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Giao Ky Duong[(1)], Nejla Nouaili[(2)] and Hatem Zaag[(3)]

[(1)] Institute of Applied Mathematics, University of Economics Ho Chi Minh City, Vietnam
[(2)] CEREMADE, Université Paris Dauphine, Paris Sciences et Lettres, France
[(3)]Université Sorbonne Paris Nord, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France

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Abstract. In this paper, we consider the complex Ginzburg Landau equation

$$\partial_t u = (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u - \alpha u \text{ where } \beta, \delta, \alpha \in \mathbb{R}.$$

The study aims to investigate the finite time blowup phenomenon. In particular, for fixed $\beta \in \mathbb{R}$, the existence of finite time blowup solutions for an arbitrary large $|\delta|$ is still unknown. Especially, Popp et al [24] formally conjectured that there is no blowup (collapse) in such case. In this work, considered as a breakthrough, we give a counter example to this conjecture. We show the existence of blowup solutions in one dimension with δ arbitrarily given and $\beta = 0$. The novelty is based on two main contributions: an investigation of a new blowup scaling (flat blowup regime) and a suitable modulation.

1. Introduction

In this paper, we are interested in the complex Ginzburg-Landau (CGL) equation

$$\begin{cases} u_t &= (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u - \alpha u, \\ u(.,0) &= u_0 \in L^\infty(\mathbb{R}^N, \mathbb{C}), \end{cases} \quad \text{(CGL)} \tag{1.1}$$

where β, δ and α are positive real numbers, $p > 1$, and $u(x,t) \in \mathbb{C}$.

Equation (1.1) is named after V. Ginzburg and L. Landau, which has a long history in mathematics and physics. In particular, the cubic case, i.e. $p = 3$, has been developed to describe the behavior of a superconductor or a superfluid near its critical temperature, where fluctuations in the order parameter become large, and the system exhibits complex, nonlinear behavior. In particular, the (CGL) captures the phenomenon of spontaneous symmetry breaking, in which the system transitions from a high-symmetry state to a low-symmetry state due to small perturbations. Additionally, the (CGL) equation also describes a variety of phenomena in physics, such as nonlinear waves, second-order phase transitions, superconductivity, superfluidity, and the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation; we refer the reader to the review by Aranson and Kramer [1] and the references therein for more detail.

The (CGL) equation can be also derived from the Navier-Stokes equations via multiple-scaling methods in several problems, most notably in convection (e.g. see ref. [22]). However, our intention here is not to treat it as a model for fluid turbulence but as an example of a nonlinear PDE which we can use to explore new methods.

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From the PDE point of view, the local Cauchy problem is well posed in a variety of functional spaces by using the semi-group theory (see [5, 11, 12]). Our framework, the functional space $L^\infty(\mathbb{R}^N)$ will be chosen as a convenient way allow us to derive the asymptotic behavior of the blowup. In particular, we say a solution to (1.1) blowing up in finite time if there exists $T \in (0, +\infty)$ such that

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty. \tag{1.2}$$

Moreover, a point $x_0 \in \mathbb{R}^N$ is said to be a blow-up point of the blowup solution if there is a sequence $\{(x_j, t_j)\}$, such that $x_j \to x_0$, $t_j \to T$ and $|u(x_j, t_j)| \to \infty$ as $j \to \infty$. The set of all blow-up points is called the blow-up set.

The study of singularity formation (such as collapse, chaotic or blowup) for equation (1.1) has been received a lot of attention in many works in the last decays. Typically, we mention to Stewartson and Stuart [29] in the description of an unstable plane Poiseuille flow; Hocking, Stewartson, Stuart and Brown [14] or in the context of binary mixtures in Kolodner and *al*, [15, 18], where the authors describe an extensive series of experiments on travelling-wave convection in an ethanol/water mixture, and they observe collapse solution that appear experimentally. We cite also the result of Turitsyn [30], who gave a harp sufficient criteria for collapse for equation (1.1) in the case of the subcritical bifurcation.

In this paper, we study the equation (1.1) as a nonlinear Partial Differential Equations (PDEs). Historically, a huge literature has been made on the blowup for PDEs in general, and on construction blowup solutions in particular.

Let's shift our focus to the literature on construction blowup solutions to the (CGL). The question of the existence of a blow-up solution for equation (2.1) remained open so far. Indeed, classical methods based on energy-type estimates break down. We cite the result of [6] and [7] which studied the CGL equation in the case $\beta = \delta$. We also point out that (2.1) may have blow-up in the focusing case, namely $\beta\delta > 0$. In [27] and [4], the authors give some evidence for the existence of a radial solution which blows up in a self-similar way.

Now, let us focus on the special case $\beta = \delta = 0$ which reduces (1.1) to the classical heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u, \text{ and } p > 1. \tag{1.3}$$

There exists an extensive literature spanning over six decades that one has investigated on blowup phenomena for (1.3). For a more comprehensive understanding of this field, we recommend referring to [28] for detailed insights into blowup studies. Regarding a specific reference, we mention to [2] (also mentioned in [20]) in which the authors constructed blowup solutions to (1.3) and described their blowup asymptotic via explicit blowup profiles. In particular, in one dimension, given a blow-up point a, we are able to construct a solution such that

- either

$$\sup_{|x-a|\leqslant K\sqrt{(T-t)\log(T-t)}} \left|(T-t)^{\frac{1}{p-1}}u(x,t) - f_{b_0}\left(\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \to 0, \tag{1.4}$$

  the authors in [2] constructed a solution such that
- or for some $k \in \mathbb{N}$, $k \geqslant 2$, and $b > 0$

$$\sup_{|x-a|<K(T-t)^{1/2k}} \left|(T-t)^{\frac{1}{p-1}}u(x,t) - f_b\left(\frac{(x-a)}{(T-t)^{1/2k}}\right)\right| \to 0, \tag{1.5}$$

as $t \to T$, for any integer $k > 0$, where

$$\begin{aligned} f_{b_0}(z) &= \left(p - 1 + b_0 z^2\right)^{-\frac{1}{p-1}} \text{ with } b_0 = \tfrac{(p-1)^2}{4p}, \\ f_b(z) &= \left(p - 1 + b|z|^{2k}\right)^{-\frac{1}{p-1}} \text{ where } b > 0. \end{aligned} \tag{1.6}$$

If $(\beta, \delta) \neq (0, 0)$, some results are available in the *subcritical* case by Zaag [31] ($\beta = 0$) and Masmoudi and Zaag [21] ($\beta \neq 0$). More precisely, if

$$p - \delta^2 - \beta\delta(p+1) > 0, \text{ so-called subcritical,} \tag{1.7}$$

then, the authors construct a solution of equation (1.1), which blows up in finite time $T > 0$ only at the origin such that for all $t \in [0, T)$,

$$\left\| \bar{\psi}_{\text{sub}}(t) u(\cdot, t) - \left(p - 1 + \frac{b_{sub} |\cdot|^2}{(T-t)|\log(T-t)|} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty} \leqslant \frac{C}{1 + \sqrt{|\log(T-t)|}}, \tag{1.8}$$

where

$$\bar{\psi}_{sub}(t) = (T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu},$$

and

$$b_{sub} = \frac{(p-1)^2}{4(p - \delta^2 - \beta\delta(1+p))} > 0 \text{ and } \mu = -\frac{2 b_{sub} \beta}{(p-1)^2}(1 + \delta^2). \tag{1.9}$$

Note that this result was previously obtained formally by Hocking and Stewartson [13] ($p = 3$) and mentioned later in Popp et *al* [25] (see those references for more blow-up results often approved numerically, in various regimes of the parameters).

For the critical case i.e. $p - \delta^2 - \beta\delta(p+1) = 0$, there are also construction of a blow-up solutions made. We mention to [23] (for the case $\beta = 0$) and [10] (for the case $\beta \neq 0$). More precisely, the authors constructed blowup solutions to equation (1.1) (see Theorem 2 in [10]) and described the blowup profile by

$$\left\| \bar{\psi}_{cri}(t) u(x, t) - \left(p - 1 + \frac{b_{cri} |x|^2}{(T-t)|\log(T-t)|^{\frac{1}{2}}} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty} \leqslant \frac{C}{1 + |\log(T-t)|^{\frac{1}{4}}},$$

where

$$\bar{\psi}_{cri}(t) = (T-t)^{\frac{1+i\delta}{p-1}} |\log(T-t)|^{-i\mu} e^{-i\nu\sqrt{T-t}},$$

with the constants $\nu = \nu(\beta, p), \mu = \mu(\beta, p)$ determined as in [10], and

$$b_{cri}^2 = \frac{(p-1)^4 (p+1)^2 \delta^2}{16(1+\delta^2)(p(2p-1) - (p-2)\delta^2)((p+3)\delta^2 + p(3p+1))}.$$

As a matter of fact, the works [21], [23], and [10] have encountered unresolved cases, where the condition

$$p - \delta^2 - \beta\delta(p+1) < 0. \tag{1.10}$$

remains unknown. Furthermore, the methodology employed in those works appears inapplicable to address (1.10). Specifically, in [25], Popp et *al* put forth a formal conjecture stating that, for a fixed β, the existence of finite time blowup solutions for arbitrarily large $|\delta|$ is not available (we refer to Remark 1.1 for further details). Contrary to this conjecture, our paper establishes a proof for the special case $\beta = 0$. More precisely, our result reads.

Theorem 1. *Let $\beta = 0$, $p > 1$ and $k \in \mathbb{N}, k \geqslant 2$, then there exist γ_0 such that for all $\gamma \in (0, \gamma_0)$, there exists $\tilde{T}(\gamma)$ such that for all $T \in (0, \tilde{T})$, there exists initial data $u_0 \in L^\infty(\mathbb{R})$ such that the corresponding solution to equation (2.1) blows up in finite time T and only at the origin. Moreover, there exists a flow $b(t) \in C^1(0,T)$ such that*
(i) For all $t \in [0,T)$, it holds that

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} u(\cdot,t) - f_{b(t)}\left(\frac{|\cdot|}{(T-t)^{\frac{1}{2k}}} \right) \right\|_{L^\infty(\mathbb{R})} \lesssim (T-t)^{\frac{\gamma}{2}(1-\frac{1}{k})} \text{ as } t \to T, \tag{1.11}$$

where $f_{b(t)}$ is defined by

$$f_{b(t)}(y) = \left(p - 1 + b(t)|y|^{2k} \right)^{-\frac{1+i\delta}{p-1}} . \tag{1.12}$$

(ii) There exists $b^* := b^*(u_0, \gamma, T) > 0$ such that $b(t) \to b^*$ as $t \to T$ and

$$|b(t) - b^*| \lesssim (T-t)^{\gamma(1-\frac{1}{k})}, \forall t \in (0,T), \tag{1.13}$$

Corollary 2. *Under the same hypothesis of Theorem 1, it holds that*

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} u(\cdot,t) - f_{b^*}\left(\frac{|\cdot|}{(T-t)^{\frac{1}{2k}}} \right) \right\|_{L^\infty(\mathbb{R})} \lesssim (T-t)^{\frac{\gamma}{2}(1-\frac{1}{k})} \text{ as } t \to T, \tag{1.14}$$

where b^* is defined in ii) of Theorem 1.

Remark 1.1 (Comments on the result)**.** *Kuznetsov and co-authors in different publications,* [17], [16], [19] *and* ([26], *section 2, page 87) made a conjecture saying that collapse of the solutions of the CGL equation (2.1) may be suppressed for suitable parameters β and δ. They suggest that the imaginary cubic term (when $p = 3$, $\beta = b$ and $\delta = -c$) in the CGLE provides a stabilization mechanism which can eventually suppress the collapse. For the understanding of this mechanism they write the CGL equation in terms of modulus and phase ($u(x,t) = A(x,t)e^{i\Phi(x,t)}$ and $k = \partial_x \Phi$) in one dimension, then we have*

$$\partial_t A = A(1 + A^2 - k^2) - 2\beta k \partial_x A - \beta A \partial_x A + \partial_x^2 A \tag{1.15}$$

$$\partial_t k = \delta \partial_x A^2 - 2\beta k \partial_x k + \beta \partial_x \left(\frac{\partial_x^2 A}{A} \right) + \partial_x \left(\frac{\partial_x \left(A^2 k \right)}{A^2} \right) \tag{1.16}$$

Then, for $\beta = 0$, $\delta >> 1$, the last term on the RHS of (1.16) is assumed to be negligibly small compared to the first one. As a result of the phase gradient mechanism (PGM) and due to the formation of sharper gradients of the amplitude the propagation speed will grow steadily during the blow up which results in a narrowing of the pulse and (eventually) a suppression of the blow up. Thus the PGM provides the comparison of the pulse up to its disappearance.
In Theorem 1, we give a counter example to this conjecture when $\beta = 0$ and for all $\delta \in \mathbb{R}$. Indeed, this conjecture is false if we consider a blow-up solution as in (1.11) with scaling equal to $\frac{x}{(T-t)^{\frac{1}{2k}}}$. Indeed, if we introduce the following self-similar variables $y = \frac{x}{(T-t)^{\frac{1}{2k}}}$, $s = -\log(T-t)$, $w(y,s) = (T-t)^{-\frac{1+i\delta}{p-1}} u(x,t)$, then w satisfy (see equation (2.3)):

$$\partial_s w = I^{-2}(s)\Delta w - \frac{1}{2k} y \cdot \nabla w - \frac{1+i\delta}{p-1} w + (1+i\delta)|w|^{p-1}w, \text{ with } I(s) = e^{-\frac{s}{2}(1-\frac{1}{k})} w.$$

Let us now write $w = We^{i\Psi}$ *and* $K = \partial_y \Psi$*, then we have*

$$\partial_s W = W\left(W^{p-1} - \frac{1}{p-1} - I^{-2}K^2\right) + I^{-2}\partial_y^2 A - \frac{y}{2k}\partial_y A \tag{1.17}$$

$$\partial_s K = \delta\partial_y(W^{p-1}) + I^{-2}\left[\partial_y\left(\frac{2K\partial_y W}{W}\right) + \partial_y^2 K\right] - \partial_y\left[\frac{y}{2k}K\right]. \tag{1.18}$$

Following the conjecture of physicists, even if K *grow, he will not be able to suppress the blow-up because the fourth term in equation* (1.17) *is coming with* $I(s)^{-2} = e^{-s(1-\frac{1}{k})}$ *which decay to* 0 *as we approach the blow-up time* T*.*

Remark 1.2. *In our paper, we focus on the construction of solutions in the case* $\beta = 0$*, but we believe that the construction of blow-up solution such in* (1.11) *is possible for the case* $\beta \neq 0$*, but there is additional difficulty coming from the fact that the linearized operator in that case is not self-adjoint and is not diagonalisable. However, we think that we could have a critical condition for the construction of such profile.*

Remark 1.3. *(Difficulty and the strategy of the proof)*

- *To prove a result such in Theorem 1, usually we consider the perturbation as the linearisation of the solution around the profile and then we prove that the perturbation goes to* 0 *as we approach the blow-up time. In this work we use a tricky linearisation introduced by Bricmont and Kupiainen in* [3]*. Indeed, we will introduce the following perturbation,modulo a phase,* $u(T-t)^{-\frac{1}{p-1}}|f_b|^{-(p-1)}f_b^{-1} - (p-1+by^{2k})$*. The study of such linearisation will simplify the computations as you will see in section 5.*
- *Our construction in this work is inspired by the work of Bricmont and Kupiainen in* [3] *and our recent result* [9]*. But this is far from being a simple adaptation of the construction made in the case of the nonlinear heat equation because of the complex structure of the CGL equation 2.1. Indeed we have a potential term* V *(see* (2.11)*) which appear in the linearized equation. We note that the computation of the projection of the potential are much more difficult to handle (see Lemma 5.9 and 5.12).*
- *The proof of Theorem 1 relies on the understanding of the dynamics of the self-similar version of* (2.1) *(see* (2.3) *below) around the profile* (2.7)*. Moreover, we proceed in two steps:*
  - *First, we reduce the question to a finite-dimensional problem: we show that it is enough to control a (2k)-dimensional variable in order to control the solution (which is infinite dimensional) near the profile.*
  - *Second, we proceed by contradiction to solve the finite-dimensional problem and conclude using a topological argument.*

Structure of the paper: To be more convenient for readers, we mention here the structure of the paper. In Section 2, we give a formal approach to our problem and setup the main linearized problem around the suitable approximation. Next, we show in Section 3 spectral properties of the linear operators. In particular, Section 4 plays a central role in our paper that reduces our problem to a finite dimensional one and the conclusion to the finite dimensional one. Finally, the conclusion yields the proof of Theorem 1 and Corollary 2 (see Section 4.5). In Section 5, we provide a priori estimates to our solution which plays an important role in our analysis. Finally, in the last Section, we give necessary estimates on the action of semigroups on the negative part of the solution. We hope this make clear the structure of the paper.

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2. Formulation of the problem

In this section, we aim to formulate our main problem. First, we formally explain how the profile in Theorem 1.11 is selected, and we then make the linearized problem around the selected approximation.

Let $\beta = \alpha = 0$, then the complex Ginzburg Landau equation (1.1) reads

$$\partial_t u = \Delta u + (1 + i\delta)|u|^{p-1}u. \tag{2.1}$$

Now, we assume that u is a solution to (2.1) on $[0, T)$ for some $T > 0$, and $k \geqslant 2$ is an integer number. We introduce the k-similarity variables as follows

$$w(y, s) = (T - t)^{\frac{1+i\delta}{p-1}} u(x, t),\ y = \frac{x}{(T - t)^{\frac{1}{2k}}},\quad s = -\ln(T - t). \tag{2.2}$$

Thanks to (2.1), w solves the following equation

$$\partial_s w = I^{-2}(s)\Delta w - \frac{1}{2k} y \cdot \nabla w - \frac{1 + i\delta}{p - 1} w + (1 + i\delta)|w|^{p-1}w, \tag{2.3}$$

where $I(s)$ is defined by

$$I(s) = e^{\frac{s}{2}\left(1-\frac{1}{k}\right)}. \tag{2.4}$$

In our paper, we are interested in a formal solution to (2.3) of the form

$$w(y, s) = \sum_{j=0}^{\infty} \frac{w_j(y)}{I^{2j}(s)} = \sum_{j=0}^{\infty} \frac{w_j(y)}{e^{sj\left(1-\frac{1}{k}\right)}},$$

where w_j is assumed to be smooth and globally bounded. Plugging this ansatz into (2.3) and looking at the leading order, we obtain

$$-\frac{1}{2k} y \cdot \nabla w_0 - (1 + i\delta)\frac{w_0}{p - 1} + (1 + i\delta)|w_0|^{p-1}w_0 = 0.$$

Since we aim to search global solutions w, w_0 must be the same. Up to modulo a phase, there exists $b > 0$ such that

$$w_0(y) = (p - 1 + by^{2k})^{-\frac{1+i\delta}{p-1}} \text{ for some } b > 0. \tag{2.5}$$

Thus, (2.5) formally explains how the profile in Theorem 1.11 arises.

