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# ON THE $(n + 3)$ -WEBS BY RATIONAL CURVES INDUCED BY THE FORGETFUL MAPS ON THE MODULI SPACES $\mathcal{M}_{0,n+3}$

LUC PIRIO<sup>†</sup>

**ABSTRACT.** For  $n \geq 2$ , we discuss the curvilinear web  $\mathcal{W}_{0,n+3}$  on the moduli space  $\mathcal{M}_{0,n+3}$  defined by the  $n + 3$  forgetful maps  $\mathcal{M}_{0,n+3} \rightarrow \mathcal{M}_{0,n+2}$ . We recall classical results (first obtained by Room) which show that this web is linearizable when  $n$  is odd, or is equivalent to a web by conics when  $n$  is even. We then turn to the abelian relations (ARs) of these webs. After recalling the classical and well-known case when  $n = 2$  (related to the famous 5-terms functional identity of the dilogarithm), we focus on the case of the 6-web  $\mathcal{W}_{0,6}$ . We show that this web is isomorphic to the web formed by the projective lines contained in Segre’s cubic primal  $\mathcal{S} \subset \mathbf{P}^4$  and that a kind of ‘Abel’s theorem’ allows to describe the ARs of  $\mathcal{W}_{0,6}$  by means of the abelian 2-forms on the Fano surface  $F_1(\mathcal{S}) \subset G_1(\mathbf{P}^4)$  of lines contained in  $\mathcal{S}$ . We deduce from this that  $\mathcal{W}_{0,6}$  has maximal rank with all its AR rational, and that these span a space which is an irreducible  $\mathfrak{S}_6$ -module. Then we take up an approach due to Damiano that we correct in the case when  $n$  is odd: it leads to an abstract description of the space of ARs of  $\mathcal{W}_{0,n+3}$  as a  $\mathfrak{S}_{n+3}$ -representation. In particular, we obtain that this web has maximal rank for any  $n \geq 2$ . Finally, we consider ‘Euler’s abelian relation  $\mathcal{E}_n$ ’, a particular AR for  $\mathcal{W}_{0,n+3}$  constructed by Damiano from a characteristic class on the grassmannian of 2-planes in  $\mathbf{R}^{n+3}$  by means of Gelfand-MacPherson theory of polylogarithmic forms. We give an explicit conjectural formula for the components of  $\mathcal{E}_n$ , which involves only rational (resp. rational and logarithmic) terms for  $n$  odd (resp. for  $n$  even). By means of direct computations, we prove that our explicit formulas are indeed correct for  $n$  less than or equal to 12.

## 1. INTRODUCTION

We work with analytic objects in the whole paper. The usual setting when studying webs with regard to their abelian relations and their rank is the complex analytic setting, but some basic constructions in [Da2] rely on real analytic objects (e.g. real grassmannians, differential forms on these) hence both cases will be considered here, but mainly the latter. In most parts, the base field will be  $\mathbf{R}$  but in some (which will be pointed out), it will be more natural to work over  $\mathbf{C}$ . In the sequel,  $\mathbf{K}$  will stand for one of these two fields.

In the Introduction, we first introduce the topic (webs, abelian relations, etc) before giving a quick overview of the content of Damiano’s paper [Da2]. We then state our results.

**1.1. Curvilinear webs.** We introduce basic notions about curvilinear webs. Our presentation below is very similar to those of [Da1] and [Da2]. For other references on webs (but where the focus is more on 1-codimensional webs), see the famous ‘Blaschke-Bol book’ [BB] and the more recent one [PP].

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1.1.1. **Webs.** Let  $U$  be a non empty domain of  $\mathbf{K}^n$  and  $d$  a positive integer.

A (**curvilinear**)  $d$ -**web** on  $U$  is a  $d$ -tuple  $\mathcal{W} = (\mathcal{F}_1, \dots, \mathcal{F}_d)$  of analytic foliations on  $U$  whose leaves are in general position, at any point of  $U$  say (one can and one often only requires that this holds true generically on  $U$ ). Since  $U$  is simply connected, each foliation  $\mathcal{F}_i$  of  $\mathcal{W}$  is defined by a global first integral, that is there exists a regular submersion  $u_i : U \rightarrow \mathbf{A}^{n-1}$  the fibers of which are connected and coincide with the leaves of  $\mathcal{F}_i$ . In this case, we also use the notation  $\mathcal{W} = \mathcal{W}(u_1, \dots, u_d)$ . For any  $i = 1, \dots, d$ , let  $X_i$  be a non-vanishing vector field on  $\Omega$  which generates the tangent distribution  $T_{\mathcal{F}_i} = \text{Ker}(du_i) \subset T_\Omega$ . The general position hypothesis which is required for the foliations of  $\mathcal{W}$  is that for any strictly increasing  $n$ -tuple  $(i_1, \dots, i_n)$  of  $\{1, \dots, d\}$ , the  $n$  associated 1-dimensional distributions  $T_{\mathcal{F}_{i_1}}, \dots, T_{\mathcal{F}_{i_n}}$  are in direct sum in  $T_\Omega$ , which results in the more analytical fact that the field of  $n$ -tangent vectors  $X_{i_1} \wedge \dots \wedge X_{i_n}$  does not vanish on  $U$ .

Another (curvilinear)  $d$ -web  $\mathcal{W}'$  defined on another domain  $U'$  is said to be **equivalent**<sup>1</sup> to  $\mathcal{W}$  if there exists a germ of analytic isomorphism  $\varphi : (U, u) \rightarrow (U', u')$  such that  $\varphi^*(\mathcal{W}')$  coincides with the germ of  $\mathcal{W}$  at  $u$  (possibly only up to relabeling the foliations). **Web geometry** consists in the study of webs up to this notion of equivalence.

As a first example of webs, but which is important for what is to come, let us consider  $n + 1$  points  $p_0, p_1, \dots, p_n$  in general position in  $\mathbf{P}^n$ . Denoting by  $\mathcal{L}_{p_i}$  the linear family of projective lines passing through  $p_i$  for  $i = 0, \dots, n$ , we get a linear web  $\mathcal{L}\mathcal{W}_{p_0, \dots, p_n} = (\mathcal{L}_{p_0}, \dots, \mathcal{L}_{p_n})$  on  $\mathbf{P}^n$ . By definition, a **quadrilateral web** is a  $(n + 1)$ -web which is isomorphic to a web of this kind. Assuming that the  $p_i$ 's are the vertices  $e_0, \dots, e_n$  of the standard  $n$ -simplex in  $\mathbf{P}^n$ , one can give a simple explicit analytic model for the **standard quadrilateral web**  $Q\mathcal{W} = \mathcal{L}\mathcal{W}_{e_0, \dots, e_n}$ : if  $x_1, \dots, x_n$  stand for the standard affine coordinates, then for  $i = 1, \dots, n$ , the  $i$ -th linear standard projection  $\pi_i : \mathbf{A}^n \rightarrow \mathbf{A}^{n-1}$ ,  $(x_s)_{s=1}^n \mapsto (x_1, \dots, \widehat{x_i}, \dots, x_n)$  is a first integral for  $\mathcal{L}_{e_i}$  whereas  $\pi_0 : (x_s)_{s=1}^n \mapsto (x_t/x_n)_{t=1}^{n-1}$  works for  $\mathcal{L}_{e_0}$ .

A  $d$ -web with  $d \geq n + 1$  is said to be quadrilateral if all its  $(n + 1)$ -subwebs are quadrilateral. Finally, for a planar 3-web, being quadrilateral can be characterized by the closure of any small ‘hexagonal figures’ which can be traced on the definition domain of the considered web by traveling along the leaves of its foliations (see for instance [PP, §1.2]). Hence the term ‘quadrilateral’ never appears when considering planar webs, one uses ‘hexagonal’ instead.

1.1.2. **Abelian relations and ranks.** An important notion for the study of webs is that of ‘abelian relation’ (ab. AR). This notion is interesting first since it is well-behaved (invariant) modulo equivalences and also because it is linked to classical objects and results of projective algebraic geometry.<sup>2</sup> In order to recall this notion, given a web  $\mathcal{W} = \mathcal{W}(u_1, \dots, u_d)$ , one denotes by  $u_i^1, \dots, u_i^{n-1}$  the components of the first integral  $u_i$  for each  $i$ . Given  $k \in \{0, \dots, n - 1\}$ , we set  $\mathbf{I}_n^k$  (or just  $\mathbf{I}$  when both  $n$  and  $k$  are unambiguously fixed) for the set of  $k$ -tuples  $I = (i_1, \dots, i_k)$  with  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . Then for any  $i = 1, \dots, d$  and any tuple  $I \in \mathbf{I}$ , we set  $\wedge^I du_i = du_i^{i_1} \wedge \dots \wedge du_i^{i_k} \in \Omega^k(U)$  and we denote by  $u_i^*(\Omega^k)$  the  $\mathbf{K}$ -vector space of differential  $k$ -forms on  $U$  spanned by  $\{\wedge^I du_i \mid I \in \mathbf{I}\}$ . We then define the **space of  $k$ -th abelian relations**

<sup>1</sup>But the general position assumption holding true only on a certain Zariski open subset of  $\mathbf{P}^n$ .

<sup>2</sup>The web-theoretic notion of abelian relation is related to that of abelian differentials and to fundamental results of algebraic geometry such as Abel’s addition theorem and its converse, etc. See [PP] and the references therein for an overview on this perspective on web geometry.

(ARs) for  $\mathcal{W}$  as the  $\mathbf{K}$ -vector space, denoted by  $AR^{(k)}(\mathcal{W})$ , of  $d$ -tuples of elements  $\omega_i^k \in u_i^*(\Omega^k)$  summing up to 0:

$$AR^{(k)}(\mathcal{W}) = \left\{ (\omega_i^k)_{i=1}^d \in \prod_{i=1}^d u_i^*(\Omega^k) \mid \sum_{i=1}^d \omega_i^k = 0 \text{ in } \Omega^k(U) \right\}.$$

It is easily seen that the above definition does not really depend on the first integrals  $u_i$  but only on the associated foliations. A  $k$ -AR can be written quite explicitly in terms of the first integrals  $u_i$ : such an object corresponds to a family  $(F_i^I)$  indexed by pairs  $(i, I) \in \underline{d} \times \mathbf{I}$ , of functions  $F_i^I : \text{Im}(u_i) \rightarrow \mathbf{K}$ , such that the following relation between differential forms holds true identically:

$$\sum_{i=1}^d \sum_{I \in \mathbf{I}} F_i^I(u_i) (\wedge^I du_i) = \sum_{i=1}^d \sum_{\substack{I=(i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq n}} F_i^I(u_i) (du_i^{i_1} \wedge \dots \wedge du_i^{i_k}) = 0.$$

By definition, the  **$k$ -th rank** of  $\mathcal{W}$  is

$$\text{rk}^k(\mathcal{W}) = \dim_{\mathbf{K}} AR^{(k)}(\mathcal{W}) \in \mathbf{N} \cup \{\infty\}.$$

It is an invariant attached to  $\mathcal{W}$  (two equivalent webs have the same  $k$ -rank).

In this text, we will essentially deal only with ARs of top degree, *i.e.* in the case when  $k = n - 1$ . Except in other situations in which the complete notation will be used, we will drop the superscript  $n - 1$  everywhere and just speak of ARs of webs.

An important property of ARs (at least of top degree) is that they extend globally but as multi-valued objects.<sup>3</sup> This is of importance when considering webs globally defined on varieties with non trivial topology, as are those under scrutiny in this paper (see §1.1.5 below). For such a web, the ARs organize themselves into a local system whose monodromy may be non trivial.

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Let us illustrate the notion of abelian relation with an explicit example, which will happen to be important in the whole article. For  $i = 0, \dots, n$ , we denote by  $\pi_i^k$  (with  $k = 1, \dots, n - 1$ ) the components of the first integrals  $\pi_i$  given above for the standard quadrilateral web  $\mathcal{L}\mathcal{W}_{p_0, \dots, p_n}$  and we set  $\Pi_i = d\pi_i^1 \wedge \dots \wedge d\pi_i^{n-1} \in \pi_i^*(\Omega^{n-1})$ . It is then not difficult to verify that the following relation is identically satisfied

$$(1) \quad \sum_{i=0}^n (-1)^i \frac{\Pi_i}{\pi_i^1 \dots \pi_i^{n-1}} = 0$$

which is equivalent to saying that  $((-1)^i \Pi_i / (\pi_i^1 \dots \pi_i^{n-1}))_{i=0}^n$  is an element of  $AR(\mathcal{L}\mathcal{W}_{p_0, \dots, p_n})$ .

As for many kinds of webs, there are universal bounds on the rank of curvilinear webs. The following result has been obtained by Damiano, and is of crucial importance considering the purpose of this paper.<sup>4</sup>

<sup>3</sup>The proof of this is standard and is an easy generalization of that in the case of planar webs (*cf.* [P1, §1.2.2]). Details are left to the reader.

<sup>4</sup>This result is only stated in [Da2]. However a detailed proof is given §3.3 in Damiano's thesis [Da1]. In the case when  $n = 3$  and  $d \leq 5$ , the majoration (1) has been obtained long before by Kähler, see **S**<sub>4</sub> in [Bla3].

**Proposition 1.1.** *For any curvilinear  $d$ -web  $\mathcal{W}_d$  on a domain of  $\mathbf{C}^n$ , one has:*

$$(2) \quad \text{rk}(\mathcal{W}_d) \leq \sum_{\sigma=0}^{d-n-1} \binom{n-2+\sigma}{\sigma} (d-n-\sigma)^+$$

where  $m^+$  stands for  $\max\{0, m\}$  for any  $m \in \mathbf{Z}$ .

The case of a quadrilateral web is interesting. When  $d = n + 1$ , the RHS of (2) is equal to 1 for any  $n \geq 2$  and since (1) corresponds to a non-trivial AR for  $\mathcal{QW}$ , it comes that  $\mathcal{QW}$  has **maximal rank**, that is is such that (2) actually is an equality.

1.1.3. **Algebraic webs.** What makes the relevance of the notion of web of maximal rank is that, by means of a classical construction relying on basic and very well-known results of algebraic geometry, one can associate such a web to any plane algebraic curve.

The construction, which goes back to the early developments of web geometry [Bla1], goes as follows: let  $C \subset \mathbf{P}^2$  be any reduced algebraic curve of degree  $d \geq 3$ . For any generic line  $L_0$  intersecting  $C$  transversally, one can find  $d$  germs of algebraic maps  $P_i : (\check{\mathbf{P}}^2, L_0) \rightarrow C$  such that, as 0-cycles, one has  $C \cdot L = \sum_{i=1}^d P_i(L)$  for any line  $L$  sufficiently close to  $L_0$  (see Figure 1 below). The  $P_i$ 's are first integrals of the (germ at  $L_0$  of the) **algebraic web associated to  $C$** , denoted by  $\mathcal{W}_C$ .

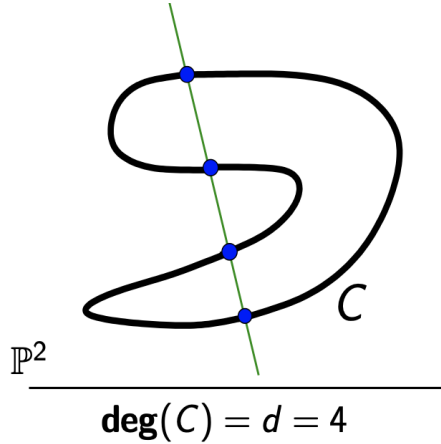


FIGURE 1.

Then it follows from Abel's addition theorem that for any abelian differential  $\omega$  on  $C$ , one has  $\sum_{i=1}^d P_i^*(\omega) = 0$  from which it can be deduced that  $\omega \mapsto (P_i^*(\omega))_{i=1}^d$  is a well-defined injective linear map  $H^0(C, \omega_C^1) \rightarrow \text{AR}(\mathcal{W}_C)$ . It follows that  $\mathcal{W}_C$  has rank at least equal to  $h^0(\omega_C^1) = p_a(C) = (d-1)(d-2)/2$ , a quantity which coincides with the RHS of (2) when  $n = 2$ . It follows that the algebraic web  $\mathcal{W}_C$  is always of maximal rank.

What makes the interest of the notions of abelian relation and rank in the case of planar webs, is that these allow, in some cases, to prove algebraization results for webs of maximal rank (cf. [PP] for an extensive exposition from this perspective). But the real interest of all these notions lies precisely in the fact that there exist planar webs of maximal rank which however do not come from a plane algebraic curve.

1.1.4. **Abel's identity of the dilogarithm and Bol's web.** Several authors of the XIXth and XXth centuries have independently discovered equivalent versions of the nowadays so-called Abel's 5-terms equation

$$(Ab) \quad R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

which is identically satisfied for any  $(x, y) \in \mathbf{R}^2$  such that  $0 < x < y < 1$ , by the famous *Rogers' dilogarithm*  $R$  defined by

$$(3) \quad R(x) = \mathbf{Li}_2(x) + \frac{1}{2} \log(x) \log(1-x) - \pi^2/6$$

for  $x \in ]0, 1[$ , where  $\mathbf{Li}_2$  stands for the classical bilogarithm.<sup>5</sup>

Abel's identity (or more precisely, its total derivative with respect to  $x$  and  $y$ ) gives rise to an AR, denoted by  $\mathcal{Ab}$ , for the so-called **Bol's web**  $\mathcal{B}$  which is the planar 5-web defined by the rational functions appearing as arguments of  $R$  in  $(Ab)$ :

$$(4) \quad \mathcal{B} = \mathcal{W}\left(x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right).$$

Of course Bol's web admits other ARs than the dilogarithmic AR  $\mathcal{Ab}$ : for instance, its 3-subweb  $\mathcal{W}(x, y, x/y)$  is hexagonal and carries the AR associated to the basic functional equation

$$\text{Log}(x) - \text{Log}(y) - \text{Log}(x/y) = 0$$

of the logarithm (the total derivative of which is exactly the identity (1) in the case when  $n = 2$ ).

Bol's web is very particular and it has been known for a long time that it enjoys the following remarkable properties as a planar web:

**[Hexagonality].** *Bol's web is hexagonal and all its 3-terms ARs are logarithmic. Moreover, the space  $\mathbf{AR}_{\text{Log}}(\mathcal{B})$  that these abelian relations span is 5-dimensional.*

**[Maximality of the rank].** *Since the 5-terms abelian relation  $\mathcal{Ab}$  is dilogarithmic, it does not belong to the space of logarithmic ARs hence Bol's web has maximal rank 6 and there is a decomposition in direct sum*

$$\mathbf{AR}(\mathcal{B}) = \mathbf{AR}_{\text{Log}}(\mathcal{B}) \oplus \langle \mathcal{Ab} \rangle.$$

**[Non linearizability].**  *$\mathcal{B}$  is not linearizable hence not equivalent to an algebraic web.*

**[Characterization].** *For  $d \geq 3$ , a hexagonal planar  $d$ -web either is linearizable and equivalent to a web formed by  $d$  pencils of lines or  $d = 5$  and it is equivalent to  $\mathcal{B}$ .*

Bol's web has been the first known example of an **exceptional web**, that is a web of maximal rank but which is not equivalent to an algebraic one. Since its discovery in the 1930s by Blaschke and Bol (see [Bla1, §4] and [Bol]), several new examples of planar exceptional webs have been discovered (see the sixth chapter of [PP] for a recent overview).

The interest of the notion of 'exceptional web' comes from the facts that first, such webs exist in numbers; and second, that one can mimic for them some very classical constructions of algebraic geometry (canonical map, etc) which ask many interesting questions about the similarity between

<sup>5</sup>We recall that  $\mathbf{Li}_2(z)$  is defined as the sum of the series  $\sum_{k \geq 0} z^k/k^2$  which converges on the unit disk  $\{z \in \mathbf{C}, |z| < 1\}$ .

plane algebraic curves and planar exceptional webs. This is possibly what motivated several authors (such as Chern and Griffiths, *cf.* [ChG, p. 83]) to qualify as one of the main problems in web geometry the following one:

(6) *Determine/classify the webs of maximal rank, especially the non-algebraic ones.*

Most of the works regarding this problem concerned webs of codimension 1. For curvilinear webs, we are only aware of old (but quite remarkable) results by Blaschke and Walberer about skew curvilinear 3-webs regarding their  $k$ -ranks [Bla2, BW], and the recent results of Damiano which we return to in this article.

