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A k -power-free morphism is a $(k + 1)$ -power-free morphism for any integer $k \geq 5$

Francis Wlazinski

September 5, 2023

Abstract

For any integer $k \geq 5$, we show that a morphism that preserves k -power-free words also preserves $(k + 1)$ -power-free words.

1 Introduction and preliminaries

The purpose of this article is to answer a question I was asked a few years ago. It will mainly interest those who asked me about it. But also maybe a few other people.

Let us recall some basic notions of Combinatorics of words.

1.1 Words

An *alphabet* A is a finite set of symbols called *letters*. Since an alphabet with one element is of limited interest to us, we always assume that the cardinality of alphabets is at least two. A *word* over A is a finite sequence of letters from A . The *empty word* ε is the empty sequence of letters. Equipped with the concatenation operation, the set A^* of words over A is a free monoid with ε as neutral element and A as set of generators.

Given a non-empty word $u = a_1 \dots a_n$, with $a_i \in A$ for every integer i from 1 to n , the *length* of u denoted by $|u|$ is the integer n , that is, the number of letters of u . By convention, we have $|\varepsilon| = 0$. The *mirror image* of u , denoted by \tilde{u} , is the word $a_n \dots a_2 a_1$.

A word u is a *factor* of a word v if there exist two (possibly empty) words p and s such that $v = pus$. We denote by $\text{Fcts}(v)$ the set of all factors of v . If $u \in \text{Fcts}(v)$, we also say that v *contains* the word u (as a factor). If $p = \varepsilon$, u is a *prefix* of v . If $s = \varepsilon$, u is a *suffix* of v . If $u \neq v$, u is a *proper factor* of v . If u , p , and s are non-empty words, u is an *internal factor* of v .

Two words u and v are *conjugated* if $u = t_1 t_2$ and $v = t_2 t_1$ for two (possibly empty) words t_1 and t_2 .

Let w be a non-empty word and let i, j be two integers such that $0 \leq i - 1 \leq j \leq |w|$. We denote by $w[i..j]$ the factor of w such that $|w[i..j]| = j - i + 1$ and $w = pw[i..j]s$ for two words s and p satisfying $|p| = i - 1$. Note that, when $j = i - 1$, we have $w[i..j] = \varepsilon$. When $i = j$, we also denote by $w[i]$ the factor $w[i..i]$, which is the i^{th} letter of w . In particular, $w[1]$ and $w[|w|]$ are respectively the first and the last letter of w .

Powers of a word are defined inductively by $u^0 = \varepsilon$, and for every integer $n \geq 1$, $u^n = uu^{n-1}$. Given an integer $k \geq 2$, since the case ε^k is of little interest, we call a k -power any word u^k with $u \neq \varepsilon$. Given an integer $k \geq 2$, a word is k -power-free if it does not contain any k -power as factor. A *primitive* word is a word that is not a k -power of another word whatever the integer $k \geq 2$. A (non-empty) k -power v^k is called *pure* if any proper factor of v^k is k -power-free. In particular, we say that v^k is a pure k -power of a word w if $v^k \in \text{Fcts}(w)$ and v^k is pure. Repeating the fact that a non-pure k -power contains a k -power, which is itself pure or not, we obtain that any k -power contains a pure k -power. Moreover, if v^k is a pure k -power then v is primitive but the converse does not hold.

Remark 1.1 *A word cannot start with two different pure k -powers.*

The following proposition gives the well-known solutions (see [3]) to two elementary equations on words and will be widely used in the following sections:

Proposition 1.2 *Let A be an alphabet and u, v, w three words over A .*

1. *If $vu = uw$ and $v \neq \varepsilon$, then there exist two words r and s over A , and an integer n such that $u = r(sr)^n$, $v = rs$ and $w = sr$.*
2. *If $vu = uv$, then there exist a word w over A , and two integers n and p such that $u = w^n$ and $v = w^p$.*

We also need a property on words that is an immediate consequence of Proposition 1.2(2).

Lemma 1.3 [1, 2] *If a non-empty word v is an internal factor of vv , i.e., if there exist two non-empty words x and y such that $vv = xvy$, then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$, and $v = t^{i+j}$.*

We also use a well-known result on combinatorics on words:

Proposition 1.4 (Fine and Wilf) [3, 4] *Let x and y be two words. If a power of x and a power of y have a common prefix of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then x and y are powers of the same word.*

As a consequence of Proposition 1.4, we get:

Corollary 1.5 (Keränen) [1] *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then there exist two words t_1 and t_2 such that x is a power of t_1t_2 and y is a power of t_2t_1 with t_1t_2 and t_2t_1 primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

Lemma 1.6 *Let $k \geq 3$ be an integer. Let v and \bar{v} be two conjugated words over A .*

If v^k is a pure k -power over A then the same holds for \bar{v}^k .

Proof.

Let us assume that $v = v_1v_2$ and $\bar{v} = v_2v_1$ for two words v_1 and v_2 over A . And, by contradiction, let us assume that \bar{v}^k is not a pure k -power, that is, $(v_2v_1)^k = \bar{v}^k = t_1t^kt_2$ with t and t_1t_2 non empty words.

If $|t_1| > |v_2|$ or if $|t_2| > |v_1|$ then t^k is a proper factor of $(v_1v_2)^{k-1}v_1$ or of $v_2(v_1v_2)^{k-1}$, that is, of v^k , i.e., v^k is not pure: a contradiction.

If $|t_1| \leq |v_2|$ and $|t_2| \leq |v_1|$ then t^k is a common factor of itself and v^{k+1} with $|v^{k-1}| \leq |t^k| < |v^k|$. It follows that $\frac{k-1}{k} \times |v| \leq |t| < |v|$. Thus $|t^k| \geq |t| + \frac{(k-1)^2}{k} \times |v| \geq |t| + |v|$. By Lemma 1.5, we get that v is not primitive, i.e., v^k is not pure: a final contradiction. \square

Lemma 1.7 *Let $k \geq 3$ be an integer, let α and β be two words, and let Y^k be a pure k -power over an alphabet B . For any pure k -power X^k over B and for any integer $0 \leq \ell \leq k-3$, if $X^k \in \text{Fcts}(\alpha Y^{k-\ell-1}\beta)$ and $X^k \notin \text{Fcts}(\alpha Y^{k-\ell}\beta)$, then $|X| > (k-\ell-2)|Y|$.*

Proof.

If $X^k \in \text{Fcts}(\alpha Y^{k-\ell-1}\beta)$, there exist two words γ and δ such that $\alpha Y^{k-\ell-1}\beta = \gamma X^k \delta$.

If $|\alpha| \leq |\gamma|$ then $X^k \in \text{Fcts}(Y^{k-\ell-1}\beta) \subset \text{Fcts}(\alpha Y^{k-\ell}\beta)$: a contradiction with the hypotheses. Therefore, we have $|\alpha| > |\gamma|$. In a same way, we get that $|\beta| > |\delta|$. It follows that $Y^{k-\ell-1}$ is an internal factor of X^k .

Having $|Y^{k-\ell-1}| \geq |X| + |Y|$ means that a power of X and a power of Y have a common factor of length at least $|X| + |Y|$. By Corollary 1.5, there exist two words X_1 and X_2 and two integers i and j such that $X = (X_1X_2)^i$ and $Y = (X_2X_1)^j$ with X_1X_2 and X_2X_1 primitive words. Since X^k and Y^k are pure k -powers, it implies that $i = j = 1$. We get that $\alpha(X_2X_1)^{k-\ell-1}\beta = \gamma(X_1X_2)^k\delta$ with $|\alpha| > |\gamma|$ and $|\beta| > |\delta|$. But X_2X_1 is not an internal factor of $(X_2X_1)^2$. Thus α ends with X_1 and β starts with X_2 . It follows that $(X_1X_2)^k = X^k \in \text{Fcts}(\alpha Y^{k-\ell}\beta)$: a contradiction with the hypotheses.

So, we necessarily have $|Y^{k-\ell-1}| < |X|$. \square

Even if we can work with the previous lemma, in fact, the different situations we will encounter are similar of the hypotheses of the following corollary. Let us recall that we denote $\prod_{j=1}^s w_j$ the concatenation of the s words w_1, w_2, \dots, w_s .

Lemma 1.8 *Let $k \geq 3$ and $s \geq 1$ be two integers, let $\alpha_0, \alpha_1, \dots, \alpha_s$ be $s+1$ words over an alphabet B , and let $Y_1^k, Y_2^k, \dots, Y_s^k$ be s pure k -powers over B of the same length $k \times L$. For any pure k -power X^k over B and for any integer $0 \leq \ell \leq k-3$, if $X^k \in \text{Fcts}(\alpha_0 \prod_{i=1}^s (Y_i^{k-\ell-1}\alpha_i))$ and $X^k \notin \text{Fcts}(\alpha_0 \prod_{i=1}^s (Y_i^{k-\ell}\alpha_i))$, then $|X| > (k-\ell-2) \times L$.*

Proof.

This proof is done by induction on s .

The property is obtain for $s = 1$ by Lemma 1.7. Let us assume that the property is satisfied for an integer s .

Let $\alpha_0, \alpha_1, \dots, \alpha_{s+1}$ be $s+2$ words over an alphabet B , and let $Y_1^k, Y_2^k, \dots, Y_{s+1}^k$ be $s+1$ pure k -powers over B of the same length $k \times L$. Let X^k be a pure k -power over B and let $0 \leq \ell \leq k - 3$ be an integer such that $X^k \in \text{Fcts} \left(\alpha_0 \prod_{i=1}^{s+1} (Y_i^{k-\ell-1} \alpha_i) \right)$ and $X^k \notin \text{Fcts} \left(\alpha_0 \prod_{i=1}^{s+1} (Y_i^{k-\ell} \alpha_i) \right)$.

There exist two words π and σ such that $\pi X^k \sigma = \alpha_0 \prod_{i=1}^{s+1} (Y_i^{k-\ell-1} \alpha_i)$.

If $|\sigma| \geq |\alpha_{s+1}|$, then $X^k \in \text{Fcts} \left(\alpha_0 \prod_{i=1}^s (Y_i^{k-\ell-1} \alpha_i) Y_{s+1}^{k-\ell-1} \right)$.

But $X^k \notin \text{Fcts} \left(\alpha_0 \prod_{i=1}^s (Y_i^{k-\ell} \alpha_i) Y_{s+1}^{k-\ell-1} \right)$ (it is a subset of $\text{Fcts} \left(\alpha_0 \prod_{i=1}^{s+1} (Y_i^{k-\ell} \alpha_i) \right)$).

Taking $\alpha_s Y_{s+1}^{k-\ell-1}$ instead of α_s , by induction hypothesis, we get that $|X| > (k - \ell - 2) \times L$.

On the same way, if $|\pi| \geq |\alpha_0|$, taking $Y_1^{k-\ell-1} \alpha_1$ for α_0 and α_i for α_{i-1} for all integers $2 \leq i \leq s + 1$, by induction hypothesis, we get that $|X| > (k - \ell - 2) \times L$.

Thus $|\sigma| < |\alpha_{s+1}|$ and $|\pi| < |\alpha_0|$, that is $\prod_{i=1}^s (Y_i^{k-\ell-1} \alpha_i) Y_{s+1}$ and, in particular, $Y_1^{k-\ell-1}$ are internal factors of X^k . As we have done in the proof of Lemma 1.7, if $|Y_1^{k-\ell-1}| \geq |X| + |Y_1|$ by Corollary 1.5 and since X^k and Y_1^k are pure k -powers, there exist two words X_1 and X_2 such that $X = X_1 X_2$ and $Y_1 = X_2 X_1$ with $X_1 X_2$ and $X_2 X_1$ primitive words. But $X_2 X_1$ is not an internal factor of $(X_2 X_1)^2$. Thus α_0 ends with X_1 and α_1 starts with X_2 . It follows that $(X_1 X_2)^k = X^k \in \text{Fcts} \left(\alpha_0 Y_1^{k-\ell} \alpha_1 \right) \subset \text{Fcts} \left(\alpha_0 \prod_{i=1}^{s+1} (Y_i^{k-\ell} \alpha_i) \right)$: a contradiction with the hypotheses.

So, we necessarily have $|X| > |Y_1^{k-\ell-2}|$. \square

1.2 Morphisms

Let A and B be two alphabets. A *morphism* f from A^* to B^* is a mapping from A^* to B^* such that $f(uv) = f(u)f(v)$ for all words u, v over A . When B has no importance, we say that f is a morphism on A or that f is defined on A .

Given an integer L , f is *L -uniform* if $|f(a)| = L$ for every letter a in A . A morphism f is *uniform* if it is L -uniform for some integer $L \geq 0$. Given a set X of words over A , and given a morphism f on A , we denote by $f(X)$ the set $\{f(w) \mid w \in X\}$.

A morphism f on A is *k -power-free* if and only if $f(w)$ is k -power-free for every k -power-free word w over A . For instance, the *empty morphism* ϵ ($\forall a \in A, \epsilon(a) = \epsilon$) or the *identity endomorphism* Id ($\forall a \in A, Id(a) = a$) are k -power-free.

We say that a morphism is *non-erasing* if, for all letters $a \in A$, $f(a) \neq \epsilon$. The empty morphism ϵ is the only morphism that is both erasing and k -power-free. Indeed, for any non-empty erasing morphism f , there exist two different letters a and b in A (remember $\text{Card}(A) \geq 2$) such that $f(a) \neq \epsilon, f(b) = \epsilon$, and so $f(aba^{k-1})$ contains a k -power.

A morphism on A is called *prefix* (resp. *suffix*) if, for all different letters a and b in A , the

word $f(a)$ is not a prefix (resp. not a suffix) of $f(b)$. A prefix (resp. suffix) morphism is non-erasing. A morphism is *bifix* if it is prefix and suffix.

Given a morphism f on A , the *mirror morphism* \tilde{f} of f is defined for all words w over A , by $\tilde{f}(w) = \widetilde{f(w)}$. In particular, $\tilde{f}(a) = \widetilde{f(a)}$ for every letter a in A . Note that f is k -power-free if and only if \tilde{f} is k -power-free.

Proofs of the three following lemmas are left to the reader.

Lemma 1.9 *Let f be a bifix morphism on an alphabet A and let u, v, w , and t be four words over A .*

The equality $f(u) = f(v)p$ where p is a prefix of $f(w)$ implies $u = vw'$ for a prefix w' of w such that $f(w') = p$.

Symmetrically, the equality $f(u) = sf(v)$ where s is suffix of $f(t)$ implies $u = t'v$ for a suffix t' of t such that $f(t') = s$.

Lemma 1.10 *Let f be a prefix morphism on an alphabet A , let u and v be words over A , and let a and b be letters in A . Furthermore, let p_1 (resp. p_2) be a prefix of $f(a)$ (resp. of $f(b)$). If $(p_1; p_2) \neq (\varepsilon; f(b))$ and if $(p_1; p_2) \neq (f(a); \varepsilon)$ then the equality $f(u)p_1 = f(v)p_2$ implies $u = v$ and $p_1 = p_2$.*

Lemma 1.11 *Let f be a suffix morphism on an alphabet A , let u and v be words over A , and let a and b be letters in A . Furthermore, let s_1 (resp. s_2) be a suffix of $f(a)$ (resp. of $f(b)$). If $(s_1; s_2) \neq (\varepsilon; f(b))$ and if $(s_1; s_2) \neq (f(a); \varepsilon)$ then the equality $s_1f(u) = s_2f(v)$ implies $u = v$ and $s_1 = s_2$.*

Definition 1.12 *A morphism f from A^* to B^* is a ps-morphism (Keränen [1] called f a ps-code) if and only if the equalities*

$$f(a) = ps, f(b) = ps' \text{ and } f(c) = p's$$

with $a, b, c \in A$ (possibly $c = b$) and $p, s, p', s' \in B^$ imply $b = a$ or $c = a$.*

Obviously, taking $c = b$, and $s = \varepsilon$ in a first time and $p = \varepsilon$ in a second time, we obtain that a ps-morphism is a bifix morphism.

Lemma 1.13 [1, 2] *If f is not a ps-morphism then f is not a k -power-free morphism for all integers $k \geq 2$.*

Lemma 1.14 [5]

Let f be a ps-morphism from A^ to B^* and let u, v and w be words over A such that $f(u) = \delta\beta$, $f(v) = \alpha\beta$, and $f(w) = \alpha\gamma$ for some non-empty words α, β, γ , and δ over B . Then it implies that $v = v_1av_2$, $u = u_1bv_2$, and $w = v_1cw_2$ for some words v_1, v_2, u_1 , and w_2 , and some letters a, b , and c . Moreover, we have either $b = a$ or $c = a$.*

Furthermore, if $|\delta| < |f(u[1])|$ then $u_1 = \varepsilon$ and if $|\gamma| < |f(w[|w|])|$ then $w_2 = \varepsilon$.

Assuming that $f(\bar{w}) = pu^ks$ for a factor \bar{w} of a word w and a non-empty word u , and assuming that \bar{w} contains a factor w_0 such that $|f(w_0)| = |u|$, if f is a ps-morphism,

Lemma 1.16 states that \bar{w} necessarily contains a k -power w'^k such that $f(w')$ is a conjugate of u . We will say that $f(w)$ contains a synchronised k -power u^k or that $f(w)$ and u^k are synchronised. More precisely:

Definition 1.15 *Let $k \geq 2$ be an integer. Let f be a morphism from A^* to B^* , w be a word over A , and u be a non-empty word over B such that $f(w)$ contains the k -power u^k . Let \bar{w} be a shortest factor of w whose image by f contains u^k , i.e., $f(\bar{w}) = pu^ks$ with $|p| < |f(\bar{w}[1])|$ and $|s| < |f(\bar{w}[\bar{w}])|$.*

We say that $f(w)$ and u^k are synchronised if there exist three words w_0, w_1 , and w_2 such that $|f(w_0)| = |u|$ and $\bar{w} = w_1w_0w_2$ with $p = \varepsilon$ if $w_1 = \varepsilon$, and $s = \varepsilon$ if $w_2 = \varepsilon$.