Next, we is motivated by [3], to introduce the linearized problem as follows

$$w(y, s) = e^{i\theta(s)} f_{b(s)}(y, s)\left(1 + e_{b(s)}(y, s)q(y, s)\right), \tag{2.6}$$

the functions $f_{b(s)}$ and $e_{b(s)}$ are given by

$$f_{b(s)}(y) = \left(p - 1 + b(s)y^{2k}\right)^{-\frac{1+i\delta}{p-1}}, \tag{2.7}$$

and

$$e_{b(s)}(y) = \left(p - 1 + b(s)y^{2k}\right)^{-1}. \tag{2.8}$$

It is worth noting that the combined influence of the two parameters $b(s)$ and $\theta(s)$ is crucial to our analysis, as they govern the dynamics of neutral modes in our construction problem. As a matter of fact, our linearized problem is not only focused on linearisation with respect to the variable q, but also extends to determine the flows (θ, b) (so-called modulations). Thanks to (2.3), which allows us to express (q, b, θ) as follows:

$$\partial_s q = \mathcal{L}_{\delta,s} q + B(q) + T(q) + N(q) + D_s(\nabla q) + R_s(q) + V(q), \tag{2.9}$$

where

$$\mathcal{L}_{\delta,s}q = I^{-2}(s)\Delta q - \frac{1}{2k}y\cdot\nabla q + (1+i\delta)\Re(q),, \tag{2.10}$$

$$V(q) = ((p-1)e_b - 1)\left[(1+i\delta)\Re q - q\right], \tag{2.11}$$

$$N(q) = (1+i\delta)\left(|1+e_bq|^{p-1}(1+e_bq) - 1 - 2e_b\Re q - \frac{p-1}{2}e_bq - \frac{p-3}{2}e_b\bar{q}\right), \tag{2.12}$$

and

$$\begin{cases} B(q) &= \frac{b'(s)}{p-1}y^{2k}\left(1+i\delta+(p+i\delta)e_bq\right), \\ T(q) &= -i\theta'(s)\left(e_b^{-1}-q\right) = -i\theta'(s)\left(p-1+by^{2k}-q\right), \\ D_s(q) &= -I(s)^{-2}\frac{p+i\delta}{p-1}4kby^{2k-1}e_b\nabla q, \\ R_s(q) &= I^{-2}(s)y^{2k-2}\left(\alpha_1+\alpha_2y^{2k}e_b+\left(\alpha_3+\alpha_4y^{2k}e_b\right)q\right), \end{cases} \tag{2.13}$$

with the explicitly determinated constants as follows

$$\begin{aligned} \alpha_1 &= -(1+i\delta)2k(2k-1)\tfrac{b}{p-1} \quad \alpha_2 = 4(1+i\delta)(p+i\delta)k^2\tfrac{b^2}{(p-1)^2}, \\ \alpha_3 &= -(p+i\delta)2k(2k-1)\tfrac{b}{p-1} \quad \alpha_4 = 4(p+i\delta)(2p-1+i\delta)k^2\tfrac{b^2}{(p-1)^2}. \end{aligned} \tag{2.14}$$

Our aim is to construct a triple $(q,b,\theta)(s), s\in[-\ln T,+\infty)$ where $\theta(\cdot), b(\cdot)\in C^1([-\log T,\infty),\mathbb{R})$ are suitably selected such that equation (2.9) has a unique solution $q(\cdot,s)$ on $[-\ln T,\infty)$ satisfying

$$\|q(s)\|_{L^\infty_M} = \left\|\frac{q(s)}{1+|y|^M}\right\|_{L^\infty} \to 0 \text{ as } s\to\infty,$$

where the constant M is defined by

$$M = \frac{2kp}{p-1}. \tag{2.15}$$

Choosing such a value for M involves a delicate process. On the one hand, it ensures that that the linearisation (2.6) is effective, providing us with a rigorous approximation. On the other hand, it allows us to effectively control the nonlinearity $\mathcal{N}$, as we showed in (5.33).

3. Spectral properties of linear operators

In this section, we aim to give some spectral properties of the linear operators appearing in our paper.

First, let us introduce

$$\rho_s = \frac{I(s)}{\sqrt{4\pi}}e^{-\frac{I^2(s)y^2}{4}}. \tag{3.1}$$

Then, we introduce $L^2_{\rho_s}(\mathbb{R},\mathbb{C})$ as the weighted Hilbert space defied by

$$L^2_{\rho_s}(\mathbb{R},\mathbb{C}) = \left\{f\in L^2_{loc}(\mathbb{R},\mathbb{C}) \text{ such that } \int_{\mathbb{R}}|f|^2\rho_s dy < +\infty\right\}. \tag{3.2}$$

For each $m\in\mathbb{N}$, we recall $h_m(z)$ as the Hermite polynomial of degree m defined by

$$h_m(z) = \sum_{\ell=0}^{[\frac{m}{2}]}\frac{m!}{\ell!(m-2\ell)!}z^{m-2\ell}. \tag{3.3}$$

By a simple computation, it is easy to check

$$\int h_nh_m\rho_s dy = 2^n n!\delta_{nm}. \tag{3.4}$$

Now, we define the scaled function H_m

$$H_m(y,s) = I^{-m}(s)h_m(I(s)y) = \sum_{\ell=0}^{[\frac{m}{2}]} \frac{m!}{\ell!(m-2\ell)!}(-I^{-2}(s))^\ell y^{m-2\ell}. \tag{3.5}$$

Thanks to (3.4), it holds that

$$(H_n(.,s), H_m(.,s))_s = \int H_n(y)H_m(y)\rho_s(y)dy = I^{-2n}2^n n!\delta_{nm}. \tag{3.6}$$

We introduce

$$\mathcal{L}_{0,s}q = I^{-2}(s)\Delta q - \frac{1}{2k}y\cdot\nabla q, \tag{3.7}$$

and

$$\mathcal{L}_s q := I^{-2}(s)\Delta q - \frac{1}{2k}y\cdot\nabla q + q = \mathcal{L}_{0,s}q + q. \tag{3.8}$$

We can easily derive the decomposition of the operator $\mathcal{L}_s$ in *Jordan block* as follows

$$\mathcal{L}_s H_m(y,s) = \begin{cases} \left(1-\frac{m}{2k}\right)H_m(y,s) + m(m-1)(1-\frac{1}{k})I^{-2}(s)f_{m-2} & \text{if} \quad m \geqslant 2, \\ \left(1-\frac{m}{2k}\right)H_m(y,s) & \text{if} \quad m \in \{0,1\}. \end{cases} \tag{3.9}$$

For some $s \geqslant \sigma$, we represent $\mathcal{K}_{0,s,\sigma}$ and $\mathcal{K}_{s,\sigma}$ as the semigroups associated with the linear operators $\mathcal{L}_{0,s}$ and $\mathcal{L}_s$, respectively. As a matter of fact, the semigroups are fundamental solutions to the following

$$\begin{cases} \partial_s \mathcal{K}_{0,s,\sigma} &= \mathcal{L}_{0,s}\mathcal{K}_{0,s,\sigma} \text{ for all } s > \sigma, \\ \mathcal{K}_{0,\sigma,\sigma} &= Id, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \partial_s \mathcal{K}_{s,\sigma} &= \mathcal{L}_s\mathcal{K}_{s,\sigma} \text{ for all } s > \sigma, \\ \mathcal{K}_{\sigma,\sigma} &= Id. \end{cases} \tag{3.11}$$

Thanks to Mehler's formula, the kernels of the semigroups are explicit (initially proved in [2]) and given by

$$\mathcal{K}_{0,s,\sigma}(y,z) = \mathcal{F}\left(e^{-\frac{s-\sigma}{2k}}y - z\right) \text{ and } \mathcal{K}_{s\sigma}(y,z) = e^{s-\sigma}\mathcal{F}\left(e^{-\frac{s-\sigma}{2k}}y - z\right) \tag{3.12}$$

where

$$\mathcal{F}(\xi) = \frac{L(s,\sigma)}{\sqrt{4\pi}}e^{-\frac{L^2\xi^2}{4}} \text{ with } L^2(s,\sigma) = \frac{I_\sigma^2}{(1-e^{-(s-\sigma)})} \text{ and } I(s) = e^{\frac{s}{2}(1-\frac{1}{k})}. \tag{3.13}$$

In particular, it holds that

$$\mathcal{K}_{0,s,\sigma}H_n(.,\sigma) = e^{-(s-\sigma)(\frac{n}{2k})}H_n(.,s) \text{ and } \mathcal{K}_{s,\sigma}H_n(.,\sigma) = e^{(s-\sigma)(1-\frac{n}{2k})}H_n(.,s). \tag{3.14}$$

Next, we are based on (3.9) to represent the *Jordan block* for $\mathcal{L}_{\delta,s}$. We define

$$\left\{\hat{H}_m = (1+i\delta)H_m(y,s), \check{H}_m = iH_m(y,s) | m \in \mathbb{N}\right\}, \tag{3.15}$$

where $H_m(y,s)$ is defined in (3.5). Thus, it holds that for each $m \geqslant 2$

$$\begin{cases} \mathcal{L}_{\delta,s}(\hat{H}_m) &= \left(1-\frac{m}{2k}\right)\hat{H}_m + m(m-1)(1-\frac{1}{k})I^{-2}(s)\hat{H}_{m-2}, \\ \mathcal{L}_{\delta,s}(\check{H}_m) &= -\frac{m}{2k}\check{H}_m + m(m-1)(1-\frac{1}{k})I^{-2}(s)\check{H}_{m-2}, \end{cases} \tag{3.16}$$

and for $m \in \{0,1\}$, we have

$$\begin{cases} \mathcal{L}_{\delta,s}(\hat{H}_m) &= \left(1-\frac{m}{2k}\right)\hat{H}_m, \\ \mathcal{L}_{\delta,s}(\check{H}_m) &= -\frac{m}{2k}\check{H}_m. \end{cases} \tag{3.17}$$

3.1. Decomposition of q

In this part, we aim to introduce decomposition of the solution q along time-dependent polynomials $\{H_n, n \geqslant 0\}$ (also $\{\hat{H}_n, \check{H}_n, n \geqslant 0\}$). Let us expand q by

$$q(y,s) = \sum_{0\leqslant n\leqslant [M]} Q_n(s)H_n(y,s) + q_-(y,s), \tag{3.18}$$

where M defined as in (2.15), and H_n is defined in (3.5), $Q_n(s) \in \mathbb{C}$, q_- satisfy

$$Q_n(q,s) = \frac{\displaystyle\int qH_n\rho_s}{\displaystyle\int H_n^2\rho_s}. \tag{3.19}$$

Additionally, we can understand that q_- is the rest part of q which is orthogonal to H_n for all $n \leqslant [M]$ i.e.

$$\int q_-(y,s)H_n(y,s)\rho_s(y)dy = 0 \text{ for all } n \leqslant [M].$$

For more convenience, we define two projectors $P_{+,[M]}$ and $P_{-,[M]}$ (P_+ and P_- for short)

(3.20)

and

$$P_{-,[M]}(q) := q - P_{+,[M]}. \tag{3.21}$$

Thus, by introducing

$$q_+ = P_{+,[M]} \text{ and } q_- = P_{-,[M]}, \tag{3.22}$$

we can express

$$q = q_+ + q_-. \tag{3.23}$$

Definition 3.1 (δ-decomposition)**.** *We define the projectors $\mathscr{Q}_{\Re,\delta}$ and $\mathscr{Q}_{\Im,\delta}$ on complex numbers by*

$$\mathscr{Q}_{\Re,\delta}(z) = \Re(z) = \hat{z} \text{ and } \mathscr{Q}_{\Im,\delta}(z) = \Im(z) - \delta\Re(z) = \check{z}, \tag{3.24}$$

for each $z \in \mathbb{C}$, then we have a unique decomposition for each $z \in \mathbb{C}$ that

$$z = \hat{z}(1 + i\delta) + i\check{z}.$$

Consequently,

Lemma 3.2. *The projectors $\mathscr{Q}_{\Re,\delta}$ and $\mathscr{Q}_{\Im,\delta}$ hold true:*

(i) It is easy to check the following properties: for all $\lambda \in \mathbb{R}, z_1, z_2 \in \mathbb{C}$

$$\mathscr{Q}_{\Re,\delta}(z_1 + z_2) = \mathscr{Q}_{\Re,\delta}(z_1) + \mathscr{Q}_{\Re,\delta}(z_2) \text{ and } \mathscr{Q}_{\Im,\delta}(z_1 + z_2) = \mathscr{Q}_{\Im,\delta}(z_1) + \mathscr{Q}_{\Im,\delta}(z_2), \tag{3.25}$$

and

$$\mathscr{Q}_{\Re,\delta}(\lambda z_1) = \lambda\mathscr{Q}_{\Re,\delta}(z_1) \text{ and } \mathscr{Q}_{\Im,\delta}(\lambda z_1) = \lambda\mathscr{Q}_{\Im,\delta}(z_1), \tag{3.26}$$

and

$$\mathscr{Q}_{\Re,\delta}((1 + i\delta)\lambda) = \lambda, \text{ and } \mathscr{Q}_{\Im,\delta}(i\lambda) = \lambda. \tag{3.27}$$

(ii) Let $V(q)$ defined as in (2.11), then

$$\widehat{V(q)} = 0 \text{ and } \widetilde{V(q)} = (1 - (p-1)e_b)\,\check{q}. \tag{3.28}$$

Proof. It directly replies on Definition 3.1. We kindly refer readers to check the details. ∎

According to Definition 3.1, we can decompose q_+ and q_- as follows

$$q_+ = \sum_{n\leqslant[M]} Q_n H_n(s) = \sum_{n\leqslant[M]} \hat{q}_n \hat{H}_n(s) + \check{q}_n \check{H}_n(s), \tag{3.29}$$

and

$$q_- = (1+i\delta)\hat{q}_- + i\check{q}_-, \tag{3.30}$$

where $\hat{q}_n, \check{q}_n \in \mathbb{R}$ and $\hat{q}_-, \check{q}_-$ are real-valued functions. In particular, these components can be explicitly expressed by

$$\hat{q}_n(s) = \hat{P}_n(q) := \mathscr{Q}_{\Re,\delta}(Q_n),\ \check{q}_n(s) = \check{P}_{n,M}(q) := \mathscr{Q}_{\Im,\delta}(Q_n). \tag{3.31}$$

and

$$\hat{q}_- = \mathscr{Q}_{\Re,\delta}(q_-) \text{ and } \check{q}_- = \mathscr{Q}_{\Im,\delta}(q_-). \tag{3.32}$$

Finally, we obtain the unique decomposition as follows

$$q(y,s) = \left(\sum_{n\leqslant[M]} \hat{q}_n(s)\hat{H}_n(y,s) + \check{q}_n(s)\check{H}_n(y,s)\right) + (1+i\delta)\hat{q}_-(y,s) + i\check{q}_-(y,s). \tag{3.33}$$

3.2. Equivalent norms

In this section, we establish equivalent norms used in our paper. Let us introduce L^∞_M defined by

$$L^\infty_M(\mathbb{R}) = \{g \text{ such that } (1+|y|^M)^{-1} g \in L^\infty(\mathbb{R})\}. \tag{3.34}$$

L^∞_M is complete with the norm

$$\|g\|_{L^\infty_M} = \|(1+|y|^M)^{-1} g\|_{L^\infty}, \tag{3.35}$$

since L^∞ is complete with $\|.\|_{L^\infty}$. In particular, we also introduce $\|.\|_s$ as follows

$$\|q\|_s = \sum_{m=0}^{[M]} |q_m| + |q_-|_s, \tag{3.36}$$

where

$$|q_-|_s = \sup_y \frac{|q_-(y,s)|}{I^{-M}(s) + |y|^M}. \tag{3.37}$$

As a matter of fact, we have the following equivalence:

$$C_1(s)\|q\|_{L^\infty_M} \leqslant \|q\|_s \leqslant C_2(s)\|q\|_{L^\infty_M} \text{ for some } C_1, C_2 \in \mathbb{R}^*_+, \tag{3.38}$$

which yields L^∞_M is also complete with $\|.\|_s$.

4. The proof assuming some technical results

In this section we give the complete proof of Theorem 1. The main idea is to reduce the problem to a finite dimensional problem ($2k$-dimensional one) which is classical and can be solved by a topological argument. We hope that the explanation of the strategy we give in this section will be more reader friendly. Below, we give the main steps:

- The first step: we construct a shrinking set $V_{A,\gamma,b_0,\theta_0}(s)$ including necessary bounds such that the belonging in this set completely implies the result in Theorem 1.
- In the second step: we construct initial data at initial time s_0 for (2.9) which is parameterized by $2k$ parameters in accordance with the $2k$ projections $\hat{q}_0, .., \hat{q}_{2k-1}$ of q on $\hat{H}_n, n \leqslant 2k$.

- In the third step: we impose two orthogonal conditions

$$\mathscr{Q}_{\Re,\delta}\left(\int q(s)\hat{H}_{2k}(y,s)\rho_s dy\right)=\mathscr{Q}_{\Im,\delta}\left(\int q(s)\check{H}_0(y,s)L^2_{\rho_s}dy\right)=0, \tag{4.1}$$

  which are responsible for nullifying two projections $\hat{q}_{2k}$ and $\check{q}_0$ of q onto $\hat{H}_{2k}$ and $\check{H}_0$. According to (3.16), the projections involve the zero modes arising as big challenges in the construction. Therefore, the appearance of these modulations is critical to our construction. Additionally, we show the locally unique existence of the solution (q,b,θ) to the coupled problem (2.9) & (4.1).
- In the fourth step: By using the spectral approach of the linear operator $\mathcal{L}_{\delta,s}$, we reduce the control of $(q,b,\theta)(s)$ (an infinite dimensional problem) to a $2k$-dimensional one involving $(\hat{q}_0,..,\hat{q}_{2k-1})$.
- In the last subsection, we solve the finite dimensional one by using a topological argument and we give the complete conclusion to Theorem 1.

4.1. Definition of a shrinking set

In this section, we define a "shrinking set" which control the behavior of the solution by some error bounds.

Definition 4.1 (Shrinking set). *Let $k\geqslant 2, k\in\mathbb{N}, \gamma>0, b_0>0, \theta_0\in\mathbb{R}, s\geqslant 1$ and $A\geqslant 1$, we define $V_{A,\gamma,b_0,\theta_0}(s)$ as the set of all (q,b,θ), where $q\in(L^\infty_M(\mathbb{R}),\mathbb{R})$ satisfying the following conditions:*

(i) The first condition: for all n satisfying $0\leqslant n\leqslant M$, we have

$$|\hat{q}_n|\leqslant I^{-\gamma}(s) \text{ and } |\check{q}_n|\leqslant I^{-\gamma}(s). \tag{4.2}$$

(ii) The second condition:

$$\left\|\frac{\hat{q}_-(y,s)}{I^{-M}+|y|^M}\right\|_{L^\infty}\leqslant I^{-\gamma}(s),\quad \left\|\frac{\check{q}_-(y,s)}{I^{-M}+|y|^M}\right\|_{L^\infty}\leqslant AI^{-\gamma}(s). \tag{4.3}$$

where $I(s)$ defined as in (2.4); $\hat{q}_n$ and $\check{q}_n$ given as in (3.31); and $\hat{q}_-=\mathscr{Q}_{\Re,\delta}(q_-)$,$\check{q}_-=\mathscr{Q}_{\Im,\delta}(q_-)$ and the negative part q_- defined as in (3.22).

(iii) The third condition:

$$\frac{b_0}{2}\leqslant b\leqslant 2b_0, \tag{4.4}$$

$$\frac{|\theta_0|}{2}\leqslant|\theta|\leqslant 2|\theta_0|. \tag{4.5}$$

In below, we show some rough bounds for functions belonging to $V_{A,\gamma,b_0,\theta_0}(s)$ for some $s\geqslant 1$.

Lemma 4.2. *Let $(q,b)\in V_{A,\gamma,b_0,\theta_0}(s)$ arbitrarily given, then the following estimates hold*

$$|q_+(y)|\leqslant CI^{-\gamma}(s)\left(\sum_{n\leqslant[M]}(I^{-n}(s)+|y|^n)\right),\forall y\in\mathbb{R}, \tag{4.6}$$

$$|q_-(y)|\leqslant CAI^{-\gamma}(s)(I^{-M}(s)+|y|^M),\forall y\in\mathbb{R}, \tag{4.7}$$

$$\left|\mathbb{1}_{\{|y|\geqslant\frac{1}{2}\}}q_+(y)\right|\leqslant CI^{-\gamma}(s)\left(I^{-M}(s)+|y|^M\right),\forall y\in\mathbb{R}, \tag{4.8}$$

$$\left|\mathbb{1}_{\{|y|\geqslant\frac{1}{2}\}}q(y)\right|\leqslant CAI^{-\gamma}(s)\left(I^{-M}(s)+|y|^M\right),\forall y\in\mathbb{R}, \tag{4.9}$$

where $C>0$ is a universal constant depending only on the nonlinear power p and k.

