})}{\varepsilon} \int \mu(dx, dy) \varphi_{\varepsilon, \varepsilon_0}^\rho(x - z, y - z) \right\} \right) \end{aligned}$$

(compare with (B.29)) and

$$(B.36) \quad \begin{aligned} \varphi_{\varepsilon, \varepsilon_0}(x, y) &:= \int_{\varepsilon_0 \varepsilon}^{(1-\varepsilon_0)\varepsilon} ds \frac{p_{s/d}(-x) p_{(\varepsilon-s)/d}(y)}{p_{\varepsilon/d}(y-x)}, \\ \bar{q}_{\ell, \varepsilon_0}(x, y; z) &:= P_{\lfloor xn^{\frac{1}{d}} \rfloor} (H_{\lfloor zn^{\frac{1}{d}} \rfloor} \in [\varepsilon_0 \ell, (1-\varepsilon_0)\ell] | S_\ell = \lfloor yn^{\frac{1}{d}} \rfloor). \end{aligned}$$

The super-index ρ in (B.35) means that the corresponding function is zero as soon as one of its arguments is in the ball of radius ρ . By using (4.19), we readily get

$$(B.37) \quad \sup_{\substack{\mu \in \mathcal{M}_1(\mathbb{R}^d \times \mathbb{R}^d) \\ n, K \geq 1}} \mathcal{E}_{2,3}(n, K, \mu) \leq 2\varepsilon_0.$$

It remains to deal with the term $\mathcal{E}_{2,4}(n, K, \mu)$. Note that we may safely restrict the integrals over x and y to $|x-y| \leq A$ in (B.36) (up to some error term not larger than ε_0) since $\mu \in \mathcal{M}(A, \varepsilon_0)$ (same argument as in Step 3).

Step 4c. Decomposition of $\phi_{n, \varepsilon, K n^{-1/d}}^{(\varepsilon_0)}(\mu)$. To lighten notations, we introduce

$$(B.38) \quad x_n := \lfloor xn^{\frac{1}{d}} \rfloor, \quad y_n := \lfloor yn^{\frac{1}{d}} \rfloor, \quad z_n := \lfloor zn^{\frac{1}{d}} \rfloor,$$

and

$$(B.39) \quad u := x_n - z_n, \quad v := y_n - z_n.$$

We may thus write

$$(B.40) \quad \bar{q}_{\ell, \varepsilon_0}(x, y; z) = \frac{Q_\ell(u, v)}{P_u(S_\ell = v)}, \quad Q_\ell(u, v) := P_u(H_0 \in [\varepsilon_0, 1-\varepsilon_0]\ell, S_\ell = v).$$

We now proceed with several approximations of $Q_\ell(u, v)$. First, by decomposing on the value of H_0 and reversing time on the interval $[0, H_0]$, we obtain

$$(B.41) \quad Q_\ell(u, v) = \sum_{i \in [\varepsilon_0, 1-\varepsilon_0]\ell} P_0(S_i = u, H_0 > i) P_0(S_{\ell-i} = v).$$

Recall that $|u| \geq K$. By decomposing on the value of $H_{\partial B_K}$ and the position of the walk at that time, we obtain

$$(B.42) \quad Q_\ell(u, v) = \sum_{i \in [\varepsilon_0, 1-\varepsilon_0]\ell} \sum_{\substack{1 \leq j \leq i \\ w \in \partial B_K}} P_0(S_j = w, H_{\partial B_K} = j < H_0) P_w(S_{i-j} = u, H_0 > i-j) P_0(S_{\ell-i} = v),$$

For any $a \in \mathbb{N}$ with $a \leq \varepsilon_0 \ell$ (to be determined later in the proof), we have $Q_\ell(u, v) \geq Q_{\ell, a}(u, v)$, where

$$(B.43) \quad Q_{\ell, a}(u, v) := \sum_{i \in [\varepsilon_0, 1-\varepsilon_0]\ell} \sum_{\substack{1 \leq j \leq a \\ w \in \partial B_K}} P_0(S_j = w, H_{\partial B_K} = j < H_0) P_w(S_{i-j} = u, H_0 > i-j) P_0(S_{\ell-i} = v).$$

