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# ALMOST SURE SCATTERING AT MASS REGULARITY FOR RADIAL SCHRÖDINGER EQUATIONS

MICKAËL LATOCCA

ABSTRACT. We consider the radial nonlinear Schrödinger equation  $i\partial_t u + \Delta u = |u|^{p-1}u$  in dimension  $d \geq 2$  for  $p \in [1, 1 + \frac{4}{d}]$  and construct a natural Gaussian measure  $\mu$  which support is almost  $L^2_{\text{rad}}$  and such that  $\mu$  - almost every initial data gives rise to a unique global solution. Furthermore, for  $p > 1 + \frac{2}{d}$  the solutions constructed scatter in a space which is almost  $L^2$ . This paper can be viewed as the higher dimensional counterpart of the work of Burq and Thomann [BT20], in the radial case.

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## 1. INTRODUCTION

We consider the Cauchy problem and the long time dynamics for the semilinear Schrödinger equation:

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \\ u(0) = u_0 \in H^s(\mathbb{R}^d), \end{cases}$$

where  $p \geq 1$  need not be an integer,  $s \in \mathbb{R}$  and  $d \geq 2$ . (NLS) is known to be invariant under the scaling symmetry:

$$u(t, x) \mapsto u_\lambda(s, y) := \lambda^{\frac{2}{p-1}} u(\lambda^2 s, \lambda y),$$

which is such that  $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s(d,p)} \|u\|_{\dot{H}^s}$  with  $s(d,p) := \frac{d}{2} - \frac{2}{p-1}$ , the *critical* regularity threshold and where  $\dot{H}^s$  stands for the homogeneous Sobolev space. (NLS) also enjoys several formal conservation laws, the most important being the conservation of the *mass* and the *energy*, that is the quantities

$$M(t) := \|u(t)\|_{L^2}^2 \text{ and } E(t) := \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{p+1} \|u(t)\|_{L^{p+1}}^{p+1}$$

are conserved under the flow of (NLS).

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When  $s < s(d, p)$ , local well-posedness and even global well-posedness are expected and when  $s > s(d, p)$  an ill-posedness behaviour is to be expected. This article deals with the exponent range  $p \leq 1 + \frac{4}{d}$  and regularity  $s = 0$ , *i.e.*, the mass regularity. Remark that  $s(d, 1 + \frac{4}{d}) = 0$ . We gather below some known results concerning this problem. We recall that a solution  $u$  to (NLS) is said to scatter in  $H^s$  (which stands for the non-homogeneous Sobolev spaces) forward in time (resp. backward in time) if there exists  $u_+$  (resp.  $u_-$ ) such that:

$$\|u(t) - e^{it\Delta}u_{\pm}\|_{H^s} \xrightarrow{t \rightarrow \pm\infty} 0.$$

This expresses that even if  $u$  is the solution of a nonlinear equation, its long time behaviour will eventually be close the linear evolution of some state  $u_{\pm}$ , which may not always be the initial data  $u_0$ .

**1.1. Scattering for the nonlinear Schrödinger equations.** The literature pertaining to the long time behaviour of (NLS) is broad and we do not claim to be exhaustive. For an introduction to the deterministic theory of the mass sub-critical and critical nonlinear Schrödinger equations we refer to [Tao06, Caz03]. The following theorem gathers some of the most important results known in the scattering theory of (NLS). We also refer to [Nak99, Dod16c] for more scattering results.

**Theorem 1.1** (Deterministic theory). *Let  $p \geq 1$  and  $d \geq 2$ . Then:*

- (i) For  $p \in [1, 1 + \frac{4}{d}]$  the Cauchy problem for (NLS) is globally well-posed in  $L^2(\mathbb{R}^d)$  and if  $p > 1 + \frac{4}{d}$  the Cauchy problem is ill-posed in  $L^2$ .
- (ii) For  $p \leq 1 + \frac{2}{d}$  and for every  $u_0 \in L^2$ , the solutions do not scatter in  $L^2$ , neither forward nor backward in time.
- (iii) For  $d \geq 2$ ,  $p \in (1 + \frac{2}{d}, 1 + \frac{4}{d-2})$  and initial data in  $H^1$  scattering in  $L^2$  holds.
- (iv) For  $d \geq 2$ ,  $p = 1 + \frac{4}{d}$  and radial initial data in  $L^2$ , scattering holds in  $L^2$ .

*Proof.* (i) is the standard local well-posedness result when  $p < 1 + \frac{4}{d}$ , see [Tao06], Chapter 3 and globality comes from the conservation of mass. In the critical case the global theory is more involved, see [Dod16c]. The ill-posedness part is proven in [CCT03, AC09]. (ii) is the content of [Bar84]. For (iii) see [TY84] and for (iv) see [TVZ07]. For dimension 2 see [Dod16a, Dod16b].  $\square$

In order to address this gap between the previous results and the scaling heuristic, one way is to study the Cauchy problem for (NLS) with random initial data. The theory of dispersive equations with random initial data can be tracked back at least to the pioneer work of Bourgain [Bou94, Bou96] and the use of formal invariant measures. See also [BT08a, BT08b] for an approach without invariant measure construction or [Tzv15, OT17, OT, OST17] for an approach with quasi-invariant measure constructions. A key feature of the random data dispersive equation theory is that it allows for proving well-posedness results below the scaling for critical problems.

In dimension 1 the gap in the theory of mass regularity scattering for (NLS) was partially closed in [BTT13]. More precisely the authors proved almost sure global existence and scattering in  $L^2$  with respect to a measure which typical regularity is  $L^2$ , as long as  $p \geq 5 = 1 + \frac{4}{d}$  and thus missed the range  $(3, 5)$ . Their method is based on the use of the *lens transform* (see Appendix B for details) which transforms the scattering problem for (NLS) into a scattering problem for an harmonic oscillator version of the nonlinear Schrödinger equation, which turns out to be more amenable. The gap was fully closed in [BT20] by studying quasi-invariant measures in a quantified manner. In both cases such results are interesting in the sense that they give large data scattering, without assuming decay at infinity.

In dimension  $d = 2$ , the counterpart of [BTT13] in the radial case is established in [Den12]. See also [PRT14, Tho09] where almost sure scattering in higher dimension was studied (although a smallness assumption is required).

Note that in another setting, scattering for the nonlinear wave equations at energy regularity has been studied, and we refer to the works [DLM19] and also also [Bri18b, Bri18a].

**1.2. Notation.** We adopt widely used notations such as  $[x]$  for the lower integer part of a real number  $x$  and  $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ . We denote  $\otimes$  the tensor product (of functions, spaces or measures) and we write  $[A, B] = AB - BA$  the commutator of  $A$  and  $B$ . The letters  $\Omega$  and  $\mathbb{P}$  will always denote a probability space and its associate probability measure with an expectation denoted by  $\mathbb{E}$ .

For inequalities we often write  $A \lesssim B$  when there exists a universal constant  $C > 0$  such that  $A \leq CB$ . In some cases we need to track explicitly the dependence of the constant  $C$  upon other constants and we denote by  $A \lesssim_a B$  to indicate that the implicit constant depends on  $a$ . In other cases we use the notation  $C$  for a constant which can change from one line to another, and write  $C(a, b)$  to explicitly recall the dependence of  $C$  on other parameters. We write  $A \sim B$  if  $A \lesssim B$  and  $A \gtrsim B$ .

$D_t$  is defined as  $-i\partial_t$ .  $\mathcal{S}$  and  $\mathcal{S}'$  respectively stand for the space of Schwartz functions and its dual the space of tempered distributions. The index rad means radial, so for example  $\mathcal{S}_{\text{rad}}$  stands for the radially symmetric functions of  $\mathcal{S}$ . The Lebesgue spaces are denoted by  $L^p$  and for  $p \in [1, \infty]$  we set  $p'$  such that  $\frac{1}{p} + \frac{1}{p'} = 1$ . If  $X$  is a Banach space then  $L_T^p X$  serves as a shortcut for  $L^p((0, T), X)$  and  $L_w^p$  denotes the Lebesgue space with weight  $w$ . The usual Sobolev spaces are denoted by  $H^s = W^{s,2}$  where  $W^{s,p} = \{u \in \mathcal{S}', (\text{id} - \Delta)^{\frac{s}{2}} \in L^p\}$ . When  $\text{id} - \Delta$  is replaced by  $H$ , the harmonic oscillator defined in Section 2, these spaces are called the harmonic Sobolev spaces and written  $\mathcal{H}^s$  and  $\mathcal{W}^{s,p}$ . The Hölder space  $C^{0,\alpha}(I, X)$ , for  $\alpha \in (0, 1)$ , is defined as the set of functions continuous functions on  $X$  that satisfy

$$\|f\|_{C^{0,\alpha}} := \|f\|_{L^\infty(I,X)} + \sup_{t \neq s \in I} \frac{\|f(t) - f(s)\|_X}{|t - s|^\alpha} < \infty.$$

Smooth Littlewood-Paley projectors at frequency  $N = 2^n$  are denoted by  $\mathbf{P}_N$  and we set  $\mathbf{S}_N := \sum_{m=0}^n \mathbf{P}_{2^m}$  and  $\mathbf{P}_{>N} = \text{id} - \mathbf{S}_N$ .

If  $a_n, b_n$  are real independent standard Gaussian random variables then  $g_n := a_n + ib_n$  is called a complex Gaussian random variable.

**1.3. Main results.** This section presents the main results we shall prove. Precise definitions for the space  $X^0$  and the measure  $\mu$  will be given in Section 2. At this stage one can picture  $X^0$  as the radially symmetric almost  $L^2$  functions and  $\mu$  a probability measure whose support is  $X^0$ .

Keeping in mind Theorem 1.1 (ii), when  $p \leq 1 + \frac{2}{d}$  there is no scattering in  $L^2$  and thus even in the probabilistic approach there is no possible scattering result. It is however possible to prove a weaker statement, which we refer to as *weak* scattering and is the content of our first result.

**Theorem 1.2** (Almost-sure global existence and weak scattering). *Let  $d \geq 2$  and  $p \in \left(1, 1 + \frac{4}{d}\right]$ . There exists  $\sigma \in (0, \frac{1}{2})$  such that for  $\mu$ -almost every initial data  $u_0 \in X^0$ , there exists a unique global solution  $u$  to (NLS) in the space*

$$e^{it\Delta}u_0 + \mathcal{C}(\mathbb{R}, \mathcal{H}^\sigma(\mathbb{R}^d)).$$

Furthermore this solution satisfies the bound

$$\|u(s)\|_{L^{p+1}} \leq C(u_0) \log^{\frac{1}{2}}(1 + |s|),$$

for  $s \in \mathbb{R}$ . Moreover there exist constants  $C, c > 0$  such that

$$\mu(u_0 \in X^0, C(u_0) > \lambda) \leq C e^{-c\lambda^2}.$$

For  $p \in \left(1 + \frac{2}{d}, 1 + \frac{4}{d}\right]$  and  $d \leq 24$  scattering holds in  $L^2$  for almost every radial initial data at mass regularity. More precisely we define the following regularity parameter:

$$(1.1) \quad \sigma(p, d) := \begin{cases} \frac{1}{2} & \text{if } p \leq 1 + \frac{3}{d-2} \\ 2 - \frac{d-2}{2}(p-1) & \text{if } p \geq \frac{3}{d-2}. \end{cases}$$

**Theorem 1.3** (Almost sure scattering at mass regularity). *Let  $d \in \{2, \dots, 24\}$  and  $p \in \left(1 + \frac{2}{d}, 1 + \frac{4}{d}\right)$ .*

(i) *For every  $\sigma \in (0, \sigma(p, d))$  and  $\mu$ -almost every initial data there exists a unique global solution to (NLS) satisfying*

$$u(s) - e^{is\Delta} u_0 \in \mathcal{C}(\mathbb{R}, H^\sigma).$$

(ii) *The solutions constructed scatter at infinity, in  $L^2$ . More precisely, there exist  $\sigma \in (0, \sigma(p, d))$  and  $\kappa > 0$  only depending on  $p$  and  $d$  such that for  $\mu$ -almost every  $u_0$  there exist  $u_\pm \in \mathcal{H}^\sigma$  such that the corresponding solution constructed in (i) satisfies*

$$(1.2) \quad \|u(s) - e^{is\Delta}(u_0 + u_\pm)\|_{H^\sigma} \lesssim C(u_0) |s|^{-\kappa} \xrightarrow{s \rightarrow \pm\infty} 0,$$

and also

$$(1.3) \quad \|e^{-is\Delta} u(s) - (u_0 + u_\pm)\|_{H^\sigma} \lesssim C(u_0) |s|^{-\kappa} \xrightarrow{s \rightarrow \pm\infty} 0.$$

In both cases there exist numerical constants  $C, c > 0$  such that  $\mu(C(u_0) > \lambda) \leq C e^{-c\lambda}$ .

*Remark 1.4.* Since the corresponding measure  $\mu$  will be such supported by the radial functions of  $\bigcap_{\sigma>0} \mathcal{H}_{\text{rad}}^{-\sigma}$  we see that this result is radial in nature. We also emphasize that the convergence rate  $\kappa$  can be made explicit by following the computations line by line.

*Remark 1.5.* First note that in dimension  $d = 2$ , the case  $p = 3$  has been treated in [Den12]. Similarly in dimension  $d > 2$  the endpoint case  $p = 1 + \frac{4}{d}$  is an adaptation of the proof in [Den12] and we do not treat this case in view of the deterministic result [TVZ07]. Then we assume that  $p \in \left(1 + \frac{2}{d}, 1 + \frac{4}{d}\right)$ .

*Remark 1.6.* As the proof will make it clear, the obstruction to generalise to dimensions higher than 24 is contained in the establishment of a powerful enough almost-sure local Cauchy theory. We did not put much effort into developing such a local theory in higher space dimensions and we believe that this is possible, using refinements of the local theory in Bourgain's type spaces, building on bilinear estimates.

*Remark 1.7.* This result is not of small data type. The measure  $\mu$  indeed satisfies that for  $p > 2$  arbitrarily close to 2 and every  $R > 0$ ,  $\mu(u \in X^0, \|u\|_{L^p} > R) > 0$ , see Section 2 for a proof.

Finally, we highlight that some algebraic computations were carried out using a computer algebra software.

**1.4. Strategy of the proof and organisation of the paper.** The following paragraphs outline the proof of the main results. There are three main features that we need to address: assuming a global theory, how can one prove scattering? How can one construct a good local theory? How to extend this local theory into a powerful enough global theory?

1.4.1. *Proving scattering.* Assume that one has access to an almost-sure local existence theory for (NLS), even a global theory in  $L^2$ .

In order to deal with the long-time behaviour of the solutions  $u$  we write, thanks to Duhamel's formula:

$$u(s) = e^{is\Delta}u_0 - i \int_0^s e^{i(d-s')\Delta} \left( |u(s')|^{p-1}u(s') \right) ds',$$

so that in order to prove the existence of  $u_+$  we only need to prove that the integral

$$\int_0^s e^{i(s-s')\Delta} \left( |u(s')|^{p-1}u(s') \right) ds'$$

converges, which is achieved by proving that this integral actually converges absolutely. From there we can see that *a priori* bounds on  $u$  in  $L_x^r$  spaces may be needed to carry such a program. Such bounds can be obtained using Sobolev embeddings if  $u$  is more regular. This heuristic suggests that we can try to construct a better local theory in a probabilistic setting, building on some stochastic smoothing. We then will need to extend the theory to a global one.

1.4.2. *Almost-sure local well-posedness.* The standard method to prove local well-posedness is to implement a fixed-point argument in some Banach spaces. In the context of dispersive equations, Strichartz estimates allow for achieving such a goal. Heuristically, Strichartz estimates tell us that if  $u_0 \in H^s$  then the free evolution  $u_L(s) = e^{is\Delta}u_0$  may not be smoother than  $u_0$  but however exhibits some gain in space-time integrability, from which we access to a wider range of  $(p, q)$  such that bounds of the form  $\|u_L\|_{L_T^p L_x^q} \lesssim \|u_0\|_{L^2}$  hold. The advantage of working with random data is that improving the  $L^p$  integrability of a random  $L^2$  function (on the torus, for example) is essentially granted by a result which appeared in the work of Paley-Zygmund. See [BT08a] Appendix A for a proof.

**Theorem 1.8** (Kolmogorov-Paley-Zygmund). *Let  $(c_n)_{n \in \mathbb{Z}}$  an  $\ell^2$  sequence. Then if  $(g_n)_n$  is a sequence of identically distributed centred and normalised complex Gaussian variables we have*

$$\left\| \sum_{n \in \mathbb{Z}} c_n g_n \right\|_{L_\Omega^p} \lesssim \sqrt{p} \| (c_n) \|_\ell^2.$$

Using this result we can prove a probabilistic Strichartz estimate which improves the classical one. Take  $u_0$  a random initial data, then with the previous remarks we can easily control the space-time norms of  $u_L(s)$ , so we seek solutions of the form  $u(s) = u_L(s) + w(s)$  where  $w$  is deterministic and can be taken in a smoother space, for example  $w \in H^s$  for some  $s > 0$ . Formally  $w$  is the solution of the fixed point problem  $\Phi(w) = w$  where

$$\Phi(w) = i \int_0^s e^{i(s-s')\Delta} \left( (u_L(s') + w(s'))^p \right) ds'$$

and thanks to the gain in controlling the space time norms of  $u_L$  we can expect to solve this problem in  $H^s$ . For an illustration of the method see [BT08a]. For probabilistic well-posedness of (NLS) below the scaling regularity see [BOP15].

In our context we will have to take care of the fact that since we will work with random initial data slightly below  $L^2$  we will not really access to all the range of Strichartz estimates. These statements are made precise by Lemma 2.4. Establishing such a good local theory is the content of Proposition 3.3.

1.4.3. *The globalisation argument.* Once one has a good local well-posedness theory, there exists a general globalisation argument which can be traced back at least to Bourgain [Bou94]. In order to clarify the arguments of Corollary 5.6 and Corollary 5.9 below we explain the argument in a simpler setting, the invariant measure setting. One wants to achieve an almost sure global theory. To this end we first remark that thanks to the Borel-Cantelli lemma it is sufficient to prove that for every  $\delta > 0$  and every  $T > 0$  there exists a set  $G_{\delta, T}$  such that  $\mu(X \setminus G_\delta) \leq \delta$

and that for every initial data in  $G_{\delta,T}$  we have existence on  $[0, T]$ . The requirements for the argument to work are roughly the following:

- (i) A local well-posedness theory of the following flavour: for initial data at time  $t_0$  of size (in a space  $X$ ) less than  $\lambda$  there exists a solution on  $[t_0, t_0 + \tau]$  with  $\tau \sim \lambda^{-\kappa}$ , in a space  $X$ .
- (ii) An invariant measure  $\mu$ , that is if  $\phi_t$  denotes the flow of the equation,  $\mu(A) = \mu(\phi_{-t}A)$  for all  $t > 0$  and every measurable set  $A$ , at least formally.

We further assume that  $\mu(u_0, \|u_0\|_X > \lambda) \lesssim e^{-c\lambda^2}$ , as this will always be the case in the following.