Lemma 1.16 [5]

Let $k \geq 2$ be an integer. If f is a ps-morphism and if $f(w)$ contains a synchronised k -power then w contains a k -power.

Remark 1.17 [5]

More precisely, the word \bar{w} starts or ends with a k -power which image by f is a conjugated of the synchronised k -power.

Lemma 1.18 [5]

Let $k \geq 4$ be an integer.

The image of a pure k -power by a k -power-free morphism is also a pure k -power.

As direct consequences of Lemmas 1.7, 1.8 and 1.18, we get the two following corollaries:

Corollary 1.19 *Let $k \geq 4$ be an integer and let f be a morphism from A^* to B^* . Let α and β be two words over B and let v^k be a pure k -power over A . For any pure k -power t^k over A and for any integer $0 \leq \ell \leq k - 3$, if $f(t)^k \in \text{Fcts}(\alpha f(v)^{k-\ell-1}\beta)$ and $f(t)^k \notin \text{Fcts}(\alpha f(v)^{k-\ell}\beta)$ then either f is not k -power free or $|f(t)| > (k - \ell - 2)|f(v)|$.*

Corollary 1.20 *Let $k \geq 4$ and $s \geq 1$ be two integers and let f be a morphism from A^* to B^* . Let $\alpha_0, \alpha_1, \dots, \alpha_s$ be $s + 1$ words over B , and let $v_1^k, v_2^k, \dots, v_s^k$ be s pure k -powers such that $|f(v_i)| = L$ for all integers $1 \leq i \leq s$. For any pure k -power t^k over A and for any integer $0 \leq \ell \leq k - 3$, if $f(t)^k \in \text{Fcts}(\alpha_0 \prod_{i=1}^s (f(v_i)^{k-\ell-1}\alpha_i))$ and $f(t)^k \notin \text{Fcts}(\alpha_0 \prod_{i=1}^s (f(v_i)^{k-\ell}\alpha_i))$ then either f is not k -power free or $|f(t)| > (k - \ell - 2) \times L$.*

2 Reduction of a power

2.1 About k -power-free morphisms

Lemma 2.1 [5] *(see Figure 1)*

Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^* . Let v and T be non-empty words over A such that v^k is a pure k -power. Let us assume that $f(T) = \pi_1 f(v)^k \sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then one of the following holds:

- (P.1) : There exist a pure k -power x^k , a word y over A , and a word Z over B such that
 - (P.1.1) : $T = x^k y$, $|y| \leq 1$, $f(y) = \pi_1 \sigma_2$, $f(x) = \pi_1 Z$, and $f(v) = Z \pi_1$
 - (P.1.2) : or $T = y x^k$, $|y| = 1$, $f(y) = \pi_1 \sigma_2$, $f(x) = Z \sigma_2$, and $f(v) = \sigma_2 Z$.
- (P.2) : There exist a pure k -power x^k and a non-empty word y over A such that
 - (P.2.1) : $T = x^k y$ with $|f(x^{k-1})| < |\pi_1 f(v)|$
 - (P.2.2) : or $T = y x^k$ with $|f(x^{k-1})| < |f(v) \sigma_2|$.
- (P.3) : f is not k -power-free.

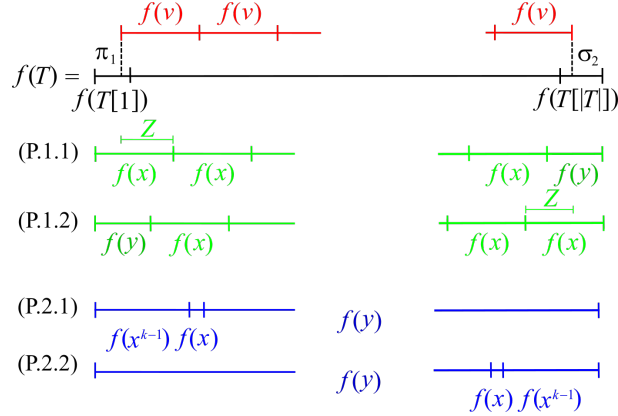


Figure 1: Different cases in Lemma 2.1

Corollary 2.2 [5]

With hypotheses and notations of Lemma 2.1, if $f(T)$ and $f(v)^k$ are synchronised (this is obviously the case when f is a uniform ps-morphism) then either f is not k -power-free or T satisfies (P.1).

Remark 2.3 If T and v^k are factor of the same word ω and if $f(v)^k$ is a shortest k -power in ω then T can not satisfy (P.2).

Corollary 2.4 [5]

Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^* . Let v^k and t^k be two pure k -powers over A . Let us assume that $f(t^k) = \pi_1 f(v)^k \sigma_2$ with $|\pi_1| < |f(t[1])|$ and $|\sigma_2| < |f(t[|t|])|$. If $\pi_1 \neq \varepsilon$ or if $\sigma_2 \neq \varepsilon$ then f is not k -power-free.

Corollary 2.5 [5]

Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^* . Let v and T be non-empty words over A such that v^k is a pure k -power. Let us assume that $f(T) = \pi_1 f(v)^{k+1} \sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then either f is not k -power-free or there exist a pure k -power x^k , a word Y over A , and a word Z over B such that

$$(P.1.1)' : T = x^{k+1}Y, |Y| \leq 1, f(Y) = \pi_1 \sigma_2, f(x) = \pi_1 Z, \text{ and } f(v) = Z\pi_1$$

$$(P.1.2)' : \text{ or } T = Yx^{k+1}, |Y| = 1, f(Y) = \pi_1 \sigma_2, f(x) = Z\sigma_2, \text{ and } f(v) = \sigma_2 Z.$$

Let α, β , and γ be three integers such that $\gamma \leq \alpha$ and $\beta \leq \alpha - \gamma + 1$. We denote $v_{(\beta)}^\gamma$ the β^{th} factor of v^γ in v^α that is $v^\alpha = v^{\beta-1} v_{(\beta)}^\gamma v^{\alpha-\gamma+1-\beta}$. In particular, when $\gamma = 1$, $v_{(\beta)}$ is the β^{th} successive factor of v in v^α .

By Corollary 2.2 and Corollary 2.5, we immediatly get:

Corollary 2.6 Let α and k be two integers such that $\alpha \geq k + 1 \geq 5$. Let f be a ps-morphism from A^* to B^* . Let v and T be non-empty words over A such that v^k is a pure k -power. Let us assume that $f(T) = \pi_1 f(v)^\alpha \sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$.

For any integer $1 \leq \beta \leq \alpha - k + 1$, the word $T_{(\beta)}$ such that $f(T_{(\beta)}) = \pi_{(\beta),1} f(v_{(\beta)}^k) \sigma_{(\beta),2}$ with $|\pi_{(\beta),1}| < |f(T_{(\beta)}[1])|$ and $|\sigma_{(\beta),2}| < |f(T_{(\beta)}[|T_{(\beta)}|])|$ satisfies (P.1).

2.2 Equations of reduction

Lemma 2.7 [5]

Let $\alpha_1, \alpha_2, \beta_1, \beta'_1, \beta_2, \gamma_1, \gamma_2$ be words over an alphabet B such that $|\beta_1| = |\beta_2| \neq 0$, β'_1 is a proper suffix of β_1 , and $0 \leq |\alpha_2| - |\alpha_1| \leq |\beta'_1|$.

Under these hypotheses, the equality $\alpha_2 \beta_2 \gamma_2 = \alpha_1 \beta'_1 \beta_1 \gamma_1$ implies $\alpha_2 \gamma_2 = \alpha_1 \beta'_1 \gamma_1$.

Lemma 2.8 [5]

Let $\kappa \geq 3$ be an integer. Let f be a morphism from A^* to B^* . Let $(w_i)_{i=1.. \kappa+1}$ and $(x_i)_{i=1.. \kappa}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i, j in $[1, \kappa]$.

We denote by w the word $w_1 x_1 \dots w_\kappa x_\kappa w_{\kappa+1}$.

We assume that there exist words $u, p, s, (X_i)_{i=1.. \kappa}$, and $(Y_i)_{i=1.. \kappa}$ over B such that $f(w_1) = pX_1$, $f(w_{\kappa+1}) = Y_\kappa s$, and $f(w_i) = Y_{i-1} X_i$ for all $2 \leq i \leq \kappa$. Moreover, we assume that, for all integers i in $[1, \kappa]$, we have $u = X_i f(x_i) Y_i$. It means that $f(w) = pu^\kappa s$.

Let us also assume that there exists an integer q such that, for every integer i in $[1, \kappa]$, $0 \leq |X_q| - |X_i| \leq |X_q''|$ where X_q'' is a common suffix of X_q and $f(x_q)$. Then the word $\tilde{w} = w_1 w_2 \dots w_\kappa w_{\kappa+1}$ satisfies $f(\tilde{w}) = p\tilde{u}^\kappa s$ with $\tilde{u} = X_i Y_i$ for every integer i in $[1, \kappa]$.

In particular, $f(\tilde{w})$ and \tilde{u}^κ are synchronised only if $f(w)$ and u^κ are synchronised.

The situation described in Figure 2 is an example of a case where the hypotheses of Lemma 2.8 hold.

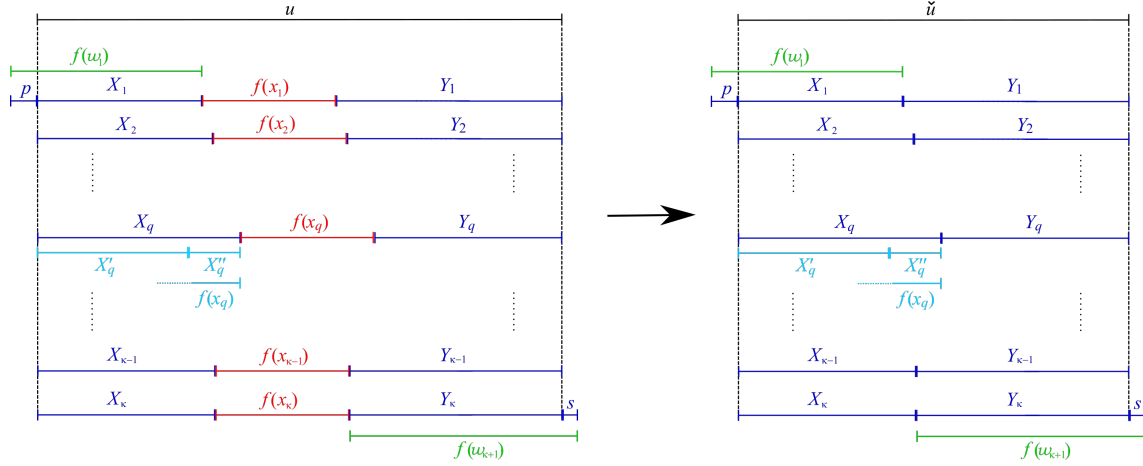


Figure 2: Reduction of a power

We say that we have reduced w .

Let us note that p is not necessarily a prefix of $f(w_1[1])$ and s is not necessarily a suffix of $f(w_\kappa[|w_\kappa|])$.

Figure 3 deals with Point 3 of Remark 2.9 and Figure 4 deals with Point 4 of Remark 2.9.

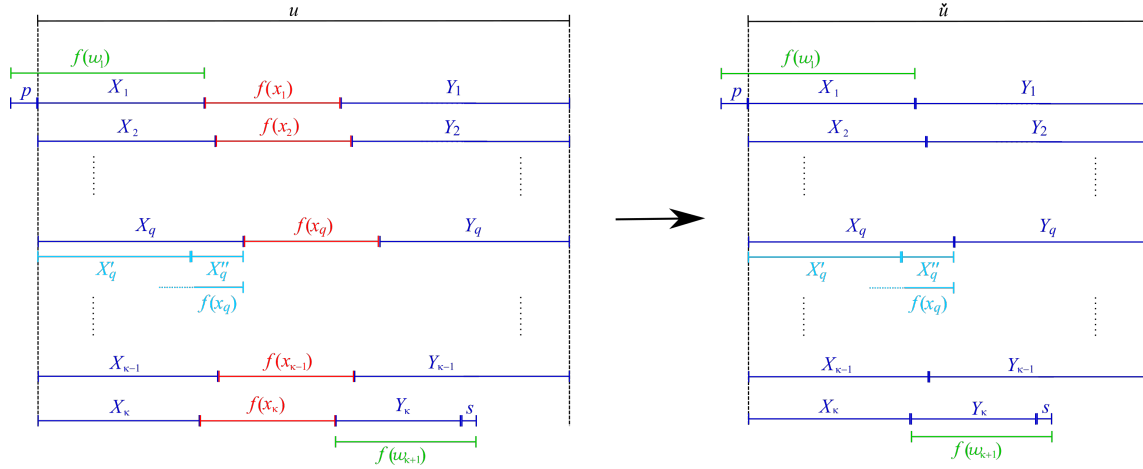


Figure 3: Point 3 of Remark 2.9

Remark 2.9 1. Using the mirror image and exchanging $|X_q|$ the maximum of $|X_i|$ by the maximum $|Y_q|$ of $|Y_i|$ (i.e., $|X_q|$ is the minimum of $|X_i|$), the condition " $0 \leq |X_q| - |X_i| \leq |X_q''|$ where X_q'' is a common suffix of X_q and $f(x_q)$ " of Lemma 2.8 can

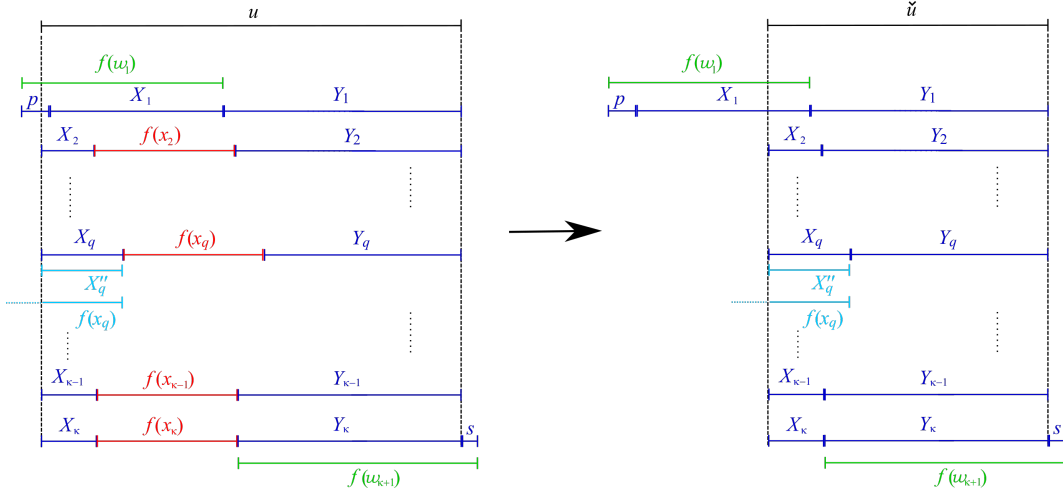


Figure 4: Point 4 of Remark 2.9

- be replaced by " $0 \leq |Y_q| - |Y_i| \leq |Y'_q|$ where Y'_q is a common prefix of Y_q and $f(x_q)$ ".
2. A prefix u_1 of u is also a prefix of \tilde{u} if $|u_1| < |X_q|$, and a suffix u_2 of u is also a suffix of \tilde{u} if $|u_2| < \max |Y_j|$.
 3. If, instead of $u = X_\kappa f(x_\kappa) Y_\kappa$, we only have that $X_\kappa f(x_\kappa) Y_\kappa$ is a prefix of u (see Figure 3) then $f(\tilde{w}) = p\tilde{u}^{\kappa-1} X_\kappa Y_\kappa s$ with $X_\kappa Y_\kappa$ prefix of \tilde{u} .
 4. If $q \neq 1$ and X_q is a suffix of $f(x_q)$, i.e., $X'_q = \varepsilon$ (see Figure 4), then we do not need x_1 and optionally not w_1 in the hypotheses of Lemma 2.8. Conclusion remains true with $u = X_1 Y_1$, $w'_2 = w_1 w_2$ or w_2 , $f(w'_2) = pX_1 Y_1 X_2$, $w = w'_2 x_2 w_3 \dots w_\kappa x_\kappa w_{\kappa+1}$, and \tilde{w} a (not necessarily proper) suffix of $w'_2 w_3 \dots w_\kappa w_{\kappa+1}$.
 5. By mirror image of Case 4, we get that, if $q \neq \kappa$ and Y_q is a prefix of $f(x_q)$ then we do not need x_κ and optionally not $w_{\kappa+1}$ in the hypotheses of Lemma 2.8. Conclusion remains true with $u = X_\kappa Y_\kappa$, $w'_\kappa = w_\kappa w_{\kappa+1}$ or w_κ , $f(w'_\kappa) = Y_{\kappa-1} X_\kappa Y_\kappa s$, $w = w_1 x_1 w_2 \dots w_{\kappa-1} x_{\kappa-1} w'_\kappa$, and \tilde{w} a (not necessarily proper) prefix of $w_1 w_2 \dots w_{\kappa-1} w'_\kappa$.
 6. As a combination of Case 4 and Case 5, we get that, if $q \neq 1$, $q \neq \kappa$, X_q is a suffix of $f(x_q)$, and Y_q is a prefix of $f(x_q)$ then we do not need neither x_1 nor x_κ in the hypotheses of Lemma 2.8. Conclusion remains true with $u = X_1 Y_1 = X_\kappa Y_\kappa$, $w'_2 = w_1 w_2$ or w_2 , $w'_\kappa = w_\kappa w_{\kappa+1}$ or w_κ , $f(w'_2) = pX_1 Y_1 X_2$, $f(w'_\kappa) = Y_{\kappa-1} X_\kappa Y_\kappa s$, $w = w'_2 x_2 w_3 \dots w_{\kappa-1} x_{\kappa-1} w'_\kappa$, and \tilde{w} a (not necessarily proper) factor of $w'_2 w_3 \dots w_{\kappa-1} w'_\kappa$.

