An exterior optimal transport problem

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AN EXTERIOR OPTIMAL TRANSPORT PROBLEM

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Abstract. This paper deals with a variant of the optimal transportation problem. Given $f \in L^1(\mathbb{R}^d, [0,1])$ and a cost function $c \in C(\mathbb{R}^d \times \mathbb{R}^d)$ of the form $c(x,y) = k(y-x)$, we minimise $\int c\, d\gamma$ among transport plans γ whose first marginal is f and whose second marginal is not prescribed but constrained to be smaller than $1-f$. Denoting by $\Upsilon(f)$ the infimum of this problem, we then consider the maximisation problem $\sup\{\Upsilon(f) : \int f = m\}$ where $m > 0$ is given. We prove that maximisers exist under general assumptions on k, and that for k radial, increasing and coercive these maximisers are the characteristic functions of the balls of volume m.

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1. Introduction

In this paper, we study the optimization problems associated with functionals which favour dispersion and are based on some Wasserstein energies. These functionals correspond to the non-local term of the energy studied in [4].

We denote by $\mathcal{M}_+(\mathbb{R}^d)$ the set of positive Radon measures on $\mathbb{R}^d$. Given a cost function c and $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$, we let $\mathcal{T}_c(\mu,\nu)$ be the c-transport cost between μ and ν (see Section 2 for the exact definition of $\mathcal{T}_c$). Given a measurable set $E \subset \mathbb{R}^d$ with finite volume, we consider the optimisation problem

$$\Upsilon_{\rm set}(E) := \inf\Big\{\mathcal{T}_c(E,F) : F \subset \mathbb{R}^d \text{ Lebesgue measurable}, |F| = |E|,\ |F \cap E| = 0\Big\} \tag{1.1}$$

where we identify E with the restriction of the Lebesgue measure on E. Given $m > 0$, we introduce the maximisation problem

$$\mathcal{E}_{\rm set}(m) := \sup_{|E|=m} \Upsilon_{\rm set}(E). \tag{1.2}$$

The main goal of the article is to investigate the existence of maximisers for this problem and to characterise these latter.

If we apply the direct method of the Calculus of Variations, we obtain that, up to extraction, any maximising sequence E_n converges weakly to some function $u_\infty \in L^1(\mathbb{R}^d, [0,1])$. However, there is no guarantee at this point that u_∞ is a characteristic function or has mass m. Our strategy is to extend the functional $\Upsilon_{\rm set}$ as a functional Υ defined on $L^1(\mathbb{R}^d, [0,1])$. Applying the bathtub principle (see Proposition 4.11) to a maximiser of the relaxed problem, we show that the supremum in (1.2) is actually reached (see Corollary 1.2). This relaxation approach is not new: it was successfully applied to several variational problems in the last few years (see for instance [6, 1, 20, 2]).

Given $f \in L^1(\mathbb{R}^d, [0,1])$, the set of admissible exterior transport plans is defined as

$$\Pi_f := \Big\{\gamma \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d) : \gamma_x = f,\ \gamma_y \le 1 - f\Big\}.$$

Here, the measures $f\,dx$ and $(1-f)\,dy$ are identified with their respective densities and γ_x and γ_y denote respectively the first and second marginals of γ. We then define the primal problem

$$\Upsilon(f) := \inf\left\{\int c\, d\gamma : \gamma \in \Pi_f\right\}.$$

We have $\Upsilon(\chi_E) = \Upsilon_{\text{set}}(E)$ under mild assumptions on c (see Theorem 4.4). Given $m > 0$, our maximisation problem is now

$$\mathcal{E}(m) := \sup\left\{\Upsilon(f) : f \in L^1(\mathbb{R}^d, [0,1]),\ \int f\,dx = m\right\}.$$

By abuse of notation and when no confusion is possible, we refer to the variational problems by the values they attain (*e.g.* we write $\Upsilon_{\text{set}}(E)$ for (1.1)).

1.1. **Main results.**

The first important result of this article is that maximisers of $\mathcal{E}(m)$ exist whenever c is of the form $c(x,y) = k(y-x)$ for some $k : \mathbb{R}^d \to \mathbb{R}_+$ and satisfies

(H1) $k \in C(\mathbb{R}^d, \mathbb{R}_+)$, $k(0) = 0$ and $k(x) \to \infty$ as $|x| \to \infty$,

(H2) $\forall x \neq 0$,

$$\limsup_{r\to 0} \frac{1}{r^d}\big|B_r(x) \cap \{y \in \mathbb{R}^d,\ k(y) < k(x)\}\big| > 0,$$

(H3) $\forall\, \sigma \in \mathbb{S}^{d-1}$, $r \mapsto k(r\sigma)$ is increasing on $\mathbb{R}_+$.

Notice that k is not assumed to be strictly convex, so that our results hold in cases where the existence of an optimal transport map is not guaranteed. Also observe that all the costs of the form $k(z) = |z|^p$ with $0 < p < \infty$ satisfy the above hypotheses. However, radial symmetry is not required and the costs $k(z) = |z|^p\, h(z/|z|)$ with h positive and Lipschitz continuous on $\mathbb{S}^{d-1}$ are also admissible.

Theorem 1.1. *Assume that $c(x,y) = k(y-x)$ for $x, y \in \mathbb{R}^d$ with k satisfying (H1),(H2)&(H3). Then, for any $m > 0$ the supremum in $\mathcal{E}(m)$ is attained. Moreover, there exists $R_* = R_*(m)$ such that (up to translation) any maximiser is supported in the ball $\overline{B}_{R_*}$.*

Once the existence of maximisers for $\mathcal{E}(m)$ is established, the bathtub principle (see Proposition 4.11) and a saturation result (see Theorem 4.4) imply that (1.2) admits solutions.

Corollary 1.2. *Assume that c satisfies the hypotheses of Theorem 1.1. Then,* (1.2) *admits a maximiser and $\mathcal{E}_{\text{set}}(m) = \mathcal{E}(m)$ for any $m > 0$.*

As a second main result, we establish that if k is furthermore radially symmetric then $\mathcal{E}(m)$ and $\mathcal{E}_{\text{set}}(m)$ are uniquely maximised by balls of volume m.

Theorem 1.3. *Assume that $c(x,y) = k(|y-x|)$ for some $k \in C(\mathbb{R}_+, \mathbb{R}_+)$ increasing and such that $k(0) = 0$ and $k(x) \to \infty$ as $x \to \infty$. Then, for any $m > 0$, the maximisers of $\mathcal{E}(m)$ (and consequently those of $\mathcal{E}_{\text{set}}(m)$) are the balls of volume m.*

We point out that cost functions satisfying the hypotheses of Theorem 1.3 also satisfy hypotheses (H1),(H2)&(H3). Let us briefly sketch the proofs of these three results. They all strongly rely on the properties of the dual problem

$$\Upsilon^*(f) := \sup\left\{\int (f\varphi + (1-f)\psi)\ dx : (\varphi,\psi) \in \Phi\right\},$$

where

$$\Phi := \left\{(\varphi,\psi) \in C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d),\ \psi \le 0,\ \varphi(x) + \psi(y) \le c(x,y)\ \forall\,(x,y) \in \mathbb{R}^d \times \mathbb{R}^d\right\}.$$

We establish Theorem 1.1 using the direct method of Calculus of Variations. The main difficulty is to establish compactness of maximising sequences. If we refer to the concentration-compactness principle [17], we have to prove that given a maximising sequence f_n, no mass escapes at infinity. To do so we establish two crucial results. The first one is that $m \mapsto \mathcal{E}(m)/m$ is increasing (see Proposition 4.7). This implies that $m \mapsto \mathcal{E}(m)$ is strictly superadditive, *i.e.* that for $m > m' > 0$,

$$\mathcal{E}(m) + \mathcal{E}(m-m') < \mathcal{E}(m). \tag{1.3}$$

Notice that this is the counterpart of the strict subbadditivity inequality (also called binding inequality) which is known to provide compactness in minimisation problems, see *e.g.* [17, 9, 10]. Using the dual formulation Υ^* of Υ, we obtain the second crucial result for Theorem 1.3: a monotonicity principle on the sum of marginals of maximisers γ of $\Upsilon(f)$ (see Corollary 4.6). Combining this and (1.3), we prove that if f is almost maximising then most of its mass must remain in a bounded region (see Proposition 4.9). This gives tightness of maximising sequences for $\mathcal{E}(m)$.

To prove Corollary 1.2, we consider a maximiser f of $\mathcal{E}(m)$ provided by Theorem 1.1 and a pair of potentials (φ, ψ) optimal for the dual problem $\Upsilon^*(f)$. Using the definition of Υ^* we see that f is a maximiser of

$$\sup\left\{\int \widetilde{f}(\varphi-\psi)\ : 0\le \widetilde{f}\le 1,\ \int \widetilde{f}=m\right\}.$$

By the bathtub principle, $f=\chi_{\{\varphi-\psi>t\}}+\theta$ for some $t\in\mathbb{R}$ and some $\theta\in L^1(\mathbb{R}^d,[0,1])$ supported in $\{\varphi-\psi=t\}$. Then for any measurable subset $G\subset\{\varphi-\psi=t\}$ with $|G|=\int\theta$, the characteristic function of $E:=\{\varphi-\psi>t\}\cup G$ is also a maximiser for $\Upsilon(f)$. By Theorem 4.4 and Corollary 4.5 applied to E, there exists $F\subset\mathbb{R}^d$ such that any minimiser γ of $\Upsilon(\chi_E)$ satisfies $\gamma_y=\chi_F$. This finally implies that E maximises (1.2).

Regarding Theorem 1.3, as explained in Section 5, we may assume without loss of generality that $m=\omega_d$, the volume of the unit ball. Combining Theorem 1.1 and Lemma 4.10 yields that

$$\sup_{\int f=\omega_d}\left\{\sup_{(\psi^c,\psi)\in\Phi}\left\{\int f(\psi^c-\psi)+\int\psi\right\}\right\}, \tag{1.4}$$

coincides with $\mathcal{E}(m)$ and admits a solution (f,ψ^c,ψ), where ψ^c is the c-transform of ψ (see Definition 2.2). To show that balls are maximisers of $\mathcal{E}(m)$, we establish that each term in (1.4) is improved by replacing f by χ_{B_1} and ψ by its symmetric increasing rearrangement ψ_* (see Definition 5.1). As $\int\psi=\int\psi_*$, the third term in (1.4) does not change under rearrangement. Regarding the second term, combining the Hardy-Littlewood inequality (see [16, Theorem 3.4]) and the bathtub principle yields (recall that $\psi\le 0$)

$$-\int f\psi\le-\int f^*\psi_*\le-\int\chi_{B_1}\psi_*,$$

where f^* is the symmetric decreasing rearrangement of f (see Definition 5.1). The study of the first term $\int f\psi^c$ is more involved. Using the Brunn-Minkowski inequality, we obtain the following crucial comparison:

$$(\psi^c)^*\le(\psi_*)^c.$$

Combining this inequality with the Hardy-Littlewood inequality yields

$$\int f\psi^c\le\int f^*(\psi^c)^*\le\int f^*(\psi_*)^c. \tag{1.5}$$

Additionally, as $(\psi_*)^c$ is non-increasing, χ_{B_1} is a maximiser of

$$\sup\left\{\int\widetilde{f}(\psi_*)^c\ : 0\le\widetilde{f}\le 1,\ \int\widetilde{f}=\omega_d\right\}, \tag{1.6}$$

so that $\int f^*(\psi_*)^c\le\int(\psi_*)^c\chi_{B_1}$. Lastly, by (1.5), $\int f\psi^c\le\int\chi_{B_1}(\psi_*)^c$. This eventually proves that unit balls maximise $\mathcal{E}(\omega_d)$.

As for uniqueness, the key property to establish is that $(\psi_*)^c$ is decreasing on B_1 (see Lemma 5.3). Indeed, by [16, Theorem 3.4], this implies that χ_{B_1} is the unique maximiser of (1.6). Combining this with the fact that the inequalities in (1.5) are now equalities, we obtain that $f^*=\chi_{B_1}$, so that $f=\chi_E$ for some $E\subset\mathbb{R}^d$. Using the equality case of the Brunn-Minkowski inequality, we then show that (up to a translation) $f=\chi_{B_1}$, concluding the proof.

1.2. **Motivation.**

In [3], the following variational problem was introduced:

$$\inf_{|E|=\omega_d} \{P(E) + \alpha \Upsilon_p(E)\}, \tag{1.7}$$

where $\alpha > 0$ and where Υ_p is the functional Υ defined in (1.1) with the cost $c(x, y) = |x - y|^p$. Such a variational problem may be used to model the formation of bi-layer biological membranes (see [21, 18]). Existence of minimisers were obtained in the series of work [3, 23, 19, 4].

Notice that (1.7) is an isoperimetric problem with a non-local term Υ_p. One of the best-known examples of this type of problem is Gamow's liquid drop model for the atomic nucleus. Since the beginning of the 2010s (see [5] for an historical perspective), this model has received a lot of attention from the mathematical community, and several versions of it have been studied, see for instance [15, 13, 14, 12]. In this framework, the perimeter term represents the local attractive forces while the repulsive non-local term is given by the Riesz potential

$$V_\beta(E) := \int_E \int_E \frac{dxdy}{|x - y|^{d-\beta}},$$

where $\beta \in (0, d)$. A consequence of Riesz's rearrangement inequality is that balls are the volume constrained maximisers of V_β. This illustrates the competition between the perimeter and the Riesz potential. It is thus natural to investigate whether similar properties hold for (1.7). In our case, the proof is much more involved since the rearrangement argument does not seem to work well for the primal problem. We consider instead the dual problem Υ^* and study the (fortunately favourable) interplay between rearrangement and $c-$transforms.

As a closing remark, we point out that the functional Υ is a particular case of the optimal partial transport problem studied in [8, 7].

1.3. **Organization of the article.**

The paper is structured as follows. In Section 2, we introduce the notation and review standard facts related to optimal transport in complete separable metric spaces. In Section 3, we obtain preliminary results on the functional Υ defined in compact spaces. In Section 4, we establish Theorem 1.1. Eventually, in Section 5, we prove Theorem 1.3.

2. Notation and preliminary results

2.1. **Notation.**

Let (X, d_X) be a Polish space endowed with a positive Radon measure λ.

Given a function $f : X \to \mathbb{R}$, we decompose it as:

$$f = f_+ + f_- \quad \text{with} \quad f_+ := \max(0, f) := 0 \vee f \quad \text{and} \quad f_- := \min(0, f) := 0 \wedge f.$$

Let us stress that f_- is non-positive, contrary to the classical decomposition of a function into its positive and negative parts.

We endow $\mathcal{M}_+(X)$ with the topology of weak-$*$ convergence, that is the topology induced by duality with $C_b(X)$. The convergence of a sequence $\mu_n \in \mathcal{M}_+(X)$ to $\mu \in \mathcal{M}_+$ is written: $\mu_n \overset{*}{\rightharpoonup} \mu$ as $n \to \infty$.

Given a measure $\mu \in \mathcal{M}_+(X)$ and a set $A \subset X$, the restriction of μ to A is the measure $\mu \llcorner A$ defined as $\mu \llcorner A(B) := \mu(B \cap A)$ for every Borel set B of X. The support of μ, denoted by $\operatorname{supp} \mu$, is the closed set defined by

$$\operatorname{supp} \mu := \{x \in X : \mu(A) > 0 \text{ for all open set A containing } x\}.$$

Given $f \in L^1(X, \lambda)$ the support of f is defined as the support of the measure $f d\lambda$ and denoted by $\operatorname{supp} f$. We identify the measure $f d\lambda$ with its density f and write $f_n \overset{*}{\rightharpoonup} f$ as $n \to \infty$ to signify that $\int f_n \xi$ converges to $\int f \xi$ for every $\xi \in C_b(X)$.

Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$, we denote by $\text{Leb}(f)$ the set of its Lebesgue points.

Given $x \in \mathbb{R}^d$ and $r > 0$, $B_r(x)$ denotes the open ball of radius r centred at x, and B_r denotes the open ball of radius r centred at 0. The closed ball of radius r centred at x is denoted by $\overline{B}_r(x)$. The volume of the unit ball in $\mathbb{R}^d$ is denoted by ω_d.

Given two sets A, B of $\mathbb{R}^d$, we define their sum $A + B := \{a + b,\, a \in A,\, b \in B\}$. The gap between A and B is $d(A, B) := \inf\{|a - b|,\, a \in A,\, b \in B\}$.

2.2. **Optimal transport theory.**

In this subsection, we recall some results regarding standard optimal transport theory. Most of the material presented here comes from [22, Chapter 1].

Let (X, d_X) be a complete separable metric space (*i.e.* a Polish space) and let $c : X \times X \to \mathbb{R}$ be measurable. Given $\mu, \nu \in \mathcal{M}_+(X)$ such that $\mu(X) = \nu(X)$, the Kantorovitch problem with marginals μ and ν and cost c is

$$\mathcal{T}_c(\mu, \nu) := \inf\left\{\int c\, d\gamma :\, \gamma \in \Pi(\mu, \nu)\right\}, \tag{2.1}$$

where $\Pi(\mu, \nu)$ is the set of transport plans between μ and ν, *i.e.*

$$\Pi(\mu, \nu) := \{\gamma \in \mathcal{M}_+(X \times X) :\, \gamma_x = \mu,\, \gamma_y = \nu\}.$$

Problem (2.1) admits a dual formulation given by

$$\mathcal{T}_c^*(\mu, \nu) := \sup\left\{\int \varphi\, d\mu + \int \psi\, d\nu :\, \varphi, \psi \in C_b(X),\, \varphi \oplus \psi \le c\right\}, \tag{2.2}$$

where the function $\varphi \oplus \psi$ is defined on $X \times X$ by $(\varphi \oplus \psi)(x, y) := \varphi(x) + \psi(y)$.

Theorem 2.1 (Theorem 1.7 of [22])**.** *Let $c : X \times X \to \mathbb{R}$ be lower semi-continuous and bounded from below and let $\mu, \nu \in \mathcal{M}_+(X)$ with $\mu(X) = \nu(X)$. Then (2.1) admits a solution and*

$$\mathcal{T}_c(\mu, \nu) = \mathcal{T}_c^*(\mu, \nu).$$

Using the notion of c-transform of a function, the maxima of (2.2) can be further characterised.

Definition 2.2. Given a function $\xi : X \to \mathbb{R} \cup \{+\infty\}$, we define its c-transform (or c-conjugate) $\xi^c : X \to \mathbb{R} \cup \{-\infty\}$ by

$$\xi^c(y) := \inf_{x \in X} \{c(x, y) - \xi(x)\}.$$

Denoting $\bar{c}(y, x) := c(x, y)$, the $\bar{c}$-transform of $\zeta : X \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\zeta^{\bar{c}}(x) := \inf_{y \in X} \{\bar{c}(y, x) - \zeta(y)\}.$$

A function $\psi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be $\bar{c}$-concave if there exists $\xi : X \to \mathbb{R} \cup \{+\infty\}$ such that $\psi = \xi^c$ (the definition of c-concavity is analogous).