Let  $\lambda > 0$  to be chosen later. Let  $G_\lambda$  be the set of *good* initial data  $u_0$  which give rise to solutions defined on  $[0, T]$  and such that  $\|u(t)\|_X \leq 2\lambda$  for every  $t \leq T$ . Let also  $A_\lambda := \{u_0, \|u_0\| \leq \lambda\}$ . Choose  $\tau$  small enough (which amounts to enlarging  $\lambda$ ) so that the local theory constructed implies that a solution on  $[n\tau, (n+1)\tau]$  does not grow more than doubling its size. Then we immediately see that

$$\bigcap_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \phi_{n\tau}^{-1}(A_\lambda) \subset G_\lambda.$$

Taking the complementary we see that

$$\{u_0 \notin G_\lambda\} \subset \bigcup_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \phi_{n\tau}^{-1}(X \setminus A_\lambda),$$

and taking into account the expression of  $\tau$  and the fact that  $\mu$  is invariant under the flow this gives

$$\mu(X \setminus G_\lambda) \leq CT\lambda^\kappa e^{-c\lambda^2},$$

which is smaller than  $\delta$  for  $\lambda \sim \log^{\frac{1}{2}}\left(\frac{T}{\delta}\right)$ .

From there we obtain the global existence and a logarithmic bound for  $\|u(t)\|_X$ . However let us recall that there is no non-trivial invariant measure for the linear Schrödinger equation, or the nonlinear equation in presence of scattering in  $L^2$ .

**Proposition 1.9.** *Let  $d \geq 2$ . Then,*

- (1) *For  $\sigma \in \mathbb{R}$  the only measure supported in  $H^\sigma$  which is invariant by the flow of the linear Schrödinger equation  $i\partial_s u + \Delta_y u = 0$  with initial condition  $u(0) = u_0 \in H^\sigma(\mathbb{R}^d)$  is  $\delta_0$ .*
- (2) *Let  $p$  such that scattering holds in  $L^2$  for solutions of (NLS). Then for any  $\sigma \in \mathbb{R}$  the only measure supported in  $L^2$  which is invariant by the flow of (NLS) is  $\delta_0$ .*

*Proof.* We refer to [BT20] where a proof is given in dimension 1. The proof in dimension  $d$  is a straightforward adaptation.  $\square$

In order to overcome this difficulty, the authors in [BTT13] use the *lens transform* (see Appendix B) to transform solutions of (NLS) with time interval  $\mathbb{R}$  to solutions of

$$(HNLS) \quad \begin{cases} i\partial_t v - H v = \cos(2t) \frac{d}{2}(p-1)-2|v|^{p-1}v \\ v(0) = v_0 \in \mathcal{H}^s, \end{cases}$$

with time interval  $[-\frac{\pi}{4}, \frac{\pi}{4}]$  where  $H := -\Delta + |x|^2$  is the harmonic oscillator, and  $\mathcal{H}^s$  stands for the associate Sobolev spaces. (HNLS) admits quasi-invariant measures with quantified evolution bounds, see Proposition 4.3, which makes it possible to use a similar globalisation argument to the one presented. For this reason most of this paper studies properties of (HNLS), and we will eventually get back to (NLS) using the *lens transform* in Section 6.

For the global theory, the main difference when compared to [BT20] lies in the proof of Lemma 5.7. Let us recall that the proof in [BT20] uses:

- (i) A pointwise bound in  $L^{p+1}$  with large probability for the solutions.
- (ii) A Sobolev embedding to control the  $L^{p+1}$  norm variation by the  $\mathcal{H}^s$  norm variation, controlled by the local theory.

In dimension  $d \geq 3$  such a program would not yield the full proof of Theorem 1.3, due to the use of the Sobolev embedding which would only yield a result for  $p \leq 1 + \frac{2}{d-1}$ . Instead we control the variation of the  $L^{p+1}$  norm, writing:

$$(1.4) \quad v(t') = e^{-i(t'-t_n)H} v(t_n) - i \int_{t_n}^{t'} e^{-i(t'-s)H} (|v(s)|^{p-1} v(s)) ds.$$

for  $t'$  in some interval  $[t_n, t_{n+1}]$ . The first term is controlled using linear methods. For the second term we need to control its  $L^\infty L^{p+1}$  norm, which is not always Schrödinger admissible (terminology defined in Section 3). Instead we use the dispersive estimate  $L^{\frac{p+1}{p}} \rightarrow L^{p+1}$ .

Once we have these  $L^{p+1}$  bounds, scattering in  $\mathcal{H}^\varepsilon$  for (HNLS) is obtained through interpolation between scattering in  $\mathcal{H}^{-\sigma}$  (obtained by Sobolev embeddings and the  $L^{p+1}$  bounds) and a bound for the solution in  $\mathcal{H}^\varepsilon$ . See Proposition 6.1 for the details.

**1.4.4. Organisation of the paper.** The remaining of this paper is organised as follows: Section 2 introduces the space  $X^0$ , the Gaussian measure  $\mu$  and gathers their properties. Then the proof of Theorem 1.2 and Theorem 1.3 starts in Section 3 with a large probability local well-posedness theory. The evolution of the quasi-invariant measures are studied in Section 4 which allows to extend the local theory into an almost-sure global theory in Section 5 and the proof of Theorem 1.2 and Theorem 1.3 is finished in Section 6. Finally technical estimates are gathered in Appendix A and Appendix B for results concerning the *lens transform*.

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## 2. THE HARMONIC OSCILLATOR AND QUASI-INVARIANT MEASURES

**2.1. The harmonic oscillator in the radial setting.** We recall some facts concerning the radial harmonic oscillator, and refer to [Sze39] for proofs.

The radial harmonic oscillator is defined as the harmonic oscillator  $H := -\Delta + |x|^2$  acting on the space  $\mathcal{S}_{\text{rad}}(\mathbb{R}^d)$  or radial Schwartz functions. It is a symmetric operator and admits a self-adjoint extension on  $\mathcal{H}_{\text{rad}}^1(\mathbb{R}^d)$ . It is known that the spectrum of  $H$  acting on  $\mathcal{H}^1(\mathbb{R})$  (that is the standard one-dimensional harmonic oscillator) is discrete and made of the non-degenerate eigenvalues  $2n + 1$ , for  $n \geq 0$ . Even if we do not need the exact expression of the associate eigenfunctions, that we denote by  $f_n$ , we recall that the  $f_n$  are explicitly computable. Then we deduce that the spectrum of  $H$  acting on  $\mathcal{H}^1(\mathbb{R}^d)$  is made of the degenerate eigenvalues  $2n + d$ , for  $n \geq 0$  with eigenfunctions

$$e_{n_1} \otimes \cdots \otimes e_{n_d}, \text{ where } \sum_{i=1}^d n_i = n.$$

Finally, the spectrum of  $H$  acting on  $\mathcal{H}_{\text{rad}}^1(\mathbb{R}^d)$  is made of non-degenerate eigenvalues

$$\lambda_n^2 = 4n + d \text{ for } n \geq 0,$$

and eigenfunctions denoted by  $e_n$  which can be written explicitly, but again we do not use their expression. All that will be used in the sequel is that the sequence  $(e_n)_{n \geq 0}$  satisfies the following  $L^p$  estimates. We also refer to [IRT16] for details on the harmonic oscillator acting on radial functions.

**Lemma 2.1** ([IRT16], Proposition 2.4). *Let  $d \geq 2$ . Then*



- (i)  $\|e_n\|_{L^p} \lesssim \lambda_n^{-d(\frac{1}{2}-\frac{1}{p})}$  for  $p \in [2, \frac{2d}{d-1}]$ ;
- (ii)  $\|e_n\|_{L^p} \lesssim \lambda_n^{-\frac{1}{2}} \log^{\frac{1}{p}} \lambda_n$  for  $p = \frac{2d}{d-1}$ ;
- (iii)  $\|e_n\|_{L^p} \lesssim \lambda_n^{d(\frac{1}{2}-\frac{1}{p})-1}$  for  $p \in (\frac{2d}{d-1}, \infty]$ .

In the above inequalities the implicit constant may depend on  $d$  but not  $p$  nor  $n$ .

*Remark 2.2.* From this lemma we see that in the scale of  $L^p$  regularity, as long as  $p < \frac{2d}{d-2}$ , then eigenfunctions  $e_n$  exhibit some decay, which is used to prove probabilistic smoothing estimates, see for example [BTT13], Appendix A for such estimates.

**2.2. Measures and probabilistic smoothing.** Consider a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and let  $(g_n)_{n \geq 0}$  be an identically distributed sequence of centred, normalised complex Gaussian variables. Then we define the random variable  $f : \Omega \rightarrow \mathcal{S}'(\mathbb{R}^d)$  by

$$(2.1) \quad f^\omega(x) := \sum_{n \geq 0} \frac{g_n(\omega)}{\lambda_n} e_n(x),$$

where  $\lambda_n, e_n$  are the eigenvalues and eigenfunctions of the radial harmonic oscillator previously defined. Moreover we consider the approximations

$$f_N := \sum_{n=0}^N \frac{g_n}{\lambda_n} e_n,$$

which almost surely converge to  $f$  in  $\mathcal{S}'$ . Moreover  $(f_N)_{N \geq 1}$  is a Cauchy sequence in  $L^2(\Omega, \mathcal{H}^{-\sigma})$  for every  $\sigma > 0$ . For a proof of this fact see [BTT13], Lemma 3.3. More precisely, if we denote by  $\mu := f_*\mathbb{P}$ , the law of the random variable  $f$ , then we have the following.

**Lemma 2.3.** *The measure  $\mu$  is supported by  $X^0 := \bigcap_{\sigma > 0} \mathcal{H}_{rad}^{-\sigma}$  and moreover:*

- (i) For  $\mu$  almost every  $u$  one has  $u \notin L^2$ .
- (ii) For any  $p$  close enough to 2 and any  $R > 0$ ,  $\mu(u \in X^0, \|u\|_{L^p} > R) > 0$ .

*Proof.* (i) The first part is a straightforward adaptation of Lemma 3.3 in [BTT13]. For the proof of the fact that for  $\mu$ -almost every  $u$ , there holds  $u \notin L^2$ , see [Poi12], Section 4, Lemma 53 and Proposition 54. The argument is easily adapted to our case and mostly relies on a careful application of an inequality of Paley-Kolmogorov which states that for  $\lambda \geq 0$  and  $X \in L^2(\Omega)$ :

$$(2.2) \quad \mathbb{P}(X \geq \lambda \mathbb{E}[X]) \geq (1 - \lambda)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

(ii) This result is claimed in [BTT13] and we provide a proof in full details. We construct large data in the following way. Let  $E \subset \Omega$  be the set defined by:

$$E := \left\{ g_0 > 2R, (g_n)_{n \geq 1} \left\| \sum_{n \geq 1} \frac{g_n}{\lambda_n} e_n \right\|_{L^p} \leq R \right\},$$

which by independence has probability  $\mathbb{P}(E) = \mathbb{P}(g_0 > 2R) \mathbb{P}\left(\left\| \sum_{n \geq 1} \frac{g_n}{\lambda_n} e_n \right\|_{L^p} \leq R\right)$ . Since  $g_0$  is a Gaussian,  $\mathbb{P}(g_0 > 2R) > 0$ . Moreover, since  $p > 2$  we have

$$\mathbb{P}\left(\left\| \sum_{n \geq 1} \frac{g_n}{\lambda_n} e_n \right\|_{L^p} \leq R\right) \geq 1 - e^{-cR^2}.$$

Then on the set  $E$  we have  $\|u\|_{L^p} \geq R$ . Hence we have proven the claim.  $\square$

Note that  $X^0$  is a topological space endowed with its intersection topology. Moreover the norm

$$\|u\|_{X^0} := \sup_{\sigma > 0} \|u\|_{\mathcal{H}^{-\sigma}},$$

is such that  $X^0$  endowed with this norm is a Banach space. Finally  $X^0$  is separable. Under these hypothesis and according to Theorem 7.1.4 in [Dud02], the Borelian measure  $\mu$  is regular, that is, for every Borelian set  $A$ :

$$(2.3) \quad \mu(A) = \sup_{\substack{K \text{ compact} \\ K \subset A}} \mu(K) = \inf_{\substack{O \text{ open} \\ A \subset O}} \mu(O).$$

We define an infinite dimensional re-normalised Lebesgue measure:

$$du := \bigotimes_{n \geq 0} du_n := \bigotimes_{n \geq 0} \lambda_n \mathcal{L},$$

where  $\mathcal{L}$  is the Lebesgue measure on  $\mathbb{C}$  and the sequence  $(u_n)_{n \geq 0}$  stands for the coordinates of an  $u \in X^0$  such that  $u = \sum_{n=0}^{\infty} u_n e_n$ . The measure  $\mu$  can be explicitly computed using the definition of  $f$  and this ensures that  $\mu$  has a density with respect to the infinite Lebesgue measure  $du$ , given by

$$d\mu = \exp\left(-\frac{1}{2} \sum_{n \geq 0} \lambda_n^2 u_n^2\right) \bigotimes_{n \geq 0} du_n = e^{-\frac{1}{2} \|u\|_{\mathcal{H}^1}^2} du.$$

From now, we shall write for every  $\mu$ -measurable set  $A$ :

$$\mu(A) = C_\mu \int_A e^{-\frac{1}{2} \|u\|_{\mathcal{H}^1}^2} du,$$

where  $C_\mu > 0$  is a constant chosen so that  $\mu$  is a probability measure (the measure  $\mu$  being finite since its density with respect to  $du$  is integrable).

We have seen that for  $u$  on the support of  $\mu$ , almost surely  $u \notin L^2$  which proves that the measure  $\mu$  is non-smoothing. A gain in the  $L^p$  scale Sobolev spaces can however be exhibited as follows.

**Lemma 2.4.** *Let  $d \geq 2$  and  $\varepsilon > 0$  arbitrarily small. Define*

$$s_p = \begin{cases} d \left( \frac{1}{2} - \frac{1}{p} \right) & \text{if } p \in \left( 2, \frac{2d}{d-1} \right] \\ 1 - d \left( \frac{1}{2} - \frac{1}{p} \right) & \text{if } p \in \left( \frac{2d}{d-1}, \frac{2d}{d-2} \right] \end{cases}$$

and  $s_p^- := s_p - \varepsilon$ . Then:

- (i) For  $\mu$ -almost every  $u \in X^0$  one has  $u \in \mathcal{W}^{s_p^-, p}$ .
- (ii) More precisely there exists  $C, c > 0$  such that for every  $p \geq 2$  and  $\lambda > 0$  we have

$$\mu\left(u \in X^0, \|u\|_{\mathcal{W}^{s_p^-, p}} > \lambda\right) \leq C e^{-c\lambda^2}.$$

*Proof.* The proof is very similar to the proof of Lemma 3.3 in [BTT13] with  $N_0 = 0$  and  $N = \infty$ , and using the bounds 2.1. See also [AT08].  $\square$

The probabilistic smoothing gain is represented on Figure 1.

We introduce the following family of sets:

$$(2.4) \quad E_{(r,q,\sigma)}(\lambda) := \left\{ u \in X^0, \|e^{-itH} u\|_{L^r_{[-\pi,\pi]} \mathcal{W}^{\sigma,q}} > \lambda \right\}.$$

Lemma 2.4 immediately implies the following corollary. For details see [Tho09]. Note that we take norms in time on  $[-\pi, \pi]$  although we work only on  $[-\frac{\pi}{4}, \frac{\pi}{4}]$ . This is due to the need of room for the estimates in Lemma A.2.

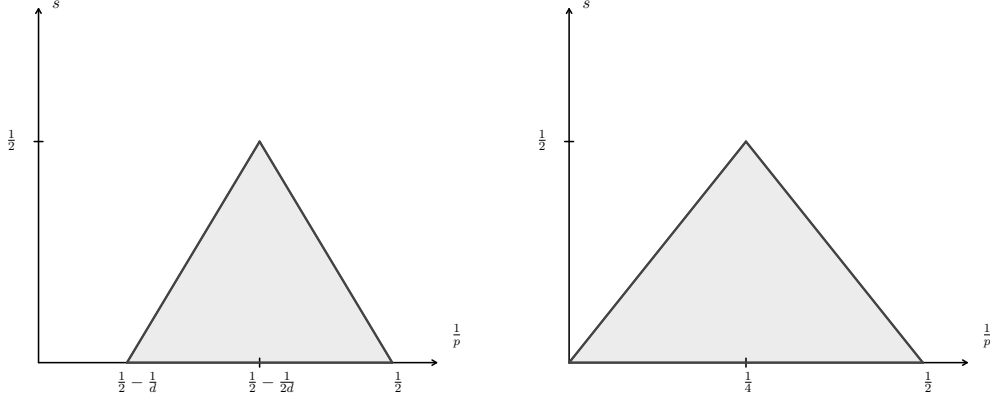


FIGURE 1. Probabilistic gain on the grey zone. The right picture describes the case  $d = 2$  and the picture on the left is the case  $d \geq 3$ , where we can see that the probabilistic gain holds for sufficiently large  $\frac{1}{p}$ .

**Corollary 2.5.** *There exists  $C, c > 0$  such that for every  $r \geq 1$ ,  $q \in [2, \frac{2d}{d-2}]$  and  $\sigma \in [0, s_q^-)$  one has*

$$\mu \left( E_{(r,q,\sigma)}(\lambda) \right) \leq C e^{-c\lambda^2} \text{ for all } \lambda \geq 1$$

### 3. THE PROBABILISTIC LOCAL CAUCHY THEORY

In this section we construct a local Cauchy theory for (HNLS).

A pair  $(q, r) \in [2, \infty]^2$  is Schrödinger admissible in dimension  $d \geq 1$  if  $(q, r, d)$  satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \text{ and } (q, r, d) \neq (2, \infty, 2).$$

*Remark 3.1.* We should check on many occasions that a given pair  $(q, r)$  is admissible. One should keep in mind that the condition  $q, r \geq 2$  is required. Taking into account the relation between  $q$  and  $r$ , the requirement  $q, r \geq 2$  is equivalent to  $q \geq 2$ . In the same spirit,  $(q', r')$  is such that  $(q, r)$  is admissible if and only if  $\frac{1}{2} \leq \frac{1}{r'} \leq \frac{1}{2} + \frac{1}{d}$ .

We define the Strichartz space

$$Y_{[t_0, t]}^\sigma := \bigcap_{(q,r) \text{ admissible}} L^q([t_0, t], \mathcal{W}^{\sigma, r})$$

and its dual  $\tilde{Y}_{[t_0, t]}^\sigma$  that we respectively endow with the norms:

$$\|u\|_{Y_{[t_0, t]}^\sigma} := \sup_{(q,r) \text{ admissible}} \|u\|_{L^q([t_0, t], \mathcal{W}^{\sigma, r})} \text{ and } \|u\|_{\tilde{Y}_{[t_0, t]}^\sigma} := \inf_{(q,r) \text{ admissible}} \|u\|_{L^{q'}([t_0, t], \mathcal{W}^{\sigma, r'})}.$$

Let  $\sigma > 0$ . Using the Sobolev embedding, if  $(q, \tilde{r})$  is Schrödinger admissible and  $r$  is such that  $\frac{d}{\tilde{r}} = \frac{d}{r} + \sigma$  then

$$\|u\|_{L_T^q L^r} \lesssim \|u\|_{L_T^q \mathcal{W}^{\sigma, \tilde{r}}} \lesssim \|u\|_{Y_T^\sigma}.$$

Thus we often refer to  $(q, r)$  being  $\sigma$ -Schrödinger admissible if  $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \sigma$ , and  $q \geq 2$ .

