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ARITHMETIC SUBGROUPS OF CHEVALLEY GROUP SCHEMES OVER FUNCTION FIELDS II: CONJUGACY CLASSES OF MAXIMAL UNIPOTENT SUBGROUPS

Claudio Bravo ¹ Benoit Loisel ²

ABSTRACT. Let \mathcal{C} be a smooth, projective, geometrically integral curve defined over a perfect field \mathbb{F} . Let $k = \mathbb{F}(\mathcal{C})$ be the function field of \mathcal{C} . Let \mathbf{G} be a split simply connected semisimple \mathbb{Z} -group scheme. Let \mathcal{S} be a finite set of places of \mathcal{C} . In this paper, we investigate on the conjugacy classes of maximal unipotent subgroups of \mathcal{S} -arithmetic subgroups. These are parameterized thanks to the Picard group of $\mathcal{O}_{\mathcal{S}}$ and the rank of \mathbf{G} . Furthermore, these maximal unipotent subgroups can be realized as the unipotent part of natural stabilizer, that are the stabilizers of sectors of the associated Bruhat-Tits building. We decompose these natural stabilizers in terms of their diagonalisable part and unipotent part, and we precise the group structure of the diagonalisable part.

MSC Codes: 20G30, 20E45 (primary) 11R58, 20E42 (secondary)

Keywords: Arithmetic subgroups, algebraic function fields, Chevalley groups, maximal unipotent subgroups, Bruhat-Tits buildings.

CONTENTS

1. Introduction	1
2. The Bruhat-Tits building and the action of $\mathbf{G}(k)$	3
3. Main results	5
4. Numbering of conjugacy classes of maximal unipotent subgroups	6
5. Maximal unipotent subgroups	7
6. $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -conjugacy classes of Borel k -subgroups	10
7. Fixing groups of germs at infinity of chambers	11
8. Comparison between the stabilizers of a vector chamber and of its germ	15
9. Applications to principal congruence subgroups	19
Acknowledgements	21
References	21

1. INTRODUCTION

In the following, we always denote by \mathbf{G}_k a split simply connected semisimple k -group scheme and by \mathbf{G} a semisimple \mathbb{Z} -group scheme such that $\mathbf{G} \otimes_{\mathbb{Z}} k \cong \mathbf{G}_k$ [DG70b, Exp. XXV 1.3]. All the considered group schemes are assumed to be linear, smooth and connected. For such a group scheme \mathbf{H} defined over a (commutative) ring R , we denote by $\mathbf{H}(R)$ the group of R -points of \mathbf{H} .

Along this work, we always denote by \mathcal{C} a smooth, projective, geometrically integral curve over a perfect field \mathbb{F} . The function field k of \mathcal{C} is a separable extension of $\mathbb{F}(x)$, where $x \in k$ is transcendental over \mathbb{F} . Hence, it follows from [Sti09, I.1.5, p.6] that the closure $\tilde{\mathbb{F}}$ of \mathbb{F} in k is a finite extension of \mathbb{F} . In all that follows, without loss of generality,

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we assume that $\tilde{\mathbb{F}} = \mathbb{F}$, i.e. \mathbb{F} is algebraically closed in k . Let \mathcal{S} be a finite set of closed point of \mathcal{C} . We denote by $\mathcal{O}_{\mathcal{S}}$ the ring of functions of \mathcal{C} that are regular outside \mathcal{S} , so that $\text{Quot}(\mathcal{O}_{\mathcal{S}}) = k$.

In the following, a subgroup $G \subset \mathbf{G}(k)$ which is commensurable with the group $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ of $\mathcal{O}_{\mathcal{S}}$ -points of \mathbf{G} is called an \mathcal{S} -arithmetic subgroup. In [Rag76, §1], \mathcal{S} -arithmetic subgroups are defined as follows: considering a faithful linear representation $\rho : \mathbf{G}_k \rightarrow \text{SL}_{n,k}$ defined over k , an \mathcal{S} -arithmetic subgroup is a subgroup of $\mathbf{G}(k)$ which is commensurable with $\mathbf{G}(k) \cap \rho^{-1}(\text{SL}_n(\mathcal{O}_{\mathcal{S}}))$. According to [BT84, 1.4.5], \mathbf{G} admits a faithful linear representation defined over \mathbb{Z} which is a closed embedding. Thus, the natural inclusion $\mathbf{G}(\mathcal{O}_{\mathcal{S}}) \subset \mathbf{G}(k) \cap \rho^{-1}(\text{SL}_n(\mathcal{O}_{\mathcal{S}}))$ is an equality. Hence, the definition of \mathcal{S} -arithmetic subgroups given by [Rag76, §1] coincides with that of commensurable subgroups of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$. Both depends on either the choice of a representation or of a model of \mathbf{G} over \mathbb{Z} .

As the rational points of an algebraic group over a field, by Jordan decomposition, there are elements of $\mathbf{G}(k)$ that are either semisimple or unipotent. We say that a subgroup U of $\mathbf{G}(k)$ is unipotent (resp. diagonalisable) if any element $u \in U$ is unipotent (resp. semisimple) in $\mathbf{G}(k)$. In this paper, we focus on maximal unipotent subgroups of \mathcal{S} -arithmetic subgroups.

Let F be a field of characteristic 0 or a field of positive characteristic $p > 0$ such that $[F : F^p] \leq p$. For instance, F can be a global field. Let U be a unipotent subgroup of $\mathbf{G}(F)$. There exists a (maximal) unipotent subgroup-scheme \mathbf{U}_F of \mathbf{G}_F such that $U \subseteq \mathbf{U}_F(F)$. This result is a consequence of [BT71, Corollary 3.7], when F has characteristic zero, and of Theorem [Gil02, Theorem 2], in positive characteristic.³ In particular, if U is a maximal unipotent subgroup of $\mathbf{G}(F)$, then $U = \mathbf{U}_F(F)$. Since $\mathbf{U}_F(F)$ is solvable, it is contained in the group $\mathbf{B}_F(F)$, for a Borel subgroup \mathbf{B}_F of \mathbf{G}_F .⁴ Moreover, the group $\mathbf{U}_F(F)$ is a unipotent radical of $\mathbf{B}_F(F)$. Since the Borel subgroups of \mathbf{G}_F are $\mathbf{G}(F)$ -conjugated, we conclude that all the maximal unipotent (abstract) subgroups of $\mathbf{G}(F)$ are $\mathbf{G}(F)$ -conjugated.

When R is an arbitrary commutative ring, maximal unipotent subgroups are not always $\mathbf{G}(R)$ -conjugate. Moreover, in the context of \mathcal{S} -arithmetic subgroups, Corollary 4.1 will provide families of counter-examples. Nevertheless, when the arithmetic properties of the ring R are closed to be that of a field (e.g. R is a PID), one will observe that the maximal unipotent subgroups are $\mathbf{G}(R)$ -conjugate (c.f. Corollary 4.2).

This work is devoted to understand the conjugacy classes of maximal unipotent subgroups contained in any \mathcal{S} -arithmetic subgroup G of $\mathbf{G}(k)$. As we say above, for simply connected semisimple groups over F , there is a unique conjugacy class of maximal unipotent subgroups. This is a consequence of the $\mathbf{G}(F)$ -conjugation of Borel subgroups. This $\mathbf{G}(F)$ -conjugation result corresponds to the $\mathbf{G}(F)$ -transitive action on chambers of the spherical Tits building associated to (\mathbf{G}, F) [AB08, Thm. 6.56]. Thanks to this combinatorial interpretation, in § 5 we describe the maximal unipotent subgroups of G , as well as its G -conjugacy classes, in terms of certain unipotent subgroups $U(G_D)$ of the G -stabilizers G_D of chambers $\partial_{\infty}D$ of the spherical building defined from (\mathbf{G}, k) . These properties are summarized in Theorem 3.1. Theorem 3.2 provides a parametrization of the G -conjugacy classes of maximal unipotent subgroups of G thanks to the Étale cohomology, as developed in § 6.

In some cases (c.f. [MS13], [MS15], [Ser03, Ch. II, § 2.5], [Beh04, §4 (d)] and [Stu80, §3]), the groups G_D (or G -stabilizers of simplices in the affine building which are related with G_D) have been directly computed. So, it appears to be more natural to compute directly the G_D than the $U(G_D)$. Then, in order to understand the groups $U(G_D)$, in § 7 and § 8, we compare each G_D with its (unique) maximal unipotent subgroup $U(G_D)$. In order to do this, in Theorem 3.4, we find a diagonalizable subgroup of $\mathbf{G}(k)$ that is isomorphic to $G_D/U(G_D)$ and that isomorphic to the direct product of an

³This also arises when the splitness assumption of \mathbf{G} is omitted.

⁴This also arises when \mathbf{G} is assumed quasi-split instead of split.

“arithmetic bounded torus” T with a finitely generated free \mathbb{Z} -module. The group T is finite whenever \mathbb{F} is finite, so that the preceding result describes the decomposition in the torsion and the free part of the abelian group $G_D/U(G_D)$. For a principal congruence subgroup Γ of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, when for instance \mathbb{F} is a finite field and $\mathcal{S} = \{P\}$, the application of Theorems 3.1, 3.2 and 3.4 gives in § 9 that the maximal unipotent subgroups of Γ are exactly the stabilizers in Γ of chambers of the spherical building.

Consider a (non necessarily split) semisimple group \mathbf{G} of rank 1 defined over a local field K . Since the group of rational points $\mathbf{G}(K)$ of \mathbf{G} is a locally compact unimodular group, it has a Haar measure μ which is $\mathbf{G}(K)$ -invariant. A lattice Λ of $\mathbf{G}(K)$ is a subgroup of finite μ -covolume, i.e. $\mu(\mathbf{G}(K)/\Lambda)$ is finite. In [Bau03, Th. 2.3], Baumgartner characterizes the maximal unipotent subgroups of any lattice Λ of $\mathbf{G}(K)$.

Assume that \mathbb{F} is finite and that \mathbf{G} has rank 1 (i.e. $\mathbf{G} = \mathrm{SL}_2$ since it is assume split simply connected and semisimple). It follows from [Ser03, Ex. 2, Ch. II, § 2.9, Pag. 110] that $\mathbf{G}(\mathcal{O}_{\{P\}})$ is a lattice of $\mathbf{G}(k_P)$. Since $\mathbf{G}(\mathcal{O}_{\{P\}}) \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, when $P \in \mathcal{S}$, the group $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ is also a lattice of $\mathbf{G}(k_P)$. In particular, any \mathcal{S} -arithmetic subgroup G of $\mathbf{G}(k_P)$ is a lattice of $\mathbf{G}(k_P)$. Thus, the results of Baumgartner describe the maximal unipotent subgroups contained in G when \mathbb{F} is finite and \mathbf{G} has rank 1. Still in the context of finite fields, Serre proves in [Ser03, Ch. II, § 2.9] that the maximal unipotent subgroups contained in finite index subgroups G of $\mathrm{SL}_2(\mathcal{O}_{\{P\}})$, whose torsion is p -primary, are exactly the stabilizers of chambers of the spherical building of (SL_2, k) . In other words, Serre proves that the aforementioned unipotent subgroups are the G -stabilizers defined from the action of G on $\mathbb{P}^1(k)$ by Moebius transformations. In the same work, Serre characterizes the homology of G modulo a representative system of the conjugacy classes of its maximal unipotent subgroups in terms of the Euler-Poincaré characteristic of G . These results are summarized in [Ser03, Th. 14, Ch. II, § 2.9]. In [Ser71, § 3, Pag. 155], Serre conjectures that these results can be extended to the context where \mathbf{G} has an arbitrary rank.

In the following, we describe the groups involved in the preceding conjecture due to Serre. For simplicity, this study is limited to split groups because they admit a Chevalley pinning defined over \mathbb{Z} . In particular, points of tori and root groups naturally make sense over arbitrary commutative rings. This may not happen for non-split groups, since root groups may not be defined over \mathbb{Z} . Some investigation for quasi-split groups of rank 1 are developed in [Bra23]. In the method we follow, it appears crucial to restrict to simply connected semisimple groups. Indeed, this hypothesis allows us to apply results on injections of unipotent groups such as [BT71, Corollary 3.7] and Theorem [Gil02, Theorem 2]. It also allows us to obtain certain identification using étale cohomology due to the triviality of certain étale cohomological sets. More precisely, with the method followed in this work, we cannot obtain a parametrization of conjugacy classes of maximal unipotent subgroups when \mathbf{G} is not simply connected.

2. THE BRUHAT-TITS BUILDING AND THE ACTION OF $\mathbf{G}(k)$

2.1. Recall on algebraic groups and Bruhat-Tits buildings. In the following, we consider a given Killing couple (\mathbf{T}, \mathbf{B}) of \mathbf{G} defined over \mathbb{Z} [DG70b, Exp. XXII, 5.3.13]. The group \mathbf{B} defines a subset of positive roots Φ^+ of the set of root $\Phi(\mathbf{T}, \mathbf{B})$ as in [Bor91, 20.4]. This induces a basis of simple roots $\Delta = \Delta(\mathbf{B})$ [Bou81, VI.1.6] of \mathbf{G} relatively to the Borel k -subgroup \mathbf{B} . For any $\alpha \in \Phi$, let \mathbf{U}_{α} be the \mathbf{T} -stable unipotent subgroup of \mathbf{G} defined from this. Since \mathbf{G} is assumed to be split, it admits a Chevalley pinning [DG70b, Exp. XXIII, 1.1]. We denote by $\theta_{\alpha} : \mathcal{G}_{\alpha} \rightarrow \mathbf{U}_{\alpha}$ the \mathbb{Z} -isomorphism given by the Chevalley pinning. We denote by \mathbf{U}^+ the subgroup of \mathbf{G} generated by the \mathbf{U}_{α} for $\alpha \in \Phi^+$. The group \mathbf{U}^+ is the unipotent radical of \mathbf{B} [DG70b, Exp. XXIV, 1.13(i)].

Let $P \in \mathcal{S}$. Let $\nu_P : k^* \rightarrow \mathbb{Z}$ be the valuation induced by P , and let k_P be the completion of k with respect to ν_P . Let \mathcal{O}_P be the ring of integers of the local field k_P . The datum of \mathbf{T} and the root groups \mathbf{U}_{α} , together with the valuation ν_P induces a root group datum on $\mathbf{G}(k_P)$ [BT72, 6.2.3(b)]. We denote by $X_P = X(\mathbf{G}, k, P)$ the affine

Bruhat-Tits building of the split semisimple k -group \mathbf{G}_k over the valued field (k, ν_P) , associated to this valued root group datum, and by $\mathbb{A}_{0,P}$ the apartment of X_P associated to this root group datum [BT72, 7.4.2]. The apartment $\mathbb{A}_{0,P}$ is an Euclidean space over a vector space $V_{0,P}$. We denote by $D_{0,P} \subset V_{0,P}$ the vector chamber in $V_{0,P}$ associated to (\mathbf{T}, \mathbf{B}) , that is $D_{0,P} = \{x \in V_{0,P}, \alpha(x) > 0, \forall \alpha \in \Delta\}$. Since \mathbf{G} is semisimple and simply connected, the pointwise stabilizer of $\mathbb{A}_{0,P}$ is $\mathbf{T}(\mathcal{O}_P)$ [BT72, 6.4.16(b)]. Since \mathbf{G}_{k_P} is split, semisimple and simply connected, it follows from [BT84, 4.6.31 and 4.6.32] that there exists a special vertex $x_{0,P} \in X_P$ such that its stabilizer in $\mathbf{G}(k_P)$ is $\mathbf{G}(\mathcal{O}_P)$. In fact, since $\mathbf{T}(\mathcal{O}_P) \subset \mathbf{G}(\mathcal{O}_P)$, the vertex $x_{0,P}$ belongs to $\mathbb{A}_{0,P}$ [BT72, 7.4.10].