We note that what might be an ‘algebraic curvilinear web’ has still not been precisely defined, even if it is known since [BW] that some webs certainly deserve to be characterized as such. We will shed more light (but just a little) about this later on in the Introduction.

1.1.5. **The webs  $\mathcal{W}_{0,n+3}$  for  $n \geq 2$ .** It is well-known and very easy to verify that Bol’s web admits also the following nice geometric description: the web defined by the five rational first integrals in (4) is nothing else but a model in local coordinates, of the web on the 2-dimensional moduli space  $\mathcal{M}_{0,5}$  defined by the five rational fibrations  $\mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4} \simeq \mathbf{P}^1 \setminus \{0, 1, \infty\}$  induced by forgetting a point among the five of any configuration element of  $\mathcal{M}_{0,5}$ .

Describing Bol’s web in such a geometric way is interesting since it suggests immediately the following generalization: for  $n \geq 2$  fixed and any  $i = 1, \dots, n+3$ , the forgetful map

$$\varphi_i : \mathcal{M}_{0,n+3} \rightarrow \mathcal{M}_{0,n+2}$$

consisting in forgetting the  $i$ -th point of a configuration of  $n+3$  points on  $\mathbf{P}^1$ , is a surjective rational map whose fibers are smooth rational curves with  $n+2$  punctures. The corresponding foliations of dimension 1 satisfy the ‘general position assumption’ of §1.1.1 hence form a curvilinear  $(n+3)$ -web on  $\mathcal{M}_{0,n+3}$ , which we will denote by

$$(7) \quad \mathcal{W}_{0,n+3} = \mathcal{W}(\varphi_1, \dots, \varphi_{n+3}).$$

As explained just above,  $\mathcal{W}_{0,5}$  is a geometric model for Bol’s web  $\mathcal{B}$ . The latter being so important regarding the study of non-linearizable webs of maximal rank, it appears quite natural to study the whole family of webs  $\mathcal{W}_{0,n+3}$ ’s from the same perspective.

The webs  $\mathcal{W}_{0,n+3}$ ’s have been first mentioned by Burau in [Bu], a paper on which we will come back further in §2.2. The study of the  $\mathcal{W}_{0,n+3}$ ’s for  $n \geq 2$  arbitrary with regard to their abelian relations, their ranks, etc, is much more recent and is due to Damiano hence we are going to describe Damiano’s work in some detail in the next subsection.

To end this short presentation, let us mention the following obvious generalizations of the webs  $\mathcal{W}_{0,n+3}$ ’s: for any fixed  $k \in \{1, \dots, n-1\}$ , there are  $\binom{n+3}{k}$  ways to forget  $k$  points among all those of a configuration of  $n+3$  points on  $\mathbf{P}^1$  hence the corresponding forgetful maps  $\mathcal{M}_{0,n+3} \rightarrow \mathcal{M}_{0,n+3-k}$  are the first integrals of a  $\binom{n+3}{k}$ -web of dimension  $k$  on  $\mathcal{M}_{0,n+3}$ , denoted by  $\mathcal{W}_{0,n+3}^k$ . The 1-codimensional webs  $\mathcal{W}_{0,n+3}^{n-1}$  have been studied in [Per] where the author proved that they carry many logarithmic and dilogarithmic ARs and that these are sufficiently many to obtain that, in some sense, each such web has maximal rank.<sup>6</sup>

<sup>6</sup>Using the terminology introduced in [P4], one can say that each web  $\mathcal{W}_{0,n+3}^{n-1}$  is ‘AMP’.

It would be interesting to study and even quite better, to give a complete description of the  $\ell$ -ARs and the  $\ell$ -ranks of the webs  $\mathcal{W}_{0,n+3}^k$ , this for any  $n \geq 2$ ,  $k \leq n - 1$  and  $\ell \leq n - k$ . The present paper can be seen as a contribution to this wide open question, in the specific case when  $k = 1$  and  $\ell = n - 1$ .

**1.2. Damiano's work.** In [GM], Gelfand and MacPherson describe a geometric construction of some differential forms on the spaces of projective configurations in (real) projective spaces from characteristic classes. Their general construction admits as a particular and very interesting case (the differential version of) Abel's dilogarithmic identity ( $\mathcal{Ab}$ ).

In his thesis [Da1] under the supervision of MacPherson (published in condensed form as the paper [Da2]), Damiano applies Gelfand-Macpherson's approach to study the webs  $\mathcal{W}_{0,n+3}$ 's for  $n$  arbitrary, especially from the point of view of their rank and abelian relations. After having established several basic results about curvilinear webs such as Proposition 1.1 and studied carefully quadrilateral  $(n + 1)$ -webs of curves and their abelian relations, Damiano focused on the web  $\mathcal{W}_{0,n+3}$ 's for  $n \geq 2$ . We give below a short overview of the very interesting results he got/claims he got about these webs.

He started by noticing that any  $(n + 1)$ -subweb  $\mathcal{W}_{ij}$  of  $\mathcal{W}_{0,n+3}$  given by disregarding only the  $i$ -th and  $j$ -th foliations of  $\mathcal{W}_{0,n+3}$ , is quadrilateral and consequently has rank 1 (according to the remark just after Proposition 1.1 above). The corresponding  $\binom{n+3}{2} = (n + 3)(n + 2)/2$  ARs are called 'combinatorial abelian relations' of  $\mathcal{W}_{0,n+3}$  and their span is denoted by  $AR_C(\mathcal{W}_{0,n+3})$ . This is an a priori quite big subspace of the total space of ARs of  $\mathcal{W}_{0,n+3}$  on which the symmetric group  $\mathfrak{S}_{n+3}$  naturally acts linearly. In the case when  $n = 2$  and up to the equivalence of webs  $\mathcal{W}_{0,5} \simeq \mathcal{B}$ ,  $AR_C(\mathcal{W}_{0,5})$  coincides with the space  $AR_{Log}(\mathcal{B})$  considered above in (5).

Next, Damiano applies Gelfand-MacPherson approach to the construction of another AR for  $\mathcal{W}_{0,n+3}$ , not a combinatorial one. The geometric starting point is a special case of the *Gelfand-MacPherson correspondence*, namely that the moduli space  $\mathcal{M}_{0,n+3}$  identifies naturally with the quotient of a certain dense open subset  $\widetilde{G}_2(\mathbf{R}^{n+3})$  of the grassmannian variety of 2-planes in  $\mathbf{R}^{n+3}$  by the linear action of the identity component  $H_{n+3}^0 \simeq (\mathbf{R}_{>0})^{n+2}$  of the Cartan torus  $H_{n+3} \subset GL_{n+3}(\mathbf{R}^{n+3})$ . Actually (and this is important for what is to come) likely in order to work uniformly regarding the parity of  $n$ , Damiano works with the corresponding open subset  $\widetilde{G}_2^{or}(\mathbf{R}^{n+3})$  of the grassmannian variety of oriented 2-planes in  $\mathbf{R}^{n+3}$  and the associated quotient  $\mathcal{EM}_{0,n+3} = \widetilde{G}_2^{or}(\mathbf{R}^{n+3})/H_{n+3}^0$  which is seen as a space of 'enhanced configurations' of  $n + 3$  points on  $\mathbf{P}^1$  (coming with a covering map  $\mathcal{EM}_{0,n+3} \rightarrow \mathcal{M}_{0,n+3}$ ).

The forgetful maps  $\varphi_i : \mathcal{M}_{0,n+3} \rightarrow \mathcal{M}_{0,n+2}$  admit natural lifts between the corresponding spaces of enhanced configurations (denoted by  $\varphi_i$  as well) and also between the (open subsets of) corresponding oriented grassmannians, denoted by  $\tilde{\varphi}_i : \widetilde{G}_2^{or}(\mathbf{R}^{n+3}) \rightarrow \widetilde{G}_2^{or}(\mathbf{R}_i^{n+2})$  where  $\widetilde{G}_2^{or}(\mathbf{R}_i^{n+2})$  stands for the grassmann subvariety formed by oriented 2-planes contained in the  $i$ -th coordinate hyperplane  $\mathbf{R}_i^{n+2} = \{x_i = 0\}$  of  $\mathbf{R}^{n+3}$ .

Denoting by  $\pi_{n+3}$  the quotient map  $\widetilde{G}_2^{or}(\mathbf{R}^{n+3}) \rightarrow \mathcal{EM}_{0,n+3} = \widetilde{G}_2^{or}(\mathbf{R}^{n+3})/H_{n+3}^0$ , it can be verified that the latter factors through  $\pi_{n+2}$  when restricted along  $\widetilde{G}_2^{or}(\mathbf{R}_i^{n+2})$  and these quotient maps



together with the forgetful maps defined in the previous paragraph all fit into the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{G}_2^{or}(\mathbf{R}^{n+3}) & \xrightarrow{\widetilde{\varphi}_i} & \widetilde{G}_2^{or}(\mathbf{R}_i^{n+2}) \\
\pi_{n+3} \downarrow & & \downarrow \pi_{n+2} \\
\mathcal{EM}_{0,n+3} & \xrightarrow{\varphi_i} & \mathcal{EM}_{0,n+2} \\
\downarrow & & \downarrow \\
\mathcal{M}_{0,n+3} & \xrightarrow{\varphi_i} & \mathcal{M}_{0,n+2}.
\end{array}$$

Let  $\mathcal{T}$  be the tautological rank 2 vector bundle over  $G_2^{or}(\mathbf{R}^{n+3})$ . Its Euler class is a characteristic class element of  $H^2(G_2^{or}(\mathbf{R}^{n+3}), \mathbf{R})$  which can be represented by a closed  $SO_{n+3}$ -invariant 2-form, denoted by  $e_n$ . For any  $k \geq 1$ , the integral along the fibers of  $\pi_{n+3}$  of the  $k$ -th wedge power  $e_n^k$  exists and is a  $(2k - n - 2)$ -differential form on  $\mathcal{EM}_{0,n}$ , denoted by  $\widetilde{e}_n^k = \pi_*(e_n^k)$ . It is an example of a ‘generalized dilogarithm form’ according to the terminology introduced in [GM].

Damiano proves that (1)  $\widetilde{e}_n^k$  vanishes, except when  $k = n + 1$  since in this case  $e_n^{n+1}$  is an invariant volume form on  $G_2^{or}(\mathbf{R}^{n+3})$ ; (2) for any  $i$ , the restriction of  $e_n$  along  $G_2^{or}(\mathbf{R}_i^{n+2})$  coincides (up to a sign corresponding to the compatibility between natural orientations) with the 2-form  $e_{n-1}$  intrinsically associated to this oriented grassmannian. Combining this (in the case when  $k = n$ ) with the key technical result of [GM] which is a differential identity between generalized dilogarithms, Damiano obtains that

$$\sum_{i=1}^{n+3} (-1)^i \varphi_i^*(\widetilde{e}_{n-1}^n) = 0$$

holds true identically on any connected components of  $\mathcal{EM}$ . This allows to consider the  $(n + 3)$ -tuple of  $(n - 1)$ -differential forms  $((-1)^i \varphi_i^*(\widetilde{e}_{n-1}^n))_{i=1}^{n+3}$  as an AR for  $\mathcal{W}_{0,n+3}$ , called the **Euler abelian relation** in [Da2] and which we will denote by  $\mathcal{E}_n$  here. When  $n = 2$ , it can be verified (see [GM], [Da1, §8.4] or §5.1.4 below) that  $\mathcal{E}_2$  coincides with Abel’s abelian relation  $\mathcal{Ab}$  up to the natural identification of  $\mathcal{W}_{0,5}$  with Bol’s web  $\mathcal{B}$ .

The material above being introduced, it is now possible to state the main results claimed by Damiano about the webs  $\mathcal{W}_{0,n+3}$ ’s as the following ones:

**Main claims in [Da2].** *Let  $n$  be an integer bigger than or equal to 2.*

1. *The web  $\mathcal{W}_{0,n+3}$  is quadrilateral and non linearizable.*
2.  *$\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  has dimension bigger than or equal to  $n(n + 3)/2$ .*
3. *The Euler abelian relation  $\mathcal{E}_n$  is a non-trivial element of  $\mathbf{AR}(\mathcal{W}_{0,n+3})$ .*
4. *There is a decomposition in direct sum*

$$(8) \quad \mathbf{AR}(\mathcal{W}_{0,n+3}) = \mathbf{AR}_C(\mathcal{W}_{0,n+3}) \oplus \langle \mathcal{E}_n \rangle$$

*and consequently  $\dim \mathbf{AR}_C(\mathcal{W}_{0,n+3}) = n(n + 3)/2$  and  $\mathcal{W}_{0,n+3}$  has maximal rank, i.e.*

$$\mathrm{rk}(\mathcal{W}_{0,n+3}) = (n + 1)(n + 2)/2.$$

5. The natural action of  $\mathfrak{S}_{n+3}$  on  $\mathcal{M}_{0,n+3}$  induces an action on the space of abelian relations of  $\mathcal{W}_{0,n+3}$  whose decomposition in irreducible factors corresponds to the decomposition in direct sum (8). Moreover:
- (i).  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  is the  $\mathfrak{S}_{n+3}$ -representation with Young diagram  $[221^{n-1}]$ ;
  - (ii).  $\langle \mathcal{E}_n \rangle$  is the sign  $\mathfrak{S}_{n+3}$ -representation (with Young diagram  $[1^{n+3}]$ ).
6. A quadrilateral curvilinear  $d$ -web either is equivalent to a web formed by the lines passing through  $d$  points in  $\mathbf{P}^n$  or  $d = n + 3$  and the considered web is equivalent to  $\mathcal{W}_{0,n+3}$ .

These remarkable statements are generalizations (with precisions/refinements) of the well-known properties of Bol's web listed in (5). However, some of them were previously known<sup>7</sup> and even worse, some are not correct. It is what we are going to explain by stating our results in the next subsection.

1.3. **Results.** The purpose of this text is to revisit the statements discussed above.

In short, mainly we

- recall a classical result implying that any web  $\mathcal{W}_{0,n+3}$  is linearizable when  $n$  is odd;
- discuss in depth the web  $\mathcal{W}_{0,6}$  (case when  $n = 3$ ): realizing that it is isomorphic to the web by lines on Segre's cubic hypersurface  $\mathcal{S} \subset \mathbf{P}^4$ , we describe its ARs algebraically in terms of the global sections of the dualizing sheaf  $\omega_{\Sigma}^2$  of the associated Fano surface  $\Sigma = F_1(\mathcal{S}) \subset G_1(\mathbf{P}^4)$ . In particular, we get that it is quite justified to say that  $\mathcal{W}_{0,6}$  is (equivalent to) an 'algebraic web'.
- prove that some of the claims (8) are not correct for any odd integer  $n \geq 3$ . In this case, we give corrected versions of them;
- give an explicit expression for the (components of the) Euler abelian relation  $\mathcal{E}_n$ , which is rational when  $n$  is odd, whereas it involves logarithmic terms as well when  $n$  is even. This formula is conjectural in full generality but we have verified that it indeed holds true for  $n$  less than or equal to 12.

We describe below in more detail the results obtained (or just conjectured) in this text and how they are related to the claims (8).

1.3.1. **Linearization.** First we revisit the linearization problem for  $\mathcal{W}_{0,n+3}$  by recalling a nice and rather simple (but seemingly forgotten) construction and results by Room rediscovered independently by Burau more than thirty years later (precise references will be given in §2.2).

For any integer  $n$ , we set  $\delta_n = 1$  if  $n$  is odd and  $\delta_n = 2$  if it is even.

**Theorem A.** (Room-Burau) **1.** *For any  $n \geq 2$ , the web  $\mathcal{W}_{0,n+3}$  is realizable as a web of rational curves of degree  $\delta_n$  on a certain projective variety  $V_n$ .*

**2.** *In particular,  $\mathcal{W}_{0,n+3}$  is linearizable for any odd integer  $n \geq 3$ .*

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<sup>7</sup>For instance, **6.** was already known to Blaschke in the case when  $n = 3$ , as the reading of [Bla4, §51] shows.

The second point of this theorem shows that the second part of assertion (8).1. is wrong when  $n$  is odd (see §2.2.2 below for a description of the main flaw in the arguments advanced in [Da1] for proving (8).1). We believe that  $\mathcal{W}_{0,n+3}$  is indeed non linearizable when  $n$  is even but this is still conjectural.

An interesting feature of the preceding theorem (which was of course stated by Room and by Burau only in terms of projective algebraic geometry), is that it is constructive: there exists an explicit linear system  $\mathcal{L}_n$  on  $\mathbf{P}^n$  giving rise to a map  $\varphi_n : \mathbf{P}^n \dashrightarrow \mathbf{P}^{N_n}$  birational onto its image  $V_n = \text{Im}(\varphi_n)$  such that

$$\mathcal{W}_{0,n+3} = (\varphi_n)_*(\mathcal{W}_{0,n+3})$$

is a web by rational curves of degree  $\delta_n$  on  $V_n$ .

The  $V_n$ 's together with the associated  $\mathcal{L}_n$ 's form an interesting family of projective varieties and linear systems, studied in classical papers and in a few recent publications as well. The first two cases  $n = 2$  and  $n = 3$  are very classical and have really been studied a lot: for instance,  $V_2$  is the *del Pezzo quintic surface* in  $\mathbf{P}^5$  and  $\mathcal{W}_{0,5} = (\varphi_2)_*(\mathcal{W}_{0,5})$  is the web formed by the five fibrations by conics on it. The case  $n = 3$  is not less classical than the previous one, and given the importance it will have regarding our approach, it deserves to be stated as the following

**Proposition B.** *The variety  $V_3$  is Segre's cubic primal  $\mathcal{S}$ , that is the (projectively unique) irreducible cubic hypersurface in  $\mathbf{P}^4$  with 10 nodes. And the push-forward web  $\mathcal{W}_{0,6} = (\varphi_3)_*(\mathcal{W}_{0,6})$  coincides with the linear web on  $\mathcal{S}$  formed by the six covering families of lines contained in it.*

**1.3.2. The web  $\mathcal{W}_{0,6}$ , its abelian relations and Segre's cubic primal.** What makes the previous result important for us is that once aware of it, it becomes unavoidable to relate the web-theoretic questions we are interested in to some very nice and nowadays well-know results about 3-dimensional cubic hypersurfaces. We recall this material below and explain how it is related to curvilinear webs, referring to §3.1 further for more details and references.

Let  $X \subset \mathbf{P}^4$  be a cubic hypersurface which we assume here to be smooth (essentially for simplicity). It is known that through a general point  $x$  of  $X$  pass six pairwise distinct lines included in  $X$ . Since three such lines necessarily span a 3-plane<sup>8</sup>, the general position assumption of §1.1.1 is satisfied hence these lines are the leaves of a linear 6-web on (a certain Zariski open subset  $X^0$  of)  $X$ , denoted by  $\mathcal{LW}_X$ , and which is canonically defined on  $X$ . Looking at the global geometric picture is quite relevant here, and can be better understood by considering the Fano scheme  $F = F_1(X)$  of lines contained in  $X$ , which is a surface naturally embedded in  $G_1(\mathbf{P}^4)$ .

First, remark that  $F$  is naturally the space of leaves of the foliations composing  $\mathcal{LW}_X$  locally: at any  $x_0 \in X^0$ , one can define regular germs  $L_i : (X, x_0) \rightarrow F$  for  $i = 1, \dots, 6$ , such that for any  $x \in X$  sufficiently close to  $x_0$ , the  $L_i(x)$ 's correspond to the six lines contained in  $X$  and passing through  $x$ . These six (germs of) maps are local first integrals for  $\mathcal{LW}_X$ : locally at  $x_0$ , one has

$$\mathcal{LW}_X = \mathcal{W}(L_1, \dots, L_6).$$

In general (for instance when  $X$  is non singular),  $F$  is a smooth irreducible surface with remarkable properties some of which allow to describe the ARs of  $\mathcal{LW}_X$  quite nicely. The properties which will be crucial for our purpose are the following two:

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<sup>8</sup>This follows easily from Bezout's theorem.