**Proposition 3.2** (Dispersion and Strichartz estimates). *Given  $\sigma \geq 0$  and  $t_0, T \in (0, \frac{\pi}{4})$ , the following estimates hold.*

- (i) (Dispersion) For  $r \geq 2$  and  $t > 0$  holds  $\|e^{itH}\|_{L^{r'} \rightarrow L^r} \leq C(d, r)|t|^{-d(\frac{1}{2} - \frac{1}{r})}$ ;
- (ii) (Homogeneous)  $\|e^{itH}\|_{\mathcal{H}^\sigma \rightarrow Y_{[t_0, T]}^\sigma} \leq C(d)$ ;

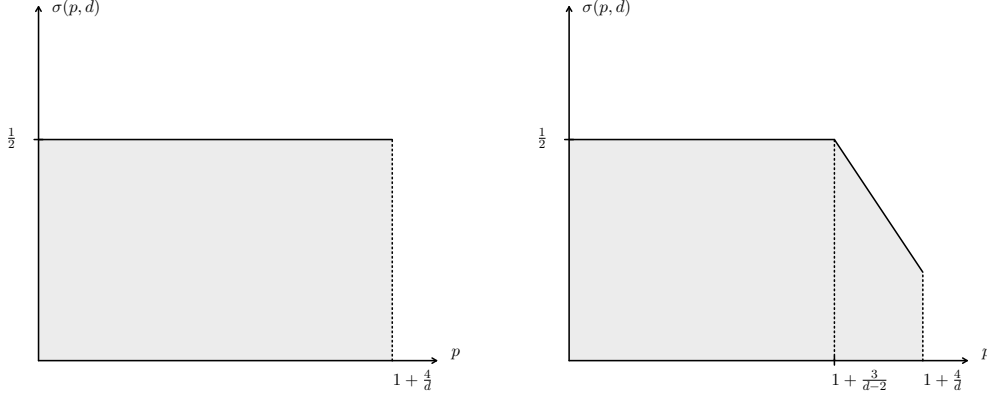


FIGURE 2. The maximum regularity in the local theory. On the left is the case  $d \leq 8$  and on the right is the case  $d > 8$ .

$$(iii) \text{ (Non-homogeneous) } \left\| \int_0^t e^{i(t-s)H} ds \right\|_{\tilde{Y}_T^\sigma \rightarrow Y_T^\sigma} \leq C(d).$$

*Proof.* The proof is carried out in detail in dimension two in [Den12], Proposition 2.7. For dimension  $d \geq 3$  we refer to [Poi12]. In both cases the proof is an application of the  $TT^*$  argument and the Christ-Kiselev lemma along with a dispersive estimate for  $e^{-itH} : L^1 \rightarrow L^\infty$  which is obtained through stationary phase estimates.  $\square$

It will be convenient to set  $Y_{t_0, \tau}^\sigma := Y_{[t_0, t_0 + \tau]}^\sigma$ . We introduce the following set of initial data: for  $p \in (1, 1 + \frac{4}{d})$ , if  $d \leq 8$ ,

$$(3.1) \quad A_\lambda^\circ := E_{(4, \frac{2d}{d-1}, \sigma)}(\lambda) \cap E_{(a(p-1), b(p-1), \sigma)}(\lambda),$$

where  $a, b$  depend on  $p, d$  and will be chosen in the proof of Proposition 3.3. Similarly if  $d > 8$  we set

$$(3.2) \quad A_\lambda^\circ := E_{(q_0, r_0, \sigma(p, d)^-)}(\lambda) \cap E_{(a(p-1), b(p-1), \sigma(p, d))}(\lambda),$$

with  $q_0 := \frac{4}{(d-2)(p-1)-2}$ ,  $r_0 := \frac{2d}{d+2-(d-2)(p-1)}$ ,  $\sigma(p, d) = 2 - \frac{d-2}{2}(p-1)$  and where  $a, b$  depend on  $p, d$  and will be chosen in the proof of Proposition 3.3. Note that once such  $a, b$  are fixed and provided  $b(p-1) < \frac{2d}{d-2}$ , these sets satisfy

$$\mu(A_\lambda^\circ) \leq C e^{-c\lambda^2},$$

for numerical constants  $c, C > 0$ , according to Lemma 2.5.

We will establish a local theory in  $\mathcal{H}^\sigma$  for  $\sigma < \sigma(d, p)$  where  $\sigma(p, d)$  is defined by (1.1) and represented as a function of  $p$  in the case  $d \leq 8$  and  $d > 8$  on Figure 2.

The main result of this section is a flexible local well-posedness result, where the initial data is given at a time  $t_0$ . We state only a forward in time result in  $[0, \frac{\pi}{4})$ , a similar statement holds for negative times.

**Proposition 3.3** (Local Cauchy theory). *Let  $d \geq 2$ ,  $t_0 \in [0, \frac{\pi}{4})$ ,  $p \in [1, 1 + \frac{4}{d})$  and  $\sigma \in [0, \sigma(p, d)]$ . Given  $\lambda > 0$ , there exists  $\tau > 0$  such that for  $v_0 \in X^0 \setminus A_\lambda^\circ$ , there exists a unique solution*

$$v \in e^{-i(t-t_0)H} v_0 + Y_{[t_0, t_0 + \tau]}^\sigma \hookrightarrow \mathcal{C}^0([t_0, t_0 + \tau], \mathcal{H}^\sigma)$$

to (HNLS) on  $[t_0, t_0 + \tau]$  such that  $v(t_0) = v_0$ , where uniqueness hold for  $v - e^{-i(t-t_0)H} v_0$  in  $Y_{[t_0, t_0 + \tau]}^\sigma$ . Moreover there exist  $\alpha, \beta > 0$  only depending on  $p$  and  $d$  such that  $\tau \sim \lambda^{-\alpha} (\frac{\pi}{4} - t_0)^\beta$ .

*Remark 3.4.* The key point that we will use in the sequel is that given an initial data at time  $t_0$  of typical size  $\sim \lambda$  (in the set  $X^0 \setminus A_\lambda^\circ$ , which is of measure  $\geq 1 - Ce^{-c\lambda^2}$ ) a solution to (HNLS) can be constructed on  $[t_0, t_0 + \tau]$  where  $\tau \sim \lambda^{-\alpha} (\frac{\pi}{4} - t_0)^\beta$  for positive constants  $\alpha, \beta$ , the exact expression of which is not needed, the key feature being the polynomial dependence on the distance to  $\frac{\pi}{4}$ . Observe that for  $t_0 \leq t < \frac{\pi}{4}$ , up to reducing the local time  $\tau$ , we can choose  $\tau$  not depending on  $t_0 \in [0, t)$ . We say that  $\tau$  is *uniform* in  $[0, t)$ .

The strategy for proving local well-posedness is a fixed-point argument, which turns out to be easier than the one in [BT20]. In the latter case, the lack of sufficient regularisation in the stochastic linear term requires a more refined analysis, whereas Lemma 2.4 is already powerful enough for our purpose. We write  $\Phi : Y_{t_0, \tau}^\sigma \rightarrow \mathcal{S}'$  for the map defined by

$$(3.3) \quad \Phi(w)(t) := -i \int_{t_0}^t e^{-i(t-s)H} \left( \cos(2s)^{\frac{d}{2}(p-1)-2} |v_L(s) + w(s)|^{p-1} (v_L(s) + w(s)) \right) ds,$$

where  $v_L(t) := e^{-i(t-t_0)H} v_0$  is the linear evolution of the initial data  $v_0$ . The crucial estimates needed to run the fixed point argument are gathered in the following lemma.

**Lemma 3.5.** *Let  $\Phi$  defined by (3.3). Let  $d \geq 2$ . Let also  $p \in [1, 1 + \frac{4}{d})$ ,  $\sigma \in [0, \sigma(p, d))$  and  $\tau \leq \frac{1}{2} (\frac{\pi}{4} - t_0)$ . There exist constants  $\alpha > 0, \beta > 0$  which only depend on  $p, d$  and  $\sigma$ , such that for  $v_0 \notin A_\lambda^\circ$  and for  $w_1, w_2, w_3$  in the ball  $B(0, \lambda)$  of  $Y_{t_0, \tau}^\sigma$ , one has:*

$$(3.4) \quad \|\Phi(w)\|_{Y_{t_0, \tau}^\sigma} \lesssim_{d,p} \left( \frac{\pi}{4} - t_0 \right)^{-\beta} \tau^\alpha \lambda^p$$

and

$$(3.5) \quad \|\Phi(w_1) - \Phi(w_2)\|_{Y_{t_0, \tau}^\sigma} \lesssim_{d,p} \left( \frac{\pi}{4} - t_0 \right)^{-\beta} \tau^\alpha \lambda^{p-1} \|w_1 - w_2\|_{Y_{t_0, \tau}^\sigma}.$$

*Proof of Proposition 3.3.* Lemma 3.5 proves that  $\Phi$  is a contraction on the ball  $B(0, \lambda)$  of  $Y_{[t_0, t_0 + \tau]}^\sigma$  as soon as  $\tau \lesssim (\frac{\pi}{4} - t_0)^{\beta/\alpha} \lambda^{-p/\alpha}$ . Then the Picard fixed point theorem provides existence and uniqueness in this ball. Iterating the uniqueness argument finitely many times yields uniqueness in  $Y_{[t_0, t_0 + \tau]}^\sigma$ .  $\square$

*Proof of Lemma 3.5.* Let  $u_0 \in X^0 \setminus A_\lambda^\circ$ , the parameters  $a, b$  of the latter will be chosen in the proof. From (3.3) and by the non-homogeneous Strichartz estimates with an admissible pair  $(q, r)$ , one has

$$\begin{aligned} \|\Phi(v)\|_{Y_{[t_0, t_0 + \tau]}^\sigma} &\leq \left\| \cos(2s)^{\frac{d}{2}(p-1)-2} |v_L(s) + w(s)|^{p-1} (v_L(s) + w(s)) \right\|_{L_{[t_0, t_0 + \tau]}^{q'} \mathcal{W}^{\sigma, r'}} \\ &\leq \left( \int_{t_0}^{t_0 + \tau} \left( \frac{\pi}{4} - s \right)^{\left( \frac{d}{2}(p-1)-2 \right) q_1} \right)^{\frac{1}{q_1}} \| |v_L + w|^{p-1} (v_L + w) \|_{L_{[t_0, t_0 + \tau]}^{\tilde{q}} \mathcal{W}^{\sigma, r'}} \\ &\lesssim \left( \frac{\pi}{4} - t_0 \right)^{\frac{d}{2}(p-1)-2} \tau^{\frac{1}{q_1}} \| |v_L + w|^{p-1} (v_L + w) \|_{L_{[t_0, t_0 + \tau]}^{\tilde{q}} \mathcal{W}^{\sigma, r'}}, \end{aligned}$$

where we used Hölder's inequality with  $q_1, \tilde{q}$  to be adjusted and such that  $\frac{1}{q_1} + \frac{1}{\tilde{q}} = \frac{1}{q}$ . We also used that  $|\cos(2s)| \gtrsim (\frac{\pi}{4} - s)$  and  $\tau \leq \frac{1}{2} (\frac{\pi}{4} - t_0)$ .

Then by the Sobolev product estimates from Lemma A.1 and the triangle inequality one can write

$$\begin{aligned} \|\Phi(w)\|_{Y_{[t_0, t_0+\tau]}^\sigma} &\lesssim \left(\frac{\pi}{4} - t_0\right)^{\frac{d}{2}(p-1)-2} \tau^{\frac{1}{q_1}} \|v_L + w\|_{L^{q_2} \mathcal{W}^{\sigma, r_2}} \| |v_L + w|^{p-1} \|_{L^{q_3} L^{r_3}} \\ &\lesssim \left(\frac{\pi}{4} - t_0\right)^{\frac{d}{2}(p-1)-2} \tau^{\frac{1}{q_1}} \left( \|v_L\|_{L^{q_2} \mathcal{W}^{\sigma, r_2}} + \|w\|_{L^{q_2} \mathcal{W}^{\sigma, \frac{2d}{d-1}}} \right) \\ &\quad \times \left( \|v_L\|_{L^{q_3(p-1)} L^{r_3(p-1)}}^{p-1} + \|w\|_{L^{q_3(p-1)} L^{r_3(p-1)}}^{p-1} \right), \end{aligned}$$

where the parameters  $q_1, q_2, q_3$  and  $r_3$  need to satisfy

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q'},$$

and

$$\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r'}.$$

We recall that  $q', r'$  need to satisfy  $q', r' \in [1, 2]$  and

$$\frac{2}{q'} + \frac{d}{r'} = \frac{d}{2} + 2.$$

Assume that we are able to choose the parameters  $q_1, q_2, q_3, q', r_2, r_3, r'$  so that the norm  $L^{q_2} L^{r_2}$  is Schrödinger admissible,  $L^{q_3(p-1)} L^{r_3(p-1)}$  is  $\sigma$ -Schrödinger admissible and  $\|v_L\|_{L^{q_2} \mathcal{W}^{\sigma, r_2}} < \infty$ ,  $\|v_L\|_{L^{q_3(p-1)} L^{r_3(p-1)}} < \infty$ . The previous estimates show that for  $v_0 \in X^0 \setminus A_\lambda^\circ$  (with  $(a, b) := (q_3, r_3)$ ) and  $w$  in the ball  $B(0, \lambda)$  of  $Y_{[t_0, t_0+\tau]}^\sigma$  we have

$$\|\Phi(w)\|_{Y_{[t_0, t_0+\tau]}^\sigma} \lesssim \left(\frac{\pi}{4} - t_0\right)^{\frac{d}{2}(p-1)-2} \tau^{\frac{1}{q_1}} \lambda^p,$$

so we obtain (3.4). Similar computations yield (3.5), using the inequality (A.2) to estimate the term  $|v_L + w_1|^{p-1}(v_L + w_1) - |v_L + w_2|^{p-1}(v_L + w_2)$ , thus we omit the details.

Now we need to explain how to choose the parameters. We distinguish several cases.

*Case 1.* We assume  $p \leq \min\{1 + \frac{3}{d-2}, 1 + \frac{4}{d}\}$  so that  $\sigma(p, d) = \frac{1}{2}$ . To control the  $\mathcal{W}^{\sigma, r_2}$  norm we need to take  $r_2 = \frac{2d}{d-1}$  in order to use the estimates from Lemma 2.4 so that since  $u_0 \notin A_\lambda^\circ$   $\|v_L\|_{L^{q_2} \mathcal{W}^{\sigma, \frac{2d}{d-1}}} \leq \lambda$  because  $\sigma < s_c(\frac{2d}{d-1})$ . This forces us to take  $q_2 = 4$  in order to get:

$$\|w\|_{L^4 \mathcal{W}^{\sigma, \frac{2d}{d-1}}} \leq \|w\|_{Y_{[t_0, t_0+\tau]}^\sigma}.$$

We also have  $\|v_L\|_{L_3^q L^{r_3(p-1)}} \leq \lambda$ , provided the condition  $r_3(p-1) \leq \frac{2d}{d-2}$  which is implied by  $r'$  satisfying  $\frac{1}{r'} \geq p\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1}{2d}$ . Note that in order to ensure that  $q' \in [1, 2]$  we need  $\frac{1}{2} \leq \frac{1}{r'} \leq \frac{1}{2} + \frac{1}{d}$ . The above restriction on  $r'$  makes it possible to choose such an  $r'$  if and only if

$$p\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1}{2d} \leq \frac{1}{2} + \frac{1}{d},$$

which gives the condition  $p \leq 1 + \frac{3}{d-2}$  satisfied by hypothesis. Then we choose such an  $r'$  (which fixes  $r_3$  as desired, and satisfies  $r' \in [1, 2]$ ) and an associate  $q'$ , which is also in the range  $[1, 2]$ . To make sure that we can control the term  $\|w\|_{L^{q_3(p-1)} L^{r_3(p-1)}}$  by the Strichartz norm  $\|w\|_{Y_{t_0, \tau}^\sigma}$ , we use the Sobolev embedding which reads

$$\|v_L\|_{L^{q_3(p-1)} L^{r_3(p-1)}} \lesssim \|v_L\|_{L^{q_3(p-1)} W^{\sigma, \tilde{r}_3(p-1)}}$$

where  $\tilde{r}_3$  satisfies the Sobolev condition  $\frac{1}{\tilde{r}_3(p-1)} \leq \frac{\sigma}{d} + \frac{1}{r_3(p-1)}$ . If moreover  $(q_3(p-1), \tilde{r}_3(p-1))$  is Schrödinger admissible, that is  $\frac{2}{q_3(p-1)} + \frac{d}{\tilde{r}_3(p-1)} = \frac{d}{2}$  and  $q_3(p-1), \tilde{r}_3(p-1) \geq 2$ , then we

can control  $\|w\|_{L^{q_3(p-1)}L^{r_3(p-1)}}$  by the Strichartz norm  $\|w\|_{Y_{t_0,\tau}^\sigma}$ . We remark that the Sobolev condition can be written in the form

$$\frac{1}{r_3(p-1)} \geq -\frac{\sigma}{d} + \frac{1}{2} - \frac{2}{dq_3(p-1)},$$

which can be written in variables  $q_1, p, d$  as

$$\frac{1}{q_1} \leq 1 - \frac{(p-1)(d-2\sigma)}{4},$$

which is always satisfied for large  $q_1$ , since the quantity in the right-hand side is always bounded by below by  $1 - \frac{\min\{\frac{3}{d-2}, \frac{4}{d}\}}{4}(d-2\sigma) > 0$ . Now we can choose freely  $q_3$  such that  $q_3(p-1) \geq 2$  large enough so that  $\frac{1}{q_1} + \frac{1}{q_3} < \frac{1}{q'} - \frac{1}{4}$ .