Any apartment \mathbb{A} of X_P is endowed with an affine Coxeter complex structure associated to Φ . It induces a spherical Coxeter complex structure on the underlying vector space V of \mathbb{A} . The vector chambers of those V are called the vector chambers of X_P .

2.2. The diagonal action of $\mathbf{G}(k)$. We denote by $X_{\mathcal{S}}$ the direct product of the buildings X_P , for $P \in \mathcal{S}$. It is an Euclidean building of type $\Phi^{\mathcal{S}}$, as a finite product of Euclidean buildings all of type Φ . The abstract group $\hat{G}_{\mathcal{S}} := \prod_{P \in \mathcal{S}} \mathbf{G}(k_P)$ acts on $X_{\mathcal{S}}$ via

$$(g_P)_{P \in \mathcal{S}} \cdot (x_P)_{P \in \mathcal{S}} = (g_P \cdot x_P)_{P \in \mathcal{S}}, \quad \forall (g_P)_{P \in \mathcal{S}} \in \hat{G}_{\mathcal{S}}, \forall (x_P)_{P \in \mathcal{S}} \in X_{\mathcal{S}}.$$

By definition, an apartment (resp. a chamber, a vector chamber) of $X_{\mathcal{S}}$ is a product of apartments (resp. chambers, vector chambers), vector of the X_P , for $P \in \mathcal{S}$. Since $\mathbf{G}(k_P)$ acts strongly transitively on $X_{\mathcal{S}}$ [BT72, 2.2.6], we know that $\hat{G}_{\mathcal{S}}$ acts strongly transitively on $X_{\mathcal{S}}$.

We define an apartment of $X_{\mathcal{S}}$ by $\mathbb{A}_0 := \prod_{P \in \mathcal{S}} \mathbb{A}_{0,P}$. Since $\hat{G}_{\mathcal{S}}$ acts transitively on the set of apartments of $X_{\mathcal{S}}$, the apartments of $X_{\mathcal{S}}$ are the $g \cdot \mathbb{A}_0$, for $g \in \hat{G}_{\mathcal{S}}$. Hence, the group $\hat{G}_{\mathcal{S}}$ acts transitively on the vector chambers of $X_{\mathcal{S}}$, that are the vector chambers of the vector spaces associated to its apartments.

The conical cell $D_0 = \prod_{P \in \mathcal{S}} D_{0,P}$ is a vector chamber of the underlying vector space V_0 of \mathbb{A}_0 . By transitivity of $\hat{G}_{\mathcal{S}}$, the vector chambers of $X_{\mathcal{S}}$ are the $D = g \cdot D_0 \subset g \cdot V_0$, for $g \in \hat{G}_{\mathcal{S}}$. For any $x \in \mathbb{A}_0$, the subset $Q(x, D_0) = x + D_0 \subset \mathbb{A}_0$ is a sector chamber as a product of sector chambers on each component. As in [BT72, 7.4.12], we define the sector chambers of X as the subsets $g \cdot Q(x, D_0) \subset g \cdot \mathbb{A}_0$, for $g \in \hat{G}_{\mathcal{S}}$ and $x \in \mathbb{A}_0$.

We define a point $x_0 = (x_{0,P})_{P \in \mathcal{S}}$ of the apartment \mathbb{A}_0 . It is a special vertex since so are the $x_{0,P} \in \mathbb{A}_{0,P}$, for all $P \in \mathcal{S}$.

Since $\mathbf{G}(k)$ embeds in $\hat{G}_{\mathcal{S}}$ via $g \mapsto (g)_{P \in \mathcal{S}}$, the group $\mathbf{G}(k)$ acts diagonally on $X_{\mathcal{S}}$ via

$$g \cdot (x_P)_{P \in \mathcal{S}} = (g \cdot x_P)_{P \in \mathcal{S}},$$

for $(x_P)_{P \in \mathcal{S}} \in X_{\mathcal{S}}$ and $g \in \mathbf{G}(k)$.

2.3. Rational chambers at infinity. We denote by $\partial_{\infty} X_{\mathcal{S}}$ the spherical building at infinity of $X_{\mathcal{S}}$ as defined in [AB08, § 11.8]. It consists of parallelism classes of geodesical rays of $X_{\mathcal{S}}$. Its apartments (resp. chambers) are in one-to-one correspondence with the apartments (resp. vector chambers) of $X_{\mathcal{S}}$ [AB08, 11.75, 11.79]. Since $\hat{G}_{\mathcal{S}}$ acts simplicially and by isometries on $X_{\mathcal{S}}$, it sends geodesical rays onto geodesical rays. Thus, it induces an action of $\hat{G}_{\mathcal{S}}$ on $\partial_{\infty} X_{\mathcal{S}}$. We denote by $\partial_{\infty} \mathbb{A}$ (resp. $\partial_{\infty} D$) the image in $\partial_{\infty} X_{\mathcal{S}}$ of an apartment \mathbb{A} (resp. a vector chamber D). Since $\hat{G}_{\mathcal{S}}$ acts transitively on the pairs consisting of an apartment and a vector chamber of this apartment of $X_{\mathcal{S}}$, the group $\hat{G}_{\mathcal{S}}$ also acts strongly transitively on $\partial_{\infty} X_{\mathcal{S}}$.

Lemma 2.1. *The stabilizer in $\mathbf{G}(k)$ of $\partial_{\infty} D_0$ is $\mathbf{B}(k)$.*

Proof. For each $P \in \mathcal{S}$, the stabilizer of $\partial_{\infty} D_{0,P}$ in $\mathbf{G}(k_P)$ is $\mathbf{B}(k_P)$. Hence the stabilizer of $\partial_{\infty} D_0$ in $\hat{G}_{\mathcal{S}}$ is $\prod_{P \in \mathcal{S}} \mathbf{B}(k_P)$. Thus, by diagonal action, the stabilizer of $\partial_{\infty} D_0$ in $\mathbf{G}(k)$ is the intersection of the groups $\mathbf{B}(k_P) \cap \mathbf{G}(k) \subset \mathbf{G}(k_P)$, for $P \in \mathcal{S}$. Since the algebraic group \mathbf{B} is closed in \mathbf{G} , we have that $\mathbf{B}(k_P) \cap \mathbf{G}(k) = \mathbf{B}(k)$, for each $P \in \mathcal{S}$. \square

We denote by $\partial_\infty^k X_{\mathcal{S}}$ the $\mathbf{G}(k)$ -orbit of $\partial_\infty D_0$ in the set of chambers of $\partial_\infty X_{\mathcal{S}}$. An element of $\partial_\infty^k X_{\mathcal{S}}$ is called a rational chamber at infinity of $X_{\mathcal{S}}$. Since $\text{Stab}_{\mathbf{G}(k)}(\partial_\infty D_0)$ equals $\mathbf{B}(k)$, there exists a $\mathbf{G}(k)$ -equivariant one-to-one correspondence between $\partial_\infty^k X_{\mathcal{S}}$ and the Borel variety $\mathbf{G}(k)/\mathbf{B}(k)$. Thus, the set $\partial_\infty^k X_{\mathcal{S}}$ corresponds to the set of chambers in the usual spherical building defined from $\mathbf{G}(k)$.

3. MAIN RESULTS

The main results of this work is the following theorem that describes maximal unipotent subgroups of G in terms of its action on $\partial_\infty^k X_{\mathcal{S}}$.

Theorem 3.1. *Recall that \mathbf{G}_k denotes a split simply connected semisimple k -group, \mathbf{B} a Borel subgroup with unipotent radical \mathbf{U}^+ . Recall that the base field \mathbb{F} is assumed to be perfect. Let G be an \mathcal{S} -arithmetic subgroup of $\mathbf{G}(k)$.*

For each $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, let $h_D \in \mathbf{G}(k)$ be an arbitrary element such that $h_D \cdot \partial_\infty D = \partial_\infty D_0$. Let us write:

$$(1) \quad G_D = \text{Stab}_G(\partial_\infty D) = h_D^{-1} \mathbf{B}(k) h_D \cap G, \quad \text{and} \quad U(G_D) := h_D^{-1} \mathbf{U}^+(k) h_D \cap G.$$

Then:

- (1) $U(G_D)$ is the subgroup of all the unipotent elements in G_D , and
- (2) $\mathfrak{U} := \{U(G_D) : \partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}\}$ is the set all the maximal unipotent subgroups of G .

Fix a set $\{\partial_\infty D_\sigma : \sigma \in \Sigma\}$ of representatives of the G -orbits of $\partial_\infty^k X_{\mathcal{S}}$, and write $h_\sigma := h_{D_\sigma}$, $G_\sigma = G_{D_\sigma}$ and $U(G_\sigma) = U(G_{D_\sigma})$. Then:

- (3) $\mathfrak{U}/G := \{U(G_\sigma) : \sigma \in \Sigma\}$ is a set of representatives of the conjugacy classes of maximal unipotent subgroups of G .

The isomorphism classes of vector bundles of rank 1 on $\text{Spec}(\mathcal{O}_{\mathcal{S}})$ (resp. \mathcal{C}) form a group $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ (resp. $\text{Pic}(\mathcal{C})$) with the tensor product as composition law. Moreover, if $\overline{\mathcal{S}}$ is the image of \mathcal{S} in $\text{Pic}(\mathcal{C})$, then, we have $\text{Pic}(\mathcal{O}_{\mathcal{S}}) \cong \text{Pic}(\mathcal{C})/\langle \overline{\mathcal{S}} \rangle$. Thank to étale cohomology, a numbering of the maximal unipotent subgroups described in Theorem 3.1 can be done for $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ in terms of $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ and of the rank $\mathfrak{t} = \text{rk}(\mathbf{G})$ of \mathbf{G} , which is the dimension of \mathbf{T} . This also has interesting consequences on the \mathcal{S} -arithmetic subgroups G of $\mathbf{G}(k)$. For instance, this implies that the number of G -conjugacy classes of maximal unipotent subgroup is finite whenever \mathbb{F} is finite.

Theorem 3.2. *There exists a bijection between the set of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbits in $\partial_\infty^k X_{\mathcal{S}}$ and the group $\text{Pic}(\mathcal{O}_{\mathcal{S}})^{\mathfrak{t}}$.*

Remark 3.3. If \mathbf{G}_k is a split reductive k -group scheme (non necessarily simply connected semisimple), then there exists a bijection between the set of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbits in $\partial_\infty^k X_{\mathcal{S}}$ and $\ker(H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G}))$. It is proven in § 6.

Let p be a prime number. We say that the torsion of a group $G \subset \mathbf{G}(k)$ is p -primary if each finite order element in G has a p -power order. For instance, if \mathbb{F} is finite of characteristic p , then the torsion of any principal congruence subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ is p -primary (c.f. Lemma 9.1).

The following result describes the group $G_D/U(G_D)$ as the direct product of an ‘‘arithmetic bounded torus’’ T with a finitely generated free \mathbb{Z} -module. The group T is finite whenever \mathbb{F} is finite, so that the following result describes the decomposition in the free part and torsion part of the abelian group $G_D/U(G_D)$.

Theorem 3.4. *Let G be an \mathcal{S} -arithmetic subgroup of $\mathbf{G}(k)$. For each $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, let h_D , G_D and $U(G_D)$ as in Theorem 3.1. Then, for each $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, there exists $T(G_D) \subset h_D^{-1} \mathbf{T}(k) h_D$ such that:*

- (1) G_D is an extension of the maximal unipotent group $U(G_D)$ by $T(G_D)$.

- (2) $T(G_D) \cong T \times \mathbb{Z}^r$, where T is commensurable with a subgroup of $\mathbf{T}(\mathbb{F})$, and $r = r(\mathbf{G}, \mathcal{S}, D)$ is less or equal than $\mathfrak{t} \cdot \#\mathcal{S}$.
- (3) Moreover, if $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, then $T \subseteq \mathbf{T}(\mathbb{F})$,
- (4) if $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ and $\mathcal{S} = \{P\}$, then $r = 0$, and
- (5) if $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, $\mathcal{S} = \{P\}$, \mathbb{F} is finite of characteristic p and the torsion of G is p -primary, then $T(G_D) = \{\text{id}\}$. In other words, the group G_D is unipotent.

4. NUMBERING OF CONJUGACY CLASSES OF MAXIMAL UNIPOTENT SUBGROUPS

By using the Theorem 3.1 and 3.2, we can count the number of conjugacy classes of maximal unipotent subgroups as follows.

Corollary 4.1. *Recall that the ground field \mathbb{F} is perfect and \mathbf{G}_k is assumed to be a split simply connected semisimple group. There exists a bijective map between the set of conjugacy classes of maximal unipotent subgroups of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ and $\text{Pic}(\mathcal{O}_{\mathcal{S}})^{\mathfrak{t}}$.*

Proof. This is an immediate consequence of Theorem 3.1 (3) and Theorem 3.2. \square

The following result shows that the characterization of the conjugacy classes of maximal unipotent subgroups in $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, when $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain, is similar than the description in the field context.

Corollary 4.2. *The function ring $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain if and only if any maximal unipotent subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ is conjugate to $\mathbf{U}^+(\mathcal{O}_{\mathcal{S}})$.*

Proof. It is well known that $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain if and only if $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ is trivial. But, Theorem 3.2 shows that $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ is trivial exactly when $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ acts transitively on $\partial_{\infty}^k X_{\mathcal{S}}$. Thus, $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain precisely when there exists a unique conjugacy class of maximal unipotent subgroups in $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$. As a consequence, if each maximal unipotent subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ is conjugate to $\mathbf{U}^+(\mathcal{O}_{\mathcal{S}})$, then $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain.

Conversely, if $\mathcal{O}_{\mathcal{S}}$ is principal, we can choose $\partial_{\infty} D_0$ as a representative of the unique $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbit. Theorem 3.1, applied with $h_{\sigma} = 1$, shows that each unipotent subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ is conjugate to $U(\mathbf{G}(\mathcal{O}_{\mathcal{S}})) = \mathbf{U}^+(k) \cap \mathbf{G}(\mathcal{O}_{\mathcal{S}}) = \mathbf{U}^+(\mathcal{O}_{\mathcal{S}})$. \square

Corollary 4.3. *Assume that the field \mathbb{F} is finite. Then, each \mathcal{S} -arithmetic subgroup G of $\mathbf{G}(k)$ has finitely many conjugacy classes of maximal unipotent subgroups. Moreover, if $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, then G has at most $[\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G] \cdot \text{Card}(\text{Pic}(\mathcal{O}_{\mathcal{S}}))^{\mathfrak{t}}$ conjugacy classes of maximal unipotent subgroups.*

Proof. Assume that $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$. Then, the number of G -orbits in $\partial_{\infty}^k X_{\mathcal{S}}$ is less or equal than the set of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbits in $\partial_{\infty}^k X_{\mathcal{S}}$ multiplied by the index $[\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G]$. Then, the second statement follows from Corollary 4.1. Note that, since $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ is finite whenever \mathbb{F} is finite (c.f. [Ser03, Ch. II, § 2.2]), the group $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ has finitely many conjugacy classes of maximal unipotent subgroups.