- (i). the space  $\mathbf{H}^0(F, \Omega_F^2)$  of global holomorphic 2-forms on  $F$  has dimension 10;
- (ii). for any  $\omega \in \mathbf{H}^0(F, \Omega_F^2)$ , its trace, defined locally by  $\text{Tr}(\omega) = \sum_{i=1}^6 L_i^*(\omega)$ , vanishes on  $X$ .

The second property has to be seen as an analogue, for the incidence between point and lines included in the hypercubic  $X$ , of the classical Abel's addition theorem for the abelian differentials on algebraic curves mentioned in §1.1.3 above. As in the case of any algebraic planar web, one first deduce from (ii) that the following 'Trace map'

$$(9) \quad \begin{aligned} \text{Tr} : \mathbf{H}^0(F, \Omega_F^2) &\longrightarrow \mathbf{AR}(\mathcal{LW}_X) \\ \omega &\longmapsto (L_i^*(\omega))_{i=1}^6 \end{aligned}$$

is well-defined. Since it is obviously injective, it follows that the rank of  $\mathcal{LW}_X$  is bigger than or equal to  $h^0(F, \Omega_F^2)$ , which is 10 according to (i). Hence considering the majoration (2) in case  $n = 3$  and  $d = 6$ , we deduce that  $\text{rk}(\mathcal{LW}_X) = h^0(F, \Omega_F^2) = 10$  is maximal.

What has been obtained above can be stated in condensed form as the following:

**Proposition C.** *For any sufficiently general (e.g. smooth) cubic hypersurface  $X \subset \mathbf{P}^4$ :*

1. *The map (9) induces a linear isomorphism  $\mathbf{H}^0(F, \Omega_F^2) \simeq \mathbf{AR}(\mathcal{LW}_X)$ ;*
2. *Consequently  $\mathcal{LW}_X$  is a linear 6-web of maximal 2-rank  $10 = h^0(F, \Omega_F^2)$ .*

Considering the way a web of the form  $(\mathcal{LW}_X)$  is defined and in view of the preceding result, it is more than reasonable to say that such webs are 'algebraic'.<sup>9</sup>

The preceding proposition shows in particular that, as a linear web,  $\mathbf{W}_{0,6} = \mathcal{LW}_S$  is just a special element of a family of algebraic webs, which generically are of maximal rank with all their ARs coming from holomorphic 2-forms on the corresponding Fano surfaces. A natural question which immediately arises is whether the isomorphism (9) also has a specialization for  $\mathcal{S}$ .

It is known that many results/constructions concerning smooth hypercubics in  $\mathbf{P}^4$  generalize to some singular cubics. In particular, some cases of nodal cubics have been considered by several authors, especially the case of 1-nodal cubics threefolds. The case of Segre's cubic is quite specific. For instance, the Fano surface  $\Sigma = F_1(\mathcal{S})$  of Segre's cubic is a rational surface with 21 irreducible components, which is in sharp contrast to the case of the Fano surface of a smooth hypercubic. We have not been able to localize in the huge existing literature on the subject a place where the suitable generalization of property (ii) above which could apply to the case of Segre's cubic is proved. However, by explicit computations, we have verified that it is indeed the case, which implies in particular that  $\mathbf{AR}(\mathcal{W}_{0,6})$  is isomorphic to the space  $\mathbf{H}^0(\Sigma, \omega_\Sigma^2)$  of global abelian 2-forms on  $\Sigma$ .

Actually, it is easily seen that the isomorphism of complex vector spaces

$$(10) \quad \mathbf{H}^0(\Sigma, \omega_\Sigma^2) \simeq \mathbf{AR}(\mathcal{W}_{0,6})$$

induced by (9) is an isomorphism of  $\mathfrak{S}_6$ -modules, a fact from which interesting consequences can be deduced. For instance, it is classically known that  $\mathbf{H}^0(\Sigma, \omega_\Sigma^2)$  hence  $\mathbf{AR}(\mathcal{W}_{0,6})$ , is irreducible as a  $\mathfrak{S}_6$ -representation, an observation which immediately appears as contradicting statement **5**. in

<sup>9</sup>Cf: the generalization of the notion of 'algebraic web' introduced in [P3, §1.3].

(8). But actually even more can be obtained: from an explicit description that one can give of a basis of  $H^0(\Sigma, \omega_\Sigma^2)$  and using the fact that  $\mathcal{W}_{0,6} = \varphi_3^*(\mathcal{LW}_S)$ , one obtains the following

**Theorem D. 1.** *One has  $\mathcal{W}_{0,6} = \varphi_3^*(\mathcal{LW}_S)$  hence  $\mathcal{W}_{0,6}$  is an algebraizable (hence linearizable) web of maximal rank 10, and (10) actually is an isomorphism of  $\mathfrak{S}_6$ -modules.*

2. *Regarding the abelian relations of  $\mathcal{W}_{0,6}$ , the following assertions hold true:*

- a. *One has  $\text{AR}_C(\mathcal{W}_{0,6}) = \text{AR}(\mathcal{W}_{0,6})$ ; in particular Euler's AR  $\mathcal{E}_3$  is combinatorial;*
- b. *As a  $\mathfrak{S}_6$ -module,  $\text{AR}_C(\mathcal{W}_{0,6}) = \text{AR}(\mathcal{W}_{0,6})$  is irreducible with Young diagram  $[31^3]$ .*

Assertions **a.** and **b.** above contradict statements **4.** and **5.** of (8) in the case when  $n = 3$ .

1.3.3. **The general case.** Motivated by the preceding result, we have investigated the general case taking up and verifying Damiano's approach. We correct his main theorem by establishing

**Theorem E.** *Let  $n$  be any integer bigger than or equal to 2.*

- 1. *When  $n$  is even, all the claims in (8) hold true.*
- 2. *On the other hand, this is not true when  $n$  is odd since then the following holds true:*
  - a. *The web  $\mathcal{W}_{0,n+3}$  is linearizable.*
  - b. *The space of combinatorial ARs has dimension  $(n+1)(n+2)/2$  hence the rank of  $\mathcal{W}_{0,n+3}$  is indeed maximal but one has  $\text{AR}(\mathcal{W}_{0,n+3}) = \text{AR}_C(\mathcal{W}_{0,n+3})$ .*
  - c. *The Euler abelian relation is combinatorial: i.e. one has  $\mathcal{E}_n \in \text{AR}_C(\mathcal{W}_{0,n+3})$ .*
  - d. *The representation of  $\mathfrak{S}_{n+3}$  on  $\text{AR}_C(\mathcal{W}_{0,n+3}) = \text{AR}(\mathcal{W}_{0,n+3})$  is irreducible with associated Young symbol  $[3, 1^n]$ .*

Of course, the first part of this theorem is fully due to Damiano. Only the second one (when  $n$  is odd) is new. However, it is fair to mention that it is proved just by adding an elementary (but new) fact to Damiano's argumentation.

1.3.4. **Explicit formulas for Euler's abelian relation.** We have also investigated more in depth Euler's abelian relation  $\mathcal{E}_n$ . Here are the main results we have obtained about it, relatively to a certain explicit choice of rational first integrals for  $\mathcal{W}_{0,n}$ :

- *We give a simple closed integral formula for the components of  $\mathcal{E}_n$  for any  $n \geq 2$ ;*
- *For any odd integer  $n \geq 3$ , we give an explicit conjectural rational formula for the components of  $\mathcal{E}_n$ . We prove this formula for  $n$  sufficiently small (e.g. for  $n \leq 11$ );*
- *For any even integer  $n \geq 2$ , we give an explicit conjectural formula for the components of  $\mathcal{E}_n$ , involving only rational and logarithmic quantities. We prove that this formula is valid for  $n$  small enough (e.g. for  $n \leq 12$ ).*

We refer respectively to Proposition 5.9, Proposition 5.22 and Proposition 5.28 for more precise statements, and to (84), (118) and (125) for the corresponding formulas.

1.4. **Plan of the paper.** The current Introduction constitutes the first section of this paper.

The sequel is organized as follows:

- In Section §2, we start by discussing a few basic facts about the moduli spaces  $\mathcal{M}_{0,n+3}$  and the webs  $\mathcal{W}_{\mathcal{M}_{0,n+3}}$  for  $n \geq 2$  arbitrary. We recall Room and Burau's results in §2.2, from which one deduces Theorem A. The corresponding flaw in the argumentation of [Da1] is briefly discussed in §2.2.2.
- Section §3 is entirely devoted to the case  $n = 3$  which is studied in depth. We start by discussing in §3.1 the 6-web by projective lines carried by a general cubic hypersurface in  $\mathbf{P}^4$ . We explain that Proposition C follows from well-known results of Clemens and Griffiths about cubic threefolds. The case of Segre's cubic  $\mathcal{S}$  is discussed in §3.3. By means of elementary explicit computations, we prove that, as for smooth cubics, the trace gives rise to the isomorphism (10) for  $\mathcal{S}$  as well. From this, we deduce an explicit basis of  $AR(\mathcal{W}_{0,6})$ , all the elements of which are rational abelian relations. The structure of  $AR(\mathcal{W}_{0,6})$  as a  $\mathfrak{S}_6$ -module is studied in §3.3.5. Theorem D is proved there.
- In Section §4, Damiano's approach for studying  $AR(\mathcal{W}_{0,n+3})$  as a  $\mathfrak{S}_{n+3}$ -module is taken up in detail. Essentially all the material here is taken from [Da1], the single novelty being Lemma 4.10. Although its statement as well as its proof are elementary, the second part of Theorem E follows quite easily from it. We get an explicit basis for the space  $AR_C(\mathcal{W}_{0,n+3})$  of combinatorial ARs.
- Section §5.1 is about Euler's abelian relation, the construction of which is taken up in detail. After having given a concise integral formula for the components of  $\mathcal{E}_n$  for  $n$  arbitrary in §5.1.4,  $\mathcal{E}_n$ , we deduce from some dihedral invariance properties of  $\mathcal{E}_n$  two transformations formulas that its components must satisfy (in §5.1.5). We then turn to the case when  $n$  is odd: in §5.1.8, using the two just mentioned transformation formulas, we give an explicit rational expression for the components of  $\mathcal{E}_n$ , which is conjectural in full generality but is proved to be the right one for any odd integer  $n \leq 11$ . The case when  $n$  is even is considered just after in §5.1.9. We start by dealing with the case  $n = 4$ . We first remark that Abel's method for solving abelian functional equations applies quite well as well for determining the ARs of any given curvilinear web. Applying this approach, we are able to give an explicit formula for the components of  $\mathcal{E}_4$  (cf. (121)), from which we conjecture a closed formula involving only rational and logarithmic quantities for any even integer  $n \geq 2$  (see (125)). We prove that this formula is indeed valid for any even integer  $n \leq 12$ .
- In the last section §6 of the paper, we formulate some questions that we find interesting about curvilinear webs. Some are in relation to what has been discussed before, others are not. A few of them concern the study of some projective varieties a better understanding of which could enlighten us on the abelian relations of the  $\mathcal{W}_{0,n+3}$ 's when  $n$  is odd.

Two appendices have been added at the end:

- In Appendix A, we investigate the 1-abelian relations of  $\mathcal{W}_{0,6}$  corresponding to the abelian 1-differentials on (a certain desingularization  $\widetilde{\Sigma}$  of) the Fano surface  $\Sigma$  of Segre's cubic  $\mathcal{S}$ . After explaining conceptually why the global sections of  $\omega_{\widetilde{\Sigma}}^1$  give rise to 1-abelian relations for  $\mathcal{W}_{0,6}$  (via the specialization to Segre's cubic of a certain version of 'Abel's

theorem' for the holomorphic 1-forms on Fano surfaces of smooth cubics in  $\mathbf{P}^4$ ), we explicitly determine these latter using the  $\mathfrak{S}_6$ -structure of the space  $\mathbf{AR}^{(1)}(\mathcal{W}_{0,6})$  they span.

- Finally in Appendix B, we study the web  $\mathcal{LW}_\mathcal{C}$  formed by the lines contained in the so-called 'chordal cubic'  $\mathcal{C} \subset \mathbf{P}^4$ . This is a degenerate case in which  $\mathcal{LW}_\mathcal{C}$  is a 3-web (and not a 6-web as in the case of a generic cubic threefold). Using Abel's method, we determine the 1-abelian relations of  $\mathcal{LW}_\mathcal{C}$ . It turns out that for this specific example, this web is equivalent to Blaschke-Walberer web defined on the variety of triangles included in  $\mathcal{C}$ . We take the opportunity offered by this coincidence to recall here the very nice classical but seemingly forgotten results by Blaschke and Walberer about curvilinear 3-webs in dimension 3 with maximal rank.

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Enfin, et bien sûr, l'auteur est également très reconnaissant et redevable à Brubru pour ses relectures nombreuses et ses corrections passionnées.

## 2. The moduli spaces $\mathcal{M}_{0,n+3}$ and the webs $\mathcal{W}_{0,n+3}$ on them

We start by recalling some basic facts about the moduli spaces  $\mathcal{M}_{0,n+3}$  before describing in several ways the webs  $\mathcal{W}_{0,n+3}$  on them. In §2.2, we recall Room-Burau's result and its consequence (namely Theorem A) for the webs  $\mathcal{W}_{0,n+3}$ 's. We finish in §2.2.2 by briefly discussing the flawed reasoning used in [Da1] to wrongly conclude that none of webs  $\mathcal{W}_{0,n+3}$  is linearizable.

All the material presented in this section is fairly standard and well known or well referenced in the literature. For this reason, no proof is given below, it seemed preferable to give specific references instead.

**2.1. Basic facts.** We start by discussing  $\mathcal{M}_{0,n+3}$  (for  $n \geq 1$  arbitrary) over the field of complex numbers before discussing the web  $\mathcal{W}_{0,n+3}$  on it. We end by considering its version over the reals, this in order to fit better with the setting of [Da2]. Everything in §2.1 is standard and well-known.

In what follows,  $n$  stands for a fixed integer bigger than or equal to 1.

**2.1.1. The moduli space  $\mathcal{M}_{0,n+3}$ .** Working over  $\mathbf{C}$ ,  $\mathcal{M}_{0,n+3}$  stands for the moduli space of projective configurations of  $n + 3$  pairwise distinct points on the complex projective line  $\mathbf{P}^1$ :

$$\mathcal{M}_{0,n+3} = \left\{ (z_i)_{i=1}^{n+3} \in (\mathbf{P}^1)^{n+3} \mid z_i \neq z_j \text{ for all } i \neq j \right\} / \text{PGL}_2(\mathbf{C}).$$

As is well known, it is a smooth irreducible rational affine complex variety of dimension  $n$ . In order to get an explicit rational coordinates system on  $\mathcal{M}_{0,n+3}$ , let us consider the affine arrangement in  $\mathbf{C}^n$ , denoted by  $A_n$ , defined as the union of the  $2n + n(n-1)/2 = n(n+3)/2$  affine hyperplanes cut out by the equations  $X_i = 0$ ,  $X_i - 1 = 0$  and  $X_i - X_j = 0$  for  $i, j = 1, \dots, n$  with  $i < j$ . Then, as is well known, the following rational map

$$(11) \quad \begin{aligned} \psi_n : \mathbf{C}^n \setminus A_n &\longrightarrow \mathcal{M}_{0,n+3} \\ (x_i)_{i=1}^n &\longmapsto [0, 1, \infty, x_1, \dots, x_n] \end{aligned}$$

is an isomorphism of affine varieties whose inverse map can be made explicit quite easily: for  $\mathbf{z} = [z_1, \dots, z_{n+3}] \in \mathcal{M}_{0,n+3}$ , let  $g_{\mathbf{z}}$  be the projective automorphism of  $\mathbf{P}^1$  such that  $g_{\mathbf{z}}(z_1) = 0$ ,  $g_{\mathbf{z}}(z_2) = 1$  and  $g_{\mathbf{z}}(z_3) = \infty$ , namely

$$g_{\mathbf{z}}(\zeta) = \frac{(\zeta - z_1)(z_2 - z_3)}{(\zeta - z_3)(z_2 - z_1)}$$

for any  $\zeta \in \mathbf{P}^1$  (where this formula must be suitably interpreted when one of the  $z_i$ 's involved is equal to  $\infty$ ). Then all the  $g_{\mathbf{z}}(z_k)$ 's for  $k = 4, \dots, n+3$  belong to  $\mathbf{C} \setminus \{0, 1\}$  and are pairwise distinct. Consequently, the map

$$\begin{aligned} \phi_n : \mathcal{M}_{0,n+3} &\longrightarrow \mathbf{C}^n \setminus A_n \\ \mathbf{z} = [z_i]_{i=1}^{n+3} &\longmapsto (g_{\mathbf{z}}(z_k))_{k=4}^{n+3} \end{aligned}$$

is well-defined and the composition  $\phi_n \circ \psi_n$  obviously is the restriction of the identity to  $\mathbf{C}^n \setminus A_n$ .



**2.1.2. Action of  $\mathfrak{S}_{n+3}$  by automorphisms and birational transformations.** For any  $\sigma \in \mathfrak{S}_{n+3}$  and  $\zeta = (\zeta_i)_{i=1}^{n+3} \in (\mathbf{P}^1)^{n+3}$ , one sets  $\sigma\zeta = (\zeta_{\sigma(i)})_{i=1}^{n+3}$ . Clearly, the map  $\zeta = (\zeta_i)_{i=1}^{n+3} \mapsto \sigma\zeta = (\zeta_{\sigma(i)})_{i=1}^{n+3}$  on  $(n+3)$ -tuples of points of  $\mathbf{P}^1$  is  $\mathrm{PGL}_2(\mathbf{C})$ -equivariant hence induces an automorphism of  $\mathcal{M}_{0,n+3}$ , denoted by  $\gamma_\sigma$ . One gets a morphism of groups

$$\begin{aligned} \mathfrak{S}_{n+3} &\longrightarrow \mathrm{Aut}(\mathcal{M}_{0,n+3}) \\ \sigma &\longmapsto \gamma_\sigma \end{aligned}$$

which turns out to be an isomorphism.<sup>10</sup>

Conjugating by the isomorphism  $\phi_n$ , one gets a group embedding of  $\mathfrak{S}_{n+3}$  into the group  $\mathbf{Bir}_n$  of birational maps in  $n$  variables

$$(12) \quad \begin{aligned} \mathfrak{S}_{n+3} &\hookrightarrow \mathbf{Bir}_n \\ \sigma &\longmapsto G_\sigma = \phi_n \circ \gamma_\sigma \circ \psi_n \end{aligned}$$

which is easy to describe explicitly: given a permutation  $\sigma \in \mathfrak{S}_{n+3}$ , for any  $x = (x_i)_{i=1}^n \in \mathbf{C}^n$  we set  $\xi = (\xi_s)_{s=1}^{n+3} = (0, 1, \infty, x_1, \dots, x_n)$  and  $\sigma\xi = (\xi_{\sigma(s)})_{s=1}^{n+3}$ . It is then immediate to verify that the birational map  $G_\sigma : \mathbf{C}^n \dashrightarrow \mathbf{C}^n$  is given by

$$(13) \quad G_\sigma(x) = \left( \frac{(\xi_{\sigma(k)} - \xi_{\sigma(1)})(\xi_{\sigma(2)} - \xi_{\sigma(3)})}{(\xi_{\sigma(k)} - \xi_{\sigma(3)})(\xi_{\sigma(2)} - \xi_{\sigma(1)})} \right)_{k=4}^{n+3}.$$

**Example 1.** *1. By way of illustration, let us consider the case of the  $(n+3)$ -cycle  $c = (1 \dots n+3)$ . For  $\zeta = (0, 1, \infty, x_1, \dots, x_n)$  with  $x = (x_1, \dots, x_n) \in \mathbf{C}^n$ , one has  $c\zeta = (x_n, 0, 1, \infty, x_1, \dots, x_{n-1})$  hence  $g_{c\zeta}$  is given by  $g_{c\zeta}(y) = (y - x_n)/(x_n(y - 1))$  (for  $y \in \mathbf{P}^1$ ). It follows that  $G_c$  is given by*

$$G_c(x) = \left( \frac{1}{x_n}, \frac{x_1 - x_n}{x_n(x_1 - 1)}, \dots, \frac{x_{n-1} - x_n}{x_n(x_{n-1} - 1)} \right).$$

*2. Some other explicit examples of birational maps  $G_\sigma$  are given in Table 1 of Appendix A.*

Formula (13) is of importance for our purpose since at many places in the sequel, it will allow us to explicitly compute pull-backs of abelian relations on  $\mathcal{M}_{0,n+3}$  under automorphisms  $\gamma_\sigma$  for  $\sigma \in \mathfrak{S}_{0,n+3}$ , in the rational coordinates  $x_i$ 's defined by means of (11).