We decompose the latter as

$$(B.44) \quad Q_{\ell, a}(u, v) = Q_{\ell, a}^{(1)}(u, v) - \mathfrak{E}_{\ell, a}^{(1)}(u, v),$$

where

$$(B.45) \quad Q_{\ell,a}^{(1)}(u, v) := \sum_{i \in [\varepsilon_0, 1-\varepsilon_0]\ell} \sum_{\substack{1 \leq j \leq a \\ w \in \partial B_K}} P_0(S_j = w, H_{\partial B_K} = j < H_0) P_w(S_{i-j} = u) P_0(S_{\ell-i} = v),$$

and

$$(B.46) \quad \mathfrak{E}_{\ell,a}^{(1)}(u, v) := \sum_{i \in [\varepsilon_0, 1-\varepsilon_0]\ell} \sum_{\substack{1 \leq j \leq a \\ w \in \partial B_K}} P_0(S_j = w, H_{\partial B_K} = j < H_0) P_w(S_{i-j} = u, H_0 \leq i-j) P_0(S_{\ell-i} = v)$$

Note that so far we have ignored the fact that $P_0(S_m = q)$ is non-zero only if m and q have the same parity. We write $m \leftrightarrow q$ in that case.

We now bound $Q_{\ell,a}^{(1)}(u, v)$ from below as follows:

$$(B.47) \quad Q_{\ell,a}^{(1)}(u, v) \geq Q_{\ell,a}^{(2)}(u, v) - \mathfrak{E}_{\ell,a}^{(2)}(u, v)$$

where

$$(B.48) \quad Q_{\ell,a}^{(2)}(u, v) := P_0(H_{\partial B_K} < H_0) \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ i-\ell \leftrightarrow v}} \inf_{\substack{w \in \partial B_K \\ 1 \leq j \leq a \\ i-j \leftrightarrow u-w}} P_w(S_{i-j} = u) P_0(S_{\ell-i} = v)$$

and

$$(B.49) \quad \mathfrak{E}_{\ell,a}^{(2)}(u, v) := P_0(H_{\partial B_K} \geq a) \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ i-\ell \leftrightarrow v}} \inf_{\substack{w \in \partial B_K \\ 1 \leq j \leq a \\ i-j \leftrightarrow u-w}} P_w(S_{i-j} = u) P_0(S_{\ell-i} = v).$$

The term $Q_{\ell,a}^{(2)}(u, v)$ is the term that will give the main contribution to $\mathcal{E}_{2,4}(n, K, \mu)$. We will first control the terms $\mathfrak{E}_{\ell,a}^{(1)}(u, v)$ and $\mathfrak{E}_{\ell,a}^{(2)}(u, v)$ and then come back to $Q_{\ell,a}^{(2)}(u, v)$.

Step 4d. Control of $\mathfrak{E}_{\ell,a}^{(1)}(u, v)$. Let us now deal with the error term $\mathfrak{E}_{\ell,a}^{(1)}(u, v)$. We use the following lemma, the proof of which is postponed to the end of the section

Lemma B.2. *We have*

$$(B.50) \quad \sup_{\substack{|u| \wedge |w| \geq K \\ k \geq 1}} k^{\frac{d}{2}} P_w(S_k = u, H_0 \leq k) \leq (\text{cst}) K^{2-d}.$$

By Lemma B.2, the contribution from $\mathfrak{E}_{\ell,a}^{(1)}(u, v)$ is at most:

$$(B.51) \quad \begin{aligned} & \int_{|x-y| \leq A} \mu(dx, dy) \int dz \frac{1}{\varepsilon} n^{1-2/d} \frac{\mathfrak{E}_{\ell,a}^{(1)}(u, v)}{P_u(S_\ell = v)} \\ & \leq (\text{cst}) K^{2-d} \int_{|x-y| \leq A} \mu(dx, dy) \frac{1}{\varepsilon n^{\frac{2}{d}}} \int ndz \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ i-\ell \leftrightarrow v}} (i-a)^{-\frac{d}{2}} \frac{P_0(S_{\ell-i} = v)}{P_u(S_\ell = v)} \end{aligned}$$

By using the local limit theorem (Proposition B.1) on $P_u(S_\ell = v)$ and noticing that

$$(B.52) \quad \int ndz P_0(S_{\ell-i} = v) = \sum_{z \in \mathbb{Z}^d} P_0(S_{\ell-i} = y_n - z) = 1,$$

we obtain

$$(B.53) \quad (B.51) \leq C(A, \varepsilon_0, \varepsilon) \left(\frac{\ell}{\varepsilon_0 \ell - a} \right)^{d/2} K^{2-d} \leq C(A, \varepsilon_0, \varepsilon) \left(\frac{2}{\varepsilon_0} \right)^{d/2} K^{2-d},$$

provided n is larger than some $n_1(\varepsilon_0, \varepsilon, a)$. We may now choose $K \geq K_2(\varepsilon_0, \varepsilon, A)$ such that

$$(B.54) \quad (B.51) \leq \varepsilon_0.$$

In the following we set $K \geq K_2(\varepsilon_0, \varepsilon, A) \vee K_1(\varepsilon_0)$ (recall (B.32)) once and for all.