Now, let G be a arbitrary \mathcal{S} -arithmetic subgroup of $\mathbf{G}(k)$. Let G^{\natural} be a subgroup of $G \cap \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ such that the indices $[G : G^{\natural}]$ and $[\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G^{\natural}]$ are finite. Since $[\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G^{\natural}]$ is finite, the group G^{\natural} has finitely many conjugacy classes of maximal unipotent subgroups. Equivalently, the number of G^{\natural} -orbits in $\partial_{\infty}^k X_{\mathcal{S}}$ is finite (c.f. Theorem 3.1 (3)). Since $G^{\natural} \subseteq G$, the number of G -orbits in $\partial_{\infty}^k X_{\mathcal{S}}$ is also finite. Thus, it follows from Theorem 3.1 (3) that G has finitely many conjugacy classes of maximal unipotent subgroups. This proves the first statement. \square

Let $\text{Div}(\mathcal{C})$ be the group of divisor of the curve \mathcal{C} . We denote by $\langle \mathcal{S} \rangle$ the subgroup of $\text{Div}(\mathcal{C})$ generated by a finite set \mathcal{S} of closed point of \mathcal{C} . The following result shows that, for some large enough set of places \mathcal{S} , the group $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ has a unique conjugacy class of maximal unipotent subgroups, since $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain, for such sets \mathcal{S} . This holds even when \mathcal{C} does not have any rational point.

Corollary 4.4. *Assume that the field \mathbb{F} is finite. Then, for any smooth, projective, geometrically integral curve \mathcal{C} over \mathbb{F} , there exists a finite set of closed points \mathcal{S} , such that:*

(C1) *each maximal unipotent subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ is conjugate to $\mathbf{U}^+(\mathcal{O}_{\mathcal{S}})$.*

Moreover, if \mathcal{S} and \mathcal{S}' satisfy (C1), $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$ and $\langle \mathcal{S} \cup \mathcal{S}' \rangle$ does not contain a non-trivial principal divisor, then $\mathcal{S} \cap \mathcal{S}'$ also satisfies (C1).

Proof. Let $\mathcal{S}_0 = \{P\}$, where P is a closed point of \mathcal{C} . Let us write $A = \mathcal{O}_{\mathcal{S}_0}$. Since \mathbb{F} is finite, the Picard group $\text{Pic}(A) = \text{Pic}(\mathcal{C})/\langle \overline{P} \rangle$ is finite (c.f. [Ser03, Ch. II, § 2.2]). Let P_1, \dots, P_n be a finite set of closed point of \mathcal{C} such that $\overline{P}_1, \dots, \overline{P}_n$ generates $\text{Pic}(A)$. Let us write $\mathcal{S} = \{P, P_1, \dots, P_n\}$. Then, the Picard group $\text{Pic}(\mathcal{O}_{\mathcal{S}}) = \text{Pic}(\mathcal{C})/\langle \overline{\mathcal{S}} \rangle$ is isomorphic to $\text{Pic}(A)/(\langle \overline{\mathcal{S}} \rangle / \langle \overline{P} \rangle)$, which is trivial, since $\overline{\mathcal{S}}$ contains a set that generates $\text{Pic}(A)$. In other words, the ring $\mathcal{O}_{\mathcal{S}}$ is principal. Thus, it follows from Corollary 4.2 that \mathcal{S} satisfies (C1). Hence, the first statement follows.

Now, assume that \mathcal{S} and \mathcal{S}' satisfy (C1) and $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$. Then, it follows from Corollary 4.2 that $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ and $\text{Pic}(\mathcal{O}_{\mathcal{S}'})$ are trivial. We have the exact sequence

$$(2) \quad 0 \rightarrow (\langle \overline{\mathcal{S}} \rangle / \langle \overline{\mathcal{S} \cap \mathcal{S}'} \rangle) \rightarrow \text{Pic}(\mathcal{O}_{\mathcal{S} \cap \mathcal{S}'}) \rightarrow \text{Pic}(\mathcal{O}_{\mathcal{S}}) \rightarrow 0.$$

Let $\pi : \text{Pic}(\mathcal{C}) \rightarrow \text{Pic}(\mathcal{O}_{\mathcal{S}'})$ be the projection. Then $\pi(\langle \overline{\mathcal{S}} \rangle) = (\langle \overline{\mathcal{S}} \rangle + \langle \overline{\mathcal{S}'} \rangle) / \langle \overline{\mathcal{S}'} \rangle \cong \langle \overline{\mathcal{S}} \rangle / (\langle \overline{\mathcal{S}} \rangle \cap \langle \overline{\mathcal{S}'} \rangle)$. Since $\langle \mathcal{S} \cup \mathcal{S}' \rangle$ does not contain a non-trivial principal divisor, we have $\langle \overline{\mathcal{S}} \rangle \cap \langle \overline{\mathcal{S}'} \rangle = \langle \overline{\mathcal{S} \cap \mathcal{S}'} \rangle$, whence $\pi(\langle \overline{\mathcal{S}} \rangle) = (\langle \overline{\mathcal{S}} \rangle / \langle \overline{\mathcal{S} \cap \mathcal{S}'} \rangle)$. Hence, since $\text{Pic}(\mathcal{O}_{\mathcal{S}'})$ is trivial, we have that $\langle \overline{\mathcal{S}} \rangle / \langle \overline{\mathcal{S} \cap \mathcal{S}'} \rangle$ is trivial. Therefore, since $\text{Pic}(\mathcal{O}_{\mathcal{S}})$ is trivial, we conclude from Equation (2) that $\text{Pic}(\mathcal{O}_{\mathcal{S} \cap \mathcal{S}'})$ is trivial, whence $\mathcal{S} \cap \mathcal{S}'$ satisfies (C1) according to Corollary 4.2. \square

5. MAXIMAL UNIPOTENT SUBGROUPS

The main goal of this section is to prove Theorem 3.1. Before that we need some preparation, which precises the structure of all maximal unipotent subgroups of G .

Lemma 5.1. *Assume that \mathbb{F} is a perfect field of characteristic $p > 0$. Then $[k : k^p] = p$.*

Proof. Recall that, since k is the function field of a curve, k is a finite extension of $\mathbb{F}(u)$, where $u \in k$ is transcendental over \mathbb{F} . Let us write $k = \mathbb{F}(u, \theta_1, \dots, \theta_r)$, where θ_i is algebraic over $\mathbb{F}(u)$. Since \mathbb{F} is perfect of characteristic $p > 0$, we have $k^p = \mathbb{F}(u^p, \theta_1^p, \dots, \theta_r^p)$. Let $f_i(T)$ (resp. $g_i(T)$) be the irreducible polynomial of θ_i (resp. θ_i^p) in $L_i := \mathbb{F}(u, \theta_1, \dots, \theta_{i-1})$ (resp. in $F_i := \mathbb{F}(u^p, \theta_1^p, \dots, \theta_{i-1}^p)$). On the one hand, since $0 = g_i(\theta_i^p) = (\tilde{g}_i(\theta_i))^p$, for some $\tilde{g}_i \in L_i[T]$ with the same degree of g_i , we have $\deg(g_i) \geq \deg(f_i)$. On the other hand, since $0 = f_i(\theta_i)^p = \tilde{f}_i(\theta_i^p)$, for $\tilde{f}_i \in F_i[T]$ with the same degree of f_i , we have $\deg(g_i) \leq \deg(f_i)$. Thus

$$[\mathbb{F}(u, \theta_1, \dots, \theta_r) : \mathbb{F}(u)] = \prod_{i=1}^r \deg(f_i) = \prod_{i=1}^r \deg(g_i) = [\mathbb{F}(u^p, \theta_1^p, \dots, \theta_r^p) : \mathbb{F}(u^p)],$$

whence $[k : k^p] = [\mathbb{F}(u) : \mathbb{F}(u^p)] = p$. \square

Lemma 5.2. *Assume that the ground field \mathbb{F} is perfect (of arbitrary characteristic). Then, for each unipotent subgroup U of $\mathbf{G}(k)$, there exists $h \in \mathbf{G}(k)$ such that $U \subseteq h^{-1}\mathbf{U}^+(k)h$.*

Proof. Let U be a unipotent subgroup of $\mathbf{G}(k)$. Since \mathbf{G}_k is a simply connected semisimple k -group, the group U is k -embeddable in the unipotent radical $\mathcal{R}_u(\mathbf{P})$ of a k -parabolic subgroup \mathbf{P} of \mathbf{G} . This result is a consequence of [BT71, Corollary 3.7], when \mathbb{F} has characteristic zero, and of Theorem [Gil02, Theorem 2] and Lemma 5.1, in positive characteristic. In other words, if \mathbb{F} is a perfect field, then U is contained in $\mathcal{R}_u(\mathbf{P})(k)$. Moreover, since $\mathcal{R}_u(\mathbf{P})$ is contained in the unipotent radical of some Borel subgroup of \mathbf{G} , and two Borel subgroups are $\mathbf{G}(k)$ -conjugate, we conclude that $U \subseteq h^{-1}\mathbf{U}^+(k)h$, for some $h \in \mathbf{G}(k)$. \square

Let $G \subset \mathbf{G}(k)$ be an \mathcal{S} -arithmetic subgroup of $\mathbf{G}(k)$, i.e. a group that is commensurable with $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$. For each $\alpha \in \Phi^+$ and each $h \in \mathbf{G}(k)$, we denote by G_{α} and by $N(h)$, the groups:

$$\begin{aligned} G_{\alpha} &:= h^{-1}\mathbf{U}_{\alpha}(k)h \cap G, \\ N_{\alpha}(h) &:= \theta_{\alpha}^{-1}(hG_{\alpha}h^{-1}) = \theta_{\alpha}^{-1}(\mathbf{U}_{\alpha}(k) \cap hGh^{-1}). \end{aligned}$$

For $G = \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ we write $M_{\alpha}(h) := N_{\alpha}(h)$.

Lemma 5.3. *For any $h \in \mathbf{G}(k)$ and any $\alpha \in \Phi^+$, the group $N_{\alpha}(h)$ is infinite.*

Proof. By commensurability of G with $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, there exists $G^{\natural} \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}}) \cap G$ such that the indices $[G : G^{\natural}]$ and $[\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G^{\natural}]$ are finite. Let us write $M_{\alpha}^{\natural}(h) := \theta_{\alpha}^{-1}(\mathbf{U}_{\alpha}(k) \cap hG^{\natural}h^{-1})$. Let $\{x_i\}_{i \in \mathcal{I}}$ be a system of representatives of $M_{\alpha}(h)/M_{\alpha}^{\natural}(h)$. By definition of $M_{\alpha}(h)$, the element $g_i = h^{-1}\theta_{\alpha}(x_i)h$ belongs to $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$. If $g_i g_j^{-1}$ belongs to G^{\natural} , then $h^{-1}\theta_{\alpha}(x_i - x_j)h = g_i g_j^{-1} \in G^{\natural}$, or equivalently $x_i - x_j \in M_{\alpha}^{\natural}(h)$, whence $i = j$. Thus $[M_{\alpha}(h) : M_{\alpha}^{\natural}(h)] \leq [\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G^{\natural}]$. The same argument shows that $[N_{\alpha}(h) : M_{\alpha}^{\natural}(h)] \leq [G : G^{\natural}]$. Thus, we have that $M_{\alpha}(h)$ and $N_{\alpha}(h)$ are commensurable. Since $\mathcal{O}_{\mathcal{S}}$ is an integral domain, it follows from [BL23, Prop. 6.5] that $M_{\alpha}(h)$ contains a nonzero $\mathcal{O}_{\mathcal{S}}$ -ideal. Thus, we conclude that $N_{\alpha}(h)$ is infinite. \square

Lemma 5.4. *Let $h_1, h_2 \in \mathbf{G}(k)$. Then, the group $h_1^{-1}\mathbf{U}^+(k)h_1 \cap G$ is contained in $h_2^{-1}\mathbf{U}^+(k)h_2 \cap G$ if and only if $h_2 h_1^{-1} \in \mathbf{B}(k)$.*

In particular, if the equivalent conditions are satisfied, then

$$h_1^{-1}\mathbf{U}^+(k)h_1 \cap G = h_2^{-1}\mathbf{U}^+(k)h_2 \cap G.$$

Proof. Firstly, assume that $h_2 h_1^{-1} \in \mathbf{B}(k)$. Since $\mathbf{B}(k)$ normalizes $\mathbf{U}^+(k)$, we have $h_2 h_1^{-1} \mathbf{U}^+(k) h_1 h_2^{-1} = \mathbf{U}^+(k)$. This implies that $h_2^{-1} \mathbf{U}^+(k) h_2 = h_1^{-1} \mathbf{U}^+(k) h_1$. Thus, by intersecting with G the both sides of the preceding equation, we get

$$h_1^{-1}\mathbf{U}^+(k)h_1 \cap G = h_2^{-1}\mathbf{U}^+(k)h_2 \cap G.$$

Now, we prove the converse implication. Let us write $\tau = h_2 h_1^{-1}$, so that $\tau(\mathbf{U}^+(k) \cap h_1 G h_1^{-1})\tau^{-1}$ is contained in $\mathbf{U}^+(k) \cap h_2 G h_2^{-1}$. For each $\alpha \in \Phi^+$, we can write $\mathbf{U}_{\alpha}(k) \cap h_1 G h_1^{-1} = \{\theta_{\alpha}(z) : z \in N_{\alpha}(h_1)\}$. Then, we have

$$\tau \left\{ \prod_{\alpha \in \Phi^+} \theta_{\alpha}(x_{\alpha}) : x_{\alpha} \in N_{\alpha}(h_1) \right\} \tau^{-1} \subseteq \mathbf{U}^+(k) \cap h_2 G h_2^{-1}.$$

By the Bruhat decomposition, let us write $\tau = b' w b$, where $b, b' \in \mathbf{B}(k)$ and $w \in N^{\text{sph}}$. Since b' normalizes $\mathbf{U}^+(k)$, we deduce that

$$b \left(\prod_{\alpha \in \Phi^+} \theta_{\alpha}(x_{\alpha}) \right) b^{-1} \in \prod_{\alpha \in \Phi^+} \mathbf{U}_{w^{-1}\alpha}(k), \quad \forall (x_{\alpha})_{\alpha \in \Phi^+} \in \prod_{\alpha \in \Phi^+} N_{\alpha}(h_1).$$

Assume by contradiction that $w \neq \text{id}$. Let α_0 be a positive root such that the root $w^{-1} \cdot \alpha_0$ is negative. Then, the α_0 -coordinate of $b \left(\prod_{\alpha \in \Phi^+} \theta_{\alpha}(x_{\alpha}) \right) b^{-1}$ equals zero, for all $(x_{\alpha})_{\alpha \in \Phi^+} \in \prod_{\alpha \in \Phi^+} N_{\alpha}(h_1)$. Since the linear algebraic k -group \mathbf{B} normalizes the subgroup \mathbf{U}^+ , the conjugacy map defined from $b \in \mathbf{B}(k)$ induces a scheme automorphism ψ_b on \mathbf{U}^+ . Moreover, since \mathbf{U}^+ has a parametrization (as k -variety) $\prod_{\beta \in \Phi^+} \theta_{\beta} : \prod_{\beta \in \Phi^+} \mathbb{G}_{a,k} \rightarrow \mathbf{U}^+$, the automorphism ψ_b induces an automorphism ϕ_b on $\prod_{\beta \in \Phi^+} \mathbb{G}_{a,k}$. Let ϕ_b^* be the automorphism ring $\phi_b^* : \mathbb{F}[X_{\beta} | \beta \in \Phi^+] \rightarrow \mathbb{F}[X_{\beta} | \beta \in \Phi^+]$ induced by ϕ_b . For each $\alpha \in \Phi^+$, let us denote by P_{α} the polynomial in $\text{Card}(\Phi^+)$ indeterminates defined as the image of X_{α} by the ring automorphism ϕ_b^* . This polynomial

depends on b and on the ordering of factors of $\beta \in \Phi^+$. Since each X_α is nonzero, we get that each polynomial P_α is nonzero. Moreover, by definition of P_α , we have that

$$b \left(\prod_{\alpha \in \Phi^+} \theta_\alpha(z_\alpha) \right) b^{-1} = \prod_{\alpha \in \Phi^+} \theta_\alpha \left(P_\alpha((z_\alpha)_{\alpha \in \Phi^+}) \right), \quad \forall (z_\alpha)_{\alpha \in \Phi^+} \in \prod_{\alpha \in \Phi^+} \mathbb{G}_\alpha.$$

Thus $P_{\alpha_0}((x_\alpha)_{\alpha \in \Phi^+}) = 0$, for all $(x_\alpha)_{\alpha \in \Phi^+} \in \prod_{\alpha \in \Phi^+} N_\alpha(h_1)$. In other word, we get that P_{α_0} vanishes on the product $\prod_{\alpha \in \Phi^+} N_\alpha(h_1)$, where any $N_\alpha(h_1)$ is infinite (c.f. Lemma 5.3). Since $P_{\alpha_0} \neq 0$, we get a contradiction. Therefore, we conclude that $w = \text{id}$, whence $\tau = h_2 h_1^{-1}$ belongs to $\mathbf{B}(k)$. \square

The following result directly proves Statement (1) and (2) in Theorem 3.1.