For any positive integer ℓ , since $|f(x_i)| = |f(x_j)|$ is equivalent to $|f(x_i^\ell)| = |f(x_j^\ell)|$ and since a prefix (resp. a suffix) of $f(x_i)$ is a prefix (resp. a suffix) of $f(x_i^\ell)$, we immediately obtain the following Corollary that will be the central point of proof of Lemma 4.5.

Corollary 2.10 (method of reduction)[5]

Let $\kappa \geq 3$ and $\ell \geq 1$ be two integers, let α be an integer in $\{1, 2\}$ and let β be an integer in $\{\kappa - 1, \kappa\}$

Let f be a morphism from A^* to B^* and let $(w_i)_{i=\alpha.. \beta+1}$, $(x_i)_{i=\alpha.. \beta}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i, j in $[\alpha, \beta]$.

We denote by w the word $w_\alpha x_\alpha^\ell \dots w_\beta x_\beta^\ell w_{\beta+1}$.

We assume that there exist u, p, s , $(X_i)_{i=\alpha.. \beta}$ and $(Y_i)_{i=\alpha.. \beta}$ words over B such that $f(w_i) = Y_{i-1} X_i$ for all integers i in $[1 + \alpha; \beta]$. Furthermore, we also assume that $f(w_\alpha) = pu^{\alpha-1} X_1$ and $f(w_{\beta+1}) = Y_\kappa u^{\kappa-\beta} s$ where $u = X_i f(x_i^\ell) Y_i (\neq \varepsilon)$ for all integers i in $[\alpha, \beta]$: it means that $f(w) = pu^\kappa s$.

Finally, we assume that there exists an integer q such that, for any integer i in $[\alpha, \beta]$, $0 \leq |X_q| - |X_i| \leq |X_q''|$ where X_q'' is a common suffix of X_q and $f(x_q)$, $0 \leq |X_q| - |X_i| \leq |f(x_q)|$ when $\alpha = 2$, or $0 \leq |Y_i| - |Y_q| \leq |f(x_q)|$ when $\beta = \kappa - 1$.

Then, for any integer $0 \leq \phi < \ell$, the word $w' = w_\alpha x_\alpha^\phi \dots w_\beta x_\beta^\phi w_{\beta+1}$ satisfies $f(w') = pu'^\kappa s$ with $u' = X_i f(x_i^\phi) Y_i$ for any integer i in $[1; \kappa]$.

In particular, $f(w')$ and u'^κ are synchronised only if $f(w)$ and u^κ are synchronised.

2.3 Situations of reduction

Let $k \geq 4$ be an integer. Let f be a morphism from A^* to B^* and let ω be a word over A such that $f(\omega) = \varrho(\chi\gamma)^k \chi \varsigma$ for some words ϱ, χ, γ and ς over B such that $\chi\gamma \neq \varepsilon$, $|\varrho| < |f(\omega[1])|$ and $|\varsigma| < |f(\omega[|\omega|])|$. We denote $U = \chi\gamma$ and $S = \chi\varsigma$. In particular, when $\gamma = \varepsilon$, we get that $f(\omega) = \varrho U^{k+1} \varsigma$.

For any integer j in $[1, k+1]$, let i_j the smallest integer such that $\varrho(\chi\gamma)^{j-1}$ is a prefix of $f(\omega[1..i_j])$. And let i'_j be the smallest integer such that $\varrho(\chi\gamma)^{j-1} \chi$ is a prefix of $f(\omega[1..i'_j])$. In particular, we have $i_1 = 1$, and $i'_{k+1} = |\omega|$ (see Figure 5). By convention, let $i_{k+2} = i'_{k+1}$. Furthermore, there exist words p_j, p'_j, s_j and s'_j such that, $f(\omega[i_j]) = p_j s_j$, $f(\omega[i'_j]) = p'_j s'_j$, $p_1 = \varrho$, $s_{k+2} = \varsigma p_j \neq \varepsilon$ if $j \neq 1$ and $s_1 \neq \varepsilon$, $f(\omega[1..i_j]) = \varrho(\chi\gamma)^{j-1} s_j$ and $f(\omega[1..i'_j]) = \varrho(\chi\gamma)^{j-1} \chi s'_j$ for any integer j in $[1, k+1]$.

Moreover, we assume that $i_j < i_{j+1}$. In particular, we get that $\chi\gamma = s_j f(\omega[i_j + 1..i_{j+1} - 1]) p_{j+1}$ when $1 \leq j \leq k$.

Since a factor of ω can appear many times in ω , it is necessary to indicate which exact factor we are going to work with: if $\omega[n..m] = z$, we denote $n = n_z$ and $m = m_z$ fixing by this way the considered occurrence of z in ω . For any positive integer α , if $\omega[n..m] = z^\alpha$, we also denote $n = n_z$ and $m = m_z$ without specifying α . It is the same notation as the case $\alpha = 1$: we will precise only if necessary.

To simplify notations, let us recall that, given two integers $1 \leq n_z \leq m_z \leq |\omega|$, the word $\omega[n_z..m_z] = z^\alpha$ define two words we denote z_p and z_s such that $\omega = z_p z^\alpha z_s$, with $n_z = |z_p| + 1$ and $m_z = |z_p z^\alpha|$. This means that $z_p = \omega[1..n_z - 1]$ and $z_s = \omega[m_z + 1..|\omega|]$.

Given two integers $1 \leq n_z \leq m_z \leq |\omega|$, we also define a word D_z and three integers λ_z, d_z and c_v (even if c_v is not used in this section). Eventually, we will precise $D_{z,\omega}, \lambda_{z,\omega}, d_{z,\omega}$ and

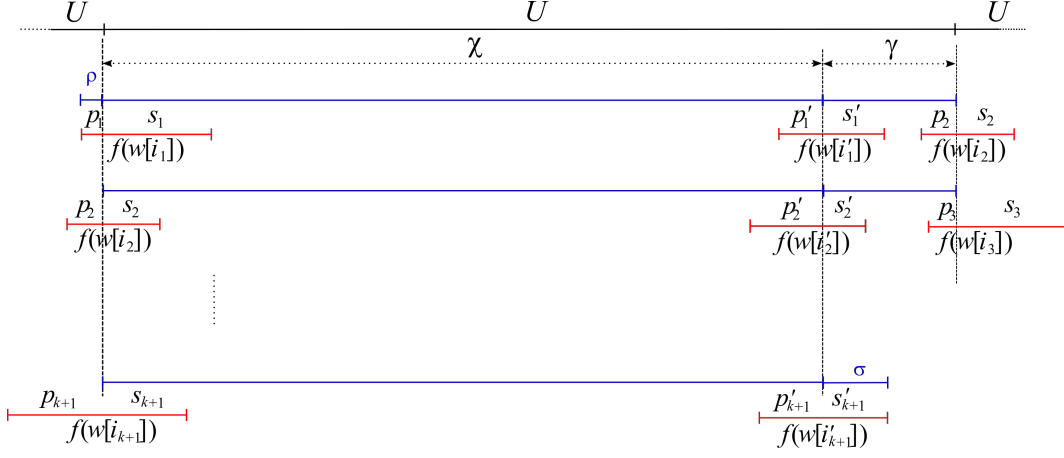


Figure 5: Example of a decomposition of $f(\omega)$

$c_{z,\omega}$ if a doubt may occur. Briefly, λ_z is the integer such that $f(\omega[n_z..m_z]) = f(z^\alpha)$ starts in the λ_z^{th} occurrence of $\chi\gamma$; d_z indicates if the first occurrence of $f(z)$ in $f(\omega[n_z..m_z])$ straddles or not two consecutive occurrences of $\chi\gamma$; c_z is the lowest integer such that $(\chi\gamma)^{c_z-1}\chi$ and $f(\omega[n_z..m_z])$ overlap and D_z is a prefix of $\chi\gamma$ such that $f(z_p z)$ ends with D_z or $D_z f(z)$.

More precisely, if $n_z = 1$ and $\varrho \neq \varepsilon$ then $\lambda_z = 0$, $d_z = 1$ and D_z is the word such that $f(z) = \varrho D_z$. If $n_z = 1$ and $\varrho = \varepsilon$ then $\lambda_z = 1$, $d_z = 0$ and $D_z = \varepsilon$. When $n_z \geq 2$, let λ_z be the integer such that $n_z \in]i_{\lambda_z}; i_{\lambda_z+1}]$, i.e., $|\varrho(\chi\gamma)^{\lambda_z-1}| \leq |f(\omega[1..n_z-1])| = |f(z_p)| < |\varrho(\chi\gamma)^{\lambda_z}|$. If $|f(z_p z)| \leq |\varrho(\chi\gamma)^{\lambda_z}|$ then let $d_z = 0$ otherwise let $d_z = 1$. Let D_z be the word such that $f(z_p z^{d_z}) = \varrho(\chi\gamma)^{\lambda_z-1+d_z} D_z$. It means that $D_z = s_{\lambda_z} f(\omega[i_{\lambda_z} + 1..n_z - 1])$ when $d_z = 0$ and $s_{\lambda_z} f(\omega[i_{\lambda_z} + 1..n_z - 1])f(z) = \chi\gamma D_z$ when $d_z = 1$. In particular, D_z is a proper suffix of $f(z)$ when $d_z = 1$. Finally, c_z is the lowest integer such that $|f(\omega[1..m_z])| < |\varrho(\chi\gamma)^{\lambda_z+c_z-1}\chi|$ when $m_z \neq |\omega|$ and $c_z = k + 1 - \lambda_z$ otherwise.

It is important to remark that, if $\omega[n_z..m_z] = z^\alpha$, the integers n_z and m_z define z^α and z . But, since we may have several occurrences of z^α in ω , we do not have the contrary. In other words, the equality $z = z'$ not necessarily implies $n_z = n_{z'}$ or $m_z = m_{z'}$. In the same vein, λ_z , d_z , c_z and D_z depend on n_z and m_z but not directly of z . But if no question exists over the considered factor of ω or if the choice of the considered factor does not matter, we will write z^α instead of $\omega[n_z..m_z]$.

For any integer $\alpha \geq 2$ and for any word $\omega[n_z..m_z] = z^\alpha$ with $n_z, m_z \in [1, |\omega|]$, the word $f(\omega[n_y..m_y]) = f(y^\alpha) = f(y)^\alpha$ with $n_y, m_y \in [1, i_{k+2}]$ is a *conjugated shift to the left* of $f(\omega[n_z..m_z]) = f(z^\alpha) = f(z)^\alpha$ (in $f(\omega)$) if there exist two words $t_1 \neq \varepsilon$ and t_2 such that $f(y) = t_2 t_1$, $f(z) = t_1 t_2$ and if we have one of the following conditions:

1. $D_z = D_y t_2$ when $d_y = d_z$
2. $D_y = D_z t_1$ when $d_y = 1$ and $d_z = 0$
3. $D_y f(y) t_2 = \chi\gamma D_z$ when $d_y = 0$ and $d_z = 1$

Let us remark that conditions (2) and (3) imply $|D_z| < |t_2|$. Moreover, taking $t_2 = \varepsilon$, let us also note that $f(z^\alpha)$ is a conjugated shift to the left of itself.

We say that $f(y)^\alpha$ is a *conjugated shift to the right* of $f(z)^\alpha$ if $f(z)^\alpha$ is a conjugated shift to the left of $f(y)^\alpha$. We simply say that $f(y)^\alpha$ is a *conjugated shift* of $f(z)^\alpha$ if it is a conjugated shift to the left or to the right of $f(z)^\alpha$.

For a general use of conjugated shifts of $f(z)^\alpha$, we will switch the roles of t_1 and t_2 in definition and conditions (1) to (3) for a conjugated shift to the right.

Lemma 2.11 *For any integer $\alpha \geq 2$, if $f(x)^\alpha$ is a conjugated shift to the right of $f(z)^\alpha$ and if $f(y)^\alpha$ is a conjugated shift to the left of $f(z)^\alpha$ then $f(y)^\alpha$ is a conjugated shift to the left of $f(x)^\alpha$.*

Proof.

We will only prove Lemma 2.11 when $d_x = d_y = d_z$. The other cases are left to the reader.

Let $t_1 (\neq \varepsilon)$, t_2 , t'_1 and $t'_2 (\neq \varepsilon)$ be the words such that $f(y) = t_2 t_1$, $f(x) = t'_2 t'_1$ and $f(z) = t_1 t_2 = t'_1 t'_2$. We have $D_z = D_y t_2$ and $D_x = D_z t'_1$. And, since $f(x)^\alpha$ is a factor of $f(\omega)$, there exists an integer β such that $D_x f(x)^\alpha$ is a prefix of $(\chi\gamma)^\beta \chi\varsigma$.

If $|t'_1| \leq |t_1|$, let T be the word such $t_1 = t'_1 T$. We get that $t'_2 = T t_2$, $D_x = D_y t_2 t'_1$, $f(y) = (t_2 t'_1) T$, $f(x) = T (t_2 t'_1)$ and $t_2 t'_1 \neq \varepsilon$: it ends the proof.

If $|t'_1| > |t_1|$, let T be the word such $t'_1 = t_1 T$. We get that $t_2 = T t'_2$, $D_x = D_y T t'_2 t'_1 = D_y T f(x)$. It means that $f(x)^\alpha$ is preceded by $f(x)$ in $f(\omega)$, i.e., $(\chi\gamma)^\beta \chi\varsigma$ starts with $D_y T f(x)^{\alpha+1}$. We will consider the first occurrence of $f(x)^\alpha$ in $f(x)^{\alpha+1}$ denoted by $f(\bar{x})^\alpha$ to avoid confusion. We get $D_{\bar{x}} = D_y T$, $f(y) = T (t'_2 t_1)$, $f(\bar{x}) = (t'_2 t_1) T$ and $t'_2 t_1 \neq \varepsilon$. \square

Proofs of Lemmas 2.12 and 2.13 and Corollary 2.14 are left to the reader.

Lemma 2.12 *For any integer $\alpha \geq 2$, if $f(x)^\alpha$ and $f(y)^\alpha$ are two conjugated shifts to the right of $f(z)^\alpha$ then $f(y)^\alpha$ is a conjugated shift of $f(x)^\alpha$.*

Lemma 2.13 *For any integer $\alpha \geq 2$, if $f(x)^\alpha$ and $f(y)^\alpha$ are two conjugated shifts to the left of $f(z)^\alpha$ then $f(y)^\alpha$ is a conjugated shift of $f(x)^\alpha$.*

Corollary 2.14 *The relation \mathfrak{R} , defined on $\text{Fcts}(f(\omega))$ by $x\mathfrak{R}y$ if there exists an integer $\alpha \geq 2$ such that x^α is a conjugated shift of y^α , is a relation of equivalence.*

For any pure k -power $\omega[n_v..m_v] = v^k$ of ω , there are $k-2$ choices for the factor v^3 in v^k . Let us recall that we denote $v_{(\beta)}^3$ the β^{th} factor of v^3 in v^k that is $\omega[n_v..m_v] = v^{\beta-1} v_{(\beta)}^3 v^{k-\beta-2}$ with $1 \leq \beta \leq k-2$.

We will focus on these different cubes v^3 but without specifying β in this section.

Simplifications will not always be made in the occurrence of $\chi\gamma$ where a word $f(x^3)$ begins, i.e., the λ_x -th. It will happen that we must consider the next occurrence of $\chi\gamma$ according

to the values of d_x and d_v . In order to specify the selected occurrence of $\chi\gamma$, we define the sets $L_{j,v}$ and $R_{j,v}$.

For any factor $\omega[n_v..m_v] = v^3$ of a pure k -power $v^k \in \text{Fcts}(\omega)$, and for any integer $j \in [1; k+1]$, let $L_{j,v}$ be the set of the words $\omega[n_x..m_x] = x^3$ such that $f(x)^k$ is a conjugated shift of $f(v)^k$, $f(\omega[n_x..m_x]) = f(x)^3$ is a conjugated shift to the left of $f(\omega[n_v..m_v]) = f(v)^3$ with $j = \lambda_x$ if $d_x = d_v = 0$ and $j = \lambda_x + 1$ otherwise. In particular, if $v \in L_{j,v}$ and $d_v = 1$ then $j = \lambda_v + 1$.

We also denote $R_{j,v}$ the set of the words $\omega[n_x..m_x] = x^3$ such that $f(x)^k$ is a conjugated shift of $f(v)^k$, $f(\omega[n_x..m_x]) = f(x)^3$ is a conjugated shift to the right of $f(\omega[n_v..m_v]) = f(v)^3$ with $j = \lambda_x + d_v \times d_x$ (see Figure 6).

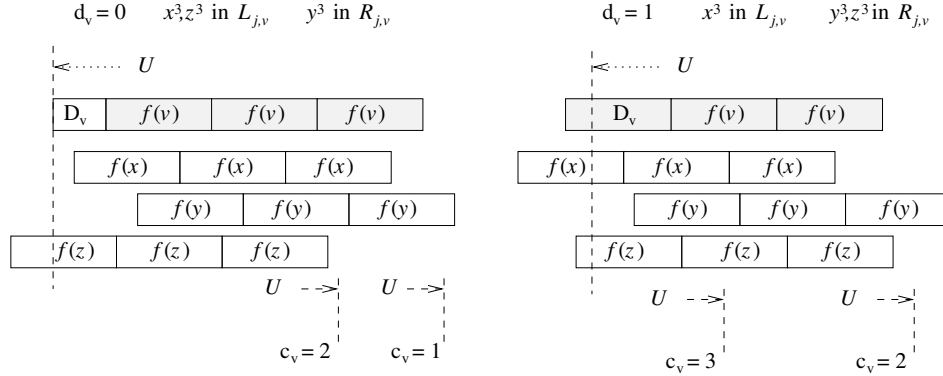


Figure 6: Examples of $L_{j,v}$ and $R_{j,v}$ depending on d_v and c_v when $k = 3$

If $\omega[n_{x_j}..m_{x_j}] = x_j^3$ is a word in $L_{j,v} \cup R_{j,v}$, we denote $t_{1,j}, t_{2,j}$ the words such that $f(v) = t_{1,j} t_{2,j}$ and $f(x_j) = t_{2,j} t_{1,j}$.