Step 4e. Control of $\mathfrak{E}_{\ell,a}^{(2)}(u, v)$. Let $w \in \partial B_K$ and $1 \leq j \leq a$. Similarly to Step 4d, the total contribution from $\mathfrak{E}_{\ell,a}^{(2)}(u, v)$ is at most

$$(B.55) \quad P_0(H_{\partial B_K} \geq a) \int_{|x-y| \leq A} \mu(dx, dy) \int dz \frac{1}{\varepsilon} n^{1-\frac{2}{d}} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ i-\ell \leftrightarrow v}} P_w(S_{i-j} = u) \frac{P_0(S_{\ell-i} = v)}{P_u(S_\ell = v)}.$$

By the local limit theorem (Proposition B.1), there exists $C(A, \varepsilon_0, \varepsilon)$ such that

$$(B.56) \quad \frac{P_0(S_{\ell-i} = v)}{P_u(S_\ell = v)} \leq C(A, \varepsilon_0, \varepsilon),$$

uniformly in $i \leq (1 - \varepsilon_0)\ell$, $|x - y| \leq A$ and $z \in \mathbb{R}^d$. Therefore, the quantity in (B.55) is bounded from above by

$$(B.57) \quad C(A, \varepsilon_0, \varepsilon) P_0(H_{\partial B_K} \geq a) \int \mu(dx, dy) \frac{1}{\varepsilon n^{\frac{2}{d}}} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ i-\ell \leftrightarrow v}} \int ndz P_w(S_{i-j} = u).$$

Since

$$(B.58) \quad \int ndz P_w(S_{i-j} = u) = \sum_{z \in \mathbb{Z}^d} P_w(S_{i-j} = x_n - z) = 1,$$

we get that

$$(B.59) \quad (B.55) \leq C(A, \varepsilon_0, \varepsilon) P_0(H_{\partial B_K} \geq a).$$

Since $A, \varepsilon_0, \varepsilon$ and K are fixed, we may now pick a large enough such that

$$(B.60) \quad (B.55) \leq \varepsilon_0.$$

Step 4f. Control of $\mathcal{E}_{2,4}(n, K, \mu)$. We define

$$(B.61) \quad \tilde{Q}_{\ell,a}^{(2)}(u, v) := \frac{Q_{\ell,a}^{(2)}(u, v)}{P_0(H_{\partial B_K} < H_0)}.$$

By the local limit theorem (Proposition B.1), we get for all $i \leftrightarrow \ell - v$,

$$(B.62) \quad \sup_v |P_0(S_{\ell-i} = v) - 2p_{(\ell-i)/d}(v)| \leq \frac{(\text{cst})}{(\ell-i)^{\frac{d}{2}+1}} \leq \frac{c(\varepsilon_0)}{\ell^{\frac{d}{2}+1}}.$$

We may thus write

$$(B.63) \quad \tilde{Q}_{\ell,a}^{(2)}(u, v) = \tilde{Q}_{\ell,a}^{(3)}(u, v) - \mathfrak{E}_{\ell,a}^{(3)}(u, v),$$

with

$$(B.64) \quad \tilde{Q}_{\ell,a}^{(3)}(u, v) := 2 \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ \ell-i \leftrightarrow v}} \inf_{\substack{w \in \partial B_K \\ 1 \leq j \leq a \\ i-j \leftrightarrow u-w}} P_w(S_{i-j} = u) p_{(\ell-i)/d}(v).$$

and

$$(B.65) \quad \mathfrak{E}_{\ell,a}^{(3)}(u, v) \leq c(\varepsilon_0) \frac{1}{\ell} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ \ell-i \leftrightarrow v}} \inf_{\substack{w \in \partial B_K \\ 1 \leq j \leq a \\ i-j \leftrightarrow u-w}} P_w(S_{i-j} = u) \ell^{-d/2}.$$