Definition 2.3. Let (X, d_X) be a metric space and $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be increasing and such that $\omega(0) = 0$. A function $\varphi : X \to \mathbb{R}$ is ω-continuous if for all $x, x' \in X$,

$$|\varphi(x) - \varphi(x')| \le \omega(d_X(x, x')).$$

Similarly, we say that $c : X \times X \to \mathbb{R}$ is ω-continuous if for all $x, x', y, y' \in X$,

$$|c(x, y) - c(x', y')| \le \omega(d_X(x, x') + d_X(y, y')).$$

Proposition 2.4. *Let $\varphi, \psi : X \to \mathbb{R}$ be fixed and assume that φ^c and $\psi^{\bar{c}}$ take real values. The following statements hold:*

(i) If c is ω-continuous, then φ^c is also ω-continuous,
(ii) $\varphi^{c\bar{c}} \ge \varphi$, and $\varphi^{c\bar{c}} = \varphi$ if and only if φ is c-concave,
(iii) φ^c is the largest function ψ compatible with the constraint $\varphi \oplus \psi \le c$ and $\psi^{\bar{c}}$ is the largest function φ compatible with the constraint $\varphi \oplus \psi \le c$.

Remark that if X is compact and φ, ψ and c are bounded then φ^c and $\psi^{\bar{c}}$ take real values. Moreover, if c is continuous, say ω-continuous, the proposition states that φ^c and $\psi^{\bar{c}}$ are ω-continuous. This yields the following existence result for (2.2).

Theorem 2.5 (Proposition 1.11 of [22])**.** *Let X be a compact metric space and $c : X \times X \to \mathbb{R}$ be continuous. Then there exists a solution (φ, ψ) to (2.2), where φ is c-concave and $\psi = \varphi^c$. In particular,*

$$\mathcal{T}_*^c(\mu,\nu) = \max\left\{\int \varphi\, d\mu + \int \varphi^c\, d\nu : \varphi \ \ c\text{-concave}\right\}.$$

A pair of functions maximising (2.2) is called a pair of Kantorovitch potentials.

3. Study of Υ in compact metric spaces

Let (X, d_X) be a compact metric space and let $c : X \times X \to \mathbb{R}$ be a continuous cost function. We endow (X, d_X) with a measure $\lambda \in \mathcal{M}_+(X)$ such that $\lambda(X) > 0$ and denote by $L^1(X)$ the set of $\mathbb{R}$-valued functions integrable with respect to λ. Given $f \in L^1(X)$, we define the set of admissible transport plans

$$\Pi_f := \{\gamma \in \mathcal{M}_+(X \times X) : \gamma_x = f,\, \gamma_y \le 1 - f\}$$

and the primal problem

$$\Upsilon(f) := \inf\left\{\int c\, d\gamma : \gamma \in \Pi_f\right\}. \tag{3.1}$$

Notice that Π_f is empty whenever f does not satisfy $0 \le f \le 1$ or when $\int f\, d\lambda > \lambda(X)/2$. In the other cases, there exists $g \in L^1(X)$ such that $g \ge 0$, $f + g \le 1$ and $\int g\, d\lambda = \int f\, d\lambda$. Thus,

$$\gamma := \frac{1}{\int f\, d\lambda}(f\, d\lambda) \otimes (g\, d\lambda) \ \in \Pi_f,$$

and Π_f is not empty. We now fix $0 < m \le \lambda(X)/2$ and define

$$L^1_m := \left\{f \in L^1(X, [0,1]) : \int f \ \le m\right\}.$$

Given $f \in L^1_m$ and $\varphi, \psi \in C(X)$, we set

$$K_f(\varphi,\psi) := \int (f\varphi + (1-f)\psi)\ d\lambda \tag{3.2}$$

and define the dual problem

$$\Upsilon^*(f) := \sup\left\{K_f(\varphi,\psi) : (\varphi,\psi) \in \Phi\right\}, \tag{3.3}$$

where

$$\Phi := \{(\varphi,\psi) \in C(X) \times C(X),\, \psi \le 0,\, \varphi \oplus \psi \le c\}.$$

For the remainder of the section we fix $f \in L^1_m$. As in the classical theory of optimal transport, a simple application of the direct method of Calculus of Variations shows that (3.1) admits a minimiser.

Proposition 3.1. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then, the infimum in* (3.1) *is a minimum.*

Remark 3.2. If we let $f \in L^1_m$, by Proposition 3.1, there exists $\gamma \in \mathcal{M}_+(X \times X)$ optimal for $\Upsilon(f)$. Notice that γ solves the classical optimal transport problem from f towards $g := \gamma_y$ defined by (2.1). Moreover, we have the identity $\Upsilon(f) = \mathcal{T}_c(f, g)$.

Let us now show that $\Upsilon^*(f) = \Upsilon(f)$ and that (3.3) admits a maximising pair (φ, ψ). We first establish that we can reduce the set of competitors for (3.3). To simplify the notation we denote by φ^c_- the function $(\varphi^c)_- := \varphi^c \wedge 0$.

Lemma 3.3. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then, there holds*

$$\Upsilon^*(f) = \sup\{K_f(\psi^{\bar{c}}, \psi) : \psi = \varphi^c_- \text{ for some } \varphi \in \Phi'\}, \tag{3.4}$$

where

$$\Phi' := \left\{\varphi \in C(X),\, \varphi = (\varphi^c_-)^{\bar{c}},\, \max \varphi^c \geq 0\right\}. \tag{3.5}$$

Proof.
Step 1. We can replace ψ by φ^c_- and assume that $\max \varphi^c \geq 0$.

Let $(\varphi, \psi) \in \Phi$. By Proposition 2.4 (iii), $\psi \leq \varphi^c$, so that $\psi \leq \varphi^c \wedge 0 = \varphi^c_-$. As $1 - f \geq 0$, $K_f(\varphi, \varphi^c_-) \geq K_f(\varphi, \psi)$. Therefore, we can restrict the maximisation to the pairs (φ, φ^c_-) in the supremum (3.3). Now, if $\max \varphi^c = -t < 0$ we set $\widetilde{\varphi} := \varphi - t$ so that $\widetilde{\varphi}^c = \varphi^c + t$. Consequently, $\max \widetilde{\varphi}^c = 0$ and in particular, $\widetilde{\varphi}^c_- = \widetilde{\varphi}^c$ so that $(\widetilde{\varphi}, \widetilde{\varphi}^c) \in \Phi$. We then compute

$$\begin{aligned} K_f(\widetilde{\varphi}, \widetilde{\varphi}^c) &= \int f(\widetilde{\varphi} - \widetilde{\varphi}^c)\, d\lambda + \int \widetilde{\varphi}^c\, d\lambda \\ &= \int f(\varphi - \varphi^c)\, d\lambda + \int \varphi^c\, d\lambda + t(\lambda(X) - 2m) \\ &= K_f(\varphi, \varphi^c_-) + t(\lambda(X) - 2m). \end{aligned}$$

As $2m \leq \lambda(X)$ we obtain $K_f(\widetilde{\varphi}, \widetilde{\varphi}^c_-) \geq K_f(\varphi, \varphi^c_-)$. Hence

$$\Upsilon^*(f) = \sup\left\{K_f(\varphi, \varphi^c_-) : \varphi \in C(X),\, \max \varphi^c \geq 0\right\}.$$

Step 2. There holds $\varphi = (\varphi^c_-)^{\bar{c}}$.

Let us introduce the mapping $P : C(X) \to C(X)$ defined by $P(\varphi) := (\varphi^c_-)^{\bar{c}}$. For $\varphi \in C(X)$, $\varphi^c \geq \varphi^c_-$, so that $P(\varphi) = (\varphi^c_-)^{\bar{c}} \geq \varphi^{c\bar{c}}$. By Proposition 2.4 (ii), $\varphi^{c\bar{c}} \geq \varphi$, hence

$$P(\varphi) \geq \varphi. \tag{3.6}$$

By Proposition 2.4 (ii) again there holds $P(\varphi)^c = (\varphi^c_-)^{\bar{c}c} \geq \varphi^c_-$. Taking the negative part yields

$$P(\varphi)^c_- \geq \varphi^c_-. \tag{3.7}$$

We deduce from (3.6) and (3.7) that

$$K_f(P(\varphi), P(\varphi)^c_-) \geq K_f(\varphi, \varphi^c_-).$$

Now, we observe that if $\max \varphi^c \geq 0$ we also have $\max \varphi^c_- = 0$ and, by (3.7), $\max P(\varphi)^c_- = 0$ which implies that $\max P(\varphi)^c \geq 0$. Hence,

$$\Upsilon^*(f) = \sup\left\{K_f(\widetilde{\varphi}, \widetilde{\varphi}^c_-) : \widetilde{\varphi} \in C(X),\, \max \widetilde{\varphi}^c \geq 0,\, \widetilde{\varphi} = P(\varphi) \text{ for some } \varphi \in C(X)\right\}. \tag{3.8}$$

To conclude, we show that $P(P(\varphi)) = P(\varphi)$ for any $\varphi \in C(X)$. By (3.6), $P(P(\varphi)) \geq P(\varphi)$. Taking the $\bar{c}$-transform in (3.7) yields $P(P(\varphi)) \leq P(\varphi)$ and we have indeed $P(P(\varphi)) = P(\varphi)$. Hence we have $\tilde{\varphi} \in \Phi'$ in (3.8) and we get

$$\Upsilon^*(f) = \sup\left\{K_f(\tilde{\varphi}, \tilde{\varphi}^c_-) : \tilde{\varphi} \in \Phi'\right\}. \tag{3.9}$$

Finally, by definition $\tilde{\varphi} = (\tilde{\varphi}^c_-)^{\bar{c}}$ for $\tilde{\varphi} \in \Phi'$ and (3.4) follows from (3.9) by letting $\psi := \tilde{\varphi}^c_-$. □

We can now establish that the supremum in (3.4) is reached.

Proposition 3.4. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then, the set Φ' is compact in $(C(X), \|\cdot\|_\infty)$ and the suprema in* (3.4) *and* (3.3) *are attained.*

Proof.
Let us show that Φ' is compact. Let φ_n be a sequence in Φ'. The function c is ω-continuous for some modulus of continuity $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, so that by Proposition 2.4 (ii) for every $n \geq 0$, $\varphi^c_n := (\varphi_n)^c$ and $\varphi^c_{n-} := ((\varphi_n)^c)_-$ are ω-continuous. By definition of Φ', $\varphi_n = (\varphi^c_{n-})^{\bar{c}}$, so that φ_n is also ω-continuous for every $n \geq 0$. Let us show that the sequences φ_n and φ^c_{n-} are uniformly

bounded in $(C(X), \|\cdot\|_\infty)$. We observe that for every $n \geq 0$, $\max \varphi_n^c \geq 0$. Denoting by x_n a point of X such that $\varphi_{n-}^c(x_n) = 0$, by ω-continuity we have for $x \in X$ and $n \geq 0$,

$$-\omega(\mathrm{diam}(X)) \leq -\omega(|x - x_n|) \leq \varphi_{n-}^c(x) - \varphi_{n-}^c(x_n) = \varphi_{n-}^c(x) \leq 0.$$

Thus the sequence φ_{n-}^c is uniformly bounded in $(C(X), \|\cdot\|_\infty)$. By definition of the c-transform

$$\min_{X\times X} c - \max_X \varphi_{n-}^c \leq (\varphi_{n-}^c)^{\bar{c}} \leq \max_{X\times X} c - \min_X \varphi_{n-}^c.$$

Hence the sequence φ_n is also uniformly bounded. By Arzelá-Ascoli's theorem, there exists a pair $(\varphi, \psi) \in C(X)\times C(X)$ such that, up to extraction, $(\varphi_n, \varphi_{n-}^c)$ converges uniformly to (φ, ψ).

Let us show that $\varphi \in \Phi'$. By Proposition 2.4 (iii) and by uniform convergence $\varphi_n^c \to \varphi^c$ as $n \to \infty$ so that

$$\varphi_{n-}^c \to \varphi_-^c \quad \text{uniformly as } n \to \infty, \tag{3.10}$$

which yields $\psi = \varphi_-^c$. From (3.10) and the uniform continuity of c, we deduce that

$$(\varphi_{n-}^c)^{\bar{c}} = \varphi_n \to (\varphi_-^c)^{\bar{c}} \quad \text{uniformly as } n \to \infty.$$

Since $\varphi_n \to \varphi$ as $n \to \infty$, we obtain $\varphi = (\varphi_-^c)^{\bar{c}}$. Lastly, by uniform convergence, the fact that $\max \varphi_n^c \geq 0$ for all $n \geq 0$ implies that $\max \varphi^c \geq 0$, so that $\varphi \in \Phi'$. This shows that Φ' is a compact subset of $(C(X), \|\cdot\|_\infty)$.

Let now ψ_n be a maximising sequence for (3.4). For all $n \geq 0$, there exists $\varphi_n \in \Phi'$ such that $\psi_n = \varphi_{n-}^c$. By compactness of Φ', $\varphi_n \to \varphi$ as $n \to \infty$ for some $\varphi \in \Phi'$. Setting $\psi = \varphi_-^c$, we have $\psi_n \to \psi$ and $\psi_n^{\bar{c}} \to \psi^{\bar{c}}$ as $n \to \infty$. The functional K_f being continuous with respect to uniform convergence, we obtain

$$K_f(\psi^{\bar{c}}, \psi) = \lim K_f(\psi_n^{\bar{c}}, \psi_n) = \Upsilon^*(f).$$

This proves that ψ is a maximiser for (3.4) and by Lemma 3.3, ψ also maximises (3.3). □

We are now ready to prove that there is no duality gap between (3.1) and (3.3). The proof is an adaptation of [22, Section 1.6.3].

Proposition 3.5. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Then,*

$$\Upsilon^*(f) = \Upsilon(f).$$

Proof.

Step 1. Definition of H and first properties.

For $p \in C(X \times X)$, we define

$$H(p) := -\sup\left\{ \int (f\varphi + (1-f)\psi)\, d\lambda : (\varphi, \psi) \in \Phi_p \right\}$$

where

$$\Phi_p := \{(\varphi, \psi) \in C(X) \times C(X),\ \psi \leq 0,\ \varphi \oplus \psi \leq c - p\}.$$

We first observe that $c - p$ is continuous and bounded from below. Thus, by applying Proposition 3.4 with $c - p$ in place of c, we see that the above supremum is a maximum.

Let us now show that H is convex. Let $p_0, p_1 \in C(X \times X)$ and $\theta \in [0, 1]$ and let us set $p := (1-\theta)p_0 + \theta p_1$. We denote by (φ_0, ψ_0) and (φ_1, ψ_1) two maximising pairs associated with p_0 and p_1 and set $\varphi := (1-\theta)\varphi_0 + \theta\varphi_1$, $\psi := (1-\theta)\psi_0 + \theta\psi_1$. We see that (φ, ψ) is an admissible pair ($\psi \leq 0$ and $\varphi \oplus \psi \leq c - p$), so that

$$H(p) \leq -\int (f\varphi + (1-f)\psi)\, d\lambda = (1-\theta)H(p_0) + \theta H(p_1).$$

This proves that H is convex.

Next, we establish that H is lower semi-continuous in $(C(X \times X), \|\cdot\|_\infty)$. Let p_n and p be elements of $C(X \times X)$ such that $p_n \to p$ uniformly as $n \to \infty$. The sequence $c - p_n$ is uniformly

equi-continuous. Therefore, proceeding as in the proof of Proposition 3.4, there exists a sequence of uniformly bounded and equi-continuous admissible pairs (φ_n, ψ_n) such that

$$H(p_n) = -\int (f\varphi_n + (1-f)\psi_n)\, d\lambda \qquad \text{for every } n \geq 0.$$

We first extract a subsequence $p_{n'}$ such that $\lim_{n'} H(p_{n'}) = \liminf_n H(p_n)$. By Arzelà-Ascoli's theorem, there exists $(\varphi, \psi) \in C(X) \times C(X)$ such that $\varphi_{n'} \to \varphi$ and $\psi_{n'} \to \psi$ uniformly as $n' \to \infty$. By pointwise convergence, $\psi \leq 0$ and $\varphi \oplus \psi \leq c - p$. Passing to the limit yields

$$H(p) \leq -\int (f\varphi + (1-f)\psi)\, d\lambda = -\lim_{n'} \int (f\varphi_{n'} + (1-f)\psi_{n'})\, d\lambda = \liminf_n H(p_n).$$

Thus H is lower semi-continuous.

Step 2. Absence of duality gap.

Since H is convex and lower semi-continuous on the Banach space $(C(X \times X), \|\cdot\|_\infty)$, there holds $H = H^{**}$ on $C(X \times X)$. Here, for a Banach space $\mathcal{X}$ and a function $F : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, F^* denotes the Legendre transform of F defined on the topological dual $\mathcal{X}^*$ of $\mathcal{X}$ by

$$F^*(x^*) := \sup \{x^*(x) - F(x) : x \in \mathcal{X}\}.$$

In particular,

$$\Upsilon^*(f) = -H(0) = -H^{**}(0) = \inf\{H^*(\gamma) : \gamma \in \mathcal{M}(X \times X)\}. \tag{3.11}$$

We now compute H^*. Let $\gamma \in \mathcal{M}(X \times X)$. By definition,

$$H^*(\gamma) = \sup_{p \in C(X \times X)} \left\{ \int p\, d\gamma + \sup_{(\varphi,\psi) \in \Phi_p} \left\{ \int (f\varphi + (1-f)\psi)\, d\lambda \right\} \right\}.$$

Let us first assume that there exists $q \in C(X \times X, \mathbb{R}_+)$ such that $t := -\int q\, d\gamma > 0$. We set $\varphi = \min c$, $\psi = 0$ and $p_n := -nq$ for $n \geq 1$. We obtain

$$H^*(\gamma) \geq nt - m \min c \ \to \infty \qquad \text{as } n \to \infty.$$

Thus, when computing $H^*(\gamma)$, we may assume that $\gamma \geq 0$. We rewrite $H^*(\gamma)$ as

$$\begin{aligned} H^*(\gamma) = \int c\, d\gamma + \sup_{p \in C(X \times X)} \sup_{(\varphi,\psi) \in \Phi_p} \Big\{ & \int (p - c + \varphi \oplus \psi)\, d\gamma \\ & + \int \varphi\, d(f\lambda - \gamma_x) + \int \psi\, d((1-f)\lambda - \gamma_y)) \Big\}. \end{aligned} \tag{3.12}$$

Let us set

$$G(\gamma) := \sup \left\{ \int \varphi\, d(f\lambda - \gamma_x) + \int \psi\, d((1-f)\lambda - \gamma_y) : (\varphi, \psi) \in C(X) \times C(X),\ \psi \leq 0 \right\}.$$

On the one hand, given $(\varphi, \psi) \in \Phi_p$ and $\gamma \geq 0$,

$$\int (p - c + \varphi \oplus \psi)\, d\gamma \leq 0.$$

Therefore, $H^*(\gamma) \leq \int c\, d\gamma + G(\gamma)$. On the other hand, given (φ, ψ) admissible for $G(\gamma)$, setting $p = c - \varphi \oplus \psi$ yields the converse inequality thanks to (3.12). Hence

$$H^*(\gamma) = \int c\, d\gamma + G(\gamma). \tag{3.13}$$

Given $\gamma \in \mathcal{M}_+(X \times X)$, we have $G(\gamma) = 0$ if $\gamma \in \Pi_f$ and $G(\gamma) = +\infty$ otherwise. Combining this with (3.13), we obtain that for $\gamma \in \mathcal{M}(X \times X)$,

$$H^*(\gamma) = \begin{cases} \displaystyle\int c\, d\gamma & \text{if } \gamma \in \Pi_f, \\ +\infty & \text{in the other cases.} \end{cases}$$

Taking the infimum with respect to $\gamma \in \mathcal{M}(X \times X)$ and recalling (3.11), we get

$$\Upsilon^*(f) = \inf \{H^*(\gamma) : \gamma \in \mathcal{M}(X \times X)\} = \inf \left\{ \int c \, d\gamma : \gamma \in \Pi_f \right\} = \Upsilon(f),$$

which concludes the proof. □

Remark 3.6. There is still no duality gap between (3.1) and (3.3) if we only assume c to be lower semi-continuous. This result can be obtained by approximating c pointwise from below by a non-decreasing sequence of continuous functions.