Theorem 5.5. *Let G be an \mathcal{S} -arithmetic subgroup of $\mathbf{G}(k)$. For each $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, let $h_D \in \mathbf{G}(k)$ be an arbitrary element such that $h_D \cdot \partial_\infty D = \partial_\infty D_0$. Let us write $G_D = \text{Stab}_G(\partial_\infty D) = h_D^{-1} \mathbf{B}(k) h_D \cap G$, and $U(G_D) = h_D^{-1} \mathbf{U}^+(k) h_D \cap G$. Then:*

- (1) $U(G_D)$ is the subgroup of all the unipotent elements in G_D , and
- (2) $\mathfrak{U} := \{U(G_D) : \partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}\}$ is the set of all the maximal unipotent subgroups of G .

Proof. Let u be a unipotent element of G_D . Since u belongs to $h_D^{-1} \mathbf{B}(k) h_D$, there exists a unique pair $(\tilde{t}, \tilde{u}) \in h_D^{-1} \mathbf{T}(k) h_D \times h_D^{-1} \mathbf{U}^+(k) h_D$ such that $u = \tilde{t} \cdot \tilde{u}$, since a Borel subgroup is a semi-direct product over the ground field k [Bor91, 21.13]. Moreover, since u is unipotent $\tilde{t} = \text{id}$, whence $u \in h_D^{-1} \mathbf{U}^+(k) h_D$. Thus $u \in U(G_D)$. Hence, we conclude that $U(G_D)$ is the subgroup of all the unipotent elements in G_D . This proves Statement (1).

Let $U \subseteq G$ be a unipotent subgroup containing $U(G_D)$. It follows from Lemma 5.2 that U is contained in $h^{-1} \mathbf{U}^+(k) h \cap G$, for some $h \in \mathbf{G}(k)$. Hence

$$h_D^{-1} \mathbf{U}^+(k) h_D \cap G = U(G_D) \subseteq U \subseteq h^{-1} \mathbf{U}^+(k) h \cap G.$$

Then, it follows from Lemma 5.4 that the preceding inclusion becomes the equality

$$U(G_D) = U = h^{-1} \mathbf{U}^+(k) h \cap G.$$

We conclude that $U(G_D)$ is a maximal unipotent subgroup of G .

Let U be a maximal unipotent subgroup of G . It follows from Lemma 5.2 that U is contained in $h^{-1} \mathbf{U}^+(k) h \cap G$, for some $h \in \mathbf{G}(k)$. We set $\partial_\infty D = h^{-1} \cdot \partial_\infty D_0$, so that $U(G_D)$ equals $h^{-1} \mathbf{U}^+(k) h \cap G$. In particular, we get $U \subseteq U(G_D)$. Moreover, since U is maximal, we conclude $U = U(G_D)$. Thus, Statement (2) follows. \square

End of the proof of Theorem 3.1. It remains to prove Statement (3) of Theorem 3.1. Let $\mathfrak{U}/G = \{U(G_\sigma) : \sigma \in \Sigma\}$ as in Theorem 3.1. Let $U(G_D) = h_D^{-1} \mathbf{U}^+(k) h_D \cap G$ be an element of \mathfrak{U} . Then, by definition of the set $\{\partial_\infty D_\sigma\}_{\sigma \in \Sigma}$, there exists $\sigma \in \Sigma$ such that $\partial_\infty D = g \cdot \partial_\infty D_\sigma$, for some $g \in G$. Therefore $U(G_D) = g U(G_\sigma) g^{-1}$. This proves that each G -conjugacy class in \mathfrak{U} contains an element of \mathfrak{U}/G .

Conversely, let $\sigma_1, \sigma_2 \in \Sigma$ such that there exists an element $g \in G$ satisfying $g U(G_{\sigma_1}) g^{-1} = U(G_{\sigma_2})$. Let us write $D = g \cdot D_{\sigma_1}$. Let h be an element of $\mathbf{G}(k)$ such that $\partial_\infty D = h^{-1} \cdot \partial_\infty D_0$. Then $U(G_D) = h^{-1} \mathbf{U}^+(k) h \cap G$. Since $g h_{D_1}^{-1} \cdot \partial_\infty D_0 = \partial_\infty D = h^{-1} \cdot \partial_\infty D_0$, we get $h g h_{D_1}^{-1} \in \text{Stab}_{\mathbf{G}(k)}(\partial_\infty D_0)$. Thus, it follows from Lemma 2.1 that $h g h_{D_1}^{-1}$ belongs to $\mathbf{B}(k)$. Since $\mathbf{B}(k)$ normalizes $\mathbf{U}^+(k)$, we get:

$$(3) \quad U(G_D) = h^{-1} \mathbf{U}^+(k) h \cap G = (g h_{D_1}^{-1}) \mathbf{U}^+(k) (h_{D_1} g^{-1}) \cap G.$$

Moreover, since g belongs to G , we have $g^{-1} G g = G$. Thus, Equality (3) becomes

$$U(G_D) = g \left(h_{D_1}^{-1} \mathbf{U}^+(k) h_{D_1} \cap G \right) g^{-1} = g U(G_{\sigma_1}) g^{-1} = U(G_{\sigma_2}).$$

Therefore, Lemma 5.4 applied to $U(G_D) = U(G_{\sigma_2})$ implies that $h_{D_2} h^{-1} \in \mathbf{B}(k)$, whence we deduce that $\partial_\infty D = h^{-1} \cdot \partial_\infty D_0 = h_{D_2}^{-1} \cdot \partial_\infty D_0 = \partial_\infty D_{\sigma_2}$. In other words, we conclude

$g \cdot \partial_\infty D_{\sigma_1} = \partial_\infty D_{\sigma_2}$. This implies that $\sigma_1 = \sigma_2$, by definition of the set of representatives $\{\partial_\infty D_\sigma\}_{\sigma \in \Sigma}$. Thus, Statement (3) in Theorem 3.1 follows. \square

6. $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -CONJUGACY CLASSES OF BOREL k -SUBGROUPS

Here we prove Theorem 3.2. By definition of $\partial_\infty^k X_{\mathcal{S}}$, for any $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, there exists $h \in \mathbf{G}(k)$ such that $\partial_\infty D = h^{-1} \cdot \partial_\infty D_0$. Since $\text{Stab}_{\mathbf{G}(k)}(\partial_\infty D_0) = \mathbf{B}(k)$, we have that $\partial_\infty^k X_{\mathcal{S}}$ is in bijection with the Borel variety $\mathbf{G}(k)/\mathbf{B}(k)$. In particular, the set of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbits in $\partial_\infty^k X_{\mathcal{S}}$ is in bijection with the double quotient $\mathbf{G}(\mathcal{O}_{\mathcal{S}}) \backslash \mathbf{G}(k)/\mathbf{B}(k)$.

We start this section by realizing the double quotient $\mathbf{G}(\mathcal{O}_{\mathcal{S}}) \backslash \mathbf{G}(k)/\mathbf{B}(k)$ as the kernel of an homomorphism between two Étale cohomology groups. This result is valid even when \mathbf{G}_k is a quasi-split reductive k -group.

Proposition 6.1. *Let $\phi : H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G})$ be the map defined from the natural exact sequence $1 \rightarrow \mathbf{T} \rightarrow \mathbf{G} \rightarrow \mathbf{G}/\mathbf{T} \rightarrow 1$. There exists a bijective map from $\mathbf{G}(\mathcal{O}_{\mathcal{S}}) \backslash \mathbf{G}(k)/\mathbf{B}(k)$ to $\ker(\phi)$.*

Proof. Firstly, we claim that $(\mathbf{G}/\mathbf{B})(\mathcal{O}_{\mathcal{S}}) = (\mathbf{G}/\mathbf{B})(k)$ by using a classical argument via the valuative criterion for properness and patching (c.f. [Liu02, Prop. 1.6, § 4]). Indeed, set $x \in (\mathbf{G}/\mathbf{B})(k)$. By the valuative criterion of properness, for any prime ideal P' of $\mathcal{O}_{\mathcal{S}}$ there exists a unique element $x_{P'} \in (\mathbf{G}/\mathbf{B})((\mathcal{O}_{\mathcal{S}})_{P'})$ such that $x = \text{Res}(x_{P'})$. Since the functor of points \mathfrak{h}_V of $V = \mathbf{G}/\mathbf{B}$ is faithfully flat, we can suppose that $x_{P'} \in V((\mathcal{O}_{\mathcal{S}})_{f_{P'}})$ where $f_{P'} \notin P'$. Moreover, $\text{Spec}(\mathcal{O}_{\mathcal{S}})$ can be covered by a finite set $\left\{ \text{Spec}((\mathcal{O}_{\mathcal{S}})_{f_{P'_i}}) \right\}_{i=1}^n$. Hence, by a patching argument, we find $\bar{x} \in V(\mathcal{O}_{\mathcal{S}})$ such that $x_{P'_i} = \text{Res}(\bar{x})$, and then $x = \text{Res}(\bar{x})$. This element is unique by local considerations. Thus, we conclude the claim.

Now, let us consider the exact sequence of algebraic varieties

$$1 \rightarrow \mathbf{B} \xrightarrow{\iota} \mathbf{G} \xrightarrow{\pi} \mathbf{G}/\mathbf{B} \rightarrow 1.$$

It follows from [DG70a, § 4, 4.6] that there exists a long exact sequence

$$1 \rightarrow \mathbf{B}(\mathcal{O}_{\mathcal{S}}) \rightarrow \mathbf{G}(\mathcal{O}_{\mathcal{S}}) \rightarrow (\mathbf{G}/\mathbf{B})(\mathcal{O}_{\mathcal{S}}) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{B}) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G}).$$

Moreover, it follows from [DG70b, Exp. XXVI, Cor. 2.3] that

$$H_{\text{fppf}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{B}) = H_{\text{fppf}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}).$$

But, since \mathbf{B} and \mathbf{T} are both smooth over $\text{Spec}(\mathcal{O}_{\mathcal{S}})$, we have

$$\begin{aligned} H_{\text{fppf}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{B}) &= H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{B}), \\ H_{\text{fppf}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) &= H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}). \end{aligned}$$

This implies that $H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{B}) = H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T})$. Thus, there exists a long exact sequence

$$1 \rightarrow \mathbf{B}(\mathcal{O}_{\mathcal{S}}) \rightarrow \mathbf{G}(\mathcal{O}_{\mathcal{S}}) \rightarrow (\mathbf{G}/\mathbf{B})(k) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G}).$$

This implies that

$$\ker(H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G})) \cong \mathbf{G}(\mathcal{O}_{\mathcal{S}}) \backslash (\mathbf{G}/\mathbf{B})(k).$$

According to [BT65, 4.13(a)], the quotient $(\mathbf{G}/\mathbf{B})(k)$ equals $\mathbf{G}(k)/\mathbf{B}(k)$, whence the result follows. \square

Corollary 6.2. *Assume that \mathbf{G}_k is a split simply connected semisimple k -group. Denote by \mathfrak{t} its semisimple rank which is the dimension of its maximal split torus \mathbf{T} . Then, there is a one-to-one correspondence between $\mathbf{G}(\mathcal{O}_{\mathcal{S}}) \backslash \mathbf{G}(k)/\mathbf{B}(k)$ and $\text{Pic}(\mathcal{O}_{\mathcal{S}})^{\mathfrak{t}}$.*

Proof. Since \mathbf{T} is split over \mathbb{Z} , we have that $\mathbf{T} \cong \mathbb{G}_{m,\mathbb{Z}}^{\mathbf{t}}$. It follows from Hilbert's Theorem 90 (c.f. [Mil80, Ch. III, Prop. 4.9]) that

$$H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbb{G}_m) = H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbb{G}_m) \cong \text{Pic}(\mathcal{O}_{\mathcal{S}}).$$

Thus, we get that

$$H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) = H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \cong \text{Pic}(\mathcal{O}_{\mathcal{S}})^{\mathbf{t}}.$$

Since $\phi\left(H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T})\right) \subseteq H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G})$ and $H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T})$ equals $H_{\text{ét}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T})$, we get that

$$\ker(\phi) = \ker\left(H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \rightarrow H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G})\right).$$

Moreover, since \mathbf{G}_k is a simply connected semisimple k -group scheme and $\mathcal{O}_{\mathcal{S}}$ is a Dedekind domain, it follows from [Har67, Th. 2.2.1 and Cor. 2.3.2] that $H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{G})$ is trivial, for any integral model \mathbf{G} of \mathbf{G}_k . We conclude that

$$\ker(\phi) = H_{\text{Zar}}^1(\text{Spec}(\mathcal{O}_{\mathcal{S}}), \mathbf{T}) \cong \text{Pic}(\mathcal{O}_{\mathcal{S}})^{\mathbf{t}},$$

whence the result follows. \square

7. FIXING GROUPS OF GERMS AT INFINITY OF CHAMBERS

We want to decompose the semisimple group $G_D/U(G_D)$ as the product of a free \mathbb{Z} -module of finite rank and a ‘‘arithmetic bounded torus’’ T , which is a torsion group whenever \mathbb{F} is finite (c.f. Theorem 3.4 (2)). The group T can be described as the stabilizer of a filter.

The sector chambers of \mathbb{A}_0 with direction D_0 form a basis of a filter $\gamma(D_0)$, as in [BT72, 7.2.3]. The germ associated to this basis is called the germ of \mathbb{A}_0 with direction $D_0 = (D_{0,P})_{P \in \mathcal{S}}$. For each $P \in \mathcal{S}$, it induces a germ of $\mathbb{A}_{0,P}$ with direction $D_{0,P}$. Since the set of pointwise stabilizer subgroups is a directed set according to [BT72, 7.2.2], the union of the pointwise stabilizers over a basis of the filter $\gamma(D_{0,P})$ forms a group. This group is the pointwise stabilizer in $\mathbf{G}(k_P)$ of the filter $\gamma(D_{0,P})$, that is:

$$\hat{P}_{\gamma(D_{0,P})}^P := \bigcup_{x \in \mathbb{A}_{0,P}} \text{Fix}_{\mathbf{G}(k_P)}(Q(x_P, D_{0,P})).$$

We have

$$(4) \quad \hat{P}_{\gamma(D_{0,P})}^P = \mathbf{T}(\mathcal{O}_P)\mathbf{U}^+(k_P),$$

whenever \mathbf{G} is semisimple [BT72, 7.2.3] and split⁵. Since the family of pointwise stabilizers in $\mathbf{G}(k_P)$ form a directed set, also does the family of pointwise stabilizers in any subgroup of $\mathbf{G}(k_P)$.