*Case 2.* We assume  $d \geq 8$ ,  $p \in [1, 1 + \frac{4}{d})$  and thus  $\sigma(p, d) = 2 - \frac{d-2}{2}(p-1)$ . We take the initial data in  $\tilde{A}_\lambda^\circ$ , defined by (3.2). We choose

$$q_2 := \frac{4}{(d-2)(p-1)-2} \geq 2 \text{ and } r_2 := \frac{2d}{d+2-(d-2)(p-1)} \geq 2.$$

Observe that according to Lemma 2.4 we have  $\|v_L\|_{L^{q_2}\mathcal{W}^{\sigma,r_2}} \leq \lambda$  and  $\|w\|_{L^{q_2}\mathcal{W}^{\sigma,r_2}} \leq \lambda$  because  $(q_2, r_2)$  is Schrödinger admissible. As above we need to make sure we can find  $r_3$  such that

$$r_3(p-1) \leq \frac{2d}{d-2}$$

and

$$\frac{1}{r_3} + \frac{d+2-(d-2)(p-1)}{2d} = \frac{1}{r'}.$$

The condition that  $1 \leq q' \leq 2$  translates into  $\frac{1}{2} \leq \frac{1}{r'} \leq \frac{1}{2} + \frac{1}{d}$ . The condition on  $r_3$  is written:

$$\frac{1}{r'} \geq \frac{(p-1)(d-2)}{2d} + \frac{d+2-(d-2)(p-1)}{2d}.$$

The two previous conditions can be simultaneously satisfied provided  $p \leq 1 + \frac{4}{d}$  which is our hypothesis. Then we choose  $q'$  accordingly. To finish the proof we need to control the norm  $\|w\|_{L^{q_3(p-1)}L^{r_3(p-1)}}$  and with the same computations as above this translates into a condition on  $q_1$  which reads

$$\frac{1}{q_1} \leq 1 - \left(\frac{d}{4} - \frac{\sigma(d)}{2}\right)(p-1),$$

which is satisfied for large  $q_1$  as soon as

$$1 - \left(\frac{d}{4} - \frac{\sigma(d)}{2}\right)(p-1) > 0,$$

that is  $p-1 \leq \frac{-d+4+\sqrt{d^2+8d-16}}{2(d-2)}$ . This is satisfied since  $p-1 \leq \frac{4}{d} \leq \frac{-d+4+\sqrt{d^2+8d-16}}{2(d-2)}$ . Then we choose  $q_3$  large enough so that  $q_3(p-1) \geq 2$  and we conclude.  $\square$

#### 4. QUASI-INVARIANT MEASURES AND THEIR EVOLUTION

Our next task is to globalise the local statement of Proposition 3.3. In order to do so we need to keep track of the measures of the good sets of well-posedness, *i.e* we need to estimate  $\mu(\phi_t(A))$  where  $A$  is a  $\mu$ -measurable set and  $\phi_t$  stands for the flow of (HNLS). Due to explicit time dependence in the equation (HNLS) we do not expect the measure formally defined by

$$(4.1) \quad \nu_t(A) = C_\mu \int_A e^{-\frac{1}{2}\|u\|_{\mathcal{H}^1}^2 - \frac{\cos(2t)t^{p-3}}{p+1}\|u\|_{L^{p+1}}^{p+1}} du = \int_A e^{-\frac{\cos(2t)t^{p-3}}{p+1}\|u\|_{L^{p+1}}^{p+1}} d\mu,$$

to be invariant, but only quasi-invariant. This is one of the main ideas in [BT20] and we will carry out the same program. The quasi-invariance will be obtained using a Liouville theorem on finite dimensional approximations of (HNLS) and a limiting argument.

With the smooth projections  $\mathbf{S}_N$  defined in the introduction, we introduce the following approximate equations:

$$(HNLS_N) \quad \begin{cases} i\partial_t v - Hv = \cos(2t)^{p-3} \mathbf{S}_N(|\mathbf{S}_N v|^{p-1} \mathbf{S}_N v) \\ v(0) = u_0, \end{cases}$$

and denote by  $\phi_t^N$  its flow. Writing  $u = u_N + u^N$ , a solution to (HNLS\_N) with  $u_N = \mathbf{S}_N u$  we observe that from equation (HNLS\_N),  $u_N$  satisfies

$$(4.2) \quad \begin{cases} i\partial_t v_N - Hv_N = \cos(2t)^{p-3} \mathbf{S}_N(|\mathbf{S}_N v_N|^{p-1} \mathbf{S}_N v_N) \\ v(0) = \mathbf{S}_N u_0, \end{cases}$$

which is a finite dimensional ordinary differential equation in  $\mathbb{C}^{N+1} \simeq \mathbb{R}^{2(N+1)}$ .

We recall the following useful fact, which is proved in [BGT04] for instance:

$$(4.3) \quad \sup_{N \geq 1} \|\mathbf{S}_N\|_{L^q \rightarrow L^q} \leq C(q) \text{ for } q \geq 2,$$

from which we infer that the local Cauchy theory of Proposition 3.3 applies to (HNLS\_N) without modification.

From equation (HNLS\_N), we observe that  $v^N$  satisfies

$$(4.4) \quad \begin{cases} i\partial_t v^N - Hv^N = 0 \\ v(0) = (\text{id} - \mathbf{S}_N)u_0, \end{cases}$$

so that if we identify  $v^N$  to the sequence  $(v_n)_{n>N}$  such that  $v^N = \sum_{n>N} v_n e_n$ , we can explicitly solve (4.4) and find that for every  $n \geq N+1$ ,

$$(4.5) \quad v_n(t) = e^{-it\lambda_n^2} (u_0)_n,$$

and denote by  $\phi_t^{\perp, N}$  its flow.

We start with a lemma which is nothing but the Liouville theorem, which proof is recalled.

**Lemma 4.1.** *The measure  $du$  is invariant under the flow  $\phi_t^N$  of (HNLS\_N).*

*Proof.* First we recall that (4.2) is locally well-posed in  $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^{2(N+1)})$ , thanks to the Cauchy theory for ordinary differential equations and globally well-posed since  $t \mapsto \|u_N(t)\|_{L^2}$  is conserved. Moreover this equation admits a Hamiltonian structure. In order to see it, we write that for every  $n$ ,  $u_n = p_n + iq_n$  where  $p_n, q_n$  are real numbers. Then we claim that there exists a function  $E_N = E_N(t, p_1, \dots, p_N, q_1, \dots, q_N)$  such that if  $p = (p_1, \dots, p_N)$  and  $q := (q_1, \dots, q_N)$ , (4.2) takes the form

$$(4.6) \quad \begin{cases} p'(t) = \partial_q E_N(t, p(t), q(t)) \\ q'(t) = -\partial_p E_N(t, p(t), q(t)). \end{cases}$$

In order to find  $E_N$ , write  $u = p + iq$  and equate real and imaginary parts of (4.2) to obtain that a Hamiltonian satisfying (4.6) is given by

$$E_N(t, p_1, \dots, p_N, q_1, \dots, q_N) := \frac{1}{2} \sum_{n=0}^N \lambda_n^2 (a_n^2 + b_n^2) + \frac{\cos(2t)^{\frac{d}{2}(p-1)-2}}{p+1} \left\| \sum_{n=0}^N (a_n + ib_n) e_n \right\|_{L^{p+1}(\mathbb{R}^d)}^{p+1}.$$

More details are given in [BTT13], Lemma 8.1. For a solution  $(p, q)$  to (4.6) we write  $(p(t), q(t)) = \tilde{\phi}_t^N(p_0, q_0)$ . Let

$$du_N = \bigotimes_{n=0}^N du_n = \bigotimes_{n=0}^N (dp_n \otimes dq_n),$$



then for every smooth function  $f$  with compact support we have

$$\frac{d}{dt} \int f(\tilde{\phi}_t^N(p_0, q_0)) du_N(p_0, q_0) = \int (\partial_p E_N \partial_q f - \partial_q E_N \partial_p f) dp dq = 0,$$

where we integrated by parts in the last equality. Finally by density this shows that the measure  $du_N$  is invariant under  $\tilde{\phi}_t^N$ . Moreover by (4.5) and recalling that the Lebesgue measure is invariant under rotation we see that the measure  $\otimes_{n>N} du_n$  is conserved by the flow  $\phi_t^{\perp, N}$ . To conclude, observe that  $\phi_t^N = \tilde{\phi}_t^N \otimes \phi_t^{\perp, N}$  and use the Fubini theorem.  $\square$

The Hamiltonian we have found takes the form:

$$E_N(t) := \frac{1}{2} \|v(t)\|_{\mathcal{H}^1}^2 + \frac{\cos(2t)^{\frac{d}{2}(p-1)-2}}{p+1} \|\mathbf{S}_N v(t)\|_{L^{p+1}}^{p+1}.$$

It is *not conserved* under the flow  $\phi_t^N$  and more precisely

$$(4.7) \quad E'_N(t) = \frac{d(p-1)-4}{p+1} \tan(2t) \cos(2t)^{\frac{d}{2}(p-1)-2} \|\mathbf{S}_N v(t)\|_{L^{p+1}}^{p+1}.$$

For  $t \geq 0$  and  $N \geq 0$  we define the finite measures, which are not necessarily probability measures, associated to the unconserved energies  $E_N$  by:

$$\nu_t^{(N)}(A) := C_\mu \int_A e^{-\frac{1}{2} \|\sqrt{H}u\|_{L^2}^2 - \frac{\cos(2t)^{\frac{d}{2}(p-1)-2}}{p+1} \|\mathbf{S}_N u\|_{L^{p+1}}^{p+1}} du.$$

From the definition it follows that for all  $\mu$ -measurable sets  $A$  one has  $\nu_t^{(N)}(A) \leq \mu(A)$ . Moreover we have the following convergence result.

**Lemma 4.2.** *Let  $t \geq 0$ . Then the measure  $\nu_t$  is not trivial, i.e its density with respect to  $\mu$  does not vanish almost surely. Moreover we have the strong convergence  $\nu_t^{(N)} \rightarrow \nu_t$ , that is for every measurable set  $A$ ,  $\nu_t^{(N)}(A) \rightarrow \nu_t(A)$ .*

*Proof.* To see the first claim, just observe that for  $u$  in the support of the measure  $\mu$ , since  $p+1 \in \left(2, 2 + \frac{4}{d}\right)$  we have  $p+1 < \frac{2d}{d-2}$  and then  $\|u\|_{L^{p+1}} < \infty$  thanks to Lemma 2.1. Moreover for such  $u$  in the support of  $\mu$  we have  $\mathbf{S}_N u \rightarrow u$  in  $L^{p+1}$  as  $N \rightarrow \infty$  thanks to Lemma 4.3. The domination

$$e^{-\frac{\cos(2t)^{p-3}}{p+1} \|\mathbf{S}_N u\|_{L^{p+1}}^{p+1}} \leq 1$$

and Lebesgue's convergence theorem ensures the strong convergence  $\nu_t^{(N)} \rightarrow \nu_t$ .  $\square$

The quantitative quasi-invariance property which we are going to state and prove are exactly the same as in [BT20]. However for convenience we recall the proof.

**Proposition 4.3** (Measure Evolution). *Let  $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ ,  $N \geq 1$  and a  $\mu$ -measurable set  $A$ .*

- (i)  $(\phi_t^N)_* \mu$  and  $\mu$  are mutually absolutely continuous with respect to each other;
- (ii)  $\nu_t \left( \phi_t^N(A) \right) \leq \nu_0(A)^{\cos(2t)^{\left| \frac{d}{2}(p-1)-2 \right|}};$
- (iii)  $\nu_0(A) \leq \nu_t \left( \phi_t^N(A) \right)^{\cos(2t)^{\left| \frac{d}{2}(p-1)-2 \right|}}.$

*Proof.* Note that once we have proved (ii) and (iii) then we immediately can conclude the proof for (i) since the two previous points give us that

$$(\phi_t^N)_* \nu_t \ll \nu_0 \ll (\phi_t^N)_* \nu_t.$$

As by definition  $\mu \ll \nu_t$  and  $\nu_t \ll \mu$ , which proves (i).

To prove (ii) we start by studying the measure  $\nu_t^N$ , writing

$$\begin{aligned} \frac{d}{dt} \nu_t^{(N)}(\phi_t^N A) &= C_\mu \frac{d}{dt} \left( \int_{v \in \phi_t^N A} e^{-\frac{1}{2} \|\sqrt{H}v\|_{L^2}^2 - \frac{\cos(2t) \frac{d}{2}(p-1)-2}{p+1} \|\mathbf{S}_N v\|_{L^{p+1}}^{p+1}} dv \right) \\ &= C_\mu \frac{d}{dt} \left( \int_A e^{-\frac{1}{2} \|\sqrt{H}\phi_t^{(N)} u_0\|_{L^2}^2 - \frac{\cos(2t) \frac{d}{2}(p-1)-2}{p+1} \|\mathbf{S}_N \phi_t^{(N)} u_0\|_{L^{p+1}}^{p+1}} du_0 \right) \\ &= C_\mu \frac{d(p-1)-4}{2} \tan(2t) \int_A \alpha(t, u) e^{-E_N(t)} du_0, \end{aligned}$$

with  $\alpha(t, u) = \frac{\cos(2t) \frac{d}{2}(p-1)-2}{p+1} \|\mathbf{S}_N u(t)\|_{L^{p+1}}^{p+1}$ . In the first inequality we have used the change of variable  $v = \phi_t^N u_0$ , which leaves  $du$  invariant according to Lemma 4.1.

Now apply Hölder's inequality for a  $k \geq 1$  to be chosen later, and use that for all positive  $\alpha$  we have  $\alpha^k e^{-\alpha} \leq k^k e^{-k}$  which gives

$$\begin{aligned} \frac{d}{dt} \nu_t^{(N)}(\phi_t^N A) &\leq |d(p-1)-4| \tan(2t) \left( C_0 \int_A \alpha^k(t, u(t)) e^{-E_N(t)} du_0 \right)^{\frac{1}{k}} \left( \nu_t^{(N)}(\phi_t^N A) \right)^{1-\frac{1}{k}} \\ &= |d(p-1)-4| \tan(2t) \left( C_0 \int_A \alpha^k(t, u(t)) e^{-\alpha(t, u) - \frac{1}{2} \|\sqrt{H}u(t)\|_{L^2}^2} du_0 \right)^{\frac{1}{k}} \\ &\quad \times \left( \nu_t^{(N)}(\phi_t^N A) \right)^{1-\frac{1}{k}} \\ &\leq |d(p-1)-4| \tan(2t) \frac{k}{e} \left( \nu_t^{(N)}(\phi_t^N A) \right)^{1-\frac{1}{k}}, \end{aligned}$$

where we used the backward change of variable that leaves the measure invariant again. Now we choose  $k$  to optimise this inequality, namely  $k := -\log \left( \nu_t^{(N)}(\phi_t^N A) \right)$  so that

$$\frac{d}{dt} \nu_t^{(N)}(\phi_t^N A) \leq |d(p-1)-4| \tan(2t) \log \left( \nu_t^{(N)}(\phi_t^N A) \right) \nu_t^{(N)}(\phi_t^N A).$$

We rewrite it as:

$$\begin{aligned} -\frac{d}{dt} \left( \log \left( -\log \left( \nu_t^{(N)}(\phi_t^N A) \right) \right) \right) &\leq |d(p-1)-4| \tan(2t) \\ &= -\left| \frac{d}{2}(p-1)-2 \right| \frac{d}{dt} (\log(\cos(2t))), \end{aligned}$$

which after integration reads

$$-\log \left( \nu_t^{(N)}(\phi_t^N A) \right) \leq \left( \nu_0^{(N)}(A) \right)^{\cos(2t) \left| \frac{d}{2}(p-1)-2 \right|}.$$

Then we observe that for  $M \geq N$  and for every  $\mu$ -measurable set  $A$ , one has

$$\nu_t^{(N)}(\phi_t^N A) = \nu_t^{(M)}(\phi_t^N A) \rightarrow \nu_t(\phi_t^N A) \text{ as } M \rightarrow \infty$$

so that finally get the result passing to the limit.

The estimate (iii) is obtained by similar means, observing first that

$$\frac{d}{dt} \nu_t^{(N)}(\phi_t^N A) \geq -|d(p-1)-4| \tan(2t) \frac{k}{e} \left( \nu_t^{(N)}(\phi_t^N A) \right)^{1-\frac{1}{k}},$$

for every  $k \geq 1$ , optimising in  $k$  and integrating as before.  $\square$

## 5. GLOBAL THEORY

Before stating global results we need an approximation lemma which quantifies to what extent  $(\text{HNLS}_N)$  is a good approximation of  $(\text{HNLS})$ .

**Lemma 5.1** (Approximation). *Let  $d \geq 2$ ,  $p \in \left(1, 1 + \frac{4}{d}\right)$  and  $\sigma < 0$ . Let also  $t_0, t \in \left(0, \frac{\pi}{4}\right)$  such that  $t_0 < t$ . Consider  $v_0 \in X^0 \setminus A_\lambda^\circ$  an initial data. We assume that  $\lambda > 0$  is large enough so that the local solution  $w$  (resp.  $w_N$ ) associated to  $v_0$  to the problem  $(\text{HNLS})$  (resp.  $(\text{HNLS}_N)$ ) constructed in Proposition 3.3 on  $[t_0, t]$  exist in some  $Y_{[t_0, t]}^{\tilde{\sigma}}$  for some  $\tilde{\sigma} > 0$ , and that the estimates (3.4) and (3.5) hold.*

*Then there exists a constant  $C(t_0, t) > 0$  such that for every  $\sigma' < \sigma$ , there holds*

$$\|w - w_N\|_{L^\infty([t_0, t], \mathcal{H}^{\sigma'})} \leq C(t_0, t)N^{\sigma' - \sigma}.$$

*Proof.* We follow the lines of [Bou94] where a similar result is proven. Let us write

$$\begin{aligned} w - w_N &= z_N - (\text{id} - \mathbf{S}_N)(v_N) \\ &= z_N - \mathbf{P}_{>N}(v_L) - \mathbf{P}_{>N}(w_N), \end{aligned}$$

where

$$z_N(t) := w(t) - \mathbf{S}_N w_N(t) + (\text{id} - \mathbf{S}_N)v_L,$$

and satisfies

$$z_N(t_0) = \mathbf{P}_{>N}v(t_0).$$

We recall that the local well-posedness theory developed in Proposition 3.3 applies *verbatim* to equation  $(\text{HNLS}_N)$  uniformly in  $N$ . For  $\lambda$  large enough, the  $v_N = v_L + w_N$  exist on the time interval  $[t_0, t]$  and we have  $\|w_N\|_{Y_{[t_0, t]}^\sigma} \leq C(t_0, t)$ . Then the Bernstein inequality ensures that  $\|\mathbf{P}_{>N}w_N\|_{L^\infty([t_0, t], \mathcal{H}^{\sigma'})} \leq C(t_0, t)N^{\sigma' - \sigma}$ .

From the Bernstein inequality it follows that since  $v(t_0) \in \mathcal{H}^\sigma$  we have  $\|\mathbf{P}_{>N}v_L\|_{L^\infty([t_0, t], \mathcal{H}^{\sigma'})} \leq C(t_0, t)N^{\sigma' - \sigma}$ .