More specifically, if j_0 is an integer such that $\omega[n_v..m_v] = v^3 \in L_{j_0,v}(\cup R_{j_0,v})$, we may assume that $n_{x_{j_0}} = n_v$ and $m_{x_{j_0}} = m_v$, i.e., $x_{j_0} = v$. Let us remark that, by this choice, $j_0 = \lambda_v + d_v$. But j_0 can take other values.

Remark 2.15 *If there exist two integers i, j , and a pure k -power x^k such that $x^3 \in L_{i,v} \cup R_{i,v}$ with $d_x = d_v$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$, then $L_{j,x} \cup R_{j,x} \neq \emptyset$. Indeed, if y^k be a word in $L_{j,v} \cup R_{j,v}$, then, by Corollary 2.14, y^k is a conjugated shift of x^k .*

3 Simplification

In all this section, $k \geq 4$ is an integer, f is a morphism from A^* to B^* and ω is a word over A such that $f(\omega) = \varrho(\chi\gamma)^k \chi \varsigma$ for some words ϱ, χ, γ and ς over B such that $\chi\gamma \neq \varepsilon$, $|\varrho| < |f(\omega[1])|$ and $|\varsigma| < |f(\omega[|\omega|])|$. We denote $U = \chi\gamma$ and $S = \chi\varsigma$.

Let us recall that, for any integer j in $[1, k+1]$, i_j is the smallest integer such that $\varrho(\chi\gamma)^{j-1}$ is a prefix of $f(\omega[1..i_j])$ and i'_j is the smallest integer such that $\varrho(\chi\gamma)^{j-1} \chi$ is a prefix of

$f(\omega[1..i'_j])$. Moreover, $f(\omega[i_j]) = p_j s_j$ and $f(\omega[i'_j]) = p'_j s'_j$ with $f(\omega[1..i_j]) = \varrho(\chi\gamma)^{j-1} s_j$ and $f(\omega[1..i'_j]) = \varrho(\chi\gamma)^{j-1} \chi s'_j$.

For any pure k -power $\omega[n_v..m_v] = v^k$ such that $f(v)^k$ is a factor of $(\chi\gamma)^k \chi$, there are (at least) $k + 2 - c_v$ different occurrences of $(\chi\gamma)^{c_v-1} \chi$ in $f(\omega)$. Since, for any integer $j \in [1; k + 2 - c_v]$, $f(v)^k$ is a factor of $p_j(\chi\gamma)^{c_v-1} \chi s'_{j+c_v-1} = f(\omega[i_j..i'_{j+c_v-1}])$, let n_j be the greatest integer and m_j be the lowest integer such that $i_j \leq n_j \leq m_j \leq i'_{j+c_v-1}$ and $f(v)^k$ is factor of $f(\omega[n_j..m_j])$. Let us denote \widehat{v}_j the word $\omega[n_j..m_j]$. In other words, for any integer $j \in [1; k + 2 - c_v]$, the word \widehat{v}_j is the shortest factor of $\omega[i_j..i'_{j+c_v-1}]$ such that $f(\widehat{v}_j)$ contains $f(v)^k$. More precisely, $f(\widehat{v}_j) = \pi_{1,j} f(v)^k \sigma_{2,j}$ for two words $\pi_{1,j}$ and $\sigma_{2,j}$ such that $|\pi_{1,j}| < |f(\omega[n_j])| = |f(\widehat{v}_j[1])|$ and $|\sigma_{2,j}| < |f(\omega[m_j])| = |f(\widehat{v}_j[|\widehat{v}_j|])|$.

If $j > 1$ then $\pi_{1,j}$ is a suffix of the image of a factor of ω . If $j < k + 2 - c_v$ then $\sigma_{2,j}$ is a prefix of the image of a factor of ω .

Remark 3.1 *By Lemma 2.1, either f is not k -power-free or the word \widehat{v}_j satisfies one of the following properties:*

- (P.1) : *There exist a pure k -power $x_{j,v}^k$, a word $y_{j,v}$ over A and a word Z_j over B such that*

$$(P.1.1) : \widehat{v}_j = (x_{j,v})^k y_{j,v}, f(y_{j,v}) = \pi_{1,j} \sigma_{2,j}, f(x_{j,v}) = \pi_{1,j} Z_j \text{ and } f(v) = Z_j \pi_{1,j}$$

$$(P.1.2) : \text{or } \widehat{v}_j = y_{j,v} (x_{j,v})^k, f(y_{j,v}) = \pi_{1,j} \sigma_{2,j}, f(x_{j,v}) = Z_j \sigma_{2,j} \text{ and } f(v) = \sigma_{2,j} Z_j.$$

- (P.2) : *There exist a pure k -power v_j^k and a non-empty word $y_{j,v}$ over A such that*

$$(P.2.1) : \widehat{v}_j = (x_{j,v})^k y_{j,v} \text{ with } |f(x_{j,v}^{k-1})| < |\pi_{1,j} f(v)| < |f(x_{j,v})| + |f(v)|$$

$$(P.2.2) : \text{or } \widehat{v}_j = y_{j,v} (x_{j,v})^k \text{ with } |f(x_{j,v}^{k-1})| < |f(v) \sigma_{2,j}| < |f(x_{j,v})| + |f(v)|.$$

It particularly means that, when $|\chi\gamma| > 2|f(v)|$, we have $i_j < i_{j+1}$ for any integer j in $[1, k + 1]$.

In other words, if \widehat{v}_j satisfies (P.1.1) then $f(x_{j,v})^k$ is a conjugated shift to the left of $f(v)^k$. And, if \widehat{v}_j satisfies (P.1.2) then $f(x_{j,v})^k$ is a conjugated shift to the right of $f(v)^k$.

Let us recall that we denote $z_{(\beta)}^3$ the β^{th} factor of z^3 in z^k .

If $\beta = 1$ and \widehat{v}_j satisfies (P.1.1) then $f(v_{j(\beta)}^3) = f(v_j)^3$ is a conjugated shift to the left of $f(v_{(\beta)}^3) = f(v)^3$. If $\beta = k - 2$ and \widehat{v}_j satisfies (P.1.2) then $f(v_{j(\beta)}^3) = f(v_j)^3$ is a conjugated shift to the right of $f(v_{(\beta)}^3) = f(v)^3$. Otherwise $f(v_j)^3$ is both a conjugated shift to the right and a conjugated shift to the left of $f(v)^3$. For instance, if $2 \leq \beta \leq k - 3$ and \widehat{v}_j satisfies (P.1.1) then $f(v_{j(\beta)}^3)$ is a conjugated shift to the left of $f(v_{(\beta)}^3)$ and a conjugated shift to the right of $f(v_{(\beta-1)}^3)$.

If \widehat{v}_{j_1} and \widehat{v}_{j_2} satisfy (P.1.1) (resp. (P.1.2)) then $f(v_{j_2(\beta)}^3)$ is a conjugated shift (to the left or to the right) of $f(v_{j_1(\beta)}^3)$.

Moreover, if \widehat{v}_{j_1} satisfies (P.1.1) and \widehat{v}_{j_2} satisfies (P.1.2) then $f(v_{j_2(\beta)}^3)$ is a conjugated shift of $f(v_{j_1(\beta+1)}^3)$.

We have to match these possibilities with the ones of the position of a factor $f(v)^3$ of $f(v)^k$ in $(\chi\gamma)^k$.

Lemma 3.2 *If \widehat{v}_{j_1} and \widehat{v}_{j_2} satisfy (P.1) then $(\widehat{v}_{j_1})_{j_2}$ satisfies (P.1) and, consequently, $\|D_{x_{j_1},v} - |D_{x_{j_2},v}|\| < |f(v)|$.*

Proof.

It is a consequence of Corollary 2.14. \square

Two consecutive $(x_{j,v})^k$ can overlap but under some conditions, the length of this overlap is bounded.

Proposition 3.3 *Let us assume that f is a k -power morphism and that, for all integers $i \in [1, k+2-c_v]$, any power of $f(x_{i,v})$ and of $\chi\gamma$ do not have any common factor of length greater than $|f(x_{i,v})| + |\chi\gamma|$.*

For any integer $j \in [1, k+1-c_v]$, if \widehat{v}_j and \widehat{v}_{j+1} satisfy (P.1) then there exist two words $\tau_{1,j}$ and $\tau_{2,j}$ such that $w[n_j..m_{j+1}]$ contains the factors $(x_{j,v})^{k-1}\tau_{1,j}(x_{j+1,v})^k$ and $(x_{j,v})^k\tau_{2,j}(x_{j+1,v})t^{k-1}$.

In particular, when $k \geq 4$, the word $w[n_j..m_{j+1}]$ contains $(x_{j,v})^3\tau_{1,j}(x_{j+1,v})^3$.

Proof.

Since any power of $f(x_{j,v})$ (resp. $f(x_{j+1,v})$) and of $\chi\gamma$ do not have any common factor of length greater than $|f(x_{j,v})| + |\chi\gamma|$ (resp. $|f(x_{j+1,v})| + |\chi\gamma|$), we get that $i_j < i_{j+1}$.

Since \widehat{v}_j and \widehat{v}_{j+1} satisfy (P.1), let $w[n'_j..m'_{j+1}]$ the factor of $w[n_j..m_{j+1}]$ that starts with $(x_{j,v})^k$ and ends with $(x_{j+1,v})^k$.

If $|f(w[n'_j..m'_{j+1}])| \leq (2k-1)|f(x_{j,v})|$ then $f(x_{j,v}^k)$ and $f(x_{j+1,v}^k)$ have a common factor of length at least $|f(x_{j,v})| = |f(x_{j,v})| + |f(x_{j+1,v})| - \gcd(|f(x_{j,v})|; |f(x_{j+1,v})|)$. By Corollary 1.5 and Lemma 1.18, there exist two words t_1 and t_2 such that $f(x_{j,v}) = t_1t_2$ and $f(x_{j+1,v}) = t_2t_1$ with t_1t_2 and t_2t_1 primitive words. It means that $f(w[n'_j..m'_{j+1}])$ is a common factor of a power of $f(x_{j,v})$ and $\chi\gamma$. Since $i_j \leq n_j < i_{j+1} \leq n_{j+1}$, we get that $|f(w[n'_j..m'_{j+1}])| \geq |\chi\gamma| + |f(x_{j+1,v}^k)| - \|D_{x_{j,v}} - |D_{x_{j+1,v}}|\|$. By Lemma 3.2, it implies that $|f(w[n'_j..m'_{j+1}])| \geq |\chi\gamma| + |f(x_{j,v}^{k-1})|$: a contradiction with the hypotheses.

It follows that $|f(w[n'_j..m'_{j+1}])| > (2k-1)|f(x_{j,v})|$. \square

Let $Int.Pwrs(\omega)$ be the set of the k -powers v^k of ω such that $f(v^k) \in \text{Fcts}((\chi\gamma)^k\chi)$. More precisely, $Int.Pwrs(\omega)$ is the set of the k -powers of $\omega[2..|\omega|-1]$ when $\varrho \neq \varepsilon$ and $\varsigma \neq \varepsilon$, of $\omega[1..|\omega|-1]$ when $\varrho = \varepsilon$ and $\varsigma \neq \varepsilon$, of $\omega[2..|\omega|]$ when $\varrho \neq \varepsilon$ and $\varsigma = \varepsilon$ and of ω when $\varrho = \varsigma = \varepsilon$. We denote $Pure.Int.Pwrs(\omega)$ the set of pure k -powers of $Int.Pwrs(\omega)$.

Let $\ell_{\min} = \min\{|f(t)| \mid t^k \in \text{Fcts}(\omega)\}$ and let $Min.Pwrs(\omega) = \{v^k \in \text{Fcts}(\omega) \mid |f(v)| = \ell_{\min}\}$, i.e., $Min.Pwrs(\omega)$ is the set of the k -powers of ω whose image have a minimal length. When f is k -power-free, a k -power in $Min.Pwrs(\omega)$ is necessarily a pure k -power.

In a similar way, let $\ell_{\min \text{ int}} = \min\{|f(t)| \mid t^k \in \text{Int.Pwrs}(\omega)\}$ and let $\text{Min.Int.Pwrs}(\omega) = \{v^k \in \text{Int.Pwrs}(\omega) \mid |f(v)| = \ell_{\min \text{ int}}\}$, i.e., $\text{Min.Int.Pwrs}(\omega)$ is the set of the k -powers of $\text{Int.Pwrs}(\omega)$ those image have a minimal length. As for $\text{Min.Pwrs}(\omega)$, when f is k -power-free, a k -power in $\text{Min.Int.Pwrs}(\omega)$ is necessarily a pure k -power.

Note that $\ell_{\min \text{ int}} \neq \ell_{\min}$ implies that ω starts or ends with a pure k -power those image by f is minimal and that $\text{Min.Int.Pwrs}(\omega) \neq \text{Min.Pwrs}(\omega)$.

Let $Sp_1(\omega)$ be the set $\{v^k \mid v^k \in \text{Pure.Int.Pwrs}(\omega) \text{ and } \widehat{v}_j \text{ satisfies (P.1) for all integers } j \in [1; k+2-c_v]\}$. Let $Sp_2(\omega)$ (resp. $Sp_{2.1}(\omega)$ and $Sp_{2.2}(\omega)$) be the set $\{v^k \mid v^k \in \text{Pure.Int.Pwrs}(\omega) \setminus Sp_1(\omega) \mid \text{there exists an integer } j \in [1; k+2-c_v] \text{ such that } \widehat{v}_j \text{ satisfies (P.2) (resp. (P.2.1) and (P.2.2))}\}$.

Let $Sp_{1,\min}(\omega)$ be the set $\{v^k \mid v^k \in \text{Min.Int.Pwrs}(\omega) \text{ and } \widehat{v}_j \text{ satisfies (P.1) for all integers } j \in [1; k+2-c_v]\}$. Let $Sp_{2,\min}(\omega)$ (resp. $Sp_{2.1,\min}(\omega)$ and $Sp_{2.2,\min}(\omega)$) be the set $\{v^k \mid v^k \in \text{Min.Int.Pwrs}(\omega) \setminus Sp_1(\omega) \mid \text{there exists an integer } j \in [1; k+2-c_v] \text{ such that } \widehat{v}_j \text{ satisfies (P.2) (resp. (P.2.1) and (P.2.2))}\}$.

Remark 3.4 *If v^k belongs to $Sp_1(\omega)$ then, for any integer $2 \leq \ell \leq k-2$, we have $L_{j,v(\ell)} \neq \emptyset$ and $R_{j,v(\ell)} \neq \emptyset$ for any integer $j \in [1, k+2-c_v]$.*

Remark 3.5 *If $\ell_{\min \text{ int}} = \ell_{\min}$ then $\text{Pure.Int.Pwrs}(\omega) = Sp_1(\omega)$.*

Remark 3.6 *If f is k -power-free and if $v^k \in Sp_{2.1,\min}(\omega)$ (resp. $Sp_{2.2,\min}(\omega)$), then the only integer j such that \widehat{v}_j satisfies (P.2.1) (resp. (P.2.2)) is $j = 1$ (resp. $j = k+2-c_v$). More precisely, $(x_{1,v})^k$ (resp. $(x_{k+2-c_v,v})^k$) is a prefix (resp. a suffix) of ω .*

Remark 3.7 *Let us note that, by Corollary 2.6, if a $(k+1)$ -power v^{k+1} is an internal factor of ω and if $v_{(1)}^k$ and $v_{(2)}^k$ are pure k -powers such that $v^{k+1} = v_{(1)}^k v = v v_{(2)}^k$ then $v_{(1)}^k$ or $v_{(2)}^k$ belongs to $Sp_1(\omega)$.*

4 Almost $(k+1)$ -power

In all this section, $k \geq 4$ is an integer, f is a morphism from A^* to B^* and ω is a word over A such that $f(\omega) = \varrho(\chi\gamma)^k \chi \varsigma$ for some words ϱ, χ, γ and ς over B such that $\chi\gamma \neq \varepsilon$, $|\varrho| < |f(\omega[1])|$ and $|\varsigma| < |f(\omega[|\omega|])|$. We denote $U = \chi\gamma$ and $S = \chi\varsigma$.

For any integer j in $[1, k+1]$, i_j is the smallest integer such that $\varrho(\chi\gamma)^{j-1}$ is a prefix of $f(\omega[1..i_j])$ and i'_j is the smallest integer such that $\varrho(\chi\gamma)^{j-1}\chi$ is a prefix of $f(\omega[1..i'_j])$.

Lemma 4.1 *For any pure k -power v^k of ω , if f is k -power-free and if the words $(\chi\gamma)^k \chi$ and $f(v)^k$ have a common factor of length at least $|\chi\gamma| + |f(v)|$ then $f(\omega)$ and $(\chi\gamma)^k$ are synchronised.*

Proof.

See Step 1 of the proof of Proposition 4.1 in [5].

Let us assume that f is k -power-free and that $(\chi\gamma)^k\chi$ and $f(v)^k$ have a common factor of length at least $|\chi\gamma| + |f(v)|$.

Let v_1 and v_2 be the words such that $\omega = v_1v^kv_2$.

Since $(\chi\gamma)^k\chi$ is a factor of U^{k+1} , by Corollary 1.5, there exist two words t_1 and t_2 and two integers r and q such that $f(v) = (t_1t_2)^r$ and $U = (t_2t_1)^q$ with t_1t_2 and t_2t_1 primitive words.