**2.1.3. The web  $\mathcal{W}_{0,n+3}$ .** We now assume that  $n \geq 2$ . In the Introduction,  $\mathcal{W}_{0,n+3}$  has been defined as the curvilinear  $(n+3)$ -web on  $\mathcal{M}_{0,n+3}$  whose  $i$ -th foliation is the fibration (by  $(n+3)$ -punctured smooth rational curves) induced by the map consisting in forgetting the  $i$ -th point

$$(14) \quad \varphi_i : \mathcal{M}_{0,n+3} \longrightarrow \mathcal{M}_{0,n+2}.$$

Here we want to make explicit an affine birational model of  $\mathcal{W}_{0,n+3}$ .

Let  $p_1, \dots, p_{n+2}$  be  $n+2$  fixed points in general position in  $\mathbf{P}^n$ . A natural choice for the  $p_i$ 's is to take the vertices of the standard simplex in  $\mathbf{P}^n$ , namely

$$(15) \quad p_i = [\delta_i^1 : \dots : \delta_i^{n+2}] \quad \text{for } i = 1, \dots, n+2 \quad \text{and} \quad p_{n+2} = [1 : \dots : 1].$$

Given a generic point  $p \in \mathbf{P}^n$ , there exists a unique rational normal curve (RNC) of degree  $n$  in  $\mathbf{P}^n$  passing through all the  $p_i$ 's and through  $p$ . Denoting by  $\nu_p : \mathbf{P}^1 \rightarrow \mathbf{P}^n$  a projective parametrization

<sup>10</sup>This is a classical fact of Teichmüller theory, see the remark at the very end of [N, §4] for instance.

of this curve, one obtains a  $(n+3)$ -tuple of points  $\mu(p) = (v_p^{-1}(p_1), \dots, v_p^{-1}(p_{n+2}), v_p^{-1}(p)) \in (\mathbf{P}^1)^{n+3}$  whose class modulo  $\mathrm{PGL}_2(\mathbf{C})$  only depends on  $p$ . This gives us a well-defined map

$$(16) \quad \begin{aligned} \mathbf{P}^n &\dashrightarrow \mathcal{M}_{0,n+3} \\ p &\longmapsto [\mu(p)]. \end{aligned}$$

Conversely, given  $\mathbf{z} = [z_1 : \dots : z_{n+3}] \in \mathcal{M}_{0,n+3}$ , the map  $v_{\mathbf{z}} : \mathbf{P}^1 \rightarrow \mathbf{P}^n$ ,  $t \mapsto [(z_{n+2} - z_i)/(t - z_i)]_{i=1}^{n+1}$  is a bijective parametrization of a RNC of degree  $n$  which sends  $z_i$  to  $p_i$  for any  $i$  ranging from 1 to  $n+2$ . We get that way a rational map  $\mathcal{M}_{0,n+3} \dashrightarrow \mathbf{P}^n$ ,  $\mathbf{z} \mapsto v_{\mathbf{z}}(z_{n+3}) = [(z_{n+2} - z_i)/(z_{n+3} - z_i)]_{i=1}^{n+1}$  which is easily verified to be the inverse map of (16) which, as a result, is proved to be birational.

We will now give an explicit description of the pull-back of  $\mathcal{W}_{0,n+3}$  on  $\mathbf{P}^n$  under (16), that we will denote (a bit abusively) using the same notation. First remark that the projection from one of its points of a degree  $n$  RNC in  $\mathbf{P}^n$ , is a RNC of degree one less (namely  $n-1$ ) in  $\mathbf{P}^{n-1}$ . From this, one deduces easily that for  $i = 1, \dots, n+2$ , what corresponds on  $\mathbf{P}^n$  (modulo (16)) to the  $i$ -th forgeful map (14) is nothing else but the linear projection  $\pi_{p_i} : \mathbf{P}^n \dashrightarrow \mathbf{P}^{n-1}$  from  $p_i$ . Finally, one verifies easily that the leaf of the  $(n+3)$ -th foliation of  $\mathcal{W}_{0,n+3}$  on  $\mathbf{P}^n$  through a generic point is precisely the RNC of degree  $n$  passing through this point as well as through all the  $p_i$ 's.

We thus get the following geometric description of the pull-back of  $\mathcal{W}_{0,n+3}$  under (16):

$$(17) \quad \textit{considered on } \mathbf{P}^n, \textit{ the web } \mathcal{W}_{0,n+3} \textit{ is formed by the families of lines passing through one of the } p_i \textit{'s (for } i = 1, \dots, n+2) \textit{ plus the family of degree } n \textit{ rational normal curves containing all these points.}$$

From this description of  $\mathcal{W}_{0,n+3}$ , it is straightforward to verify that in the affine coordinates associated to the embedding  $\mathbf{C}^n \hookrightarrow \mathbf{P}^n$ ,  $u = (u_i)_{i=1}^n \mapsto [u_1 : \dots : u_n : 1]$ , this web admits the rational functions  $U_i$  as first integrals, where  $U_i(u) = (u_1, \dots, \widehat{u_i}, \dots, u_n)$  for  $i = 1, \dots, n$  and

$$(18) \quad U_{n+1}(u) = \left( \frac{u_j}{u_n} \right)_{j=1}^{n-1}, \quad U_{n+2}(u) = \left( \frac{u_j - 1}{u_n - 1} \right)_{j=1}^{n-1} \quad \text{and} \quad U_{n+3}(u) = \left( \frac{u_n(u_j - 1)}{u_j(u_n - 1)} \right)_{j=1}^{n-1}.$$

It is obvious that  $\mathcal{W}_{0,n+3}$  is invariant (as an unordered web) by the whole automorphism group  $\mathrm{Aut}(\mathcal{M}_{0,n+3}) \simeq \mathfrak{S}_{n+3}$  and that its action on the set of foliations of this web is isomorphic to the standard action of the symmetric group on  $\{1, \dots, n+3\}$ .

**2.1.4. Combinatorial abelian relations.** For any distinct elements  $i, j \in \{1, \dots, n+3\}$  (say such that  $i < j$ ), we denote by  $\mathcal{W}_{i,j}$  the  $(n+1)$ -subweb of  $\mathcal{W}_{0,n+3}$  formed by all its foliations except the  $i$ -th and the  $j$ -th ones. Considering the last paragraph of the preceding subsection, it follows that all the subwebs  $\mathcal{W}_{i,j}$ 's are equivalent (up to global automorphisms of  $\mathcal{M}_{0,n+3}$ ).

Working with the affine coordinates  $u_i$ 's, it follows from (18) that  $\mathcal{W}_{n+2,n+3}$  coincides with the standard quadrilateral web considered in §1.1.2. Hence it has rank equal to 1 (*cf.* (1)) and since all the  $(n+1)$ -subwebs of  $\mathcal{W}_{0,n+3}$  are equivalent, any of them has rank 1 too: for any  $i, j$  with  $i < j$ ,  $\mathcal{W}_{i,j}$  carries a nontrivial abelian relation, which will be denoted by  $AR_{i,j}$ . This AR is unique up to multiplication by a nonzero scalar (hence there is some ambiguity in its definition, but that doesn't really matter and therefore we will not pay attention to it).

The  $(n+3)(n+2)/2$  abelian relations  $AR_{i,j}$ 's are all rational<sup>11</sup> and span what Damiano calls the subspace of ‘combinatorial abelian relations’ of  $\mathcal{W}_{0,n+3}$ , denoted by

$$AR_C(\mathcal{W}_{0,n+3}) = \left\langle AR_{i,j} \mid 1 \leq i < j \leq n+3 \right\rangle \subset AR(\mathcal{W}_{0,n+3}).$$

The term ‘combinatorial’ comes from the fact that in their span, the  $AR_{i,j}$ 's satisfy relations of combinatorial nature which give rise to a matroid (see §4.1 and §8.2 in [Da1]). Although this aspect is interesting, we will not consider it here.

**2.1.5. Real locus.** While most of the considerations in this paper are within the complex framework (which according to us is most natural), it is the real framework that was considered in both [Da1] and [Da2]. Working over  $\mathbf{R}$  requires some subtleties, for instance regarding the domain on which the webs we are working with are considered, which were not very precisely explicated according to us. Hence we discuss briefly this aspect below.<sup>12</sup> For some recent references, we mention [Br], [FG, §2.3] as well as the recent preprint [AHL].

The real moduli space  $\mathcal{M}_{0,n+3}(\mathbf{R})$  is the set of  $(n+3)$ -tuples  $(x_1, \dots, x_{n+3})$  of pairwise distinct points on the real projective line  $\mathbf{P}_{\mathbf{R}}^1$  (that we identify with a circle  $S^1$ ), modulo the diagonal action of  $\mathrm{PSL}_2(\mathbf{R})$ . Its is a smooth manifold of dimension  $n$ , but with several connected components contrarily to its complex version  $\mathcal{M}_{0,n+3}$ . Clearly,  $\mathcal{M}_{0,n+3}(\mathbf{R})$  is invariant by the action of  $\mathfrak{S}_{n+3}$  discussed in §2.1.2 and this group acts by real-analytic diffeomorphisms on it.

By definition, the ‘positive part’  $\mathcal{M}_{0,n+3}^{>0}$  of  $\mathcal{M}_{0,n+3}(\mathbf{R})$  is the subset formed by ‘positive configurations’ that is configurations coming from tuples  $(x_1, \dots, x_{n+3})$  positively cyclically ordered on the circle  $S^1$  (with respect to a chosen orientation of  $S^1$ , previously fixed once for all). It is a connected component of  $\mathcal{M}_{0,n+3}(\mathbf{R})$  which is isomorphic to a ball whose topological closure in a certain compactification of the full real moduli space, is isomorphic (as a stratified real-analytic manifold) to a polytope, the well-known  $n$ -th ‘associahedron’ (or ‘Stasheff polytope’).

The positive part  $\mathcal{M}_{0,n+3}^{>0}$  is a connected component of  $\mathcal{M}_{0,n+3}(\mathbf{R})$ , from which all the others can be constructed. Indeed, first one verifies that the set of permutations elements of  $\mathfrak{S}_{n+3}$  letting  $\mathcal{M}_{0,n+3}^{>0}$  globally invariant is a subgroup  $D_{0,n+3} \subset \mathfrak{S}_{n+3}$  isomorphic to the dihedral group of order  $2(n+3)$ . Consequently, for each element  $\sigma$  of the coset space  $\mathfrak{R}_n = \mathfrak{S}_{n+3}/D_{0,n+3}$ , there exists a connected component  $\mathcal{M}_{0,n+3}^{>0}(\sigma)$  of  $\mathcal{M}_{0,n+3}(\mathbf{R})$ , defined as the image of  $\mathcal{M}_{0,n+3}^{>0}$  by any representative of  $\sigma$  in  $\mathfrak{S}_{n+3}$  (it is obviously well-defined): one has

$$\mathcal{M}_{0,n+3}^{>0}(\sigma) = \sigma \left( \mathcal{M}_{0,n+3}^{>0} \right) \subset \mathcal{M}_{0,n+3}(\mathbf{R}).$$

When  $n$  is fixed and that there is no risk of any ambiguity, we will denote  $\mathcal{M}_{0,n+3}^{>0}(\sigma)$  by  $\mathcal{M}(\sigma)$ .

For  $\sigma$  ranging in  $\mathfrak{S}_{n+3}$ , one gets all the connected components of  $\mathcal{M}_{0,n+3}(\mathbf{R})$ . In other terms, there is a disjoint union

$$(19) \quad \mathcal{M}_{0,n+3}(\mathbf{R}) = \bigsqcup_{\sigma \in \mathfrak{R}_n} \mathcal{M}_{0,n+3}^{>0}(\sigma)$$

<sup>11</sup>By this we mean that when expressed in the  $u_i$ 's, the components of these ARs are rational  $(n-1)$ -forms.

<sup>12</sup>Note that it has some importance, for instance for properly defining the  $\mathfrak{S}_{n+3}$ -action on  $AR(\mathcal{W}_{0,n+3})$  or, as explained in §1.2 (see also §5.1 further), when performing integration along fibers to obtain generalized dilogarithmic forms à la Gelfand and McPherson.

where any of the  $|\mathfrak{R}_n| = (n+2)!/2$  components  $\mathcal{M}_{0,n+3}^{>0}(\sigma)$  is isomorphic to the positive part  $\mathcal{M}_{0,n+3}^{>0}$  hence is topologically trivial.

**Example 2.** By way of illustration, let us consider the case when  $n = 2$  in more detail. Let  $\mathcal{A}_2$  be the arrangement of five lines in  $\mathbf{R}^2$  cut out by  $xy(x-1)(y-1)(x-y) = 0$ . Then  $F : \mathbf{R}^2 \setminus \mathcal{A}_2 \rightarrow \mathcal{M}_{0,5}(\mathbf{R})$ ,  $(x, y) \mapsto [0 : x : y : 1 : \infty]$  is an isomorphism and the 12 connected components of  $\mathcal{M}_{0,5}(\mathbf{R})$  correspond to the 12 regions  $U_L$  with label  $L$  ranging from I to XII on Figure 2 below.

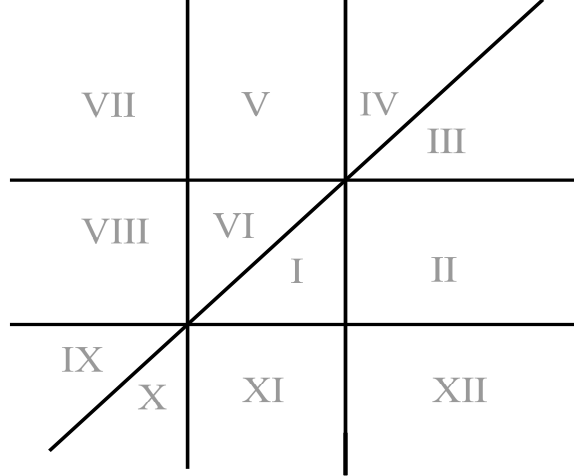


FIGURE 2. Affine realization of the real moduli space  $\mathcal{M}_{0,5}(\mathbf{R})$  with a labeling from I to XII of its connected components.

The region  $U_I$  is the triangle  $\{(x, y) \in \mathbf{R}^2 \mid 0 < x < y < 1\}$  and it corresponds to (that is,  $F(U_I)$  is) the positive part  $\mathcal{M}_{0,5}^{>0}$ . Both the cyclic shift  $c : p = [p_1, \dots, p_5] \mapsto [p_5, p_1, \dots, p_4]$  and the dihedral reflection  $r : [p_1, \dots, p_5] \mapsto [p_4, p_3, p_2, p_1, p_5]$  let  $\mathcal{M}_{0,5}^{>0}$  invariant and their pull-backs under  $F$  are respectively the order 5 birational map  $C : (x, y) \mapsto (1-y, (1-y)/(1-x))$  and the affine involution  $R : (x, y) \mapsto (1-y, 1-x)$ . The two maps  $C$  and  $R$  generate a subgroup of  $\mathbf{Bir}_2$  isomorphic to the automorphism group of a pentagon, that is the dihedral group of order 10.

Being aware of the decomposition (19) makes some of the subtleties arising when studying the real version of  $\mathcal{W}_{0,n+3}$  on  $\mathcal{M}_{0,n+3}(\mathbf{R})$  more apparent, which was not sufficiently discussed in [Da2] in our opinion. Indeed, (19) indicates that there is no global web in the real setting but several, namely one for each  $\sigma \in \mathfrak{R}_n$ , denoted by  $\mathcal{W}_{0,n+3}^\sigma$ , which is the one defined by the (restriction of the) forgetful maps (14) on the component  $\mathcal{M}_{0,n+3}^{>0}(\sigma)$ . For each such  $\sigma$ , let  $\mathbf{AR}(\sigma) = \mathbf{AR}(\mathcal{W}_{0,n+3}^\sigma)$  be the space of abelian relations of  $\mathcal{W}_{0,n+3}^\sigma$  which are globally defined on the whole definition domain of this web. Moreover, since this domain is (homeomorphic to) a ball,  $\mathbf{AR}(\sigma)$  is naturally identified with the space of ARs of the restriction of  $\mathcal{W}_{0,n+3}$  on any open subset of  $\mathcal{M}_{0,n+3}^{>0}(\sigma)$ .

Since  $\mathcal{M}_{0,n+3}^{>0}(\sigma)$  is stable under the action of a subgroup  $D_{0,n+3}(\sigma)$  conjugate to  $D_{0,n+3}$ , each  $\mathbf{AR}(\sigma)$  carries a structure of  $D_{0,n+3}$ -representation (well-defined up to conjugations). But since there is no a priori natural identification between  $\mathbf{AR}(\sigma)$  and  $\mathbf{AR}(\mathbf{1})$ , it is not clear at all whether the structure of  $D_{0,n+3}$ -module of  $\mathbf{AR}(\mathbf{1})$  extends to a representation of the whole symmetric

group. To put it briefly, without any supplementary argument, it is not clear how to make  $\mathfrak{S}_{n+3}$  act on a space of ARs (eg.  $\mathbf{AR}(\mathbf{1})$ ) in order to get a linear representation of this group.

But the situation has some subtleties even in the complex setting. In this case indeed, the ARs of  $\mathcal{W}_{0,n+3}$  are sections of a local system on  $\mathcal{M}_{0,n+3}$  (cf. page 3). But since its topology is not trivial and because there is no configurations invariant by all elements of  $\mathfrak{S}_{n+3}$ , one cannot straightaway define an action of  $\mathfrak{S}_{n+3}$  (induced by pull-backs) on a vector space constructed from some ARs of  $\mathcal{M}_{0,n+3}$ .

★

Note that, in contrast with the whole space of abelian relations (whether in a real or complex setting), the situation is quite simpler if one restricts to the subspace of combinatorial ones. Indeed, since all the components of (1) are rational functions, all the combinatorial abelian relations  $AR_{i,j}$ 's are rational hence  $\mathfrak{S}_{n+3}$  naturally acts on  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ . One verifies easily that this representation is defined over  $\mathbf{Q}$  (but we will not use this arithmetic fact in the sequel).

In the complex setting, a possibly bigger  $\mathfrak{S}_{n+3}$ -representation (actually obtained from the one of the previous paragraph by successive extensions), may be obtained by considering the local system of ARs of  $\mathcal{W}_{0,n+3}$  with unipotent monodromy, denoted by  $\mathbf{AR}_U(\mathcal{W}_{0,n+3})$ . The latter comes with an increasing filtration  $\mathbf{AR}_U^\bullet(\mathcal{W}_{0,n+3})$  whose 0-th piece is the vector space  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  and where it is not difficult to figure how are defined the pieces of higher degree.<sup>13</sup> Then one verifies easily that the associated graded space  $\mathbf{Gr}^k \mathbf{AR}_U^\bullet(\mathcal{W}_{0,n+3})$  is a  $\mathfrak{S}_{n+3}$ -module for any  $k \geq 0$  hence one obtains a representation

$$(20) \quad \mathbf{GrAR}_U^\bullet(\mathcal{W}_{0,n+3}) = \bigoplus_{k \geq 0} \mathbf{Gr}^k \mathbf{AR}_U^\bullet(\mathcal{W}_{0,n+3})$$

which is a priori bigger than the representation  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  discussed in the preceding paragraph since obviously the latter coincides with the 0-th piece of (20).