We are left with estimating the  $Y_{[t_0, t]}^{\sigma'}$  norm of  $z_N$ . In order to do so, we observe that  $z_N$  satisfies:

$$\begin{aligned} (5.1) \quad i\partial_t z_N - Hz_N &= \cos(2t)^{\frac{d}{2}(p-1)-2} (F(v_L + w) - \mathbf{S}_N F(v_L + w - z_N)) \\ &= \cos(2t)^{\frac{d}{2}(p-1)-2} (\text{id} - \mathbf{S}_N)F(v_L + w) \\ &\quad + \cos(2t)^{\frac{d}{2}(p-1)-2} \mathbf{S}_N (F(v_L + w) - F(v_L + w - z_N)), \end{aligned}$$

where  $F$  is defined by  $F(X) = X|X|^{p-1}$ , the nonlinear term. Using the decomposition in the right-hand side of (5.1) and the Strichartz estimates from Proposition 3.2 and a local existence time associated to  $w(t_0)$  given by Lemma 3.5, we obtain that for  $\delta \leq \tau$ :

$$(5.2) \quad \|z_N\|_{Y_{[t_0, t_0+\delta]}^{\sigma'}} \leq \|\mathbf{P}_{>N}w(t_0)\|_{\mathcal{H}^{\sigma'}}$$

$$(5.3) \quad + \|\cos(2t)^{\frac{d}{2}(p-1)-2} (\text{id} - \mathbf{S}_N)F(v_L + w)\|_{\tilde{Y}_{[t_0, t_0+\delta]}^{\sigma'}}$$

$$(5.4) \quad + \|\cos(2t)^{\frac{d}{2}(p-1)-2} \mathbf{S}_N (F(v_L + w) - F(v_L + w - z_N))\|_{\tilde{Y}_{[t_0, t_0+\delta]}^{\sigma'}}.$$

We now deal with each term (5.2), (5.3), (5.4). For (5.2), by Bernstein's inequality we have

$$\|\mathbf{P}_{>N}w(t_0)\|_{\mathcal{H}^{\sigma'}} \leq CN^{\sigma' - \sigma} \|w(t_0)\|_{\mathcal{H}^\sigma}.$$

Using the local theory estimates from Lemma 3.5 and Bernstein's inequality, we obtain

$$\|\cos(2t)^{\frac{d}{2}(p-1)-2} (\text{id} - \mathbf{S}_N)F(v_L + w)\|_{\tilde{Y}_{[t_0, t_0+\delta]}^{\sigma'}} \leq CN^{\sigma' - \sigma} \left(\frac{\pi}{4} - t\right)^{-\beta} \delta^\alpha \lambda^p,$$

and finally with the estimates from Lemma 3.5 again we also have:

$$\begin{aligned} & \|\cos(2t)^{\frac{d}{2}(p-1)-2} \mathbf{S}_N (F(v_L + w) - F(v_L + w - z_N))\|_{\dot{Y}_{[t_0, t_0+\delta]}^{\sigma'}} \\ & \leq C \lambda^{p-1} \left(\frac{\pi}{4} - t\right)^{-\beta} \delta^\alpha \|z_N\|_{Y^{\sigma'}([t_0, t_0+\delta])}. \end{aligned}$$

Combining these estimates and for  $\delta$  small enough we have

$$\|z_N\|_{Y_{t_0, \delta}^\sigma} \leq \frac{1}{2} \|z_N\|_{Y_{t_0, \delta}^\sigma} + \frac{C(t) \|w(t_0)\|_{\mathcal{H}^\sigma}}{2} N^{\sigma' - \sigma}.$$

Finally we have proved that  $\|z_N\|_{Y^{\sigma'}([t_0, t_0+\delta])} \leq N^{\sigma' - \sigma} C(t) \|w(t_0)\|_{\mathcal{H}^\sigma}$ . We need to iterate this estimate in time until we reach time  $t$  and we need to check that only a finite number of steps are required. We remark that the local existence time  $\tau$  can be chosen uniformly in  $[t_0, t]$  thanks to the remark following Proposition 3.3 and that  $\delta$  can be chosen only depending on  $t$  and  $\lambda$ , thus is uniform in  $[t_0, t]$ . Repeating the argument  $\lfloor \frac{t}{\tau} \rfloor$  times gives

$$\begin{aligned} \|z_N\|_{Y_{[t_0, t]}^\sigma} & \leq N^{\sigma' - \sigma} C(t) \sum_{n=1}^{\lfloor \frac{t}{\tau} \rfloor} \|w(t_0 + n\tau)\|_{\mathcal{H}^\sigma} \\ & \leq N^{\sigma' - \sigma} C(t) \sum_{n=1}^{\lfloor \frac{t}{\tau} \rfloor} 2^n \|w(t_0)\|_{\mathcal{H}^\sigma}, \end{aligned}$$

where we used that on an interval of local well-posedness the size of the function in  $\mathcal{H}^\sigma$  does not grow more than the double. Since  $\tau$  does only depend only on  $t$  and the parameters we obtain the required estimate on  $z_N$  so that finally

$$\|w - w_N\|_{L^\infty([t_0, t], \mathcal{H}^{\sigma'})} \leq \|(\text{id} - \mathbf{S}_N)(v_L + w_N)\|_{Y_{t_0, t}^{\sigma'}} + \|z_N\|_{Y_{t_0, t}^{\sigma'}} \leq N^{\sigma' - \sigma} C(t_0, t)$$

as claimed.  $\square$

**Corollary 5.2.** *Let  $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$ , and  $\varepsilon > 0$ . Let  $B_\varepsilon$  denotes the ball of radius  $\varepsilon$  in the space  $X^0$ . Let  $A$  be a Borelian set of  $X^0$ . Then for sufficiently large  $N$ , there holds*

$$(5.5) \quad \phi_t^N(\phi_t^{-1}A) \subset A + B_\varepsilon.$$

*Proof.* The proof is essentially contained in Lemma 5.1. Observe that for any  $u_0 \in \phi_t^{-1}(A)$ , we can write that  $\phi_t u_0 = v_L(t) + w(t)$  and  $\phi_t^N u_0 = v_L(t) + w_N(t)$ . Let  $\sigma$  be as in Lemma 5.1 and  $\sigma' < \sigma$ . For  $\sigma > \sigma' > 0$ , the result of Lemma 5.1 writes

$$\begin{aligned} \|\phi_t(u_0) - \phi_t^N(u_0)\|_{\mathcal{H}^{-\sigma}} & = \|w(t) - w_N(t)\|_{\mathcal{H}^{-\sigma}} \\ & \leq \|w - w_N\|_{L^\infty([0, t], \mathcal{H}^{-\sigma})} \\ & \leq C(t) N^{-\sigma + \sigma'}, \end{aligned}$$

so that for  $N$  large enough we have  $\|\phi_t u_0 - \phi_t^N u_0\|_{\mathcal{H}^{-\sigma}} \leq \varepsilon$  for all  $\sigma < 0$ , which means that  $\|\phi_t(u_0) - \phi_t^N(u_0)\|_{X^0} \leq \varepsilon$ , and since  $\phi_t(u_0) \in A$  this implies  $\phi_t^N(u_0) \in A + B_\varepsilon$ .  $\square$

We use this approximation theorem to the following consequence of Proposition 4.3.

**Corollary 5.3.** *Let  $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$  and a  $\mu$ -measurable Borelian set  $A$ . Then one has*

$$(5.6) \quad \nu_0(A) \leq \nu_t(\phi_t(A))^{\cos(2t) \lfloor \frac{d}{2}(p-1)-2 \rfloor}.$$

Moreover if we define

$$(5.7) \quad A_{t, \lambda} := \{u_0 \in X^0, \|\phi_t u_0\|_{L^{p+1}} > \lambda\},$$

then

$$(5.8) \quad \nu_0(A_t) \leq e^{-\frac{\lambda^{p+1}}{p+1}}.$$

*Proof.* Let  $A$  be a Borelian set of  $X^0$  and  $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$ . Applying Proposition 4.3 we have:

$$\nu_0(\phi_t^{-1}A) \leq \nu_t\left(\phi_t^N(\phi_t^{-1}A)\right)^{\cos(2t)^{\left|\frac{d}{2}(p-1)-2\right|}}.$$

Given  $\varepsilon > 0$ , thanks to (5.5) we fix  $N$  such that  $\phi_t^N(\phi_t^{-1}A) \subset A + B_\varepsilon$ . For such  $N$  we can write

$$\nu_0(\phi_t^{-1}A) \leq \nu_t(A + B_\varepsilon)^{\cos(2t)^{\left|\frac{d}{2}(p-1)-2\right|}}.$$

Letting  $\varepsilon \rightarrow 0$  and using the regularity of the measure  $\mu$  established in Section 2.2, and the Lebesgue convergence theorem we have  $\nu_0(A + B_\varepsilon) \rightarrow \nu_0(A)$ . Changing  $A$  in  $\phi_t A$  this gives (5.6).

Let us prove (5.8). Using (5.6) (assuming that  $A_{t,\lambda}$  is a Borelian set), and remarking that by definition of  $A_t$  we have  $\|\phi_t u_0\| > \lambda$  for every  $u_0 \in A_t$  so that:

$$\begin{aligned} \nu_0(A) &\leq \nu_t(\phi_t A)^{\cos(2t)^{\frac{d}{2}(1-p)+2}} \\ &\leq \left( e^{-\cos(2t)^{\frac{d}{2}(p-1)-2} \frac{\lambda^{p+1}}{p+1}} \mu(\phi_t A) \right)^{\cos(2t)^{\frac{d}{2}(1-p)+2}} \\ &\leq e^{-\frac{\lambda^{p+1}}{p+1}}, \end{aligned}$$

where we used that  $\mu(\phi_t A) \leq 1$  in the last line.

The set  $A_{t,\lambda}$  is indeed a Borelian set.  $A_{t,\lambda}$  is indeed an open set: let  $u_n \in X^0 \setminus A_{t,\lambda}$ , a converging sequence, that is  $u_n \rightarrow u$  in any  $\mathcal{H}^{-\sigma}$  and  $\|u_n\|_{L^{p+1}} \leq \lambda$ . We observe that the sequence  $(u_n)_{n \geq 0}$  being bounded in  $L^{p+1}$ , up to extraction we have  $u_n \rightharpoonup^* u$  in  $L^{p+1}$  so that

$$\|u\|_{L^{p+1}} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^{p+1}},$$

and thus the claim.  $\square$

*Remark 5.4.* The estimate (5.8) is better than the bound given by  $\nu_0(A) \leq \mu(A)^{\cos(2t)^{\frac{d}{2}(p-1)-2}}$ , which amounts to saying  $\nu(A) \leq 1$  when  $p \sim 1 + \frac{4}{d}$ .

We are now ready to state our first global result. The first version is an arbitrary large measure existence result of solutions defined on  $[-t, t]$  for arbitrary  $t$ .

**Lemma 5.5** (Quasi almost sure global existence). *Let  $d \geq 2$ ,  $p \in (1, 1 + \frac{4}{d})$  and  $\sigma \in (0, \sigma(p, d))$ . Let  $\eta > 0$  and  $0 < t < \frac{\pi}{4}$ . There exists a set  $G_{\eta,t}$  such that  $\mu(X^0 \setminus G_{\eta,t}) \leq \eta$  and such that for every initial data  $u_0 \in G_{\eta,t}$  there exist a unique solution  $v(t') = e^{-it'H}u_0 + w(t')$  to (HNLS) which is defined on  $[-t, t]$  and satisfies*

$$(5.9) \quad \|w(t')\|_{\mathcal{H}^\sigma} \lesssim \log^{\frac{1}{2}}\left(\frac{1}{\eta}\right) \left(\frac{\pi}{4} - t\right)^{\frac{d}{4}(p-1)-1} \left| \log\left(\frac{\pi}{4} - t\right) \right|^{\frac{1}{2}},$$

for every  $|t'| \leq t$ , and where the implicit constant only depends on  $p$  and  $d$ . Moreover:

$$(5.10) \quad \|w\|_{L^q((0,t), L^r)} \lesssim \log^{\frac{1}{2}}\left(\frac{1}{\eta}\right) \left(\frac{\pi}{4} - t\right)^{\frac{d}{4}(p-1)-1} \left| \log\left(\frac{\pi}{4} - t\right) \right|^{\frac{1}{2}},$$

for every  $\sigma$ -Schrödinger admissible pair  $(q, r)$  and every  $|t'| \leq t$ .

*Proof.* Let  $t, \eta$  as in the lemma and  $\lambda > 0$ . We denote by  $G_\lambda$  the set of good initial data made of  $u_0 \in X^0$  which give rise to local solutions  $v = v_L + w$  of the Cauchy problem (HNLS) until time  $t$  and which furthermore satisfy the bounds  $\|w\|_{L^q([0,t], \mathcal{V}^{\sigma,r})} \leq 2\lambda$  for  $0 \leq t' \leq t$  and any Schrödinger admissible pair  $(q, r)$ . Let  $B_\lambda := X^0 \setminus G_\lambda$ , which is the set of bad initial data.

First, observe that on the compact interval  $[0, t]$  a uniform local existence time  $\tau$  can be picked. Thanks to Proposition 3.3 this local existence time has the form  $\tau \sim \lambda^{-\alpha} \left(\frac{\pi}{4} - t\right)^\beta$  for irrelevant constants  $\alpha, \beta > 0$ . Then, for the solution  $v = v_L + w$  to exist on  $[n\tau, (n+1)\tau]$ , Proposition 3.3 proves that it is sufficient for  $u(n\tau)$  to belong to  $X^0 \setminus A_\lambda^\circ$ . Under this condition the Cauchy problem can be solved until time  $(n+1)\tau$ . Thus we have obtained that

$$\bigcap_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} (\phi_{n\tau})^{-1} (X^0 \setminus A_\lambda^\circ) \subset G_\lambda,$$

and taking the complementary set gives

$$B_\lambda \subset \bigcup_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} (\phi_{n\tau})^{-1} A_\lambda^\circ.$$

We use Corollary 5.3 to infer

$$\nu_0(B_\lambda) \leq \sum_{n=0}^{\lfloor \frac{t}{\tau} \rfloor} \nu_0 \left( (\phi_{n\tau})^{-1} A_\lambda^\circ \right) \leq \sum_{n=0}^{\lfloor \frac{t}{\tau} \rfloor} \nu_t(A_\lambda^\circ)^{\cos(2n\tau)} \left| \frac{d}{2}(p-1)-2 \right|.$$

Using that  $\nu_t(A_\lambda^\circ) \leq \mu(A_\lambda^\circ)$  and the bound  $n\tau \leq t$  leads to

$$\nu_0(B_\lambda) \leq \sum_{n=0}^{\lfloor \frac{t}{\tau} \rfloor} \mu(A_\lambda^\circ)^{\left(\frac{\pi}{4}-t\right)^{\left|\frac{d}{2}(p-1)-2\right|}} \leq t\tau^{-1} e^{-c\lambda^2 \left(\frac{\pi}{4}-t\right)^{2-\frac{d}{2}(p-1)}},$$

where we used Corollary 2.5 in the last inequality. Recalling the expression of  $\tau$  in terms of  $\left(\frac{\pi}{4} - t\right)$  and  $\lambda$ , proves that in order to ensure  $\nu_0(B_\lambda) \leq \eta$  one can choose

$$\lambda \sim \log^{\frac{1}{2}} \left( \frac{1}{\eta} \right) \left( \frac{\pi}{4} - t \right)^{\frac{d}{4}(p-1)-1} \left| \log \left( \frac{\pi}{4} - t \right) \right|^{\frac{1}{2}},$$

which ends the proof.  $\square$

Now we transform the previous quasi almost-sure lemma into an almost sure global existence result using a Borel-Cantelli argument.

**Corollary 5.6.** *Let  $d \geq 2$ ,  $p \in \left(1, 1 + \frac{4}{d}\right)$  and  $\sigma \in [0, \sigma(p, d))$ . There exists a set  $G$  of full measure, i.e.,  $\mu(G) = 1$  such that for  $u_0 \in G$  there exists a unique global solution  $v = v_L + w$  to (HNLS), where  $v_L(t) = e^{-itH}u_0$  and  $w(t) \in \mathcal{H}^\sigma$  satisfies the bound*

$$(5.11) \quad \|w\|_{L^q([-t, t], \mathcal{W}^{\sigma, r})} \leq C(u_0) \left( \frac{\pi}{4} - |t| \right)^{\frac{d}{4}(p-1)-1} \left( 1 + \left| \log \left( \frac{\pi}{4} - |t| \right) \right| \right)^{\frac{1}{2}},$$

for every  $|t| < \frac{\pi}{4}$  and every Schrödinger admissible pair  $(q, r)$ . Moreover there exist constants  $c, C > 0$  such that:

$$(5.12) \quad \mu(u_0, C(u_0) > \lambda) \leq C e^{-c\lambda^2}.$$

*Proof.* First note that by time reversibility of (HNLS) it is sufficient to prove the result on  $[0, \frac{\pi}{4})$ . Let  $t_n \rightarrow \frac{\pi}{4}$ . For  $n \geq 1$  we set  $B_n := B_{\frac{1}{n^2}, t_n} = X^0 \setminus G_{\frac{1}{n^2}, t_n}$  and observe that  $\mu(B_n) \leq \frac{1}{n^2}$  which forms a convergent sequence, thus

$$\mu \left( \bigcup_{k \geq n} B_k \right) \leq \sum_{k \geq n} \mu(B_k) \xrightarrow{n \rightarrow \infty} 0.$$

Letting  $B := \limsup_{n \rightarrow \infty} B_k = \bigcap_{n \geq 1} \bigcup_{k \geq n} B_k$  we have:

$$\mu \left( \limsup_{n \rightarrow \infty} B_n \right) \leq \limsup_{n \rightarrow \infty} \mu \left( \bigcup_{k \geq n} B_k \right) \rightarrow 0,$$

thus  $G := X^0 \setminus B$  is a set of full measure and moreover by definition of  $G$ , for every  $u_0 \in G$  there exist an  $n$  such that  $u_0 \in G_k$  for all  $k \geq n$ . Thus we obtain that such  $u_0$  give rises to global solutions satisfying (5.11).

Finally in order to prove (5.12) we observe that for a given  $t > 0$ ,  $\{C(u_0) > \lambda\} \subset B_{e^{-c\lambda^2}, t}$  which yields the result.  $\square$

Next we want to estimate the  $L^{p+1}$  norm of the solutions. We will distinguish two cases and introduce a real number  $p_{\max}$  defined by

$$p_{\max}(d) := \frac{5 - d + \sqrt{9d^2 - 2d + 9}}{2(d-1)} < 1 + \frac{3}{d-2}$$

if  $d \leq 7$  and if  $d \geq 8$  then  $p_{\max}(d)$  is the only real root of the polynomial

$$P_d = (d-2)X^3 + (d-4)X^2 - 6X - 2d - 4.$$

Note that for  $d \geq 8$  one has

$$p_{\max}(d) \geq \min \left\{ \frac{5 - d + \sqrt{9d^2 - 2d + 9}}{2(d-1)}, 1 + \frac{3}{d-2} \right\}.$$

To prove this fact, we observe that the discriminant of  $P_d$  is negative, at least for  $d \geq 8$  and thus  $P_d$  has a unique real root. Note that  $\min \left\{ \frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)}, 1+\frac{3}{d-2} \right\} = 1+\frac{3}{d-2}$  as soon as  $d \geq 9$ . To conclude we need to show that  $P_d(1+\frac{3}{d-2}) < 0$  which is equivalent to  $d^2 - 10d + 7 > 0$ , satisfied for  $d \geq 9$ . Similarly one has  $P_d(\frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)}) < 0$  for  $d = 8$ . We also have  $p_{\max}(d) \geq 1 + \frac{4}{d}$  if  $d \leq 7$  with similar computations.