In the remainder of the section, we focus on the properties of the potentials $(\psi^{\bar{c}}, \psi)$ maximising (3.4). We first show that the sign of $\psi^{\bar{c}c}$ enforces constraints on the local values of the marginals of any plan γ optimal for $\Upsilon(f)$.

Proposition 3.7. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Let $(\psi^{\bar{c}}, \psi)$ be a maximiser of (3.3) and let γ be a minimiser of (3.1). We set $g := \gamma_y$. Then, γ is a minimiser of (2.1) with $(\mu, \nu) = (f, g)$ and $(\psi^{\bar{c}}, \psi^{\bar{c}c})$ is a pair of Kantorovitch potentials realising the maximum in (2.2). Moreover, up to λ-negligible sets,*

$$f + g \equiv 1 \quad \text{on } \{\psi^{\bar{c}c} < 0\} \qquad \text{and} \qquad g \equiv 0 \quad \text{on } \{\psi^{\bar{c}c} > 0\}. \tag{3.14}$$

Proof.
By Remark 3.2, γ realises the minimum in (2.1) and $\Upsilon(f) = \mathcal{T}_c(f, g)$. As there is no duality gap in (3.1) nor in (2.1), $\Upsilon^*(f) = \mathcal{T}_c^*(f, g)$. Additionally, $(\psi^{\bar{c}}, \psi^{\bar{c}c})$ is admissible for $\mathcal{T}_c^*(f, g)$ and $\psi = (\psi^{\bar{c}c})_-$. Thus

$$K_f(\psi^c, \psi) = \int f\psi^{\bar{c}} \, d\lambda + \int (1 - f)(\psi^{\bar{c}c})_- \, d\lambda = \mathcal{T}_c^*(f, g) \geq \int f\psi^{\bar{c}} \, d\lambda + \int g\psi^{\bar{c}c} \, d\lambda. \tag{3.15}$$

Hence

$$\int (1 - f - g)(\psi^{\bar{c}c})_- \, d\lambda \geq \int g(\psi^{\bar{c}c})_+ \, d\lambda.$$

Since $(1 - f - g)(\psi^{\bar{c}c})_- \leq 0$ and $g(\psi^{\bar{c}c})_+ \geq 0$, the integrands must vanish λ-almost everywhere: we deduce (3.14). Additionally, the inequality in (3.15) is an equality. Consequently, $(\psi^{\bar{c}}, \psi^{\bar{c}c})$ is a pair of Kantorovitch potentials for (2.2). □

To end this section, we establish a comparison principle on the potentials maximising (3.4). We say that a set $\Psi \subset C(X)$ admits a minimal (respectively maximal) element for the relation $\leq$ if there exists $\psi_0 \in \Psi$ such that for any $\psi \in \Psi$, $\psi_0 \leq \psi$ (respectively $\psi_0 \geq \psi$).

Proposition 3.8. *Assume that X is a compact metric space and that $c \in C(X \times X, \mathbb{R})$. Let $f \in L^1_m$ and let us define*

$$\Psi_f := \left\{ \psi, \, \psi = \varphi^c_- \text{ for some } \varphi \in \Phi' \text{ and } K_f(\psi^{\bar{c}}, \psi) = \Upsilon^*(f) \right\},$$

where K_f is defined in (3.2) and Φ' in Lemma 3.3.
Then:

(i) Ψ_f admits a maximal element for the relation $\leq$, denoted by ψ_f in the sequel,
(ii) For $f_1, f_2 \in L^1_m$, there holds $f_1 \leq f_2 \implies \psi_{f_1} \geq \psi_{f_2}$.

Proof.
Step 1. Sufficient condition and preliminary claim.
Notice that Ψ_f is not empty by Proposition 3.4. To obtain (i), we prove that the set

$$\Phi_f := \left\{ \varphi \in \Phi', \, K_f(\varphi, \varphi^c_-) = \Upsilon^*(f) \right\}$$

admits a minimal element φ_f and then $\psi_f := (\varphi_f^c)_-$ is the desired maximal element of Ψ_f. Let us make a preliminary observation.

Claim. *Let $f_1, f_2 \in L^1_m$ with $f_1 \leq f_2$ and set $\varphi_i \in \Phi_{f_i}$ for $i \in \{1, 2\}$. Then $\varphi_\wedge := \varphi_1 \wedge \varphi_2 \in \Phi_{f_1}$.*

Let us first prove that $\varphi_\wedge \in \Phi'$. In the sequel we write $\varphi_i^c := (\varphi_i)^c$ and $\varphi_{i-}^c := ((\varphi_i)^c)_-$ for $i \in \{1, 2, \wedge\}$. We observe that $\varphi_\wedge \in C(X)$. By definition of the c-transform, we obtain

$$\varphi_\wedge^c = (\varphi_1 \wedge \varphi_2)^c \geq \varphi_1^c \vee \varphi_2^c \tag{3.16}$$

and

$$(\varphi_1 \vee \varphi_2)^c = \varphi_1^c \wedge \varphi_2^c. \tag{3.17}$$

Since for $i \in \{1, 2\}$, $\max \varphi_i^c \geq 0$ we have by (3.16) that $\max \varphi_\wedge^c \geq 0$.

We now prove that $(\varphi_{\wedge-}^c)^{\bar{c}} = \varphi_\wedge$. We observe that $\varphi_{\wedge-}^c \leq \varphi_\wedge^c$ which implies $(\varphi_{\wedge-}^c)^{\bar{c}} \geq \varphi_\wedge^{c\bar{c}}$. By Proposition 2.4 (ii), $\varphi_\wedge^{c\bar{c}} \geq \varphi_\wedge$ so that $(\varphi_{\wedge-}^c)^{\bar{c}} \geq \varphi_\wedge$. Conversely, taking the negative part of (3.16), we have $\varphi_{\wedge-}^c \geq (\varphi_1^c \vee \varphi_2^c)_- = \varphi_{1-}^c \vee \varphi_{2-}^c$. Taking the $\bar{c}$-transform and using (3.17) yields

$$(\varphi_{\wedge-}^c)^{\bar{c}} \leq (\varphi_{1-}^c \vee \varphi_{2-}^c)^{\bar{c}} = (\varphi_{1-}^c)^{\bar{c}} \wedge (\varphi_{2-}^c)^{\bar{c}} = \varphi_1 \wedge \varphi_2 = \varphi_\wedge.$$

Hence $(\varphi_{\wedge-}^c)^{\bar{c}} = \varphi_\wedge$ and $\varphi_\wedge \in \Phi'$.

We now show that the pair $(\varphi_\wedge, \varphi_{\wedge-}^c)$ maximises $\Upsilon^*(f_1)$. We set

$$\Delta_K := K_{f_1}(\varphi_\wedge, \varphi_{\wedge-}^c) - K_{f_1}(\varphi_1, \varphi_{1-}^c) = \int f_1(\varphi_\wedge - \varphi_1) + \int (1 - f_1)(\varphi_{\wedge-}^c - \varphi_{1-}^c)\,.$$

By optimality of φ_1, $\Delta_K \leq 0$. Let us prove the converse inequality. Substituting $f_1 = f_2 + f_1 - f_2$ in the definition of Δ_K, we obtain

$$\Delta_K = \int f_2(\varphi_\wedge - \varphi_1) + \int (1 - f_2)(\varphi_{\wedge-}^c - \varphi_{1-}^c) + \int (f_2 - f_1)(\varphi_1 - \varphi_\wedge + \varphi_{\wedge-}^c - \varphi_{1-}^c)\,.$$

We have $f_2 - f_1 \geq 0$ and $\varphi_1 - \varphi_\wedge \geq 0$. Additionally, $\varphi_\wedge^c \geq \varphi_1^c$, so that $\varphi_{\wedge-}^c \geq \varphi_{1-}^c$. Thus the last integral in Δ_K is non-negative. Adding and subtracting $f_2\varphi_2$ in the first integral yields

$$\Delta_K \geq \int f_2\varphi_2 + \int f_2(\varphi_\wedge - \varphi_1 - \varphi_2) + \int (1 - f_2)(\varphi_{\wedge-}^c - \varphi_{1-}^c)\,. \tag{3.18}$$

Let us set $\varphi_\vee := \varphi_1 \vee \varphi_2$. By optimality of φ_2, we have $K_{f_2}(\varphi_2, \varphi_{2-}^c) \geq K_{f_2}(\varphi_\vee, \varphi_{\vee-}^c)$, which rewrites as

$$\int f_2\varphi_2 \geq \int f_2\varphi_\vee + \int (1 - f_2)(\varphi_{\vee-}^c - \varphi_{2-}^c)\,.$$

Injecting this inequality in the first term of the right-hand side of (3.18) yields

$$\Delta_K \geq \int f_2(\varphi_\wedge + \varphi_\vee - \varphi_1 - \varphi_2) + \int (1 - f_2)(\varphi_{\vee-}^c + \varphi_{\wedge-}^c - \varphi_{1-}^c - \varphi_{2-}^c)\,. \tag{3.19}$$

The integrand in the first integral of (3.19) vanishes. Regarding the second term, using (3.17) and (3.16) we obtain

$$\varphi_{\vee-}^c + \varphi_{\wedge-}^c \geq \varphi_{1-}^c \wedge \varphi_{2-}^c + \varphi_{1-}^c \vee \varphi_{2-}^c = \varphi_{1-}^c + \varphi_{2-}^c.$$

Hence the integrand in the second integral is non-negative. We conclude that $\Delta_K \geq 0$ and finally that $\Delta_K = 0$ so that the claim is proved.

Step 2. Construction of the minimal element of Φ_f.

By Lemma 3.4, Φ' is compact. As $\varphi \mapsto K_f(\varphi, \varphi^{c}{}_-)$ is continuous for the norm of uniform convergence, Φ_f is compact as well. Let $(\varphi_j)_{j\geq 0}$ be a dense subset of Φ_f. For $x \in X$ and $j \geq 0$, we define $\widetilde{\varphi}_j$ and φ_f by

$$\widetilde{\varphi}_j(x) := \min(\varphi_0(x), \dots, \varphi_j(x)) \qquad \text{and} \qquad \varphi_f(x) := \inf\{\varphi(x),\, \varphi \in \Phi_f\}.$$

Using our preliminary claim with $f_1 = f_2 = f$ recursively, we obtain that for any $j \geq 0$, $\widetilde{\varphi}_j \in \Phi_f$. As Φ_f is compact and $\widetilde{\varphi}_j \to \varphi_f$ pointwise, we obtain that $\widetilde{\varphi}_j \to \varphi_f$ uniformly and $\varphi_f \in \Phi_f$, so that φ_f is the desired minimal element of Φ_f.

Step 3. Conclusion.

Taking $\psi_f := \varphi_{f-}^c$ proves (i). Let $f_1 \leq f_2$ as given in the statement of (ii). By the previous step, there exist φ_1, φ_2 respective minimal elements for Φ_{f_1} and Φ_{f_2} such that $\psi_1 := \varphi_{1-}^c$

and $\psi_2 := \varphi_{2-}^c$ are respective maximal elements for Ψ_{f_1} and Ψ_{f_2}. By the preliminary claim, $\varphi_1 \wedge \varphi_2 \in \Phi_{f_1}$ and by minimality of φ_1 we have $\varphi_1 \leq \varphi_1 \wedge \varphi_2$, so that $\varphi_2 \geq \varphi_1$. Hence $\psi_2 \leq \psi_1$. □

4. Existence of maximisers of (4.2) for translation invariant costs in $\mathbb{R}^d$

We now assume that $X = \mathbb{R}^d$, that λ is the Lebesgue measure and that $c(x,y) = k(y-x)$, with $k : \mathbb{R}^d \to \mathbb{R}_+$. We recall the following hypotheses on k.

(H1) $k \in C(\mathbb{R}^d, \mathbb{R}_+)$, $k(0) = 0$ and $k(x) \to \infty$ as $|x| \to \infty$,

(H2) $\forall x \neq 0$,

$$\limsup_{r \to 0} \frac{1}{r^d} \left| B_r(x) \cap \{y \in \mathbb{R}^d,\, k(y) < k(x)\} \right| > 0,$$

(H3) $\forall\, \sigma \in \mathbb{S}^{d-1}$, $r \mapsto k(r\sigma)$ is increasing on $\mathbb{R}_+$.

Notice that under hypotheses (H1)&(H2), there holds $k(x) > 0$ for $x \neq 0$.

The primal problem is now defined as

$$\Upsilon(f) := \inf \left\{ \int c \, d\gamma : \gamma \in \mathit{\Pi}_f \right\}, \tag{4.1}$$

where

$$\mathit{\Pi}_f := \left\{ \gamma \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d) : \gamma_x = f,\, \gamma_y \leq 1 - f \right\}.$$

The goal of this section is to prove that for every $m > 0$ the energy

$$\mathcal{E}(m) := \sup \left\{ \Upsilon(f) : f \in L^1(\mathbb{R}^d, [0,1]), \int f \, = m \right\} \tag{4.2}$$

admits a maximiser.

4.1. First properties of Υ and saturation theorem.

In this subsection, we collect some properties of the functional Υ defined in $\mathbb{R}^d$ and establish a saturation property (Theorem 4.4), namely that if γ is a minimiser for $\Upsilon(f)$ then $\gamma_y(x) \in \{f(x), 1 - f(x)\}$ for almost every $x \in \mathbb{R}^d$.

We start by proving that minimisers of (4.1) exist. The proof of this result is similar to the proof of [4, Proposition 2.1], but with weaker assumptions on the cost c and in the context of functions taking values in $[0,1]$ rather than in $\{0,1\}$.

Proposition 4.1. *Assume that k satisfies (H1). Then, for any $m > 0$ and $f \in L^1_m$, the infimum in (4.1) is attained. Additionally, given any minimiser γ of (4.1) we have $\Upsilon(f) = \mathcal{T}_c(f,g)$, where $g := \gamma_y$.*

Lastly, there exists $R = R(m)$ non-decreasing in m such that for any $f \in L^1_m$,

$$\Upsilon(f) = \min \left\{ \int c \, d\gamma : \gamma \in \mathit{\Pi}_f,\, \forall\, (x,y) \in \operatorname{supp} \gamma,\, |x-y| \leq R \right\} \tag{4.3}$$

and for any minimiser γ of (4.1), there holds $|x-y| \leq R$ on $\operatorname{supp}\gamma$.

Proof.
The strategy of the proof is to first establish (4.3) with an infimum in place of the minimum. Then we use this property to derive compactness for (4.1).

Step 1. Restricting the set of competitors for (4.1).

We let $\gamma \in \mathit{\Pi}_f$ and set $g := \gamma_y$. We want to build a competitor $\widetilde{\gamma}$ for $\Upsilon(f)$ such that for some $R > 0$, $|x-y| \leq R$ for every $(x,y) \in \operatorname{supp}\widetilde{\gamma}$. For $R > 0$, we define

$$\Gamma_R := \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\, |x-y| \geq R \right\}.$$

We consider a standard partition of $\mathbb{R}^d$ into cubes $(Q_i)_{i\geq 0}$ with side-length $\rho_1(m) := (3m)^{1/d}$. We define

$$I := \{i \geq 0,\, m_i := \gamma(\Gamma_R \cap (Q_i \times \mathbb{R}^d)) > 0\},$$

and for $i \in I$, we set

$$\gamma_{\mathrm{bad},i} := \chi_{\Gamma_R \cap (Q_i \times \mathbb{R}^d)}\gamma.$$

As $|Q_i| - \int f - \int g \geq m \geq m_i$, there exists a positive measure $\mu_i \ll \chi_{Q_i}\lambda$ such that

$$\mu_i \leq \chi_{Q_i}(1 - f - g) \qquad \text{and} \qquad \mu_i(\mathbb{R}^d) = \gamma_{\mathrm{bad},i}(\mathbb{R}^d \times \mathbb{R}^d).$$

Denoting by θ_i the first marginal of $\gamma_{\mathrm{bad},i}$ we set

$$\widetilde{\gamma}_i := \chi_{Q_i \times \mathbb{R}^d}\gamma - \gamma_{\mathrm{bad},_i} + \frac{1}{m_i}\theta_i \otimes \mu_i \ \geq 0.$$

For $i \in I^c$, we simply define $\widetilde{\gamma}_i := \chi_{(Q_i \times \mathbb{R}^d)}\gamma$. As a consequence, $\widetilde{\gamma} := \sum_{i\geq 0}\widetilde{\gamma}_i$ is a transport plan whose first marginal is f and second marginal $\widetilde{g}$ verifies $\widetilde{g} \leq g \leq 1 - f$. By construction, for $R > \sqrt{d}\rho_1(m)$, we have $\widetilde{\gamma}(\Gamma_R) = 0$.