**Example 3.** *Let us again consider the web  $\mathcal{W}_{0,5}$ , but this time within the complex setting (compare with Example 2 just above). In this case, up to the natural identification of  $\mathcal{W}_{0,5}$  with Bol's web  $\mathcal{B}$  already mentioned above,  $\mathbf{AR}_C(\mathcal{W}_{0,5})$  corresponds to what was noted by  $\mathbf{AR}_{\text{Log}}(\mathcal{B})$  in the Introduction and the filtration  $\mathbf{AR}_U^\bullet(\mathcal{W}_{0,5})$  has only two pieces, with  $\mathbf{AR}_U^{\leq 1}(\mathcal{W}_{0,5})$  being obtained by adjoining the AR corresponding to the Abel's identity ( $\mathcal{A}\mathbf{b}$ ) to the combinatorial ARs.*

*In the case under scrutiny, we then have  $\mathbf{GrAR}_U^\bullet(\mathcal{W}_{0,5}) = \mathbf{AR}_C(\mathcal{W}_{0,5}) \oplus \langle \mathcal{A}\mathbf{b} \rangle$  which shows that the decomposition in direct sum  $\mathbf{AR}(\mathcal{B}) = \mathbf{AR}_{\text{Log}}(\mathcal{B}) \oplus \langle \mathcal{A}\mathbf{b} \rangle$  mentioned in (5) can actually be interpreted as a decomposition of  $\mathfrak{S}_5$ -representations.*

**2.2. Room-Bureau's linear systems and consequences regarding the linearizability of the  $\mathcal{W}_{0,n+3}$ 's.** Here we recall some nice constructions/results of classical algebraic geometry. Then we explain some rather immediate consequences of them for the webs  $\mathcal{W}_{0,n+3}$ 's, in what concerns their realizability as webs by rational curves of low degree.

Albeit different in certain aspects, the cases when  $n$  is even or  $n$  is odd share some similarities as well, henceforth both will be discussed alongside. As before, the material below already appeared in several references, hence no proof is given. Classical references are: the paper by Room [R1],

<sup>13</sup>Very summarily, for any  $k \geq 1$ , the stalk  $\mathbf{AR}_{U,z}^{\leq k}(\mathcal{W}_{0,n+3})$  of the  $k$ -th piece of this filtration at any configuration  $z \in \mathcal{M}_{0,n+3}$  is the space of (germs of) abelian relations  $\Omega$  at this point whose analytic continuation along any loop  $\gamma$  based at  $z$ , denoted by  $\Omega^\gamma$ , is such that  $\Omega^\gamma - \Omega \in \mathbf{AR}_{U,z}^{\leq k-1}(\mathcal{W}_{0,n+3})$ . Details are left to the reader.

the content of which has been generalized in arbitrary dimension in his book [R2] and Burau's article [Bu]. For much more recent papers (without any reference to Room's or Burau's classical works), see [K] and [BM].

2.2.1. We continue to use the notation introduced before:  $p_1, \dots, p_{n+2}$  are  $n+2$  points in general position in  $\mathbf{P}^n$  with  $n \geq 2$ , etc. Our goal is to recall the definition of a linear system  $|\mathcal{L}_n|$  on  $\mathbf{P}^n$ , first considered by Room and Burau for  $n$  arbitrary, the associated rational map of which will allow to give a nice projective model for  $\mathcal{W}_{0,n+3}$ .

We set  $P = \sum_{i=1}^{n+2} p_i$  (as a 0-cycle of degree  $n+2$  on  $\mathbf{P}^n$ ). The linear system  $\mathcal{L}_n$  we are interested in is defined as follows, according to the parity of  $n$ : one has

$$\mathcal{L}_n = \left| d_n H - \nu_n P \right| = \left\{ \begin{array}{l} \text{hypersurfaces of degree } d_n \text{ in } \mathbf{P}^n \text{ vanishing} \\ \text{at the order } \nu_n \text{ at } p_i \text{ for all } i = 1, \dots, n+2 \end{array} \right\} \subset |\mathcal{O}_{\mathbf{P}^n}(d_n)|,$$

with

$$(d_n, \nu_n) = \begin{cases} (n+1, n-1) & \text{if } n \text{ is even;} \\ ((n+1)/2, (n-1)/2) & \text{if } n \text{ is odd.} \end{cases}$$

We denote by  $N_n$  (or just by  $N$  when there is no ambiguity) the projective dimension of  $|\mathcal{L}_n|$  and by  $\varphi_n$  the associated rational map:

$$\varphi_n = \varphi_{\mathcal{L}_n} : \mathbf{P}^n \dashrightarrow \mathbf{P}^N \simeq |\mathcal{L}_n|^\vee.$$

Finally, we set

$$V_n = \text{Im}(\varphi_n) = \varphi_n(\mathbf{P}^n) \subset \mathbf{P}^N.$$

It is a non degenerate projective variety in  $\mathbf{P}^N$  which satisfies several nice properties, that we have all gathered in the following result (recall that  $\delta_n$  is 1 if  $n$  is odd, and 2 if  $n$  is even).

**Proposition 2.1.** *The following assertions hold true for any  $n \geq 2$ .*

1. *The map  $\varphi_n : \mathbf{P}^n \dashrightarrow V_n$  is birational and its image  $V_n$  is isomorphic to  $(\mathbf{P}^1)^{n+3} // \text{PGL}_2(\mathbf{C})$  (the GIT quotient of  $n+3$  copies of the projective line by the diagonal action of  $\text{PGL}_2(\mathbf{C})$ ).*
2. *For any foliation  $\mathcal{F}$  of  $\mathcal{W}_{0,n+3}$ , its image  $(\varphi_n)_*(\mathcal{F})$  by  $\varphi_n$  is a curvilinear foliation on  $V_n$ , whose general leaf is an irreducible rational curve of degree  $\delta_n$ . These are exactly the covering families of  $V_n$  by lines (for  $n$  odd) or by conics (for  $n$  even).*
3. *For any  $n \geq 2$ , the automorphism group  $\text{Aut}(V_n)$  of  $V_n$  identifies with  $\mathfrak{S}_{n+3}$ .*
4. *When  $n$  is odd, one has*

$$N = N_n = \binom{n+1}{(n+1)/2} - \binom{n+1}{(n+1)/2 - 2} - 1$$

*and the following statements are satisfied:*

- (i).  *$V_n$  is singular, with singular set consisting of  $\binom{n+2}{(n+1)/2}$  isolated singular points.*
- (ii).  *$\text{Pic}(V_n)$  is free of rank 1. Moreover,  $K_{V_n} \simeq \mathcal{O}_{V_n}(-1)$  hence  $V_n$  is a Fano variety.*

5. *When  $n$  is even one has*

$$(21) \quad N = N_n = \sum_{s=0}^{\lfloor (n+1)/3 \rfloor} (-1)^s \binom{n+2}{s} \binom{2n-3s+1}{n} - 1$$

*and the following statements are satisfied:*

- (i).  $V_n$  is smooth;
- (ii). one has  $\text{Pic}(V_n) \simeq \mathbf{Z}^{n+3}$ . Moreover  $K_{V_n} \simeq \mathcal{O}_{V_n}(-2)$  hence  $V_n$  is a Fano manifold.

The results above are quite nice generalizations for  $n$  even and odd respectively, of some well-known properties of the del Pezzo quintic surface in  $\mathbf{P}^5$  (case  $n = 2$ ) and of Segre's cubic hypersurface  $S$  in  $\mathbf{P}^4$  (case  $n = 3$ , discussed in more details §3.3 further). If we will only use the first two points of this proposition in this article, we have mentioned the others to show that the  $V_n$ 's form an interesting family of Fano varieties generalizing very classical examples, which should be studied further in our opinion. Focusing especially on the case  $n = 5$ , we will briefly come back on this at the end in §6.3.2.

In order to help the reader who might be interested in the statements above, let us indicate quickly where these can be found in the existing (classical or more recent) literature. First, the case  $n = 5$  is studied by Room in [R1] by classical projective methods. For  $n$  arbitrary, the fact that  $\varphi_n$  is birational onto its image is not explicitly stated in [R2] or [Bu] but we think that it is because this fact was more or less obvious to these authors. Another classical reference for the case when  $n$  is odd is [Cob]. For a modern treatment, see Theorem 3.1 (for  $n$  odd) and Theorem 3.4 (for all cases) in [K]. Point 2. is rather immediate once it is known that  $\varphi_n$  is birational onto its image. Regarding the determination of the dimension  $N$  as a function of  $n$ , see [R2, p. 363], [Bu, §4.a] and [K, §3.1] for  $n$  odd. A formula for  $N$  in the case when  $n$  is even is a bit more involved and can be found in [Bu, §4.b] or in [K, §3.3]. The other mentioned statements, namely that  $V_n$  coincides with  $(\mathbf{P}^1)^{n+3}/\text{PGL}_2(\mathbf{C})$ , that  $\text{Aut}(V_n)$  is naturally isomorphic to  $\mathfrak{S}_{n+3}$ , or the description of the singularities, of the Picard group and of the canonical class of  $V_n$ , all have been obtained quite recently, in [BM].

2.2.2. From the first two points of Proposition 2.1, one immediately deduces the

**Corollary 2.2. 1.** *For any  $n \geq 2$ , the push-forward  $\mathcal{W}_{0,n+3} = (\varphi_n)_*(\mathcal{W}_{0,n+3})$  is a web on  $V_n$  which is equivalent to  $\mathcal{W}_{0,n+3}$  and whose leaves are rational curves of degree  $\delta_n$ .*

**2.** *In particular,  $\mathcal{W}_{0,n+3}$  is linearizable when  $n$  is odd.*

The second point contradicts a result claimed by Damiano, namely that  $\mathcal{W}_{0,n+3}$  is not linearizable for any  $n \geq 2$ , which is one of the points stated in Proposition 1.6 of [Da2]. But not even an indication of proof appears in the whole paper, except in the classical case  $n = 2$  for which standard references are given (namely §29, §30 and §31 of [BB]). More details about the reasoning leading Damiano to state this can be found in his thesis. Since realizing that  $\mathcal{W}_{0,6}$  actually is linearizable has been the departure point of the work leading to the present paper, we believe it is worth explaining precisely where Damiano's reasoning is not correct according to our understanding.

Actually, only a few lines are devoted in [Da1] to establish that  $\mathcal{W}_{0,n+3}$  is not linearizable for any  $n$  and can be found pages 20 and 21 of Damiano's dissertation. His reasoning is by *reductio ad absurdum* by means of an induction on  $n$  and can be decomposed as follows:

1. *If  $\mathcal{W}_{0,n+3}$  were linearizable, it would admit a model  $\mathfrak{B}_n$  in  $\mathbf{R}^n$  with linear leaves;*

2. Then the  $(n + 2)$ -web  $\mathfrak{W}'_n$  in  $H \simeq \mathbf{R}^{n-1}$  obtained by projecting  $\mathfrak{W}_n$  onto a hyperplane  $H$  transverse to one of the foliations (and then disregarding this foliation) would also be linearizable;
3. But  $\mathfrak{W}'_n$  is equivalent to  $\mathfrak{W}_{0,n+2}$  (case  $n - 1$ );
4. Hence assuming that  $\mathfrak{W}_{0,n+3}$  is linearizable would allow to deduce by induction that  $\mathfrak{W}_{0,5}$  is linearizable as well, but this is not the case.

The idea for arguing by recurrence on  $n$  (step 3.) is justified by the following fact which is easy to prove (details are left to the reader): if  $\pi_{\mathcal{F}_i}$  stands for the map corresponding to quotienting by the  $i$ -th foliation  $\mathcal{F}_i$  of  $\mathfrak{W}_{0,n+3}$ , then  $(\pi_{\mathcal{F}_i})_*(\mathfrak{W}_{0,n+3}) = ((\pi_{\mathcal{F}_i})_*(\mathcal{F}_j))_{j \neq i}$  is a  $(n+2)$ -web in dimension  $n - 1$  which is equivalent to  $\mathfrak{W}_{0,n+2}$ . The problem in the reasoning above lies in the second step: the projection onto a hyperplane  $H$  which is considered therein is taken for a model of a quotient map with respect to a foliation  $\mathfrak{F}_i$  of the linear model  $\mathfrak{W}_n$  of  $\mathfrak{W}_{0,n+3}$ . But for that to be the case, it is necessary that this foliation be precisely that of lines through a point (possibly at infinity) and it is not obvious at all that  $\mathfrak{W}_{0,n+3}$  admits (even locally) a linear model in  $\mathbf{R}^n$  whose (at least) one of the foliations enjoys this property. And it turns out that it is not the case for instance when  $n = 3$  since  $\mathfrak{W}_{0,6}$  is linearizable (according to Corollary 2.2) whereas  $\mathfrak{W}_{0,5}$  is not.

We believe that what might have been the source of the mistake discussed in the previous paragraph is the model (17) of  $\mathfrak{W}_{0,n+3}$  in  $\mathbf{P}^n$ : at the exception of the last one, all its foliations are formed by the lines going through one of the  $p_i$ 's hence for this model, quotienting by such a foliation (which in this case makes sense even globally and corresponds to the  $i$ -th forgeful map (14)) is indeed given by the linear projection from the corresponding  $p_i$  and onto a hyperplane. But of course, this projection does not linearize the last foliation since the generic leaf of the latter is a RNC of degree  $n$ .

All our remarks above show that, although it is a very appealing approach, one has to be careful when trying to investigate the webs  $\mathfrak{W}_{0,n+3}$ 's inductively by quotienting along one of its foliations.

A general idea that emerges from the results of the present article is that the  $\mathfrak{W}_{0,n+3}$ 's seem to form two distinct classes of webs according to the parity of  $n$ , and those belonging to the same class seem to share most of their properties. For this reason, the validity of the following conjecture (already mentioned in [Bu]) seems more than plausible:

**Conjecture 2.3** (Bureau). *For any  $n \geq 2$  even, the web  $\mathfrak{W}_{0,n+3}$  is not linearizable.*

That  $\mathfrak{W}_{0,5}$  is not linearizable is very classical and goes back to the early beginning of web geometry (cf. [BB, p. 262]). Since then, no progress has been done regarding this conjecture. We confess having no idea how to handle it, even in the special case  $n = 4$ .

**Remark 2.4.** *As we finished writing this article, we realized that  $\mathfrak{W}_{0,6}$  is linearizable is explicitly mentioned in the booklet on webs [Bla4] published by Blaschke in 1955, see the last paragraph of §54 in it. It is unfortunate that Damiano missed this reference as well as Bureau's paper [Bu].*



### 3. A thorough study of the case $n = 3$ .

Our goal in this section is to study in depth the web  $\mathcal{W}_{0,6}$  by showing it can be seen as a particular case of a quite large class of interesting curvilinear webs coming from algebraic geometry.

We first recall below some well-known facts about lines on cubic threefolds and then discuss how these facts are interpreted in terms of webs. The study of the Fano surface of lines in a cubic hypersurface  $X$  of  $\mathbf{P}^4$  is a very nice and nowadays classical part of algebraic geometry when  $X$  is smooth. In subsection §3.1 below, we first recall some points of this theory and explain their consequences for web geometry. Most of these results have been extended to the case of singular cubic hypersurfaces, especially nodal cubics. But as far as we are aware of, only the case of cubics with only a few nodes (typically one node) has been concretely considered in the literature.<sup>14</sup> For this reason, we discuss in detail the case of Segre's cubic (which carries 10 nodes, which is the maximal number of nodes that a hypercubic of  $\mathbf{P}^4$  can have) in a separate subsection, namely in §3.3.

**3.1. Webs by lines on smooth cubic hypersurfaces of  $\mathbf{P}^4$ .** The theory of lines on smooth cubic threefolds has been considered in many papers. In addition of Fano's original paper [Fa2], we mention [Ghe], [BS], [CIG], [T], [Mur] and [AK]. Among these, our main references are [CIG] and [AK], to which the reader may refer for justifications of the assertions below.

Here  $X$  stands for a smooth cubic hypersurface in  $\mathbf{P}^4$ . The underlying set of its Fano scheme  $F_1(X)$ , denoted below just by  $F$  to simplify, is the subset of the grassmannian variety of lines in the ambient  $\mathbf{P}^4$  formed by the lines contained in  $X$ : one has

$$F = F_1(X) = \{ \ell \in G_1(\mathbf{P}^4) \mid \ell \subset X \}.$$

It is not difficult to exhibit polynomial equations cutting  $F$  in affine charts of  $G_1(\mathbf{P}^4)$ , which shows that  $F$  is naturally an algebraic subvariety of this grassmannian (cf. [AK, (1.14)]). Defining the associated incidence variety as the algebraic subvariety  $T = \{(x, \ell) \in X \times F \mid x \in \ell\}$  of  $X \times F$ , both  $X$  and  $F$  naturally fit into the following incidence diagram:

$$(22) \quad \begin{array}{ccc} & T & \\ p \swarrow & & \searrow q \\ X & & F \end{array}$$

where  $p$  and  $q$  stand respectively for the restrictions to  $T$  of the standard projections of  $X \times F$  onto its first and second factors. It will be useful to use the following notation below: we assume that the ambient projective space is the projectivization of a 5-dimensional vector space  $V$ :  $\mathbf{P}^4 = \mathbf{P}(V)$ . To simplify, we will write  $G$  for the grassmannian variety  $G_1(\mathbf{P}^4) = G_2(V)$ . We denote by  $\mathcal{T}_G$  the rank 2 tautological bundle over  $G$  and  $\rho : G \hookrightarrow \mathbf{P}(\wedge^2 V) = \mathbf{P}^9$  stands for the Plücker embedding. Finally,  $\iota : F \hookrightarrow G$  denotes the natural inclusion of  $F$  into the grassmannian and we write  $\varrho = \rho \circ \iota : F \hookrightarrow \mathbf{P}^9$  for the corresponding embedding of  $F$  into  $\mathbf{P}^9$ .

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<sup>14</sup>However we would not be surprised if all nodal cases, from zero (smooth) to ten nodes (Segre's cubic) were considered as fully understood by experts.

3.1.1. Here is a list of properties of  $F$  and of the incidence correspondance (22), many of which will have interesting consequences when interpreted in terms of webs:

- i. A line  $\ell \subset X$  can be of two types, according to decomposition of its normal bundle in  $X$ : up to the identification  $\ell \simeq \mathbf{P}^1$ , either  $N_{\ell/X} = \mathcal{O}_{\mathbf{P}^1}(0)^{\oplus 2}$  and  $\ell$  is of the ‘*first type*’, or  $N_{\ell/X} = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$  and  $\ell$  is of the ‘*second type*’ (cf. [CIG, Proposition 6.19]). In both cases, one has  $h^0(N_{\ell/X}) = 2$  and  $h^1(N_{\ell/X}) = 0$ .
- ii. The Fano variety  $F$  is a smooth irreducible surface (see [CIG, Lemma 7.7]). Hence  $T$  is smooth as well. Moreover  $q$  makes of  $T$  a  $\mathbf{P}^1$ -bundle over  $F$  hence  $T$  has dimension 3.
- iii. Through a general point  $x$  of  $X$  pass exactly 6 pairwise distinct lines included in  $X$ , which we will denote (with arbitrary labels) by  $\ell_1(x), \dots, \ell_6(x)$  (see [AK, (1.7)]). In other terms, the map  $p : T \rightarrow X$  in (22) is generically 6 to 1.
- iv. The embedding  $\varrho : F \hookrightarrow \mathbf{P}^9$  is canonical: if  $\mathcal{O}_F(1) = \varrho^*(\mathcal{O}_{\mathbf{P}^9}(1))$  stands for the corresponding line bundle, one has an isomorphism:

$$K_F = \Omega_F^2 \simeq \mathcal{O}_F(1)$$

(see [AK, (1.8)]). Moreover,  $\iota(F)$  is non degenerate in  $\mathbf{P}^9$  (i.e. no hyperplane of  $\mathbf{P}^9$  contains  $\iota(F)$ , see [CIG, Lemma 10.2]) hence there is a natural identification of vector spaces

$$(23) \quad H^0(F, \Omega_F^2) \simeq H^0(\mathbf{P}^9, \mathcal{O}_{\mathbf{P}^9}(1)) = (\wedge^2 V)^\vee$$

from which it follows that  $h^0(\Omega_F^2) = 10$  (see [CIG, Lemma 10.2]).

- v. The pull-back under  $\iota$  of the dual of the tautological bundle  $\mathcal{T} = \mathcal{T}_G$  identifies with  $\Omega_F^1$ :

$$(24) \quad \Omega_F^1 = \iota^*(\mathcal{T}^\vee)$$

(see [AK, (1.10)-(ii)]).<sup>15</sup> Moreover, the restriction map  $H^0(G, \mathcal{T}^\vee) \rightarrow H^0(F, \mathcal{T}^\vee|_F)$  induces an isomorphism

$$V^\vee = H^0(G, \mathcal{T}^\vee) \simeq H^0(F, \Omega_F^1).$$

In particular, one has  $h^0(F, \Omega_F^1) = 5$ .