Proof. The result immediately follows by the definition of $V_{A,\gamma,b_0,\theta_0}(s)$. We kindly refer the reader to check these estimates. ∎

4.2. Preparation of initial data

Let us now prepare initial data to our problem. We consider $\hat{\mathbf{d}} = (\hat{d}_0, .., \hat{d}_{2k-1}) \in \mathbb{R}^{2k}$ and we introduce

$$\psi(\hat{\mathbf{d}}, y, s_0) = I^{-\gamma}(s_0) \sum_{j=0}^{2k-1} \hat{d}_j \hat{H}_j(y, s_0). \tag{4.10}$$

By the orthogonality of $\{\hat{H}_j, \check{H}_k\}$, we can easily check that

$$\hat{P}_{2k}(\psi) = 0, \text{ and } \check{P}_0(\psi) = 0, \tag{4.11}$$

which satisfy (4.1) at s_0.
In below, we show important properties for initial data defined in (4.10).

Lemma 4.3 (Reparing of initial data). *Let $\hat{d} = (\hat{d}_i)_{0\leqslant i\leqslant 2k-1} \in \mathbb{R}^{2k}$ satisfying $\max_{0\leqslant i\leqslant 2k-1} \left|\hat{d}_i\right| \leqslant 2$ and $b_0 > 0$ arbitrarily given. Then, there exists $\gamma_1(b_0) > 0$ such that for all $\gamma \in (0, \gamma_1)$, there exists $s_1(\gamma, b_0) \geqslant 1$ such that for all $s_0 \geqslant s_1$, the following properties are valid with $\psi(\hat{d}, y, s_0)$ defined in (4.10):*

(i) There exits a quadrilateral $\mathbb{D}_{s_0} \subset [-2, 2]^{2k}$ such that the mapping

$$\begin{array}{c} \Gamma : \mathbb{D}_{s_0} \to \mathbb{R}^{2k} \\ (\hat{d}_0, ..., \hat{d}_{2k-1}) \mapsto (\hat{\psi}_0, ..., \hat{\psi}_{2k-1}) \end{array}, \tag{4.12}$$

is linear one to one from $\mathbb{D}_{s_0}$ to $\hat{\mathcal{V}}(s_0)$, with

$$\hat{\mathcal{V}}(s) = \left[-I^{-\gamma}(s), I^{-\gamma}(s)\right]^{2k}, \tag{4.13}$$

where $(\hat{\psi}_0, ..., \hat{\psi}_{2k-1})$ are the coefficients of initial data $\psi(\hat{d}_0, ..., \hat{d}_{2k-1})$ corresponding to the decomposition (3.33). In addition to that, we have

$$\Gamma|_{\partial\mathbb{D}_{s_0}} \subset \partial\hat{\mathcal{V}}(s_0) \text{ and } deg\left(\Gamma|_{\partial\mathbb{D}_{s_0}}\right) \neq 0. \tag{4.14}$$

(ii) For all $(\hat{d}_0, ..., \hat{d}_{2k-1}) \in \mathbb{D}_{s_0}$, the following estimates are valid

$$\left|\hat{\psi}_0\right| \leqslant I^{-\gamma}(s_0), ..., \left|\hat{\psi}_{2k-1}\right| \leqslant I^{-\gamma}(s_0), \quad \hat{\psi}_{2k} = .. = \hat{\psi}_M = 0 \text{ and } \check{\psi} \equiv 0, \psi_- \equiv 0. \tag{4.15}$$

Proof.
- *The proof of item (i):* From (4.10), definition's $\hat{H}_n$ as in (3.5) and (3.31), and by a direct computation we can prove that there exists a square matrix $\mathscr{A}_{s_0}$ satisfying

$$\mathscr{A} = Id + \tilde{\mathscr{A}}_{s_0} \text{ with } \tilde{\mathscr{A}}_{s_0} = (\tilde{a}_{ij}), |\tilde{a}_{ij}| \leqslant CI^{-2}(s_0),$$

and we have

$$\begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{2k-1} \end{pmatrix} = I^{-\gamma}(s_0)\mathscr{A}_{s_0} \begin{pmatrix} \hat{d}_0 \\ \hat{d}_1 \\ \vdots \\ \hat{d}_{2k-1} \end{pmatrix}, \tag{4.16}$$

which immediately concludes item (i).
- *The proof of item (ii):* From (4.10), $\psi(\hat{d}_0, ..., \hat{d}_{2k-1}, s_0)$ is a polynomial of order $2k-1$, so it follows that

$$\hat{\psi}_n = 0, \forall n \in \{2k, .., M\}, \check{\psi}_n = 0, \forall n \in \{0, .., M\} \text{ and } \psi_- \equiv 0.$$

In addition, since $(\hat{d}_0, ..., \hat{d}_{2k-1}) \in \mathbb{D}_{s_0}$, we apply item (i) to deduce that $(\psi_0, ..., \psi_{2k-1}) \in \hat{\mathcal{V}}(s_0)$ and

$$\left|\hat{\psi}_n\right| \leqslant I^{-\delta}(s_0), \forall n \in \{0, ..., 2k-1\}$$

which concludes item (ii) and the proof Lemma 4.3.∎

Remark 4.4. *Note that $s_0 = -\ln(T)$ is the master constant. In almost every argument in this paper it is almost to be sufficiently depending on the choice of all other constants (γ_0 and b_0). In addition, we denote C as the universal constant which is independent to b_0 and s_0.*

4.3. Local in time solution of problem (2.9) & (4.1)

In this part, we prove the local existence of solution to the problem (2.9) & (4.1). The result reads.

Proposition 4.5 (Local existence of the coupled problem (2.9) & (4.1))**.** *Let $\hat{\boldsymbol{d}} \in \mathbb{R}^{2k}$ satisfying $\left|\hat{\boldsymbol{d}}\right| \leqslant 2$, $\gamma > 0$, $b_0 > 0, \theta_0 \in \mathbb{R}$ and $A \geqslant 1$. Then, there exits $s_2(\gamma, b_0, \theta_0) \geqslant 1$, such that for all $s_0 \geqslant s_2$, the following holds: if we take initial data (ψ, b_0, θ_0) with ψ defined as as in (4.10), then there exists $s_{loc} > s_0$ such that the coupled problem (2.9) & (4.1) has a unique solution $(q, b, \theta)(s) \in V_{A,\gamma,b_0,\theta_0}(s)$ on $[s_0, s_{loc}]$.*

Proof. First, it is classical that equation (2.3) is locally well-posed in $L^\infty(\mathbb{R})$. So, with initial data (ψ, b_0, θ_0) and the transformation in (2.6), there exists $\bar{s}_1 > s_0$ such that equation (2.3) uniquely exists on $[s_0, \bar{s}_1]$. Now, let us introduce $\vec{\mu} = (b, \theta)$ and define

$$\mathcal{F}(s, \vec{\mu}) = \begin{pmatrix} \mathscr{Q}_{\Re,\delta}\left(\int \left[w(y,s)(f_b e_b)^{-1} e^{-i\theta} - \left(p - 1 + by^{2k}\right)\right] H_{2k}\rho_s dy\right) \\ \mathscr{Q}_{\Im,\delta}\left(\int \left[w(s)(f_b e_b)^{-1} e^{-i\theta} - \left(p - 1 + by^{2k}\right)\right] H_0\rho_s dy\right) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(s, \vec{\mu}) \\ \mathcal{F}_2(s, \vec{\mu}) \end{pmatrix}. \tag{4.17}$$

By (3.19) and (4.11), we have

$$\mathcal{F}(s_0, \vec{\mu}_0) = 0 \text{ where } \vec{\mu}_0 = (b_0, \theta_0).$$

So, the result will be a direct consequence of the implicit function theorem. Let us recall the Jacobian matrix of $\mathcal{F}$ in accordance with the variable $\vec{\mu}$

$$\mathbf{J}_{\vec{\mu}}[\mathcal{F}](s, \vec{\mu}) = \begin{pmatrix} \partial_b \mathcal{F}_1(s, \mu) & \partial_\theta \mathcal{F}_1(s, \vec{\mu}) \\ \partial_b \mathcal{F}_2(s, \vec{\mu}) & \partial_\theta \mathcal{F}_2(s, \vec{\mu}) \end{pmatrix}, \tag{4.18}$$

where

$$\begin{aligned}
\partial_\theta \mathcal{F}_1(s, \vec{\mu}) &= \mathscr{Q}_{\Re,\delta}\left(-i\int w(s) f_b e_b)^{-1} e^{-i\theta} H_{2k}\rho_s dy\right), \\
\partial_b \mathcal{F}_1(s, \vec{\mu}) &= \mathscr{Q}_{\Re,\delta}\left(\int \left[\frac{p + i\delta}{p-1} w(s) y^{2k} (f_b)^{-1} e^{-i\theta} - y^{2k}\right] H_{2k}\rho_s dy\right), \\
\partial_\theta \mathcal{F}_2(s, \vec{\mu}) &= \mathscr{Q}_{\Im,\delta}\left(-i\int w(s)(f_b e_b)^{-1} e^{-i\theta} H_0\rho_s dy\right), \\
p a_b \mathcal{F}_2(s, \vec{\mu}) &= \mathscr{Q}_{\Im,\delta}\left(\int \frac{p + i\delta}{p-1}\left[w(s) y^{2k} (f_b)^{-1} e^{-i\theta} - y^{2k}\right] H_0\rho_s\right).
\end{aligned}$$

The main goal is to prove

$$\mathbf{Det}\left(\mathbf{J}_{\vec{\mu}}[\mathcal{F}](s, \vec{\mu})\right)|_{(s,\vec{\mu})=(s_0,\vec{\mu}_0)} \neq 0. \tag{4.19}$$

- *Expansion for $\partial_b \mathcal{F}_1(s_0, \vec{\mu})|_{\vec{\mu}=\vec{\mu}_0}$:* Thanks to (2.6) and (4.10), it follows that

$$w(s_0) f_{b_0}^{-1} e^{-i\theta} = 1 + e_{b_0}\Psi, \tag{4.20}$$

which yields

$$\frac{p+i\delta}{p-1}w(s_0)y^{2k}(f_{b_0})^{-1}e^{-i\theta_0}-y^{2k}=\frac{1+i\delta}{p-1}y^{2k}+\frac{1+i\delta}{p-1}y^{2k}e_{b_0}\Psi. \tag{4.21}$$

So, we derive from (4.10) that

$$\int\frac{p+i\delta}{p-1}\left[w(s)y^{2k}(f_b)^{-1}e^{-i\theta}-y^{2k}\right]H_{2k}\rho_s dy=\frac{1+i\delta}{p-1}I^{-4k}(s_0)2^{4k}(2k)!\left(1+O(I^{-\gamma}(s_0))\right).$$

Therefore, we get

$$\partial_b\mathcal{F}_1(s_0,\vec{\mu})\mid_{\vec{\mu}=\vec{\mu}_0}=\frac{1}{p-1}I^{-4k}(s_0)2^{4k}(2k)!\left(1+O(I^{-\gamma}(s_0))\right).$$

- *Expansion for* $\partial_\theta\mathcal{F}_1(s_0,\vec{\mu})\mid_{\vec{\mu}=\vec{\mu}_0}$*:* Using (4.20) again, we get

$$-i\int w(s_0)(f_{b_0}e_{b_0})^{-1}e^{-i\theta_0}H_{2k}\rho_s dy=-ib_0I^{-4k}(s_0)2^{4k}(2k)!.$$

Thus, it follows that

$$\partial_\theta\mathcal{F}_1(s_0,\vec{\mu})\mid_{\vec{\mu}=\vec{\mu}_0}=0. \tag{4.22}$$

- *Expansion for* $\partial_b\mathcal{F}_2(s_0,\vec{\mu})\mid_{\vec{\mu}=\vec{\mu}_0}$*:* By using (4.21), we obtain

$$\left|\partial_b\mathcal{F}_2(s_0,\vec{\mu})\mid_{\vec{\mu}=\vec{\mu}_0}\right|\leqslant CI^{-2k}(s_0).$$

- *Expansion for* $\partial_\theta\mathcal{F}_2(s_0,\vec{\mu})\mid_{\vec{\mu}=\vec{\mu}_0}$*:* By the same way, we get

$$\begin{array}{rcl}\partial_\theta\mathcal{F}_2(s_0,\mu_0) & = & (-(p-1)-I^{-\gamma}(s_0)\hat{d}_0)-\delta I^{-\gamma}(s_0)\left(\delta\hat{d}_0+\check{d}_0\right)+C(b)I^{-2k}(s_0)\\ & = & (1-p)+O\left(I^{-\gamma}(s_0)\right).\end{array} \tag{4.23}$$

By combining the previous expansions, we obtain

$$\mathbf{Det}\left(\mathbf{J}_{\vec{\mu}}[\mathcal{F}](s,\vec{\mu})\right)\mid_{(s,\vec{\mu})=(s_0,\vec{\mu}_0)}=(1-p)2^{4k}(2k)!I^{-4k}(s_0)\left(1+O(I^{-\gamma}(s_0))\right)\neq 0, \tag{4.24}$$

provided that $s_0\geqslant s_{2,1}$. Thus, it is a direct consequence of the implicit function theorem that there exists $\vec{\mu}(s)=(\theta(s),b(s))\in C\left([s_0,s_{\rm loc}],\mathbb{R}^2\right)\cap C^1\left((s_0,s_{\rm loc}),\mathbb{R}^2\right)$ with $s_{\rm loc}\in(0,\bar{s}_1)$ such that

$$\mathcal{F}(s,\vec{\mu}(s))\equiv 0,\forall s\in[s_0,s_{\rm loc}],$$

and

$$\frac{b_0}{2}<b(s)\leqslant 2b_0 \text{ and } \frac{|\theta_0|}{2}<\theta(s)<2|\theta_0|.$$

In particular, $(q,b,\theta)(s)\in V_{A,\gamma,b_0,\theta_0}(s)\quad\forall s\in[s_0,s_{\rm loc}]$, thanks to the continuity of the solution. Finally, we get the conclusion of the proposition. ∎

Remark 4.6 (Propagating the existence)**.** *By the way in the proof of Proposition 4.5 we can prove that if the solution* (q,b,θ) *exists on* $[s_0,\bar{s}]$ *for some* $\bar{s}>s_0$ *and* $(q,b,\theta)(\bar{s})\in V_{A,\gamma,b_0,\theta_0}(\bar{s})$*. Then, there exists* $\epsilon>0$ *such that the solution* (q,b,θ) *to problem (2.9) & (4.1), uniquely exists on* $[s_0,\bar{s}+\epsilon]$*. Since we can prove*

$$\boldsymbol{Det}\left(\boldsymbol{J}_{\vec{\mu}}[\mathcal{F}](s,\vec{\mu})\right)\mid_{(s,\vec{\mu})=(\bar{s},\vec{\mu})}=(1-p)2^{4k}(2k)!I^{-4k}(\bar{s})\left(1+O(AI^{-\gamma}(\bar{s}))\right)\neq 0,\vec{\mu}=(b,\theta)(\bar{s}).$$

4.4. Reduction to a finite dimensional problem

As we defined shrinking set $V_{A,\gamma,b_0,\theta_0}(s)$ in Definition 4.1, it is sufficient to prove there exists a unique global solution (q,b,θ) on $[s_0,+\infty)$ for some s_0 sufficient large that

$$(q,b,\theta)(s)\in V_{A,\gamma,b_0,\theta_0}(s),\forall s\geqslant s_0.$$

In particular, we show in this part that the control of infinite problem is reduced to a finite dimensional one. As an important step to get the conclusion of our result, we first show the following a *priori* estimates.

Proposition 4.7 (A *priori* estimates)**.** *Let $b_0>0$ and $k\in\mathbb{N},k\geqslant 2,b_0>0$, then there exists $\gamma_3(k,b_0)>0$ such that for all $\gamma\in(0,\gamma_3)$, there exists $s_3(\gamma,b_0)$ such that for all $s_0\geqslant s_3$, the following property holds: Assume (q,b,θ) is a solution to problem (2.9) & (4.1) that $(q,b,\theta)(s)\in V_{A,\gamma,b_0,\theta_0}(s)$ for all $s\in[\bar{s},s_0]$ for some $\bar{s}\geqslant s_0$, and $q_{2k}(s)=0$ for all $s\in[\bar{s},s_0]$, then for all $s\in[\tau,\bar{s}]$, with $s_0\leqslant\tau\leqslant\bar{s}$, the following properties hold:*

(i) (Smallness of the modulation parameter $\theta(s)$). It holds that

$$\left|\theta'(s)\right|\leqslant CA^2I^{-2\gamma}(s),\quad \forall s\in[s_0,\bar{s}].$$

(ii) (Oscillation the modulation flow $b(s)$). It holds that

$$\left|b'(s)\right|\leqslant CA^2I^{-2\gamma}(s)\ \text{and}\ \frac{3}{4}b_0\leqslant b(s)\leqslant\frac{5}{4}b_0,\quad \forall s\in[s_0,\bar{s}].$$

(iii) (ODEs of the finite modes). For all $j\in\{0,...,[M]\}$, we have

$$\left|\hat{q}_j'(s)-\left(1-\frac{j}{2k}\right)\hat{q}_j(s)\right|\leqslant CA^2I^{-2\gamma}(s),\quad \forall s\in[s_0,\bar{s}].$$

$$\left|\check{q}_j'(s)+\frac{j}{2k}\check{q}_j(s)\right|\leqslant CI^{-2\gamma}(s),\quad \forall s\in[s_0,\bar{s}].$$

(iv) (Control of the infinite-dimensional part q_-): For $q_-=(1+i\delta)\hat{q}_-+i\check{q}_-$, we have

$$|\hat{q}_-(s)|_s\leqslant e^{-\frac{s-\sigma}{p-1}}|\hat{q}_-(\sigma)|_\sigma+C\left(I^{-\frac{\bar{p}+1}{2}\gamma}(s)+e^{-\frac{s-\sigma}{p-1}}I^{-\frac{\bar{p}+1}{2}\gamma}(\sigma)\right),$$

and

$$|\check{q}_-(s)|_s\leqslant e^{-\frac{s-\sigma}{p-1}}|\check{q}_-(\sigma)|_\sigma+C\left(I^{-\gamma}(s)+e^{-\frac{s-\sigma}{p-1}}I^{-\gamma}(\sigma)\right),$$

where $\bar{p}=\min(p,2)$.

Proof of Proposition 4.7. This result plays an important role in the proof of Theorem 1. For the reader's convenience, we put the complete proof of Proposition 4.7 in Section 5.∎

Consequently, we have the following result.

Proposition 4.8 (Reduction to a finite dimensional problem)**.** *Let $k\in\mathbb{N},k\geqslant 2,b_0,\theta_0>0$ and $A\geqslant 1$, then there exists $\gamma_4(b_0)$ such that for all $\gamma\in(0,\gamma_4)$, there exists $s_4(b_0,\gamma)$ such that for all $s_0\geqslant s_4$, the following property holds: Assume that (q,b,θ) is a solution to problem (2.9) & (4.1) in accordance with initial data $(q,b,\theta)(s_0)=(\psi(d_0,...,d_{2k-1},s_0),b_0,\theta_0)$ where $\psi(d_0,...,d_{2k-1}),s_0)$ defined as in (4.10) with $\max_{0\leqslant i\leqslant 2k-1}|d_i|\leqslant 2$; and $(q,b)(s)\in V_{A,\gamma,b_0,\theta_0}(s)$ for all $s\in[s_0,\bar{s}]$ for some $\bar{s}>s_0$ that $(q,b)(\bar{s})\in\partial V_{A,\gamma,b_0,\theta_0}(\bar{s})$, then the following properties are valid:*

*(i) **(Reduction to some finite number of modes)**: Consider $\hat{q}_0,...,\hat{q}_{2k-1}$ be projections of q corresponding to (3.31) then, we have*

$$(\hat{q}_0,..,\hat{q}_{2k-1})(\bar{s})\in\partial\mathcal{V}(\bar{s}),$$

where $\mathcal{V}(\bar{s})=[-I^{-\gamma}(\bar{s}),I^{-\gamma}(\bar{s})]^{2k}$ and $I(s)$ is given by (2.4) .