If $r \geq 2$ then $f(v^{k-1}) = (t_1t_2)^{(k-1) \times r}$. Since $k \geq 3$, we have $(k-1) \times r \geq 2k-2 \geq k$. It means that $f(v^{k-1})$ contains a k -power with v^{k-1} a k -power-free word by definition of v , i.e., f is not k -power-free.

From now, we assume that $r = 1$. If $q \leq k-2$ then v^q would be an internal factor of v^k that is of ω with $|f(v)^q| = |U|$: $f(\omega)$ and U^k are synchronised.

Thus $q \geq k-1$. Since χ is a prefix of $U = \chi\gamma$, let ℓ be the greatest integer such that $(t_2t_1)^\ell$ is a prefix of χ . There exists a prefix T of t_2t_1 different from t_2t_1 such that $\chi = (t_2t_1)^\ell T$. We have $f(\omega) = f(v_1)(t_1t_2)^k f(v_2) = \varrho U^k \chi \varsigma = \varrho (t_2t_1)^{q \times k + \ell} T \varsigma$ with $q \geq k-1$.

Let x be the greatest integer such that $\varrho(t_2t_1)^x$ is a prefix of $f(v_1v)$ and let y be the greatest integer such that $(t_2t_1)^y T \varsigma$ is a suffix of $f(v^2v_2)$. There exist four words $t'_p, t''_p \neq \varepsilon, t'_s \neq \varepsilon$ and t''_s such that $t_2t_1 = t'_p t'_s = t''_p t''_s$, $f(v_1v) = \varrho(t_2t_1)^x t'_p$, $f(v^2v_2) = t''_s (t_2t_1)^y T \varsigma$ and $f(v^{k-3}) = t'_s (t_2t_1)^{qk+\ell-x-y-2} t''_p$.

If $x = 0$ then $|f(v_1v)| = |f(v_1)t_1t_2| = |\varrho t'_p| < |\varrho t_1t_2|$, i.e., $|f(v_1)| < |\varrho| < |f(\omega[1])|$: it implies that $v_1 = \varepsilon$. So, we get that $f(\omega)$ starts with $f(v^3) = (t_1t_2)^3$ and with $\varrho(t_2t_1)^3$. Since t_2t_1 is a primitive word, by Lemma 1.3, (t_2t_1) is not an internal factor of $(t_2t_1)^2$. Since $|\varrho| < |t_1t_2| = |f(v)|$, it implies that $\varrho = t_1$ and $t'_p = t_2$. Moreover $f(\omega) = (t_1t_2)^k f(v_2) = t_1(t_2t_1)^{q \times k + \ell} T \varsigma$ and so $t''_s = t_1$. In the same way, if $y = 0$ we get $\varsigma = t_2$, $t''_s = t_1$ and $t'_p = t_2$. When $x \neq 0$ and $y \neq 0$, since $f(v_1v)$ ends with t_1t_2 and since $f(v^2v_2)$ starts with t_1t_2 , if $t'_p \neq t_2$ or if $t''_s \neq t_1$ then (t_1t_2) is an internal factor of $(t_1t_2)^2$: a contradiction with Lemma 1.3 and the fact that t_1t_2 is a primitive word.

Thus $t'_p = t_2 = t''_p$, $t'_s = t_1 = t''_s$, $f(v_1v) = \varrho t_2 f(v)^x$ and $f(v^2v_2) = f(v)^y t_1 T \varsigma$. Since f is bifix, it follows that $f(v_1v)$ ends with $f(v)^x$ and $f(v^2v_2)$ starts with $f(v)^y$. So, we get that $v^{q \times k + \ell - 1} = v^{x+y+k-3}$ is an internal factor ω with $q \times k + \ell - 1 \geq q$. It implies that v^q is an internal factor of ω with $|f(v)^q| = |U|$, i.e., $f(\omega)$ and U^k are synchronised. \square

Remark 4.2 For any pure k -power v^k of ω , if $f(v)^k$ is an internal factor of $(\chi\gamma)^k\chi$ and if $f(\omega)$ and $(\chi\gamma)^k$ are not synchronised, by Lemma 4.1, then either f is not k -power-free or $|\chi\gamma| > |f(v)^{k-1}| > 3|f(v)|$.

As a corollary of Lemma 1.7 and of Lemma 3.9 in [5], we get :

Corollary 4.3 Let us assume that $f(\omega)$ and $(\chi\gamma)^k\chi$ are not synchronised and that $|f(t)| > |\gamma|$, for all pure k -power $t^k \in \text{Fcts}(\omega)$.

Let v^k be a k -power in $\text{Min.Int.Pwrs}(\omega)$.

For any integer i , if $x_i^3 = \omega[n_{x_i}..m_{x_i}]$ is a word in $L_{i,v} \cup R_{i,v}$, we assume that $m_{x_i} < n_{x_{i+1}}$.

When one of the four following situations holds either f is not k -power-free or there exists a word ω' such that $f(\omega') = \varrho'(\chi'\gamma')^k\chi'\varsigma'$ for some words ϱ' , ς' , χ' and γ' ($= \gamma$) over B satisfying $\chi'\gamma' \neq \varepsilon$, $|\varrho'| < |f(\omega'[1])|$, $|\varsigma'| < |f(\omega'[[\omega']])|$, and $0 < |\chi'\gamma'| < |\chi\gamma|$.

We also get that $f(\omega')$ and $(\chi'\gamma')^k\chi'$ are not synchronised.

Furthermore, for all pure k -powers $(t')^k \in \text{Fcts}(\omega')$, we have $|f(t')| > |\gamma'|$.

1. $d_v = 1$, $|D_v f(v)^2| < |\chi|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, k+1]$.
2. $d_v = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, k]$ and there exists a positive integer ϕ such that $\omega[n_v..|\omega|]$ starts with $v^{\phi+2}$ and $\sup\{2|f(v)|; |D_v f(v)^\phi|\} \leq |\chi| < |D_v f(v)^{\phi+1}|$.
3. $d_v = 0$, $|D_v f(v)^2| \leq |\chi|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k+1]$.
4. $d_v = 0$, $|D_v f(v)| \leq |\chi| < |D_v f(v)^2|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k]$.

Proof of Corollary 4.3 is almost the same than Proof of Lemma 3.9 in [5]. Condition $m_{x_i} < n_{x_{i+1}}$ prevents possible overlaps between consecutive x_j^3 .

Proof.

See Appendix A for figures of different cases.

Let us first note that, either f is not k -power-free or $v^k \in \text{Pure.Int.Pwrs}(\omega)$.

Since $f(x_2)^k$ is an internal factor of $(\chi\gamma)^k\chi$, by Remark 4.2, either f is not k -power-free or $|\chi\gamma| > 3|f(x_2)| = 3|f(v)|$. And, since $|\gamma| < |f(v)|$, we have $|\chi| > 2|f(v)|$.

Case (1): $d_v = 1$, $|D_v f(v)^2| < |\chi|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, k+1]$.

If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 1$ (including $x_{j_0} = v$) or if $x_j^3 \in R_{j,v}$, let X_j be the word D_{x_j} and let e_j be the integer d_{x_j} . If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 0$, let X_j be the suffix of $f(x_j)$ such that $D_{x_j} f(x_j)^2 = \chi\gamma X_j$ and let $e_j = 2$. Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [2; k+1]\}$. For all integers $j \in [2, k+1]$, if $d_{x_j} = 0$ with $x_j^3 \in L_{j,v}$, or if $d_{x_j} = 1$, then, by definition, we have that X_j is a suffix of $f(x_j)$. If $d_{x_j} = 0$ with $x_j^3 \in R_{j,v}$ then it means that $D_v = D_{x_j} t_{2,j}$. But D_v is a suffix of $f(v) = t_{1,j} t_{2,j}$. So, it implies that $X_j = D_{x_j}$ is a suffix of $t_{1,j}$ and of $f(x_j) = t_{2,j} t_{1,j}$.

In particular, X_q is a suffix of $f(x_q)$. It follows that $0 \leq |X_q| - |X_j| \leq |X_q| \leq |f(x_q)|$ for all integers $j \in [2, k+1]$. Furthermore, if $d_{x_j} = 0$ with $x_j^3 \in R_{j,v}$ then $\lambda_{x_j} = j$, and $\lambda_{x_j} = j - 1$ otherwise. It follows that $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = \varrho U^{j-1} X_j = \varrho(\chi\gamma)^{j-1} X_j$.

Since $|X_j f(x_j)| \leq 2|f(x_j)| = |f(v)^2| \leq |\chi|$, it follows that $X_j f(x_j)$ is a prefix of χ . Hence, there exists a word \bar{Y}_j such that $\chi = X_j f(x_j) \bar{Y}_j$ for all integers $j \in [2, k+1]$. Let w_2 be the prefix of ω such that $f(w_2) = \varrho\chi\gamma X_2$, i.e., $w_2 = \omega[1..n_{x_2} - 1]x_2^{e_2}$ and let w_{k+2} be the suffix of ω such that $f(w_{k+2}) = \bar{Y}_{k+1}\varsigma$, i.e., $\omega = \omega[1..n_{x_{k+1}} - 1]x_{k+1}^{1+e_{k+1}}w_{k+2}$. In particular, we have $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j})\bar{Y}_j\gamma X_{j+1}$ for all integers $j \in [2, k]$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = \bar{Y}_{j-1}\gamma X_j$ for all integers $j \in [3, k+1]$.

In summary, we obtain that $\omega = w_2x_2w_3x_3..w_{k+1}x_{k+1}w_{k+2}$, $f(\omega) = \varrho(\chi\gamma)^k\chi\varsigma$ with $\chi\gamma = X_jf(x_j)\bar{Y}_j\gamma$ for all integers $j \in [2, k]$ and $\chi\gamma = X_{k+1}f(x_{k+1})\bar{Y}_{k+1}\gamma$. Moreover, there exists an integer $q \in [2, k+1]$ such that $0 \leq |X_q| - |X_j| \leq |X_q| \leq |f(x_q)|$ and X_q is a suffix of $f(x_q)$.

Taking $Y_{k+1} = \bar{Y}_{k+1}$ and $Y_j = \bar{Y}_j\gamma$ for all integers $j \in [2, k]$, by Corollary 2.10 (or Lemma 2.8 and using Remark 2.9 (Points 3 and 4)), in particular the property of synchronised words, we can reduce $f(\omega)$. More precisely, let χ' be the non-empty word $X_q\bar{Y}_q$, i.e., χ' is the suffix of χ of length $|\chi| - |f(v)|$. And let \bar{w}_2 be the shortest suffix of w_2 such that $f(\bar{w}_2)$ ends with $\chi'X_2$ and let ω' be the word $\bar{w}_2w_3..w_{k+1}w_{k+2}$.

We obtain that $f(\omega') = \varrho'(\chi'\gamma)^k\chi'\varsigma$ for a word $|\varrho'| < |f(\omega'[1])| = |f(\bar{w}_2[1])|$. And we denote $\gamma' = \gamma$ and $\varsigma' = \varsigma$.

Let us recall that, by hypothesis, $m_{x_i} < n_{x_{i+1}}$, for all integers $i \in [2, k]$. Briefly, it means that either w_i ends by x_i and w_{i+1} starts with x_i , or w_i ends by x_i^2 , or w_{i+1} starts with x_i^2 . More precisely, let $\tau_2 = \omega[1..n_{x_2} - 1]$, $\tau_{k+2} = \omega[m_{x_{k+1}} + 1..|\omega|]$ and, for all integers $i \in [3, k+1]$, let $\tau_i = \omega[m_{x_{i-1}} + 1..n_{x_i} - 1]$. We get that $\omega = \tau_2x_2^3\tau_3x_3^3.. \tau_{k+1}x_{k+1}^3\tau_{k+2}$ and that ω' is a suffix of $\tau_2x_2^2\tau_3x_3^2.. \tau_{k+1}x_{k+1}^2\tau_{k+2}$, i.e., $w_i = x_{i-1}^{\alpha_i-1}\tau_ix_i^{\beta_i}$, with $\alpha_i + \beta_i = 2$ and $\alpha_1 = \beta_{k+2} = 0$. In fact, since x_1 and x_{k+2} are not defined, we have $w_2 = \tau_2x_2^{\beta_2}$ and $w_{k+2} = x_{k+1}^{\alpha_{k+1}}\tau_{k+2}$. In particular, for all integers $i \in [3, k]$, we have $w_{i-1}w_iw_{i+1}w_{i+2} = x_{i-2}^{\alpha_{i-2}}\tau_{i-1}x_{i-1}^2\tau_ix_i^2\tau_{i+1}x_{i+1}^2\tau_{i+2}x_{i+2}^{\beta_{i+2}}$.

Let $(t')^k$ be a pure k -power of ω' , i.e., $f(t')^k \in \text{Fcts}(f(w_2w_3 \dots w_{k+1}w_{k+2}))$.

If $(t')^k \in \text{Fcts}(\omega)$ then $|f(t')| > |\gamma| = |\gamma'|$.

If $(t')^k \notin \text{Fcts}(\omega)$ and, for all integers $i \in [3, k]$, $f(t')^k \notin \text{Fcts}(f(w_{i-1}w_iw_{i+1}w_{i+2}))$, it implies that there exists an integer j such that $|f(t')^k| > |f(w_jw_{j+1}w_{j+2})| \geq 2|\chi'\gamma'| - |f(v)| \geq 2|\chi| - 3|f(v)| > |f(v)| > |\gamma| = |\gamma'|$.

If $(t')^k \notin \text{Fcts}(\omega)$ and $f(t')^k \in \text{Fcts}(f(w_{i-1}w_iw_{i+1}w_{i+2}))$ for an integer $i \in [3, k]$, let us denote $\tau'_{i-1} = x_{i-2}^{\alpha_{i-2}}\tau_{i-1}$, $\tau'_i = \tau_i$, $\tau'_{i+1} = \tau_{i+1}$, $\tau'_{i+2} = \tau_{i+2}x_{i+2}^{\beta_{i+2}}$. Thus, we have $f(t')^k \in \text{Fcts}\left(f(\tau'_{i-1})\prod_{j=1}^3f(x_{i-2+j})^2f(\tau'_{i-1+j})\right)$. Let T be the word $f(\tau'_{i-1})\prod_{j=1}^3f(x_{i-2+j})^3f(\tau'_{i-1+j})$.

If $f(t')^k \notin \text{Fcts}(T)$, by Corollary 1.20 with $\ell = k - 3$ and $s = 3$, we get that either f is not k -power-free or $|f(t')| > |f(x_i)| = |f(v)| > |\gamma| = |\gamma'|$. If $f(t')^k \in \text{Fcts}(T) \subset \text{Fcts}(f(\omega))$, let T' be the shortest factor of T that contains $f(t')^k$. By Lemma 2.1, either f is not k -power-free or $T' = x^ky$ or yx^k with $|f(t')| \geq |f(x)| > |\gamma| = |\gamma'|$.

Case (2): $d_v = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, k]$ and there exists a positive integer ϕ such that $\omega[n_v..|\omega|]$ starts with $v^{\phi+2}$ and $\sup\{2|f(v)|; |D_v f(v)^\phi|\} \leq |\chi| < |D_v f(v)^{\phi+1}|$.

In this case, χ is an internal factor of $f(v)^{\phi+2}$. More precisely, χ is a prefix of $D_v f(v)^{\phi+1}$.

For every integer $j \in [2, k]$, we define X_j and e_j as Case (1) and we obtain that X_j is also a suffix of $f(x_j)$.

If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, then χ is a prefix of the word $D_v f(v)^{\phi+1} = X_j t_{2,j}(t_{1,j}t_{2,j})^{\phi+1}$ and so of $X_j f(x_j)^{\phi+2}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$, since $(\chi\gamma)^2$ is a prefix of $\chi\gamma D_v f(v)^{\phi+1} = \chi\gamma D_v(t_{1,j}t_{2,j})^{\phi+1} = D_{x_j}(t_{2,j}t_{1,j})^{\phi+2}t_{2,j} = \chi\gamma X_j f(x_j)^\phi t_{2,j}$, it follows that χ is a prefix of

$X_j f(x_j)^{\phi+1}$.

In the same way, we show that χ is a prefix of $X_j f(x_j)^{\phi+1}$ when $x_j^3 \in R_{j,v}$.

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [2; k]\}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or if $x_j^3 \in R_{j,v}$ with $d_{x_j} = 0$ then $|X_j| \leq |X_{j_0}| = |X_v|$. Thus, if $q \neq j_0$, either $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$, or $x_j^3 \in R_{j,v}$ with $d_{x_j} = 1$. Let δ be the greatest integer such that $|X_q f(x_q)^\delta| \leq |\chi| < |X_q f(x_q)^{\delta+1}|$.

For every integer $j \in [2, k]$, since $|X_j f(x_j)^\delta| \leq |X_q f(x_q)^\delta| \leq |\chi|$, there exists a word \bar{Y}_j such that $\chi = X_j f(x_j)^\delta \bar{Y}_j$. Since χ is a prefix of $X_q f(x_q)^{\phi+2}$, we obtain $\chi = X_q f(x_q)^\delta \bar{Y}_q$ with \bar{Y}_q a prefix of $f(x_q)$.

Let w_2 be the prefix of ω such that $f(w_2) = \varrho \chi \gamma X_2$, let w_{k+1} be the suffix of ω such that $f(w_{k+1}) = \bar{Y}_k \gamma \chi \varsigma$ and, for all integers $j \in [3, k]$, let w_j be the word such that $f(w_j) = \bar{Y}_{j-1} \gamma X_j$.