- vi. It follows that the Albanese variety  $\text{Alb}(F) = H^0(F, \Omega_F^1)^\vee / H_1(F, \mathbf{Z})$  is of dimension 5. Moreover (for any base point  $x_0 \in F$ ), the Albanese map  $a = a_{x_0} : F \rightarrow \text{Alb}(F)$ ,  $x \mapsto (a^x = a_{x_0}^x : H^0(F, \Omega_F^1)) \rightarrow \mathbf{C}$ ,  $\omega \mapsto \int_{x_0}^x \omega$ ) is an embedding.
- vii. The cup-product mapping  $H^0(F, \Omega_F^1) \wedge H^0(F, \Omega_F^1) \rightarrow H^0(F, \Omega_F^2)$  is injective ([CIG, Lemma 9.13]) hence it is an isomorphism (for dimensional reasons).
- viii. Actually, for  $k = 1, 2$ , the Albanese mapping  $a : F \rightarrow \text{Alb}(F)$  induces an isomorphism

$$a^* : H^0(\text{Alb}(F), \Omega_{\text{Alb}(F)}^k) \simeq H^0(F, \Omega_F^k).$$

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<sup>15</sup>The isomorphism (24) is known as the ‘*tangent bundle theorem*’. It is considered as one of the deepest results about the hypercubic  $X$  and its Fano variety  $F$  and has been obtained by several authors: originally by Fano (in a rough form). Modern proofs have been given independently by Clemens-Griffiths (see [CIG, Corollary 12.38]) and Tyurin.

- ix. A line  $\ell \subset X$  is of the second type if and only if there exists a 2-plane  $P \subset \mathbf{P}^4$  such that  $X \cdot P = 2\ell + \ell'$  as 1-cycles on  $X$ , for another line  $\ell' \subset X$  (cf. [Mur, Lemma (1.14)]). The lines  $\ell \subset X$  of second type form a non-singular subset  $C$  of  $F$ , of dimension 1 (see [Mur, Corollary (1.9)]).
- x. The set  $\mathcal{E}$  of ‘Eckardt points’ on  $X$ , that is points  $x \in X$  such that  $p^{-1}(x) = \{\ell \in F \mid x \in \ell\}$  has positive dimension, is finite (cf. [CIG, Lemma 8.1]).<sup>16</sup>
- xi. One sets  $X' = X \setminus \mathcal{E}$  and  $T' = p^{-1}(X')$ . Then the restriction  $p' : T' \rightarrow X'$  of  $p$  over  $X'$  is unramified outside  $S' = q^{-1}(C) \cap T'$  and has simple ramification along a Zariski open subset of each irreducible component of  $S'$  (see [CIG, Lemma 10.18]).
- xii. The map  $\mathcal{T} : X' \rightarrow \text{Alb}(F)$ ,  $x \mapsto \sum_{i=1} a(\ell_i(x))$  is well-defined and holomorphic. Since  $\mathcal{E}$  is finite, it extends holomorphically to the whole  $X$ . Since the latter is simply-connected, it follows that the map  $\mathcal{T}$  is constant (cf. (11.10) in [CIG]).
- xiii. For  $k = 1, 2$ , there is a global trace map

$$\text{Tr}^{(k)} = p_* \circ q^* : \Omega_{\mathbf{C}(F)}^k \rightarrow \Omega_{\mathbf{C}(X)}^k$$
<sup>17</sup>

which locally (on a neighborhood of any  $x \in X'$ ) is written  $\text{Tr}^{(k)}(\omega) = \sum_{i=1}^6 \ell_i^*(\omega)$ .

- xiv. The map  $\mathcal{T} : X' \rightarrow \text{Alb}(F)$ ,  $x \mapsto \sum_{i=1} a(\ell_i(x))$  is well-defined and holomorphic. Since  $\mathcal{E}$  is finite, it extends holomorphically to the whole  $X$ . Since the latter is simply-connected, it follows that the map  $\mathcal{T}$  is constant (cf. (11.10) in [CIG]).
- xv. For  $k = 1, 2$  and for any global holomorphic  $k$ -differential  $\omega \in H^0(F, \Omega_F^k)$ , the  $k$ -form  $\text{Tr}^{(k)}(\omega)$  (which can be written  $\text{Tr}^{(k)}(\omega) = \sum_{i=1}^6 \ell_i^*(\omega)$  locally) vanishes identically on  $X$ .
- xvi. Let  $\overline{\Delta}_X$  be the variety of triangles in  $X$ , namely the 3-dimensional subvariety of  $F^3$  formed by the 3-tuples of pairwise distinct lines  $(l_1, l_2, l_3)$  such that  $l_1 + l_2 + l_3 = X \cdot P$  (equality between 1-cycles) for a certain 2-plane  $P \subset \mathbf{P}^4$ . Then there exists  $\alpha_1 \in \text{Alb}(F)$  such that  $a(l_1) + a(l_2) + a(l_3) = \alpha_1$  for any generic triangle  $(l_1, l_2, l_3) \in \overline{\Delta}_X$  (cf. [CIG, (11.9)]).

From xi., it follows that  $X^* = X' \setminus p(S')$  is a Zariski open set of  $X$  (it is the complement of a finite union of surfaces contained in  $X$ ) such that through any  $x^* \in X^*$ , pass exactly six pairwise distinct lines included in  $X$ . Consequently, the linear 6-web  $\mathcal{LW}_X$  defined by the lines included in  $X$  is well-defined on  $X^*$ . Now, it is straightforward to state in web-theoretic terms for the web  $\mathcal{LW}_X$ , some of the sixteen properties i. to xvi. stated just above. We get the following

**Proposition 3.1.** 1.  $\mathcal{LW}_X$  is an irreducible 6-web which is skew and linearizable.

2. For  $k = 1, 2$  and  $\omega \in H^k(F, \Omega_F^k)$ , one has  $\text{Tr}^{(k)}(\omega) = 0$  hence the linear map

$$\begin{aligned} \text{Tr}^{(k)} : H^0(F, \Omega_F^k) &\longrightarrow \text{AR}^{(k)}(\mathcal{LW}_X) \\ \omega &\longmapsto \left( \ell_i^*(\omega) \right)_{i=1}^6 \end{aligned}$$

is well-defined and injective.

3. Consequently, the following statements hold true:

<sup>16</sup>Actually, it is known that  $\text{Card}(\mathcal{E}) \leq 30$  and that  $X$  is the Fermat cubic threefold in case of equality.

<sup>17</sup>Here for any algebraic variety  $Z$  and any  $k \geq 1$ ,  $\Omega_{\mathbf{C}(Z)}^k$  stands for the  $\mathbf{C}(Z)$ -module of global rational  $k$ -forms on  $Z$ .

a. The second trace map  $\text{Tr}^{(2)}$  induces an isomorphism  $\mathbf{H}^0(F, \Omega_F^2) \simeq \mathbf{AR}^{(2)}(\mathcal{LW}_X)$ . These spaces are of dimension 10 hence  $\mathcal{LW}_X$  has maximal 2-rank;

b. The image  $\mathbf{AR}_{ab}^{(1)}(\mathcal{LW}_X) = \text{Im}(\text{Tr}^{(1)})$  of the first trace map is a 5-dimensional subspace of  $\mathbf{AR}^{(1)}(\mathcal{LW}_X)$ , which is such that the wedge map

$$\begin{aligned} \wedge^2 \mathbf{AR}_{ab}^{(1)}(\mathcal{LW}_X) &\longrightarrow \mathbf{AR}^{(2)}(\mathcal{LW}_X) \\ (\omega_i)_{i=1}^6 \wedge (\varpi_i)_{i=1}^6 &\longmapsto (\omega_i \wedge \varpi_i)_{i=1}^6 \end{aligned}$$

is well-defined and is an isomorphism.

*Proof.* The first point of **1** is rather obvious: the irreducibility of  $\mathcal{LW}_X$  as a web follows immediately from that of  $F$  as an algebraic surface. Regarding the linearizability of  $\mathcal{LW}_X$ , it suffices to consider its local image in  $\mathbf{P}^3$  by a generic linear projection from a point of  $\mathbf{P}^4$ . As for the skewness of  $\mathcal{LW}_X$ , it is also easy to establish: assuming that this web is not skew would imply that through a general point of  $X$  passes a doubly-ruled surface  $\Sigma$  included in  $X$ . Since  $\Sigma$  is not a plane (otherwise the set  $\mathcal{E}$  of Eckardt points of  $X$  would be infinite), it has to be a quadric surface spanning a hyperplane. But then  $\langle \Sigma \rangle \cap X = \Sigma \cup P$  for a 2-plane  $P \subset X$  residual to  $\Sigma$ , which is not possible since, again,  $\mathcal{E}$  is known to be finite.<sup>18</sup>

To end the proof of **1**, it remains to show that the lines included in  $X$  passing through a general point  $x$ , denoted by  $\ell_1(x), \dots, \ell_6(x)$ , are in general position. To do so, we recall that the tangent directions at  $x$  of the  $\ell_i(x)$ 's all six belong to the second fundamental form  $II_{X,x}$  of  $X$  at the considered point, that we see here as a conic in the projectivized tangent space  $\mathbf{P}(T_x X) \simeq \mathbf{P}^2$ . Then assume that  $x \in X$  is generic and such that there are three pairwise distinct lines in  $X$  through  $x$  which are coplanar. Then their associated tangent directions at this point span a line in  $\mathbf{P}(T_x X)$ , denoted by  $L_x$ . Because  $L_x \cap II_{X,x}$  has cardinality (at least) three, it follows that  $L_x \subset II_{X,x}$ , which gives us that the conic  $II_{X,x}$  is not reducible hence has rank  $\leq 2$ . But this would contradict (in the case when  $n = 3$ ) the well-known fact that, given a  $n$ -dimensional smooth complex projective hypersurface with  $n \geq 2$ , its second fundamental form at a generic point, viewed as a quadric in  $\mathbf{P}^{n-1}$ , has maximal rank equal to  $n$ .<sup>19</sup> It follows that  $\mathcal{LW}_X$  satisfies the general position assumption of §1.1 which finishes the proof of **1**.

The second and third points of the proposition are nothing else but reinterpretations in terms of  $\mathcal{LW}_X$  of some the points stated above (mainly the points **iv**, **vi**, **vii** and **xv**).  $\square$

Let  $\mathcal{LW}_{X,x}$  be the germ of  $\mathcal{LW}_X$  at a generic point  $x \in X^*$ . Using linear projections centered at sufficiently generic points of  $\mathbf{P}^4$ , one gets a five dimensional family of linear models for  $\mathcal{LW}_{X,x}$ . It seems more than likely that two such local linear models are generically not projectively equivalent hence that this (germ of) web admits a 4-dimensional family of non projectively equivalent linearizations. A rigorous proof that it is indeed the case would be welcome.

<sup>18</sup>A more direct way to conclude may have been to recall that a cubic containing a 2-plane is necessarily singular (see the paragraph just before Proposition 2.2 in [Do1]).

<sup>19</sup>It is worth mentioning that this 'well-known fact' is rather subtle and in any case less obvious than it seems at first sight. For instance, all the considered hypotheses are necessary: there exist complex affine or real projective hypersurfaces which are smooth but with a degenerate second fundamental form at any point.

Another property of  $\mathcal{LW}_X$  which is not completely clear yet concerns its 1-rank: from **3.a.** in the above proposition, we know that  $\text{rk}^{(1)}(\mathcal{LW}_X) \geq h^0(\Omega_F^1) = 5$ . Is this majoration actually an equality? In other terms, does the space  $\text{AR}_{ab}^{(1)}(\mathcal{LW}_X)$  coincide with the whole space  $\text{AR}^{(1)}(\mathcal{LW}_X)$  of 1-ARs of the web  $\mathcal{LW}_X$ ? We believe that the answer is affirmative (still assuming that  $X$  is smooth) but we don't have any argument to offer justifying this.

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In view of Proposition 3.1 and considering to which extent it is very similar to the very classical corresponding statement holding true for classical ‘algebraic webs’, we believe it is justified to say that  $\mathcal{LW}_X$  is ‘algebraic as well’ (see also our discussion in §1.3 of [P3]).

Since the web  $\mathcal{W}_{0,6}$  is (birationally) equivalent to the web  $\mathcal{LW}_S$  formed by the six covering families of lines in Segre’s cubic  $S$  and because the latter can be seen as a degeneration of smooth cubic hypersurfaces, it is natural to wonder whether it is justified to say that it is an algebraic web as well. If this is obvious in what concerns its geometric definition (see §3.3 just below) this is not as clear with regard to the ARs of  $\mathcal{LW}_S$  since it is not evident that the statements in §3.1.1 used to obtain Proposition 3.1 admit counterparts for a singular cubic. Indeed, some of these properties are not satisfied by singular cubics, as the first example considered in the next subsection shows.

**3.2. The case of (some) singular cubics.** We first discuss the case of a special cubic threefold with a singular set of dimension 1. Then we say a few words about those cubics with only nodal singularities. We will focus on the case of Segre’s cubic just after in §3.3.

**3.2.1. The chordal cubic.** Let  $[Z_0 : \cdots : Z_4]$  stand for fixed homogeneous coordinates on  $\mathbf{P}^4$  and denote by  $[\cdot] : \mathbf{C}^5 \setminus \{0\} \rightarrow \mathbf{P}^4$  the projectivisation map. We fix a rational normal quartic curve  $\Gamma \subset \mathbf{P}^4$ . Then the 2-secants to  $\Gamma$  (which are the lines intersecting  $\Gamma$  in two points counted with multiplicity) fill out a cubic hypersurface  $\mathcal{C} \subset \mathbf{P}^4$ , known as the ‘*chordal cubic threefold*’ (see [Se, Coll]). Taking the Zariski closure of the projectivisation of the image of  $\nu_4 : \mathbf{C} \ni t \mapsto (1, t, t^2, t^3, t^4) \in \mathbf{C}^5$  for the quartic curve  $\Gamma$ , one gets that  $\mathcal{C}$  is cut out by the following cubic equation

$$(25) \quad \det \begin{bmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \\ Z_2 & Z_3 & Z_4 \end{bmatrix} = Z_0(Z_2Z_4 - Z_3^2) - (Z_2^3 - 2Z_1Z_2Z_3 - Z_2^3 + Z_1^2Z_4) = 0.$$

The hypersurface  $\mathcal{C}$  is singular along  $\Gamma$  and is the unique cubic threefold of  $\mathbf{P}^4$  singular along a rational quartic curve (up to projective equivalence). Note also that  $\mathcal{C}$  is semistable and that it appears as a special degenerate point in the GIT moduli space of cubic threefolds (see [Al]). The chordal cubic is discussed in the paragraphs §44 and §45 of C. Segre’s memoir [Se] and in the first part of the recent paper [Coll], two references the reader interested by more details is invited to consult.

It is known that the Fano surface  $F = F_1(\mathcal{C})$  of lines included in  $\mathcal{C}$  has two components denoted by  $F'$  and  $F''$ : the former  $F'$  is the family of lines which are 2-secant with  $\Gamma$  and the latter  $F''$  is the set of lines  $\ell \subset \mathbf{P}^4$  such that the restriction to  $\Gamma$  of the linear projection  $\pi_\ell : \mathbf{P}^4 \dashrightarrow \mathbf{P}^2$  from  $\ell$  has a ramification divisor of degree 2 on  $\Gamma$  (or equivalently,  $\pi_\ell(\Gamma)$  is a conic). Both  $F'$  and  $F''$

are isomorphic to the symmetric square of  $\Gamma$  hence both are isomorphic to  $\Gamma^{[2]} \simeq (\mathbf{P}^1)^{[2]} \simeq \mathbf{P}^2$ .<sup>20</sup> The two components  $F'$  and  $F''$  intersect along the smooth rational curve  $\mathcal{K} \subset G_1(\mathbf{P}^4)$  whose points are the lines tangent to  $\Gamma$  ( $\mathcal{K}$  is the image of the Gauss map  $\Gamma \rightarrow G_1(\mathbf{P}^4)$ ,  $\gamma \mapsto T_\gamma \Gamma$ ). Of course, one has  $\mathcal{K} \simeq \Gamma \simeq \mathbf{P}^1$ .

As a scheme, it can be verified that  $F_1(\mathcal{C})$  contains  $F'$  and  $F''$  with multiplicity 4 and 1 respectively (cf. the bottom of p. 214 in [Col1]) from which it follows that through a general point  $c$  of  $\mathcal{C}$  pass exactly three pairwise distinct lines included in  $\mathcal{C}$ : two lines  $\ell'_1(c)$  and  $\ell''_2(c)$  corresponding to two distinct points of  $F''$  and the 2-secant  $\ell'(c)$  to  $\Gamma$  passing through  $c$ , which comes with multiplicity 4. From this it follows that when smooth cubics degenerate to  $\mathcal{C}$ , the associated 6-webs by lines degenerate onto a web that it is justified to denote by  $\mathcal{LW}_\mathcal{C}$  but which is a 3-web.

It is not difficult to make  $\mathcal{LW}_\mathcal{C}$  explicit. Indeed, from the very definition of  $\mathcal{C}$  it follows that

$$(26) \quad \mu : \mathbf{C}^3 \longrightarrow \mathbf{P}^4, \quad (s, t) = (s, t_1, t_2) \longmapsto [\nu_4(t_1) + s^2\nu_4(t_2)]$$

is an affine parametrization<sup>21</sup> of (a Zariski-open subset of)  $\mathcal{C}$  and it can be verified that the three lines contained in  $\mathcal{C}$  and passing through a generic point  $c = \mu(s, t)$  are

- the line  $\ell'(c)$  with parametrization  $\mathbf{P}^1 \ni \sigma \mapsto \mu(\sigma, t) \in \mathcal{C} \subset \mathbf{P}^4$  which is the 2-secant line  $\langle \nu_4(t_1), \nu_4(t_2) \rangle$ . It corresponds to a point of  $F' = \Gamma^{[2]}$  hence comes with multiplicity 4;
- two lines elements of  $F''$ , denoted by  $\ell'_+(c)$  and  $\ell''_-(c)$ . These lines are those joining  $c$  to a point  $C_\pm(s, t) \in \mathbf{P}^4$  whose coordinates are in  $\mathbf{Z}[i](s, t)$  and can be explicitated (moreover, one can manage things in such a way that  $C_-(s, t) = \overline{C_+(s, t)}$ ).

From the explicit formulas one can get for the lines  $\ell'_\pm(c)$  and  $\ell'(c)$  for  $c = \mu(s, t)$  generic in  $\mathcal{C}$ , it is not difficult to deduce some explicit rational first integrals on  $\mathbf{C}^3$  for the pull-back under  $\mu$  of the web  $\mathcal{LW}_\mathcal{C}$ . Indeed, it can be verified that this pull-back (again denoted with the same notation to simplify), admits the following three rational first integrals in the coordinates  $s, t_1, t_2$ :

$$(27) \quad U' = (t_1, t_2), \quad U''_+ = \left( \frac{s t_2 + i t_1}{s + i}, \frac{s t_2^2 + i t_1^2}{s t_2 + i t_1} \right) \quad \text{and} \quad U''_- = \overline{U''_+} = \left( \frac{s t_2 - i t_1}{s - i}, \frac{s t_2^2 - i t_1^2}{s t_2 - i t_1} \right).$$

From these explicit formulas, one gets that  $\mathcal{LW}_\mathcal{C}$  is a 3-web by rational curves on  $\mathbf{C}^3$  and it is straightforward to verify that some rational vector fields defining it are  $X', X''_+$  and  $X''_- = \overline{X''_+}$  with

$$X' = \frac{\partial}{\partial s} \quad \text{and} \quad X''_+ = 2 \frac{\partial}{\partial s} + \left( \frac{t_1 - t_2}{s + i} \right) \frac{\partial}{\partial t_1} + i \left( \frac{t_1 - t_2}{s(s + i)} \right) \frac{\partial}{\partial t_2}.$$

By direct computations, one gets the following formulas for the Lie brackets of these vector fields:

$$(28) \quad \begin{aligned} [X''_+, X''_-] &= \frac{3s + i}{s(s + i)} X''_+ - \frac{3s - i}{s(s - i)} X''_- - \frac{8i}{1 + s^2} X' \\ [X''_+, X'] &= \frac{3s + i}{2s(s + i)} X''_+ - \frac{s - i}{2s(s + i)} X''_- - \frac{2}{s} X' \\ \text{and} \quad [X''_-, X'] &= \frac{3s - i}{2s(s - i)} X''_+ - \frac{s + i}{2s(s - i)} X''_- - \frac{2}{s} X' \end{aligned}$$

<sup>20</sup>For any set  $A$ ,  $A^{[2]}$  stands for its symmetric product, that is  $A^{[2]} = (A \times A)/\mathfrak{S}_2$ .