(ii) ***(Transverse crossing)*** *There exists* $m \in \{0, .., 2k-1\}$ *and* $\omega \in \{-1, 1\}$ *such that*

$$\omega \hat{q}_m(\bar{s}) = I^{-\gamma}(\bar{s}) \text{ and } \omega \frac{d\hat{q}_m}{ds} \mid_{s=\bar{s}} > 0.$$

Remark 4.9. *In (ii) of Proposition 4.8, we show that the solution* $q(s)$ *crosses the boundary* $\partial V_{A,\gamma,b_0,\theta_0}(s)$ *at* $\bar{s}$ *with positive speed, in other words, that all points on* $\partial V_{A,\gamma,b_0,\theta_0}(\bar{s})$ *are strict exit points in the sense of* [8, Chapter 2].

Proof. Let us start the proof Proposition 4.8 assuming Proposition 4.7. Let us consider $\delta \leqslant \delta_3$ and $s_0 \geqslant s_3$ such that Proposition 4.7 holds.

- *Proof of item (i):* From item (i) in Proposition 4.7, it is sufficient to show that for all $s \in [s_0, \bar{s}]$ the following are valid

$$|\hat{q}_j(s)| \leqslant \frac{1}{2} I^{-\gamma}(s), \forall j \in \{2k+1, ..., [M]\} \quad \text{(note that } q_{2k} \equiv 0\text{)}, \tag{4.25}$$

$$|\check{q}_j(s)| \leqslant \frac{1}{2} I^{-\gamma}(s), \forall j \in \{1, ..., [M]\} \quad \text{(note that } \check{q}_0 \equiv 0\text{)}, \tag{4.26}$$

and

$$|\hat{q}_-(s)|_s \leqslant \frac{1}{2} I^{-\gamma}(s), \quad |\check{q}_-(s)|_s \leqslant \frac{1}{2} I^{-\gamma}(s). \tag{4.27}$$

+ For (4.25): Using item (ii) in Proposition 4.7, we arrive at

$$\left[\hat{q}_j(s) \pm \frac{1}{2} I^{-\gamma}(s)\right]' = \left(1 - \frac{j}{2k}\right) \hat{q}_j(s) \pm \frac{\gamma}{2} \left(\frac{1}{2k} - \frac{1}{2}\right) I^{-\gamma}(s) + O(I^{-2\gamma}(s)). \tag{4.28}$$

Hence, with $j > 2k, \gamma \leqslant \gamma_{4,1}$ and initial data $\hat{q}_j(s_0) = 0$ that $\hat{q}_j(s_0) \in \left(-\frac{1}{2} I^{-\gamma}(s_0), \frac{1}{2} I^{-\gamma}(s_0)\right)$, it follows from (4.28) that

$$\hat{q}_j(s) \in \left(-\frac{1}{2} I^{-\gamma}(s), \frac{1}{2} I^{-\gamma}(s)\right), \forall s \in [s_0, \bar{s}],$$

which concludes (4.25). We proceed in a similar fashion to prove (4.26).

+ For (4.27): Since, the proof for $\hat{q}_-$ is the same as for $\check{q}_-$, we only prove the inequality satisfied by $\hat{q}$ in (4.27). We divide into two cases that $s - s_0 \leqslant s_0$ and $s - s_0 \geqslant s_0$. According to the first case, we apply item (iii) of Proposition 4.7 by $\tau = s_0$ that

$$|\hat{q}_-(s)|_s \leqslant C \left(I^{-\frac{\bar{p}+1}{2}\gamma}(s) + e^{-\frac{s-s_0}{p-1}} I^{-\frac{\bar{p}+1}{2}\gamma}(s_0)\right) \leqslant \frac{1}{2} I^{-\gamma}(s),$$

provided that $\gamma \leqslant \gamma_{4,2}$ and $s_0 \geqslant s_{4,2}(\gamma)$. In the second case, we use item (iii) again with $\tau = s - s_0$, and we obtain

$$\begin{aligned} |\hat{q}_-(s)|_s &\leqslant e^{-\frac{s_0}{p-1}} I^{-\gamma}(\tau) + C \left(I^{-\frac{\bar{p}+1}{2}\gamma}(s) + e^{-\frac{s_0}{p-1}} I^{-\frac{\bar{p}+1}{2}\gamma}(\tau)\right) \\ &\leqslant C(e^{-\frac{s_0}{p-1}} I^{\gamma}(s) I^{-\gamma}(\tau) + I^{-\frac{1}{2}\gamma}(s)) I^{-\gamma}(s) \leqslant \frac{1}{2} I^{-\gamma}(s). \end{aligned}$$

Thus, (4.27) completely follows. Finally, using the definition of $V_{A,\gamma,b_0,\theta_0}(s)$; the fact $(q, b)(\bar{s}) \in \partial V_{A,\gamma,b_0,\theta_0}(\bar{s})$; estimates (4.25), (4.26) and (4.27); and item (ii) of Proposition 4.8, we get the conclusion of item (i).

- *Proof of item (ii)*: As a consequence of item (i), there exist $m = 0, ..2k-1$ and $\omega = \pm 1$ such that $\hat{q}_m(\bar{s}) = \omega I^{-\delta}(\bar{s})$. By item (ii) in Proposition 4.7, we see that for $\delta > 0$

$$\omega \hat{q}_m'(\bar{s}) \geqslant \left(1 - \frac{m}{2k}\right) \omega \hat{q}_m(\bar{s}) - C I^{-2\gamma}(\bar{s}) \geqslant C \left((1 - \frac{m}{2k}) I^{-\gamma}(\bar{s}) - I^{-2\gamma}(\bar{s})\right) > 0,$$

which concludes the proof of Proposition 4.8. ∎

4.5. Conclusion of Theorem 1

In this part we aim to give the complete proof to Theorem 1 by using a topological *shooting argument*:

The proof of Theorem 1. First, we aim to prove that there exist $(\hat{d}_0,..,\hat{d}_{2k-1}) \in \mathbb{D}_{s_0}$ such that problem (2.9) & (4.1) with initial data $(\psi(\hat{d}_0,...,\hat{d}_{2k-1},s_0),b_0,\theta_0)$ and $\psi(\hat{d}_0,...,\hat{d}_{2k-1},s_0)$ defined as in (4.10), has a solution $(q_{\hat{d}_0,..,\hat{d}_{2k-1}},b,\theta)(\cdot)$ defined for all $s \in [s_0,\infty)$ such that

$$(q_{\hat{d}_0,..,\hat{d}_{2k-1}},b,\theta)(s) \in V_{A,\gamma,b_0,\theta_0}(s) \text{ for all } s \geqslant s_0.$$

Now, let us start to the proof the existence. Let $b_0>0,\theta_0>0,\gamma \geqslant 0$ and s_0 such that Lemma 4.3 and Propositions 4.7-4.8 hold, and we denote $T=e^{-s_0}>0$ (small since s_0 is large enough). We proceed by contradiction, we assume that for all $(\hat{d}_0,...,\hat{d}_{2k-1}) \in \mathbb{D}_{s_0}$ (the set defined in Lemma 4.3) there exists $s_*=s_*(\hat{d}_0,..,\hat{d}_{2k-1}) < +\infty$ such that

$$\begin{aligned} &q_{\hat{d}_0,..,\hat{d}_{2k-1}}(s) \in V_{A,\gamma,b_0,\theta_0}(s), \qquad \forall s \in [s_0,s_*],\\ &q_{\hat{d}_0,..,\hat{d}_{2k-1}}(s_*) \in \partial V_{A,\gamma,b_0,\theta_0}(s_*). \end{aligned}$$

By using item (i) of Proposition 4.8, we get $(\hat{q}_0,..,\hat{q}_{2k-1})(s_*) \in \partial V(s_*)$ and we introduce Φ by

$$\Phi: \begin{array}{l} \mathbb{D}_{s_0} \to \partial[-1,1]^{2k} \\ (\hat{d}_0,..\hat{d}_{2k-1}) \to I^{\gamma}(s)(\hat{q}_0,..,\hat{q}_{2k-1})(s_*), \end{array}$$

which is well defined and satisfies the following properties:

(*i*) Φ is continuous from $\mathbb{D}_{s_0}$ to $\partial[-1,1]^{2k}$ thanks to the continuity in time of q on the one hand, and the continuity of s_* in $(\hat{d}_0,...,\hat{d}_{2k-1})$ on the other hand, which is a direct consequence of the transversality in item (ii) of Proposition 4.8.

(ii) It holds that $\Phi\mid_{\partial\mathbb{D}_{s_0}}$ has nonzero degree. Indeed, for all $(\hat{d}_0,...,\hat{d}_{2k-1}) \in \partial\mathbb{D}_{s_0}$, we derive from item (i) of Lemma 4.3 that $s_*(\hat{d}_0,...,\hat{d}_{2k-1})=s_0$ and

$$\deg\left(\Phi\mid_{\partial\mathbb{D}_{s_0}}\right) \neq 0.$$

From Wazewski's principle in degree theory such a Φ cannot exist. Thus, we can prove that there exists $(\hat{d}_0,...,\hat{d}_{2k-1}) \in \mathbb{D}_{s_0}$ such that the corresponding solution $(q,b)(s) \in V_{A,\gamma,b_0,\theta_0}(s), \forall s \geqslant s_0$.

and by (iii) of Proposition 4.8, Φ is continuous.

In the following we will prove that Φ has nonzero degree, which mean by the degree theory (Wazewski's principle) that for all $s \in [s_0,\infty)$ $q(s)$ remains in $V_{A,\gamma,b_0,\theta_0}(s)$, which is a contradiction with the Exit Proposition.

Indeed Using Lemma 4.3, and the fact that $q(-\ln T)=\psi_{\hat{d}_0,..,\hat{d}_{2k-1}}$, we see that when $(\hat{d}_0,..,\hat{d}_{2k-1})$ is on the boundary of the quadrilateral $\mathbb{D}_T$, $\hat{q}_0,..,\hat{q}_{2k-1}(-\ln T) \in \partial[-I^{-2\gamma}(s),I^{-2\gamma}(s)]^{2k}$ and $q(-\ln T) \in V_{A,\gamma,b_0,\theta_0}(-\ln T)$ with strict inequalities for the other components.

By Proposition 4.8, $q(s)$ leaves $V_{A,\gamma,b_0,\theta_0}$ at $s_0=-\ln T$, hence $s_*=-\ln T$.

Using (ii) of Proposition 4.8, we get that the restriction of Φ on the boundary of $\mathbb{D}_{s_0}$ is of degree 1, which means by the shooting method that for all $s \in [s_0,\infty)$ $q(s)$ remains in $V_{A,\gamma,b_0,\theta_0}(s)$, which is a contradiction.

We conclude that there exist $(\hat{d}_0,..,\hat{d}_{2k-1}) \in \mathbb{D}_{s_0}$ and $(b,\theta)(\cdot) \in (C^1(-\ln T,+\infty))^2$ such that for all $s \geqslant -\ln T = s_0$, $(q_{\hat{d}_0,..,\hat{d}_{2k-1}},b,\theta)(s) \in V_{A,\gamma,b_0,\theta_0}(s)$ for all $s \geqslant s_0$. In particular, we obtain

$$\left\| \frac{q}{1+|y|^M} \right\|_{L^\infty} \leqslant CI^{-\gamma}(s). \tag{4.29}$$

Now, we use the above existence to conclude the proof of items in Theorem 1.

- *The proof of item (i) in Theorem 1:* We derive from (2.6), $M = \frac{2kp}{p-1}$, and the following estimate

$$|f_b e_b| = |f_b|^p \leqslant C(1+|y|^{-\frac{2kp}{p-1}}) = C(1+|y|^{-M})$$

that

$$\|w(s) - f_{b(s)}\|_{L^\infty} = \|f_{b(s)} e_b q(s)\|_{L^\infty} \leqslant C I^{-\gamma}(s), \text{ with } I(s) = e^{\frac{s}{2}\left(1-\frac{1}{k}\right)}.$$

Now, we still write $b(t) = b(s)$ with $s = -\ln(T-t)$ and, we hope no risk of confusion to arise here, then using (2.2), we get

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} u(\cdot,t) - f_{b(t)}\left(\frac{|\cdot|}{(T-t)^{\frac{1}{2k}}}\right)\right\|_{L^\infty} \leqslant C(T-t)^{\frac{\gamma}{2}\left(1-\frac{1}{k}\right)}, \forall t \in (0,T), T = e^{-s_0},$$

which concludes item (i).
- *The proof of item (ii) in Theorem 1:* Since $(q,b)(s) \in V_{A,\gamma,,b_0,\theta_0}(s)$ for all $s \geqslant s_0$, we derive from item (i) in Proposition 4.7

$$|b'(\tau)| \leqslant C e^{-\gamma\tau\left(1-\frac{1}{k}\right)} \tag{4.30}$$

that the integral

$$\int_{s_0}^{\infty} b'(\tau) d\tau \text{ converges.}$$

Let $b^* = b(s_0) + \int_{s_0}^{\infty} b'(\tau) d\tau$, then we have

$$b(s) \to b^* \text{ as } s \to +\infty.$$

In particular, we again use (4.30) that

$$|b(s) - b_*| \leqslant C e^{-s\gamma\left(1-\frac{1}{k}\right)}.$$

By using that fact that $t = T - e^{-s}$, we obtain

$$|b(t) - b^*| \leqslant C(T-t)^{\gamma\left(1-\frac{1}{k}\right)}, \forall t \in [0,T),$$

which completely concludes item (ii). Finally, we get the conclusion of Theorem 1. ∎

Additionally, we end this part by completing the proof of Corollary 2.

Proof of Corollary 2. Let us introduce the function $F(a) = \left(p-1+(ab(t)+(1-a)b^*)\, y^{2k}\right)^{-\frac{1+i\delta}{p-1}}$, where $a \in [0,1]$. We can easily derive

$$|F'(a)| \leqslant C(b_0)\,|b(t)-b^*| \leqslant C(T-t)^{\gamma\left(1-\frac{1}{k}\right)},$$

then, we obtain

$$|f_{b(t)} - f_{b^*}| = \left|\int_0^1 F'(a) da\right| \leqslant C(T-t)^{\gamma\left(1-\frac{1}{k}\right)},$$

which concludes the proof of Corollary 2. ∎

5. A priori estimates

In this section, we aim to give the complete proof to Proposition 4.7. We divide the section into three parts:

- Subsection 5.1: we project the terms in the equation (2.9) on $\{\hat{H}_n, \check{H}_n\}$ for all $n \in \{0, .., [M]\}$ and derive a *priori* estimates for $\tilde{q}_n$ and q_n.
- Subsection 5.2: we provide the estimation to the infinite parts i.e. P_- (defined in (3.21)) of the terms in equation (2.9).
- Subsection 5.3: we use the established estimates in Parts 1 and 2 and we derive the conclusion of Proposition 4.7.

5.1. The finite dimensional part q_+

In this part we give projection of equation (2.9) on the eigenfunctions of the operator $\mathcal{L}_{\delta,s}$. More precisely, we will find the main contribution in the projections $\hat{P}_{n,M}$ and $\check{P}_{n,M}$ of the eight terms appearing in equation (2.9): $\partial_s q$, $\mathcal{L}_{\delta,s}q$, Vq, $B(q)$, $T(q)$, $N(q)$, $D_s(\nabla q)$ and R_s.

Let $A \geqslant 1, b_0 > 0, \theta_0 > 0, \gamma \in (0, \frac{1}{2})$ and $s_0 \geqslant 1$ and we also assume that $(q, b, \theta)(s) \in V_{A,\gamma,b_0,\theta_0}(s)$ for all $s \in [s_0, \bar{s}]$ for some $\bar{s} > s_0$. Then, the following results hold true.

+ First term: $\partial_s q$.

Lemma 5.1. *For all $0 \leqslant n \leqslant M$, we have*

$$\begin{aligned} \hat{P}_{n,M}(\partial_s q) &= \partial_s \hat{q}_n + (1 - \tfrac{1}{k})(n+1)(n+2)I^{-2}(s)\hat{q}_{n+2}, \\ \check{P}_{n,M}(\partial_s q) &= \partial_s \check{q}_n + (1 - \tfrac{1}{k})(n+1)(n+2)I^{-2}(s)\check{q}_{n+2}. \end{aligned} \tag{5.1}$$

Proof. The result follows by Definition 3.1 and the identities in [9, Lema 5.1]. ∎

+ Second term: $\mathcal{L}_{\delta,s}q$.

Lemma 5.2. *For all $0 \leqslant n \leqslant M$, we have*

$$\begin{aligned} \hat{P}_n(\mathcal{L}_{\delta,s}q) &= (1 - \frac{n}{2k})\hat{q}_n + (1 - \frac{1}{k})(n+1)(n+2)I^{-2}(s)\hat{q}_{n+2}, \\ \check{P}_n(\mathcal{L}_{\delta,s}q) &= -\frac{n}{2k}\check{q}_n + (1 - \frac{1}{k})(n+1)(n+2)I^{-2}(s)\check{q}_{n+2}. \end{aligned} \tag{5.2}$$

Proof. It follows by Definition 3.1 and [9, Lemma 5.2] ∎

+ Third term $B(q) = \dfrac{b'(s)}{p-1} y^{2k}(1 + i\delta + (p + i\delta)e_b q)$.

Lemma 5.3. *There exists $s_1(A) \geqslant 1$, such that for all $s \geqslant s_1$, we have*

a) For $0 \leqslant n \leqslant [M]$, $n \neq 2k$,

$$\left|\hat{P}_n(B(q))\right| \leqslant C|b'(s)|I^{-\gamma}(s),$$

b) For $0 \leqslant n \leqslant [M]$,

$$\left|\check{P}_n(B(q))\right| \leqslant Cb'(s)I^{-\gamma}(s),$$

c) For $n = 2k$,

$$\left|\hat{P}_{2k}(B(q)) - \frac{1}{p-1}b'(s)\right| \leqslant C|b'(s)|I^{-\gamma}(s).$$

Proof. First, we use the orthogonality of $\{H_n, n \geqslant 1\}$ to derive

$$\int y^{2k} H_n(s)\rho_s dy = \begin{cases} 0 & \text{if} \quad n > 2k, \\ \|H_{2k}(s)\|^2_{L^2_{\rho_s}} + O(I^{-2k-2}) & \text{if} \quad n = 2k, \\ O(I^{-2j-2}) & \text{if} \quad n < 2k. \end{cases} \tag{5.3}$$

Thanks to (3.6), (3.19) and Definition 3.1, the estimates in the lemma immediatly follow by

$$\left| \int y^{2k} H_n(s) q \rho_s dy \right| \leqslant C I^{-\gamma-2n}(s), \text{ for all } n \leqslant [M]. \tag{5.4}$$

To proof (5.4), we are based on (3.18) that we obtain

$$\begin{aligned} \int y^{2k} e_b q H_n \rho_s dy &= \sum_{j \leqslant [M]} Q_j(q) \int y^{2k} H_n(s) H_j(s) \rho_s dy + \int y^{2k} e_b q_- H_n \rho_s dy \\ &= Q_n(s) \|H_n\|^2_{L^2_{\rho_s}} + \int y^{2k} e_b q_- H_n \rho_s dy \end{aligned}$$

Accordingly Definition 3.1, it follows that

$$|Q_n(s)| \, \|H_n(s)\|^2_{L^2_{\rho_s}} \leqslant C I^{-2n-\gamma}(s).$$

Additionally, we apply (4.7) to estimate

$$\left| \int y^{2k} e_b(y) H_n(s) q_- \rho_s dy \right| \leqslant CAI^{-\gamma}(s) \int |y|^{2k} (I^{-M}(s) + |y|^M)(I^{-n} + |y|^n) \rho_s(y) dy.$$

Now, we aim to prove that for all $n \leqslant [M]$

$$\left| \int |y|^{2k} (I^{-M}(s) + |y|^M)(I^{-n}(s) + |y|^n) \rho_s(y) dy \right| \leqslant C I^{-2k-M-n}(s). \tag{5.5}$$

By changing variable $z = I(s)y$ and ρ_s's definition in (3.1), we get

$$\left| \int |y|^{2k} (I^{-M}(s) + |y|^M)(I^{-n}(s) + |y|^n) \rho_s(y) dy \right| \leqslant C I^{-2k-M-n}(s) \int |z|^{2k} (1+|z|)^{M+n} e^{-\frac{|z|^2}{4}} dz,$$

which concludes (5.5). In particular, it also follows

$$\left| \int y^{2k} e_b(y) H_n(s) q_- \rho_s dy \right| \leqslant CAI^{-\gamma-2k-2n}(s) \leqslant CI^{-\gamma-2n}(s),$$

provided that $s \geqslant s_1(A,k)$. Thus, (5.4) holds true. Finally, we finish the proof of the lemma. ∎