**Lemma 5.7.** *Let  $d \geq 2$ ,  $p \in (1, p_{\max}(d))$  and  $\sigma \in (0, \sigma(p, d))$ . Let also  $\eta > 0$  and  $|t| < \frac{\pi}{4}$ . There exists a set  $G_{\eta, t} \subset X^0$  such that  $\mu(X^0 \setminus G_{\eta, t}) \leq \eta$  and such that for all  $u_0 \in G_{\eta, t}$ , there exists a unique solution to (HNLS) with initial data  $u_0$  which writes  $v(t') = e^{-it'H}u_0 + w(t')$  where  $w \in Y^\sigma$ . Furthermore this solution satisfies*

$$(5.13) \quad \|w(t')\|_{L^{p+1}} \lesssim \log^{\frac{1}{2}} \left( \frac{1}{\eta} \right) \left| \log \left( \frac{\pi}{4} - |t| \right) \right|^{\frac{1}{2}},$$

for  $t' \leq t$  with the implicit constant only depending on  $p$  and  $d$ .

*Proof.* Again we only deal with the forward in time part of the estimate, by time reversibility. Let  $0 < t < \frac{\pi}{4}$ ,  $\eta > 0$  and  $p$  as in the lemma. Let also  $\tau > 0$  be a local existence time, given by Proposition 3.3, which we choose uniform in  $[0, t]$ , and which we may shrink in the sequel. We also claim that we can prove the result on  $v$  rather than  $w$  since estimates of this kind on the linear part  $v_L$  are granted by Lemma 2.5.

We start with the case  $p > 1 + \frac{2}{d-1}$ . We set  $t_n := n\tau$  for  $n = 0, \dots, \lfloor \frac{t}{\tau} \rfloor$ . Let  $t' \in [t_n, t_{n+1})$  and write  $v(t')$  in terms of  $v(t_n)$  as the solution of the initial value problem at  $t_n$  using the Duhamel formula:

$$(5.14) \quad v(t') = e^{-i(t'-t_n)H} v(t_n) - i \int_{t_n}^{t'} e^{-i(t'-s)H} \left( \cos(2s) \right)^{\frac{d}{2}(p-1)-2} |v(s)|^{p-1} v(s) \, ds.$$

The triangle inequality and the dispersion estimates  $L^{p+1} \rightarrow L^{\frac{p+1}{p}}$  given by Proposition 3.2 yield

$$(5.15) \quad \begin{aligned} \|v(t')\|_{L^{p+1}} &\leq \|v(t_n)\|_{L^{p+1}} + \underbrace{\|(e^{-i(t'-t_n)} - \text{id})v(t_n)\|_{L^{p+1}}}_I \\ &\quad + \underbrace{\int_{t_n}^{t'} \frac{1}{|t-s|^{d(\frac{1}{2}-\frac{1}{p+1})}} \cos(2s)^{\frac{d}{2}(p-1)-2} \|v(s)\|_{L^{p+1}}^p ds}_{II}. \end{aligned}$$

In the sequel we estimate the terms  $I$  and  $II$  differently.

For  $II$  we use that

$$\cos(2s) \gtrsim \left(\frac{\pi}{4} - t\right)$$

for  $s \in [t_n, t_{n+1}]$ . Then for parameters  $\gamma, \gamma' \geq 1$  satisfying  $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$  and Hölder's inequality:

$$\begin{aligned} II &\lesssim \left(\frac{\pi}{4} - t\right)^{\frac{d}{2}(p-1)-2} \int_{t_n}^{t'} \frac{1}{|t'-s|^{d(\frac{1}{2}-\frac{1}{p+1})}} \|v(s)\|_{L^{p+1}}^p ds \\ &\lesssim \left(\frac{\pi}{4} - t\right)^{\frac{d}{2}(p-1)-2} \left( \int_{t_n}^{t'} |t'-s|^{-d\gamma(\frac{1}{2}-\frac{1}{p+1})} ds \right)^{\frac{1}{\gamma}} \left( \int_{t_n}^{t'} \|v(s)\|_{L^{p+1}}^{p\gamma'} ds \right)^{\frac{1}{\gamma'}} \\ &\lesssim \left(\frac{\pi}{4} - t\right)^{\frac{d}{2}(p-1)-2} \tau^{\frac{1}{\gamma}-d(\frac{1}{2}-\frac{1}{p+1})} \|v\|_{L_{[t_n, t_{n+1}]}^{p\gamma'} L^{p+1}}^p, \end{aligned}$$

provided the integrability condition  $\gamma d \left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$  which we write conveniently in the form

$$(5.16) \quad \frac{1}{\gamma} > d \left(\frac{1}{2} - \frac{1}{p+1}\right).$$

Now we deal with the case  $d \leq 8$  and  $d \geq 8$  separately.

*Case 1.* Assume that  $d \leq 8$  so that  $\sigma(p, d) = \frac{1}{2}$ . Recall that by the local Cauchy-theory and for every Schrödinger admissible pairs  $(q, r)$ ,  $v$  also enjoys bounds of the form

$$\|v\|_{L_{[t_n, t_{n+1}]}^q L^r} \lesssim \left(\frac{\pi}{4} - t_n\right)^{-\alpha} \tau^\beta \lambda^p$$

for some  $\alpha, \beta > 0$  which only depend on  $d$  and  $p$  and are given by Lemma 3.5. Indeed such a bound can be obtained for  $w = v - w_L$  associated to initial data  $v(t_n)$  in  $X^0 \setminus A_\lambda^\circ$  where  $A_\lambda^\circ$  is given by Proposition 3.3; and the same estimates are obtained for  $v_L$  as soon as  $r < 2 + \frac{4}{d}$ .

Thus the norm  $L^{p\gamma'} L^{p+1}$  is  $\frac{1}{2}$ -Schrödinger admissible<sup>1</sup> if and only if

$$(5.17) \quad p\gamma' > 2,$$

$$(5.18) \quad \frac{1}{p\gamma'} > \frac{d-1}{4} - \frac{d}{2(p+1)},$$

and

$$(5.19) \quad \frac{d-1}{4} - \frac{d}{2(p+1)} > 0.$$

Condition (5.19) is equivalent to  $p \geq 1 + \frac{2}{d-1}$  which is satisfied, since  $p \geq 1 + \frac{2}{d}$  by hypothesis. Now observe that once conditions (5.16), (5.17) and (5.18) are satisfied, we obtain

$$(5.20) \quad II \leq \left(\frac{\pi}{4} - t\right)^{-\alpha} \tau^\beta \lambda^p,$$

<sup>1</sup>Note that  $p+1 < 2 + \frac{4}{d}$



for  $\alpha = \alpha(p, d) > 0$  and  $\beta = \beta(p, d) > 0$ .

We remark that Conditions (5.16) and (5.17) are satisfied as soon as

$$(5.21) \quad \begin{cases} p < \frac{\frac{d+3}{d-3}}{2} \\ p \leq \frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)}. \end{cases}$$

Then one has  $\frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)} \leq \frac{d+3}{d-3}$ . We obtain that the conditions (5.16), (5.17) and (5.18) are satisfied for  $p < p_{\max}$ , which is satisfied by hypothesis.

*Case 2.* Assume that  $d \geq 8$  and  $p \geq 1 + \frac{3}{d-2}$ . With  $\sigma = \sigma(p, d)^-$ , conditions (5.16), (5.17) and (5.18) become:

$$(5.22) \quad \begin{cases} \frac{1}{\gamma} > d \left( \frac{1}{2} - \frac{1}{p+1} \right) \\ p\gamma' > \frac{1}{2} \\ \frac{1}{\gamma'} > \frac{p}{2} \left( \frac{d}{2} - \sigma(p, d) - \frac{d}{p+1} \right), \end{cases}$$

which is equivalent to

$$(5.23) \quad \begin{cases} (d-2)p^2 + (d-6)p - 2d - 4 < 0 \\ (d-2)p^3 + (d-4)p^2 - 6p - 2d - 4 < 0 \\ (d-2)p^2 + (d-4)p - 2d - 2 > 0. \end{cases}$$

We observe that the second condition is precisely  $p < p_{\max}(d)$  and that the first is equivalent to  $p \leq \frac{d+2}{d-2} = 1 + \frac{4}{d-2}$  which is satisfied. The last condition is equivalent to  $p > \frac{4-d+\sqrt{9d^2-16d}}{2(d-2)}$  which is smaller than  $1 + \frac{3}{d-2}$ , so that the previous conditions are satisfied if and only if  $p < p_{\max}(d)$ .

We turn to estimating  $I$ . In order to do so, applying a Sobolev embedding in time and switching derivatives from time to space and provided we fix  $\varepsilon > 0$  sufficiently small, gives the existence of constants  $C := C_\varepsilon$ ,  $\beta := \beta(\varepsilon)$  and an integer  $q := q(\varepsilon)$  satisfying

$$I \leq C\tau^\beta \|S(\cdot - t_n)v(t_n)\|_{L^q_{(\cdot)^{-2}dt}(\mathbb{R}, \mathcal{W}^{\varepsilon, p+1})}.$$

A complete proof of this claim is postponed to Lemma A.2 which we refer to for the details.

We introduce the sets

$$B_{n,\lambda} := \left\{ u_0 \in X^0, \|S(\cdot - t_n)\phi_{t_n}v_0\|_{L^q_{(\cdot)^{-2}dt}(\mathbb{R}, \mathcal{W}^{\varepsilon, p+1})} > \lambda \left( \frac{\pi}{4} - t_n \right)^{-\frac{\alpha(p,d)}{2}} \right\}$$

where  $\alpha(p, d) = \frac{d}{2}(p-1) - 2$ . Let also

$$B'_{n,\delta} := \{v_0 \in X^0, \|S(\cdot - t_n)v_0\|_{L^q_{(\cdot)^{-2}dt}(\mathbb{R}, \mathcal{W}^{\varepsilon, p+1})} > \delta\}.$$

Then applying Proposition 4.3 gives

$$\begin{aligned} \nu_0(B_{n,\lambda}) &\leq \nu_{t_n}(\phi_{t_n}B_{n,\lambda})^{\cos(2t_n)^{\alpha(p,d)}} \\ &\leq \mu(\phi_{t_n}B_{n,\lambda})^{\cos(2t_n)^{\alpha(p,d)}} \\ &\leq \mu\left(B'_{n,\lambda\left(\frac{\pi}{4}-t_n\right)^{-\alpha(p,d)}}\right)^{\cos(2t_n)^{\alpha(p,d)}}. \end{aligned}$$

Now we claim that there exists constants  $C, c > 0$  only depending on  $d, p$  such that that

$$(5.24) \quad \mu(B'_{n,\delta}) \leq e^{-c\delta^2}.$$

Assuming (5.24), we conclude that:

$$(5.25) \quad \nu_0(B_{n,\lambda}) \leq e^{-c\lambda^2}.$$

For

$$u_0 \in (\phi_{t_n})^{-1} \left( X^0 \setminus A_{\lambda\left(\frac{\pi}{4}-t_n\right)^{-\alpha(p,d)}} \right) \cap \left( X^0 \setminus B_{n,\lambda} \right) \cap \left( X^0 \setminus A_{t_n,\lambda} \right),$$

where we recall that  $A_{t_n, \lambda}$  is defined by (5.7), we have  $\phi_{t_n} u_0 \in X^0 \setminus A_{\lambda(\frac{\pi}{4} - t_n) - \alpha(p, d)}^\circ$  and then:

$$(5.26) \quad I \lesssim \tau^\beta \left( \frac{\pi}{4} - t \right)^{-\frac{\alpha}{2}} \lambda.$$

Taking into account (5.26) and (5.20) in (5.15) imply that we can adjust  $\tilde{\alpha}, \tilde{\beta} > 0$  such that

$$(5.27) \quad \tau \sim \lambda^{-\tilde{\beta}} \left( \frac{\pi}{4} - t \right)^{-\tilde{\alpha}} \quad \text{and} \quad \|v(t')\|_{L^{p+1}} \leq 2\lambda \quad \text{for } t' \in [t_n, t_{n+1}],$$

Let us prove (5.24). It is sufficient to prove that

$$(5.28) \quad \|S(\cdot)f\|_{L_{\Omega}^{p_0} L_{\langle \cdot \rangle}^q L_{(t)}^{-2} dt(\mathbb{R}, \mathcal{W}^{\varepsilon, p+1})} \lesssim \sqrt{p_0},$$

at least for  $p_0 \geq \max\{q, r\}$ . Assuming such a bound gives the claim using the Markov inequality, and we skip the details, see Appendix A of [BT08a] for instance.

It remains to prove (5.28). Let  $p_0 \geq \max\{q, r\}$ . By Minkowski's inequality and Theorem 1.8 we get

$$\begin{aligned} \|S(\cdot)f\|_{L_{\Omega}^{p_0} L_{\langle \cdot \rangle}^q L_{(t)}^{-2} dt(\mathbb{R}, \mathcal{W}^{\varepsilon, p+1})} &\lesssim \left\| \left\| \sum_{n \geq 0} \frac{e^{it\lambda_n^2}}{\lambda_n^{1-\varepsilon}} g_n e_n \right\|_{L_{\Omega}^{p_0} L_{\langle \cdot \rangle}^q L_{(t)}^{-2} dt(\mathbb{R}, \mathcal{W}^{\varepsilon, p+1})} \right\| \\ &\lesssim \sqrt{p_0} \|(\lambda_n^{-1+\varepsilon} e_n)\|_{L_x^{p+1} \ell_{\mathbb{N}}^2}. \end{aligned}$$

Remark that we used the finiteness of the measure  $\langle t \rangle^{-2} dt$ . Another use of the Minkowski inequality, recalling that  $p+1 \in (2, \frac{2d}{d-2})$  and Lemma 2.4 finally prove the claim.

To conclude the proof, we repeat again the usual argument. The set of *good* initial data  $G_\lambda$  is defined as the set of initial data  $u_0 \in X^0$  which give rise to solutions defined on  $[0, t]$  and such that  $\|v(t')\|_{L^{p+1}} \leq 2\lambda$  for every  $t' \leq t$ . We then observe that the analysis developed to get the bound (5.27) leads to the inclusion:

$$\bigcap_{n=0}^{\lfloor \frac{t}{\tau} \rfloor} \left( (\phi_{n\tau})^{-1} (X^0 \setminus A_{\lambda(\pi/4-t) - \alpha(p, d)}^\circ) \cap (X^0 \setminus B_{n, \lambda}) \cap (X^0 \setminus A_{t_n, \lambda}) \right) \subset G_\lambda.$$

The set of *bad* initial data  $B_\lambda := X^0 \setminus G_\lambda$  then satisfies

$$\nu_0(B_\lambda) \lesssim t \lambda^{\tilde{\beta}} \left( \frac{\pi}{4} - t \right)^{-\tilde{\alpha}} e^{-c\lambda^2},$$

where we used Lemma 4.3 to bound

$$\begin{aligned} \nu_0 \left( (\phi_{n\tau})^{-1} (X^0 \setminus A_{\lambda(\pi/4-t) - \alpha(p, d)}^\circ) \right) &\leq \nu_{n\tau} (X^0 \setminus A_{\lambda(\pi/4-t) - \alpha(p, d)}^\circ)^{\cos(2t_n)^\alpha(p, d)} \\ &\leq e^{-c\lambda^2}, \end{aligned}$$

and the expression of  $\tau$ . Finally the measure of this set is made smaller than  $\eta$  by taking  $\lambda \sim \log^{\frac{1}{2}} \left( \frac{1}{\eta} \right) \left| \log \left( \frac{\pi}{4} - |t| \right) \right|^{\frac{1}{2}}$ , which ends the proof.

Finally we treat the case  $p < 1 + \frac{2}{d-1}$ , which is much simpler, and sufficient for proving the lemma in dimension  $d = 2$ . Then for  $\sigma < \frac{1}{2}$  close enough to  $\frac{1}{2}$  we have  $\mathcal{H}^\sigma \hookrightarrow L^{p+1}$ . Then

$$(5.29) \quad \|v(t_n) - v(t)\|_{L^{p+1}} \lesssim \|v(t_n) - v(t)\|_{\mathcal{H}^\sigma}.$$

Then we adjust  $\tau$  such that if  $v(t_n) \in X^0 \setminus A_\lambda^\circ$  and  $|t - t_n| \leq \tau$  then by the local Cauchy theory of Proposition 3.3, there exists  $\alpha, \beta > 0$  depending on  $p, d$  such that  $\|v(t_n) - v(t)\|_{\mathcal{H}^\sigma} \lesssim \lambda \tau^\beta \lambda \left( \frac{\pi}{4} - t_n \right)^{-\alpha}$ , which gives the control needed on the variation of  $\|v(t)\|_{L^{p+1}}$ . Then the same globalising argument as above finishes the proof. Since the proof in this case is very similar to the one in [BT20] we refer to the latter article for details.  $\square$

*Remark 5.8.* Note that  $p_{\max}(d)$  is greater than  $1 + \frac{4}{d}$  as soon as  $d \leq 7$ . This is indeed easily checked on a computer, but can also be proved taking advantage of the fact that  $d \mapsto p_{\max}(d) - 1 - \frac{4}{d}$  is monotone, thus it is sufficient to check that  $p_{\max}(7)p_{\max}(8) < 0$  which is true by elementary computations. This proves that for  $d \leq 7$  we have access to logarithmic growth bounds for  $\|v(t)\|_{L^{p+1}}$  and will be crucial in the proof of Theorem 1.3.

From there we can deduce exactly as in Corollary 2.5 the following global estimate.

**Corollary 5.9.** *Let  $d \geq 2$  and  $p \in (1, p_{\max}(d))$ . There exists  $\varepsilon_0 > 0$  and a set  $G \subset X^0$  of full measure, i.e.,  $\mu(G) = 1$  such that for all  $u_0 \in G$  and all  $\varepsilon \in [0, \varepsilon_0]$  there exists a unique global solution  $v(t) = e^{-itH}u_0 + w(t)$  to (HNLS) with initial data  $u_0$  which furthermore satisfies*

$$(5.30) \quad \|w(t)\|_{\mathcal{W}^{\varepsilon, p+1}} \lesssim C(u_0) \left| \log \left( \frac{\pi}{4} - |t| \right) \right|^{\frac{1}{2}},$$

for every  $|t| < \frac{\pi}{4}$ . Moreover there exist  $c, C > 0$  such that

$$\mu(C(u_0) > \lambda) \lesssim e^{-c\lambda^2}.$$

*Proof.* We only rapidly explain how we obtain the  $\mathcal{W}^{\varepsilon, p+1}$  estimates. This follows from the proof of Lemma 5.7, where we can multiply equation (5.14) by  $H^{\frac{\varepsilon}{2}}$ , then all the estimates are essentially the same provided  $\varepsilon > 0$  is small enough. We use in particular that we use non-endpoints estimates, for which there is always room for an  $\varepsilon$  modification of the parameters.  $\square$

## 6. SCATTERING FOR (NLS) AND END OF THE PROOF

*Proof of Theorem 1.2.* This is a consequence of the bound (5.13) of Corollary 5.9 and an application of the *lens transform*, see Remark B.3.  $\square$

In order to prove Theorem 1.3 we need an  $\mathcal{H}^\varepsilon$  scattering result for (HNLS). The main ingredient in the proof is the following proposition.