Analogously, given an apartment \mathbb{A} and a vector chamber D , we define a filter $\gamma(D)$ given by the basis of sector chambers of \mathbb{A} with direction D . The pointwise stabilizer of $\gamma(D)$ is defined as the union of the pointwise stabilizers of sector chambers in \mathbb{A} with direction D .

Let $\partial_{\infty}D \in \partial_{\infty}^k X_{\mathcal{S}}$ and $h_D \in \mathbf{G}(k)$ such that $\partial_{\infty}D = h_D^{-1} \cdot \partial_{\infty}D_0$. Let \mathbb{A} be the apartment of $X_{\mathcal{S}}$ defined as

$$(5) \quad \mathbb{A} = h_D^{-1} \cdot \mathbb{A}_0.$$

We want to consider the pointwise stabilizer in G of the filter $\gamma(D) = (\gamma(D_P))_{P \in \mathcal{S}}$ generated by the germ of \mathbb{A} with direction D .

⁵The groups $H = \mathbf{T}(k_P)_b$, H^0 and $H^1 = \mathbf{T}(\mathcal{O}_P)$ defined in [BT84, 4.6.3] are the same since \mathbf{T} is a smooth connected integral model over \mathcal{O}_P of a maximal (split) torus as being defined over \mathbb{Z} (see [BT84, 4.6.32]).

For each $P \in \mathcal{S}$, the pointwise stabiliser of $\gamma(D_P)$ in G is

$$\bigcup_{x_P \in \mathbb{A}_P} \text{Fix}_G(Q(x_P, D_P)) = h_D^{-1} \left(\bigcup_{y_P \in \mathbb{A}_{0,P}} \text{Fix}_{\mathbf{G}(k_P)}(Q(y_P, D_{0,P})) \cap h_D G h_D^{-1} \right) h_D.$$

Thus, the pointwise stabiliser of $\gamma(D)$ in G is

$$G_{\gamma(D)} := \bigcap_{P \in \mathcal{S}} h_D^{-1} \left(\bigcup_{y_P \in \mathbb{A}_{0,P}} \text{Fix}_{\mathbf{G}(k_P)}(Q(y_P, D_{0,P})) \cap h_D G h_D^{-1} \right) h_D.$$

Since \mathcal{S} is finite,

$$G_{\gamma(D)} = h_D^{-1} \left(\bigcup_{y=(y_P)_{P \in \mathcal{S}} \in \mathbb{A}_0} \bigcap_{P \in \mathcal{S}} \text{Fix}_{\mathbf{G}(k_P)}(Q(y_P, D_{0,P})) \cap h_D G h_D^{-1} \right) h_D.$$

Moreover, since

$$\bigcap_{P \in \mathcal{S}} \left(\text{Fix}_{\mathbf{G}(k_P)}(Q(y_P, D_{0,P})) \cap h_D G h_D^{-1} \right) = \text{Fix}_{h_D G h_D^{-1}} \left(Q((y_P)_{P \in \mathcal{S}}, D_0) \right),$$

we deduce that

$$(6) \quad G_{\gamma(D)} = \bigcup_{y \in \mathbb{A}} \text{Fix}_G(Q(y, D)).$$

For any $y \in \mathbb{A}$, the pointwise stabilizer $\text{Fix}_G(Q(y, D))$ is contained in $\text{Stab}_G(\partial_\infty D)$. Thus, the group $G_{\gamma(D)}$ is contained in $\text{Stab}_G(\partial_\infty D)$, which equals $h_D^{-1} \mathbf{B}(k) h_D \cap G$. By the writing in semi-direct product of the Borel subgroup, for any $g \in G_{\gamma(D)}$, there exists a unique pair $(t, u) \in (h_D^{-1} \mathbf{T}(k) h_D) \times (h_D^{-1} \mathbf{U}^+(k) h_D)$ such that $g = t \cdot u$. Then, we define

$$(7) \quad T(G_{\gamma(D)}) := \{t \in h_D^{-1} \mathbf{T}(k) h_D : \exists u \in h_D^{-1} \mathbf{U}^+(k) h_D, t \cdot u \in G_{\gamma(D)}\},$$

and

$$(8) \quad U(G_{\gamma(D)}) := h_D^{-1} \mathbf{U}^+(k) h_D \cap G_{\gamma(D)}.$$

The main goal of this section is to describe the groups $T(G_{\gamma(D)})$ and $U(G_{\gamma(D)})$ in order to decompose $G_{\gamma(D)}$ as in Equation (4).

Lemma 7.1. *The map $f : G_{\gamma(D)} \rightarrow h_D^{-1} \mathbf{T}(k) h_D$ defined by $f(tu) = t$ is a group homomorphism and it induces a group isomorphism $G_{\gamma(D)}/U(G_{\gamma(D)}) \cong T(G_{\gamma(D)})$.*

Proof. Let $g_1 = t_1 \cdot u_1$ and $g_2 = t_2 \cdot u_2$, as above. Since $h_D^{-1} \mathbf{T}(k) h_D$ normalizes $h_D^{-1} \mathbf{U}^+(k) h_D$, we have $g_1 g_2^{-1} = t_1 u_1 u_2^{-1} t_2^{-1} = t_1 t_2^{-1} \tilde{u}_1 \tilde{u}_2$, for some $\tilde{u}_1, \tilde{u}_2 \in h_D^{-1} \mathbf{U}^+(k) h_D$. Thus, by uniqueness of the writing, we deduce that f is a group homomorphism. Note that, by definition, $\ker(f) = h_D^{-1} \mathbf{U}^+(k) h_D \cap G_{\gamma(D)} = U(G_{\gamma(D)})$, and $\text{Im}(f) = T(G_{\gamma(D)})$. Thus, the result follows. \square

The following lemmas explain why any unipotent element contained in $U(G_D)$ belongs to $G_{\gamma(D)}$.

Lemma 7.2. *For any $u \in \mathbf{U}^+(k)$ there exists a k -sector chamber $Q_P = Q(y_P, D_{0,P})$ in $\mathbb{A}_{0,P}$ such that $u \in \text{Fix}_{\mathbf{G}(k)}(Q_P)$.*

Proof. Let us write $u \in \mathbf{U}^+(k)$ as $u = \prod_{\alpha \in \Phi^+} \theta_\alpha(x_\alpha)$, for some $(x_\alpha)_{\alpha \in \Phi^+} \in \prod_{\alpha \in \Phi^+} k$. Let v be a vertex in the standard apartment $\mathbb{A}_{0,P}$ of $X(\mathbf{G}, k, P)$ defined in § 2.1. According to [BT72, 6.4.9], the group $\prod_{\alpha \in \Phi} \theta_\alpha \left(\pi_P^{-\alpha(v)} \mathcal{O}_P \right)$ is contained in $\text{Stab}_{\mathbf{G}(k_P)}(v)$. In particular, the unipotent element u is contained in $\text{Stab}_{\mathbf{G}(k_P)}(v)$, for any vertex $v \in \mathbb{A}_{0,P}$ such that $\alpha(v) \geq -\nu_P(x_\alpha)$. This implies that u fixes the complex

$$E_P := \{z \in \mathbb{A}_{0,P} : \alpha(z) > -\nu_P(x_\alpha), \forall \alpha \in \Phi^+\}.$$

Let $y_P \in E_P$. Then $Q(y_P, \partial_\infty D_{0,P})$ is contained in E_P . Thus, we conclude that u fixes $Q(y_P, \partial_\infty D_{0,P})$, as wished. \square

Lemma 7.3. *Let $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$. For any $g \in h_D^{-1} \mathbf{U}^+(k) h_D$ there exists a k -sector chamber $Q = Q(y, D)$, with $y \in \mathbb{A} = h_D^{-1} \cdot \mathbb{A}_0$, such that $g \in \text{Fix}_{\mathbf{G}(k)}(Q)$.*

Proof. Let $u = h_D g h_D^{-1} \in \mathbf{U}^+(k)$. We have to show that there exists $Q' = Q(y', D_0)$ with $y' \in \mathbb{A}_0$ such that $u \in \text{Fix}_{\mathbf{G}(k)}(Q')$. Indeed, it follows from Lemma 7.2 that there exists a sector chamber $Q_P = Q(y_P, D_{0,P})$ in $\mathbb{A}_{0,P}$ such that $u \in \text{Fix}_{\mathbf{G}(k)}(Q_P)$. Let $Q' = \prod_{P \in \mathcal{S}} Q_P$ be a k -sector chamber of $X_{\mathcal{S}}$, which is contained in \mathbb{A}_0 by definition of \mathbb{A}_0 (c.f. § 2.1). Then, the direction of Q' is D_0 . Moreover, since for any point $z = (z_P)_{P \in \mathcal{S}}$ we have $\text{Stab}_{\mathbf{G}(k)}(z) = \bigcap_{P \in \mathcal{S}} \text{Stab}_{\mathbf{G}(k)}(z_P)$, we get

$$\text{Fix}_{\mathbf{G}(k)}(Q') = \bigcap_{P \in \mathcal{S}} \text{Fix}_{\mathbf{G}(k)}(Q_P),$$

whence we conclude that u belongs to $\text{Fix}_{\mathbf{G}(k)}(Q')$. Therefore, if we set $Q = h_D^{-1} \cdot Q'$, then we have $Q = Q(h_D^{-1} \cdot y', h_D^{-1} \cdot D_0) = Q(y, D)$, for $x = h_D^{-1} \cdot y' \in \mathbb{A}$, and $g \in \text{Fix}_{\mathbf{G}(k)}(Q)$. \square

Proposition 7.4. *For any $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, we have*

$$U(G_{\gamma(D)}) = h_D^{-1} \mathbf{U}^+(k) h_D \cap G = U(G_D).$$

Moreover, $U(G_{\gamma(D)})$ is the subgroup of all unipotent elements in $G_{\gamma(D)}$.

Proof. On one hand, by definition, the group $U(G_{\gamma(D)})$ is contained in $h_D^{-1} \mathbf{U}^+(k) h_D \cap G$. On the other hand, let $u \in h_D^{-1} \mathbf{U}^+(k) h_D \cap G$ be an arbitrary element. It follows from Lemma 7.3 that u fixes a k -sector chamber $Q(y, D)$, where $y \in \mathbb{A}$. Thus, it follows from Equation (6) that g belongs to $G_{\gamma(D)}$. Hence, we conclude that u belongs to $h_D^{-1} \mathbf{U}^+(k) h_D \cap G_{\gamma(D)}$, which equals $U(G_{\gamma(D)})$.

Let u be a unipotent element of $G_{\gamma(D)}$. Since u is a unipotent element of the Borel subgroup $h_D^{-1} \mathbf{B}(k) h_D$, it is contained in its unipotent radical $h_D^{-1} \mathbf{U}^+(k) h_D$ (since it is defined over k [Bor91, 15.4]). Thus $u \in U(G_{\gamma(D)})$. Since $U(G_{\gamma(D)})$ is unipotent, we conclude that $U(G_{\gamma(D)})$ is the subgroup of all the unipotent elements in $G_{\gamma(D)}$. \square

Proposition 7.5. *Assume that $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$. Then, for any $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, we have $T(G_{\gamma(D)}) \subseteq h_D^{-1} \mathbf{T}(\mathbb{F}) h_D$.*

Proof. Let $t \in T(G_{\gamma(D)})$ be an arbitrary element. By definition of $T(G_{\gamma(D)})$, there is $u \in h_D^{-1} \mathbf{U}^+(k) h_D$ such that $tu \in G_{\gamma(D)}$. Moreover, by definition of $G_{\gamma(D)}$, the element tu pointwise stabilizes a sector chamber $Q = Q(y, D)$ of \mathbb{A} . In particular, tu stabilizes a vertex $z \in Q$. Let us write $z = (z_P)_{P \in \mathcal{S}}$, where $z_P \in X(\mathbf{G}, k, P)$. Since $\mathbf{G}(k)$ acts diagonally on $X_{\mathcal{S}}$, the element tu stabilizes each coordinate z_P of z .

Consider the element $b = h_D (tu) h_D^{-1} \in \mathbf{G}(k)$. Since $t \in h_D^{-1} \mathbf{T}(k) h_D$ and $u \in h_D^{-1} \mathbf{U}^+(k) h_D$, we have that $b \in \mathbf{B}(k)$. Moreover, since tu stabilizes each z_P , the element b belongs to the parahoric subgroup $\hat{P}_{z'_P} = \text{Stab}_{\mathbf{G}(k_P)}(z') \subseteq \mathbf{G}(k_P)$, where $z'_P = h_D \cdot z_P$. In other words, if we write $z' = h_D \cdot z$, then b belongs to $\bigcap_{P \in \mathcal{S}} \left(\hat{P}_{z'_P} \cap \mathbf{B}(k) \right)$. In the Borel subgroup, write $b = sv$ with $s \in \mathbf{T}(k)$ and $v \in \mathbf{U}^+(k)$. By uniqueness of the pair (s, v) , we have $s = h_D t h_D^{-1}$ and $v = h_D u h_D^{-1}$.

Recall that there is a canonical isomorphism $X^*(\mathbf{T}) = X^*(\mathbf{B})$ since $\mathcal{R}_u(\mathbf{B}) = \mathbf{U}^+$ and $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}^+$. In particular, for any $\chi \in X^*(\mathbf{B})$, we have that $\chi(b) = \chi(s)$. According to [BT72, § 8.1], the parahoric subgroups of $\mathbf{G}(k)$ are bounded⁶. Hence, so is $\hat{P}_{z'_P} \cap \mathbf{B}(k) \subset \mathbf{G}(k_P)$. Since \mathbf{G}_k is semisimple, for each $P \in \mathcal{S}$, the set of

⁶In fact, the parahoric subgroups are compact whenever \mathbb{F} is finite, since $\mathbf{G}(k)$ acts properly on its building \mathcal{X}_k whenever \mathbf{G} is semisimple and k is locally compact.

values $\nu_P\left(\chi\left(\widehat{P}_{z'_P} \cap \mathbf{B}(k)\right)\right)$ is lower bounded⁷ according to [BT84, 4.2.19]. Thus, $\chi\left(\widehat{P}_{z'_P} \cap \mathbf{B}(k)\right)$ is a subgroup of \mathcal{O}_P^\times . Hence, we conclude

$$\chi\left(\widehat{P}_{z'} \cap \mathbf{B}(k)\right) = \chi\left(\bigcap_{P \in \mathcal{S}} \widehat{P}_{z'_P} \cap \mathbf{B}(k)\right) \subseteq \bigcap_{P \in \mathcal{S}} \chi\left(\widehat{P}_{z'_P} \cap \mathbf{B}(k)\right) \subseteq \bigcap_{P \in \mathcal{S}} \mathcal{O}_P^\times.$$

Up to conjugate by an element in $\mathrm{GL}_n(k)$, there is a faithful linear representation $\rho : \mathbf{G} \rightarrow \mathrm{GL}_{n,k}$ embedding \mathbf{T} in the maximal torus of diagonal matrices \mathbf{D}_n and embedding \mathbf{B} in the standard Borel subgroup \mathbf{B}_n of $\mathrm{GL}_{n,k}$ consisting in upper triangular matrices. Thus, the restriction of ρ to the injective group homomorphism $\mathbf{T} \rightarrow \mathbf{D}_n$ induces a surjective homomorphism $\rho^* : X^*(\mathbf{D}_n) \rightarrow X^*(\mathbf{T})$ (c.f. [BT65, 1.2]). For any character $\chi' \in X^*(\mathbf{B}_n) = X^*(\mathbf{D}_n)$, we have $\rho^*(\chi') \in X^*(\mathbf{T}) = X^*(\mathbf{B})$, whence $\chi'(\rho(b)) \in \bigcap_{P \in \mathcal{S}} \mathcal{O}_P^\times$. Thus, the eigenvalues of $\rho(b)$ belong to $\bigcap_{P \in \mathcal{S}} \mathcal{O}_P^\times$.