+ Fourth term: $T(q) = -i\theta'(s)(e_b^{-1} + q) = -i\theta'(s)(p - 1 + by^{2k} + q)$**.**

Lemma 5.4 (Projection of $T(q)$ on $\tilde{H}_n$ and $\hat{H}_n$.)**.** *For $0 \leqslant n \leqslant [M]$, the projection on $\tilde{H}_n$ is given by*

$$\hat{P}_n(T(q)) \quad = \quad \begin{cases} \theta' \check{q}_{2k} & \text{if } n = 2k, \\ -\theta' \left((1+\delta^2)\hat{q}_n - \delta \check{q}_n \right) & \text{else.} \end{cases}$$

and the projection on $\check{H}_n$ is given by

$$\check{P}_n(T(q)) \quad = \quad \begin{cases} -\theta' \left((p-1) + (1+\delta^2)\hat{q}_0 \right) & \text{if } n = 0, \\ -\theta' \left(b - \delta \check{q}_{2k} \right) & \text{if } n = 2k, \\ -\theta' \left((1+\delta^2)\hat{q}_n - \delta \check{q}_n \right) & \text{else.} \end{cases}$$

Proof. First, we apply (3.25) and (3.31) express as follows

$$\hat{P}_n(T(q)) = -\theta'(s)\left(\hat{P}_n(i(p-1)) + b\hat{P}_n(iy^{2k}) + \hat{P}_n(iq)\right),$$
$$\check{P}_n(T(q)) = -\theta'(s)\left(\check{P}_n(i(p-1)) + b\check{P}_n(iy^{2k}) + \check{P}_n(iq)\right),$$

the first two term are easy to derive. To obtain the projection $\check{P}_n(iq)$ and $\hat{P}_n(iq)$ we can first write

$$\begin{aligned} i\hat{H}_n &= \delta\hat{H}_n - \check{H}_n, \\ i\check{H}_n &= (1+\delta^2)\hat{H}_n - \delta\check{H}_n. \end{aligned}$$

and it follows then

$$\begin{aligned} \hat{P}_n(iq) &= \delta\hat{q}_n - \check{q}_n, \\ \check{P}_n(iq) &= (1+\delta^2)\hat{q}_n - \delta\check{q}_n. \end{aligned}$$

Using the modulation condition (4.1) we get

$$\begin{aligned} \hat{P}_n(T) &= -\theta'\left(\hat{P}_n(i(p-1)) + b\hat{P}_n(iy^{2k}) + \hat{P}_n(iq)\right), \\ &= \begin{cases} \theta'\check{q}_{2k} & \text{if } n = 2k, \\ -\theta'\left((1+\delta^2)\hat{q}_n - \delta\check{q}_n\right) & \text{else.} \end{cases} \end{aligned}$$

$$\begin{aligned} \check{P}_n(T) &= -\theta'\left(\check{P}_n(i(p-1)) + b\check{P}_n(iy^{2k}) + \check{P}_n(iq)\right), \\ &= \begin{cases} -\theta'\left((p-1) + (1+\delta^2)\hat{q}_0\right) & \text{if } n = 0, \\ -\theta'\left(b - \delta\check{q}_{2k}\right) & \text{if } n = 2k, \\ -\theta'\left((1+\delta^2)\hat{q}_n - \delta\check{q}_n\right) & \text{else.} \end{cases} \end{aligned}$$

Finally, we finish the proof of the lemma. ∎

+ Fifth term: $N(q) = (1+i\delta)\left(|1+e_bq|^{p-1}(1+e_bq) - 1 - 2e_b\Re q - \frac{p-1}{2}e_bq - \frac{p-3}{2}e_b\bar{q}\right)$.

Lemma 5.5. *Let $s \geqslant 1$, $L \geqslant 1, L \in \mathbb{N}$. Then, the term $N = N(q)$ defined in above, satisfies*

$$\sup_{|y|<1}\left|N - \sum_{l=0}^{L}\sum_{\substack{0\leqslant j,k\leqslant L \\ 2\leqslant j+k\leqslant L}} B_{j,k}(y)q^j\bar{q}^k + \tilde{B}_{j,k}(y)q^j\bar{q}^k\right| \leqslant C\left(|q|^{L+1}\right),$$

where $B^l_{j,k}$ is an even polynomial of degree less or equal to K and the rest $\tilde{B}^l_{j,k}$ satisfies

$$\forall |y|<1\, |\tilde{B}_{j,k}| \leqslant C(1+|y|^{2kL}).$$

Moreover

$$\forall |y|<1\, |B_{j,k} + \tilde{B}_{j,k}| \leqslant C.$$

On the other hand, in the region $|y| \geqslant 1$, we have

$$|N(q)| \leqslant C\,|e_bq|^{\bar{p}}, \quad \text{or } C|1+e_bq|^p$$

for some constant C and $\bar{p} = \min(p,2)$.

Proof. We notice that in the region $|y| \leqslant 1$ and for $s \geqslant C$ where C is a fixed constant, e_b is bounded from above and from below. Using a Taylor expansion in terms of $e_b q$ and $e_b \bar{q}$, we see that N can be written as

$$\forall s \geqslant 1 \text{ and } |y| < 1, \left| N - \sum_{\substack{0 \leqslant j,m \leqslant K \\ 2 \leqslant j+m \leqslant K}} c_j e_b^{j+m} q^j \bar{q}^m \right| \leqslant C|e_b q|^{K+1}.$$

Using (3.33) and the fact that

$$e_b(y,s) = \sum_{l=0}^{L} E_l b^l y^{2kl}$$

we can expand $(e_b)^{j+m}$ in terms of y

$$\left| e_b^{j+m} - \sum_{i=0}^{L} A_{j,m}^i |y|^{2ki} \right| \leqslant C|y|^{2k(L+1)}, \tag{5.6}$$

where $A_{j,m}^i$ and $\tilde{A}_{j,m}^i$ depends on b. By the definition of the set we can easily see that $|A_{j,m}^i| + |\tilde{A}_{j,m}^i| \lesssim C$.

If we introduce $B_{j,m} = \sum_{=0}^{L} A_{j,m}^i |y|^{2ki}$ and $\tilde{B}_{j,m} = e_b^{j+m} - \sum_{i=0}^{L} A_{j,m}^i |y|^{2ki}$ Then we get from the above computation

$$\sup_{|y|<1} \left| N - \sum_{l=0}^{L} \sum_{\substack{0 \leqslant j,k \leqslant K \\ 2 \leqslant j+k \leqslant K}} B_{j,k}(y) q^j \bar{q}^k + \tilde{B}_{j,k}(y) q^j \bar{q}^k \right| \leqslant C\left(|q|^{K+1} + 1\right),$$

where $B_{j,k}^l$ is an even polynomial of degree less or equal to K and the rest $\tilde{B}_{j,k}^l$ satisfies

$$\forall |y| < 1 \, |\tilde{B}_{j,k}| \leqslant C(1 + |y|^{2kL}).$$

Moreover

$$\forall |y| < 1 \, |B_{j,k} + \tilde{B}_{j,k}| \leqslant C.$$

Hence the Lemma is proved. ∎

Using Lemma 5.5, we have the following lemma.

Lemma 5.6. *There exits $s_3(A, b_0, \theta_0) \geqslant 1$ such that for all $s \in [s_0, \bar{s}]$, for some $\bar{s} > s_0$, and $0 \leqslant n \leqslant [M]$, it holds that*

$$\left| \hat{P}_n(N(q)) \right| + \left| \check{P}_n(N(q)) \right| \leqslant CA^2 I^{-2\gamma}(s). \tag{5.7}$$

Proof. It is enough to prove (5.7) for the projection on H_n, it implies the same for $\hat{P}_n(N)$ and $\check{P}_n(N)$. We write

$$\int H_n N \rho_s(y) dy = \int_{|y|<1} H_n N \rho_s(y) dy + \int_{|y|>1} H_n N \rho_s(y) dy.$$

Using Lemma 5.5, we deduce that

$$\left|\int_{|y|<1} H_n N\rho_s(y)N - \int_{|y|<1} H_n\rho_s(y)\sum_{l=0}^{L}\sum_{\substack{0\leqslant j,k\leqslant L\\ 2\leqslant j+k\leqslant L}} B_{j,k}(y)q^j\bar{q}^k + \tilde{B}_{j,k}(y)q^j\bar{q}^k\right| \leqslant C\int_{|y|<1}|H_n||e_bq|^{L+1}\rho_s. \tag{5.8}$$

Let us write

$$q^j = \left(\sum_{l=0}^{[M]}\hat{q}_l\hat{H}_l + \check{q}_l\hat{H}_l + q_-\right)^j,$$

Using the fact that $\|q\|_{L^\infty}\leqslant 1$ (which holds for large s from the definition of the shrinking set $V_{A,\gamma,b_0,\theta_0}(s)$) , then we have

$$\left|q^j - q_+^j\right| \leqslant C\left(|q_-|^j + |q_-|\right).$$

Using the fact that $q(s)\in V_{A,\gamma,b_0,\theta_0}(s)$, we have $|\hat{q}_-| + |\check{q}_-|\leqslant CAI^{-\gamma}(I^{-M}+|y|^M)$, we obtain

$$\left|q^j - q_+^j\right| \leqslant CAI^{-\gamma}(I^{-M}+|y|^M),$$

and

$$\left|\bar{q}^k - \bar{q}_+^k\right| \leqslant CAI^{-\gamma}(I^{-M}+|y|^M),$$

thus give us the desired estimation in (5.7) for the second integral in (5.8).
Let us focus on the L.H.S of inequality (5.8), we note that H_n satisfies for all $|y|<1$, $|H_n(y,s)|\leqslant C(1+|y|^n)$, we have also by the definition of the shrinking set

$$|e_bq(y,s)|^{L+1}\leqslant CI^{-\gamma(L+1)},$$

it follows that

$$\int_{|y|<1}|H_n||e_bq|^{L+1}\rho_s \leqslant CI^{-\gamma(L+1)}\int_{|y|<1}(1+|y|^n)\rho_s dy \leqslant CI^{-\gamma(L+1)-2n},$$

which gives the good estimation $|\int_{|y|<1}H_nN\rho_s|\leqslant CI^{-2\gamma-2n}$.
To end the proof we estimate $\int_{|y|>1}H_nN\rho_s dy$. By Lemma 5.5, the definition of $V_{A,\gamma,b_0,\theta_0}(s)$ and using Lemma A.2 from [9], we get

$$\left|\int_{|y|>1}H_nN\rho_s\right| \leqslant C\left|\int_{|y|>1}H_n(1+I^{-p-\delta}(1+|y|^{Mp}))\rho_s\right| \leqslant Ce^{-\frac{I}{8}}.$$

Thus end the proof of the Lemma. ∎

+ Sixth term: $D_s(q) = -\dfrac{p+i\delta}{p-1}4kby^{2k-1}I^{-2}(s)e_b\nabla q$**.**

Lemma 5.7. *For $\gamma\in(0,\frac{1}{2})$, there exists $s_4(A,b_0,\theta_0,\gamma)\geqslant 1$ such that for all $s\in[s_0,\bar{s}]$ for some $\bar{s}\geqslant s_0$, and $0\leqslant n\leqslant[M]$, it holds that*

$$|\hat{P}_n(D_s(q))|\leqslant CI^{-2\gamma}(s) \text{ and } |\check{P}_n(D_s(q))|\leqslant CI^{-2\gamma}(s)$$

Proof. By (3.19) and Definition 3.1, it is sufficient to prove

$$|Q_n(q)(s)| \leqslant CI^{-2\gamma}(s) \text{ for all } s \in [s_0, \bar{s}],$$

provided that $s \geqslant s_4(A, b_0, \theta_0, \gamma)$. Additionally, we reply on (3.36), we only need to verify

$$\left| \int D_s(q) H_n \rho_s dy \right| \leqslant CI^{-2\gamma}(s).$$

By $D_s(q)$'s definition, we can express

$$\int D_s H_n \rho_s dy = -\frac{p+i\delta}{p-1} 4bkI^{-2}(s) \int y^{2k-1} e_b \nabla q H_n \rho_s dy.$$

So, it follows by integration by parts which is quite the same as [9, Lemma 5.5]. We kindly refer the reader to check the details. Finally, we finish the proof of the lemma. ∎

+ Seventh term: $R_s(q) = I^{-2}(s) y^{2k-2} \left(\alpha_1 + \alpha_2 y^{2k} e_b + \left(\alpha_3 + \alpha_4 y^{2k} e_b\right) q\right)$**.**

Lemma 5.8. *For $\gamma \in (0, \frac{1}{2})$, there exists $s_5(A, b_0, \theta_0, \gamma) \geqslant 1$ such that for all $s_0 \geqslant s_5, s \in [s_0, \bar{s}]$, and $n \leqslant [M]$*

$$|\hat{P}_n(R_s(q))| \leqslant CI^{-2\gamma}(s) \text{ and } |\check{P}_n(R_s(q))| \leqslant CI^{-2\gamma}(s).$$

Proof. We refer to the proof of Lemma 5.3. ∎

Eighth term: $V(q) = ((p-1)e_b - 1)\left[(1+i\delta)\Re q - q\right]$

Lemma 5.9. *For $\gamma \in (0, \frac{1}{2})$, there exists $s_6(A, b_0, \theta_0, \gamma) \geqslant 1$ such that for all $s_0 \geqslant s_6, s \in [s_0, \bar{s}]$, we have*

$$\hat{P}_n(V(q)) = 0 \text{ and } |\check{P}_n(V(q))| \leqslant CI^{-2\gamma}(s).$$

Proof. First, we reply on the fact that $q = (1+i\delta)\hat{q} + i\check{q})$

$$V(q) \quad = \quad -i\left(1 - e_b(p-1)\right)\left(\Im q - \delta \Re q\right) = i\left((p-1)e_b - 1\right)\check{q}.$$

Thanks to Definition 3.1, it immediately follows that

$$\forall n \in \mathbb{N}, \quad \hat{P}_n(V) = 0.$$

We write, for $L \in \mathbb{N}$,

$$e_b(y,s) = (p-1)^{-1}\left[\sum_{j=0}^{L}\left(-\frac{b}{p-1}y^{2k}\right)^j + \left(-\frac{b}{p-1}y^{2k}\right)^{L+1} e_b(y,s)\right], \tag{5.9}$$

then we deduce that

$$(p-1)e_b - 1 = -by^{2k}e_b = \sum_{j=1}^{L}\left(-\frac{b}{p-1}y^{2k}\right)^j + \left(-\frac{b}{p-1}y^{2k}\right)^{L+1} e_b(y,s)$$

Let us first estimate $\int y^{2kj}(\delta\Re q - \Im q) H_n \rho_s dy$, we write

$$\int y^{2kj}(\delta\Re q - \Im q) H_n \rho_s dy = \int_{|y|\leqslant 1} y^{2kj}(\delta\Re q - \Im q) H_n \rho_s dy + \int_{|y|\geqslant 1} y^{2kj}(\delta\Re q - \Im q) H_n \rho_s dy.$$

We recall that for all $q \in V_{A,\gamma,b_0,\theta_0}$, by Claim 4.2

$$|V(q)| \leqslant CI^{-\gamma}(1 + |y|^M),$$

then by Lemma A.1 from [DNZ22], we obtain

$$\left|\int_{|y|\geqslant 1} y^{2kj}(\delta\Re q-\Im q)H_n\rho_s dy\right|\leqslant CI^{-\gamma}e^{-\frac{1}{8}I(s)}\leqslant CI^{-2\gamma-2n}(s),\forall s\in[s_0,s^*]. \tag{5.10}$$

Using (3.33), we write

$$\int_{|y|\leqslant 1} y^{2kj}(\delta\Re q-\Im q)H_n\rho_s dy=\int_{|y|\leqslant 1} y^{2kj}(\delta\Re q_+-\Im q_+)H_n\rho_s dy+\int_{|y|\leqslant 1} y^{2kj}(\delta\Re q_--\Im q_-)H_n\rho_s dy \tag{5.11}$$

Using the bound on q_- given by (4.3), the second integral can be bounded as follows

$$\left|\int_{|y|\leqslant 1} y^{2kj}(\delta\Re q_--\Im q_-)H_n\rho_s dy\right|\leqslant C\int_{|y|\leqslant 1} y^{2kj}(I^{-M}(s)+|y|^M)H_n\rho_s dy \tag{5.12}$$

If we introduce the change of variabe $z=yI(s)$ and $\rho(z)dz=\rho_s(y)dy$, we can write

$$\begin{aligned}\int_{|y|\leqslant 1} y^{2kj}(I^{-M}(s)+|y|^M)H_n\rho_s dy &= I^{-M-n-2kj}(s)\int_{|z|\leqslant I(s)}|z|^{2kj}(1+|z|^M)|h_n(z)|e^{-\frac{|z|^2}{4}}dz.\\ &\leqslant CI^{-M-n-2k}(s).\end{aligned}$$

we conclude that there exist γ_6, such that for all $0<\gamma\leqslant\gamma_6$, we have

$$\left|\int_{|y|\leqslant 1} y^{2kj}(\delta\Re q_--\Im q_-)H_n\rho_s dy\right| \leqslant CI^{-2\gamma-2n}(s)\ . \tag{5.13}$$

Let us focus on the first integral of equation (5.11), using estimation given by (4.2), we obtain

$$\int_{|y|\leqslant 1} y^{2kj}(\delta\Re q_+-\Im q_+)H_n\rho_s dy = \delta\Re\left(\int_{|y|\leqslant 1} y^{2kj}q_+H_n\rho_s dy\right)-\Im\left(\int_{|y|\leqslant 1} y^{2kj}q_+H_n\rho_s dy\right)\ .$$

we will just give the estimate on $\displaystyle\int_{|y|\leqslant 1} y^{2kj}q_+\rho_s H_n dy$:

$$\begin{aligned}\left|\int_{|y|\leqslant 1} y^{2kj}q_+H_n\rho_s dy\right| &= \left|\sum_{l=0}^{[M]}Q_l\int_{|y|\leqslant 1} y^{2kj}H_lH_n\rho_s(y)dy\right|\\ &\begin{cases}= & 0 \text{ if } 2kj+l<n\\ \leqslant & CI^{-2\gamma}(s)I^{-2kj-l-n}(s)\leqslant CI^{-2\gamma-2n}(s) \text{ if } 2kj+l\geqslant n\end{cases}\end{aligned} \tag{5.14}$$

By (5.10), (5.13) and (5.14), we conclude that

$$|\check{P}_n(V)|\leqslant CI^{-2\gamma}(s).$$

∎

5.2. Estimates of the infinite dimensional parts of the terms in equation (2.9)

Let (q,θ,b) be a solution to the problem (2.9) & (4.1) on $[s_0,\bar{s}]$ for some $\bar{s}>0$. Under the assumption $(q,\theta,b)(s)\in V_{A,\gamma,b_0,\theta_0}(s)$ for $s\in[s_0,\bar{s}]$. Then, we aim to provide the following results.

+ First term: $P_-\left(\partial_s q\right)$ where $P_-(\cdot)$ defined as in (3.21).