Taking $Y_j = \bar{Y}_j \gamma$ for all integers $j \in [2, k]$, by Corollary 2.10 (or Lemma 2.8 and using Remark 2.9(6)), we can reduce $f(\omega)$. More precisely, let χ' be non-empty the word $X_q \bar{Y}_q$. Accordingly, χ' is both a prefix and a suffix of χ so χ' is an internal factor of $f(v)^{\phi+1}$. Let \bar{w}_2 be the shortest suffix of w_2 such that $f(\bar{w}_2)$ ends with $\chi' \gamma X_2$. Let \bar{w}_{k+1} be the shortest prefix of w_{k+1} such that $f(\bar{w}_{k+1})$ starts with $\bar{Y}_k \gamma \chi'$. There exists a word ς' such that $f(\bar{w}_{k+1}) = \bar{Y}_k \gamma \chi' \varsigma'$. Let ω' be the word $\bar{w}_2 w_3 \dots w_k \bar{w}_{k+1}$. If we denote $\gamma' = \gamma$, we obtain $f(\omega') = \varrho (\chi' \gamma')^k \chi' \varsigma'$ where $|\chi'| = |\chi| - |f(x_q)| < |\chi|$.

For any pure k -power $(t')^k \in \text{Fcts}(\omega')$, as in Case (1), we show that either f is not k -power-free or $|f(t')| > |\gamma'|$.

Case (3): $d_v = 0$, $|D_v f(v)^2| \leq |\chi|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k+1]$.

For every integer $j \in [1, k+1]$, let X_j be the word $D_{x_j} f(x_j)$ if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ (including $x_{j_0} = v$), or the word D_{x_j} if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or if $x_j^3 \in R_{j,v}$.

If $x_j^3 \in L_{j,v}$, let $e_j = 1$, and if $x_j^3 \in R_{j,v}$, let $e_j = 0$.

For any word $x_j^3 \in R_{j,v}$, since $|D_v f(v)^2| \leq |\chi| \leq |\chi \gamma|$, we necessarily have $d_{x_j} = 0$. Furthermore, $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})| = |f(v)|$.

If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 0$, we have $X_v = D_v f(v) = D_{x_j} t_{2,j} t_{1,j} t_{2,j} = X_j t_{2,j}$ and so $0 \leq |X_v| - |X_j| = |t_{2,j}| < |f(v)|$. If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 1$, we have $X_j t_{2,j} = D_{x_j} t_{2,j} = D_v t_{1,j} t_{2,j} = X_v$ and so $0 \leq |X_v| - |X_j| = |t_{2,j}| < |f(v)|$.

We have $|X_v| = \max\{|X_j|; j \in [1; k+1]\}$ and $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = \varrho (\chi \gamma)^{j-1} X_j$ for all integers $j \in [1, k+1]$.

Since $|X_j f(x_j)| \leq |D_v f(v)^2| \leq |\chi|$, the word $X_j f(x_j)$ is a prefix of χ . Thus, there exist words \bar{Y}_j such that $\chi = X_j f(x_j) \bar{Y}_j$ for all j in $[1, k+1]$. Let w_1 be the word $\omega[1..n_{x_1} - 1]x_1^{e_1}$ and let w_{k+2} be the word such that $\omega[n_{x_{k+1}}..|\omega|] = x_{k+1}^{1+e_{k+1}} w_{k+2}$. In particular, we have $f(w_1) = pX_1$, $f(w_{k+2}) = \bar{Y}_{k+1} \varsigma$ and, for every integer $j \in [1, k]$, $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j} \bar{Y}_j \gamma X_{j+1})$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = \bar{Y}_{j-1} \gamma X_j$ for all integers $j \in [2, k+1]$.

By Corollary 2.10 (or Lemma 2.8), we can reduce $f(\omega)$. More precisely, $\omega' = w_1 w_2 \dots w_{k+1} w_{k+2}$, $\varrho' = \varrho$, $\varsigma' = \varsigma$, and $\chi' = X_i \bar{Y}_i (\neq \varepsilon)$ for all integers $i \in [1, k+1]$. We obtain $f(\omega') = \varrho' (\chi' \gamma)^k \chi' \varsigma'$ with $|\varrho'| < |f(\omega[1])| = |f(w_1[1])| = |f(\omega'[1])|$, $|\varsigma'| < |f(\omega[[\omega]])| = |f(\omega'[[\omega']])|$, $\gamma' = \gamma$ and $|\chi'| = |\chi| - |f(v)| < |\chi|$.

For any pure k -power $(t')^k \in \text{Fcts}(\omega')$, as in Case (1), we show that either f is not k -power-free or $|f(t')| > |\gamma'|$.

Case (4): $d_v = 0$, $|D_v f(v)| \leq |\chi| < |D_v f(v)^2|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k]$.

Let S_2 be the set of integers j such that there exists a word x_j^3 in $R_{j,v}$ with $d_{x_j} = 1$ but no word in $R_{j,v}$ with $d_{x_j} = 0$ and no word in $L_{j,v}$.

Case 4.1: $S_2 = \emptyset$

If $x_j^3 \in R_{j,v}$ (with $d_{x_j} = 0$), if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| \geq |\chi|$ then let X_j be the word D_{x_j} and let $e_j = d_{x_j}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| < |\chi|$ then let X_j be the word $D_{x_j} f(x_j)$ and let $e_j = 1$. For all integers $j \in [1, k]$, we have $f(\omega[1..n_{x_j} - 1]) f(x_j^{e_j}) = \varrho (\chi \gamma)^{j-1} X_j$.

For all integers $j \in [1, k]$, $X_j f(x_j)$ is a prefix of χ . Consequently, there exists a word \bar{Y}_j such that $\chi = X_j f(x_j) \bar{Y}_j$. Since $|\chi f(x_j)| > |X_j f(x_j)^2| \geq |\chi|$, we obtain that $X_j f(x_j)^2$ is a prefix of $(\chi \gamma)^2$. It follows that \bar{Y}_j is a prefix of $f(x_j)$.

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [1, k]\}$. In particular, we have $|\bar{Y}_q| \leq |f(x_q)|$ and $0 \leq |X_q| - |X_j| = |\bar{Y}_j| - |\bar{Y}_q| \leq |f(x_j)| = |f(x_q)|$ for every integer j in $[1, k]$.

Let w_1 be the word $\omega[1..n_{x_1} - 1] x_1^{e_1}$ and let w_{k+1} be the word such that $\omega[n_{x_k} \dots |\omega|] = x_k^{1+e_k} w_{k+1}$. We have $f(w_1) = \varrho X_1$, $f(w_{k+1}) = \bar{Y}_k \gamma \chi \varsigma$. We obtain that $f(\omega[n_{x_j} \dots n_{x_{j+1}} - 1]) f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j}) \bar{Y}_j \gamma X_{j+1}$ for all integers $j \in [1, k-1]$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = \bar{Y}_{j-1} \gamma X_j$ for all integers $j \in [2, k]$.

By Lemma 2.8 and using Remark 2.9(5), we can reduce $f(\omega)$.

The non-empty word $\chi' = X_{k+1} \bar{Y}_{k+1}$ is a prefix of χ . Moreover, any suffix of χ of length at most $\max\{|\bar{Y}_i|\}$ is also a prefix of χ' . Let \bar{w}_{k+1} be the shortest prefix of w_{k+1} such that $f(\bar{w}_{k+1})$ starts with χ' . There exists a word ς' such that $f(\bar{w}_{k+1}) = \chi' \varsigma'$. We take $\omega' = w_1 w_2 \dots \bar{w}_{k+1}$, $\gamma' = \gamma$, $\varrho' = \varrho$. Hence, $f(\omega') = \varrho (\chi' \gamma)^k \chi \varsigma$ starts with $\varrho' (\chi' \gamma)^k \chi' \varsigma'$. Moreover, $|\varrho'| = |\varrho| < |f(W[1])| = |f(w_1[1])| = |f(\omega'[1])|$ and $|\chi'| < |\chi|$.

For any pure k -power $(t')^k \in \text{Fcts}(\omega')$, as in Case (1), we show that either f is not k -power-free or $|f(t')| > |\gamma'|$.

Case 4.2: $S_2 \neq \emptyset$

Let $j'_0 \in S_2$ and let $v' = x_{j'_0}$.

If $j \in S_2$, let X_j be the word D_{x_j} and let $e_j = 1$.

If $j \notin S_2$, we assume that if $x_j^3 \in R_{j,v}$ then $d_{x_j} = 0$ else we take $x_j^3 \in L_{j,v}$. If $x_j^3 \in R_{j,v}$ (with $d_{x_j} = 0$), or if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| > |\chi|$ (for instance x_{j_0}), let X_j be the word such that $D_{x_j} f(x_j)^2 = \chi X_j$ and let $e_j = 2$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| \leq |\chi|$, let X_j be the word such that $D_{x_j} f(x_j)^3 = \chi X_j$ and let

$e_j = 3$. For all integers $j \in [1, k]$, we have $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = \varrho(\chi\gamma)^j X_j$. Especially, the word X_j is a suffix of $f(x_j)$ for every integer $j \in [1, k]$.

Let j_1 be an integer in S_2 , i.e., $x_{j_1}^3 \in R_{j,v}$ and $d_{x_{j_1}} = 1$. Hence, $(\chi\gamma)^2$ starts with $X_{j_1}f(x_{j_1}^2)$. By definition, we have $\chi X_{j_0} = D_v f(v)^2 = \chi X_{j_1} t_{2,j}$. For any word $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or with $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| \leq |\chi|$, always by definitions, we obtain $\chi X_j = \chi X_{j_0} t_{1,j}$. It follows that $|f(x_{j_1})| = |f(x_j)| \geq |X_j| > |X_{j_0}| \geq |X_{j_1}|$. Furthermore, the words $f(x_j)$ and $f(x_{j_1})$ are conjugated. Let $\tau_{2,j}$ be the non-empty suffix of X_j (and of $f(x_j)$) such that $X_j = X_{j_1} \tau_{2,j}$ and let $\tau_{1,j}$ be the word such that $f(x_j) = \tau_{1,j} \tau_{2,j}$. Since χX_{j_1} ends with $\tau_{1,j}$, we obtain $f(x_{j_1}) = \tau_{2,j} \tau_{1,j}$. Thus, $(\chi\gamma)^2$ starts with $X_{j_1} (\tau_{2,j} \tau_{1,j})^2 = X_j f(x_j) \tau_{1,j}$. Since f is bifix, it implies that $\omega[m_{x_j} + 1..|\omega|]$ also starts with x_j . In other words, x_j^3 is followed by x_j in ω .

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [1, k]\}$. In particular, $0 \leq |X_q| - |X_j| \leq |f(x_q)|$.

Since $|\chi| > 2|f(v)|$, the word $X_j f(x_j)$ is a prefix of χ , for all integers $j \in [1, k]$. Consequently, there exists a word \bar{Y}_j such that $\chi = X_j f(x_j) \bar{Y}_j$.

Let w_2 be the prefix of ω such that $f(w_2) = \varrho \chi X_1$, that is, $w_2 = \omega[1..n_{x_1} - 1] x_1^{e_1}$ and let w_{k+2} be the suffix of ω such that $f(w_{k+2}) = \bar{Y}_{k+1} S$, that is, $\omega = \omega[1..n_{x_{k+1}} - 1] x_{k+1}^{1+e_{k+1}} w_{k+2}$. Accordingly, for all integers $j \in [1, k-1]$, we have $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j}) \bar{Y}_j \gamma X_{j+1}$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = \bar{Y}_{j-1} \gamma X_j$ for all integers $j \in [3, k+1]$.

By Lemma 2.8 and using Remark 2.9(4), we can reduce $f(\omega)$. Reduction is almost the same that case where $d_v = 1$, $|D_v f(v)^2| < |\chi|$, and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [2, k+1]$. Let us note that χ' is a suffix of χ and that any prefix of χ of length at most $\max\{|X_j|\}$ is also a prefix of χ' .

For any pure k -power $(t')^k \in \text{Fcts}(\omega')$, as in Case (1), we show that either f is not k -power-free or $|f(t')| > |\gamma'|$. \square

Proposition 4.4 *Let $k \geq 5$, let f be a morphism from A^* to B^* and let ω be a word over A .*

We assume that $f(\omega) = \varrho(\chi\gamma)^k \chi \varsigma$ for some words ϱ, ς, χ and γ such that $|\varrho| < |f(\omega[1])|$, $|\varsigma| < |f(\omega[|\omega|])|$, $\chi\gamma \neq \varepsilon$.

If $f(\omega)$ and $(\chi\gamma)^k \chi$ are not synchronised and if $|\gamma| < |f(t)|$, for all pure k -powers $t^k \in \text{Fcts}(\omega)$ then f is not k -power-free.

Proof of Proposition 4.4 is done using iteratively Lemma 4.5. By induction, if f was not k -power-free, we could find an infinite sequence $(\omega_i, \chi_i, \gamma_i)_{i \geq 0}$ of words starting with $(\omega_0, \chi_0, \gamma_0) = (\omega, \chi, \gamma)$ such that $0 < |\chi_{i+1} \gamma_{i+1}| < |\chi_i \gamma_i|$: this is impossible.

Lemma 4.5 *Let $k \geq 5$, let f be a morphism from A^* to B^* and let ω be a word over A .*

We assume that $f(\omega) = \varrho(\chi\gamma)^k \chi \varsigma$ for some words ϱ, ς, χ and γ such that $|\varrho| < |f(\omega[1])|$, $|\varsigma| < |f(\omega[|\omega|])|$ and $\chi\gamma \neq \varepsilon$. We also assume that $f(\omega)$ and $(\chi\gamma)^k \chi$ are not synchronised,

and that $|\gamma| < |f(t)|$, for all pure k -powers $t^k \in \text{Fcts}(\omega)$.

Then either f is not k -power-free or there exist a word $\tilde{\omega}$ such that $f(\tilde{\omega}) = \tilde{\varrho}(\tilde{\chi}\tilde{\gamma})^k\tilde{\chi}\tilde{\zeta}$ for some words $\tilde{\varrho}$, $\tilde{\zeta}$, $\tilde{\chi}$ and $\tilde{\gamma}$ satisfying $|\tilde{\varrho}| < |f(\tilde{\omega}[1])|$, $|\tilde{\zeta}| < |f(\tilde{\omega}[\tilde{\omega}])|$, $\tilde{\chi}\tilde{\gamma} \neq \varepsilon$, and $0 < |\tilde{\chi}\tilde{\gamma}| < |\chi\gamma|$. Moreover, $f(\tilde{\omega})$ and $(\tilde{\chi}\tilde{\gamma})^k\tilde{\chi}$ are not synchronised, and $|\tilde{\gamma}| < |f(\tilde{t})|$ for all pure k -powers $(\tilde{t})^k \in \text{Fcts}(\tilde{\omega})$.

Proof.

If f is not k -power-free, in particular, if f is not a ps-morphism then it ends the proof. Hence, we assume that f is a ps-morphism. In particular, f is injective.

Let us denote $U = \chi\gamma$, $\bar{U} = \gamma\chi$ and $S = \chi\varsigma$: we get that $f(\omega) = \varrho U^k S$.

The words ω necessarily contains a k -power. Indeed, the contrary ends the proof: f would not be k -power-free.

For any pure k -power $v^k \in \text{Fcts}(\omega)$, since $f(\omega)$ and $(\chi\gamma)^k\chi$ are not synchronised, by Lemma 4.1, $(\chi\gamma)^k\chi$ and $f(v)^k$ do not have any common factor of length at least $|\chi\gamma| + |f(v)|$.

Step 1 : $\omega[2..|\omega| - 1]$ contains a k -power and so a pure- k -power.

The proof is almost the same as the corresponding step in the proof of Proposition 4.1 in [5].

By contradiction, let us assume that $\omega[2..|\omega| - 1]$ is k -power-free. It implies that ω starts or ends with a pure k -power. Let s_1 and p_{k+2} be the words such that $f(\omega[1]) = \varrho s_1$ and $f(\omega[|\omega|]) = p_{k+2}\varsigma$, that is, $(\chi\gamma)^k\chi = s_1 f(\omega[2..|\omega| - 1]) p_{k+2}$.

If $|s_1| \leq |(\chi\gamma)^{k-1}\chi|$ then there exists a word U_c such that $s_1 U_c$ is the prefix of $s_1 f(\omega[2..|\omega| - 1]) p_{k+2} = (\chi\gamma)^k\chi$ of length $|s_1(\chi\gamma)|$. Trivially, the word U_c is a conjugate of $\chi\gamma$ (and $|U_c| = |\chi\gamma|$).

If $|s_1| + |p_{k+2}| \leq |\chi|$, we naturally have $|s_1| \leq |(\chi\gamma)^{k-1}\chi|$. Moreover $|s_1| + |U_c^k| + |p_{k+2}| \leq |(\chi\gamma)^k\chi|$. It means that $f(\omega[2..|\omega| - 1])$ starts with U_c^k . Since $\omega[2..|\omega| - 1]$ is a k -power-free word, it ends the proof, f is not k -power-free.

Let us now study the case where $|s_1| + |p_{k+2}| > |\chi|$.

Let us recall that, since we assume that $\omega[2..|\omega| - 1]$ is k -power-free, any pure k -power of $\omega = \omega[1..|\omega|]$ is necessarily a prefix or a suffix of it.

If ω starts with a pure k -power t^k , let W_{com} be the greatest prefix of $s_1 f(t[2..|t|]) f(t^{k-1})$ that is a factor of $(\chi\gamma)^k\chi$ so a common factor of a power of $f(t)$ and a power of $\chi\gamma$. Let us note that if $\omega = t^k$ then $W_{com} = (\chi\gamma)^k\chi$ otherwise $W_{com} = s_1 f(t[2..|t|]) f(t^{k-1})$.

If $|W_{com}| \geq |\chi\gamma| + |f(t)|$, by Corollary 1.5, there exist two words t_1 and t_2 , and two integers r and q such that $f(t) = (t_1 t_2)^r$ and $\chi\gamma = (t_2 t_1)^q$ with $t_1 t_2$ and $t_2 t_1$ primitive words. Since t^k is a pure k -power, it follows that $r = 1$. Otherwise, f is not k -power-free. Since $f(\omega)$ and $\chi\gamma$ are not synchronised, we have $|f(t)| \neq |\chi\gamma|$, i.e., $q \geq 2$. Thus $f(\omega[2..|\omega|])$ contains $(t_2 t_1)^{qk-1}$ with $qk - 1 \geq k$. Either f is not k -power-free or $\omega[2..|\omega|]$ ends with a k -power. In this second case, by a length criterion, $f(\omega[2..|\omega| - 1])$ necessarily contains $(t_2 t_1)^{qk-1-q}$ with $qk - 1 - q \geq k$: f is not k -power-free.