Let us now compare the transportation cost of γ and $\widetilde{\gamma}$. We compute:

$$\begin{aligned}\int c\, d\widetilde{\gamma} - \int c\, d\gamma &= \sum_{i\in I}\int_{Q_i\times\mathbb{R}^d} c\, d\left(\frac{\theta_i \otimes \mu_i}{m_i} - \gamma_{\mathrm{bad},i}\right)\\ &\leq \left(\sum_{i\in I} m_i\right)\left(\max_{z\in\overline{Q}_{\rho_1(m)}} k(z) - \inf_{|z|\geq R} k(z)\right).\end{aligned}$$

Let us set $M := \max\{k(z),\, z \in \overline{Q}_{\rho_1(m)}\}$. By (H1), there exists $R > \sqrt{d}\rho_1(m)$ such that if $|z| > R$, then $k(z) > M$. With this choice of R we have $\int c\, d\widetilde{\gamma} \leq \int c\, d\gamma$. Lastly, whenever $\gamma(\Gamma_R) > 0$,

$$\int c\, d\widetilde{\gamma} < \int c\, d\gamma. \tag{4.4}$$

Step 2 : Lower semi-continuity of the transportation cost.

This step is classical. To prove that $\gamma \mapsto \int c\, d\gamma$ is lower semi-continuous with respect to weak convergence, we proceed by approximation. Let us assume that $\gamma_n \xrightarrow{*} \gamma$ as $n \to \infty$. For $j \geq 0$, we define $c_j := c \wedge j$. The sequence c_j is non-decreasing and converges pointwise to c. For every $j \geq 0$, $c_j \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, so that

$$\int c_j\, d\gamma = \lim_n \int c_j\, d\gamma_n \leq \liminf_n \int c\, d\gamma_n.$$

By the monotone convergence theorem,

$$\int c\, d\gamma = \lim_j \int c_j\, d\gamma \leq \liminf_n \int c\, d\gamma_n,$$

which concludes the second step of the proof.

Step 3. $\Upsilon(f)$ admits a minimiser.

Let γ_n be a minimising sequence for (4.1). Let us show that the sequence γ_n is tight. By the first step, we can assume that there exists $R = R(m)$ such that for any $n \geq 0$ there holds $|x - y| \leq R$ on $\operatorname{supp}\gamma_n$. Now, because $\int f \leq m < \infty$, there exists $R' = R'(m) > 0$ such that $\int_{\mathbb{R}^d\setminus B_{R'}} f \leq \varepsilon$. Hence,

$$\gamma_n\left(\mathbb{R}^d \times \mathbb{R}^d \setminus (B_{R'} \times B_{R+R'})\right) = \gamma_n\left(B_{R'} \times (\mathbb{R}^d \setminus B_{R+R'})\right) + \gamma_n\left((\mathbb{R}^d \setminus B_{R'}) \times \mathbb{R}^d\right) \leq 0 + \varepsilon$$

which proves that the sequence γ_n is tight. Together with the second step, this shows that (4.1) admits a minimiser. Moreover, by (4.4) for any minimiser γ of (4.1) there holds $|x - y| \leq R$ on $\operatorname{supp}\gamma$. Lastly, setting $g := \gamma_y$ the identity $\Upsilon(f) = \mathcal{T}_c(f, g)$ is immediate. □

We now establish some basic properties of the functional Υ. The results here are similar to [4, Proposition 2.2 & Lemma 2.4].

Proposition 4.2. *Assume that k satisfies (H1). Given $m > 0$ and $f_1, f_2 \in L^1_m$ we have:*

(i) If $f_1 + f_2 \leq 1$, then
$$\Upsilon(f_1 + f_2) \geq \Upsilon(f_1) + \Upsilon(f_2).$$
As a consequence, if $f_1 \leq f_2$, then $\Upsilon(f_1) \leq \Upsilon(f_2)$.

(ii) There exists $R = R(m)$ such that if $d(\operatorname{supp} f_1, \operatorname{supp} f_2) \geq R$, then
$$\Upsilon(f_1 + f_2) = \Upsilon(f_1) + \Upsilon(f_2).$$

(iii) There exists $C = C(m) > 0$ such that
$$|\Upsilon(f_1) - \Upsilon(f_2)| \leq C\|f_1 - f_2\|_{L^1}.$$

(iv) Let $f, f_n \in L^1(\mathbb{R}^d, [0,1])$ be such that the sequence $f_n\lambda$ is tight and $f_n \overset{*}{\rightharpoonup} f$. Then $\Upsilon(f_n) \to \Upsilon(f)$.

Proof.
Step 1. Proof of (i)&(ii).

To prove (i), we consider a transport plan γ optimal for $\Upsilon(f_1 + f_2)$ whose existence is guaranteed by Proposition 4.1. We would like to extract from γ two plans γ^1 and γ^2 admissible for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. Using the convention $0/0 = 0$, we define γ^1 and γ^2 through
$$d\gamma^1(x,y) := \frac{f_1(x)}{(f_1+f_2)(x)}\, d\gamma(x,y) \qquad \text{and} \qquad d\gamma^2(x,y) := \frac{f_2(x)}{(f_1+f_2)(x)}\, d\gamma(x,y).$$
By construction, $\gamma^1_x = f_1$ and $\gamma^2_x = f_2$. We also have $\gamma^1 \leq \gamma$, so that
$$\gamma^1_y \leq \gamma_y \leq 1 - (f_1 + f_2) \leq 1 - f_1.$$
Likewise, $\gamma^2_y \leq 1 - f_2$. Therefore, γ^1 and γ^2 are admissible for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. Moreover
$$\Upsilon(f_1) + \Upsilon(f_2) \leq \int c\, d\gamma^1 + \int c\, d\gamma^2 = \int c\, d\gamma = \Upsilon(f_1 + f_2),$$
which is the desired conclusion.

To prove (ii), we consider transport plans γ^1 and γ^2 which are optimal for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. We define $g_1 := \gamma^1_y$ and $g_2 := \gamma^2_y$. If we set $\gamma := \gamma^1 + \gamma^2$, we have $\gamma_x = f_1 + f_2$ and $\gamma_y = g_1 + g_2$. Moreover, by Proposition 4.1, if $d(\operatorname{supp} f_1, \operatorname{supp} f_2) \geq R$ for $R = R(m)$ large enough, then the supports of g_1 and g_2 are also disjoint. Consequently, $g_1 + g_2 \leq 1 - (f_1 + f_2)$, so that γ is admissible for $\Upsilon(f_1 + f_2)$ and we have the desired converse inequality
$$\Upsilon(f_1 + f_2) \leq \int c\, d(\gamma^1 + \gamma^2) \leq \Upsilon(f_1) + \Upsilon(f_2).$$

Step 2. Proof of (iii).

Exchanging the roles of f_1 and f_2, it is enough to prove the estimate
$$\Upsilon(f_2) - \Upsilon(f_1) \leq C\|f_2 - f_1\|_{L^1}. \tag{4.5}$$
Let γ^1 be a minimiser of $\Upsilon(f_1)$ and let us set $g_1 := \gamma^1_x$. In the next substeps, we build from γ^1 an exterior transport plan γ^2 for f_2 with controlled cost.

Step 2.a. Transporting most of $f_1 \wedge f_2$.

Using the convention $0/0 = 0$, we define a plan γ' by
$$d\gamma'(x,y) := \frac{(f_1 \wedge f_2)(x)}{f_1(x)} d\gamma_1(x,y).$$
We set $g' := \gamma'_y$. Notice that $\gamma' \leq \gamma^1$, which implies $g' \leq g_1$. Additionally, $\gamma'_x = f_1 \wedge f_2$, so that
$$\gamma^1(\mathbb{R}^d \times \mathbb{R}^d) - \gamma'(\mathbb{R}^d \times \mathbb{R}^d) = \int (f_1 - f_1 \wedge f_2) = \int (f_1 - f_2)_+ . \tag{4.6}$$

Heuristically, γ' corresponds to sending through γ^1 as much mass from f_2 as possible. However, we have to remove some of this mass because the constraint $g' \leq 1 - f_2$ might not hold true everywhere. Let

$$u := (f_2 + g' - 1)_+$$

and define γ'' as

$$d\gamma''(x,y) := \frac{g'(y) - u(y)}{g'(y)}\, d\gamma'(x,y).$$

We set $f'' := \gamma''_x$ and $g'' := \gamma''_y$. By construction, $g'' = g' - u$ so $g'' \leq 1 - f_2$ as desired. Since $\gamma'' \leq \gamma'$, we also have $f'' \leq f_1 \wedge f_2 \leq f_2$. Now since $g' \leq g_1 \leq 1 - f_1$ we have $u \leq (f_2 - f_1)_+$ from which we infer

$$\gamma'(\mathbb{R}^d \times \mathbb{R}^d) - \gamma''(\mathbb{R}^d \times \mathbb{R}^d) = \int (g' - g'') = \int u \leq \int (f_2 - f_1)_+ .$$

Summing this and (4.6) yields

$$\gamma^1(\mathbb{R}^d \times \mathbb{R}^d) - \gamma''(\mathbb{R}^d \times \mathbb{R}^d) \leq \|f_2 - f_1\|_{L^1}. \tag{4.7}$$

Eventually since $\gamma'' \leq \gamma' \leq \gamma^1$ and $c \geq 0$ we have

$$\int c\, d\gamma'' - \int c\, d\gamma^1 \leq 0. \tag{4.8}$$

Step 2.b. Final construction.

We are now ready to build an admissible transport plan γ^2 for $\Upsilon(f_2)$. Noticing that $f_2 - f'' \geq 0$ we write $f_2 = f'' + (f_2 - f'')$. By (4.7) we have

$$\int (f_2 - f'') = \int (f_2 - f_1) + \int (f_1 - f'') \leq 2\|f_2 - f_1\|_{L^1}. \tag{4.9}$$

Arguing as in the proof of Proposition 4.1, we can find a function $0 \leq g''' \leq 1 - f_2 - g''$ with $\int g''' \leq \int (f_2 - f'')$ and a transport plan γ''' between $f_2 - f''$ and g''' such that for some $C = C(m) > 0$,

$$\int c\, d\gamma''' \leq C \int (f_2 - f'') \overset{(4.9)}{\leq} C\|f_2 - f_1\|_{L^1}. \tag{4.10}$$

Finally we define $\gamma^2 := \gamma'' + \gamma'''$ which is admissible for $\Upsilon(f_2)$ by construction. Summing (4.8) and (4.10), we get

$$\Upsilon(f_2) \leq \int c\, d\gamma^2 \leq \int c\, d\gamma_1 + C\|f_2 - f_1\|_{L^1} = \Upsilon(f_1) + C\|f_2 - f_1\|_{L^1}.$$

This proves (4.5) and thus point (iii).

Step 3. Proof of (iv).

Let f_n and f_n be as in the statement of the proposition. By weak convergence, we have $f_n, f \in L^1_m$ for some $m > 0$. Using the Lipschitz continuity of Υ with respect to L^1 convergence, we may assume without loss of generality that f_n (and thus also f) are supported in $\overline{B}_{R_0}$ for some $R_0 > 0$. Applying Proposition 4.1 we get that minimisers of $\Upsilon(f_n)$ and $\Upsilon(f)$ are supported in $\overline{B}_R \times \overline{B}_R$ for some $R > R_0 > 0$. We may thus restrict these problems to the compact set $\overline{B}_R$. Using Proposition 3.5 we have $\Upsilon(f_n) = \Upsilon^*(f_n)$ and it is thus enough to prove the continuity of Υ^* with respect to the weak-$*$ topology.

By Proposition 3.4, for every $n \geq 0$ there exists a pair of potentials (φ_n, ψ_n) maximising $\Upsilon^*(f_n)$. Since for every n, φ_n belongs to Φ' (where Φ' is defined by (3.5)) and since this set is compact by Proposition 3.4 we have that a subsequence $\varphi_{n'}$ of φ_n converges in $C(\overline{B}_R)$ to some $\varphi \in \Phi'$. Arguing as in the proof of Proposition 3.4 we see that $\psi_{n'}$ also converges to ψ with (φ, ψ) admissible for $\Upsilon^*(f)$. By weak-strong convergence we then have

$$\limsup \Upsilon^*(f_{n'}) = \limsup \int f_{n'}\varphi_{n'} + (1 - f_{n'})\psi_{n'} = \int f\varphi + (1 - f)\psi \leq \Upsilon^*(f).$$

Similarly, if (φ, ψ) are optimal potentials for $\Upsilon^*(f)$, they are admissible for $\Upsilon^*(f_n)$ and thus

$$\liminf \Upsilon^*(f_n) \geq \liminf \int f_n\varphi + (1-f_n)\psi = \int f\varphi + (1-f)\psi = \Upsilon^*(f).$$

We then have $\lim \Upsilon^*(f_{n'}) = \Upsilon^*(f)$ and by uniqueness of the limit we see that the extraction was not necessary. This establishes (iv) and ends the proof of the proposition. □

The next lemma and theorem state very important saturation properties satisfied by the optimal exterior transport plan. These results extend [7, Lemma 5.1 & Proposition 5.2] to more general costs c.

Lemma 4.3. *Assume that k satisfies (H1)&(H2). For $f \in L^1_m$ let γ be optimal for $\Upsilon(f)$. Then for every $(x_0, y_0) \in \operatorname{supp}\gamma$ there holds $f + \gamma_y \equiv 1$ almost everywhere on the saturation set*

$$S(x_0, y_0) := \{y \in \mathbb{R}^d,\, k(y - x_0) < k(y_0 - x_0)\}.$$

Proof.
In the proof we set $g := \gamma_y$ and $h := f + g$. Let $(x_0, y_0) \in \operatorname{supp}\gamma$ and assume without loss of generality that $x_0 = 0$. We suppose by contradiction that there exists $\varepsilon > 0$ such that the set

$$S_\varepsilon := \{h < 1\} \cap \{y \in \mathbb{R}^d,\, k(y) < k(y_0) - \varepsilon\}$$

has positive Lebesgue measure. Notice that by (H1), $k(x) \to \infty$ as $|x| \to \infty$ so that S_ε is bounded. Therefore,

$$m_\varepsilon := \int_{S_\varepsilon} (1-h) \quad \in (0, \infty).$$

We now exhibit an exterior transport plan $\overline{\gamma}$ whose transportation cost is strictly smaller than the one of γ. Given $r > 0$, we define the measure $\gamma^0 := \gamma \llcorner (B_r \times B_r(y_0))$. As $(0, y_0) \in \operatorname{supp}\gamma$, for every $r > 0$,

$$0 < \gamma^0(\mathbb{R}^d \times \mathbb{R}^d) \leq \int_{B_r} f \, \leq |B_r|. \tag{4.11}$$

Thus, by the last inequality in (4.11) there exists $r_\varepsilon > 0$ such that for every $r \in (0, r_\varepsilon]$,

$$\gamma^0(\mathbb{R}^d \times \mathbb{R}^d) = \alpha m_\varepsilon$$

for some $0 < \alpha \leq 1$. Let us fix $r \in (0, r_\varepsilon]$. We define a competitor $\widetilde{\gamma}$ for $\Upsilon(f)$ by setting $\widetilde{\gamma} := \gamma - \gamma^0 + \eta$, where

$$\eta := \gamma^0_x \otimes \frac{1-h}{m_\varepsilon} \chi_{S_\varepsilon}.$$

By construction, $\widetilde{\gamma}_x = \gamma_x = f$. We also have

$$f + \widetilde{\gamma}_y \leq f + g + \alpha(1-h)\chi_{S_\varepsilon} \leq h + \alpha(1-h) = 1 - (1-h)(1-\alpha) \leq 1,$$

so that $\widetilde{\gamma}$ is admissible for $\Upsilon(f)$. We compute

$$\int c\, d\widetilde{\gamma} - \int c\, d\gamma = \int c\, d\eta - \int c\, d\gamma^0 \leq \alpha m_\varepsilon \left(\max_{\overline{B}_r \times \overline{S}_\varepsilon} c(x,y) - \min_{\overline{B}_r \times \overline{B}_r(y_0)} c(x,y) \right).$$

By continuity of c there exists $r_\varepsilon > 0$ such that for $0 < r \leq r_\varepsilon$,

$$\max_{\overline{B}_r \times \overline{S}_\varepsilon} c(x,y) \leq k(y_0) - \varepsilon/2 \qquad \text{and} \qquad \min_{\overline{B}_r \times \overline{B}_r(y_0)} c(x,y) \geq k(y_0) - \varepsilon/4.$$

Thus for $0 < r \leq r_\varepsilon$,

$$\int c\, d\widetilde{\gamma} - \int c\, d\gamma \leq -\alpha\varepsilon m_\varepsilon/4 < 0,$$

which contradicts the fact that γ is a minimiser for $\Upsilon(f)$. □

Theorem 4.4. *Assume that k satisfies (H1)&(H2). For $f \in L^1_m$, let $\gamma \in \Pi_f$ be a minimiser of (4.1) and set $g := \gamma_y$. Then, defining*

$$E := \{x : \exists\, y \neq x \text{ such that } (x,y) \in \operatorname{supp}\gamma \text{ or } (y,x) \in \operatorname{supp}\gamma\},$$

the set E is Lebesgue measurable and we have the identity $g = (1-f)\chi_E + f\chi_{E^c}$.

Proof.
Step 1. A preliminary claim.

We first prove the following. Let $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$ be such that $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$, and let $\gamma \in \Pi(\mu,\nu)$. Assume that μ and ν are absolutely continuous with respect to the Lebesgue measure. If we define the set

$$\mathcal{A}(\gamma) := \{x : \exists\, y \neq x \text{ such that } (x,y) \in \operatorname{supp}\gamma\},$$

then $\mu \leq \nu$ on $\mathcal{A}(\gamma)^c$.

To prove the claim, let us first show that $\mathcal{A}(\gamma)$ is Lebesgue measurable. We define

$$\mathcal{D}(\gamma) := \operatorname{supp}\gamma \setminus \{(x,x) :\ x \in \mathbb{R}^d\},$$

which is a Borel set of $\mathbb{R}^d \times \mathbb{R}^d$. If we denote by $p_x : X \times X \to X$ the canonical projection on the first variable, we have

$$p_x(\mathcal{D}(\gamma)) = \{x \in \mathbb{R}^d :\ \exists y \neq x,\ (x,y) \in \operatorname{supp}\gamma\} = \mathcal{A}(\gamma).$$

We deduce that $\mathcal{A}(\gamma)$ is Lebesgue measurable as the continuous image of a Borel set.

We now show that $\mu \leq \nu$ on $\mathcal{A}(\gamma)^c$. Let $\phi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$. By definition of $\mathcal{A}(\gamma)$, if $(x,y) \in \operatorname{supp}\gamma$ and $x \in \mathcal{A}(\gamma)^c$ then $x = y$. Therefore

$$\begin{aligned}\int_{\mathcal{A}(\gamma)^c} \phi\, d\mu = \int \phi(x)\chi_{\mathcal{A}(\gamma)^c}(x)\, d\gamma(x,y) &= \int \phi(y)\chi_{\mathcal{A}(\gamma)^c}(y)\chi_{\mathcal{A}(\gamma)^c}(x)\, d\gamma(x,y)\\ &\leq \int \phi(y)\chi_{\mathcal{A}(\gamma)^c}(y)\, d\gamma(x,y) = \int_{\mathcal{A}(\gamma)^c} \phi\, d\nu,\end{aligned}$$

and the claim is proved.