Since $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, the element $b \in h_D \mathbf{G}(\mathcal{O}_{\mathcal{S}}) h_D^{-1}$. Then, $\rho(b)$ is conjugate in $\mathrm{GL}_n(k)$ to a matrix in $\mathrm{GL}_n(\mathcal{O}_{\mathcal{S}})$. Thus, the characteristic polynomial $P_{\rho(b)}$ of $\rho(b)$ has coefficients in $\mathcal{O}_{\mathcal{S}}$. Since the eigenvalues are in $\bigcap_{P \in \mathcal{S}} \mathcal{O}_P$, we deduce that $P_{\rho(b)}$ has coefficients in $\mathcal{O}_{\mathcal{S}} \cap \bigcap_{P \in \mathcal{S}} \mathcal{O}_P = \mathbb{F}$. Since \mathbb{F} is assumed to be algebraically closed in k (c.f. § 1) and $P_{\rho(b)}$ is split over k , we deduce that the eigenvalues of $\rho(b)$ are in \mathbb{F}^\times , whence $\chi'(\rho(b)) \in \mathbb{F}^\times$ for any $\chi' \in X^*(\mathbf{B}_n)$.

Since ρ^* is surjective, we deduce that $\chi(s) = \chi(b) \in \mathbb{F}^\times$ for all $\chi \in X^*(\mathbf{B})$. Thus, since \mathbf{T} is split and defined over \mathbb{Z} , we deduce from the perfect dual pairing (c.f. [Bor91, 8.6]) that $s \in \mathbf{T}(\mathbb{F})$. Hence $t = h_D^{-1} s h_D \in h_D^{-1} \mathbf{T}(\mathbb{F}) h_D$. \square

Corollary 7.6. *For any \mathcal{S} -arithmetic subgroup G of $\mathbf{G}(k)$, the group $T(G_{\gamma(D)})$ is commensurable with a subgroup of $h_D^{-1} \mathbf{T}(\mathbb{F}) h_D$.*

Proof. Since G is commensurable with $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, there exists a subgroup G^\sharp of $G \cap \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ such that $[G : G^\sharp]$ and $[\mathbf{G}(\mathcal{O}_{\mathcal{S}}) : G^\sharp]$ are finite. Since $G^\sharp \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, it follows from Proposition 7.5 that $T(G_{\gamma(D)}^\sharp) \subseteq h_D^{-1} \mathbf{T}(\mathbb{F}) h_D$. Thus, in order to prove the result, we have to check that $T(G_{\gamma(D)}^\sharp)$ has finite index in $T(G_{\gamma(D)})$. It follows from Equation (6) that $G_{\gamma(D)}$ equals

$$\bigcup_{y \in \mathbb{A}} \mathrm{Fix}_G(Q(y, D)) = \bigcup_{y \in \mathbb{A}} (G \cap \mathrm{Fix}_{\mathbf{G}(k)}(Q(y, D))) = G \cap \left(\bigcup_{y \in \mathbb{A}} \mathrm{Fix}_{\mathbf{G}(k)}(Q(y, D)) \right)$$

In other words, the group $G_{\gamma(D)}$ is the intersection of G with the group $\mathbf{G}(k)_{\gamma(D)} := \bigcup_{y \in \mathbb{A}} \mathrm{Fix}_{\mathbf{G}(k)}(Q(y, D))$. An analogous argument shows that $G_{\gamma(D)}^\sharp$ is the intersection of G^\sharp with $\mathbf{G}(k)_{\gamma(D)}$. Since, for any $H \subseteq \mathbf{G}(k)$ we have $[G \cap H : G^\sharp \cap H] \leq [G : G^\sharp]$, we get, by taking $H = \mathbf{G}(k)_{\gamma(D)}$, that $G_{\gamma(D)}^\sharp$ has finite index in $G_{\gamma(D)}$. Moreover, since $G_{\gamma(D)}^\sharp \subseteq G_{\gamma(D)}$, it follows from Equation (8) that $U(G_{\gamma(D)}^\sharp)$ equals

$$h_D^{-1} \mathbf{U}^+(k) h_D \cap G_{\gamma(D)}^\sharp = (h_D^{-1} \mathbf{U}^+(k) h_D \cap G_{\gamma(D)}) \cap G_{\gamma(D)}^\sharp = U(G_{\gamma(D)}) \cap G_{\gamma(D)}^\sharp.$$

In other words, we have $U(G_{\gamma(D)}^\sharp) = U(G_{\gamma(D)}) \cap G_{\gamma(D)}^\sharp$. Since, for any normal subgroup N of $G_{\gamma(D)}$ we have $[G_{\gamma(D)}/N : G_{\gamma(D)}^\sharp/(N \cap G_{\gamma(D)}^\sharp)] \leq [G_{\gamma(D)} : G_{\gamma(D)}^\sharp]$, we obtain, by taking $N = U(G_{\gamma(D)})$, that $G_{\gamma(D)}^\sharp/U(G_{\gamma(D)}^\sharp)$ has finite index in $G_{\gamma(D)}/U(G_{\gamma(D)})$. Thus, the result follows from Lemma 7.1. \square

In the context where $\mathcal{S} = \{P\}$ and $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, the G -stabilizer of the germ of \mathbb{A} with direction D is the G -stabilizer of $\partial_\infty D$, as follows:

⁷We are in the case where $\mathbf{G} = \mathbf{G}^1$ since \mathbf{G}_k is assumed to be semisimple (c.f. [BT84, 4.2.16]).

Proposition 7.7. *Assume that $\mathcal{S} = \{P\}$ and $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$. Then, for any $\partial_{\infty}D \in \partial_{\infty}^k X_{\mathcal{S}}$, we have*

$$G_{\gamma(D)} = h_D^{-1} \mathbf{B}(k) h_D \cap G = G_D.$$

Proof. Let $\partial_{\infty}D$ be an arbitrary element in $\partial_{\infty}^k X_{\mathcal{S}}$. Note that, since $G \subseteq \mathbf{G}(\mathcal{O}_{\{P\}})$, if $\mathbf{G}(\mathcal{O}_{\{P\}})_{\gamma(D)} = h_D^{-1} \mathbf{B}(k) h_D \cap \mathbf{G}(\mathcal{O}_{\{P\}}) = \mathbf{G}(\mathcal{O}_{\{P\}})_D$, then $G_{\gamma(D)} = h_D^{-1} \mathbf{B}(k) h_D \cap G = G_D$. Thus, in that follows, we assume that $G = \mathbf{G}(\mathcal{O}_{\{P\}})$.

For any $y \in \mathbb{A}$, the pointwise stabilizer $\text{Fix}_{\mathbf{G}(\mathcal{O}_{\{P\}})}(Q(y, D))$ is contained in $\text{Stab}_{\mathbf{G}(\mathcal{O}_{\{P\}})}(\partial_{\infty}D)$. Thus, the group $\mathbf{G}(\mathcal{O}_{\{P\}})_{\gamma(D)}$ is contained in $\mathbf{G}(\mathcal{O}_{\{P\}})_D$, which equals $h_D^{-1} \mathbf{B}(k) h_D \cap \mathbf{G}(\mathcal{O}_{\{P\}})$. Hence, it remains to prove $\mathbf{G}(\mathcal{O}_{\{P\}})_D \subseteq \mathbf{G}(\mathcal{O}_{\{P\}})_{\gamma(D)}$. Let $Q = h_D^{-1} \cdot Q(x_0, D_0)$ be a sector chamber in \mathbb{A} with direction D . In particular, the sector chamber Q equals $Q(x, D)$, for certain $x \in \mathbb{A}$. It follows from [BL23, Theorem 2.2] that there exists a sector chamber $Q' = Q(x', D) \subseteq Q$ which embeds in the quotient $\mathbf{G}(\mathcal{O}_{\{P\}}) \backslash \mathcal{X}(\mathbf{G}, k, P)$. Let $g \in \mathbf{G}(\mathcal{O}_{\{P\}})_D$. The complex $Q' \cap g \cdot Q'$ is the intersection of two sector chamber with same direction. In particular, it follows from [AB08, 11.77] that $Q' \cap g \cdot Q'$ contains a sector chamber Q'' , which has the form $Q'' = Q(x'', D)$. Since $Q'' \subseteq Q'$, we have $Q'' \subseteq \mathbb{A}$, whence $x'' \in \mathbb{A}$. Let z be a point in Q'' . Then z belongs to $Q' \cap g \cdot Q'$, whence there exists $w \in Q'$ satisfying $z = g \cdot w$. Moreover, since g belongs to $\mathbf{G}(\mathcal{O}_{\{P\}})$ and since Q' does not have two points in the same $\mathbf{G}(\mathcal{O}_{\{P\}})$ -orbit, we have that $z = w$. Thus, we conclude that $g \in \text{Stab}_{\mathbf{G}(\mathcal{O}_{\{P\}})}(z)$, for all $z \in Q''$. Hence g belongs to $\mathbf{G}(\mathcal{O}_{\{P\}})_{\gamma(D)}$, whence the result follows. \square

8. COMPARISON BETWEEN THE STABILIZERS OF A VECTOR CHAMBER AND OF ITS GERM

As above, let $\partial_{\infty}D \in \partial_{\infty}^k X_{\mathcal{S}}$ and $h_D \in \mathbf{G}(k)$ be such that $h_D \cdot \partial_{\infty}D = \partial_{\infty}D_0$. Recall that $G_D = h_D^{-1} \mathbf{B}(k) h_D \cap G$ is the stabilizer in G of the vector chamber D , or equivalently of $\partial_{\infty}D$. We define a diagonalisable group by

$$(9) \quad T(G_D) = \{t \in h_D^{-1} \mathbf{T}(k) h_D : \exists u \in h_D^{-1} \mathbf{U}^+(k) h_D, tu \in G_D\}.$$

Using the writing in a semi-direct product of the Borel subgroup $h_D^{-1} \mathbf{B}(k) h_D$ containing G_D , any element $g \in G_D$ can be written uniquely $g = tu$ where $t \in h_D^{-1} \mathbf{T}(k) h_D$ and $u \in h_D^{-1} \mathbf{U}^+(k) h_D$. Thus, one can define a map:

$$(10) \quad \begin{aligned} f : G_D &\rightarrow h_D^{-1} \mathbf{T}(k) h_D \\ g = tu &\mapsto t \end{aligned}$$

which is, in fact, a group homomorphism.

Lemma 8.1. *The group homomorphism f induces an isomorphism $G_D/U(G_D) \cong T(G_D)$.*

Proof. The kernel of f is $U(G_D)$ and its image is $T(G_D)$. \square

Lemma 8.2. *Let $G_{\gamma(D)}$ be the group defined in Equation (6). Then, $G_{\gamma(D)}$ is a normal subgroup in G_D .*

Proof. Let $\tau \in G_D$ and let $g \in G_{\gamma(D)}$ be arbitrary elements. Then, by definition of $G_{\gamma(D)}$, the element g belongs to $\text{Fix}_G(Q)$, where Q is contained in \mathbb{A} . Since, by definition G_D equals $\text{Stab}_G(\partial_{\infty}D)$, we have $\tau \cdot \partial_{\infty}D = \partial_{\infty}D$. In particular, it follows from [AB08, 11.77] that the intersection $Q \cap \tau \cdot Q$ contains a sector chamber Q' . In particular, since $Q' \subseteq Q$, the direction of the sector chamber Q' is D and Q' is contained in \mathbb{A} . Since $\tau \in G$, the element $\tau g \tau^{-1}$ belongs to $\text{Fix}_G(\tau \cdot Q)$. But, since $Q' \subseteq \tau \cdot Q$, we have $\text{Fix}_G(\tau \cdot Q) \subseteq \text{Fix}_G(Q')$. Thus, we conclude that $\tau g \tau^{-1}$ belongs to $\text{Fix}_G(Q')$, where $Q' \subseteq \mathbb{A}$. This implies that $\tau g \tau^{-1} \in G_{\gamma(D)}$, whence the result follows. \square

It follows from Lemma 8.2 that the quotient groups $G_D/G_{\gamma(D)}$ and $T(G_D)/T(G_{\gamma(D)})$ make sense. The following proposition describe their structure. As in § 3, we denote by \mathbf{t} the rank of \mathbf{G} .

Proposition 8.3. *We have $G_D/G_{\gamma(D)} \cong T(G_D)/T(G_{\gamma(D)})$ is isomorphic to \mathbb{Z}^r , for some $r = r(\mathbf{G}, \mathcal{S}, D) \in \mathbb{Z}_{\geq 0}$ such that $r \leq \mathbf{t} \cdot \#\mathcal{S}$.*

Proof. At first we prove that $G_D/G_{\gamma(D)}$ and $T(G_D)/T(G_{\gamma(D)})$ are isomorphic. Indeed, we have $G_D/U(G_D) \cong T(G_D)$ according to Lemma 8.1 and that $G_{\gamma(D)}/U(G_{\gamma(D)}) \cong T(G_{\gamma(D)})$ according to Lemma 7.1. Then, since Proposition 7.4 shows that $U(G_{\gamma(D)}) = h_D^{-1}\mathbf{U}^+(k)h_D \cap G = U(G_D)$, we conclude $G_D/G_{\gamma(D)} \cong T(G_D)/T(G_{\gamma(D)})$.

Let $h \in \mathbf{G}(k)$ be such that $\partial_\infty D = h^{-1} \cdot \partial_\infty D_0$. Then, by Equality (4), we have that the pointwise stabilizer of $\gamma(D)$ in $\widehat{G}_\mathcal{S}$ is $\prod_{P \in \mathcal{S}} h\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)h^{-1}$. Thus, by diagonal action of G on x , the pointwise stabilizer $G_{\gamma(D)}$ in G of $\gamma(D)$ can be obtained as:

$$G \cap \bigcap_{P \in \mathcal{S}} \mathbf{G}(k) \cap (h\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)h^{-1}) = G \cap h \left(\bigcap_{P \in \mathcal{S}} \mathbf{G}(k) \cap (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)) \right) h^{-1}.$$

Since $\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P) \subset \mathbf{B}(k_P)$ and $\mathbf{G}(k) \cap \mathbf{B}(k_P) = \mathbf{B}(k)$, in $\mathbf{G}(k_P)$, we have that

$$\mathbf{G}(k) \cap (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)) = \mathbf{B}(k) \cap (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)).$$

Consider the quotient group

$$\Lambda_h := h\mathbf{B}(k)h^{-1}/h \left(\bigcap_{P \in \mathcal{S}} \mathbf{B}(k) \cap (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)) \right) h^{-1}$$

By restricting the quotient homomorphism $\pi_h : h\mathbf{B}(k)h^{-1} \rightarrow \Lambda_h$ to G_D , we deduce that $G_D/G_{\gamma(D)}$ is isomorphic to a subgroup of Λ_h .