So, we have $|W_{com}| < |\chi\gamma| + |f(t)|$. By definition of W_{com} , if $\omega = t^k$ then $W_{com} = (\chi\gamma)^k\chi =$

$s_1 f(t[2..|t|])f(t^{k-2})f(t[1..|t|-1])p_{k+2}$ would be a common factor of $f(t)^k$ and $(\chi\gamma)^k\chi$ with $|W_{com}| \geq 2|f(t)|$ and $|W_{com}| > 2|\chi\gamma|$. That is $|W_{com}| > |f(t)| + |\chi\gamma|$: a contradiction. It follows that $\omega \neq t^k$ and $|W_{com}| = |f(t)| + |s_1 f(t[2..|t|])f(t^{k-2})|$. Since $|\gamma| < |f(t)|$ and $|s_1| \leq |f(t)|$, we get that $|W_{com}| > |f(t)| + |\gamma| + 2|s_1|$. So it implies $|s_1| < |\chi|/2$. If ω does not end with a k -power, we get that $\omega[2..|\omega|]$ is k -power-free and $f(\omega[2..|\omega|])$ contains the k -power $(\gamma\chi)^k$: f is not k -power free.

In the case where ω ends with a k -power $(t')^k$, we similarly obtain $|p_{k+2}| < |\chi|/2$ and ω starts with a k -power t^k . It follows that $|s_1| + |p_{k+2}| < |\chi|$: a final contradiction.

Step 2: For any pure k -power $v^k \in \text{Fcts}(\omega[2..|\omega|-1])$, $|f(v^{k-2})| < |\chi|$ and the word $f(v)^k$ is an internal factor of $(\chi\gamma)^2\chi$, i.e., $c_v = 1, 2$ or 3 .

For any pure k -power $v^k \in \text{Fcts}(\omega[2..|\omega|-1])$, the word $f(v)^k$ is an internal factor of $(\chi\gamma)^k\chi$. So, by Lemma 4.1, $|f(v)^k| < |\chi\gamma| + |f(v)|$, i.e., $|f(v)^{k-1}| < |\chi\gamma| < |\chi| + |f(v)|$. It follows that $|f(v)^{k-2}| < |\chi|$ and $|f(v)^k| < |\chi\gamma\chi|$. That is, $f(v)^k$ is an internal factor of $(\chi\gamma)^2\chi$. It implies $c_v = 1, 2$ or 3 .

Let us recall that, for every integer $j \in [1; k+2-c_v]$, $f(v)^k$ is an internal factor of $p_j(\chi\gamma)^{c_v} s_{j+c_v}$ and \hat{v}_j is the shortest factor of $W[i_j..i_{j+c_v}]$ such that $f(\hat{v}_j)$ contains $f(v)^k$.

Step 3: Case $Sp_1(\omega) \neq \emptyset$

Let $v^k \in Sp_{1,min}(\omega)$, i.e., v^k is a pure k -power such that $|f(v)|$ is minimal, and \hat{v}_j satisfies *P.1*, for all integers $j \in [1, k+2-c_v]$. See Remark 3.1 for the notations. In particular, let us recall that there exist a letter y_j and a word $x_{j,v}$ such that $|f(v)| = |f(x_{j,v})|$, and $\hat{v}_j = x_{j,v}^k y_j$ or $\hat{v}_j = y_j x_{j,v}^k$.

We are going to see that it implies that either f is not k -power-free or $f(\omega)$ can be reduced. These reductions using Corollary 4.3 create news words $\tilde{\omega}$, $\tilde{\chi}$ and $\tilde{\gamma}$ that satisfy all the necessary conditions.

Let us also recall that we denote by $z_{(\beta)}^3$ the β^{th} factor of z^3 in a k -power z^k , that is, $z^k = z^{\beta-1} z_{(\beta)}^3 z^{k-\beta-2}$ with $1 \leq \beta \leq k-2$.

Case 3.1: $c_v = 3$

We have $2|f(v)| \leq |f(v)^{k-2}| < |\chi|$. Moreover, since $f(v)^k$ and $(\chi\gamma)^k\chi$ do not have any common factor of length at least $|\chi\gamma| + |f(v)|$, we necessarily have $d_v = 1$ and $|D_v f(v^{k-2})| < |\chi\gamma| < |D_v f(v^{k-1})|$. That is, $|D_v f(v^\phi)| \leq |\chi| < |D_v f(v^{\phi+1})|$ with $\phi = k-3$ or $k-2$.

For every integer $j \in [1; k-1]$, since $f(x_{j,v}^k)$ and $(\chi\gamma)^k\chi$ do not have any common factor of length at least $|\chi\gamma| + |f(x_{j,v})|$, we necessarily have $d_{x_{j,v}} = 1$. For every integer $j \in [1; k-1]$, if \hat{v}_j satisfies *(P.1.1)* then $(x_{j,v}^3)_{(1)} \in L_{j+1, v_{(1)}}$ and if \hat{v}_j satisfies *(P.1.2)* then $(x_{j,v}^3)_{(1)} \in R_{j+1, v_{(1)}}$. In other words, we have $L_{j+1, v_{(1)}} \cup R_{j+1, v_{(1)}} \neq \emptyset$ with $j+1 \in [2; k]$. By Proposition 3.3 and Corollary 4.3(2), either f is not k -power-free or we can reduce ω .

Case 3.2: $c_v \neq 3$ and there exists a positive integer $\beta (\leq k-2)$ such that $d_{v_{(\beta)}} = 1$

We necessarily have $c_v = 2$ and thus $k+2-c_v = k$. Since $f(v)^k$ and $(\chi\gamma)^k\chi$ do not have any common factor of length at least $|\chi\gamma| + |f(v)|$ and since $c_v = 2$, we have $|D_{v_{(\beta)}} f(v_{(\beta)})^2| <$

$|\chi|$.

For every integer $j \in [1; k]$, if \widehat{v}_j satisfies (P.1.1) then $(x_{j,v}^3)_{(\beta)} \in L_{j+1, v(\beta)}$ and if \widehat{v}_j satisfies (P.1.2) then $(x_{j,v}^3)_{(\beta)} \in R_{j+1, v(\beta)}$. That is, $L_{j, v(\beta)} \cup R_{j, v(\beta)} \neq \emptyset$ for every integer $j \in [2; k+1]$. By Corollary 4.3(1), either f is not k -power-free or a reduction can be done.

Case 3.3: $c_v \neq 3$ and, for every positive integer $\beta (\leq k-2)$, we have $d_{v(\beta)} = 0$

It means that $|D_v f(v)^{k-2}| \leq |\chi\gamma|$.

If $c_{v(1)} = 1$ then $|D_{v(1)} f(v(1))^2| \leq |\chi|$ and $L_{j, v(1)} \cup R_{j, v(1)} \neq \emptyset$ for every integer $j \in [1; k+1]$. By Corollary 4.3(3), either f is not k -power-free or a reduction can be done.

If $c_{v(1)} = 2$, there exists an integer $\phi \geq \beta$ such that $|D_{v(\phi)} f(v(\phi))| \leq |\chi| < |D_{v(\phi)} f(v(\phi))^2|$ and $L_{j, v(\phi)} \cup R_{j, v(\phi)} \neq \emptyset$ for every integer $j \in [1; k]$. By Corollary 4.3(4), either f is not k -power-free or a reduction can be done.

Step 4: Case $Sp_1(\omega) = \emptyset$ (and therefore $\ell_{\min int} \neq \ell_{\min}$)

By Corollary 2.5 and Corollary 2.6 (see also Remark 3.7), it follows that ω does not contain any $(k+1)$ -power, otherwise, we would have $Sp_1(\omega) \neq \emptyset$.

Supposing that f is k -power-free, let us recall that, if v^k is a k -power of $Min.Int.Pwrs(\omega)$ and then v^k is a pure k -power, i.e., $v^k \in Pure.Int.Pwrs(\omega)$. Moreover, we necessarily have $v^k \in Sp_{2, min}(\omega) = Sp_{2.1, min}(\omega) \cup Sp_{2.2, min}(\omega)$. Let us also recall that if $v^k \in Sp_{2.1, min}(\omega)$ (resp. $Sp_{2.2, min}(\omega)$), then the only integer j such that \widehat{v}_j satisfies (P.2.1) (resp. (P.2.2)) is $j = 1$ (resp. $j = k + 2 - c_v$) and $x_{1,v}^k$ (resp. $x_{k+2-c_v, v}^k$) is a prefix (resp. a suffix) of ω . Finally, let us recall that, if $\varrho = \varepsilon$ (resp. $\varsigma = \varepsilon$), any k -power prefix (resp. suffix) of ω is an internal factor of ω .

As in Step 3, we are going to see that it implies that either f is not k -power-free or $f(\omega)$ can be reduced. And again, we obtain words $\tilde{\omega}$, $\tilde{\chi}$ and $\tilde{\gamma}$ satisfying all the necessary conditions.

Case 4.1: $Sp_{2.2, min}(\omega) = \emptyset$

We use the notations of Remark 3.1.

Let v^k a pure k -power in $Sp_{2.1, min}(\omega)$. We have $\widehat{v}_1 = x_{1,v}^k y_1$ with $|f(x_{1,v})^{k-1}| < |\pi_{1,1}| + |f(v)|$ and $\widehat{v}_j = x_{j,v}^k y$ or $\widehat{v}_j = y_j x_{j,v}^k$ for all integers $2 \leq j \leq k + 2 - c_v$ where $f(x_{j,v})^k$ is a conjugated shift of $f(v)^k$.

Let ω_1 the word such that $\omega = x_{1,v} \omega_1$ and let X_1 be the word such that $f(x_{1,v}) = \pi_{1,1} X_1$.

Case 4.1.1: ω does not end with a pure k -power or ω ends with a pure k -power t^k such that $|f(t)| > |\gamma| + |X_1|$.

Let χ_1 be the word such that $\chi = X_1 \chi_1$ and let γ_1 be the word γX_1 . We have $f(\omega_1) = (\chi_1 \gamma_1)^k \chi_1 \varsigma$ and $|f(v)| > |X_1| + (k-2)|f(x_{1,v})| > |X_1| + (k-2)|\gamma| \geq |\gamma_1|$. Since ω_1 is a proper suffix of ω , $v^k \in Min.Int.Pwrs(\omega_1)$. Moreover, since $f(x_{1,v})^{k-1}$ and $f(v)^k$ are internal factor of $(\chi\gamma)^k \chi$ and since $f(v)^k$ (resp. $f(x_{1,v})^{k-1}$) and $(\chi\gamma)^k \chi$ do not have any common factor of length at least $|\chi\gamma| + |f(v)|$ (resp. $|\chi\gamma| + |f(x_{1,v})|$), we necessarily have $d_{v, \omega} = 0$ and $c_{v, \omega} \leq 2$.

Case 4.1.1.1: $c_{v, \omega} = 1$

We have $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [2; k+1]$ in $f(\omega)$. But, in $f(\omega_1)$, we get $d_{v,\omega_1} = 1$, $|d_{v,\omega_1}f(v)^2| < |\chi_1|$ and also $L_{j,v} \cup R_{j,v} \neq \emptyset$. Any pure k -power prefix of ω_1 is in $Pure.Int.Pwrs(\omega_1)$. And, for any pure k -power $(v')^k$ in $Pure.Int.Pwrs(\omega_1)$, we have $|f(v')| \geq |f(v)| > |\gamma_1|$. If ω_1 ends with a not-pure k -power z^k , let t^k be a pure k -power factor of z^k . Since ω_1 is a suffix of ω , either t^k is a suffix of ω or $t^k \in Int.Pwrs(\omega)$. In both cases, we have $|f(z)| > |f(t)| > |\gamma_1|$.

By Corollary 4.3(1), either f is not k -power-free or a reduction can be done in $f(\omega_1)$.

Case 4.1.1.2: $c_{v,\omega} = 2$

We have $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [2; k]$ in $f(\omega)$. As previous case, either f is not k -power-free or a reduction can be done in $f(\omega_1)$ using Corollary 4.3(2).

Case 4.1.2: ω ends with a pure k -power t^k such that $|f(t)| \leq |\gamma| + |X_1|$.

Let ω_1 the word such that $\omega = x_{1,v}\omega_1 t[[t]]$, let X_1 be the word such that $f(x_{1,v}) = \pi_{1,1}X_1$, let Y_1 be the word such that $f(t[[t]]) = Y_1\zeta$, let χ_1 be the word such that $\chi = X_1\chi_1Y_1$ and let γ_1 be the word $Y_1\gamma X_1$. We have $f(\omega_1) = (\chi_1\gamma_1)^k\chi_1$, $f(v)^k \in Min.Pwrs(\omega_1) = Min.Int.Pwrs(\omega_1)$ and $|f(v)| > |X_1| + (k-2)|f(x_{1,v})|$.

As previously, we necessarily have $d_{v,\omega} = 0$ and $c_{v,\omega} \leq 2$.

Since $k \geq 5$, we have $|f(v)| > 2|X_1| + 2|\gamma| \geq |X_1| + |f(t)| + |\gamma| \geq |X_1| + |Y_1| + |\gamma| = |\gamma_1|$.

Let us remark that, if $c_{v,\omega} = 2$, we do have $|f(v)| > |\gamma_1|$ when $k = 4$. Indeed, let T be the common factor of $f(t)^{k-1}Y_1$ and $f(v)^k$. If $|T| \geq |f(t)| + |f(v)|$, by Lemma 1.5, it implies that $f(v) = (t_1t_2)^r$ and $f(t) = (t_2t_1)^q$ for two positive integers r and q and two words t_1 and t_2 . If $r \geq 2$ or $q \geq 2$ then f is not k -power-free. If $q = r = 1$ then $|f(t)| = |f(v)| > |X_1| + (k-2)|f(x_{1,v})| > |X_1| + 2|\gamma|$: a contradiction with the hypothesis $|f(t)| \leq |\gamma| + |X_1|$. If $|T| < |f(t)| + |f(v)|$, then $|f(v)| > |Y_1| + (k-2)|f(t)| > 2|Y_1| + |\gamma|$. Since $|f(v)| > 2|X_1| + |\gamma|$, we get that $|f(v)| > |\gamma_1|$.

As in Case 4.1.1, either f is not k -power-free or a reduction can be done in $f(\omega_1)$ using Corollary 4.3(1) when $c_{v,\omega} = 1$ or Corollary 4.3(2) when $c_{v,\omega} = 2$.

Case 4.2: $Sp_{2.1,min}(\omega) = \emptyset$

This case is the mirror image of $Sp_{2.2,min}(\omega) = \emptyset$.

Case 4.3: $Sp_{2.1,min}(\omega) \cap Sp_{2.2,min}(\omega) \neq \emptyset$

Let v^k a pure k -power in $Sp_{2.1,min}(\omega) \cap Sp_{2.2,min}(\omega)$.

Since $f(v)^k$ (resp. $f(x_{1,v})^k$ and $f(x_{k+2-c_v,v})^k$) and $(\chi\gamma)^k\chi$ do not have any common factor of length at least $|\chi\gamma| + |f(v)|$ (resp. $|\chi\gamma| + |f(x_{1,v})|$ and $|\chi\gamma| + |f(x_{k+2-c_v,v})|$), we necessarily have $d_{v,\omega} = 0$ and $c_{v,\omega} = 1$.

Let ω_1 the word such that $\omega = x_{1,v}\omega_1 x_{k+1,v}$, let X_1 be the word such that $f(x_{1,v}) = \pi_{1,1}X_1$, let Y_1 be the word such that $f(x_{k+1,v}) = Y_1\sigma_{2,k+1}$, let χ_1 be the word such that $\chi = X_1\chi_1Y_1$ and let γ_1 be the word $Y_1\gamma X_1$. We have $f(\omega_1) = (\chi_1\gamma_1)^k\chi_1$ and $|f(v)| > \max\{|f(x_{1,v}^{k-1})| - |\pi_{1,1}|; |f(x_{k+1,v}^{k-1})| - |\sigma_{2,k+1}|\} > \max\{|X_1| + |f(x_{1,v})|; |Y_1| + |f(x_{k+1,v})|\} + (k-3)|\gamma| \geq |X_1| + |Y_1| + |\gamma| = |\gamma_1|$.

Since $x_{1,v}^{k+1}$ and $x_{k+1,v}^{k+1}$ are not factor of ω , we get that $f(v)^k \in Min.Int.Pwrs(\omega_1)$.

We have $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [2; k]$ in $f(\omega)$. In $f(\omega_1)$, we get $d_{v,\omega_1} = 1$, $2|f(v)| < |D_{v,\omega_1} f(v)^{k-2}| < |\chi_1| < |D_{v,\omega_1} f(v)^{k-1}|$ and also $L_{j,v} \cup R_{j,v} \neq \emptyset$. Either f is not k -power-free or a reduction can be done in $f(\omega_1)$ using Corollary 4.3(2).

Case 4.4: $Sp_{2.1,min}(\omega) \cap Sp_{2.2,min}(\omega) = \emptyset$ with $Sp_{2.1,min}(\omega) \neq \emptyset$ and $Sp_{2.2,min}(\omega) \neq \emptyset$

Let v^k a pure k -power in $Sp_{2.1,min}(\omega)$ and let $(v')^k$ a pure k -power in $Sp_{2.2,min}(\omega)$. By definition, we have $|f(v)| = |f(v')|$.