Step 2. Construction of E.

We now consider an optimal exterior transport plan γ for $\Upsilon(f)$ and set $g := \gamma_y$, $h := f + g$. By Proposition 4.1, γ is an optimal transport plan from f to g. Let $\overline{\gamma}$ be the image of γ through the map $(x,y) \mapsto (y,x)$ and define

$$E := \mathcal{A}(\gamma) \cup \mathcal{A}(\overline{\gamma}).$$

We have $E^c = \mathcal{A}(\gamma)^c \cap \mathcal{A}(\overline{\gamma})^c$ and by the first step there holds $f \leq g$ and $g \leq f$ almost everywhere on E^c. Hence,

$$g\chi_{E^c} = f\chi_{E^c}.$$

To conclude the proof, we have to show that $g \equiv 1 - f$ on E or equivalently that up to Lebesgue negligible sets $\mathcal{A}(\gamma)$ and $\mathcal{A}(\overline{\gamma})$ are included in $\{g = 1 - f\}$.

On the one hand, if $x_0 \in \mathcal{A}(\gamma)$ there exists $y_0 \neq x_0$ such that $(x_0, y_0) \in \operatorname{supp}\gamma$. By Lemma 4.3, denoting

$$S(x_0,y_0) := \{y \in \mathbb{R}^d,\ k(y - x_0) < k(y_0 - x_0)\},$$

we have $g = 1 - f$ on $S(x_0,y_0)$. Notice that $S(x_0,y_0)$ is an open set and that $x_0 \in S(x_0,y_0)$ (since for $x \neq 0$, $k(x) > 0 = k(0)$). Hence $g(x_0) = 1 - f(x_0)$ and $\mathcal{A}(\gamma) \subset \{g = 1 - f\}$.

On the other hand, if $y_0 \in \mathcal{A}(\overline{\gamma})$ there exists $x_0 \neq y_0$ such that $(x_0,y_0) \in \operatorname{supp}\gamma$. Let us assume by contradiction that $g(y_0) < 1 - f(y_0)$. Without loss of generality, we can assume that y_0 is a point of Lebesgue density one of $\{g < 1 - f\}$. Then

$$\lim_{r\to 0} \frac{1}{r^d}\Big|\, \{g = 1 - f\} \cap B(y_0, r)\Big| = 0.$$

Thus by Lemma 4.3,

$$\lim_{r\to 0} \frac{1}{r^d} |S(x_0, y_0) \cap B(y_0, r)| = 0,$$

which contradicts (H2) as $y_0 \neq x_0$. Hence $g(y_0) = 1 - f(y_0)$ and $\mathcal{A}(\overline{\gamma}) \subset \{g = 1 - f\}$. This concludes the proof of the theorem. □

An important corollary is the uniqueness of the second marginal of minimisers of (4.1).

Corollary 4.5. *Assume that k satisfies (H1)&(H2). Let $f \in L^1(\mathbb{R}^d)$. Then all minimisers γ of $\Upsilon(f)$ have the same second marginal γ_y.*

Proof.
We let γ, γ' be two minimisers of (4.1) and define $\widetilde{\gamma} := (\gamma + \gamma')/2$ which also minimises (4.1). We denote $g := \gamma_y$, $g' := \gamma'_y$ and $\widetilde{g} := \widetilde{\gamma}_y$ and introduce the set

$$F := \{x \in \mathbb{R}^d : g(x), g'(x), \widetilde{g}(x) \in \{f(x), 1 - f(x)\}\}.$$

Assuming by contradiction that $g(x) \neq g'(x)$ for some $x \in F$, we have $1/2 = \widetilde{g}(x) \in \{f(x), 1 - f(x)\}$ so that $f(x) = 1/2$ and $g(x) = g'(x) = 1/2$, which is absurd. Hence $g = g'$ on F and since F is of full measure by Theorem 4.4, the proof is complete. □

4.2. **Preliminary results for the existence of a maximiser of** (4.12)**.**

We now gather results which, combined with Theorem 4.4, allow us to prove existence of a maximiser for both $\mathcal{E}_{\rm set}(m)$ and

$$\mathcal{E}(m) := \sup \left\{ \Upsilon(f) : f \in L^1(\mathbb{R}^d, [0,1]), \int f = m \right\}. \tag{4.12}$$

We first establish a corollary of Theorem 4.4 regarding the monotonicity of the sum of the marginals of solutions to (4.1).

Corollary 4.6. *Assume that k satisfies (H1)&(H2). Let $m > 0$, let $f_1, f_2 \in L^1_m$ be such that $f_1 \leq f_2$ and let γ^1, γ^2 be respective minimisers of $\Upsilon(f_1)$ and $\Upsilon(f_2)$. Then setting $g_1 := \gamma^1_y$ and $g_2 := \gamma^2_y$, we have $f_1 + g_1 \leq f_2 + g_2$.*

Proof.
Let $f_1, f_2 \in L^1_m$ be such that $f_1 \leq f_2$. In the first three steps of the proof, we additionally assume that they are compactly supported. This condition is relaxed in the fourth and final step.
By Proposition 4.1, we can assume that the ambient space is a compact ball $\overline{B}_R$. Let γ^1, γ^2 be minimisers for $\Upsilon(f_1)$ and $\Upsilon(f_2)$ respectively. For $i \in \{1, 2\}$ we define $g_i := \gamma^i_y$, $h_i := f_i + g_i$ and set

$$F := \{h_1 > h_2\}.$$

We shall prove that $|F| = 0$. By Theorem 4.4, there exists $E_1, E_2 \subset \overline{B}_R$ such that

$$h_1 = \chi_{E_1} + 2f_1\chi_{E_1^c} \qquad \text{and} \qquad h_2 = \chi_{E_2} + 2f_2\chi_{E_2^c}.$$

Since $h_2 \geq 0$, $h_1 \leq 1$ and $h_2 \geq f_2 \geq f_1$ we have

$$h_1 > 0, \qquad h_2 = 2f_2 < 1 \qquad \text{and} \qquad f_1 < 1 \quad \text{on } F. \tag{4.13}$$

Step 1. $|E_1^c \cap F| = 0$.

By definition of E_1 we have $h_1 = 2f_1$ on E_1^c and by (4.13) we have $h_2 = 2f_2$ on F and since $f_1 \leq f_2$ we get $h_1 \leq h_2$ on $E_1^c \cap F$. This contradicts the definition of F, hence $E_1^c \cap F = \emptyset$ and in particular $|E_1^c \cap F| = 0$.

Step 2. Intermediate claim.

Let ψ_1 be the maximal potential for $\Upsilon^*(f_1)$ given by Proposition 3.8. We define

$$G := \{\psi_1^{\bar{c}c} = 0\} \cap E_1 \cap F$$

and claim that $|G| = 0$. Let us assume by contradiction that $|G| > 0$. First notice that on F,

$$f_1 + g_1 = h_1 > h_2 = 2f_2,$$

so that

$$g_1 > 2f_2 - f_1 \geq f_2 \geq f_1.$$

Thus

$$G \subset E_1 \cap F \subset \{g_1 > f_1\}. \tag{4.14}$$

Now recall that by Theorem 4.4,

$$E_1 = \{x : \exists y \neq x \text{ such that } (x,y) \in \operatorname{supp}\gamma^1 \text{ or } (y,x) \in \operatorname{supp}\gamma^1\}.$$

Together with (4.14) we obtain that for almost every $y_0 \in G$ there exists $x_0 \neq y_0$ with $(x_0, y_0) \in \operatorname{supp}\gamma^1$. Without loss of generality, we assume that y_0 is a point of density of G and we set

$$S(x_0,y_0) := \left\{ y \in \mathbb{R}^d : \, k(y-x_0) < k(y_0-x_0) \right\}.$$

By (H2), we have $|G \cap S(x_0,y_0)| > 0$. Let now $\widetilde{y} \in G \cap S(x_0,y_0)$. By Proposition 3.7, $(\psi_1^{\bar{c}}, \psi_1^{\bar{c}c})$ forms a pair of Kantorovitch potentials for the optimal transport from f_1 to g_1. Thus

$$\psi_1^{\bar{c}}(x_0) + \psi_1^{\bar{c}c}(y_0) = k(y_0 - x_0) \qquad \text{and} \qquad \psi_1^{\bar{c}}(x_0) + \psi_1^{\bar{c}c}(\widetilde{y}) \leq k(\widetilde{y} - x_0).$$

However, $y_0, \widetilde{y} \in G$, so that $\psi_1^{\bar{c}c}(y_0) = \psi_1^{\bar{c}c}(\widetilde{y}) = 0$, hence

$$\psi_1^{\bar{c}}(x_0) = k(y_0 - x_0) \qquad \text{and} \qquad \psi_1^{\bar{c}}(x_0) \leq k(\widetilde{y} - x_0).$$

Eventually, as $\widetilde{y} \in S(x_0,y_0)$, we conclude that

$$\psi_1^{\bar{c}}(x_0) \leq k(\widetilde{y} - x_0) < k(y_0 - x_0) = \psi_1^{\bar{c}}(x_0),$$

obtaining a contradiction. Thus $|G| = 0$, which is the claim.

Step 3. $|E_1 \cap F| = 0$.

By Proposition 3.7,

$$\{\psi_1^{\bar{c}c} > 0\} \subset \{g_1 = 0\} \quad \text{ so that } \quad \{g_1 > 0\} \subset \{\psi_1^{\bar{c}c} \leq 0\}.$$

We observe that $g_1 = 1 - f_1$ on $E_1 \cap F$. By (4.13), $E_1 \cap F \subset \{g_1 > 0\}$ and by the previous step, $E_1 \cap F \subset \{\psi_1^{\bar{c}c} \neq 0\}$, hence $\psi_1^{\bar{c}c} < 0$ almost everywhere on $E_1 \cap F$.

Let ψ_2 be the maximal potential for $\Upsilon^*(f_2)$ given by Proposition 3.8. As $f_1 \leq f_2$, we have $\psi_1 \geq \psi_2$ so that $\psi_1^{\bar{c}c} \geq \psi_2^{\bar{c}c}$. Thus

$$\psi_2^{\bar{c}c} < 0 \text{ on } E_1 \cap F.$$

By Proposition 3.7 we deduce that

$$h_2 = g_2 + f_2 = 1 \text{ on } E_1 \cap F.$$

But since $h_2 < 1$ on F we get that $|E_1 \cap F| = 0$ and with the first step we conclude that $|F| = 0$.

Step 4. Extension to the non-compact case.

Let $f_1, f_2 \in L^1_m$ be such that $f_1 \leq f_2$. For $i \in \{1,2\}$, we set $f_{i,R} = f_i \chi_{B_R}$, consider γ_R^i an optimal exterior transport plan for $\Upsilon(f_{i,R})$ and set $g_{i,R} := (\gamma_R^i)_y$. Applying the previous steps to $f_{1,R}$ and $f_{2,R}$, we obtain

$$f_{1,R} + g_{1,R} \leq f_{2,R} + g_{2,R}. \tag{4.15}$$

For $i \in \{1,2\}$, $f_{i,R}$ L^1-converges to f_i as $R \to \infty$. By Proposition 4.2 (iii), $\Upsilon(f_{i,R}) \to \Upsilon(f_i)$ as $R \to \infty$. Additionally, γ_R^i admits a subsequence converging weakly-$*$ to some $\widetilde{\gamma}^i$ admissible for $\Upsilon(f_i)$. By lower semi-continuity of $\gamma \mapsto \int c\, d\gamma$ with respect to weak convergence, we get

$$\int c\, d\widetilde{\gamma^i} \leq \liminf_R \int c\, d\gamma_R^i = \liminf_R \Upsilon(f_{i,R}) = \Upsilon(f_i).$$

Hence $\widetilde{\gamma}^i$ is optimal for $\Upsilon(f_i)$, so that by Corollary 4.5, $\widetilde{\gamma}^i_y = g_i$. Finally, as $\gamma^i_R \overset{*}{\rightharpoonup} \widetilde{\gamma}^i$ as $R \to \infty$, $g_{j,R}$ converges in duality with $C_c(\mathbb{R}^d)$ to g_i as $R \to \infty$. Multiplying (4.15) by $\phi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$, integrating and passing to the limit we obtain that for any $\phi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$,

$$\int (f_1 + g_1)\phi \le \int (f_2 + g_2)\phi\,.$$

Hence $f_1 + g_1 \le f_2 + g_2$ which completes the proof. □

We now prove that $\mathcal{E}$ is strictly superadditive.

Proposition 4.7. *Assume that k satisfies (H1)&(H3). Let $m \in (0,\infty)$ and define $e(m) := \mathcal{E}(m)/m$. Then, e is increasing on $(0,\infty)$. As a consequence, given $0 < m' < m$,*

$$\mathcal{E}(m') + \mathcal{E}(m - m') < \mathcal{E}(m).$$

Proof.
Let $M > m > 0$. We have to establish that $\mathcal{E}(m) < (m/M)\mathcal{E}(M)$.

Step 1. $\mathcal{E}(m) \le (m/M)\mathcal{E}(M)$.
For $R > 0$ we set

$$\Gamma_R := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| > R\}.$$

Let $0 \le \varepsilon < \mathcal{E}(m)/2$ and $f \in L^1_m$ of mass exactly m and such that $\Upsilon(f) \ge \mathcal{E}(m) - \varepsilon$. We denote $\lambda := (M/m)^{1/d} > 1$ and we set

$$f_\lambda(x) := f(x/\lambda) \qquad \text{for } x \in \mathbb{R}^d,$$

so that $\int f_\lambda = M$. Let γ^λ be an optimal transport plan for $\Upsilon(f_\lambda)$. We define a Radon measure γ by

$$\int \xi(x,y)\,d\gamma(x,y) := \frac{m}{M}\int \xi(x/\lambda, y/\lambda)\,d\gamma^\lambda(x,y) \qquad \text{for } \xi \in C_c(\mathbb{R}^d \times \mathbb{R}^d).$$

Observe that γ is admissible for $\Upsilon(f)$. By Proposition 4.1, there exists $R_\lambda = R_\lambda(M)$ such that $\gamma^\lambda(\Gamma_{R_\lambda}) = 0$. Setting $R := R_\lambda/\lambda$, we then have

$$\gamma(\Gamma_R) = \frac{m}{M}\gamma^\lambda(\Gamma_{R_\lambda}) = 0. \tag{4.16}$$

Let us define

$$\kappa(r) := \min\{k(z) - k(z/\lambda) : r \le |z| \le R_\lambda\}\,.$$

As $\lambda > 1$ we have by (H3) that $\kappa(r) > 0$ for $0 < r < R_\lambda$. Additionally, $k(z/\lambda) \le k(z)$ for any $z \in \mathbb{R}^d$. Consequently, for any $0 < r < R_\lambda$,

$$\begin{aligned}
\Upsilon(f) &\le \int k(y-x)\,d\gamma(x,y)\\
&= \frac{m}{M}\int k\left(\frac{y-x}{\lambda}\right)d\gamma^\lambda(x,y)\\
&= \frac{m}{M}\int \left[k\left(\frac{y-x}{\lambda}\right) - k(y-x)\right]d\gamma^\lambda(x,y) + \frac{m}{M}\int k(y-x)\,d\gamma^\lambda(x,y)\\
&\le \frac{m}{M}\int_{\Gamma_r}\left[k\left(\frac{y-x}{\lambda}\right) - k(y-x)\right]d\gamma^\lambda(x,y) + \frac{m}{M}\mathcal{E}(M).
\end{aligned}$$

In the integral over Γ_r, the term in brackets is smaller than $-\kappa(r)$. Hence, for every $0 < r < R_\lambda$

$$\mathcal{E}(m) - \varepsilon \le \Upsilon(f) \le \frac{m}{M}\mathcal{E}(M) - \frac{m}{M}\kappa(r)\gamma^\lambda(\Gamma_r). \tag{4.17}$$

At this point we can send ε to 0 and deduce that $\mathcal{E}(m) \le (m/M)\mathcal{E}(M)$. However we need to establish a strict inequality. For this we prove in the next step that there exist $r^*, \delta > 0$ not depending on ε or f such that $\gamma^\lambda(\Gamma_{r^*}) \ge \delta$.

Step 2. Conclusion.

For $r \geq 0$, we set

$$\overline{k}(r) := \max\{k(z) : |z| \leq r\}.$$

This function is increasing, continuous and there holds $\overline{k}(0) = 0$. Notice that using a ball of mass m as a candidate for the energy $\mathcal{E}(m)$, we see that $\mathcal{E}(m) > 0$ for any $m > 0$. Let us fix $0 < r^* < R$ such that

$$m\overline{k}(r^*) \leq \mathcal{E}(m)/4. \tag{4.18}$$

By (4.16) we have $|x - y| \leq R$ for $(x, y) \in \operatorname{supp}\gamma$ and by definition $|x - y| \leq r^*$ for $(x, y) \notin \Gamma_{r^*}$. We deduce

$$\begin{aligned}\frac{\mathcal{E}(m)}{2} < \Upsilon(f) = \int_{\Gamma_{r^*}} c\, d\gamma + \int_{\Gamma_{r^*}^c} c\, d\gamma &\leq \gamma(\Gamma_{r^*})\overline{k}(R) + (m - \gamma(\Gamma_{r^*}))\,\overline{k}(r^*) \\ &\overset{(4.18)}{\leq} \gamma(\Gamma_{r^*})\overline{k}(R) + (m - \gamma(\Gamma_{r^*}))\,\frac{\mathcal{E}(m)}{4m}.\end{aligned}$$

This implies

$$\gamma(\Gamma_{r^*})\left(\overline{k}(R) - \frac{\mathcal{E}(m)}{4m}\right) > \frac{\mathcal{E}(m)}{4}.$$

Thus $4m\overline{k}(R) > \mathcal{E}(m)$ and

$$\frac{m}{M}\gamma^\lambda(\Gamma_{\lambda r^*}) = \gamma(\Gamma_{r^*}) \geq \frac{m\mathcal{E}(m)}{4m\overline{k}(R) - \mathcal{E}(m)} =: m^* > 0.$$

Plugging this in (4.17) with $r = \lambda r^* < R_\lambda$ we obtain

$$\mathcal{E}(m) - \varepsilon \leq \frac{m}{M}\mathcal{E}(M) - m^*\kappa(r^*).$$

Since $\varepsilon \in [0, \mathcal{E}(m)/2]$ is arbitrary and $m^*\kappa(r^*) > 0$, this proves the proposition. □

We close this subsection with a lemma establishing that if a function f nearly maximises $\mathcal{E}(m)$ for some $m > 0$ then there exists a cube which is at least half filled by f.