<sup>21</sup>The reason why the square of  $s$  is used for parametrizing the secant between  $\nu_4(t_1)$  and  $\nu_4(t_2)$  is that it allows to get rational first integrals for the web  $\mathcal{LW}_\mathcal{C}$  in the variables  $t_1, t_2$  and  $s$  (cf. (27)). Note that  $\mu$  is a 2-1 ramified covering onto its image  $\mu(\mathbf{C}^3) \subset \mathcal{C}$  with ramification locus the coordinate hyperplane  $s = 0$ .

from which one deduces that the web  $\mathcal{LW}_{\mathcal{C}}$  is skew. Its pull-back by  $\mu$  is formed by three global foliations on  $\mathbf{C}^3$  but it is not the same for  $\mathcal{LW}_{\mathcal{C}}$  itself which is the union on (its definition domain in)  $\mathcal{C}$  of the foliation with global rational first integral  $\ell' : X \dashrightarrow F'$ ,  $c \mapsto \ell'(c)$  and of the global irreducible 2-web whose leaves through a general point  $c \in \mathcal{C}$  are the two lines  $\ell'_{\pm}(c)$ .

The remarks above indicate which statement corresponds to the first point of Proposition 3.1 when the cubic threefold  $X$  specializes to the singular one  $\mathcal{C}$ . The other points of this Proposition concern abelian relations and clearly everything stated in Proposition 3.1 about 2-ARs does not admit a counterpart for  $\mathcal{LW}_{\mathcal{C}}$  since it is known that the 2-rank of any curvilinear 3-web in dimension 3 is zero. This example shows that what has to be the statement corresponding to Proposition 3.1 when the considered cubic is singular is not self-evident.

Even if it is a bit off topic, it is interesting to consider the 1-ARs of  $\mathcal{LW}_{\mathcal{C}}$ , since it turns out that this web has maximal 1-rank and that its 1-ARs as well come from the abelian 1-forms on the Fano surface of  $\mathcal{C}$ . We will discuss this a little further in Appendix B.

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We have taken time to discuss the case of the chordal cubic  $\mathcal{C}$  first because we find it interesting per se<sup>22</sup>, but also because it is not self-evident to show how to extend the nice theory discussed above in §3.1.1 to singular cubics as well as its web-theoretic consequence, namely Proposition 3.1. However we believe that in what concerns the webs associated to cubic threefolds, there is not much difference between the smooth case and that of cubics with only finitely many singular points. This is briefly discussed in the next subsection before considering more thoroughly the case of Segre's cubic in §3.3.

**3.2.2. Cubic threefolds with finitely many singular points.** It is known that cubic hypersurfaces in  $\mathbf{P}^4$  with only finitely many singularities share many properties with the smooth cubics, but may also present some distinct features.

The first and perhaps main difference between smooth and singular cubics is about their rationality. The smooth cubics are known to be irrational according to a famous result of the field (due to Clemens and Griffiths [CIG])<sup>23</sup>, contrarily to any singular cubic for which the projection from one of its singular points induces a birational map from it onto  $\mathbf{P}^3$ . Another example of distinct feature depending on the singularities a cubic threefold can have, is given by considering the case of cubics with finitely many ordinary nodes. Indeed, for such a cubic  $X$ , it is known that the number  $k$  of nodes is less than or equal to 10 and that the Fano surface of lines  $F_1(X)$  is generally irreducible when  $k \leq 5$  but is not for any  $k \geq 6$ .

If classical geometers considered the case of cubic threefolds with arbitrary many ordinary (or even more involved) nodes<sup>24</sup> this is not the case in more recent papers. The results of Hodge-theoretic nature listed in 3.1.1 were established in view of proving the main result of the field, namely the non rationality of a smooth cubic threefold and therefore were essentially considered

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<sup>22</sup>It is worth mentioning that in [ColI], Collino has used degenerations towards the chordal cubic  $\mathcal{C}$  in order to deduce informations about the fundamental group of Fano surfaces of smooth cubic threefolds.

<sup>23</sup>In several publications circa 1945, Fano claimed having established that a general cubic hypersurface in  $\mathbf{P}^4$  is not rational. But his proof does not seem to have been accepted as correct (*cf.* the discussion in [Pu, §2.1]).

<sup>24</sup>See for instance Segre's memoir [Se] (from §12 to §27), Fano's papers [Fa1] and [Fa2, §9] or Snyder's article [Sn], where some of the assertions mentioned above are proved by means of classical geometric methods.

for smooth cubic threefolds. Actually, most of the techniques used by modern authors actually apply to the so-called 'Lefschetz cubic hypersurfaces', that is cubics with at most one ordinary node as singularity. And indeed, in the recent papers where the material of 3.1.1 is discussed in the case of a singular cubic, only the case of cubic threefolds with one ordinary node has been really considered (in addition to [CIG]) where the case of Lefschetz cubics of  $\mathbf{P}^4$  is explicitly considered, see for instance [CM] or [GK]). Recently, cubic threefolds with more involved singularities have been considered (for instance in some papers by Allock (2003) or Casalaina-Martin & Laza (2009)) but with regard to their stability in the aim of studying their moduli theory. In particular, as far as we know, the theory thoroughly described in 3.1.1 is not discussed in a systematic way for singular cubics with more than one ordinary node in the existing literature, especially some points (such as xv. page 26, which has to be seen as a kind of Abel's theorem within this context) which are crucial in view of establishing a result similar to Proposition 3.1 for the webs of lines on singular cubics.

We consider below the case of singular cubics with only finitely many singular points. We do not assume that the singular points are ordinary nodes, just that they are isolated. We deduce the 'Abel's addition result for lines included in such a cubic' from the corresponding one holding true for smooth cubics first by means of a very natural deformation argument and also by using the fact that the Fano surfaces of such cubics satisfy properties similar to some listed in §3.1.1 for smooth cubics (*cf.* (29)). Our arguments below are rather standard and easy to follow hence we believe that very likely the 'Abel's addition result for lines on a cubic threefold with isolated singularities' we prove below will appear to the experts as a well-known folkloric generalization of the corresponding result for smooth cubics, which is classical. Proposition 3.3 and its proof have been included here not for claiming any kind of originality but just for the sake of completeness.

**Abel's theorem for abelian 2-forms on the Fano surface.** Let us set precisely the notation and assumptions we will deal with. Here  $X \subset \mathbf{P}^4$  stands for a fixed irreducible cubic threefold with finitely many singularities. We moreover assume that  $\mathcal{LW}_X$  is a well-defined 6-web on  $X$ : for  $x \in X$  generic, the six lines  $\ell_1(x), \dots, \ell_6(x)$  passing through  $x$  and contained in  $X$  are pairwise distinct and are in general position, namely any three of them span a hyperplane in the ambient  $\mathbf{P}^4$ . Here again  $F = F_1(X)$  stands for the Fano scheme parametrizing the lines included in  $X$ . The key fact behind the proof of Proposition 3.3 discussed below is the following generalization of iv. of §3.1.1 to the case of cubic threefolds with isolated singularities. Indeed, according to [AK] (*cf.* (1.4), (1.8) and (1.15) therein), although not smooth,  $F$  satisfies the following properties:

1.  $F = F_1(X) \subset G_1(\mathbf{P}^4)$  is still a reduced algebraic surface (but with singularities and even possibly several irreducible components);
- (29) 2. one has  $\omega_F^2 = \mathcal{O}_F(1)$ , *i.e.* the dualizing sheaf of  $F$  coincides with the invertible sheaf induced by  $\mathcal{O}_{\mathbf{P}^9}(1)$  via the Plücker embedding  $F \subset G_1(\mathbf{P}^4) \hookrightarrow \mathbf{P}^9$ ;
3. the canonical map  $\mathbf{H}^0(\mathbf{P}^9, \mathcal{O}_{\mathbf{P}^9}(1)) \rightarrow \mathbf{H}^0(F, \mathcal{O}_F(1)) \simeq \mathbf{H}^0(F, \omega_F^2)$  is an isomorphism.

As is well-known (see [Bar]), the canonical sheaf  $\omega_F^2$  can be identified with that of abelian differential 2-forms on  $F$ , namely the subsheaf of meromorphic 2-forms giving rise to  $\bar{\partial}$ -closed currents on their definition domain. In particular,  $\mathbf{H}^0(F, \omega_F^2)$  can naturally be seen as a subspace of the vector space  $\Omega_{C(F)}^2$  of rational 2-forms on  $F$ . Given  $\omega \in \mathbf{H}^0(F, \omega_F^2)$ , the sum  $\sum_{i=1}^6 \ell_i^*(\omega)$



(where now  $\ell_i : (X, x) \rightarrow F$  are considered as local algebraic first integrals for  $\mathcal{LW}_X$ ) is the germ at  $x$  of a global rational 2-form on  $X$ , called the 'trace of  $\omega$ ' and denoted by  $\text{Tr}(\omega)$ .

Our aim here is to establish that the points 2 and 3.a of Proposition 3.1 hold true as well for the singular cubic under scrutiny:

- Proposition 3.2.** 1. For any abelian differential 2-form  $\omega \in \mathbf{H}^0(F, \omega_F^2)$ , one has  $\text{Tr}(\omega) = 0$ .  
 2. The well-defined induced linear map  $\text{Tr} : \mathbf{H}^0(F, \omega_F^2) \rightarrow \mathbf{AR}(\mathcal{LW}_X)$  is injective.

Since  $10 = h^0(F, \omega_F^2) \leq \mathbf{rk}(\mathcal{LW}_X) \leq 10$ , one gets immediately the

**Corollary 3.3.** One has  $\mathbf{H}^0(F, \omega_F^2) \simeq \mathbf{AR}(\mathcal{LW}_X)$  hence the web  $\mathcal{LW}_X$  has maximal 2-rank.

In view of Proposition 3.1, it is natural to wonder whether the statements therein about the 1-forms of the Fano surface/the 1-ARs of the web by lines on the cubic are also verified when  $X$  has isolated singularities or not. This is more subtle than it is for 2-forms/2-ARs but we think that it is indeed the case. We will discuss this in the next paragraph.

The key fact behind Proposition 3.3 is the point **xv** of §3.1.1 which follows from the fact that when  $X$  is smooth, the trace of a global holomorphic 2-form on  $F_1(X)$  admits a holomorphic extension to the whole  $X$  hence necessarily vanishes.<sup>25</sup> It is not immediately clear in which way this approach can be extended to singular cubics. For instance, even for a cubic with simple singularities (nodes) such as Segre's cubic  $\mathcal{S}$ , proving a priori that the trace of an abelian differential 2-form on  $\Sigma = F_1(\mathcal{S})$  is abelian seems a bit delicate. We have chosen to follow another approach to prove the proposition above, which is quite natural since it relies on the smooth case via a local deformation of the considered cubic.

We continue to use the notations introduced before in the proof below:

*Proof of Proposition 3.3.* For  $\eta \in \mathbf{H}^0(F, \omega_F^2)$  arbitrary, we first want to prove that  $\sum_{i=1}^6 \ell_i^*(\eta)$  is identically zero. This can be obtained quite easily by considering a smoothing of  $(X, x)$  as a pointed cubic hypersurface. So let  $\{(X_t, x_t)\}_{t \in (\mathbf{C}, 0)}$  be a local analytic family of pointed cubic threefolds in  $\mathbf{P}^4$  such that  $X_0 = X$  and  $x_0 = x$ , with  $X_t$  smooth for  $t \neq 0$  (the local existence of such a smoothing family is obvious). Then the germ  $\mathcal{LW}_{X_t, x_t}$  of  $\mathcal{LW}_{X_t}$  at  $x_t$  is a well-defined germ of linear 6-web for  $t$  sufficiently close to 0, and the set of them form an (local) analytic deformation of  $\mathcal{LW}_{X, x} = \mathcal{LW}_{X_0, x_0}$ .<sup>26</sup> From this, we deduce that the germs of maps  $\ell_{i,t} : (X_t, x_t) \rightarrow F_t = F_1(X_t)$  which are the first integrals defining  $\mathcal{LW}_{X_t, x_t}$  for any  $t \in (\mathbf{C}, 0)$ , depend holomorphically on it.

For proving that  $\sum_{i=1}^6 \ell_i^*(\eta) = 0$ , it suffices to show that the considered  $\eta$  fits into an analytic family  $\{\eta_t\}$  where  $\eta_t$  is a global holomorphic 2-form on  $X_t$  for any  $t$  sufficiently close to 0. Indeed, in this case  $\text{Tr}(\eta_t) = \sum_{i=1}^6 \ell_i^*(\eta_t)$  is holomorphic with respect to  $t$  from one hand, but is such that  $\text{Tr}(\eta_t) \equiv 0$  for every  $t \in (\mathbf{C}, 0)$  distinct from 0 (thanks to Proposition 3.1, since all the  $X_t$ 's are

<sup>25</sup>That a smooth cubic threefold  $X$  does not carry any non trivial global holomorphic 2-form (i.e. that  $h^0(\Omega_X^2) = 0$ ) follows easily from Hodge theory (computation of the Hodge number  $h^{p,q}(X)$ 's). It is worth mentioning that the vanishing of the trace of any 2-forms on  $F_1(X)$  also follows from a famous theorem by Mumford, cf. [Mum, p. 200].

<sup>26</sup>This is not rigorously correct since the germs of webs  $\mathcal{LW}_{X_t, x_t}$  do not live on the same space. But it is easy to solve this, for instance by considering a generic linear projection  $P : \mathbf{P}^4 \rightarrow \mathbf{P}^3$  and an analytic family  $\{g_t\}_{t \in (\mathbf{C}, 0)}$  of projective transforms  $g_t \in \text{PGL}_4(\mathbf{C})$  such that  $g_0 = \text{Id}$  and  $g_t(P(x_t)) = P(x)$  for any  $t \in (\mathbf{C}, 0)$ . Then the push-forward germs of webs  $(g_t \circ P)_*(\mathcal{LW}_{X_t, x_t})$  form a genuine local holomorphic deformation of  $P_*(\mathcal{LW}_{X, x})$  on  $(\mathbf{P}^3, P(x)) \simeq (\mathbf{C}^3, 0)$ .

smooth for  $t \neq 0$ . Specializing at  $t = 0$  gives us the wanted relation. But that such an  $\eta$  can be (locally) holomorphically deformed to the smooth deformations  $X_t$  of  $X$  follows from standard facts of analytic geometry: the Fano surfaces  $F_t = F_1(X_t) \subset G_1(\mathbf{P}^4)$  for  $t$  sufficiently close to 0, organize themselves into an analytic family of surfaces  $\pi : \mathcal{F} \rightarrow (\mathbf{C}, 0)$  (cf. [AK] or [CGHL, §3]). The direct image by  $\pi$  of the relative dualizing sheaf  $\omega_{\mathcal{F}/(\mathbf{C},0)}^2$  has fiber  $\mathbf{H}^0(F_t, \omega_{F_t}^2)$  at any  $t$ . Since all these fibers are 10-dimensional (according to (29)), it follows that  $\pi_*(\omega_{\mathcal{F}/(\mathbf{C},0)}^2)$  is locally-free. Thus any  $\eta \in \mathbf{H}^0(F_0, \omega_{F_0}^2)$  can be extended into an analytic section  $t \rightarrow \eta_t \in \mathbf{H}^0(F_t, \omega_{F_t}^2)$ . As explained just above, this is sufficient to ensure that the following map is well-defined:

$$(30) \quad \text{Tr} : \mathbf{H}^0(F, \omega_F^2) \longrightarrow \mathbf{AR}(\mathcal{LW}_{X,x}), \quad \eta \longmapsto (\ell_i^*(\eta))_{i=1}^6.$$

There is a last subtlety to deal with which concerns the injectivity of this map, which is not completely obvious. Let us briefly discuss this matter. For  $i = 1, \dots, 6$ , let  $F'_i$  be the irreducible component of  $F$  which contains  $\text{Im}(\ell_i)$ . Some of the  $F'_i$ 's may coincide, hence these components can be univocally labeled  $F_j$  for  $j$  in some set  $J$  of cardinality at most 6. Let  $F^{gen}$  be the union  $\cup_{j \in J} F_j$  and denote by  $\widetilde{F}^{gen}$  its image in  $\mathbf{P}^9$  by the Plücker embedding  $G_1(\mathbf{P}^4) \subset \mathbf{P}^9$  (the notation with the upper *gen* comes from the fact that  $F^{gen}$  can equivalently be defined as the union of the irreducible components of  $F$  whose generic element meets the generic point of  $X$ ).

The following statements are easily seen to be equivalent:

- the map (30) is injective;
- (31) • the restriction map  $\mathbf{H}^0(F, \omega_F^2) \rightarrow \oplus_{j \in J} \Omega_{\mathbf{C}(F_j)}^2$ ,  $\omega \mapsto (\omega|_{F_j})_{j \in J}$  is injective;
- the surface  $\widetilde{F}^{gen}$  is non degenerate in  $\mathbf{P}^9$ , i.e.  $\langle \widetilde{F}^{gen} \rangle = \mathbf{P}^9$ .

That these equivalent properties hold true is immediate for instance when  $F$  is irreducible, but needs to be justified for an arbitrary cubic with finitely many singular points. This is the content of the lemma just below. The proposition is fully proved.  $\square$

**Lemma 3.4.** *For  $X$  as above (that is,  $X$  is an irreducible cubic threefold with finitely many singular points and such that the lines it contains form a genuine 6-web whose leaves are generically in general position), then  $\widetilde{F}^{gen}$  is non degenerate in  $\mathbf{P}^9$ .*

*Proof.* The proof is easy and goes by reduction ad absurdum. Recall that the Plücker embedding  $P : G_1(\mathbf{P}^4) \subset \mathbf{P}^9$  has for coordinates the Plücker coordinates  $\Delta_{ij}$  for  $i, j$  such that  $1 \leq i < j \leq 5$ : if  $L$  is the line passing through  $[p]$  and  $[q]$  for two distinct points  $p, q \in \mathbf{C}^5$  with coordinates  $p_i$  and  $q_i$  respectively, then by definition  $\Delta_{ij}(L) = \Delta_{ij}(p, q) = p_i q_j - p_j q_i$  for any  $i, j$  as above and the point  $[\Delta_{ij}(L)]_{1 \leq i < j \leq 5} \in \mathbf{P}^9$  is well-defined (that is depends only on  $L$ ) and corresponds to  $P(L)$ .

That  $\widetilde{F}^{gen} = P(F^{gen}) \subset \mathbf{P}^9$  is degenerate means that there exist scalars defining a hyperplane for  $[p] \in X$  generic and any  $[q]$  such that the line  $L_{p,q} = \langle [p], [q] \rangle$  joining  $[p]$  to  $[q]$  is contained in  $X$ , then the following relation is satisfied

$$(32) \quad \langle H, P(L_{p,q}) \rangle = \sum_{1 \leq i < j \leq 5} h_{ij} \Delta_{ij}(p, q) = 0.$$

Let  $F$  be a cubic homogeneous equation in five variables such that  $F = 0$  be an equation of the affine cone over  $X$  in  $\mathbf{C}^5$  and denote by  $F(\cdot, \cdot, \cdot)$  the associated symmetric trilinearization. For

$[p] \in X$  generic, the  $q$ 's such that the line  $L_{p,q}$  is included in  $X$  are those satisfying the following set of homogeneous equations (of degree 1,2 and 3 in  $q$  respectively):

$$F(p, p, q) = 2dF_p(q) = 0, \quad F(p, q, q) = 0 \quad \text{and} \quad F(q, q, q) = 6F(q) = 0.$$

Assuming that (32) is satisfied as soon as these preceding equations hold true leads to two possibilities, depending whether the two  $q$ -linear equations  $dF_p(q) = 0$  and (32) are colinear or not. If it is the case, then there exists  $\Lambda(p) \in \mathbf{C}^*$  such that the equality between linear forms  $\Lambda(p) \cdot \langle H, P(L_{p,\cdot}) \rangle = dF_p(\cdot)$  holds true. Since this identity holds for  $[p]$  generic in  $X$  which is non-degenerate in  $\mathbf{P}^4$  and because  $dF_p(\cdot)$  is homogeneous of degree 2 with respect to  $p$ , it follows that the  $\Lambda(p)$ 's can be taken to be such that  $p \mapsto \Lambda(p)$  be a linear form. But then this would imply that  $dF = 0$  on the hyperplane section  $X \cap \{\Lambda = 0\}$ , contradicting the assumption that  $X$  has isolated singularities.