The contribution of this error term is dealt with by using the same line of arguments as in Steps 4d and 4e, uniformly in $\mu \in \mathcal{M}(A, \varepsilon_0)$. Again, by the local limit theorem (Proposition B.1), we may write

$$(B.66) \quad \tilde{Q}_{\ell,a}^{(3)}(u, v) = \tilde{Q}_{\ell,a}^{(4)}(u, v) - \mathfrak{E}_{\ell,a}^{(4)}(u, v)$$

with

$$(B.67) \quad \tilde{Q}_{\ell,a}^{(4)}(u, v) := 4 \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ \ell-i \leftrightarrow v}} \inf_{\substack{w \in \partial B_K \\ 1 \leq j \leq a \\ i-j \leftrightarrow u-w}} p_{(i-j)/d}(u-w) p_{(\ell-i)/d}(v).$$

and

$$(B.68) \quad \mathfrak{E}_{\ell,a}^{(4)}(u, v) \leq c(\varepsilon_0) \frac{1}{\ell} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ \ell-i \leftrightarrow v}} p_{(\ell-i)/d}(v) \ell^{-d/2}.$$

To control the contribution from this error term, we use that

$$(B.69) \quad \int_{\mathbb{R}^d} n dz p_k(v) = \sum_{z \in \mathbb{Z}^d} p_k(z) \xrightarrow{k \rightarrow \infty} 1.$$

We now want to replace $p_{(i-j)/d}(u-w)$ by $p_{i/d}(u)$. First we replace $p_{(i-j)/d}(u-w)$ by $p_{(i-j)/d}(u)$. To this end, note that

$$(B.70) \quad |p_k(u) - p_k(u-w)| \leq (\text{cst}) \frac{|w|}{k^{\frac{1+d}{2}}}.$$

We define

$$(B.71) \quad \tilde{Q}_{\ell,a}^{(4)}(u, v) = \tilde{Q}_{\ell,a}^{(5)}(u, v) - \mathfrak{E}_{\ell,a}^{(5)}(u, v)$$

with

$$(B.72) \quad \tilde{Q}_{\ell,a}^{(5)}(u, v) := 4 \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0]\ell \\ \ell-i \leftrightarrow v}} \inf_{\substack{1 \leq j \leq a \\ i-j \leftrightarrow u}} p_{(i-j)/d}(u) p_{(\ell-i)/d}(v).$$

and

$$(B.73) \quad \mathfrak{E}_{\ell,a}^{(5)}(u, v) \leq \frac{|K|}{\sqrt{\varepsilon_0 \ell - a}} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \ell \\ \ell-i \leftrightarrow v}} (i-a)^{-d/2} p_{(\ell-i)/d}(v).$$

We can again deal with $\mathfrak{E}_{\ell,a}^{(5)}(u, v)$ in a similar way as for the previous error terms. We now replace $p_{(i-j)/d}(u)$ by $p_{i/d}(u)$ in $\tilde{Q}_{\ell,a}^{(5)}(u, v)$. By computing the derivative of $f(s) := s^{-d/2} \exp(-|u|^2/(2s))$, we see that for all $\tilde{A} > 0$:

$$(B.74) \quad \sup_{\substack{1 \leq j \leq a \\ |u| \leq \tilde{A} n^{1/d}}} |p_{i/d}(u) - p_{(i-j)/d}(u)| \leq \frac{C(\varepsilon, \varepsilon_0, a, \tilde{A})}{\ell} p_{i/d}(u).$$

Note that the restriction of the integral over $z \in \mathbb{R}^d$ to $|u| \leq \tilde{A} n^{1/d}$ (that is $z \in \mathcal{B}(x, \tilde{A})$) can be made up to an error term not larger than ε_0 provided \tilde{A} is chosen large enough (uniformly in μ , same argument as in Step 4). Therefore,

$$(B.75) \quad \tilde{Q}_{\ell,a}^{(5)}(u, v) = \tilde{Q}_{\ell,a}^{(6)}(u, v) - \mathfrak{E}_{\ell,a}^{(6)}(u, v)$$

with

$$(B.76) \quad \tilde{Q}_{\ell,a}^{(6)}(u, v) := 4 \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \ell \\ \ell-i \leftrightarrow v}} p_{i/d}(u) p_{(\ell-i)/d}(v).$$

and

$$(B.77) \quad \mathfrak{E}_{\ell,a}^{(6)}(u, v) \leq \frac{C(\varepsilon, \varepsilon_0, a, \tilde{A})}{\ell} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \ell \\ \ell-i \leftrightarrow v}} p_{i/d}(u) p_{(\ell-i)/d}(v).$$