Lemma 4.8. *Let $m > 0$. There exists a non-decreasing function $r_0 : m \mapsto r_0(m)$ such that for $m > 0$ and $f \in L^1_m$ with $\Upsilon(f) \geq \mathcal{E}(m)/2$, there exists a cube Q_0 of side-length $r_0(m)$ such that:*

$$\int_{Q_0} f \geq \frac{|Q_0|}{2}.$$

Proof.
Let $r_0 > 0$ to be fixed later and assume by contradiction that there exists a partition $\mathcal{Q}$ of $\mathbb{R}^d$ in cubes with side-length r_0 such that for every $Q \in \mathcal{Q}$,

$$\int_Q f < \frac{|Q|}{2}.$$

The strategy to get a contradiction from this hypothesis is to build an exterior transport plan for f with too small transport cost. Let $Q \in \mathcal{Q}$. Since $\int_Q (1 - f) \geq \int_Q f$ there exists a function $g_Q \geq 0$ supported in Q such that $\int g_Q = \int_Q f$ and $f\chi_Q + g_Q \leq 1$. We then set

$$\gamma_Q := f\chi_Q \otimes \frac{g_Q}{\int_Q f} \qquad \text{and} \qquad \gamma := \sum_{Q \in \mathcal{Q}} \gamma_Q.$$

Notice that γ is a valid competitor for $\Upsilon(f)$. Next for $R > 0$, we define

$$\overline{k}(R) := \max\{k(x),\, |x| \leq R\}.$$

We compute:

$$0 < \frac{\mathcal{E}(m)}{2} \leq \Upsilon(f) \leq \sum_{Q\in\mathcal{Q}} \int_{Q\times Q} k(y-x)\, d\gamma_Q \leq \overline{k}\left(\sqrt{d}r_0\right) \sum_{Q\in\mathcal{Q}} \int_Q f \; = \overline{k}\left(\sqrt{d}r_0\right) \int f \; \leq \overline{k}\left(\sqrt{d}r_0\right) m. \tag{4.19}$$

Remarking that $\overline{k} : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous at 0, increasing and with $\overline{k}(0) = 0$, we set

$$r_0 := \max\left\{ r > 0 : \overline{k}\left(\sqrt{d}r\right) \leq \frac{\mathcal{E}(m)}{4m} \right\} \; > 0$$

and obtain a contradiction with (4.19). This concludes the proof. □

4.3. **Existence of a maximiser for** (4.12)**.**

In the following subsection, we assume that (H1),(H2)&(H3) hold and prove the existence of maximisers for (4.12).

We only have to prove that maximising sequences for $\mathcal{E}(m)$ are tight. However our result is more precise. We obtain that if f nearly maximises $\mathcal{E}(m)$ then almost all its mass concentrates in a closed ball with radius $R_* = R_*(m)$. In the limit, maximisers are supported in such balls.

Proposition 4.9. *Let $m > 0$. There exist $R_* = R_*(m) > 0$, $\varepsilon_0 = \varepsilon_0(m) > 0$ non-decreasing in m with the following property. Let $0 < \varepsilon \leq \varepsilon_0$ and let $f \in L^1_m$ such that $\int f = m$ and $\Upsilon(f) \geq \mathcal{E}(m) - \varepsilon$, then up to a translation there holds*

$$\int_{\mathbb{R}^d\setminus B_{R_*}} f \; \leq \frac{2m}{\mathcal{E}(m)}\varepsilon.$$

Proof.

Outline of the proof.

(Step 1) We start by using Lemma 4.8 to get a collection $\mathcal{Q}_0$ of cubes Q of side-length $r_0 = r_0(m)$ such that $\int_Q f + g \geq |Q|/2$. We denote $\Omega_0 := \cup\mathcal{Q}_0$. We also consider the set Ω obtained by thickening Ω_0 by adding the cubes closer than some distance $R = R(m)$. The real R is chosen so that no mass of $f\chi_{\Omega_0}$ is sent outside Ω by any optimal exterior transport plan of f.

(Step 2) We build an exterior transport plan for f whose cost is very close to $\Upsilon(f\chi_\Omega)$.

(Step 3) Next, we show that Ω concentrates almost all the mass of f. Using the strict superadditivity of $m \mapsto \mathcal{E}(m)$ and the previous step, we deduce that $m_\Omega := \int f\chi_\Omega$ is close to m.

(Step 4) Eventually, we show that the distance between cubes in $\mathcal{Q}_0$ is uniformly bounded. As the cardinal of $\mathcal{Q}_0$ is also bounded, we conclude that the diameter of Ω is bounded by a distance only depending on m.

Step 1. Construction of a collection of cubes on which $\int_Q (f+g) \geq |Q|/2$.

Let $m > 0$ and f as in the statement of the proposition and assume that $\Upsilon(f) \geq \mathcal{E}(m)/2$ so that

$$\varepsilon := \mathcal{E}(m) - \Upsilon(f) \leq \mathcal{E}(m)/2. \tag{4.20}$$

Let γ be a minimiser for $\Upsilon(f)$ and let us set $g := \gamma_y$. Let r_0 and Q_0 be given by Lemma 4.8. We denote by $\hat{\mathcal{Q}}$ the regular partition of $\mathbb{R}^d$ into cubes of side-length r_0 such that $Q_0 \in \hat{\mathcal{Q}}$. For $j \geq 0$ to be fixed later, we set $r_j := 2^{-j}r_0$. Considering the partition $\mathcal{Q}$ of $\mathbb{R}^d$ into cubes of side-length r_j obtained by refining $\hat{\mathcal{Q}}$, we define $\mathcal{Q}_0$ as the subset formed by the elements $Q \in \mathcal{Q}$ such that

$$\int_Q (f+g) \; \geq \frac{|Q|}{4}.$$

We remark that $\mathcal{Q}_0$ is not empty since

$$\int_Q (f+g) \; \geq \int_Q f \; \geq \frac{|Q|}{2}$$

for at least one of the 2^j sub-cubes of Q_0 in the partition $\mathcal{Q}$.

Let us define $\Omega_0 := \cup\mathcal{Q}_0$. By Proposition 4.1, there exists $R = R(m)$ such that $|x-y| \le R$ on $\operatorname{supp}\gamma$. We denote by $\mathcal{Q}_R$ the collection of cubes $Q \in \mathcal{Q}$ such that $d(Q,\Omega_0) \le R$, and by Ω their union. By construction, there holds $\gamma(\Omega_0 \times \Omega^c) = 0$. We now define

$$f_\Omega := f\chi_\Omega, \qquad \text{and} \qquad m_\Omega := \int f_\Omega\,,$$

and we let γ_Ω be an optimal exterior transport plan for f_Ω, that is $\gamma \in \Pi_{f_\Omega}$ with $\int c\,d\gamma_\Omega = \Upsilon(f_\Omega)$.

By Proposition 4.1 again, we have (since $m_\Omega \le m$) that

$$\gamma_\Omega(\Omega_0 \times \Omega^c) = 0. \tag{4.21}$$

Step 2. Building a transport plan for f whose cost is close to $\Upsilon(f_\Omega)$.

In this step we modify γ_Ω to build an exterior transport plan γ for f with a cost close to $\Upsilon(f_\Omega)$. More precisely, we require that for some constant $C = C(r_j) > 0$ with $C(r_j) \to 0$ as $r_j \to 0$,

$$\int c\,d\gamma - \int c\,d\gamma_\Omega \le C(m - m_\Omega).$$

The proof is a refinement of the proof of the Lipschitz continuity of Υ, see Proposition 4.2 (iii). In the following we define successively the plans γ^0, γ^1, γ^2, γ^3 which satisfy in particular

$$\operatorname{supp}\gamma^0 \subset \overline{\Omega}\times\overline{\Omega}, \qquad \operatorname{supp}\gamma^1 \subset \overline{\Omega}\times\overline{\Omega^c}, \qquad \operatorname{supp}\gamma^2 \subset \overline{\Omega\setminus\Omega_0}\times\overline{\Omega\setminus\Omega_0}, \qquad \operatorname{supp}\gamma^3 \subset \overline{\Omega^c}\times\overline{\Omega^c}.$$

First we set $\gamma^0 := \gamma_\Omega \llcorner \Omega\times\Omega$ and denote $f^0 := \gamma^0_x$, $g^0 := \gamma^0_y$. We build the three remaining plans in the following substeps. These constructions will satisfy

$$(\gamma^1+\gamma^2)_x = f_\Omega - \gamma^0_x = f\chi_\Omega - f^0 \qquad \text{and} \qquad \gamma^3_x = f - f_\Omega = f\chi_{\Omega^c}.$$

We will set eventually $\widehat{\gamma} := \gamma^0+\gamma^1+\gamma^2+\gamma^3$ which will be an optimal transport plan for f. The difficulty is to preserve the constraint $f + \widehat{\gamma}_y \le 1$ while controlling the cost.

Step 2.a. Construction of γ^1.

Let us denote $\gamma^1_\Omega := \gamma_\Omega \llcorner \Omega\times\Omega^c$, $f^1_\Omega := (\gamma^1_\Omega)_x$ and $g^1_\Omega := (\gamma^1_\Omega)_y = \chi_{\Omega^c} g_\Omega$. We can not rule out the possibility that $f + g_\Omega > 1$ in some part of Ω^c so that we cannot set $\gamma^1 = \gamma^1_\Omega$. However, we will transport as much as possible mass through γ^1_Ω. Let us define

$$u := (f + g_\Omega - 1)_+,$$

which corresponds to the excess mass transported through γ^1_Ω. Using the convention $0/0 = 0$, we define γ^1 by

$$d\gamma^1(x,y) := \frac{g_\Omega(y) - u(y)}{g_\Omega(y)} d\gamma^1_\Omega(x,y).$$

At this point, we have

$$\int c\,d(\gamma^0+\gamma^1) \le \int c\,d\gamma_\Omega = \Upsilon(f_\Omega). \tag{4.22}$$

Moreover setting $f^1 := \gamma^1_x$ and $g^1 := \gamma^1_y$, there holds $\operatorname{supp} g^1 \subset \overline{\Omega^c}$. Notice that since $f_\Omega \le f$, by Corollary 4.6 we have $f_\Omega + g_\Omega \le f + g$, so that $g_\Omega \le f + g$ in Ω^c which implies $g^1 \le f + g$. Thus

$$\int_Q (f+g^1) \le \int_Q (2f+g) < \frac{|Q|}{2}, \qquad \text{for every } Q \in \mathcal{Q}\setminus\mathcal{Q}_R, \tag{4.23}$$

where we used the definition of $\mathcal{Q}_0$ and the fact that $[\mathcal{Q}\setminus\mathcal{Q}_R]\cap\mathcal{Q}_0 = \emptyset$.
Let us compute for later use the mass from Ω that still requires to be transported. By construction

$$\int_\Omega (f - f^0 - f^1) = \int d(\gamma^1_\Omega - \gamma^1) = \int \frac{u(y)}{g_\Omega(y)}\, d\gamma^1_\Omega(x,y) = \int_{\Omega^c} (f + g_\Omega - 1)_+ \le \int_{\Omega^c} f\,. \tag{4.24}$$

Step 2.b. Construction of γ^2.

We now define

$$\gamma^2_\Omega := \gamma^1_\Omega - \gamma^1 = \gamma_\Omega - \gamma^0 - \gamma^1.$$

Notice that by (4.21), $\gamma^1_\Omega(\Omega_0 \times \Omega^c) = 0$, so that

$$\operatorname{supp} \gamma^2_\Omega \subset \overline{\Omega \setminus \Omega_0} \times \overline{\Omega^c}.$$

In particular, $f^2 := f_\Omega - f^0 - f^1$ is supported in $\overline{\Omega \setminus \Omega_0}$. Let $Q \in \mathcal{Q}_R \setminus \mathcal{Q}_0$. Since $g^0 \le g_\Omega$ and $f = f_\Omega$ on Q, using Corollary 4.6 again we see that

$$\int_Q (f + g^0) \le \int_Q (f_\Omega + g_\Omega) \le \int_Q (f + g) \le \frac{|Q|}{4}.$$

Therefore for such Q there exists a function $g^2_Q : Q \to \mathbb{R}_+$ such that $f + g^0 + g^2_Q \le 1$ and $\int g^2_Q = \int_Q f^2$. Defining

$$\gamma^2_Q := \frac{1}{\int_Q f^2} \left[\chi_Q f^2\right] \otimes g^2_Q \quad \text{for } Q \in \mathcal{Q}_R \setminus \mathcal{Q}_0 \quad \text{and then} \quad \gamma^2 := \sum_{Q \in \mathcal{Q}_R \setminus \mathcal{Q}_0} \gamma^2_Q,$$

we have $\gamma^2_x = f^2$ and $g^2 := \gamma^2_y = \sum_Q g^2_Q$. Hence $f + g^0 + g^2 \le 1$ and

$$\int c \, d\gamma^2 \le \left(\int f^2 \right) \overline{k} \left(\sqrt{d} r_j \right),$$

where as in the proof of Proposition 4.7 we denote $\overline{k}(r) := \max\{k(x) : |x| \le r\}$.
By construction $f^2 = f_\Omega - f^0 - f^1$, so by (4.24) there holds $\int f^2 \le m - m_\Omega$ which leads to the cost estimate

$$\int c \, d\gamma^2 \le (m - m_\Omega) \overline{k} \left(\sqrt{d} r_j \right). \tag{4.25}$$

Step 2.c. Construction of γ^3.

We still have to transport the mass corresponding to $\chi_{\Omega^c} f$. For every $Q \in \mathcal{Q} \setminus \mathcal{Q}_R$ we have $\int_Q f \le \int_Q (f + g) \le |Q|/4$, therefore, in view of (4.23), there exists a function $g^3_Q : Q \to \mathbb{R}_+$ such that $\int g^3_Q = \int_Q f$ and $f + g^1 + g^3_Q \le 1$. As in the previous step, we define

$$\gamma^3_Q := \frac{1}{\int_Q f} \left[\chi_Q f\right] \otimes g^3_Q \qquad \text{and} \qquad \gamma^3 := \sum_Q \gamma^3_Q.$$

By construction, $(\gamma^3)_x = \chi_{\Omega^c} f$ and denoting $g^3 := (\gamma^3)_y$, we have $\operatorname{supp} g^3 \subset \overline{\Omega^c}$ as well as $f + g^1 + g^3 \le 1$. Moreover

$$\int c \, d\gamma^3 \le \left(\int_{\Omega^c} f \right) \overline{k} \left(\sqrt{d} r_j \right) = (m - m_\Omega) \overline{k} \left(\sqrt{d} r_j \right). \tag{4.26}$$

Step 2.d. Conclusion : definition and properties of $\widehat{\gamma}$.

Eventually, we set $\widehat{\gamma} := \gamma^0 + \gamma^1 + \gamma^2 + \gamma^3$ and $\widehat{g} := \widehat{\gamma}_y$. There holds $\widehat{\gamma}_x = f^0 + f^1 + f^2 + f^3 = f$ and $f + \widehat{g} \le 1$ so that $\widehat{\gamma}$ is an admissible exterior transport plan for f. Besides, collecting the estimates (4.22),(4.25)&(4.26) we get

$$\Upsilon(f) \le \int c \, d\widehat{\gamma} \le \Upsilon(f_\Omega) + 2(m - m_\Omega) \overline{k} \left(\sqrt{d} r_j \right). \tag{4.27}$$

Step 3. We show that $m - m_\Omega \le C(m)\varepsilon$ (recall the definition (4.20) of ε).

As $\Upsilon(f_\Omega) \le \mathcal{E}(m_\Omega)$ and $\mathcal{E}(m) - \varepsilon = \Upsilon(f)$, (4.27) yields

$$\mathcal{E}(m) - \varepsilon \le \mathcal{E}(m_\Omega) + 2\overline{k} \left(\sqrt{d} r_j \right) (m - m_\Omega).$$

Additionally by Proposition 4.7, $\mathcal{E}(m_\Omega) \leq \frac{m_\Omega}{m}\mathcal{E}(m)$. Hence

$$\left(\frac{\mathcal{E}(m)}{m} - 2\overline{k}\left(\sqrt{d}r_j\right)\right)(m - m_\Omega) \leq \varepsilon.$$

By continuity of k, $\overline{k}(\sqrt{d}r_j) \to 0$ as $r_j \to 0$. Recalling that $r_j = 2^{-j}r_0$, we fix $j \geq 0$ as the first integer such that $\overline{k}(\sqrt{d}r_j) \leq \mathcal{E}(m)/4m$ (notice that j does not depend on ε). Therefore

$$m - m_\Omega \leq \frac{2m\varepsilon}{\mathcal{E}(m)}. \tag{4.28}$$

This yields

$$\int_{\mathbb{R}^d\setminus\Omega} f = \int f - \int f_\Omega = m - m_\Omega \leq \frac{2m\varepsilon}{\mathcal{E}(m)}. \tag{4.29}$$

For future use, let us also notice that injecting (4.28) into (4.27) we obtain

$$\mathcal{E}(m) - \varepsilon \leq \Upsilon(f) \leq \Upsilon(f_\Omega) + \varepsilon. \tag{4.30}$$

Step 4 : Bounding the diameter of Ω.

We finally prove that Ω is uniformly bounded which would conclude the proof. For Q_-, $Q_+ \in \mathcal{Q}_0$, we write $Q_- \sim Q_+$ if there exists a finite chain

$$Q_- = Q_0, Q_1, \dots, Q_n = Q_+ \tag{4.31}$$

such that $Q_i \in \mathcal{Q}_0$ and $d(Q_{i-1}, Q_i) \leq 4R + \sqrt{d}r_j$ for $1 \leq i \leq n$. This defines an equivalence relation. Let us show that there exists only one equivalence class. We assume by contradiction that there exist at least two equivalence classes, and we let $\mathcal{C}^1$ be one of these classes and $\mathcal{C}^2$ be the union of the remaining classes. For $i \in \{1, 2\}$, we then define Ω^i to be the reunion of the cubes Q such that $d(Q, \mathcal{C}^i) \leq R$. By construction, $d(\Omega_1, \Omega_2) > 2R$. Recalling that Ω is the reunion of the cubes Q such that $d(Q, \Omega_0) \leq R$, we have $\Omega^1 \cup \Omega^2 = \Omega$.