Consider the diagonal group homomorphism

$$\varphi : \mathbf{B}(k) \rightarrow \prod_{P \in \mathcal{S}} \mathbf{B}(k_P) / (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)).$$

For any $P \in \mathcal{S}$, we have that $\mathbf{B}(k_P) / (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P)) \cong \mathbf{T}(k_P) / \mathbf{T}(\mathcal{O}_P) \cong \mathbb{Z}^{\mathbf{t}}$ where \mathbf{t} is the dimension of \mathbf{T} [Lan96, 1.3(ii)]. Since $\ker \varphi = \bigcap_{P \in \mathcal{S}} \mathbf{B}(k) \cap (\mathbf{T}(\mathcal{O}_P) \cdot \mathbf{U}^+(k_P))$, we deduce that Λ_h is isomorphic to a subgroup of $\text{im } \varphi = (\mathbb{Z}^{\mathbf{t}})^{\mathcal{S}}$. Hence Λ_h is a finitely generated free \mathbb{Z} -module, whence so is $G_D/G_{\gamma(D)}$ as a submodule of Λ_h . We denote by $r(\mathbf{G}, \mathcal{S}, D)$ its rank as free \mathbb{Z} -module, which is less or equal than $\mathbf{t} \cdot \#\mathcal{S}$ by construction. \square

End of the proof of Theorem 3.4. Statements (1) and (4) directly follows from Lemma 8.1 and Proposition 7.7, respectively. Since $T(G_D)$ and $T(G_{\gamma(D)})$ are abelian groups, it follows from Proposition 8.3 that there exists an exact sequence

$$(11) \quad 0 \rightarrow T(G_{\gamma(D)}) \rightarrow T(G_D) \rightarrow \mathbb{Z}^r \rightarrow 0,$$

where $r \leq \mathbf{t} \cdot \#\mathcal{S}$. Since \mathbb{Z}^r is \mathbb{Z} -free, the exact sequence (11) splits. Thus, the group $T(G_D)$ is isomorphic to a semi-direct product of $T(G_{\gamma(D)})$ by \mathbb{Z}^r . Note that, since $T(G_D)$ is abelian, it is isomorphic to the direct product of $T(G_{\gamma(D)})$ and \mathbb{Z}^r . Thus, Statement (2) follows from Corollary 7.6 by setting $T := h_D T(G_{\gamma(D)}) h_D^{-1}$. Moreover, Statement (3) is a direct consequence of Proposition 7.5.

Now, assume that $G \subseteq \mathbf{G}(\mathcal{O}_\mathcal{S})$, $\mathcal{S} = \{P\}$, \mathbb{F} is finite of characteristic $p > 0$ and that the torsion of G is p -primary. Since $G \subseteq \mathbf{G}(\mathcal{O}_\mathcal{S})$ and $\mathcal{S} = \{P\}$, it follows from Proposition 7.7 that $G_D = G_{\gamma(D)}$. Let $g \in G_D = G_{\gamma(D)}$ be an arbitrary element. We write it as $g = t \cdot u$, where $t \in T(G_{\gamma(D)})$ and $u \in U(G_{\gamma(D)})$ (c.f. Lemma 7.1). Since \mathbb{F} is finite, the group $T(G_{\gamma(D)})$ has cardinality q coprime with p , according to Proposition 7.5. Then $g^q = t^q \cdot \tilde{u} = \tilde{u}$, for certain $\tilde{u} \in h_D^{-1}\mathbf{U}^+(k)h_D$. Recall that, up to conjugate by an element in $\text{GL}_{n,k}(k)$, there exists a faithful k -linear representation $\mathbf{G}(k) \rightarrow \text{GL}_{n,k}(k)$ embedding $\mathbf{U}^+(k)$ in the group of unipotent upper triangular matrices

$U_n(k)$ of $GL_{n,k}(k)$. Since $\text{Char}(\mathbb{F}) = p > 0$, $U_n(k)$ is a torsion group, whence $U^+(k)$ also is a torsion group. In particular, the unipotent element $\tilde{u} \in h_D^{-1}U^+(k)h_D$ has finite order. We deduce that g has finite order. Thus, since the torsion of G is p -primary, the order of g equals p^f , for some $f \in \mathbb{Z}_{\geq 0}$. Let $a, b \in \mathbb{Z}$ be such that $1 = ap^f + bq$. Then $g = (g^{p^f})^a \cdot (g^q)^b = (g^q)^b \in h_D^{-1}U^+(k)h_D$. Hence, we deduce that $G_D \subseteq h_D^{-1}U^+(k)h_D \cap G = U(G_D)$, whence Statement (5) follows. \square

Let us denote by Σ_0 a representative system of the $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbits in $\partial_{\infty}^k X_{\mathcal{S}}$. For each $\partial_{\infty} D \in \Sigma_0$, the semisimple group $T((\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_D)$ is isomorphic to the direct product of a subgroup of $\mathbf{T}(\mathbb{F})$ and \mathbb{Z}^r , for some $r = r_D = r(\mathbf{G}, \mathcal{S}, D) \in \mathbb{Z}$ (c.f. Theorem 3.4 (2)-(3)).

Lemma 8.4. *Assume that G is a normal subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$. Let $\partial_{\infty} D' \in \partial_{\infty}^k X_{\mathcal{S}}$ be a chamber which belongs to the $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbit of $\partial_{\infty} D \in \Sigma_0$. Then:*

- (1) $G_{D'}$ is $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -conjugated to G_D ,
- (2) $U(G_{D'})$ is $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -conjugated to $U(G_D)$, and
- (3) $T(G_{D'})$ is isomorphic to the direct product of a subgroup of $\mathbf{T}(\mathbb{F})$ and a \mathbb{Z} -free group, whose rank is exactly r_D .

Proof. By definition of $\partial_{\infty} D'$, there exists $g \in \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ such that $\partial_{\infty} D' = g^{-1} \cdot \partial_{\infty} D$. In particular, we have $h_{D'} = h_D g$. Since G is normal in $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, we get

$$h_{D'}^{-1} \mathbf{B}(k) h_{D'} \cap G = (gh_D)^{-1} \mathbf{B}(k) (gh_D) \cap G = g^{-1} (h_D^{-1} \mathbf{B}(k) h_D \cap G) g.$$

In other words, we have

$$(12) \quad G_{D'} = g^{-1} G_D g.$$

By an analogous argument, we get

$$h_{D'}^{-1} U^+(k) h_{D'} \cap G = (gh_D)^{-1} U^+(k) (gh_D) \cap G = g^{-1} (h_D^{-1} U^+(k) h_D \cap G) g,$$

whence

$$(13) \quad U(G_{D'}) = g^{-1} U(G_D) g.$$

Thus, Statement (1) and (2) follows.

Now, we prove Statement (3). Let $\mathbb{A} = h_D^{-1} \cdot D_0$ and $\mathbb{A}' = h_{D'}^{-1} \cdot D_0$ as in Equation (5). Since $h_{D'} = h_D g$, we have $\mathbb{A}' = g^{-1} \cdot \mathbb{A}$. In particular, for any $z \in \mathbb{A}'$, the point $y = g^{-1} \cdot z$ belongs to \mathbb{A} . Since G is normal in $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, we have

$$\begin{aligned} \text{Fix}_G(Q(z, D)) &= \text{Fix}_{\mathbf{G}(k)}(Q(z, D)) \cap G = g^{-1} \text{Fix}_{\mathbf{G}(k)}(Q(y, D')) g \cap G \\ &= g^{-1} (\text{Fix}_{\mathbf{G}(k)}(Q(y, D')) \cap g G g^{-1}) g = g^{-1} \text{Fix}_G(Q(y, D)) g. \end{aligned}$$

Thus, it follows from Equation (6) that $G_{\gamma(D')} \subseteq g^{-1} G_{\gamma(D)} g$. An analogous argument provides the converse inclusion, whence

$$(14) \quad G_{\gamma(D')} = g^{-1} G_{\gamma(D)} g.$$

Hence, it follows from Proposition 8.3 that

$$(15) \quad T(G_D)/T(G_{\gamma(D)}) \cong G_D/G_{\gamma(D)} \cong G_{D'}/G_{\gamma(D')} \cong T(G_{D'})/T(G_{\gamma(D')}).$$

Since $G \subseteq \mathbf{G}(\mathcal{O}_{\mathcal{S}})$, it follows from Theorem 3.4 (2)-(3) that $T(G_D)$ (resp. $T(G_{D'})$) is isomorphic to the direct product of a subgroup of $\mathbf{T}(\mathbb{F})$ and a free \mathbb{Z} -module, whose rank we denote here as $r(D, G)$ (resp. $r(D', G)$). In particular, it follows from Equation (15) that $r(D, G) = r(D', G)$. Thus, in order to prove Statement (3) we have to show that $r(D, G)$ equals r_D .

Since $G_{\gamma(D)} = (\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_{\gamma(D)} \cap G$, the natural inclusion $G_D \rightarrow (\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_D$ induces an injective group homomorphism $\iota : G_D/G_{\gamma(D)} \rightarrow (\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_D / (\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_{\gamma(D)}$. We identify $G_D/G_{\gamma(D)}$ with its image via ι , so that we realize $G_D/G_{\gamma(D)}$ as a subgroup of $(\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_D / (\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_{\gamma(D)}$. Since G has finite index in $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, the group G_D has finite index in $(\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_D$, whence $G_D/G_{\gamma(D)}$ has a finite index in $(\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_D / (\mathbf{G}(\mathcal{O}_{\mathcal{S}}))_{\gamma(D)}$.

This implies that $r(D, G) = r_D$, since the respective free \mathbb{Z} -modules are commensurable. Thus, Statement (3) follows. \square

Recall that the isomorphism classes of vector bundles of rank 1 over \mathcal{C} form a group $\text{Pic}(\mathcal{C})$ with the tensor product as composition law. If e denotes the gcd of the degrees of closed points on \mathcal{C} , then we have the exact sequence:

$$0 \rightarrow \text{Pic}^0(\mathcal{C}) \rightarrow \text{Pic}(\mathcal{C}) \xrightarrow{\text{deg}} e\mathbb{Z} \rightarrow 0.$$

The group $\text{Pic}^0(\mathcal{C})$ is called the Jacobian variety of \mathcal{C} . This is finite when \mathbb{F} is finite according to Weil theorem (c.f. [Ser03, Ch. II, § 2.2]).

Example 8.5. Let $\partial_\infty D_0 \in \partial_\infty^k X_{\mathcal{S}}$ as in § 2.2. In order to simplify our calculations, we set $h_{D_0} = \text{id}$. Let $G = \mathbf{G}(\mathcal{O}_{\mathcal{S}})$ be the group of $\mathcal{O}_{\mathcal{S}}$ -points of \mathbf{G} . By definition, we have

$$(16) \quad G_{D_0} = \mathbf{B}(k) \cap \mathbf{G}(\mathcal{O}_{\mathcal{S}}) = \mathbf{B}(\mathcal{O}_{\mathcal{S}}) \text{ and } U(G_{D_0}) = \mathbf{U}^+(k) \cap \mathbf{G}(\mathcal{O}_{\mathcal{S}}) = \mathbf{U}^+(\mathcal{O}_{\mathcal{S}}).$$

Recall that $T(G_{D_0})$ is the image of G_{D_0} by the group homomorphism $f : G_{D_0} \rightarrow \mathbf{T}(k)$ defined by $f(g) = t$, where $g = t \cdot u$. Moreover, since $\ker(f) = U(G_{D_0})$ according to Lemma 8.1, it follows from Equation (16) that

$$(17) \quad T(G_{D_0}) = \mathbf{T}(\mathcal{O}_{\mathcal{S}}).$$

Now, we decompose $T(G_{D_0})$ as a direct product, as in Theorem 3.4 (2). Let $T(G_{\gamma(D_0)})$ be the semisimple group defined in Equation (7). On the one hand, Proposition 7.5 shows that $T(G_{\gamma(D_0)})$ is a subgroup of $\mathbf{T}(\mathbb{F})$. On the other hand, since $\nu_P(\mathbb{F}^*) = \{0\}$, for any place P on k , we have that $\mathbf{T}(\mathbb{F})$ fixes any point in \mathbb{A} . This implies that $\mathbf{T}(\mathbb{F})$ is contained in each $\text{Fix}_G(Q(y, D))$, for $y \in \mathbb{A}_0$. In other words, we have $\mathbf{T}(\mathbb{F}) \subseteq G_{\gamma(D_0)}$. Thus, we conclude $\mathbf{T}(\mathbb{F}) \subseteq T(G_{\gamma(D_0)})$, whence:

$$(18) \quad T(G_{\gamma(D_0)}) = \mathbf{T}(\mathbb{F})$$

Note that, from Equation (17) and Equation (18), we get that $T(G_{D_0})/T(G_{\gamma(D_0)})$ equals $\mathbf{T}(\mathcal{O}_{\mathcal{S}})/\mathbf{T}(\mathbb{F})$. Since $\mathbf{T} \cong \mathbb{G}_m^{\mathbf{t}}$, where $\mathbf{t} = \text{rk}(\mathbf{G})$, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{T}(\mathbb{F}) & \xrightarrow{\cong} & (\mathbb{F}^*)^{\mathbf{t}} \\ \downarrow \iota & \circlearrowleft & \downarrow \iota \\ \mathbf{T}(\mathcal{O}_{\mathcal{S}}) & \xrightarrow{\cong} & (\mathcal{O}_{\mathcal{S}}^*)^{\mathbf{t}} \end{array}.$$

In other words, we have

$$(19) \quad T(G_{D_0})/T(G_{\gamma(D_0)}) = \mathbf{T}(\mathcal{O}_{\mathcal{S}})/\mathbf{T}(\mathbb{F}) \cong (\mathcal{O}_{\mathcal{S}}^*)^{\mathbf{t}} / (\mathbb{F}^*)^{\mathbf{t}} \cong (\mathcal{O}_{\mathcal{S}}^*/\mathbb{F}^*)^{\mathbf{t}}$$

It follows from the Dirichlet unit theorem (c.f. [Ros02, § 14, Coro. 1]) that $\mathcal{O}_{\mathcal{S}}^*$ is isomorphic to the direct product of \mathbb{F}^* and a \mathbb{Z} -free group Λ' , whose rank is at most $\#\mathcal{S} - 1$. Moreover, [Ros02, § 14, Prop. 14.2] shows that the rank of Λ' is exactly $\#\mathcal{S} - 1$ when $\text{Pic}^0(\mathcal{C})$ is a torsion group. This is the case when \mathbb{F} is finite. We conclude that $T(G_{D_0})$ is isomorphic to the direct product of $\mathbf{T}(\mathbb{F})$ and the \mathbb{Z} -free group $\Lambda := (\mathcal{O}_{\mathcal{S}}^*/\mathbb{F}^*)^{\mathbf{t}} \cong (\Lambda')^{\mathbf{t}}$, whose rank r_{D_0} is at most $\mathbf{t} \cdot (\#\mathcal{S} - 1)$, with equality when $\text{Pic}^0(\mathcal{C})$ is a torsion group.

Remark 8.6. Assume that \mathbb{F} is an algebraic (possibly non finite) extension of a finite field \mathbb{F}_p . Since any element of $\text{Pic}^0(\mathcal{C})$ is defined over a finite extension $\mathbb{L} \subset \mathbb{F}$ of \mathbb{F}_p , the group $\text{Pic}^0(\mathcal{C})$ is a torsion group. In particular, $r_{D_0} = \text{rk}(\mathbf{G}) \cdot (\#\mathcal{S} - 1)$, for such fields.