We are now left with showing that

$$(B.78) \quad 4 \int_{\mathcal{B}(x, \tilde{A}) \cup \mathcal{B}(y, \tilde{A})} dz \frac{1}{\varepsilon} n^{1-\frac{2}{d}} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \ell \\ \ell-i \leftrightarrow v}} \frac{p_{i/d}(u) p_{(\ell-i)/d}(v)}{P_u(S_\ell = v)}$$

converges to

$$(B.79) \quad \int_{\mathcal{B}(x, \tilde{A}) \cup \mathcal{B}(y, \tilde{A})} dz \frac{1}{\varepsilon} \int_{\varepsilon_0 \varepsilon}^{(1-\varepsilon_0) \varepsilon} ds \frac{p_{s/d}(x-z) p_{(\varepsilon-s)/d}(z-y)}{p_{\varepsilon/d}(x-y)},$$

as $n \rightarrow \infty$, uniformly in $|x-y| \leq A$, by a Riemann sum approximation. By the local limit theorem (Proposition B.1),

$$(B.80) \quad \sup_{|x-y| \leq A} \left| \frac{2p_{\ell/d}(v-u)}{P_u(S_\ell = v)} - 1 \right| \leq \frac{(\text{cst})}{\ell}$$

so the proof will be complete once we prove that

$$(B.81) \quad 2n^{1-\frac{2}{d}} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \ell \\ \ell-i \leftrightarrow v}} \frac{p_{i/d}(u) p_{(\ell-i)/d}(v)}{p_{\ell/d}(v-u)} \longrightarrow \int_{\varepsilon_0 \varepsilon}^{(1-\varepsilon_0) \varepsilon} ds \frac{p_{s/d}(x-z) p_{(\varepsilon-s)/d}(z-y)}{p_{\varepsilon/d}(x-y)}$$

as $n \rightarrow \infty$, uniformly on $z \in \mathcal{B}(x, \tilde{A}) \cup \mathcal{B}(y, \tilde{A})$ and $|x - y| \leq A$. Clearly,
(B.82)

$$n^{1-\frac{2}{d}} \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \ell \\ \ell-i \leftrightarrow v}} \frac{p_{i/d}(u) p_{(\ell-i)/d}(v)}{p_{\ell/d}(v-u)} = \sum_{\substack{i \in [\varepsilon_0, 1-\varepsilon_0] \frac{\varepsilon}{n^{2/d}}, \\ \ell-i \leftrightarrow v}} \frac{p_{in^{-2/d}/d}(\frac{u}{n^{-1/d}}) p_{(\ell-i)n^{-2/d}/d}(\frac{v}{n^{-1/d}})}{p_{\varepsilon n^{-2/d}/d}(\frac{v-u}{n^{-1/d}})} \frac{1}{n^{2/d}}$$

and the rest is standard Riemann sum approximation together with the fact that the condition $\ell - i \leftrightarrow v$ reduces the number of terms by a factor $\frac{1}{2}$ (recall (B.39)).

Step 5. Conclusion. By combining (B.11), (B.13), (B.15) and (B.24), the proof is complete.

Proof of Lemma B.2. By decomposing on the value of H_0 we obtain:

$$\begin{aligned} \mathbb{P}_w(S_k = u, H_0 \leq k) &= \sum_{1 \leq r \leq k} \mathbb{P}_w(H_0 = r) \mathbb{P}_0(S_{k-r} = u) \\ (B.83) \quad &\leq \sum_{1 \leq r \leq k} \mathbb{P}_0(S_r = w) \mathbb{P}_0(S_{k-r} = u). \end{aligned}$$

By splitting the sum in two parts ($r \leq k/2$ and $r > k/2$) and using the local limit theorem (Proposition B.1) we get

$$(B.84) \quad \mathbb{P}_w(S_k = u, H_0 \leq k) \leq (\text{cst}) k^{-d/2} (\mathbb{E}_0[\ell_\infty(w)] + \mathbb{E}_0[\ell_\infty(u)]),$$

where $\ell_\infty(\cdot) := \text{card}\{i \geq 1 : S_i = \cdot\}$. Since $|u| \wedge |w| \geq K$, a standard estimate yields

$$(B.85) \quad \mathbb{E}_0[\ell_\infty(w)] \vee \mathbb{E}_0[\ell_\infty(u)] \leq (\text{cst}) K^{2-d},$$

which completes the proof. ■

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