For $i \in \{1, 2\}$, we set $f^i_\Omega := f_\Omega \chi_{\Omega^i}$ and $m^i_\Omega = \int f^i_\Omega$. We have $m^1_\Omega + m^2_\Omega = m_\Omega \leq m$ and $m^1_\Omega, m^2_\Omega \geq 2^{-jd}|Q_0|/4 = 2^{-jd-2}|Q_0|$. Additionally, by Proposition 4.2 (ii),

$$\Upsilon(f_\Omega) = \Upsilon(f^1_\Omega) + \Upsilon(f^2_\Omega) \leq \mathcal{E}(m^1_\Omega) + \mathcal{E}(m^2_\Omega).$$

Injecting this inequality into (4.30) yields

$$\mathcal{E}(m) - \varepsilon \leq \Upsilon(f) \leq \Upsilon(f_\Omega) + \varepsilon \leq \mathcal{E}(m^1_\Omega) + \mathcal{E}(m^2_\Omega) + \varepsilon.$$

By definition of e this rewrites as

$$me(m) \leq m^1_\Omega e(m^1_\Omega) + m^2_\Omega e(m^2_\Omega) + 2\varepsilon. \tag{4.32}$$

As $m^1_\Omega + m^2_\Omega \leq m$ and for $i \in \{1, 2\}$, $m^i_\Omega \geq 2^{-jd-2}|Q_0|$, we have $m^i_\Omega \leq m - 2^{-jd-2}|Q_0|$. Recall that by Proposition 4.7, e is increasing, so that $e(m^i_\Omega) \leq e(m - 2^{-jd}m_0)$. Hence

$$m^1_\Omega e(m^1_\Omega) + m^2_\Omega e(m^2_\Omega) \leq me\left(m - 2^{-jd}m_0\right).$$

With (4.32), we obtain

$$me(m) \leq me\left(m - 2^{-jd}m_0\right) + 2\varepsilon,$$

which is absurd for ε small enough because e is increasing. It follows that for $\varepsilon > 0$ small enough the relation $\sim$ has a single class. Recall that for all $Q \in \mathcal{Q}_0$, $\int_Q (f + g) \geq 2^{-jd-2}|Q_0|$. Thus the maximal length of a chain in (4.31) without any repetition is bounded by $N := \lfloor 2^{jd+3}m/|Q_0| \rfloor$. Therefore, the diameter of Ω is bounded by $(4R + 2\sqrt{d}r_j)(N + 1)$ with r_j and N only depending on m, the dimension d and the cost c. Together with (4.29) this proves the proposition. □

We can now apply the direct method of Calculus of Variations to establish the existence of a maximiser for (4.12).

Proof of Theorem 1.1.

Let f_n be a maximising sequence for (4.12) and let $R_* = R_*(m)$ be given by Proposition 4.9 so that up to translation,

$$\int_{\mathbb{R}^d \setminus B_{R_*}} f_n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $f_n \lambda$ is a tight sequence of $\mathcal{M}_+(\mathbb{R}^d)$ and up to extraction of a subsequence it converges weakly-$*$ to $f\lambda$ where f is admissible for (4.12). By Proposition 4.2 (iv),

$$\Upsilon(f) = \lim \Upsilon(f_n) = \mathcal{E}(m),$$

so that f is a maximiser for $\mathcal{E}(m)$.

Let now f be any maximiser of $\mathcal{E}(m)$. Applying Proposition 4.9 to f we have that up to a translation $\operatorname{supp} f \subset \overline{B}_{R_*}$. This concludes the proof. □

Let us show that when f is compactly supported there exist Kantorovitch potentials for the problem (4.1) (this is the situation of interest as we have just established that the maximisers of $\mathcal{E}(m)$ are compactly supported in $\mathbb{R}^d$).

Lemma 4.10. *Let $m > 0$ and assume that $f \in L^1_m$ is compactly supported. Let $R = R(m)$ be given by Proposition 4.1 such that all maximisers γ of $\Upsilon(f)$ are supported in $X := \operatorname{supp} f + \overline{B}_R$. Then, there exists a pair $(\varphi, \psi) \in C_c(\mathbb{R}^d) \times C_c(\mathbb{R}^d)$ optimal for $\Upsilon^*(f)$. Additionally, $\varphi = \psi^c$, $\psi = \varphi^c_-$ and both φ and ψ are compactly supported in X.*

Proof. Let us introduce $\tilde{c} := c_{|X \times X}$ which is a continuous cost function on the compact set X. By Proposition 3.4, there exists $\widetilde{\psi} \in C(X)$ with $\widetilde{\psi} = (\widetilde{\psi}^{\tilde{c}\tilde{c}})_-$ such that $\Upsilon^*(f) = K_f(\widetilde{\psi}^{\tilde{c}}, \widetilde{\psi})$. By Proposition 3.7,

$$\{\widetilde{\psi}^{\tilde{c}\tilde{c}} < 0\} \subset \{f + g = 1\} \subset X.$$

Combining this with $\widetilde{\psi} = (\widetilde{\psi}^{\tilde{c}\tilde{c}})_-$ and $f + g = 0$ on ∂X , we get $\widetilde{\psi} = 0$ on ∂X. We extend the potentials on $\mathbb{R}^d$ by setting

$$\psi := \begin{cases} \widetilde{\psi} & \text{in} \quad X, \\ 0 & \text{in} \quad X^c, \end{cases} \qquad \text{and for } x \in \mathbb{R}^d, \quad \varphi(x) := \psi^c(x) = \inf\{c(x,y) - \psi(y) : \, y \in \mathbb{R}^d\}.$$

We now show that the pair (φ, ψ) satisfies the conclusion of the lemma.

Observe that ψ is continuous and supported in X and that $\psi \leq 0$. Hence $\varphi \geq 0$. Moreover, for $x \in \mathbb{R}^d \setminus \operatorname{supp} \psi$,

$$\varphi(x) \leq c(x,x) - \psi(x) = 0,$$

so that φ is also supported in X.
Next, for $x \in X$,

$$\varphi(x) = \min\left(\min_{y \in \mathbb{R}^d \setminus X} c(x,y), \, \min_{y \in X} \{c(x,y) - \widetilde{\psi}(y)\} \right).$$

Let $x \in X$. For $y \in \mathbb{R}^d \setminus X$, there exists $\widetilde{y}$ in the intersection of the segment $[x,y]$ with ∂X. By continuity, $\widetilde{\psi}(\widetilde{y}) = 0 = \widetilde{\psi}(y)$ and moreover by (H3), $c(x,\widetilde{y}) \leq c(x,y)$ so that $c(x,\widetilde{y}) \leq c(x,y) - \widetilde{\psi}(y)$. We deduce that for $x \in X$ the above formula simplifies as

$$\varphi(x) = \min_{y \in X} \{c(x,y) - \widetilde{\psi}(y)\} = \widetilde{\psi}^{\tilde{c}}(x).$$

This proves that φ is continuous and that $K_f(\varphi, \psi) = K_f(\widetilde{\psi}^{\tilde{c}}, \widetilde{\psi}) = \Upsilon^*(f)$. Moreover, using the same argument as above, we have $\varphi^c(y) = 0$ for $y \notin X$. For $y \in X$,

$$\varphi^c(y) = \min\left(\min_{x \in \mathbb{R}^d \setminus X} c(x,y), \, \min_{x \in X} \{c(x,y) - \widetilde{\psi}^{\tilde{c}}(y)\} \right) = \min_{x \in X} \{c(x,y) - \widetilde{\psi}^{\tilde{c}}(y)\} = \widetilde{\psi}^{\tilde{c}\tilde{c}}(y).$$

We deduce $\varphi^c_- = 0 = \psi$ in X^c, and $\varphi^c_- = (\psi^{\tilde{c}\tilde{c}})_- = \widetilde{\psi}$ in X. Thus $\varphi^c_- = \psi$ everywhere. This ends the proof of the lemma. □

Let us now recall a variant of the bathtub principle, see [16, Theorem 1.14].

Proposition 4.11. *Let $\xi : \mathbb{R}^d \to \mathbb{R}_+$ be measurable and such that for all $t \geq 0$, $|\{\xi > t\}| < \infty$. Given $m > 0$, let*

$$t := \inf\{s \geq 0,\, |\{\xi > s\}| \leq m\}.$$

Then, the maximisers of

$$\sup_{\widetilde{f}} \left\{ \int \widetilde{f}\xi \,:\, \widetilde{f} \in L^1(\mathbb{R}^d),\, 0 \leq \widetilde{f} \leq 1,\, \int \widetilde{f} = m \right\}$$

are the functions $f := \chi_{\{\xi>t\}} + \theta$, where $\theta \in L^1(\mathbb{R}^d, [0,1])$ is supported in $\{\xi = t\}$ and satisfies

$$\int \theta = m - |\{\xi > t\}|.$$

We are now ready to establish Corollary 1.2.

Proof of Corollary 1.2.

By Theorem 1.1, the optimisation problem (4.12) admits a compactly supported solution f. Let $(\varphi, \psi) \in C_c(\mathbb{R}^d) \times C_c(\mathbb{R}^d)$ be an optimal pair for $\Upsilon^*(f)$ provided by Lemma 4.10, so that

$$\Upsilon^*(f) = \int f(\varphi - \psi) + \int \psi\,.$$

We see that f is a maximiser of:

$$\sup \left\{ \int \widetilde{f}(\varphi - \psi) \,:\, \widetilde{f} \in L^1(\mathbb{R}^d),\, 0 \leq \widetilde{f} \leq 1,\, \int \widetilde{f} = m \right\}.$$

Let us set $\xi := \varphi - \psi \geq 0$. By Proposition 4.11 there exists $t \geq 0$ and $\theta \in L^1(\mathbb{R}^d, [0,1])$ supported in $\{\xi = t\}$ such that $f = \chi_{\{\xi>t\}} + \theta$. Notice in particular that since $\theta \in [0,1]$, we have

$$|\{\xi = t\}| \geq \int \theta = m - |\{\xi > t\}|.$$

and there exist measurable subsets $G \subset \{\xi = t\}$ with $|G| = m - |\{\xi > t\}|$. For any such set, setting

$$\bar{f} := \chi_{\{\xi>t\}} + \chi_G,$$

we have $\Upsilon^*(\bar{f}) = \Upsilon^*(f)$ and $\bar{f}$ is also a maximiser of (4.12). Since $\bar{f}$ is a characteristic function, by Theorem 4.4 and Corollary 4.5, there exists $F \subset \mathbb{R}^d$ such that any minimiser γ of $\Upsilon(\bar{f})$ satisfies $\gamma_y = \chi_F$. Setting $E := \{\xi > t\} \cup G$, we deduce that

$$\Upsilon_{\mathrm{set}}(E) = \Upsilon^*(\bar{f}) = \Upsilon(\bar{f}) = \Upsilon(f)$$

so that $\mathcal{E}(m) = \mathcal{E}_{\mathrm{set}}(m)$, which concludes the proof. □

5. Maximisers of (4.12) are characteristic functions of balls

In this section we prove Theorem 1.3. We assume that $c(x,y) = k(|y-x|)$ with $k \in C(\mathbb{R}_+, \mathbb{R}_+)$ increasing and coercive and with $k(0) = 0$. In particular, we have now $c = \bar{c}$, so that the operations of c-transform and $\bar{c}$-transform coincide. Also notice that by Theorem 4.4, if $f = \chi_E$ for some Lebesgue measurable set E then $\Upsilon_{\mathrm{set}}(E) = \Upsilon(\chi_E)$. By abuse of notation, we write $\Upsilon(E)$ for $\Upsilon(\chi_E)$. Since the class of costs that we consider is invariant by scaling we assume without loss of generality that $m = \omega_d$.

We now recall the definition of symmetric rearrangement of functions with constant sign (see [16, Chapter 3] for more details on symmetric rearrangements).

Definition 5.1.

(i) Given a measurable set $A \subset \mathbb{R}^d$, we define the symmetric rearrangement of A as the open ball A^* centred at the origin and of volume $|A|$.

(ii) Let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be measurable and such that for every $t \geq 0$, $|\{\varphi > t\}| < \infty$. Its symmetric decreasing rearrangement is defined by

$$\varphi^*(x) := \int_{\mathbb{R}_+} \chi_{\{\varphi > t\}^*}(x)\, dt.$$

(iii) Let $\psi : \mathbb{R}^d \to \mathbb{R}_-$ be measurable and such that for every $t \leq 0$, $|\{\psi < t\}| < \infty$. Its symmetric increasing rearrangement is defined by

$$\psi_*(x) := -(-\psi)^*(x) = -\int_{\mathbb{R}_-} \chi_{\{\psi < t\}^*}(x)\, dt.$$

The following lemma recalls some basic properties of the symmetric increasing rearrangement ψ_* of a non-positive function ψ. All these properties but the continuity of ψ_* follow immediately from the definition. The fact that continuity is preserved by symmetric rearrangement is well-known but we have no reference for this at hand. We provide a short proof for the reader's convenience.

Lemma 5.2. *Let $\psi : \mathbb{R}^d \to \mathbb{R}_-$ be as in Definition 5.1. Then, ψ_* is non-positive, radial, non-decreasing, and for any $t \leq 0$, $\{\psi_* < t\} = \{\psi < t\}^*$. Besides, if ψ is supported in a compact set of diameter bounded by $2R > 0$ then ψ_* is supported in $\overline{B}_R$. If moreover ψ is continuous then ψ_* is also continuous.*

Proof of the last point. Let $\psi \in C_c(\mathbb{R}^d, \mathbb{R}_-)$. First, as the strict sublevels sets $\{\psi_* < t\}$ are the open balls $\{\psi < t\}^*$, ψ_* is upper semi-continuous (note that this is true even when ψ is not continuous).

Let us now establish that ψ_* is lower semi-continuous, *i.e.* that for any $t \leq 0$, $\{\psi_* \leq t\}$ is closed. We first notice that $\{\psi_* \leq 0\} = \mathbb{R}^d$ is closed. Given $t < 0$, let $t_n < 0$ be a decreasing sequence converging to t. Observe that if for some $n \geq 0$, $\{\psi < t_n\} = \emptyset$, then $\{\psi_* \leq t\} = \emptyset$ is closed. Next, we assume that for every $n \geq 0$,

$$\{\psi < t_n\} \neq \emptyset. \tag{5.1}$$

We denote by R_n the radius of the ball $\{\psi_* < t_n\}$. Notice that the sequence R_n is non-increasing and bounded by 0, so that R_n converges to some $R \geq 0$.

Let us show that the sequence R_n is decreasing. By contradiction, we assume that $R_n = R_{n+1}$ for some $n \geq 0$. Then $\{\psi < t_n\}^* = \{\psi < t_{n+1}\}^*$ and $|\{t_{n+1} \leq \psi < t_n\}| = 0$. Using (5.1) and the fact that ψ is compactly supported, there exists x such that $\psi(x) < t_{n+1}$ and y such that $\psi(y) > t_n$. Thus by continuity of ψ there exists z such that $\psi(z) = (t_{n+1} + t_n)/2$. By continuity of ψ again, there exists $\eta > 0$ such that $B_\eta(z) \subset \{t_{n+1} < \psi < t_n\}$, contradicting the fact that $|\{t_{n+1} \leq \psi < t_n\}| = 0$. As a conclusion, the sequence R_n is decreasing and

$$\{\psi_* \leq t\} = \bigcap_{n \geq 0} \{\psi_* < t_n\} = \bigcap_{n \geq 0} B_{R_n} = \overline{B}_R.$$

Hence ψ_* is lower semi-continuous and therefore continuous. □

To prove Theorem 1.3, we need a last lemma about final lemma characterising optimal potentials of $\Upsilon^*(\chi_{B_1})$.

Lemma 5.3. *Let (ψ^c, ψ) be a pair of optimal potentials for $\Upsilon^*(\chi_{B_1})$ such that ψ is radially symmetric and non-decreasing. Then ψ^c is radially symmetric and non-increasing. Besides, ψ^c is radially decreasing on B_1.*

Proof. Combining the facts that k is continuous, that $k(r) \to \infty$ as $r \to \infty$ and that ψ is bounded, we see that for any $x \in \mathbb{R}^d$,

$$\psi^c(x) = \min\{k(|y - x|) - \psi(y) : y \in \mathbb{R}^d\}.$$

As ψ is radially symmetric non-decreasing and k is increasing, we easily see that

$$\psi^c(x) = \min\{k(|y-x|) - \psi(y) : y, \exists \lambda \geq 1,\, y = \lambda x\}, \tag{5.2}$$

which in turn implies that ψ^c is radially symmetric.

From now on, for radial functions $\zeta : \mathbb{R}^d \to \mathbb{R}$, we make the abuse of notation $\zeta(r) = \zeta(r\sigma)$ for $r \geq 0$ where σ is some fixed element of $\mathbb{S}^{d-1}$. With this convention (5.2) reads

$$\psi^c(r) = \min_{s \geq r} k(s-r) - \psi(s). \tag{5.3}$$

Let us prove that ψ^c is non-decreasing. Let $0 \leq r_1 \leq r_2$. By (5.3), there exists $r \geq r_1$ such that

$$\psi^c(r_1) = k(r - r_1) - \psi(r). \tag{5.4}$$

If $r \leq r_2$, we use $\psi^c(r_2) \leq k(0) - \psi(r_2) = -\psi(r_2)$ and deduce from (5.4) and the fact that ψ is non-decreasing that

$$\psi^c(r_2) - \psi^c(r_1) \leq \psi(r) - \psi(r_2) - k(r - r_1) \leq 0.$$

If $r > r_2$, we use $\psi^c(r_2) \leq k(r - r_2) - \psi(r)$ to get

$$\psi^c(r_2) - \psi^c(r_1) \leq k(r - r_2) - k(r - r_1) \leq 0,$$

because $r_1 \leq r_2 < r$ and k is increasing. In both cases $\psi^c(r_2) - \psi^c(r_1) \leq 0$. Hence ψ^c is non-increasing on $\mathbb{R}^d$.

We finally prove that ψ^c is decreasing on B_1. Let $0 < r_1 < r_2 < 1$. Given γ a minimiser for $\Upsilon(B_1)$, there exists $y \in \mathbb{R}^d \setminus B_1$ such that $(r_1\sigma, y) \in \operatorname{supp}\gamma$. By Proposition 3.7, γ is an optimal transport plan between f and $g := \gamma_y$ and (ψ^c, ψ^{cc}) is a pair of Kantorovitch potentials for the transport between f and g. Therefore,

$$\psi^c(r_1) + \psi^{cc}(|y|) = k(|y - r_1\sigma|). \tag{5.5}$$

Let us prove by contradiction that $y \in [1, +\infty)\sigma$. Assume it is not and let $y' := |y|\sigma$. Recalling that $|y| \geq 1 > r_1$, we have $|y' - r_1\sigma| = |y| - r_1 < |y - r_1\sigma|$ and since k is increasing we deduce

$$k(|y' - r_1\sigma|) < k(|y - r_1\sigma|).$$

Then, by definition of ψ^{cc} and taking into account that it is radially symmetric we get

$$\psi^c(r_1) + \psi^{cc}(|y|) \leq k(|y' - r_1\sigma|) < k(|y - r_1\sigma|)$$

which contradicts (5.5). Therefore, $y = r\sigma$ for some $r \geq 1$. By definition of the c-transform,

$$\psi^c(r_2) + \psi^{cc}(r) \leq k(r - r_2). \tag{5.6}$$

Subtracting (5.5) to (5.6), we obtain

$$\psi^c(r_2) - \psi^c(r_1) \leq k(r - r_2) - k(r - r_1) < 0,$$

where we used $r_1 < r_2 < 1 \leq r$. This shows that ψ^c is decreasing on B_1. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3.