Lemma 5.10. *For all $s\in[s_0,\bar{s}]$, we have*

$$P_-\left(\partial_s q\right)=\partial_s q_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\left(\hat{q}_{n+2}\hat{H}_n+\check{q}_{n+2}\check{H}_n\right), \tag{5.15}$$

where $I(s)$ given as in (2.4). Consequently,

$$\mathscr{Q}_{\Re,\delta}\left(P_-(\partial_s q)\right) = \partial_s \hat{q}_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\hat{q}_{n+2}H_n(s),$$

$$\mathscr{Q}_{\Im,\delta}\left(P_-(\partial_s q)\right) = \partial_s \check{q}_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\check{q}_{n+2}H_n(s).$$

Proof. First, we based on (3.24), (3.15) and (3.32) that it is sufficient to prove (5.15). From definition (3.21), we can express

$$\begin{aligned} P_-(\partial_s q) &= \partial_s q - \sum_{n=0}^{[M]}\left(\hat{P}_n(\partial_s q)\hat{H}_n + \check{P}_n(\partial_s q)\check{H}_n\right) \\ &= \partial_s q_- + \sum_{n=0}^{[M]}\partial_s\left(\hat{q}_n\hat{H}_n + \check{q}_n\check{H}_n\right) - \sum_{n=0}^{[M]}\left(\hat{P}_n(\partial_s q)\hat{H}_n + \check{P}_n(\partial_s q)\check{H}_n\right). \end{aligned}$$

Using the fact that $\partial_s H_n \equiv 0$ if $n = 0$ or 1, and for all $n \geqslant 2$

$$\partial_s H_n(y,s) = n(n-1)\left(1-\frac{1}{k}\right)I^{-2}(s)f_{n-2}(y,s).$$

Hence, we deduce from Lemma 5.1 that

$$\begin{aligned} \hat{P}_n\left(\partial_s q\right) &= \partial_s \hat{q}_n + (1-\tfrac{1}{k})(n+1)(n+2)I^{-2}(s)\hat{q}_{n+2}, \\ \check{P}_n\left(\partial_s q\right) &= \partial_s \check{q}_n + (1-\tfrac{1}{k})(n+1)(n+2)I^{-2}(s)\check{q}_{n+2}, \end{aligned}$$

which implies

$$P_-(\partial_s q) = \partial_s q_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\left(\hat{q}_{n+2}\hat{H}_n + \check{q}_{n+2}\check{H}_n\right),$$

Finally, (5.15) follows and we conclude the proof of the lemma. ∎

+ Second term $\mathcal{L}_{\delta,s}q$.

Lemma 5.11. *For all $s \in [s_0, \bar{s}]$, it holds that*

$$P_-\left(\mathcal{L}_{\delta,s}q\right) = \mathcal{L}_{\delta,s}q_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\left(\hat{q}_{n+2}\hat{H}_n + \check{q}_{n+2}\check{H}_n\right). \tag{5.16}$$

Consequently, we have

$$\mathscr{Q}_{\Re,\delta}\left(P_-(\mathcal{L}_{\delta,s}q)\right) = \mathcal{L}_s\hat{q}_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\hat{q}_{n+2}H_n(s),$$

$$\mathscr{Q}_{\Im,\delta}\left(P_-(\mathcal{L}_{\delta,s}q)\right) = \mathcal{L}_{0,s}\check{q}_- - I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\check{q}_{n+2}H_n(s),$$

where $\mathcal{L}_{0,s}$ and $\mathcal{L}_s$ defined as in (3.7) and (3.8), respectively.

Proof. First, we are based on (3.24) to have

$$\mathscr{Q}_{\Re,\delta}\left(\mathscr{L}_{\delta,s}q_-\right)=\mathcal{L}_s\hat{q}_- \text{ and } \mathscr{Q}_{\Im,\delta}\left(\mathscr{L}_{\delta,s}q_-\right)=\mathcal{L}_{0,s}\check{q}_-.$$

So, we need only to prove (5.16) which is quite similar to the result in Lemma 5.10. Indeed, by using (3.21), we firstly obtain

$$P_-(\mathcal{L}_{\delta,s})-\mathcal{L}_{\delta,s}q_-=\left(\sum_{n=0}^{[M]}\left(\hat{P}_n(\mathcal{L}_{\delta,s}q)\hat{H}_n-\hat{q}_n\mathcal{L}_{\delta,s}\hat{H}_n\right)+\sum_{n=0}^{[M]}\left(\check{P}_n(\mathcal{L}_{\delta,s}q)\check{H}_n-\check{P}_n(\mathcal{L}_{\delta,s}q)\check{H}_n\right)\right).$$

Combining identities in (3.16) and (3.17) with Lemma 5.2, we conclude

$$\sum_{n=0}^{[M]}\left(\hat{P}_n(\mathcal{L}_{\delta,s}q)\hat{H}_n-\hat{q}_n\mathcal{L}_{\delta,s}\hat{H}_n\right)=-I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2)\hat{q}_{n+2}\hat{H}_n$$

$$\sum_{n=0}^{[M]}\left(\check{P}_n(\mathcal{L}_{\delta,s}q)\check{H}_n-\check{P}_n(\mathcal{L}_{\delta,s}q)\check{H}_n\right)=-I^{-2}(s)\left(1-\frac{1}{k}\right)\sum_{n=[M]-1}^{[M]}(n+1)(n+2),\check{q}_{n+2}\check{H}_n.$$

Finally, the conclusion of (5.16) follow by adding all above related terms, and we finish the proof of the lemma. ∎

+ Third term: $V(q)=((p-1)e_b-1)\left((1+i\delta)\Re q-q\right)$.

Lemma 5.12. *There exists $s_7(A,b_0,\theta_0)\geqslant 1$ such that for all $s_0\geqslant s_7$, it holds that for all $s\in[s_0,\bar{s}]$,*

$$\mathscr{Q}_{\Re,\delta}\left(P_-(V(q))\right)=0, \tag{5.17}$$

and

$$\left|\mathscr{Q}_{\Im,\delta}\left(P_-(V(q))\right)-(1-(p-1)e_b)\,\check{q}_-\right|\leqslant CI^{-\gamma}(s)\left(I^{-M}(s)+|y|^M\right). \tag{5.18}$$

Proof. First, we immediately conclude (5.17) by item (ii) in Lemma 3.2. It remains to prove (5.18). Using item (ii) in that Lemma again, we have

$$V(q)=(1-(p-1)e_b)i\check{q}=(1-(p-1)e_b)i(\check{q}_++\check{q}_-).$$

Regarding to (3.21), we have

$$\begin{aligned}
P_-(V(q))-(1-(p-1)e_b)i\check{q}_-&=(1-(p-1)e_b)i\check{q}_++(1-(p-1)e_b)i\check{q}_-\\
&-\sum_{n\leqslant[M]}Q_n((1-(p-1)e_b)i\check{q}_+)H_n-\sum_{n\leqslant[M]}Q_n((1-(p-1)e_b)i\check{q}_-)H_n\\
&=(1-(p-1)e_b)i\check{q}_+-\sum_{n\leqslant[M]}Q_n((1-(p-1)e_b)i\check{q}_+)H_n\\
&-\sum_{n\leqslant[M]}Q_n((1-(p-1)e_b)i\check{q}_-)H_n=i\left[(I)-(II)\right],
\end{aligned}$$

where

$$\begin{aligned}
(I)&=(1-(p-1)e_b)\check{q}_+-\sum_{n\leqslant[M]}Q_n((1-(p-1)e_b)\check{q}_+)H_n,\\
(II)&=\sum_{n\leqslant[M]}Q_n((1-(p-1)e_b)\check{q}_-)H_n.
\end{aligned}$$

- *Estimate for (I):* First, we have

$$(1-(p-1)e_b)\check{q}_+(s)=\sum_{n\leqslant[M]}\check{q}_n(1-(p-1)e_b)H_n(s)$$

For each $n\leqslant[M]$, we chose $L_n=\left[\frac{M-n}{2k}\right]$, then we apply (5.9) to derive that

$$(1-(p-1)e_b)\check{q}_+(s)=\sum_{n\leqslant[M]}\sum_{j=1}^{L_n}c_{n,j,b}\check{q}_n(s)y^{2kj}H_n(s)+\sum_{n\leqslant[M]}\tilde{c}_{n,b}\frac{y^{2k(L_n+1)}}{e_b}H_n(s),$$

which yields

$$\begin{aligned}(I)&=\sum_{n\leqslant[M]}\tilde{c}_{n,b}\frac{y^{2k(L_n+1)}}{e_b}H_n(s)-\sum_{n\leqslant[M]}Q_n\left(\sum_{m\leqslant[M]}\tilde{c}_{m,b}\frac{y^{2k(L_m+1)}}{e_b}H_m(s)\right)H_n(s)\\&=I_1-\sum_{n\leqslant[M]}Q_n(I_1)H_n(s),\ \text{where } I_1=\sum_{n\leqslant[M]}\tilde{c}_{n,b}\frac{y^{2k(L_n+1)}}{e_b}H_n(s).\end{aligned}$$

We aim to proof that

$$|(I)|\leqslant C(I^{-M}(s)+|y|^M),\tag{5.19}$$

it is sufficient to prove

$$|I_1|\leqslant C(I^{-M}(s)+|y|^M).\tag{5.20}$$

Indeed, for each $n\leqslant[M]$, we express as follows

$$\left|\frac{y^{2k(L_n+1)}}{e_b}H_n(s)\right|\leqslant\mathbb{1}_{\{|y|\leqslant1\}}\left|\frac{y^{2k(L_n+1)}}{e_b}H_n(s)\right|+\mathbb{1}_{\{|y|\geqslant1\}}\left|\frac{y^{2k(L_n+1)}}{e_b}H_n(s)\right|$$

Since $2k(L_n+1)\geqslant M-n$, we estimate

$$\mathbb{1}_{\{|y|\leqslant1\}}\left|\frac{|y|^{2k(L_n+1)}}{e_b}H_n(s)\right|\leqslant Cy^{M-n}(I^{-n}(s)+|y|^n)\leqslant C(I^{-M}(s)+|y|^M).$$

Beside that, it also holds true

$$\mathbb{1}_{\{|y|\geqslant1\}}\left|\frac{y^{2k(L_n+1)}}{e_b}H_n(s)\right|\leqslant C\frac{|y|^{2k\left(1+L_n-\frac{M-n}{2k}\right)}}{e_b}|y|^{M-n}\left(I^{-n}(s)+|y|^n\right)\leqslant C(I^{-M}(s)+|y|^M),$$

since $1+L_n-\frac{M-n}{2k}\in[0,1]$ for all $n\leqslant[M]$.
Thus, taking the sum over n to the concerning bounds, we get the conclusion of (5.19).

- *Estimate for (II):* We now notice that

$$|1-(p-1)e_b|=\left|\frac{by^{2k}}{p-1+by^{2k}}\right|\leqslant Cy^{2k}.\tag{5.21}$$

By the definition of Q_n in (3.19), we can bound as follows

$$|Q_n((1-(p-1)e_b)\check{q}_-)|=CAI^{-\gamma}(s)\int_{\mathbb{R}}y^{2k}|H_n(y,s)|(I^{-M}(s)+|y|^M)\rho_s(y)dy.$$

Since $|H_n|\leqslant C(I^{-n}(s)+|y|^n)$ and by changing variable $z=I(s)y$, we estimate the integral as follows

$$\begin{aligned}|Q_n((1-(p-1)e_b)\check{q}_-)|&=CAI^{2n-\gamma-n-M-2k}(s)\int_{\mathbb{R}}|z|^{2k}|h_n(z)|(1+|z|^M)e^{-\frac{|z|^2}{4}}dz\\&\leqslant CAI^{n-M-\gamma-2k}(s).\end{aligned}\tag{5.22}$$

Consequently,

$$|(Q_n((1-(p-1)e_b)\check{q}_-)H_n(y,s)|\leqslant CI^{-\gamma-2k+n-M}\left(I^{-n}+|y|^n\right)\leqslant CI^{-\gamma-2k}(s)\left(I^{-M}+|y|^M\right),$$

which yields

$$|(II)| \leqslant CI^{-\gamma-2k}(s)\left(I^{-M}(s)+|y|^M\right). \tag{5.23}$$

Combining (5.19) with (5.23), we conclude

$$|\mathscr{Q}_{\Im,\delta}\left(P_-(V(q))\right)-(1-(p-1)e_b)\check{q}_-| \leqslant CI^{-\gamma}(s)\left(I^{-M}(s)+|y|^M\right),$$

which concludes (5.18). Finally, the conclusion of the lemma follows. ∎

+ Fourth term $B(q)=\dfrac{b'(s)}{p-1}y^{2k}\left(1+i\delta+(p+i\delta)e_bq\right)$.

Lemma 5.13. *For all $s\in[s_0,\bar{s}]$, it holds that*

$$|P_-(B(q))(s)| \leqslant CA|b'(s)|I^{-\gamma}(s)(I^{-M}(s)+|y|^M).$$

Consequently,

$$|\mathscr{Q}_{\Re,\delta}\left(P_-(B(q))(s)\right)|+|\mathscr{Q}_{\Im,\delta}\left(P_-(B(q))(s)\right)| \leqslant CA|b'(s)|I^{-\gamma}(s)(I^{-M}(s)+|y|^M).$$

Proof. From definition of P_- as in (3.21), it immediately follows that

$$P_-\left((1+i\delta)y^{2k}\right)\equiv 0.$$

So, we obtain

$$P_-(B(q))=\frac{b'(s)}{p-1}P_-((p+i\delta)y^{2k}e_bq).$$

Accordingly Definition 3.1, it is sufficient to check that

$$\left|P_-((p+i\delta)y^{2k}e_bq)\right| \leqslant CAI^{-\gamma}(s)\left(I^{-M}(s)+|y|^M\right). \tag{5.24}$$

First, let $\chi\in C_c^\infty(\mathbb{R})$ satisfying

$$\chi(x)=1 \text{ for all } |x|\leqslant\frac{1}{2} \text{ and } \chi(x)=0 \text{ for all } |x|\geqslant 1. \tag{5.25}$$

Then, we decompose $1=\chi+(1-\chi)=\chi+\chi^c$ and we reply on (3.21) and (3.23) that

$$P_-\left((p+i\delta)y^{2k}e_bq\right)=P_-((p+i\delta)y^{2k}\chi e_bq_+)+P_-((p+i\delta)y^{2k}\chi^c e_bq_+)+P_-((p+i\delta)y^{2k}e_bq_-).$$

Since $q_-=(1+i\delta)\hat{q}_-+i\check{q}_-$ and the fact that $(q,b,\theta)(s)\in V_{A,\gamma,b_0,\theta_0}(s)$, we can bound as follows

$$|q_-| \leqslant CAI^{-\gamma}(s)(I^M(s)+|y|^M).$$

Hence, we argue in a similar fashion as in the proof of (5.23) to estimate

$$\left|Q_n((p+i\delta)y^{2k}e_bq_-)H_n\right| \leqslant CI^{-\gamma-2k}(s)(I^{-M}+|y|^M),$$

the, using (3.21), we conclude that

$$\left|P_-((p+i\delta)y^{2k}e_bq_-)\right| \leqslant CAI^{-\gamma}(s)(I^{-M}(s)+|y|^M). \tag{5.26}$$

Additionally, we use the fact that $(q,b,\theta)(s)\in V_{A,\gamma,b_0,\theta_0}(s)$ again that

$$|\chi^cq_+| \leqslant CI^{-\gamma}(s)\sum_{n=0}^{[M]}(I^{-n}(s)+|y|^n) \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M), \text{ since } |y|\geqslant\frac{1}{2}.$$

Similarly (5.23), we have

$$\left|Q_n(y^{2k}e_b\chi^cq_+)H_n\right| \leqslant CI^{-\gamma-2k}(s)(I^{-M}(s)+|y|^M).$$

Accordingly (3.21) and the fact $\|y^{2k}e_b\|_{L^\infty} \leqslant 1$, we conclude that

$$\left|P_-((p+i\delta)y^{2k}\chi^c e_b q_+)\right| \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M). \tag{5.27}$$

Let $|y| \leqslant \frac{1}{2}$ and $L \in \mathbb{N}, L \geqslant 1$, we deduce from (3.33) and the identity (5.9) that

$$\begin{aligned} y^{2k}e_b q_+ &= \sum_{\substack{j\leqslant L\\ n\leqslant[M]}} c_j(b)y^{2kj}(\hat{q}_n\widehat{H}_n + \check{q}_n\check{H}_n) + \tilde{Y} \\ &= \sum_{\substack{j\leqslant L\\ n\leqslant[M]\\ 2kj+n\leqslant[M]}} c_j(b)y^{2kj}(\hat{q}_n\widehat{H}_n + \check{q}_n\check{H}_n) + \sum_{\substack{j\leqslant L\\ n\leqslant[M]\\ 2kj+n\geqslant[M]+1}} c_j(b)y^{2kj}(\hat{q}_n\widehat{H}_n + \check{q}_n\check{H}_n) + \tilde{Y} \\ &= Y_1 + Y_2 + \tilde{Y}, \end{aligned}$$

where $\tilde{Y}$ satisfies

$$\left|\tilde{Y}\right| \leqslant CI^{-\gamma}(s)|y|^{(L+1)}, \text{ for all } |y| \leqslant 1.$$

Since $\chi = 1-\chi^c$ and $P_-(Y_1) = 0$, we have then

$$P_-(\chi Y_1) = P_-(Y_1) - P_-(\chi^c Y_1) = -P_-(\chi^c Y_1).$$

In the same way for (5.27), we obtain

$$|P_-(\chi^c Y_1)| \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M),$$

which yields

$$|P_-(\chi Y_1)| \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M).$$

Now, by changing variable $z = yI(s)$, we can prove that

$$\left|\int_{|y|\geqslant\frac{1}{2}} fH_n\rho_s dy\right| \leqslant C(K,n)\|f\|_{L^\infty_N}e^{-\frac{I(s)}{8}} \text{ with } s \geqslant 1 \text{ and for some } n \in \mathbb{N}, \text{ and } K > 0. \tag{5.28}$$

where $\|\cdot\|_{L^\infty_K}$ is similarly defined in (3.35). By applying (5.28), we have

$$|Q_n(\chi Y_2)| \leqslant CI^{-\gamma-([M]+1)-n}(s), \forall n \leqslant [M],$$

since the indices in Y_2 always satisfy that $2kj+n \leqslant [M]+1$. Hence, we arrive at

$$|Q_n(\chi Y_2)H_n| \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M), \forall n \leqslant [M],$$

In the other hand, we have

$$|\chi Y_2| \leqslant CI^{-\gamma}(s)\left(\sum_{\substack{j\leqslant L\\ n\leqslant[M]\\ 2kj+n\geqslant[M]+1}} |y|^{2kj}\left(I^{-n}(s)+|y|^n\right)\right) \leqslant CI^{-\gamma}(I^{-M}(s)+|y|^M).$$

So, we have

$$|P_-(\chi Y_2)| \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M),$$

which concludes

$$\left|P_-((p+i\delta)y^{2k}\chi e_b q_+)\right| \leqslant CI^{-\gamma}(s)(I^{-M}(s)+|y|^M). \tag{5.29}$$

Thus, (5.24) follows by (5.26), (5.27) and (5.29). Finally, we finish the proof of the lemma. ∎

+ Fifth term $T(q) = -i\theta'(s)(e_b^{-1}+q) = -i\theta'(s)(p-1+by^{2k}+q)$.

Lemma 5.14. *For all $s \in [s_0, \bar{s}]$, it holds that*

$$P_-(T(q)) = -i\theta'(s)q_-(s).$$

Consequently,

$$\mathscr{Q}_{\Re,\delta}\left(P_-(T(q))\right) = -\delta\hat{q}_- - \check{q}_- \text{ and } \mathscr{Q}_{\Im,\delta}\left(P_-(T(q))\right) = \delta(\delta\hat{q}_- + \check{q}_-).$$

Proof. Using the definition of P_- given in (3.21), we have

$$P_-(T(q)) = -\theta'(s)P_-(iq) = -i\theta'(s)q_- = -i\theta'(s)q_-(s).$$

In addition to that, we use the definition in (3.24), we have

$$iq_- = i\left\{(1+i\delta)\hat{q}_- + i\check{q}_-\right\} = (1+i\delta)\left[-\delta\hat{q}_- - \check{q}_-\right] + i\delta(\delta\hat{q}_- + \check{q}_-)$$

which yields the complete conclusion of the lemma. ∎

+ Sixth term: $N(q) = (1+i\delta)\left(|1+e_bq|^{p-1}(1+e_bq) - 1 - 2e_b\Re q - \frac{p-1}{2}e_bq - \frac{p-3}{2}e_b\bar{q}\right)$. We have the following result.