**Proposition 6.1.** *Let  $d \in \{2, 24\}$  and  $p \in \left(1 + \frac{2}{d}, 1 + \frac{4}{d}\right)$ . For almost every  $u_0 \in X^0$  there exists a unique global solution  $v$  to (HNLS). Moreover:*

(i) *There exist  $\sigma, \delta > 0$  only depending on  $p$  and  $d$ , and  $u_\pm \in \mathcal{H}^{-\sigma}$  such that*

$$(6.1) \quad \|v(t) - e^{-itH}(u_0 - u_\pm)\|_{\mathcal{H}^{-\sigma}} \lesssim C(u_0) \left( \frac{\pi}{4} - t \right)^\delta \xrightarrow[t \rightarrow \pm \frac{\pi}{4}]{} 0,$$

*which is scattering in  $\mathcal{H}^{-\sigma}$ .*

(ii) *There exists  $\varepsilon > 0$  such that:*

$$(6.2) \quad \|v(t)\|_{\mathcal{H}^\varepsilon} \lesssim_\varepsilon C(u_0)$$

*for all  $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ .*

*In both cases there exist numerical constants  $c, C > 0$  such that  $\mu(C(u_0) > \lambda) \leq Ce^{-c\lambda}$ .*

*Proof.* Again we present the proof for the forward scattering part. We work with initial data in a set of full measure  $G$  which may be taken as the intersection of the full measure sets constructed in Corollary 5.6 and Corollary 5.9. Then every  $u_0 \in G$  gives rise to a global solution to (HNLS),

$$v(t) = u_L(t) + w(t) = e^{-itH}u_0 + w(t)$$

satisfying the bounds (5.11) and (5.13). Up taking the intersection with another set of full measure we can always assume that  $\|u_L\|_{L^q \mathcal{W}^{s,r}} < \infty$  as long as  $(q, r)$  satisfies the hypothesis of Lemma 2.4, and more precisely

$$\mu(u_0, \|u_L\|_{L^q \mathcal{W}^{s,r}} > \lambda) \leq Ce^{-c\lambda^2}.$$

We start the proof with the case  $p < p_{\max}(d)$  and we will use Corollary 5.9. To start the proof we write

$$(6.3) \quad e^{itH}w(t) = -i \int_0^t e^{isH} \left( \cos(2s)^{\frac{d}{2}(p-1)-2} |u_L(s) + w(s)|^{p-1} (u_L(s) + w(s)) \right) ds.$$

By the Sobolev embedding set  $\sigma > 0$  such that  $L^{\frac{p+1}{p}} \hookrightarrow \mathcal{H}^{-\sigma}$ , one can choose for example  $\sigma := d \left( \frac{1}{2} - \frac{1}{p+1} \right)$ . We claim that there exists  $\delta > 0$  such that

$$(6.4) \quad \int_t^{\frac{\pi}{4}} |\cos(2s)|^{\frac{d}{2}(p-1)-2} \|(u_L(s) + w(s))|u_L(s) + w(s)|^{p-1}\|_{\mathcal{H}^{-\sigma}} ds \lesssim \left( \frac{\pi}{4} - t \right)^\delta.$$

Assuming (6.4) proves that the integral in (6.3) converges absolutely in  $\mathcal{H}^{-\sigma}$  and thus convergent to some  $u_+ \in \mathcal{H}^{-\sigma}$  and proves (6.1).

In order to prove (6.4) we use the Sobolev embedding and the bound  $\cos(2s) \gtrsim \left( \frac{\pi}{4} - s \right)$  to get

$$\begin{aligned} & \int_t^{\frac{\pi}{4}} |\cos(2s)|^{\frac{d}{2}(p-1)-2} \|(u_L(s) + w(s))|u_L(s) + w(s)|^{p-1}\|_{\mathcal{H}^{-\sigma}} ds \\ & \lesssim \int_t^{\frac{\pi}{4}} |\cos(2s)|^{\frac{d}{2}(p-1)-2} \|u_L(s) + w(s)\|_{L^{p+1}}^p ds \\ & \lesssim \int_t^{\frac{\pi}{4}} \left( \frac{\pi}{4} - s \right)^{\frac{d}{2}(p-1)-2} \|u_L(s)\|_{L^{p+1}}^p ds + \int_t^{\frac{\pi}{4}} \left( \frac{\pi}{4} - s \right)^{\frac{d}{2}(p-1)-2} \|w(s)\|_{L^{p+1}}^p ds \\ & \lesssim \|u_L\|_{L^{pr}L^{p+1}}^p \left( \int_t^{\frac{\pi}{4}} \left( \frac{\pi}{4} - t \right)^{r' \left( \frac{d}{2}(p-1)-2 \right)} ds \right)^{\frac{1}{r'}} \\ & \quad + C(u_0)^p \int_t^{\frac{\pi}{4}} \left( \frac{\pi}{4} - t \right)^{\frac{d}{2}(p-1)-2} \left| \log \left( \frac{\pi}{4} - t \right) \right|^{\frac{p}{2}} ds \end{aligned}$$

where in the last line we used Hölder's inequality with  $\frac{1}{r} + \frac{1}{r'} = 1$  and the bounds (5.13). As explained above, without loss of generality we can assume  $\|u_L\|_{L^{pr}L^{p+1}}$  to be finite. It remains to choose  $r'$  such that  $r' \left( \frac{d}{2}(p-1)-2 \right) > -1$  which ensures that the previous integrals are absolutely convergent and that they can be bounded by a positive power of  $\frac{\pi}{4} - t$ . This gives (6.4).

It remains to prove (6.2). First observe that  $w$  satisfies

$$i\partial_t w(t) - Hw(t) = \cos(2t)^{\frac{d}{2}(p-1)-2} (u_L(t) + w(t)) |u_L(t) + w(t)|^{p-1},$$

then apply  $H^{\frac{\varepsilon}{2}}$ , multiply by  $\overline{H^{\frac{\varepsilon}{2}} w(t)}$  and integrate in space to obtain:

$$\begin{aligned} \frac{d}{dt} \left( \|w(t)\|_{\mathcal{H}^\varepsilon}^2 \right) &= 2 \cos(2t)^{\frac{d}{2}(p-1)-2} \operatorname{Im} \left( \int_{\mathbb{R}^2} H^{\frac{\varepsilon}{2}} \left( |u_L(t) + w(t)|^{p-1} (u_L(t) + w(t)) \right) \overline{H^{\frac{\varepsilon}{2}} w(t)} dx \right) \\ &\lesssim \cos(2t)^{\frac{d}{2}(p-1)-2} \|(u_L(t) + w(t))|u_L(t) + w(t)|^{p-1}\|_{\mathcal{W}^{\varepsilon, \frac{p+1}{p}}} \|w(t)\|_{\mathcal{W}^{\varepsilon, p+1}}, \end{aligned}$$

where we used the Hölder inequality. By Sobolev's product laws and the minoration of the cos function we obtain

$$\begin{aligned} \frac{d}{dt} \left( \|w(t)\|_{\mathcal{H}^\varepsilon}^2 \right) &\lesssim \left( \frac{\pi}{4} - t \right)^{\frac{d}{2}(p-1)-2} \left( \|u_L(t)\|_{L^{p+1}}^{p-1} + \|w(t)\|_{L^{p+1}}^{p-1} \right) \\ &\quad \times \left( \|u_L(t)\|_{\mathcal{W}^{\varepsilon, p+1}} + \|w(t)\|_{\mathcal{W}^{\varepsilon, p+1}} \right) \|w(t)\|_{\mathcal{W}^{\varepsilon, p+1}}. \end{aligned}$$

Then we bound the  $L^{p+1}$  norms by the  $\mathcal{W}^{\varepsilon, p+1}$  norms and also use  $w(t) = v(t) - u_L(t)$ , which leads to

$$\frac{d}{dt} \left( \|w(t)\|_{\mathcal{H}^\varepsilon}^2 \right) \lesssim \left( \frac{\pi}{4} - t \right)^{\frac{d}{2}(p-1)-2} \left( \|u_L(t)\|_{\mathcal{W}^{\varepsilon, p+1}}^{p+1} + \|v(t)\|_{\mathcal{W}^{\varepsilon, p+1}}^{p+1} \right).$$

After integration we have

$$\begin{aligned} \|w(t)\|_{\mathcal{H}^\varepsilon}^2 &\leq \underbrace{\|w_0\|_{\mathcal{H}^\varepsilon}^2}_{=0} + C \int_0^t \left(\frac{\pi}{4} - s\right)^{\frac{d}{2}(p-1)-2} \left(\|u_L(t)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1} + \|v(t)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1}\right) ds \\ &\lesssim \|u_L\|_{L^{(p+1)r}\mathcal{W}^{\varepsilon,p+1}}^{p+1} \left(\int_0^{\frac{\pi}{4}} \left(\frac{\pi}{4} - t\right)^{r'(\frac{d}{2}(p-1)-2)} ds\right)^{\frac{1}{r'}} \\ &\quad + C(u_0)^{p+1} \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{4} - t\right)^{\frac{d}{2}(p-1)-2} \left|\log\left(\frac{\pi}{4} - t\right)\right|^{\frac{p+1}{2}} ds, \end{aligned}$$

where we used Hölder's inequality with  $\frac{1}{r} + \frac{1}{r'} = 1$  and (5.13). As above without loss of generality we can assume  $\|u_L\|_{L^{pr}\mathcal{W}^{\varepsilon,p+1}} < \infty$ . Then choosing  $r'$  such that  $r'(\frac{d}{2}(p-1)-2) > -1$  all the above integrals are convergent, which proves (6.2).

Now we explain the case  $p \geq p_{\max}(d)$ . As remarked before this case is only necessary when  $d \geq 8$ . Let  $p \in [p_{\max}(d), 1 + \frac{4}{d})$ . In order to prove (6.1) we need to prove that there exists  $\delta > 0$  such that (6.4) holds. Recalling the bounds on  $u_L$  all which is needed is the existence of  $\delta > 0$  such that

$$(6.5) \quad \int_t^{\frac{\pi}{4}} |\cos(2s)|^{\frac{d}{2}(p-1)-2} \|w(s)\|_{L^{p+1}}^p ds \lesssim \left(\frac{\pi}{4} - t\right)^\delta.$$

Similarly, in order to prove (6.2), an examination of the above proof shows that one only needs to prove that for sufficiently small  $\varepsilon > 0$  the integral

$$(6.6) \quad \int_0^{\frac{\pi}{4}} |\cos(2s)|^{\frac{d}{2}(p-1)-2} \|w(s)\|_{L^{p+1}}^{p+1} ds,$$

is finite. The proof of (6.5) and (6.6) are essentially the same, thus the proof of (6.5) is carried out in details and we only explain the modifications for (6.6). We write  $\alpha(p, d) := \frac{d}{2}(p-1) - 2$  until the end of the proof and set  $t_n := \frac{\pi}{4}(1 - 2^{-n})$ .

The key fact for proving (6.5) is the following: we claim that the real sequence  $(u_n)_{n \geq 0}$  defined by

$$u_n := \int_0^{t_n} |\cos(2s)|^{\alpha(p,d)} \|w(s)\|_{L^{p+1}}^p ds$$

is a Cauchy sequence. In order to prove this claim, we use the bound  $\cos(2s) \gtrsim (\frac{\pi}{4} - s)$  and Hölder's inequality with  $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ , where  $\gamma$  such that there exists  $\sigma \in [0, \sigma(p, d))$  such that  $(p\gamma', p+1)$  is  $(\sigma(p, d))$ -Schrödinger admissible, that is  $p\gamma' \geq 2$  and  $\frac{2}{p\gamma'} + \frac{d}{p+1} = \frac{d}{2} - \sigma$ . Note that the condition  $p\gamma' \geq 2$  can be written as  $\frac{d}{2} - \frac{d}{p+1} - \sigma(p, d) < 1$  which is equivalent to  $(d-2)p^2 + (d-6)p - 2d - 4 < 0$ , that is  $p < \frac{d+2}{d-2}$ . This last inequality is always satisfied in our case. We also mention that the  $\sigma(p, d)$  admissibility also requires that  $\frac{d}{2} - \frac{d}{p+1} - \sigma(p, d) > 0$ , which is equivalent to  $p > \frac{1}{-d+4+\sqrt{9d^2-16d}} 2(d-2)$ . Since  $p > 1 + \frac{3}{d-2} > \frac{1}{-d+4+\sqrt{9d^2-16d}} 2(d-2)$ , this last inequality is, again, always satisfied.

We can summarise the above discussion: provided  $\alpha(p, d) + \frac{1}{\gamma} > 0$ , which will be checked later, we obtain:

$$\begin{aligned} |u_{n+1} - u_n| &\leq \int_{t_n}^{t_{n+1}} \left(\frac{\pi}{4} - s\right)^{\alpha(p,d)} \|w(s)\|_{L^{p+1}}^p ds \\ &\lesssim \left(\frac{\pi}{4} - t_n\right)^{\alpha(p,d) + \frac{1}{\gamma}} \|w\|_{L_{[t_n, t_{n+1}]}^{p\gamma'} L^{p+1}}^p \end{aligned}$$

$$\lesssim \left(\frac{\pi}{4} - t_n\right)^{\alpha(p,d) + \frac{1}{\gamma}} \|w\|_{Y_{[t_n, t_{n+1}]}^{\sigma(p,d)-}}^p.$$

Then the bound obtained in Corollary 5.6 yields

$$\begin{aligned} |u_{n+1} - u_n| &\lesssim \left(\frac{\pi}{4} - t_n\right)^{\alpha(p,d) + \frac{1}{\gamma}} \left(\frac{\pi}{4} - t_{n+1}\right)^{\frac{p\alpha(p,d)}{2}} \\ &\lesssim 2^{-n(\delta + \varepsilon_0)}, \end{aligned}$$

with  $\delta := \frac{p+2}{2}\alpha(p,d) + \frac{1}{\gamma}$  and  $\varepsilon_0$  arbitrarily small. Assume that  $\delta > 0$ , then we conclude that  $(u_n)_{n \geq 0}$  is a Cauchy sequence and moreover if we choose  $n$  such that  $\frac{\pi}{4} - t \in [2^{-(n+1)}, 2^{-n}]$  then we can bound:

$$\int_t^{\frac{\pi}{4}} |\cos(2s)|^{\frac{d}{2}(p-1)-2} \|w(s)\|_{L^{p+1}}^p ds \lesssim \sum_{k \geq n} 2^{-k\delta} \lesssim 2^{-n\delta} \lesssim \left(\frac{\pi}{4} - t\right)^\delta.$$

This proves (6.5), provided we show that  $\delta > 0$ . Before checking this fact we run the same analysis for (6.6) in order to obtain the inequalities that the parameters must satisfy. This time we set

$$z_n := \int_0^{t_n} (\cos(2s))^{\alpha(p,d)} \|w(s)\|_{W^{\varepsilon, p+1}}^{p+1} ds,$$

and we only need to prove that this sequence forms a Cauchy sequence. Again we use Hölder's inequality with  $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$  such that there exists  $\sigma \in [0, \sigma(p,d)]$  such that  $(\gamma'(p+1), p+1)$  is  $\sigma(p,d)$ -Schrödinger admissible, that is  $\gamma'(p+1) \geq 2$  and  $\frac{2}{(p+1)\gamma'} + \frac{d}{p+1} = \frac{d}{2} - \sigma$ . The condition  $\gamma'(p+1) \geq 2$  is equivalent to  $p \leq \frac{d+2}{d-2}$  as above, which is satisfied. Then, with the same estimates as above and provided  $\alpha(p,d) + \frac{1}{\gamma} > 0$  we arrive at

$$|z_{n+1} - z_n| \lesssim \left(\frac{\pi}{4} - t_n\right)^{\alpha(p,d) + \frac{1}{\gamma}} \|v\|_{Y_{[t_n, t_{n+1}]}^{\sigma(p,d)-}}^{p+1}.$$

Then we use the bound

$$\|v\|_{Y_{[t_n, t_{n+1}]}^{\sigma(p,d)-}} \lesssim \left(\frac{\pi}{4} - t_{n+1}\right)^{\frac{\alpha(p,d)}{2} + \varepsilon_0}$$

obtained in Corollary 5.6, and where  $\varepsilon_0 > 0$  is arbitrarily small; and the fact that  $(\frac{\pi}{4} - t_n) \sim C2^{-n}$ , in order to get

$$|z_{n+1} - z_n| \lesssim 2^{-n(\tilde{\delta} + \varepsilon_0)}$$

where  $\tilde{\delta} := \frac{(p+3)\alpha(p,d)}{2} + \frac{1}{\gamma}$ . Assuming that  $\tilde{\delta} > 0$  proves that  $(u_n)_{n \geq 0}$  is a Cauchy sequence, and ends the proof of (6.6)

We need to find the range of  $p$  (depending on  $d$ ) for which  $\delta, \tilde{\delta} > 0$ . Note that would immediately imply  $\alpha(p,d) + \frac{1}{\gamma} > 0$  which was a necessary condition to bound  $|u_{n+1} - u_n|$  and  $|z_{n+1} - z_n|$ .

The claim  $\delta > 0$  is equivalent to:

$$\frac{p+2}{2} \left(\frac{d}{2}(p-1) - 2\right) + 1 - \frac{p}{2} \left(\frac{d}{2} - \sigma(d) - \frac{d}{p+1}\right),$$

which after simplification reads

$$Q_d(p) := 2p^3 + dp^2 + (d-6)p - 2d - 4 > 0.$$

Then we can compute that, as a polynomial in  $p$ ,  $\text{discrim}(Q_d) = 9d^4 - 76d^3 + 36d^2 - 1728d$  thus  $Q_d$  has exactly one real root if  $d \leq 9$  and exactly three for  $d \geq 10$ . In any cases we check that  $Q_d$  has exactly one positive real root for  $d \geq 8$ . Let us denote by  $p_d$  this root and

we claim that  $p_{\max}(d) > p_d$ . To verify this statement we remark that both  $P_d$  and  $Q_d$  are increasing  $[1, 1 + \frac{4}{d}]$  thus we only need to check that  $P_d \leq Q_d$  in that region, which is equivalent to  $(d-4)p^2 - 4p - d < 0$  which is satisfied for  $p \geq \frac{d}{d-4}$  (which is greater than  $1 + \frac{4}{d}$ ) and  $d \geq 8$ .

In a similar fashion,  $\tilde{\delta} > 0$  is equivalent to:

$$\frac{p+3}{2} \left( \frac{d}{2}(p-1) - 2 \right) - \frac{p+1}{2} \left( \frac{d}{2} - \sigma(p, d) - \frac{d}{p+1} \right) > 0,$$

which after simplification reads

$$R_d(p) := p^2 + \frac{d}{2}p - \frac{d}{2} - 3 > 0,$$

that is  $p > -\frac{d}{4} + \frac{1}{4}\sqrt{d^2 + 8d + 48}$ . We claim that

$$p_{\max}(d) > -\frac{d}{4} + \frac{1}{4}\sqrt{d^2 + 8d + 48} := p'_d$$

if and only if  $d \leq 24$ . This claim is equivalent to  $P_{24}(p'_{24})P_{25}(p'_{25}) < 0$ . This reduces to  $(813\sqrt{51} - 5806)(44181\sqrt{97} - 435131) < 0$  which is the case.  $\square$

*Proof of Theorem 1.3.* In order to end the proof of the theorem we assume that there exist  $\delta, \sigma, \varepsilon > 0$  such as constructed in Proposition 6.1. This immediately proves part (i) of Theorem 1.3.