Part I : Unit balls are maximisers of $\mathcal{E}(\omega_d)$.

By Theorem 1.1, there exists a compactly supported maximiser f for (4.12) with $m = \omega_d$. By Lemma 4.10, there exists an optimal pair $(\psi^c, \psi) \in C_c(\mathbb{R}^d) \times C_c(\mathbb{R}^d)$ for problem $\Upsilon^*(f)$ such that $\psi = (\psi^{cc})_-$.

Step 1. We build a radially symmetric maximiser for (4.12).

Let ψ_* be the symmetric increasing rearrangement of ψ. By Lemma 5.2, as $\psi \in C_c(\mathbb{R}^d)$, we also have $\psi_* \in C_c(\mathbb{R}^d)$. We denote by ψ_*^c the function $(\psi_*)^c$. By definition, $\psi_*^c \oplus \psi_* \leq c$. Proceeding as in the proof of Lemma 4.10, we obtain $\psi_*^c \in C_c(\mathbb{R}^d)$. Thus (ψ_*^c, ψ_*) is admissible for $\Upsilon^*(B_1)$.

Notice that (f,ψ) solves the double supremum problem (recall the definition (3.2) of K_f)

$$\sup_{f}\sup_{\psi\in C_c(\mathbb{R}^d)}\left\{K_f(\psi^c,\psi) : 0\le f\le 1,\ \int f\ =\omega_d,\ \psi\le 0\right\}.$$

Hence

$$\mathcal{E}(\omega_d)=K_f(\psi^c,\psi)=\int f(\psi^c-\psi)\ +\int\psi\ \ge\Upsilon^*(B_1)\ge K_{\chi_{B_1}}(\psi_*^c,\psi_*)=\int_{B_1}(\psi_*^c-\psi_*)\ +\int\psi_*\,.$$

In the remainder of this step, we establish the converse inequality

$$K_f(\psi^c,\psi)\le K_{\chi_{B_1}}(\psi_*^c,\psi_*), \tag{5.7}$$

so that B_1 is a maximiser of $\mathcal{E}(\omega_d)$ and the first part of Theorem 1.3 is proved. Notice that (5.7) also implies that (ψ_*^c,ψ_*) is a pair of optimal potentials for $\Upsilon^*(B_1)$. To establish (5.7), we first notice that by construction $\int\psi=\int\psi^*$ so that we only need to prove

$$\int f(\psi^c-\psi)\ \le\int_{B_1}(\psi_*^c-\psi_*)\,. \tag{5.8}$$

In Step 2 below we establish the inequality

$$(\psi^c)^*\le(\psi_*)^c=\psi_*^c, \tag{5.9}$$

where $(\psi^c)^*$ denotes the symmetric decreasing rearrangement of ψ^c. Admitting that (5.9) holds we deduce (5.8) as follows. Since f is non-negative and compactly supported we have by the Hardy-Littlewood inequality (see [16, Theorem 3.4])

$$-\int f\psi\ \le-\int f^*\psi_*\qquad\text{and}\qquad\int f\psi^c\ \le\int f^*(\psi^c)^*\ \overset{(5.9)}{\le}\ \int f^*\psi_*^c\,. \tag{5.10}$$

Using that $-\psi_*$ and ψ_*^c are radially symmetric and non-increasing, we may appeal to Proposition 4.11 and conclude that separately,

$$-\int f\psi\ \le-\int\chi_{B_1}\psi_*\qquad\text{and}\qquad\int f\psi^c\ \le\int\chi_{B_1}\psi_*^c\,. \tag{5.11}$$

Summing these inequalities gives (5.8) and thus (5.7). This proves that χ_{B_1} is a maximiser for $\mathcal{E}(\omega_d)$ and then that B_1 is a maximiser for $\mathcal{E}_{\rm set}(\omega_d)$.

Step 2. Proof of (5.9).

As ψ_*^c and $(\psi^c)^*$ are both continuous radially symmetric functions, to prove (5.9) it is sufficient to establish that for any $t>0$, $\{(\psi^c)^*>t\}\subset\{\psi_*^c>t\}$, *i.e.* that

$$|\{(\psi^c)^*>t\}|=|\{\psi^c>t\}|\le|\{\psi_*^c>t\}|. \tag{5.12}$$

Recall that as $\psi\in C_c(\mathbb{R}^d)$ and $k\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $k(x)\to\infty$ as $x\to\infty$, for any $x\in\mathbb{R}^d$ the function $k(|y-x|)-\psi(y)$ admits a minimum on $\mathbb{R}^d$. Thus for any $x\in\mathbb{R}^d$ the infimum defining $\psi^c(x)$ (see Definition 2.2) is reached. Recalling that k is also radially symmetric and increasing, we obtain

$$\begin{aligned}\{\psi^c>t\}&=\{x\in\mathbb{R}^d:\min\{k(|y-x|)-\psi(y):y\in\mathbb{R}^d\}>t\}\\&=\{x\in\mathbb{R}^d:-\psi>t-k(r)\text{ on }\overline{B}_r(x)\ \forall r\ge0\}\\&=\bigcap_{r\ge0}\{x\in\mathbb{R}^d:-\psi>t-k(r)\text{ on }\overline{B}_r(x)\}\\&=\bigcap_{r\ge0}\{-\psi>t-k(r)\}_r,\end{aligned}$$

where for $\Omega\subset\mathbb{R}^d$ and $r\ge0$, Ω_r is defined as $\Omega_r:=\{x\in\Omega:d(x,\mathbb{R}^d\setminus\Omega)>r\}$. In particular,

$$|\{\psi^c>t\}|\le\inf_{r\ge0}|\{-\psi>t-k(r)\}_r|. \tag{5.13}$$

We observe that $\{-\psi > t - k(r)\}$ is an open set for any $t > 0$ and $r \geq 0$. We also notice that (5.13) holds for all $\psi \in C_c(\mathbb{R}^d)$. In particular, it holds for ψ_*. Moreover, as ψ_* is radially non-decreasing by construction, the sets $\{-\psi_* > t - k(r)\}_r$ are open balls centred at the origin and we have in fact

$$|\{\psi_*^c > t\}| = \inf_{r\geq 0} |\{-\psi_* > t - k(r)\}_r|.$$

Let us now prove the following claim.

Claim. *Let $s > 0$ and $V > 0$.*

(i) If $V > \omega_d s^d$ then, among open sets $\Omega \subset \mathbb{R}^d$ of volume V, $|\Omega_s|$ is maximal if and only if Ω is a ball.

(ii) If $V \leq \omega_d s^d$ then $|\Omega_s| = 0$ for any set of volume V.

Let $V > 0$ and $s > 0$ and let $\Omega \subset \mathbb{R}^d$ be an open set. We assume without loss of generality that $|\Omega| = V$ and $|\Omega_s| > 0$. Notice that we always have $\Omega_s + B_s \subset \Omega$ (but the converse inclusion may fail). By the Brunn-Minkowski inequality (see for instance [11]) applied to Ω_s and B_s, there holds

$$V^{1/d} = |\Omega|^{1/d} \geq |\Omega_s + B_s|^{1/d} \geq |\Omega_s|^{1/d} + |B_s|^{1/d}. \tag{5.14}$$

If Ω_s is a ball, then Ω is a ball of volume V, $\Omega = \Omega_s + B_s$ and we have equality in (5.14). Conversely if we have equality in (5.14), by the equality case of the Brunn-Minkowski inequality and the fact that $s > 0$, Ω_s is a ball and $|\Omega| = |\Omega_s + B_s|$, so that Ω is a ball. This proves the first part of the claim.

Regarding the second part, we assume that $|\Omega| \leq \omega_d s^d$ and (by contradiction) that $|\Omega_s| > 0$. The above reasoning applies and we have $\Omega = \Omega_s + B_s$ so that $|\Omega_s| > 0$ implies $|\Omega| > |B_s| = \omega_d s^d$ and we get a contradiction. This proves the claim.

By definition, $\{-\psi > t - k(r)\}$ and $\{-\psi_* > t - k(r)\}$ have the same volume. As a consequence of the claim, for any $t > 0$ and $r > 0$,

$$|\{-\psi > t - k(r)\}_r| \leq |\{-\psi_* > t - k(r)\}_r|. \tag{5.15}$$

Notice that the previous inequality is an equality if $r = 0$, as $\Omega_0 = \Omega$ for any open set Ω. Taking the infimum on $r \geq 0$ yields

$$|\{\psi^c > t\}| \leq \inf_{r\geq 0} |\{-\psi > t - k(r)\}_r| \leq \inf_{r\geq 0} |\{-\psi_* > t - k(r)\}_r| = |\{\psi_*^c > t\}|. \tag{5.16}$$

This proves (5.12) which in turn implies (5.9).

Part II : Unit balls are the unique maximisers of $\mathcal{E}(\omega_d)$.

Step 1. Proof of $f = \chi_{\{\psi^c > \psi_*^c(1)\}}$ (exploiting the equality case in the bathtub principle).

We now show that any maximiser f is of the form $\chi_{\{\psi^c > \psi_*^c(1)\}}$. By Lemma 5.3, ψ_*^c is radially decreasing on B_1 and non-increasing on $\mathbb{R}^d$. Thus χ_{B_1} is the only function maximising

$$\sup_{\widetilde{f}} \left\{ \int \widetilde{f}\psi_*^c \, : 0 \leq \widetilde{f} \leq 1, \, \int \widetilde{f} = \omega_d \right\}.$$

As $\Upsilon^*(\chi_{B_1}) = \mathcal{E}(\omega_d)$, the inequalities in (5.10) and (5.11) are in fact equalities (and (5.9) is also an equality in B_1). Namely, there hold

$$(\psi^c)^* = \psi_*^c, \qquad -\int f\psi = -\int f^*\psi_* \qquad \text{and} \qquad \int f\psi^c = \int f^*\psi_*^c \, .$$

This leads to

$$\int f\psi^c = \int f^*(\psi^c)^* = \int f^*\psi_*^c = \int_{B_1} \psi_*^c \, ,$$

and f is a maximiser of

$$\sup_{\widetilde{f}} \left\{ \int \widetilde{f}\psi^c \,:\, 0 \le \widetilde{f} \le 1,\ \int \widetilde{f} = \omega_d \right\}.$$

Let us now prove that $|\{\psi^c > \psi_*^c(1)\}| = \omega_d$ (which with Proposition 4.11 yields $f = \chi_{\{\psi^c > \psi_*^c(1)\}}$). Since $(\psi^c)^* = \psi_*^c$ in B_1, and ψ_*^c is decreasing in B_1 by Lemma 5.3, there holds for $t \ge \psi_*^c(1)$,

$$|\{\psi^c > t\}| = |\{(\psi^c)^* > t\}| = |\{\psi_*^c > t\}|. \tag{5.17}$$

Using this for $t = \psi_*^c(1)$ we get $|\{\psi^c > \psi_*^c(1)\}| = \omega_d$ and we conclude with Proposition 4.11 that $f = \chi_{\{\psi^c > \psi_*^c(1)\}}$.

Step 2. We prove that $\{\psi^c > t\}$ is a ball for $t > \psi_*^c(1)$ (exploiting the equality case in the Brunn-Minkowski inequality).
Step 2.a.

We fix $t > \psi_*^c(1)$. Combining (5.17) and (5.16), we get that

$$|\{\psi^c > t\}| = \inf_{r\ge 0} |\{-\psi > t - k(r)\}_r| = \inf_{r\ge 0} |\{-\psi_* > t - k(r)\}_r| = |\{\psi_*^c > t\}|. \tag{5.18}$$

The following claim is established in Step 2.b below.

Claim. *There exists $r_* = r_*(t) > 0$ such that*

$$|\{\psi_*^c > t\}| = \inf_{r\ge 0} |\{-\psi_* > t - k(r)\}_r| = |\{-\psi_* > t - k(r_*)\}_{r_*}.$$

Provisionally assuming the claim let us prove that $\{\psi^c > t\}$ is a ball.
We assume without loss of generality that $|\{\psi_*^c > t\}| > 0$ (otherwise $|\{\psi^c > t\}| \le |\{\psi_*^c > t\}| = 0$ by (5.15) and the open set $\{\psi^c > t\}$ is empty). Next, the claim, (5.18) and (5.15) yield that $r_*(t)$ also minimises $\inf_{r\ge 0} |\{-\psi > t - k(r)\}_r|$. Thus by (5.18), $\{-\psi_* > t - k(r_*(t))\}_{r_*(t)}$ is a ball of positive volume. As $r_*(t) > 0$, by the equality case of the claim of Part I, Step 2, the set $\{-\psi > t - k(r_*(t))\}_{r_*(t)}$ is also a ball. As $\{\psi^c > t\} \subset \{-\psi > t - k(r_*(t))\}_{r_*(t)}$, by (5.18) the inclusion is actually an equality. Hence $\{\psi^c > t\}$ is a ball.

Step 2.b. Proof of the claim.

We first show that there exist $0 < r_t < R_t < \infty$ such that

$$\inf_{r\ge 0} |\{-\psi_* > t - k(r)\}_r| = \inf_{r_t \le r \le R_t} |\{-\psi_* > t - k(r)\}_r|. \tag{5.19}$$

We start with the upper bound on r. By (H1)&(H3), there exists R_t such that $k(R_t) = t + 1$. Hence, if $r > R_t$, $\{-\psi_* > t - k(r)\} = \mathbb{R}^d$. We can thus only consider the radii $r \le R_t$.

We now prove the lower bound on r. Recall that $t > \psi_*^c(1)$ and that ψ_*^c is decreasing in B_1. Therefore there exists $R_*(t) < 1$ such that

$$\{\psi_*^c > t\} = B_{R_*(t)}.$$

We set $r_t := \frac{1-R_*(t)}{2}$ and claim that (5.19) holds for this value. To ease notation, let us set for $r > 0$

$$S_r := \{-\psi_* > t - k(r)\}_r = \{x \in \mathbb{R}^d : -\psi_* > t - k(r) \text{ on } \overline{B}_r(x)\}.$$

We also define $\overline{R} := \frac{1+R_*(t)}{2}$. In order to prove (5.19) it is enough to show that

$$\{\psi_*^c > t\} = \cap_{r\ge r_t} S_r. \tag{5.20}$$

Recalling that the sets S_r are centred balls and that $B_{R_*(t)} \subset B_{\overline{R}}$, we have

$$\{\psi_*^c > t\} = \cap_{r\ge 0}(S_r \cap B_{\overline{R}}).$$

We now claim that

$$\cap_{r\ge r_t}(S_r \cap B_{\overline{R}}) \subset \cap_{r<r_t}(S_r \cap B_{\overline{R}}),$$

which is equivalent to

$$\cup_{r<r_t}(S_r^c \cap B_{\overline{R}}) \subset \cup_{r\ge r_t}(S_r^c \cap B_{\overline{R}}). \tag{5.21}$$

To prove this let $x \in S_r^c \cap B_{\overline{R}}$ for some $r < r_t$. By definition of S_r^c,

$$\min_{y \in \overline{B}_r(x)} k(r) - \psi_*(y) \leq t.$$

In particular since k is increasing, there exists $y \in \overline{B}_r(x)$ such that

$$k(|x-y|) - \psi^*(y) \leq t.$$

As $x \in B_{\overline{R}} \subset B_1$, and (ψ_*^c, ψ_*^{cc}) are Kantorovitch potentials for the external transport minimising $\Upsilon(B_1)$ (see Proposition 3.7) there exists $z \in B_1^c$ such that $\psi_*^{cc}(z) = \psi_*(z)$ (by (3.14)) and

$$\psi_*^c(x) = k(x-z) - \psi_*(z) = \min_y k(|x-y|) - \psi_*(y) \leq t.$$

Since $z \in B_1^c$ and $x \in B_{\overline{R}}$ we have

$$r' = |z-x| \geq 1 - \overline{R} = \frac{1 - R_*(t)}{2} = r_t$$

and thus

$$\min_{z \in \overline{B}_{r'}(x)} k(r') - \psi_*(z) \leq t$$

so that $x \in S_{r'}$. This shows (5.21) which implies

$$\{\psi_*^c > t\} = \cap_{r \geq r_t} (S_r \cap B_{\overline{R}}).$$

Eventually, we must have $S_r \subset B_{\overline{R}}$ for some $r \geq r_t$ (otherwise $\{\psi_*^c > t\} = B_{\overline{R}}$ which is absurd). This concludes the proof of (5.20) and thus of (5.19).

Next, setting

$$L(r) := |\{-\psi_* > t - k(r)\}_r|,$$

we still have to establish that the infimum of L over $[r_t, R_t]$ is reached. For this we establish that L is lower semi-continuous (together with (5.19) this will conclude the proof of the existence of $r_*(t) > 0$ minimising L over $\mathbb{R}_+$). We start by noticing that, $r \mapsto |\{-\psi_* > r\}|$ is lower semi-continuous on $\mathbb{R}_+$. Let us denote by $\rho_t(r)$ the radius of the ball $\{-\psi_* > t - k(r)\}$. As k is continuous, the function $r \mapsto \rho_t(r)$ is also lower semi-continuous. Finally, as $L(r) = \omega_d[(\rho_t(r) - r)_+]^d$, L is lower semi-continuous as well. This ends the proof of the claim.

Step 3. Conclusion.

Let now t_n be a decreasing sequence converging to $\psi_*^c(1)$. We have

$$\{\psi^c > \psi_*^c(1)\} = \bigcup_{n \geq 0} \{\psi^c > t_n\} \qquad \text{and} \qquad \{\psi_*^c > \psi_*^c(1)\} = \bigcup_{n \geq 0} \{\psi_*^c > t_n\} = B_1. \tag{5.22}$$

By (5.18), for every $n \geq 0$, $\{\psi^c > t_n\} = B_{r_n}(z_n)$, where r_n is the radius of $\{\psi_*^c > t_n\}$ and $z_n \in \mathbb{R}^d$. Since t_n is decreasing the sequence $B_{r_n}(z_n)$ is non-decreasing. Moreover, by (5.22) $r_n \to 1$ as $n \to \infty$. Hence there exists $z \in \mathbb{R}^d$ such that $\chi_{B_{r_n}} \to \chi_{B_1}(z)$ monotonically in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Eventually (5.22) implies that $\{\psi^c > \psi_*^c(1)\} = B_1(z)$. Consequently, $f = \chi_{\{\psi^c > \psi_*^c(1)\}} = \chi_{B_1(z)}$. This concludes the proof of the fact that balls are the unique maximisers to (4.12). □

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