Lemma 5.15. *There exists $s_{12}(A) \geqslant 1$ such that for all $s_0 \geqslant s_{12}$, and for all $s \in [s_0, \bar{s}]$ it holds that*

$$|P_-\left(N(q)(s)\right)| \leqslant CA^{\max(p,2)}I^{-\min(p,2)\gamma}(s)(I^{-M}(s) + |y|^M).$$

Proof. Let χ defined as in (5.25), and we decompose $N = N(q)$ as follows

$$N = \chi N + (1-\chi)N = \chi N + \chi^c N.$$

It suffices to verify the following:

$$|P_-(\chi^c N)(s)| \leqslant C(A^{\max(p,2)})I^{-\min(p,2)\gamma}(s)\left(I^{-M}(s) + |y|^M\right), \tag{5.30}$$

$$|P_-(\chi N)(s)| \leqslant CA^2I^{-2\gamma}(s)\left(I^{-M}(s) + |y|^M\right), \tag{5.31}$$

provided that $s \geqslant s_0$ with $s_0 \geqslant s_{12}(A, M)$
- *For (5.30):* First, let us prove that

$$|\chi^c\mathcal{N}(q)| \leqslant CA^{\max(2,p)}I^{-\min(2,p)\gamma}(s)(I^{-M}(s) + |y|^M), p > 1. \tag{5.32}$$

The proof of (5.32) is divided into two cases where $p \geqslant 2$ and $p \in (1,2)$.

+ Case 1: $p \geqslant 2$. By a simple expansion, we estimate

$$|\chi^c\mathcal{N}(q)| \leqslant C\chi^c(|e_bq|^2 + |e_bq|^p)$$

Since $\text{supp}(\chi^c) \subset \{|y| \geqslant \frac{1}{2}\}$, the estimate in (4.9) implies

$$|\chi^c e_bq| \leqslant CAI^{-\gamma}(s)|y|^{M-2k}.$$

Notice that $M = \frac{2kp}{p-1}$, then we get

$$|\chi^c e_bq|^p \leqslant C\left(AI^{-\gamma}(s)e_b(y)|y|^M\right)^p = C\left(I^{-\gamma}(s)(e_b(y)|y|^{2k})|y|^{\frac{2k}{p-1}}\right)^p \tag{5.33}$$

$$\leqslant CI^{-p\gamma}(s)|y|^{\frac{2kp}{p-1}} \leqslant CI^{-p\gamma}(s)|y|^{\frac{2kp}{p-1}} \leqslant CI^{-p\gamma}(s)(I^{-M}(s) + |y|^M),$$

Similarly,

$$|\chi^c e_bq|^2 \leqslant CA^2I^{-2\gamma}(s)|y|^{2(M-2k)}\chi^c \leqslant CA^2I^{-2\gamma}(s)|y|^{\frac{2k}{p-1}2}\chi^c$$

$$\leqslant CA^2I^{-2\gamma}(s)|y|^{\frac{2k}{p-1}p}\chi^c \leqslant CA^2I^{-2\gamma}(s)(I^{-M}(s) + |y|^M) \text{ since } p \geqslant 2.$$

Hence, (5.32) holds true for all $p \geqslant 2$.

- *Case 2 i.e.* $p \in (1,2)$*:* we observe that

$$|\chi^c \mathcal{N}(q)| \leqslant C\chi^c |e_b q|^p.$$

By the same way of (5.33), we deduce that

$$|\chi^c \mathcal{N}(q)| \leqslant CA^p I^{-p\gamma}(s)(I^{-M}(s) + |y|^M).$$

So, (5.32) also holds true for the case $p \in (1,2)$.

Now, we use (5.32) to establish for all $n \leqslant [M]$

$$\begin{aligned}|Q_n(\chi^c N(q)(s))| &\leqslant CA^{\max(p,2)} I^{-\min(p,2)\gamma}(s) \int_{|y|\geqslant \frac{1}{2}} (I^{-M}(s) + |y|^M)|H_n(y,s)|\rho_s(y)dy \\ &\leqslant CA^{\max(p,2)} I^{-\min(p,2)\gamma}(s) e^{-\frac{I(s)}{16}},\end{aligned}$$

which yields

$$\begin{aligned}\sum_{n\leqslant[M]} |Q_n(\chi^c N(s))|\,|H_n(y,s)| &\leqslant CA^{\max(p,2)} I^{-\min(p,2)\gamma}(s) e^{-\frac{I(s)}{16}} \sum_{n\leqslant[M]} (1+|y|^n) \\ &\leqslant CI^{-\min(p,2)\gamma}(s)(I^{-M}(s) + |y|^M),\end{aligned}$$

provided that $s \geqslant s_{12,1}(A,M)$.
Consequently,

$$\begin{aligned}|P_-(\chi^c N(s))| &\leqslant |\chi^c N(s)| + \sum_{n\leqslant[M]} |Q_n(\chi^c N(s))|\,|H_n(y,s)| \\ &\leqslant CA^{\max(2,p)} I^{-\min(p,2)}(s)(I^{-M}(s) + |y|^M),\end{aligned}$$

which concludes (5.30).

- *For* (5.31). Since $\mathrm{supp}(\chi) \subset \{|y| \leqslant 1\}$, so it is suffices to consider $|y| \leqslant 1$ and we have

$$|e_b(y)q(y,s)| \leqslant CAI^{-\gamma}(s) \quad \forall |y| \leqslant 1.$$

Therefore, we use a simple Taylor expansion to obtain the following for some $K \in \mathbb{N}, K \geqslant 1$

$$\chi N = \chi\left(N_{K,1} + N_{K,2} + \tilde{N}_K\right),$$

where

$$\begin{aligned}N_{K,1} &= \sum_{\substack{0\leqslant j,\ell\leqslant K \\ 2\leqslant j+\ell\leqslant K}} a_{K,j,\ell}(e_b)^{j+\ell} q_+^j \bar{q}_+^\ell, \quad a_{K,j,\ell} \in \mathbb{R}, \\ N_{K,2} &= \sum_{j=2}^{K} \sum_{\substack{0\leqslant \ell_1+\ell_2\leqslant j-1 \\ \ell_1\geqslant 0,\ell_2\geqslant 0}} \sum_{\substack{\ell_3+\ell_4=j-(\ell_1+\ell_2) \\ \ell_3\geqslant 0,\ell_4\geqslant 0}} d_{2,K,j,\ell_1\ell_2,\ell_3,\ell_4}(e_b)^j q_+^{\ell_1}\bar{q}_+^{\ell_2} q_-^{\ell_3}\bar{q}_-^{\ell_4},\end{aligned}$$

where $d_{2,K,j,\ell_1\ell_2,\ell_3,\ell_4} \in \mathbb{R}$ and $\tilde{N}_K$ satisfies

$$|\chi\tilde{N}_K| \leqslant C|\chi e_b q|^{K+1} \leqslant CA^{K+1} I^{-(K+1)\gamma}(s).$$

By an analogue to (5.30), it leads to

$$|P_-(\chi\tilde{N})| \leqslant CI^{-2\gamma}(s)(I^{-M}(s) + |y|^M),$$

provided that $K \geqslant K_{12,2}(A,M)$. From (3.29), we have the following decomposition

$$\begin{aligned} N_{K,1} &= \sum_{\substack{0\leqslant|\mathbf{n}|\leqslant K\\ 0\leqslant|\mathbf{m}|\leqslant K\\ 2\leqslant|\mathbf{n}|+|\mathbf{m}|\leqslant K\\ 0\leqslant\ell\leqslant K}} c_{\mathbf{n},\mathbf{m},\ell,K} b^\ell y^{2k\ell} \Pi_{j=1}^{[M]} q_j^{n_j} \bar{q}_j^{m_j} H_j^{n_j+m_j} + N_{K,1,2} \\ &:= N_{K,1,1} + N_{K,1,2}, \text{ respectively,} \end{aligned}$$

where $\mathbf{n} = (n_1, ...n_{[M]})$ and $\mathbf{m} = (m_1, ..., m_{[M]})$, $|n| = \sum n_i$ and $|m| = \sum m_i$.
Then $N_{K,1,2}$ satisfies

$$|\chi N_{K,1,2}| \leqslant CA^2 I^{-2\gamma}(s)|y|^{2k(K+1)}, \text{ povided that } s \geqslant s_{12,3}(A).$$

By the same way to (5.22), we get the following bound

$$|Q_n(\chi N_{K,1,2})| \leqslant CA^2 I^{-2\gamma+n-M-2k(K+1)}(s).$$

By repeating a similar process as for (5.30), we obtain

$$|P_-(\chi N_{K,1,2})| \leqslant CA^2 I^{-2\gamma}(s)(I^{-M}(s) + |y|^M),$$

provided that $K \geqslant K_{12,3}(M)$(fixied at the end of the proof).
For $N_{K,1,1}$, we decompose as follows

$$\begin{aligned} N_{K,1,1} = &\sum_{\substack{0\leqslant|\mathbf{n}|\leqslant K\\ 0\leqslant|\mathbf{m}|\leqslant K\\ 2\leqslant|\mathbf{n}|+|\mathbf{m}|\leqslant K\\ 0\leqslant\ell\leqslant K\\ \sum_{j=1}^{[M]} j(n_j+m_j)+2k\ell\leqslant[M]}} c_{\mathbf{n},\mathbf{m},\ell,K} b^\ell y^{2k\ell} \Pi_{j=1}^{[M]} q_j^{n_j} \bar{q}_j^{m_j} H_j^{n_j+m_j} \\ + &\sum_{\substack{0\leqslant|\mathbf{n}|\leqslant K\\ 0\leqslant|\mathbf{m}|\leqslant K\\ 2\leqslant|\mathbf{n}|+|\mathbf{m}|\leqslant K\\ 0\leqslant\ell\leqslant K\\ \sum_{j=1}^{[M]} j(n_j+m_j)+2k\ell\geqslant[M]+1}} c_{\mathbf{n},\mathbf{m},\ell,K} b^\ell y^{2k\ell} \Pi_{j=1}^{[M]} q_j^{n_j} \bar{q}_j^{m_j} H_j^{n_j+m_j} \\ = &N_{K,1,1,1} + N_{K,1,1,2}, \text{ respectively.} \end{aligned}$$

Since $N_{K,1,1,1}$ is a polynomial in y of degree less or equal than $[M]$, it follows that

$$P_-(\chi N_{K,1,1,1}) = -P((1-\chi)N_{K,1,1,1}) = -P_-(\chi^c N_{K,1,1,1}).$$

In a similar way in (5.30), we have

$$|P_-(\chi^c N_{K,1,1,1})| \leqslant |\chi^c N_{K,1,1,1}| + CA^2 I^{-2\gamma}(s) e^{-\frac{I(s)}{16}} \sum_{n\leqslant[M]} (1+|y|^n) \leqslant CA^2 I^{-2\gamma}(s)(I^{-M} + |y|^M),$$

provide that $s \geqslant s_{12,4}(A,M)$.
Estimate for $N_{K,1,1,2}$, we firstly have the fact that

$$\begin{aligned} |\chi N_{K,1,1,2}| \leqslant CA^2 I^{-2\gamma}(s) &\sum_{\substack{0\leqslant|\mathbf{n}|\leqslant K\\ 0\leqslant|\mathbf{m}|\leqslant K\\ 2\leqslant|\mathbf{n}|+|\mathbf{m}|\leqslant K\\ 0\leqslant\ell\leqslant K\\ \sum_{j=1}^{[M]} j(n_j+m_j)+2k\ell\geqslant[M]+1}} |\chi(y)||y|^{2k\ell} \left(I^{-\sum_{j=0}^{[M]} j(n_j+m_j)}(s) + |y|^{\sum_{j=0}^{[M]} j(n_j+m_j)} \right) \\ \leqslant CA^2 I^{-2\gamma}(s)(I^{-M}(s) + |y|^M), \text{ since } &\sum_{j=0}^{[M]} j(n_j+m_j) + 2k\ell \geqslant [M]+1 \text{ and } |y| \leqslant 1. \end{aligned}$$

Additionally, we have

$$\begin{aligned}
&|Q_n(\chi N_{K,1,1,2})|\\
&\leqslant CA^2I^{-2\gamma}(s) \sum_{\substack{0\leqslant|\mathbf{n}|\leqslant K\\ 0\leqslant|\mathbf{m}|\leqslant K\\ 2\leqslant|\mathbf{n}|+|\mathbf{m}|\leqslant K\\ 0\leqslant\ell\leqslant K\\ \sum_{j=1}^{[M]} j(n_j+m_j)+2k\ell\geqslant[M]+1}} \int_{\mathbb{R}} |y|^{2k\ell}\left(I^{-\sum_{j=0}^{[M]} j(n_j+m_j)}(s)+|y|^{\sum_{j=0}^{[M]} j(n_j+m_j)}\right)|H_n(y,s)|\rho_s(y)dy\\
&\leqslant CA^2I^{-2\gamma+n-[M]-1}(s),
\end{aligned}$$

where the last estimate is obtained by the same technique as in (5.22) and the fact that $\sum_{j=0}^{[M]} j(n_j+m_j)+2k\ell\geqslant[M]+1$.
Consequently,

$$\begin{aligned}
|P_-(\chi N_{K,1,1,2})| &\leqslant |\chi N_{K,1,1,2}|+\sum_{n\leqslant[M]}|Q_n(\chi N_{K,1,1,2})||H_n|\\
&\leqslant CA^2I^{-2\gamma}(s)(I^{-M}(s)+|y|^M)+CA^2I^{-2\gamma}(s)\sum_{n\leqslant[M]} I^{n-[M]-1}(s)(I^{-n}(s)+|y|^n)\\
&\leqslant CA^2I^{-2\gamma}(s)\left(I^{-M}(s)+|y|^M\right).
\end{aligned}$$

Combining the established estimates, we conclude that

$$\begin{aligned}
|P_-(\chi N)| &\leqslant |P_-(\chi N_{K,1})+|+|P_-(\chi N_{K,2})|+\left|P_-(\chi\tilde{N}_K)\right|\\
&\leqslant |P_-(\chi N_{K,1,1,1})+|+|P_-(\chi N_{K,1,1,2})+|+|P_-(\chi N_{K,1,2})+|+|P_-(\chi N_{K,2})|+\left|P_-(\chi\tilde{N}_K)\right|\\
&\leqslant CA^2I^{-2\gamma}(s)(I^{-M}(s)+|y|^M),
\end{aligned}$$

which concludes (5.31). Finally, we get the conclusion of the lemma. ∎

+ Seven term $D_s(q)=-\dfrac{p+i\delta}{p-1}4kby^{2k-1}I^{-2}(s)e_b\nabla q$.

Lemma 5.16. *For all $s\in[s_0,\bar{s}]$, and $s,\tau\in[s_0,\bar{s}], s>\tau$, it holds that*

$$\begin{aligned}
\left|\mathcal{K}_{0,s,\tau}(P_-(\widetilde{D(q)})(\tau))\right|_s &\leqslant Ce^{-\frac{p}{p-1}(s-\tau)}\left(1+\frac{1}{\sqrt{s-\tau}}\right)AI^{-1-\gamma}(\tau),\\
\left|\mathcal{K}_{s,\tau}(P_-(\widehat{D(q)})(\tau))\right|_s &\leqslant C^{-\frac{s-\tau}{p-1}}\left(1+\frac{1}{\sqrt{s-\tau}}\right)AI^{-1-\gamma}(\tau),
\end{aligned}$$

where $\widetilde{D(q)}=\mathscr{Q}_{\Im,\delta}(D_\tau(q))$ and $\widehat{D_\tau(q)}=\mathscr{Q}_{\Re,\delta}(D_\tau(q))$ with $\mathscr{Q}_{\Im,\delta}$ and $\mathscr{Q}_{\Re,\delta}$ defined in (3.24).

Proof. First, we observe that the proof of the two estimates are the same. So, it is sufficient to give the proof to the first one. According to the definition of $\mathscr{Q}_{\Im,\delta}$, we can write

$$\mathcal{K}_{0,s,\tau}(\widetilde{D_\tau(q)})=\mathscr{Q}_{\Im,\delta}\left(\mathcal{K}_{0,s,\tau}\mathcal{D}_\tau(q)\right).$$

By the same argument of [9, Lemma 5.13] in combining with Lemma 6.1, we obtain the following estimate

$$|\mathcal{K}_{0,s,\tau}(\mathcal{D}_\tau(q))|_s\leqslant CI^{-1-\gamma}(\tau)\left(1+\frac{1}{\sqrt{s-\tau}}\right),$$

which yields

$$|\mathscr{Q}_{\Im,\delta}(\mathcal{D}_\tau(q))|_s \leqslant CI^{-1-\gamma}(\tau)\left(1+\frac{1}{\sqrt{s-\tau}}\right).$$

Thus, the first estimate in the lemma follows. Finally, we conclude the proof of the lemma. ∎

+ Eighth term $R_s(q) = y^{2k-2}I^{-2}(s)e_b\left(\alpha_1+\alpha_2 y^{2k}e_b+\left(\alpha_3+\alpha_4 y^{2k}e_b\right)q\right)$**.** We have the following result.

Lemma 5.17. *For all $s\in[s_0,\bar{s}]$, it hold that we then have*

$$|P_-\left(\mathcal{R}(q)\right)| \leqslant CI^{-2\gamma}\left(I^{-M}(s)+|y|^M\right).$$

Proof. The result is quite the same Lemma as 5.13. We kindly refer the reader to check the detail. ∎

5.3. Conclusion of the proof of Proposition 4.7.

We consider $(q,b)(s)\in V_{A,\gamma,b_0,\theta_0}(s), \forall s\in[s_0,\bar{s}]$. In addition, we let $\gamma\leqslant\gamma_3(b_0)$ and $s_0\geqslant s_3(\gamma,b_0)$ such that Lemmas 5.1-5.9 are valid.

- Proof of (i) of Proposition 4.7: First we prove the smallness of the modulation parameter θ given by (i) of Proposition 4.7. We project equation (2.9) on $\check{H}_0 = iH_0$, using the fact that $\check{q}_0 = 0$ and Lemma 5.4 we get

$$\partial_s\check{q}_0 = 0 = -\theta'(s)((p-1)+(1+\delta^2)\hat{q}_0)+\check{P}_0(T(q))+\check{P}_0(N(q))+\check{P}_0(D_s(\nabla q))+\check{P}_0(R_s(q)+V(q)),$$

then, we obtain by estimations given in Lemmas 5.3, 5.6, 5.7, 5.8 and 5.9

$$|\theta'(s)| \leqslant CI^{-2\gamma}(s).$$

- Proof of (ii) of Proposition 4.7: We project equation (2.9) on $\hat{H}_{2k}$ and take on consideration that $\hat{q}_{2k}\equiv 0$ and applying the results in Lemmas 5.1-5.9, we obtain

$$|b'(s)| \leqslant CI^{-2\gamma}(s) = Ce^{-\gamma\left(1-\frac{1}{k}\right)s}. \tag{5.34}$$

Besides that, we have $b(s_0)=b_0$, then we derive

$$|b(s)-b_0| \leqslant \int_{s_0}^{s}|b'(\tau)|d\tau \leqslant C\int_{s_0}^{s} I^{-2\gamma}(\tau)d\tau,$$

which implies

$$\frac{3}{4} \leqslant b(s) \leqslant \frac{5}{4}b_0, \forall s\in[s_0,\bar{s}],$$

provided that $s_0\geqslant s_3(\gamma,b_0)$ large enough. Thus, we get the conclusion of item (ii).

- Proof of (iii) of Proposition 4.7:

By Lemmas 5.1-5.9, (i) and (ii) of Proposition 4.7, we obtain for all $n\in\{0,..[M]\}$,

$$\begin{aligned}&\left|\partial_s\hat{q}_n-\left(1-\frac{n}{2k}\right)\hat{q}_n\right| \leqslant CI^{-2\gamma}(s), \forall s\in[s_0,\bar{s}],\\ &\left|\partial_s\check{q}_n+\frac{n}{2k}\check{q}_n\right| \leqslant CI^{-2\gamma}(s), \forall s\in[s_0,\bar{s}]\end{aligned}$$

which concludes item (iii) of Proposition 4.7.