Recall that the Jacobian variety $\text{Pic}^0(\mathcal{C})$ is an Abelian variety. Moreover, recall that the group of n -torsion points $\mathcal{A}[n]$ of any abelian variety \mathcal{A} is a finite group. This implies that the group of torsion points of \mathcal{A} is countable. In particular, the group of torsion points of $\text{Pic}^0(\mathcal{C})$ is countable. Let \mathbb{F} be a non-countable perfect field. We can take, for instance, $\mathbb{F} = \mathbb{C}$, in characteristic 0, and $\mathbb{F} = \overline{\mathbb{F}_p}((T))$, in characteristic $p > 0$. Then, for an elliptic curve \mathcal{E} , the Jacobian variety $\text{Pic}^0(\mathcal{E}) \cong \mathcal{E}(\mathbb{F})$ is non-countable. This implies

that $\text{Pic}^0(\mathcal{E})$ has non-torsion elements. In particular, the integer r_{D_0} is strictly smaller than $\text{rk}(\mathbf{G}) \cdot (\#\mathcal{S} - 1)$, for such curves.

Corollary 8.7. *Assume that $\mathcal{O}_{\mathcal{S}}$ is a principal integral domain. Let G be a normal subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$. Then, for any $\partial_{\infty}D \in \partial_{\infty}^k X_{\mathcal{S}}$*

- (1) G_D is $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -conjugated to $\mathbf{B}(k) \cap G$,
- (2) $U(G_D)$ is $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -conjugated to $\mathbf{U}^+(k) \cap G$,
- (3) $T(G_D)$ is isomorphic to the direct product of a subgroup of $\mathbf{T}(\mathbb{F})$ and \mathbb{Z}^r , where r is at most $\mathfrak{t} \cdot (\#\mathcal{S} - 1)$, with equality when $\text{Pic}^0(\mathcal{C})$ is a torsion group. In particular, we have $r = \mathfrak{t} \cdot (\#\mathcal{S} - 1)$ when \mathbb{F} is finite.

Moreover, the set \mathfrak{U}/G of G -conjugacy classes of maximal unipotent subgroups in G is in bijection with the double quotient $G \backslash \mathbf{G}(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}(\mathcal{O}_{\mathcal{S}})$.

Proof. Since $\mathcal{O}_{\mathcal{S}}$ is a principal integral domain, it follows from Theorem 3.2 that $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ acts transitively on $\partial_{\infty}^k X_{\mathcal{S}}$. In other words, each $\partial_{\infty}D \in \partial_{\infty}^k X_{\mathcal{S}}$ belongs to the $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -orbit of $\partial_{\infty}D_0$. Thus, Statement (1) and (2) follows from Lemma 8.4 (1)-(2). Moreover, it follows from Lemma 8.4 (3) that $T(G_D)$ is isomorphic to the direct product of a subgroup of $\mathbf{T}(\mathbb{F})$ and a \mathbb{Z} -free group of rank r_{D_0} . Thus, Statement (3) follows from Example 8.5.

Now, we prove that \mathfrak{U}/G is in bijection with $G \backslash \mathbf{G}(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}(\mathcal{O}_{\mathcal{S}})$. Since $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ acts transitively on $\partial_{\infty}^k X_{\mathcal{S}}$, and $\text{Stab}_{\mathbf{G}(\mathcal{O}_{\mathcal{S}})}(\partial_{\infty}D_0) = \mathbf{G}(\mathcal{O}_{\mathcal{S}}) \cap \mathbf{B}(k) = \mathbf{B}(\mathcal{O}_{\mathcal{S}})$, we have that $\partial_{\infty}^k X_{\mathcal{S}}$ is in bijection with $\mathbf{G}(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}(\mathcal{O}_{\mathcal{S}})$. Therefore, the set of G -orbits on $\partial_{\infty}^k X_{\mathcal{S}}$ is in bijection with $G \backslash \mathbf{G}(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}(\mathcal{O}_{\mathcal{S}})$, whence the result follows. \square

9. APPLICATIONS TO PRINCIPAL CONGRUENCE SUBGROUPS

In this section, we present some examples on the description of maximal unipotent subgroups of some arithmetic subgroups.

Let I be a proper ideal of $\mathcal{O}_{\mathcal{S}}$. The principal congruence subgroup Γ_I defined from I is the kernel of the group homomorphism $\pi_I : \mathbf{G}(\mathcal{O}_{\mathcal{S}}) \rightarrow \mathbf{G}(\mathcal{O}_{\mathcal{S}}/I)$ induced by the projection $\pi_I : \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_{\mathcal{S}}/I$. A principal congruence subgroup is a group of the form Γ_I , for some ideal $I \subseteq \mathcal{O}_{\mathcal{S}}$.

Lemma 9.1. *Assume that \mathbb{F} is finite of characteristic $p > 0$. Then, the torsion of Γ_I is p -primary.*

Proof. According to [BT72, 9.1.19(c)], there exists an injective homomorphism of $\text{Spec}(\mathbb{Z})$ -groups $\rho : \mathbf{G} \rightarrow \text{SL}_{n,\mathbb{Z}}$. This is a faithful linear representation of \mathbf{G} . In particular, for each (commutative) ring R , the homomorphism ρ induces an injective group homomorphism $\mathbf{G}(R) \rightarrow \text{SL}_{n,\mathbb{Z}}(R)$, which, by abuse of notation, we denote ρ . Let $g \in \Gamma_I$ be a finite order element and let $P_g(T)$ be the characteristic polynomial of $\rho(g)$ over k . When we apply the ring homomorphism $\pi_I : \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_{\mathcal{S}}/I$ to each coefficient of $P_g(T)$, we obtain the a polynomial $\pi_I(P_g(T)) \in (\mathcal{O}_{\mathcal{S}}/I)[T]$. Since $\pi_I : \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_{\mathcal{S}}/I$ is a ring homomorphism, we have

$$\pi_I(P_g(T)) = \pi_I(\det(\rho(g) - T \cdot \text{id})) = \det(\pi_I(\rho(g)) - T \cdot \text{id}).$$

Moreover, since $\pi_I(\rho(g)) = \rho(\pi_I(g))$ and $\pi_I(g) = \text{id}$

$$\det(\pi_I(\rho(g)) - T \cdot \text{id}) = \det(\rho(\pi_I(g)) - T \cdot \text{id}) = \det(\text{id} - T \cdot \text{id}) = (1 - T)^n.$$

Then, we get $\pi_I(P_g(T)) = (1 - T)^n$. But, since g is a torsion element, $g^m = \text{id}$, for some $m \in \mathbb{Z}$. Then $\rho(g)^m = \text{id}$, whence we deduce that each eigenvalue of $\rho(g)$ is a root of unity. In particular, each coefficient of $P_g(T)$ lies in the algebraic closure of \mathbb{F} in k , which is \mathbb{F} itself by assumption (c.f. § 1). Thus, the polynomial $P_g(T)$ belongs to $\mathbb{F}[T]$. In particular, each coefficient of $P_g(T)$ lies in \mathbb{F} , and we have $P_g(T) = \pi_I(P_g(T)) = (T - 1)^n$, since \mathbb{F} is a subring of $\mathcal{O}_{\mathcal{S}}/I$. Hence, the matrix $\rho(g)$ is unipotent. Since k has characteristic $p > 0$, any unipotent element of $\text{SL}_{n,\mathbb{Z}}(k)$ has p -power order. Therefore, $\rho(g)^{p^t} = \text{id}$, for some $t \in \mathbb{Z}$. Thus, we conclude that $g^{p^t} = \text{id}$. \square

Corollary 9.2. *Assume that \mathbb{F} is finite of characteristic $p > 0$ and $\mathcal{S} = \{P\}$. Fix a set $\{\partial_\infty D_\sigma : \sigma \in \Sigma\}$ of representatives of the G -orbits in $\partial_\infty^k X_{\mathcal{S}}$. Then:*

- (1) $\mathfrak{U} = \{\text{Stab}_{\Gamma_I}(\partial_\infty D) : \partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}\}$ is the set of maximal unipotent subgroup of Γ_I , and
- (2) $\mathfrak{U}/\Gamma_I = \{\text{Stab}_{\Gamma_I}(\partial_\infty D_\sigma) : \sigma \in \Sigma\}$ is a set of representatives of the Γ_I -conjugacy classes in \mathfrak{U} .

Proof. Since \mathbb{F} is finite of characteristic $p > 0$, the torsion of Γ_I is p -primary according to Lemma 9.1. Since $\mathcal{S} = \{P\}$, it follows from Theorem 3.4 that, for any $\partial_\infty D \in \partial_\infty^k X_{\mathcal{S}}$, we have $U((\Gamma_I)_D) = \text{Stab}_{\Gamma_I}(\partial_\infty D)$. Then, the result follows from Theorem 3.1(2)-(3). \square

Remark 9.3. Statements (1) and (2) of Corollary 9.2 do not hold for arbitrary finite subsets \mathcal{S} . For instance, let us take $\mathcal{C} = \mathbb{P}_{\mathbb{F}}^1$ and $\mathcal{S} = \{0, \infty\}$, so that $\mathcal{O}_{\mathcal{S}} = \mathbb{F}[t, t^{-1}]$. Let $I = (t-1)\mathcal{O}_{\mathcal{S}}$, so that $\pi_I : \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_{\mathcal{S}}/I$ corresponds to the ring homomorphism defined by the evaluation at $t = 1$. Let $\mathbf{G} = \text{SL}_m$ and let Γ_I be the corresponding congruence subgroup. We identify \mathbf{T} (resp. \mathbf{B}) with the diagonal (resp. upper triangular) subgroup of SL_m . Let

$$T := \{\text{diag}(t^{n_1}, \dots, t^{n_m}) : n_1 + \dots + n_m = 0, n_1, \dots, n_m \in \mathbb{Z}\}.$$

Since $\pi_I(t) = 1$, we have $T \subseteq \Gamma_I$. Moreover, since $T \subset \mathbf{B}(k)$, we have $T \subseteq \text{Stab}_{\Gamma_I}(\partial_\infty D_0)$. This proves that $\text{Stab}_{\Gamma_I}(\partial_\infty D_0)$ is not unipotent. Thus, Statements (1) and (2) of Corollary 9.2 do not hold in this context.

Let us denote by Γ_I^+ the group $\mathbf{U}^+(k) \cap \Gamma_I$. In contrast with Corollary 9.2, the following result holds even when \mathbb{F} is an infinite perfect field.

Corollary 9.4. *Assume that $\mathcal{O}_{\mathcal{S}}$ is a principal ideal domain. Let \mathfrak{U} and \mathfrak{U}/Γ_I as in Theorem 3.1. Then*

- (1) Any $U \in \mathfrak{U}$ is $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$ -conjugated to Γ_I^+ , and
- (2) \mathfrak{U}/Γ_I is in bijection with $\Gamma_I \backslash \mathbf{G}(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}(\mathcal{O}_{\mathcal{S}})$.

Proof. Since Γ_I is a normal subgroup of $\mathbf{G}(\mathcal{O}_{\mathcal{S}})$, Statement (1) follows from Corollary 8.7 (2). Moreover, it also follows from Corollary 8.7 that \mathfrak{U}/Γ_I is in bijection with $\Gamma_I \backslash \mathbf{G}(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}(\mathcal{O}_{\mathcal{S}})$, which concludes the proof. \square

Example 9.5. Let $\mathbf{G} = \text{SL}_n$. The subgroup \mathbf{B}_n (resp. \mathbf{D}_n , resp. \mathbf{U}_n) of upper triangular (resp. diagonal, resp. unipotent upper triangular) matrices in SL_n is a Borel (resp. a maximal torus, resp. a maximal unipotent) subgroup of SL_n defined over \mathbb{Z} . Let Θ_I be the set of nilpotent upper triangular matrices with coefficients in I . The group Γ_I^+ equals $\text{id} + \Theta_I = \{\text{id} + \theta : \theta \in \Theta_I\}$. By a straightforward computation, for $\mathbf{G} = \text{SL}_n$, the group Γ_I^+ is the group generated by $\{\theta_\alpha(I) : \alpha \in \Phi^+\}$. This provides an example of the group Γ_I^+ introduced in Corollary 9.4 (1).

Since the group homomorphism $\pi_I : \text{SL}_n(\mathcal{O}_{\mathcal{S}}) \rightarrow \text{SL}_n(\mathcal{O}_{\mathcal{S}}/I)$ is surjective, we have $\Gamma_I \backslash \text{SL}_n(\mathcal{O}_{\mathcal{S}}) \cong \text{SL}_n(\mathcal{O}_{\mathcal{S}}/I)$. Moreover, since $(\Gamma_I \cap \mathbf{B}_n(\mathcal{O}_{\mathcal{S}})) \backslash \mathbf{B}_n(\mathcal{O}_{\mathcal{S}}) \cong \mathbf{B}_n(\mathcal{O}_{\mathcal{S}}/I)$, we get that $\Gamma_I \backslash \text{SL}_n(\mathcal{O}_{\mathcal{S}}) / \mathbf{B}_n(\mathcal{O}_{\mathcal{S}})$ is in bijection with $\text{SL}_n(\mathcal{O}_{\mathcal{S}}/I) / \mathbf{B}_n(\mathcal{O}_{\mathcal{S}}/I)$. Therefore, it follows from Corollary 9.4 (2) that the set \mathfrak{U}/Γ_I is in bijection with $\text{SL}_n(\mathcal{O}_{\mathcal{S}}/I) / \mathbf{B}_n(\mathcal{O}_{\mathcal{S}}/I)$. This bijection allows us to do some explicit computations on \mathfrak{U}/Γ_I . Assume that I is a prime ideal. Then, the ring $\mathbb{F}' := \mathcal{O}_{\mathcal{S}}/I$ is a field, since $\mathcal{O}_{\mathcal{S}}$ is a Dedekind domain. The exact sequence of algebraic varieties $1 \rightarrow \mathbf{B}_n \xrightarrow{\iota} \text{SL}_n \xrightarrow{\rho} \text{SL}_n / \mathbf{B}_n \rightarrow 1$, induces the following long exact sequence (c.f. [DG70a, § 4, 4.6])

$$1 \rightarrow \mathbf{B}_n(\mathbb{F}') \rightarrow \text{SL}_n(\mathbb{F}') \rightarrow (\text{SL}_n / \mathbf{B}_n)(\mathbb{F}') \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathbb{F}'), \mathbf{B}_n) \rightarrow H_{\text{ét}}^1(\text{Spec}(\mathbb{F}'), \text{SL}_n).$$

But $H_{\text{ét}}^1(\text{Spec}(\mathbb{F}'), \mathbf{B}_n) = H_{\text{ét}}^1(\text{Spec}(\mathbb{F}'), \mathbf{D}_n) \cong H_{\text{ét}}^1(\text{Spec}(\mathbb{F}'), \mathbb{G}_m)^{n-1} = \{0\}$, according to [DG70b, Exp. XXVI, Cor. 2.3] and Hilbert's Theorem 90. Then, we get that $\text{SL}_n(\mathbb{F}') / \mathbf{B}_n(\mathbb{F}')$ is in bijection with $(\text{SL}_n / \mathbf{B}_n)(\mathbb{F}')$. Thus, the set \mathfrak{U}/Γ_I is in bijection with $(\text{SL}_n / \mathbf{B}_n)(\mathbb{F}')$, in the case of a prime ideal I . In other words, the Γ_I -conjugacy classes of maximal unipotent subgroups of Γ_I are in bijection with the set of \mathbb{F}' -points of the Borel variety $\text{SL}_n / \mathbf{B}_n$.

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