Since $f(v)^k$ (resp. $f(x_{1,v})^k$) and $(\chi\gamma)^k\chi$ do not have any common factor of length at least $|\chi\gamma| + |f(v)|$ (resp. $|\chi\gamma| + |f(x_{1,v})|$), we necessarily have $d_{v,\omega} = 0$ and $c_{v,\omega} \leq 2$.

If $c_{v,\omega} = 2$, we get that a power of $f(v')^k$ is an internal factor of $f(x_{1,v})^k f(v)^k$ with $|f(v')^{k-1}| < |\chi\gamma|$. By Lemma 1.5, it implies that $f(v)^k$ and $f(v')^k$ are conjugated words. Since w does not contain v^{k+1} and $v^k \notin Sp_{2.1,min}(\omega)$, it is impossible. Thus $c_{v,\omega} = 1$.

Let ω_1 the word such that $\omega = x_{1,v}\omega_1 x_{k+1,v'}$. Let Y_1 be the word such that $f(x_{k+1,v'}) = Y_1\sigma_{2,k+1} = Y_1\varsigma$. We define the words X_1 , χ_1 and γ_1 as in previous case. We have $f(\omega_1) = (\chi_1\gamma_1)^k\chi_1$ and $f(v)^k \in Min.Int.Pwrs(\omega_1)$.

Moreover, $|f(v)| > \max\{|f(x_{1,v}^{k-1})| - |\varrho|; |f(x_{k+1,v'}^{k-1})| - |\varsigma|\} > \max\{|X_1| + |f(x_{1,v})|; |Y_1| + |f(x_{k+1,v'})|\} + (k-3)|\gamma| \geq |X_1| + |Y_1| + |\gamma| = |\gamma_1|$.

As in previous cases, either f is not k -power-free or a reduction can be done in $f(\omega_1)$ using Corollary 4.3(2). \square

Corollary 4.6 *Let A and B be two alphabets and let $k \geq 5$ be an integer. A k -power-free morphism is a $(k+1)$ -power-free morphism.*

Proof.

Let f be morphism from A^* to B^* . We assume that f is not $(k+1)$ -power-free and we want to show that f is not k -power-free.

The morphism f must be a ps-morphism. Otherwise, f is not k -power-free: it ends the proof.

Let w be a shortest $(k+1)$ -power-free word which image by f contains a $(k+1)$ -power. That is $f(w) = pu^{k+1}s$ for a non-empty word u over B and $|w| = n$ is minimal.

By the criterion of minimal length of w , p is a proper prefix of $f(w[1])$ and s is a proper suffix of $f(w[|w|])$.

If $f(w)$ and u^{k+1} are synchronised, by Lemma 1.16, w contains a $(k+1)$ -power: a contradiction with the definition of w .

Taking $\varrho = p$, $\varsigma = s$, $\gamma = \varepsilon$ and $\chi = u$, by Proposition 4.4, f is not k -power-free. \square

Remark 4.7 *The results in Proposition 4.4, Lemma 4.5 and in Corollary 4.6 are no longer true for $k = 4$ because of the case 4.1.2 of the proof of Lemma 4.5.*

References

- [1] V. Keränen. On the k -freeness of morphisms on free monoids. *Annales Academiae Scientiarum Fennicae* 61, Series A, 1986.
- [2] M. Leconte. *Codes sans répétition*. PhD thesis, LITP Université Paris 6, october 1985.
- [3] M. Lothaire. *Combinatorics on words*, volume 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983. Reprinted in 1997 by Cambridge University Press in the Cambridge Mathematical Library, Cambridge, UK, 1997.
- [4] M. Lothaire. *Algebraic Combinatorics on words*, volume 90 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK, 2002.
- [5] F. Wlazinski. Reduction in non- $(k + 1)$ power-free morphisms. *Submitted to A Special Issue of TIA dedicated to the "Journées Montoises d'Informatique Théorique 2014"*, 00:00, 2015.

A Figures for Corollary 4.3

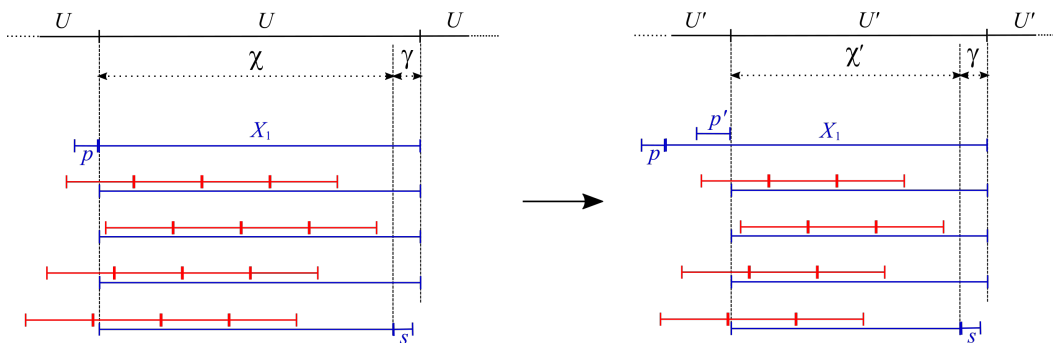


Figure 7: Example of a reduction using case 1 of Corollary 4.3 with $v_{(1)}^3$ when $k = 4$

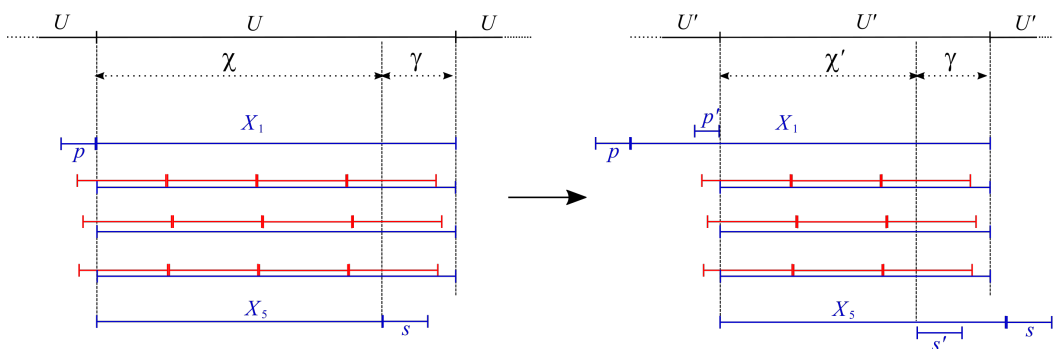


Figure 8: Example of a reduction using case 2 of Corollary 4.3 with $v_{(1)}^3$ when $k = 4$

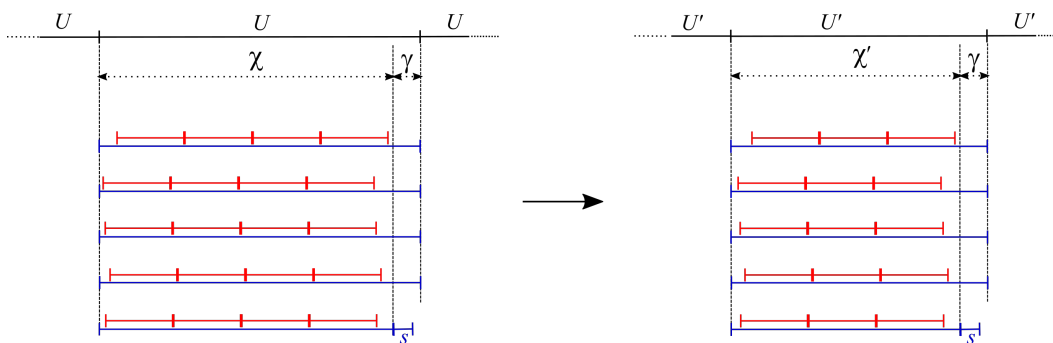


Figure 9: Example of a reduction using case 3 of Corollary 4.3 with $v_{(1)}^3$ when $k = 4$

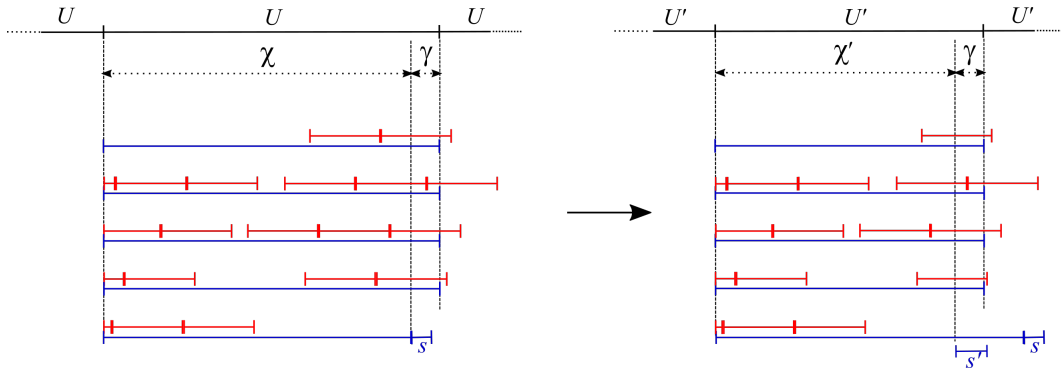


Figure 10: Example of a reduction with $v_{(1)}^3$ using case 4 of Corollary 4.3 when $k = 4$ and $S_2 = \emptyset$.

Let us remark that, in this example, a reduction with $v_{(2)}^3$ using case 1 of Corollary 4.3 can also be used.

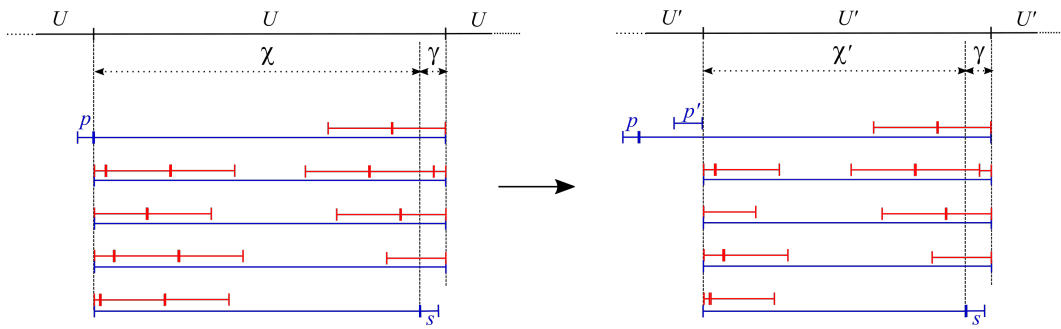
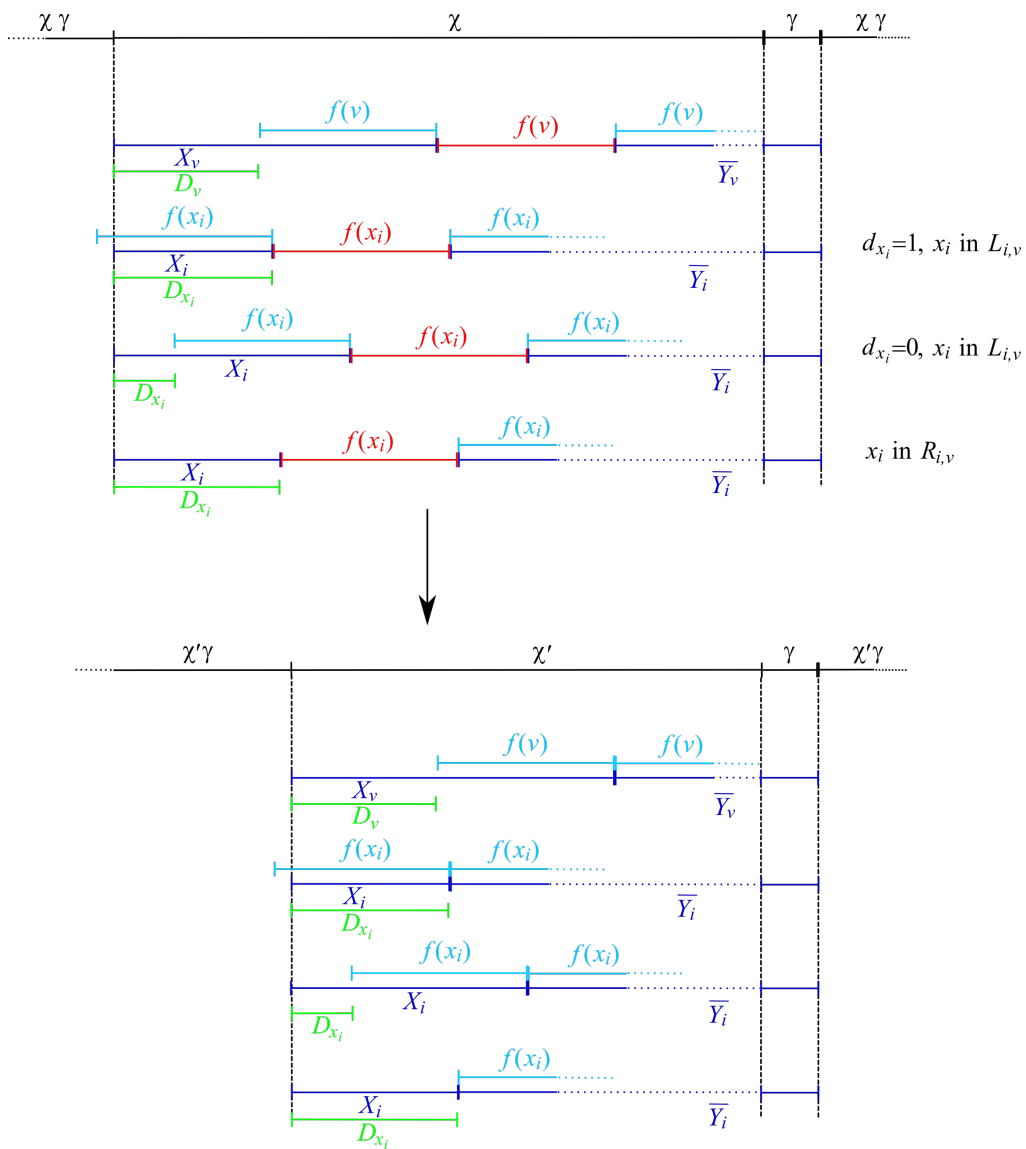


Figure 11: Example of a reduction with $v_{(1)}^3$ using case 4 of Corollary 4.3 when $k = 4$ and $S_2 \neq \emptyset$

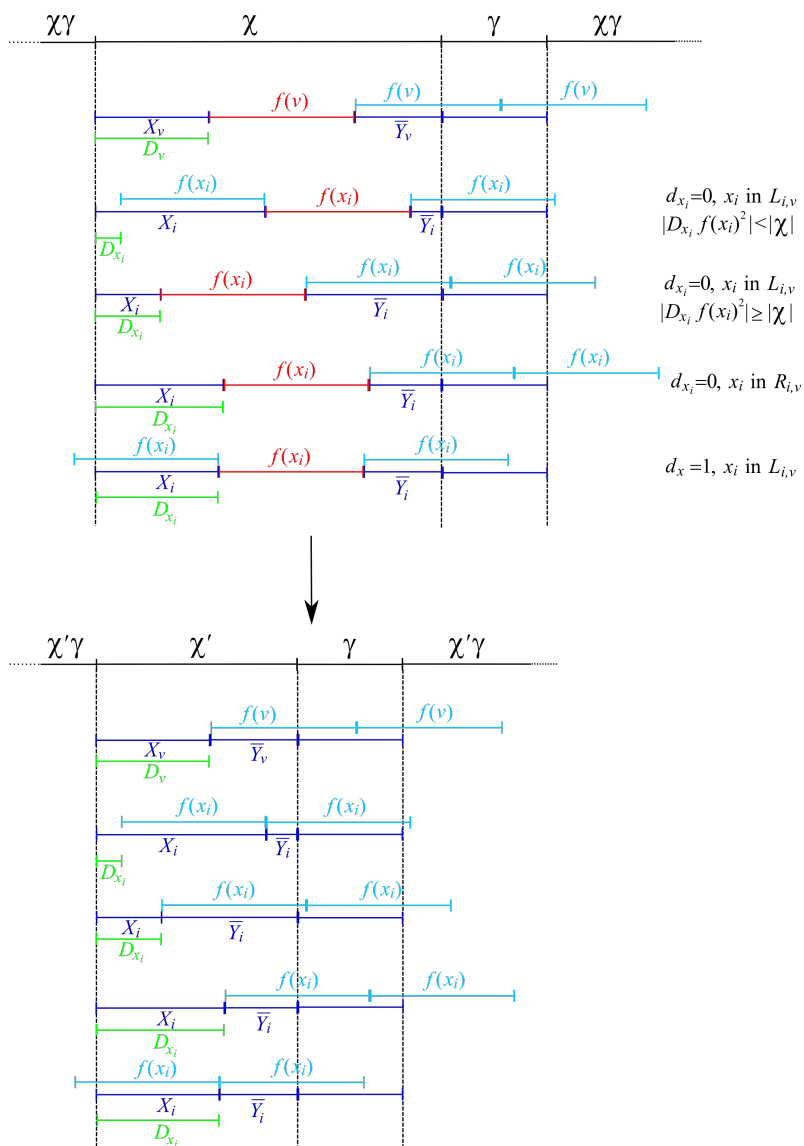
Different cases for x_j^3 in case 3 of Corollary 4.3

Let us recall that $d_v = 0$, $|D_v f(v)^2| \leq |\chi|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k+1]$.



Different cases for x_j^3 in case 4 of Corollary 4.3 when $S_2 = \emptyset$

Let us recall that $d_v = 0$, $|D_v f(v)| \leq |\chi| < |D_v f(v)|^2$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k]$.



Different cases for x_j^3 in case 4 of Corollary 4.3 when $S_2 \neq \emptyset$

Let us recall that $d_v = 0$, $|D_v f(v)| \leq |\chi| < |D_v f(v)^2|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, k]$.