- Proof of (iv) of Proposition 4.7 First, we reply on equation (2.9) and the decomposition in (3.31) to obtain the following system

$$\left\{\begin{array}{rcl}\partial_s\hat{q} &=& \mathcal{L}_s(\hat{q})+\mathscr{R}_1,\\ \partial_s\check{q} &=& \mathcal{L}_{0,s}\check{q}+\mathscr{V}\check{q}+\mathscr{R}_2,\end{array}\right.$$

where $\mathcal{L}_{0,s}$ and $\mathcal{L}_s$ respectively defined as in (3.7) and (3.8), and

$$\begin{aligned}\mathscr{V} &= 1-(p-1)e_b,\\ \mathscr{R}_1 &= \mathscr{Q}_{\Re,\delta}\left(b'(s)B(q)+i\theta'(s)T(q)+N(q)+\mathcal{D}_s(\nabla q)+\mathcal{R}_s(q)\right),\\ \mathscr{R}_2 &= \mathscr{Q}_{\Im,\delta}\left(b'(s)B(q)+i\theta'(s)T(q)+N(q)+\mathcal{D}_s(\nabla q)+\mathcal{R}_s(q)\right).\end{aligned}$$

Applying the infinite projection P_- defined as in (3.21), we get

$$\left\{\begin{array}{rcl}\partial_s\hat{q}_-(s) &=& \mathcal{L}_s\hat{q}_- + P_-\left(\mathscr{R}_1\right),\\ \partial_s\check{q}_-(s) &=& \mathcal{L}_{0,s}\check{q}_- + \mathscr{V}\check{q}_-\\ &&\quad + \left(P_-(\mathscr{V}\check{q}) - \mathscr{V}\check{q}_-\right) + P_-(\mathscr{R}_2).\end{array}\right.$$

In particular, we can write the above system in integral form as follows

$$\left\{\begin{array}{rcl}\hat{q}_-(s) &=& \mathcal{K}_{s,\sigma}\hat{q}_-(\sigma) + \int_\sigma^s \mathcal{K}_{s,\tau}\left(P_-\left(\mathscr{R}_1\right)(\tau)\right)d\tau,\\ \check{q}_-(s) &=& \mathcal{K}_{0,s,\sigma}\check{q}_-(\sigma) + \int_\sigma^s \mathcal{K}_{0,s,\tau}\left(\mathscr{V}\check{q}_-(\tau)\right)d\tau\\ &&\quad + \int_\sigma^s \mathcal{K}_{0,s,\tau}\left(P_-(\mathscr{V}\check{q}) - \mathscr{V}\check{q}_- + P_-\left(\mathscr{R}_2\right)(\tau)\right)d\tau.\end{array}\right.$$

Now, we claim the following

Claim 5.18. *Let $\bar{p}=\min(p,2)$, then it holds that*

$$\begin{aligned}\left|\mathcal{K}_{s,\tau}\left(P_-\left(\mathscr{R}_1\right)(\tau)\right)\right|_s &\leqslant Ce^{-\frac{s-\tau}{p-1}}I^{-\frac{\bar{p}+1}{2}\gamma}(\tau),\\ \left|\mathcal{K}_{0,s,\tau}\left(P_-(\mathscr{V}\check{q}) - \mathscr{V}\check{q}_- + P_-\left(\mathscr{R}_2\right)(\tau)\right)\right|_s &\leqslant Ce^{-\frac{p}{p-1}(s-\tau)}I^{-\gamma}(\tau),\end{aligned} \tag{5.35}$$

provided that $s_0 \geqslant s_{14}(A)$.

Proof. As the estimates involving to $\mathscr{R}_1$ and $\mathscr{R}_2$ are the same, we will just give the proof of the estimate involving to $\mathscr{R}_2$. Indeed, we use (5.18) to obtain

$$\left|\left(P_-(\mathscr{V}\check{q}) - \mathscr{V}\check{q}_-\right)(\tau)\right|_\tau \leqslant CI^{-\gamma}(\tau).$$

Additionally, the infinite projection P_- commutes with $\mathscr{Q}_{\Re,\delta}$ and $\mathscr{Q}_{\Im,\delta}$. Hence, we apply *a priori estimates* established in Lemmas 5.10 - 5.17 to obtain

$$\left|P_-\left(\mathscr{R}_2\right)(\tau)\right|_\tau \leqslant CI^{-\frac{\min(p,2)+1}{2}\gamma}(\tau),$$

provided that $\tau \geqslant \sigma \geqslant s_0 \geqslant s_{14,1}(A)$. Thus, we combine with the semigroup estimates in Lemma 6.1 to derive

$$\left|\mathcal{K}_{0,s,\tau}\left(P_-\left(\mathscr{R}_2\right)(\tau)\right)\right|_\tau \leqslant Ce^{-\frac{p}{p-1}(s-\tau)}I^{-\frac{\min(p,2)+1}{2}\gamma}(\tau)$$

provided that $\gamma \in \left(0,\frac{1}{2}\right)$ and we obtain (5.35). ∎

Now, let us give the proof of item (iv) of Proposition 4.7. Taking $|\cdot|_s$-norm defined in (3.37) and using Lemma 6.1, we obtain

$$\begin{aligned}|\hat{q}_-(s)|_s &\leqslant e^{-\frac{s-\sigma}{p-1}}|\hat{q}_-(\sigma)|_\sigma + \int_\sigma^s \left|\mathcal{K}_{s,\tau}(\mathscr{P}_1(\tau))\right|_s d\tau\\ &\leqslant e^{-\frac{s-\sigma}{p-1}}|\hat{q}_-(\sigma)|_\sigma + \int_\sigma^\tau e^{-\frac{s-\tau}{p-1}}I^{-\frac{\bar{p}+1}{2}\gamma}(\tau)d\tau,\\ |\check{q}_-(s)|_s &\leqslant e^{-\frac{p}{p-1}(s-\sigma)}|\check{q}_-(\sigma)|_\sigma + \int_\sigma^s \|\mathcal{V}\|_{L^\infty(\mathbb{R})}|\check{q}(\tau)|_\tau d\tau + \int_\sigma^s \left|\mathcal{K}_{s,\tau}(\mathscr{P}_2(\tau))\right|_s d\tau\\ &\leqslant e^{-\frac{p}{p-1}(s-\sigma)}|\check{q}_-(\sigma)|_\sigma + \int_\sigma^s |\check{q}_-(\tau)|_\tau d\tau + \int_\sigma^s e^{-\frac{p}{p-1}(s-\tau)}I^{-\gamma}(\tau)d\tau,\end{aligned}$$

since $\|\mathscr{V}\|_{L^\infty} \leqslant 1$. By using Grönwall's lemma, we get

$$|\hat{q}_-(s)|_s \leqslant e^{-\frac{s-\sigma}{p-1}}|\hat{q}_-(\sigma)|_\sigma + C\left(I^{-\frac{\bar{p}+1}{2}}(s) + e^{-\frac{s-\sigma}{p-1}} I^{-\frac{\bar{p}+1}{2}}(\sigma)\right),$$

and

$$|\check{q}_-(s)|_s \leqslant e^{-\frac{s-\sigma}{p-1}}|\check{q}_-(\sigma)|_\sigma + C\left(I^{-\gamma}(s) + e^{-\frac{s-\sigma}{p-1}} I^{-\gamma}(\sigma)\right).$$

which concludes the proof of item (iv) and also finish the proof of Proposition 4.7.

6. Spectral gap estimates on semigroups

In this section, we provide spectral gap estimates for semigroups $\mathcal{L}_{0,s}$ and $\mathcal{L}_s$. More precisely, the results read.

Lemma 6.1. *Let us consider $\mathcal{L}_{0,s}$ and $\mathcal{L}_s$ defined as in (3.7) and (3.8), and their semigroup be $\mathcal{K}_{0,\tau,\sigma}$ and $\mathcal{K}_{\tau,\sigma}$, respectively. It holds that*

$$|\mathcal{K}_{0,\tau,\sigma} q_-|_\tau \leqslant C e^{-\frac{p}{p-1}(\tau-\sigma)}|q_-|_\sigma, \tag{6.1}$$

$$|\mathcal{K}_{\tau,\sigma} q_-|_\tau \leqslant C e^{-\frac{1}{p-1}(\tau-\sigma)}\,|q_-|_\sigma\,, \tau \geqslant \sigma, \tag{6.2}$$

where $|\cdot|_\sigma$ defined as in (3.37).

Proof. The technique of the proof is based on [3]. First, we derive from (3.12) that

$$\mathcal{K}_{\tau,\sigma} = e^{\tau-\sigma}\mathcal{K}_{0,\tau,\sigma}.$$

Then, (6.2) is a direct consequence of (6.1). Indeed, let us assume that (6.1) holds. Hence, it follows that

$$|\mathcal{K}_{\tau,\sigma} q_-|_\tau = e^{\tau-\sigma}\,|\mathcal{K}_{0,\tau,\sigma} q_-|_\tau \leqslant C e^{\tau-\sigma} e^{-\frac{p}{p-1}(\tau-\sigma)}|q_-|_\sigma,$$

which yields (6.1). Now, it suffices to prove (6.1). Let us define

$$\Theta(z) = q_-\left(zL_\sigma^{-1}\right) \text{ and } \tilde{\Theta}(z) = \mathcal{K}_{0,\tau,\sigma} q_-(L_\tau^{-1} z). \tag{6.3}$$

Using (3.12) again, we obtain

$$\begin{aligned}
\tilde{\Theta}(z) = \mathcal{K}_{0,\tau,\sigma}(q_-)(L_\tau^{-1}z) &= \int_{\mathbb{R}} \mathcal{F}\left(e^{\frac{\tau-\sigma}{2k}} L_\tau^{-1} z - y'\right) q_-(y')dy' \\
&= \frac{1}{\sqrt{4\pi(1-e^{-(\tau-\sigma)})}} \int_{\mathbb{R}} \exp\left(-\frac{(ze^{-\frac{\tau-\sigma}{2}} - z')^2}{4(1-e^{-(\tau-\sigma)})}\right) q_-(I_\sigma^{-1} z')dz' \\
&= \int_{\mathbb{R}} e^{(\tau-\sigma)\mathcal{L}}(z,z')\Theta(z')dz',
\end{aligned}$$

where $e^{(\tau-\sigma)\mathcal{L}}(z,z')$ defined by

$$e^{(\tau-\sigma)\mathcal{L}}(z,z') = \frac{1}{\sqrt{4\pi(1-e^{-(\tau-\sigma)})}} \exp\left(-\frac{(ze^{-\frac{\tau-\sigma}{2}} - z')^2}{4(1-e^{-(\tau-\sigma)})}\right).$$

- *The case $\tau - \sigma \leqslant 1$:* From (3.37), we have

$$|\Theta(z)| \leqslant I^{-M}(\sigma)(1+|z|^M)|q_-|_\sigma.$$

Now, we apply the classical estimate in [2, Lemma 4, page 555] that we obtain

$$\left|\tilde{\Theta}(z)\right| \leqslant C I^{-M}(\sigma)(1+|z|^M)\,|q_-|_\sigma\,.$$

Returning the original variable $z = I_\tau y$, we obtain

$$|\mathcal{K}_{0,\tau,\sigma} q_-(y)| \leqslant C I^{-M}(\sigma) I^M(\tau)(I^{-M}(\tau) + |y|^M)|q_-|_\sigma,$$

which implies

$$|\mathcal{K}_{0,\tau,\sigma}q_-|_\tau \leqslant CI^{-M}(\sigma)I^M(\tau)|q_-|_\sigma \leqslant Ce^{-\frac{[M]+1}{2}(\tau-\sigma)}|q_-|_\sigma, \text{ since } \tau-\sigma\leqslant 1.$$

- *The case* $\tau-\sigma\geqslant 1$*:* We use the following decomposition

$$\tilde{\Theta}(z)=\int_{\mathbb{R}}\mathscr{N}(z,z')f(z')dz', \tag{6.4}$$

where

$$\mathscr{N}(z,z')=\frac{e^{\frac{(z')^2}{4}}}{\sqrt{4\pi(1-e^{-(\tau-\sigma)})}}\exp\left(-\frac{(ze^{-\frac{\tau-\sigma}{2}}-z')^2}{4(1-e^{-(\tau-\sigma)})}\right) \text{ and } f(z')=e^{-\frac{(z')^2}{4}}\Theta(z).$$

Since

$$\frac{(ze^{-\frac{\rho}{2}}-z')^2}{1-e^{-\rho}}-(z')^2=-z^2+\frac{(z-z'e^{-\frac{\rho}{2}})^2}{1-e^{-\rho}},$$

we can write

$$\mathscr{N}(z,z')=\frac{e^{\frac{z^2}{4}}}{\sqrt{4\pi(1-e^{-(\tau-\sigma)})}}\exp\left(-\frac{(z-z'e^{-\frac{\tau-\sigma}{2}})^2}{4(1-e^{-(\tau-\sigma)})}\right).$$

So, we have

$$\left|\partial_{z'}^n\mathscr{N}(z,z')\right|\leqslant Ce^{-\frac{n(\tau-\sigma)}{2}}(|z|+|z'|)^n e^{\frac{(z')^2}{4}}e^{(\sigma-\tau)\mathcal{L}}(z,z'), \text{ for all } n\geqslant 0, n\in\mathbb{N}. \tag{6.5}$$

Next, let us define

$$f^{(-m-1)}(z)=\int_{-\infty}^{z}f^{(-m)}(z')dz'.$$

From (6.3), we have

$$\int_{\mathbb{R}}(z')^m f(z')dz'=0 \text{ for all } m\in\{0,...,1\} \text{ and } |f(z)|\leqslant I_\sigma^{-M}(1+|z|^M)e^{-\frac{(z)^2}{2}}|q_-|_\sigma.$$

It is similar to [2, Lemma 6, page 557], we can estimate

$$\left|f^{(-m)}(z)\right|\leqslant Ce^{-\frac{(z')^2}{2}}I_\sigma^{-M}(1+|z|)^{M-m}, \text{ for all } m\leqslant[M]+1. \tag{6.6}$$

Now, by using integration by part in (6.4) then combining with estimates (6.5) and (6.6), we obtain

$$|\tilde{\Theta}(z)|=\left|\int_{\mathbb{R}}\partial_{z'}^{[M]+1}\mathscr{N}(z,z')f^{(-[M]+1)}(z')dz'\right|\leqslant Ce^{-\frac{[M]+1}{2}(\tau-\sigma)}I_\sigma^{-M}(1+|z|^M)|q_-|_\sigma.$$

Since (2.15) and the fact $M\leqslant[M]+1$, we have then

$$|\tilde{\Theta}(z)|\leqslant Ce^{\frac{M}{2}(\tau-\sigma)}I_\sigma^{-M}(1+|z|^M)\,|q_-|_\sigma=Ce^{-\frac{p}{p-1}(\tau-\sigma)}I_\tau^{-M}(1+|z|^M)|q_-|_\sigma,$$

which yields that

$$|\mathcal{K}_{0,\tau,\sigma}(q_-)(y)|_\tau\leqslant Ce^{-\frac{p}{p-1}(\tau-\sigma)}\,|q_-|_\sigma\,.$$

Thus, we get the conclusion of (6.1) for the case $\tau-\sigma\geqslant 1$. Finally, we finish the proof of the lemma. ∎

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REFERENCES

[1] I. S. Aranson and L. Kramer. The world of the complex ginzburg-landau equation. *Reviews of Modern Physics*, 74:99–143, 2002.

[2] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[3] J. Bricmont, A. Kupiainen, and G. Lin. Renormalization group and asymptotics of solutions of nonlinear parabolic equations. *Comm. Pure Appl. Math.*, 47(6):893–922, 1994.

[4] C. J. Budd, V. Rottschäfer, and J. F. Williams. Multibump, blow-up, self-similar solutions of the complex Ginzburg-Landau equation. *SIAM J. Appl. Dyn. Syst.*, 4(3):649–678, 2005.

[5] T. Cazenave. *Semilinear Schrödinger Equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[6] T. Cazenave, J. P. Dias, and M. Figueira. Finite-time blowup for a complex Ginzburg-Landau equation with linear driving. *J. Evol. Equ.*, 14(2):403–415, 2014.

[7] T. Cazenave, F. Dickstein, and F.B. Weissler. Finite-time blowup for a complex Ginzburg-Landau equation. *SIAM J. Math. Anal.*, 45(1):244–266, 2013.

[8] C. Conley. *Isolated invariant sets and the Morse index*, volume 38 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1978.

[9] G. K. Duong, N. Nouaili, and H. Zaag. Modulation theory for the flat blowup solutions of nonlinear heat equation. *Submitted*, 2022.

[10] G. K. Duong, N. Nouaili, and H. Zaag. Construction of blow-up solutions for the complex ginzburg-landau equation with critical parameters. *Mem. Amer. Math. Soc.*, 285, 2023.

[11] J. Ginibre and G. Velo. The cauchy problem in local spaces for the complex ginzburg-landau equation. differential equations, asymptotic analysis, and mathematical physics. *Differential Equations, Asymptotic Analysis, and Mathematical Physics*, 100:138–152, 1996.

[12] J. Ginibre and G. Velo. The cauchy problem in local spaces for the complex ginzburg-landau equation. ii. contraction methods. *Communications in Mathematical Physics*, 187(1):45–79, 1997.

[13] L. M. Hocking and K. Stewartson. On the nonlinear response of a marginally unstable plane parallel flow to a two-dimensional disturbance. *Proc. Roy. Soc. London Ser. A*, 326:289–313, 1972.

[14] L. M. Hocking, K. Stewartson, J. T. Stuart, and S. N. Brown. A nonlinear instability in plane parallel flow. *Journal of Fluid Mechanics*, 51:705–735, 1972.

[15] P. Kolodner, D. Bensimon, and M. Surko. Traveling wave convection in an annulus. *Physical Review Letters*, 60:1723–, 1988.

[16] Kuznetsov E.A Kaplan, E. and V. Steinberg. Burst and collapse in traveling-wave convection of a binary fluid. *Phys. Rev.*, 50:3712–3722, 1994.

[17] Kuznetsov E.A Kaplan, E. and V. Steinberg. Phase gradient mechanism of self-focusing and collapse in non-linear dispersive travelling waves. *Europhys. Lett.*, 28:237–243, 1994.

[18] P. Kolodner, S. Slimani, N. Aubry, and R. Lima. Characterization of dispersive chaos and related states of binary-fluid convection. *Physics D*, 85(1-2):165–224, 1995.

[19] Kuznetsov E.A. Rorr S. Kramer, L. and S.K. Turitsyn. Optical pulsecollapse in defocusing active medium. *JETP LETTERS*, 61:887–892, 1995.

[20] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[21] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008.

[22] A.C. Newell and J.A. Whitehead. Finite bandwidth, finite amplitude convection. *J. Fluid Mech.*, 38:279–303, 1969.

[23] N. Nouaili and H. Zaag. Construction of a blow-up solution for the complex ginzburg-landau equation in some critical case. *Arch. Rat. Mech. Anal*, 228(3), 2018.

[24] S. Popp, O. Stiller, E. Kuznetsov, and L. Kramer. The cubic complex ginzburg-landau equation for a backward bifurcation. *Physica D: Nonlinear Phenomena*, 114(1-2):81 – 107, 1998.

[25] S. Popp, O. Stiller, E. Kuznetsov, and L. Kramer. The cubic complex ginzburg-landau equation for a backward bifurcation. *Physics D*, 114(1-2):81–107, 1998.

[26] S. Popp, O. Stiller, E. Kuznetsov, and L. Kramer. The cubic complex Ginzburg-Landau equation for a backward bifurcation. *Phys. D*, 114(1-2):81–107, 1998.

[27] P. Plecháč and TITLE = On self-similar singular solutions of the complex Ginzburg-Landau equation JOURNAL = Comm. Pure Appl. Math. FJOURNAL = Communications on Pure and Applied Mathematics VOLUME = 54 YEAR = 2001 NUMBER = 10 PAGES = 1215–1242 ISSN = 0010-3640 MRCLASS = 35Q55 (35Q30) Šverák, V.
[28] P. Quittner and P. Souplet. *Superlinear parabolic problems.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.
[29] K. Stewartson and J. T. Stuart. A non-linear instability theory for a wave system in plane poiseuille flow. *Journal of Fluid Mechanics*, 48:529–545, 1971.
[30] S.K Turitsyn. Nonstable solitons and sharp criteria for wave collapse. *Phys Rev E*, 47(1):13–16, 1993.
[31] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998.