We deduce that  $u_+ \in \mathcal{H}^\varepsilon$ . In fact,  $e^{itH}w(t)$  being bounded in the Hilbert space  $\mathcal{H}^\varepsilon$  we can extract a subsequence, weakly converging to some  $\tilde{u}_+ \in \mathcal{H}^\varepsilon$  but this convergence also holds in  $\mathcal{H}^{-\sigma}$  where  $e^{itH}w(t) \rightarrow u_+$  thus by uniqueness of the weak limit,  $u_+ = \tilde{u}_+ \in \mathcal{H}^\varepsilon$ .

We claim that the bound

$$(6.7) \quad \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{\varepsilon_0}} \lesssim \left( \frac{\pi}{4} - t \right)^{(2d+1)\varepsilon_0}$$

implies the estimate (1.2). Indeed, let  $u$  be the solution to Schrödinger equation (NLS) associated to  $u_0$ . We denote by  $s = \frac{\tan t}{2}$  the time variable of  $u$  where  $t$  is the time variable of  $v := \mathcal{L}u$ . We refer to Appendix B for details. Then from Lemma B.1 we have

$$\begin{aligned} \|u(s) - e^{is\Delta_y}(u_0 + u_+)\|_{H^{\varepsilon_0}} &\lesssim \left( \frac{\pi}{4} - t(s) \right)^{-2d\varepsilon_0} \|\mathcal{L}u - \mathcal{L}(e^{is\Delta_y}(u_0 + u_+))\|_{\mathcal{H}^{\varepsilon_0}} \\ &\lesssim \left( \frac{\pi}{4} - t(s) \right)^{-2d\varepsilon_0} \|w(t(s)) - e^{-it(s)H}u_+\|_{\mathcal{H}^{\varepsilon_0}} \\ &\lesssim \left( \frac{\pi}{4} - t(s) \right)^{-2d\varepsilon_0} \|e^{it(s)H}w(t(s)) - u_+\|_{\mathcal{H}^{\varepsilon_0}} \\ &\xrightarrow{s \rightarrow \infty} 0, \end{aligned}$$

where we used that  $v(t(s)) = w(t(s)) + e^{-it(s)H}u_0$  and (6.7).

In order to prove (6.7), let  $\theta \in [0, 1]$  and introduce  $\sigma(\theta) := -\sigma\theta + (1-\theta)\varepsilon$ . Interpolating between (6.1) and (6.2) we have

$$\begin{aligned} \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{\sigma(\theta)}} &\leq \|e^{itH}w(t) - u_+\|_{\mathcal{H}^\varepsilon}^{1-\theta} \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{-\sigma}}^\theta \\ &\lesssim C(\varepsilon)^{1-\theta} \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{-\sigma}}^\theta \\ &\lesssim \left( \frac{\pi}{4} - t \right)^{\theta\delta}. \end{aligned}$$

We claim that we can find  $\varepsilon_0 \in (0, \varepsilon)$  satisfying  $\sigma(\theta) = \varepsilon_0$  and  $\delta\theta = (2d+1)\varepsilon_0$ . Indeed one can take

$$\varepsilon_0 := \frac{\varepsilon}{1 + \frac{2d+1}{\delta}(\sigma + \varepsilon)}.$$

Finally in order to prove (1.3), observe that by properties of the lens transform we have  $\mathcal{L}u(s) = e^{-it(s)H}(e^{-is\Delta_y}u(s))$  and then:

$$\begin{aligned} \|e^{-is\Delta_y}u(s) - (u_0 + u_+)\|_{\mathcal{H}^{\varepsilon_0}} &= \|e^{it(s)H}\mathcal{L}u(s) - (u_0 + u_+)\|_{H^{\varepsilon_0}} \\ &= \|\mathcal{L}u(s) - e^{-it(s)H}(u_0 + u_+)\|_{\mathcal{H}^{\varepsilon_0}} \\ &= \|v(t(s)) - e^{-it(s)H}(u_0 + u_+)\|_{\mathcal{H}^{\varepsilon_0}} \\ &\xrightarrow{s \rightarrow \infty} 0, \end{aligned}$$

where we used similar computations as above. We have thus proven the convergences (1.2) and (1.3) at a rate which is  $(\frac{\pi}{4} - t(s))^{\varepsilon_0}$  as  $s \rightarrow \infty$ . To conclude it suffices to remark that:

$$\frac{\pi}{4} - t(s) = \frac{1}{2} \arctan\left(\frac{1}{2s}\right) \sim \frac{C}{s} \text{ as } s \rightarrow \infty \quad \square$$

#### APPENDIX A. TECHNICAL ESTIMATES IN HARMONIC SOBOLEV SPACES

In this appendix we recall some well-known facts concerning the harmonic oscillator-based Sobolev spaces.

**Lemma A.1** (Harmonic Sobolev Spaces). *The following properties hold.*

- (i) For  $\sigma \in \mathbb{R}$  and  $p \in (1, \infty)$ ,  $\|u\|_{\mathcal{W}^{\sigma,p}} = \|H^{\frac{\sigma}{2}}u\|_{L^p} \sim \|\langle \nabla \rangle^\sigma u\|_{L^p} + \|\langle x \rangle^\sigma u\|_{L^p}$ .
- (ii) Sobolev embedding: in dimension  $d$ ,  $\mathcal{W}^{\sigma_1,p_1} \hookrightarrow \mathcal{W}^{\sigma_2,p_2}$  as soon as

$$\frac{1}{p_1} - \frac{\sigma_1}{d} \leq \frac{1}{p_2} - \frac{\sigma_2}{d}.$$

- (iii) For  $\sigma \geq 0$ ,  $q \in (1, \infty)$  and  $q_1, q_2, q'_1, q'_2 \in (1, \infty]$  one has

$$\|uv\|_{\mathcal{W}^{\sigma,q}} \lesssim \|u\|_{L^{q_1}} \|v\|_{\mathcal{W}^{\sigma,q'_1}} + \|u\|_{\mathcal{W}^{\sigma,q_2}} \|v\|_{L^{q'_2}},$$

$$\text{as soon as } \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q'_1} = \frac{1}{q_2} + \frac{1}{q'_2}.$$

- (iv) (Chain Rule) For  $s \in (0, 1)$ ,  $p \in (1, \infty)$  and a function  $F \in \mathcal{C}^1(\mathbb{R})$  such that  $F(0) = 0$  and such that there is a  $\mu \in L^1([0, 1])$  such that for every  $\theta \in [0, 1]$ :

$$|F'(\theta v + (1 - \theta)w)| \leq \mu(\theta) (G(v) + G(w))$$

where  $G > 0$ ; we have

$$(A.1) \quad \|F \circ u\|_{\mathcal{W}^{s,p}} \lesssim \|u\|_{\mathcal{W}^{s,p_0}} \|G \circ u\|_{L^{p_1}}$$

and

$$(A.2) \quad \|F \circ u - F \circ v\|_{\mathcal{W}^{s,p}} \lesssim \|u - v\|_{\mathcal{W}^{s,p_0}} (\|G \circ u\|_{L^{p_1}} + \|G \circ v\|_{L^{p_1}})$$

as soon as  $\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p}$ , provided  $p_0 \in (1, \infty)$  and  $p_1 \in (1, \infty)$ .

*Proof.* (i) is proved in [BTT13]. The other statements are proven for usual Sobolev spaces in [Tay07] and their readaptation to harmonic Sobolev spaces results from the use of (i). We also refer to [Den12]. For usual Sobolev spaces, (iii) and (iv) can be found in [Tay07], Chapter 2.  $\square$

Our next lemma is a technical estimate which aims at decoupling the norm in time.

**Lemma A.2.** *Let  $\varepsilon > 0$ ,  $\alpha \in (0, 1)$  and  $q, r \geq 1$  be such that  $W^{\frac{\varepsilon}{2},q}(\mathbb{R}) \hookrightarrow C^{0,\alpha}(\mathbb{R})$ . Then for every  $f \in \mathcal{W}^{\varepsilon,r}$  holds*

$$\|(e^{-i(t-t_0)H} - \text{id})f\|_{L^\infty_{[t_0, t_0+\tau]} L^r} \lesssim \tau^\alpha \|e^{itH} f\|_{L^q_{(t)-2, \text{dt}}(\mathbb{R}, \mathcal{W}^{\varepsilon,r})}.$$

where the implicit constant depends on  $\varepsilon$ ,  $q$  and  $\alpha$ .



*Proof.* Let  $\chi(t)$  a smooth function such that for  $|t| \leq \pi$ ,  $\chi(t) = \langle 2\pi \rangle^{-2}$  and for  $|t| \geq 2\pi$  one has  $\chi(t) = \langle t \rangle^{-2}$ . Set  $F(t) := e^{-i(t-t_0)H} f$  for convenience. Then we use the definition of the  $C^{0,\alpha}$  norm:

$$\begin{aligned} \|(e^{-i(t-t_0)H} - \text{id})f\|_{L^\infty_{[t_0, t_0+\tau]} L^r} &\leq |t - t_0|^\alpha \|F\|_{C^{0,\alpha}([t_0, t_0+\tau], L^r)} \\ &\leq \tau^\alpha \|F\|_{C^{0,\alpha}([-\pi, \pi], L^r)}. \end{aligned}$$

Now we use that  $\|F\|_{C^{0,\alpha}([-\pi, \pi], L^r)} \leq C \|\chi(\cdot - t_0)F\|_{C^{0,\alpha}(\mathbb{R}, L^r)}$  with a constant  $C$  which does not depend on  $\tau$ . Combined with the Sobolev embedding  $W^{\frac{\varepsilon}{2}, q}(\mathbb{R}) \hookrightarrow C^{0,\alpha}(\mathbb{R})$  we have:

$$\begin{aligned} \|(e^{-i(t-t_0)H} - \text{id})f\|_{L^\infty_{[t_0, t_0+\tau]} L^r} &\lesssim \tau^\alpha \|\chi(\cdot - t_0)F\|_{W^{\frac{\varepsilon}{2}, q}(\mathbb{R}, L^r)} \\ &\lesssim \tau^\alpha \|\chi(t)e^{-itH}f\|_{W^{\frac{\varepsilon}{2}, q}(\mathbb{R}, L^r)}. \end{aligned}$$

Now observe that in order to transfer derivatives from time to space we need to commute  $\chi$  and  $\langle D_t \rangle^{\frac{\varepsilon}{2}}$ , which follows from the following observation:

$$\begin{aligned} \langle D_t \rangle^{\frac{\varepsilon}{2}} \chi &= \left(1 + \left[\langle D_t \rangle^{\frac{\varepsilon}{2}}, \chi\right] \langle D_t \rangle^{-\frac{\varepsilon}{2}} \chi^{-1}\right) \chi \langle D_t \rangle^{\frac{\varepsilon}{2}} \\ &= (\text{id} + A) \chi \langle D_t \rangle^{\frac{\varepsilon}{2}}, \end{aligned}$$

with  $A := \left[\langle D_t \rangle^{\frac{\varepsilon}{2}}, \chi\right] \langle D_t \rangle^{-\frac{\varepsilon}{2}} \chi^{-1}$ . Since  $\chi, \chi^{-1}$  are zero order pseudodifferential operators and that  $\langle D_t \rangle^{\pm \frac{\varepsilon}{2}}$  are pseudodifferential operators of order  $\pm \frac{\varepsilon}{2}$ , the pseudodifferential calculus implies that  $A$  is of order  $-1$  which is regularising and finally  $(\text{id} + A)$  is of order zero, thus continuous on all  $L^p$  spaces, as soon as  $p \in (1, \infty)$ , see [Hö65] for instance, and the continuity constant does not depend on  $\tau$ . This gives

$$\begin{aligned} \|e^{-itH}f\|_{W^{\frac{\varepsilon}{2}, q}(\mathbb{R}, L^r)} &\lesssim \|\langle D_t \rangle^{\frac{\varepsilon}{2}} \chi(t)e^{-itH}f\|_{L^q(\mathbb{R}, L^r)} \\ &\lesssim \|\chi(t) \langle D_t \rangle^{\frac{\varepsilon}{2}} e^{-itH}f\|_{L^q(\mathbb{R}, L^r)} \\ &= \|\langle D_t \rangle^{\frac{\varepsilon}{2}} e^{-itH}f\|_{L^q_{(t)^{-2} dt}(\mathbb{R}, L^r)}. \end{aligned}$$

Since by definition  $D_t(e^{itH}f) = H(e^{itH}f)$  we deduce by the usual functional calculus that  $D_t^{\frac{\varepsilon}{2}}(e^{itH}f) = H^{\frac{\varepsilon}{2}}(e^{itH}f)$ , which ends the proof.  $\square$

## APPENDIX B. THE LENS TRANSFORM

The lens transform  $\mathcal{L} : \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d) \rightarrow \mathcal{S}'\left(\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \mathbb{R}^d\right)$  and its inverse  $\mathcal{L}^{-1}$  are defined by the following formulae

$$\begin{aligned} v(t, x) &:= \mathcal{L}u(t, x) = \frac{1}{\cos(2t)^{\frac{d}{2}}} u\left(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\right) \exp\left(-\frac{i|x|^2 \tan(2t)}{2}\right), \\ u(s, y) &= \mathcal{L}^{-1}v(s, y) = (1 - 4s^2)^{-\frac{d}{4}} v\left(\frac{\arctan(2s)}{2}, \frac{y}{\sqrt{1 - 4s^2}}\right) \exp\left(\frac{i|y|^2 s}{\sqrt{1 - 4s^2}}\right). \end{aligned}$$

Formal computations show that  $u$  solves

$$i\partial_s u + \Delta_y u = 0 \text{ on } \mathbb{R}_s \times \mathbb{R}_y^d \text{ and } u(0) = u_0$$

if and only if  $v = \mathcal{L}u$  solves

$$i\partial_t v - H v = 0 \text{ on } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \mathbb{R}_x^d \text{ and } v(0) = u_0.$$

With the variables  $s = \frac{\tan(2t)}{2}$ , or equivalently  $t = t(s) = \frac{\arctan(2s)}{2}$ , and  $y = \frac{x}{\cos(2t)}$  an elementary computation shows that  $\mathcal{L}$  maps solution of (NLS) to solution of (HNLS) with the same initial data. In particular

$$\mathcal{L}(e^{is\Delta_y} v_0) = e^{-it(s)H} v_0.$$

In the proof of Theorem 1.3 it is needed to compare the  $H^\sigma$  norms of  $u$  and the  $\mathcal{H}^\sigma$  norms of  $v$ . This will be possible thanks to the following lemma.

**Lemma B.1** (Lens Transform). *If  $u$  and  $U$  are related by*

$$u(x) = \frac{1}{\cos^{\frac{d}{2}}(2t)} U\left(\frac{x}{\cos(2t)^d}\right) \exp\left(-i \frac{x^2 \tan(2t)}{2}\right)$$

then for any  $\sigma \in [0, 1]$  and  $t \in [0, \frac{\pi}{4}]$ ,

$$\|U\|_{H^\sigma} \lesssim \|u\|_{\mathcal{H}^\sigma},$$

$$\|\langle \cdot \rangle^\sigma U\|_{L^2(\mathbb{R}^2)} \lesssim \left(\frac{\pi}{4} - t\right)^{-2d\sigma} \|u\|_{\mathcal{H}^\sigma(\mathbb{R}^2)},$$

and

$$\|U\|_{L^q} \leq \cos(2t)^{d(\frac{1}{2} - \frac{1}{q})} \|u\|_{L^q}.$$

*Proof.* See [BT20], Lemma A.2 with only minor modification to the  $d$  dimensional case.  $\square$

*Remark B.2.* This result shows that estimates at regularity  $\mathcal{H}^\sigma$  for  $u$  transfers into estimates in  $\mathcal{H}^\sigma$  for  $U$  with a loss  $(\frac{\pi}{4} - t)^{-2d\sigma}$ . This explains that in the proof of Theorem 1.3 we needed explicit decay rates on the scattering estimates for (HNLS) in order to be transferred into a scattering result for (NLS).

*Remark B.3.* The last inequality of Lemma B.1 applied to solutions  $u$  of (NLS) and the corresponding solution  $v$  to (HNLS) implies  $\|u(s)\|_{L^q} \lesssim \|v(t(s))\|_{L^q}$  as soon as  $q > 1$ .

*Remark B.4.* The ‘‘lens transform’’ may look surprising and somehow unexpected. We briefly explain how one can come up with such a transformation, only using basic insight regarding the symmetries of (NLS). In fact this heuristic is useful to derive other symmetries or pseudo-symmetries to the Schrödinger equations, see [DR15] for other instances of pseudo-symmetries and [MR19] for an application.

To simplify, take  $p = 1 + \frac{4}{d}$ . Our starting point is the scaling symmetry

$$u(s, y) \mapsto u_\lambda(s, y) := \frac{1}{\lambda^{\frac{d}{2}}} u\left(\frac{s}{\lambda^2}, \frac{y}{\lambda}\right).$$

In order to derive more general symmetries we seek for a scaling, depending on time. We change variables  $(t, x) \leftrightarrow (s, y)$  imposing a local scaling symmetry  $ds = \frac{dt}{\lambda(t)^2}$  and  $dy = \frac{dx}{\lambda(t)}$  which can be integrated as

$$y = \frac{x}{\lambda(s)} \text{ and } s = \int_0^t \frac{dt'}{\lambda(t')^2}.$$

Thus one may seek for a change of variable of the form:

$$v(t, x) = \lambda(t)^{-d/2} u\left(\int_0^t \frac{dt'}{\lambda(t')^2}, \frac{x}{\lambda(t)}\right).$$

This change of variable is not sufficient. Indeed, writing the equation satisfied by  $v$  in variables  $t, x$  writes at the first derivatives order:

$$i\partial_t v + \Delta_x v - i \frac{\lambda'(t)}{\lambda(t)} x \cdot \nabla_x v + \dots$$

which is not the linear Schrödinger equation. However it is possible to multiply by the correct exponential to eliminate the term  $x \cdot \nabla_x v$  just as in the method of variation of constants. This leads to the choice

$$v(t, x) = \lambda(t)^{-d/2} u \left( \int_0^t \frac{dt'}{\lambda(t')^2}, \frac{x}{\lambda(t)} \right) \exp \left( \frac{i|x|^2 \lambda'(t)}{4\lambda(t)} \right).$$

Now if one wants to compactify time, a usual way to do so is to chose  $t = \frac{1}{2} \arctan(2s)$  which maps  $\mathbb{R}$  to  $[-\frac{\pi}{4}, \frac{\pi}{4}]$ . Hence we arrive at the given form.

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