Thus necessarily (32) cuts a genuine line in the projectivized tangent space  $\mathbf{P}(T_p X)$  which obviously contains all the tangent directions at  $p$  of the lines  $\ell_i(p)$ 's. But this would imply in particular that these six tangent directions span a 2-plane in  $T_p X$ , contradicting the general position assumption made about the leaves of  $\mathcal{LW}_X$  at a generic point. The lemma follows.  $\square$

**Remark 3.5.** *The proof above actually shows that the conclusion of Lemma 3.4 is still satisfied under the weaker assumptions that  $\dim(X_{\text{sing}}) \leq 1$ ,  $\dim(F_1(X)) = 2$  and that among the lines contained in  $X$  through a generic point, at least three are in general position. For instance, the chordal cubic  $\mathcal{C}$  satisfies these hypotheses (but in this case  $F^{\text{gen}}$  coincides with  $F_1(\mathcal{C})$  and it was known that its Plücker image in  $\mathbf{P}^9$  is non degenerate according to the discussion in §3.2.1).*

**On Abel's theorem for abelian 1-forms on the Fano surface.** We believe that some meaningful parts of Proposition 3.1 generalize in a straightforward way to any cubic threefold  $X$  with isolated singular points, at least as soon as the lines included in it define a genuine 6-web on it.<sup>27</sup> However, as is already the case for smooth cubics (cf. [CIG, p. 338]), everything that concerns the links between some differential  $k$ -forms on the Fano surface  $F = F_1(X)$  and the  $k$ -ARs of  $\mathcal{LW}_X$  is much more subtle when  $k = 1$  than when  $k = 2$ . We discuss below several approaches which could give a proof of the extension for cubics with isolated singularities of the fact that the first trace map may induce an isomorphism between a space of 1-differential forms on the associated Fano surface and the space of 1-ARs of the corresponding web of lines on the cubic. We warn the reader that nothing is proved below, rigorous investigations on this are left for a hypothetical future work.

In what follows,  $X$  stands for a fixed cubic threefold in  $\mathbf{P}^4$  with isolated singularities.

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As a first attempt, one might naively want to extend in a straightforward manner the approach discussed in the preceding paragraph to the case when  $X$  has isolated singularities: considering a smoothing  $\pi : \mathcal{X} \rightarrow B$  of  $X = X_0 = \pi^{-1}(b_0)$  over a smooth 1-dimensional base  $B$ , one defines a coherent sheaf of 'relative abelian 1-forms'  $\omega_{\mathcal{X}/B}^1$  on the total space of the associated family

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<sup>27</sup>Note that for  $X$  a smooth cubic with isolated singularities, it may happen that  $\mathcal{LW}_X$  be a genuine 6-web but not a skew one (this precisely happens for Segre's cubic  $S$  for instance). In such a situation, the space of 1-ARs of  $\mathcal{LW}_X$  may be of infinite dimension hence one cannot expect the whole Proposition 3.1 to be generalized as it is to the case of such cubic threefolds.

of Fano surfaces  $\pi : \mathcal{F} \rightarrow B$  by requiring that the following sequence of sheaves is exact  $0 \rightarrow \pi^*(\Omega_B^1) \rightarrow \omega_{\mathcal{F}}^1 \rightarrow \omega_{\mathcal{F}/B}^1 \rightarrow 0$ .<sup>28</sup> Then setting  $F_0 = F_1(X_0) = F_1(X)$ , the same arguments as in the preceding paragraph would give that the trace induces an injective linear map  $\mathbf{H}^0(F_0, \omega_{F_0}^1) \rightarrow \mathbf{AR}^{(1)}(\mathcal{LW}_X)$ , provided that the two following facts hold true:

$$(i). \text{ one has } (\omega_{\mathcal{F}/B}^1)|_{F_0} = \omega_{F_0}^1; \quad \text{and} \quad (ii). \quad h^0(F_0, \omega_{F_0}^1) = 5.$$

If one expects that these two facts are indeed satisfied, verifying it is not obvious. Abelian differentials are not known to behave particularly well under pull-back hence (i). needs to be justified. As for (ii)., it leads to the following interesting questions: according to [AK, (1.10).(ii)], denoting by  $F_0^*$  the smooth locus of  $F_0$ , one has  $\Omega_{F_0^*}^1 \simeq (\mathcal{T}^\vee)|_{F_0^*}$  (compare with (24) in the smooth case). Does this identification extend through the singularities of  $F_0$  into an isomorphism  $\omega_{F_0}^1 \simeq (\mathcal{T}^\vee)|_{F_0}$ ? And if so, is the restriction map  $V^\vee \simeq \mathbf{H}^0(G_1(\mathbf{P}(V)), \mathcal{T}^\vee) \rightarrow \mathbf{H}^0(F_0, \omega_{F_0}^1)$  injective or even an isomorphism? It does not seem easy to answer these questions, a reason for that being that  $F_0$  has rather involved singularities (in particular, its singular locus  $F_0 \setminus F_0^*$  is 1-dimensional). It is known that  $h^1(\mathcal{O}_{F_0}) = 5$  (cf. [AK, Prop. (1.15)]) but one cannot deduce directly from this that  $h^0(\omega_{F_0}^1) = 5$  since it is not clear if/how the pure Hodge structures on the first cohomology spaces  $\mathbf{H}^1(F_b, \mathbf{C}) = \mathbf{H}^{1,0}(F_b) \oplus \mathbf{H}^{0,1}(F_b)$  for  $b \in B$  with  $F_b$  smooth degenerate when  $b$  tends to  $b_0$ .

It seems to us that one might be able to deduce from the (rather technical) Hodge-theoretic considerations in [Col1] that everything discussed above indeed holds true when considering the chordal cubic  $\mathcal{C}$  and its non irreducible hence singular Fano surface  $F_1(\mathcal{C})$  (this independently of the fact that the lines in  $\mathcal{C}$  do not define a 6-web in it, but only a 3-web). This makes us confident that everything go as sketched above in the simpler case of cubics with only finitely many singular points.

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Constructing the 1-ARs of  $\mathcal{LW}_X$  by means of the abelian 1-forms of the Fano surface  $F$  is certainly interesting conceptually, but not really necessary if mainly interested in the 1-rank of this web. Considering a smoothing  $\pi : \mathcal{X} \rightarrow B$  as above, it would already be satisfying to be able to describe an analytic family of spaces  $\mathbf{H}_b^1 \subset \Omega_{\mathbf{C}(F_b)}^1$  of rational 1-forms on  $F_b = F_1(X_b)$  with  $X_b = \pi^{-1}(b)$  for any  $b \in B$ , with  $\mathbf{H}_b^1 \simeq \mathbf{H}^0(F_b, \Omega_{F_b}^1)$  for any  $b$  such that  $X_b$  is smooth and such that the trace induces an injective map  $\mathbf{H}_b^1 \rightarrow \mathbf{AR}^{(1)}(\mathcal{LW}_{X_b})$  for every  $b \in B$ , in particular for  $b = b_0$ . We see several ways to construct such an analytic family of vector spaces  $\{\mathbf{H}_b^1\}_{b \in B}$ :

- If  $X_b$  is smooth, the map  $V^\vee \rightarrow \mathbf{H}^0(F_b, \Omega_{F_b}^1)$  of **v** page 25 above can be made rather explicit (see around (12.8) in [CIG]). If by chance the construction by means of residues might be extended to  $X_0$ , it would possibly give a map  $V^\vee \rightarrow \mathbf{H}^0(F_0^*, \Omega_{F_0^*}^1)$  whose elements in the image extend as rational 1-forms on  $F_0$  and deform into holomorphic 1-forms on the smooth nearby fibers  $X_b$  for  $b \in (B, b_0)$ . Denoting by  $\mathbf{H}_{b_0}^1 \subset \Omega_{\mathbf{C}(F_0)}^1$  their span, we would thus obtain a linear map  $\mathbf{H}_{b_0}^1 \rightarrow \mathbf{AR}^{(1)}(\mathcal{LW}_X)$  whose image is expected to be 5-dimensional.

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<sup>28</sup>Here  $\omega_{\mathcal{F}}^1$  stands for the sheaf of ‘Abelian (or Barlet [Bar]) differential 1-forms on  $\mathcal{F}$ ’, which can be characterized as the meromorphic 1-forms on (analytic open subsets of)  $\mathcal{F}$  inducing  $\bar{\partial}$ -closed currents on their definition domains.

- Instead of trying to deal with 1-forms on the singular Fano surface  $F_0$ , one can rather consider working with the global holomorphic forms on a semi-abelian (hence smooth) variety naturally associated with it. Indeed, 1. for  $b \in B$  with  $X_b$  smooth, the Albanese map  $\mathbf{alb} : F_b \rightarrow \mathrm{Alb}(F_b)$  induces an isomorphism  $\mathbf{alb}^* : \mathbf{H}^0(\Omega_{\mathrm{Alb}(F_b)}^1) \rightarrow \mathbf{H}^0(F_b, \Omega_{F_b}^1)$ ; 2. an Albanese variety with an associated Albanese map has been constructed by several authors for cubics with singularities and this construction has been proven to behave well for suitable degenerating families (see *e.g.* [Gw1] for cubic with ordinary nodes only, or [CGHL] and the reference therein for more involved isolated singularities).<sup>29</sup> In the case under scrutiny,  $\{\mathrm{Alb}(F_1(X_b))\}_{b \in B}$  would be an analytic family of 5-dimensional semi-abelian varieties from which it should be possible to construct an analytic flat family of spaces  $\mathbf{H}_b^1$  with the required properties. We believe that such an approach should work for cubics with ordinary nodes [Gw1] or for those with ‘allowable singularities’ according to the terminology used in [CGHL].
- In the case when  $X_0$  has isolated singularities of the simplest type, namely ordinary nodes, an approach might be to find a nice desingularization  $\tilde{\pi} : \tilde{\mathcal{F}} \rightarrow B$  of the relative Fano surface  $\pi : \mathcal{F} \rightarrow B$  such that the central fiber  $\tilde{F}_0 = \tilde{\pi}^{-1}(0)$  be a reduced surface with normal crossing. In such a situation, the  $(1, 0)$ -component of the Hodge structure on  $\mathbf{H}^1(\tilde{F}_0, \mathbf{C})$  obtained as the limit of the standard one of the smooth Fano surfaces  $\tilde{F}_b$  for  $b \in (B, b_0)$  coincides with the space of global sections of a certain sheaf  $\Lambda_{\tilde{F}_0}^1$  of meromorphic 1-forms on  $\tilde{F}_0$  (see [Fr, §3]). The arguments sketched at the beginning of this paragraph (*cf.* our ‘first attempt’ above), if applied not to  $\mathcal{F}$  but to the desingularized family  $\tilde{\pi} : \tilde{\mathcal{F}} \rightarrow B$ , should provide the sought-after injective morphism  $\mathbf{H}^0(\tilde{F}_0, \Lambda_{\tilde{F}_0}^1) \rightarrow \mathbf{AR}^{(1)}(\mathcal{LW}_X)$ .

In [Col2], Collino describes quite explicitly the desingularization  $\tilde{\pi} : \tilde{\mathcal{F}} \rightarrow B$  when the initial non smooth  $X_0$  is Segre’s cubic primal and the limit Hodge structure on  $\tilde{F}_0$  (see Appendix B at the end of this text for more details in the case of Segre’s cubic). We believe that most of the results in this paper may be adapted to follow the strategy sketched just above when  $X_0$  is any cubic threefold with ordinary nodes.

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Finally, as a last possible way to construct the 1-ARs of  $\mathcal{LW}_X$ , we would like to mention the old and not well-known but very interesting paper [BW]. In it (in §6 more precisely), Blaschke and Walberer construct quite directly from any cubic homogeneous form defining a given cubic threefold  $X \subset \mathbf{P}^4$ , the 1-ARs not of the web by lines on  $X$  but of an associated 3-web, namely the curvilinear 3-web  $\mathcal{TW}_X$  naturally defined on the variety  $\mathcal{T}_X$  of ‘triangles included in  $X$ ’.<sup>30</sup> It would be interesting to check whether their approach can be adapted to build by hand an explicit family of dimension 5 of 1-RAs, but for  $\mathcal{LW}_X$  this time. Note that in [BW], essentially no

<sup>29</sup>Actually, in most papers dealing with singular cubics, what is constructed is a ‘generalized intermediate Jacobian’  $J(X_0)$  associated to a suitable singular cubic threefold  $X_0$ , which is a semi-abelian variety of dimension 5 (actually, the semi-abelian intermediate Jacobian  $J(X_0)$  is constructed by relating it to a Prym variety of a non ramified 2-1 cover of a plane quintic curve associated (in a non canonical way) to  $X_0$ , see [CGHL, §3.2]). For a smooth cubic  $X$ , there is a well-understood isomorphism  $J(X) \simeq \mathrm{Alb}(F_1(X))$  and it is expected (and proved in many cases) that this extends to singular cubics, or at least to some of them. The fact that most of these constructions for singular cubics are in terms of ‘intermediate Jacobians’ and not of ‘Albanese varieties’ actually is not important for our purpose.

<sup>30</sup>We will define this variety and discuss the 3-web naturally defined on it in the case of the chordal cubic  $\mathcal{C}$  in Appendix B to which we refer the reader for more details.

assumption about the singularities of  $X$  are made; it seems to be only required that  $\mathcal{TW}_X$  is a skew curvilinear 3-web whose leaves are in general position generically. It would be interesting to revisit [BW] with a modern and rigorous approach, for instance to determine precisely under which hypothesis the results contained in it hold true.

**3.3. Segre's cubic (case  $n = 3$ ).** We now focus on the case of Segre's cubic. Since it is a very well-known and studied threefold, no proof of its properties stated below will be given, the interested reader will refer to the specific references indicated in the following subsection.

**3.3.1. References.** Segre's cubic has been discussed in many papers since its discovery by Segre in 1887. As classical references, in addition to Segre's original paper [Se] (§24 to §27 therein), one can mention §19 to §31 in 'Capitolo 8°' of Bertini's book [B], [Sn], [SR, Chap. VIII, §2.32]. As for other recent references, the following ones are interesting considering our purpose but there may exist many others we are not aware of: Segre's cubic primal is discussed in several papers by Dolgachev, such as [Do1] (in particular §2 and §4 therein) or [Do2, §5]. Other modern references are: [Hu, §3.2], [Gw1, Gw2], [Col1], [Col2] and [DFL, §2]. Among the previous references, [Col2] is perhaps the most relevant one since Collino describes there the Fano surface of lines of Segre's cubic quite explicitly and studies some of its features. It is unfortunate that such an interesting text has not been published in proper form.

**3.3.2. Segre's cubic.** We recall well-known classical facts about Segre's cubic.

First, we introduce some notation: Let  $[U_1, U_2, U_3, U_4]$  be some fixed homogeneous coordinates on  $\mathbf{P}^3$  and  $x, y, z$  stand for the affine coordinates on  $\mathbf{C}^3 = \mathbf{P}^3 \setminus \{U_4 = 0\}$  corresponding to the embedding  $(x, y, z) \mapsto [x : y : z : 1]$ . As above, the  $p_i$ 's (for  $i = 1, \dots, 5$ ) stand for the following points in  $\mathbf{P}^3$ :  $p_i = [\delta_i^j]_{j=1}^4$  for  $i = 1, \dots, 4$  and  $p_5 = [1, \dots, 1]$ . We set  $P = \{p_i\}_{i=1}^5$ .

In what follows, without any supplementary precision,  $i, j, k, l, m$  will stand for pairwise distinct elements of  $\{1, \dots, 5\}$  (hence such that  $\{i, j, k, l, m\} = \{1, \dots, 5\}$ ). For such indices, we denote by

- $L_{ij}$  the line passing through  $p_i$  and  $p_j$ : i.e.  $L_{ij} = \langle p_i, p_j \rangle \subset \mathbf{P}^3$ ;
- $P_{ijk}$  or  $P^{lm}$  the 2-plane containing  $p_i, p_j$  and  $p_k$ : i.e.  $P_{ijk} = P^{lm} = \langle p_i, p_j, p_k \rangle \subset \mathbf{P}^3$ .

Recall (cf. §2.2.1 above) that  $\mathcal{L}_3$  is the linear system of quadric surfaces in  $\mathbf{P}^3$  passing through all the  $p_i$ 's:  $\mathcal{L}_3 = |2H - \sum_{i=1}^5 p_i|$ . Its elements are exactly the surfaces in  $\mathbf{P}^3$  cut out by homogeneous equations  $\sum_{i < j} a_{ij} U_i U_j = 0$  for some coefficients  $a_{ij}$ 's whose total sum is zero.

In the coordinates  $x, y, z$ , the rational map associated to  $\mathcal{L}_3$  can be taken to be

$$(33) \quad \varphi_3 : (x, y, z) \mapsto [x(z-y) : x(1-y) : y(z-x) : y(1-x) : z-xy].$$

Its base-points in  $\mathbf{C}^3$  are origin  $\mathbf{0} = (0, 0, 0)$  and the point  $\mathbf{1} = (1, 1, 1)$  which correspond to  $p_4$  and  $p_5$  respectively. We will again denote by  $\varphi_3$  the extension to  $\mathbf{P}^3$  of the above map (as a rational map). By definition, the variety  $V_3$  is the closure of  $\varphi_3(\mathbf{P}^3 \setminus P)$  in  $\mathbf{P}^4$ . It can be verified that it is the cubic hypersurface cut out by the following equation:

$$X_1 X_2 X_3 - X_1 X_2 X_4 - X_1 X_3 X_4 + X_1 X_4 X_5 + X_2 X_3 X_4 - X_2 X_3 X_5 = 0.$$

For any distinct  $p_i$  and  $p_j$ , the line  $L_{ij} = \langle p_i, p_j \rangle$  is contracted onto a singular point of  $V_3$ , which will be denoted by  $p_{ij}$ . The  $p_{ij}$ 's are easy to determine: they are the six vertices of the canonical simplex in  $\mathbf{P}^4$  (namely the points  $[\delta_i^j]_{j=1}^5$  for  $i = 1, \dots, 5$  and  $[1 : 1 : 1 : 1 : 1]$ ) and the following

four other points  $[0 : 0 : 1 : 1 : 1]$ ,  $[0 : 1 : 0 : 1 : 1]$ ,  $[1 : 1 : 0 : 0 : 1]$  and  $[1 : 0 : 1 : 0 : 1]$ . Hence the  $p_{ij}$ 's are 10 pairwise distinct singular points of  $V_3$ . On another hand, the map  $\varphi_3$  has rank 3 outside the union of the 10 lines  $L_{ij}$ 's, hence  $V_3$  is smooth outside the  $p_{ij}$ 's. Because these singular points all are ordinary nodes (as an easy analysis shows), one deduces that  $V_3$  coincides with Segre's 10-nodal cubic hypersurface. For that reason, we will denote this threefold by  $\mathcal{S}$  from now on. We will write  $N_{\mathcal{S}} = \{p_{ij} \mid 1 \leq i < j \leq 5\}$  for the set of its nodes.

Let us describe the 2-planes and some lines contained in  $\mathcal{S}$  and their incidence with the nodes. For any  $i, j, k$  (pairwise distinct), the quadric of  $\mathcal{L}_3$  passing through a generic point  $p \in P_{ijk}$  is the union of  $P_{ijk}$  itself with the 2-plane passing through  $p_l, p_m$  and  $p$ . It follows that  $\varphi_3$  maps  $P_{ijk}$  onto a plane in  $\mathcal{S}$ , that we will denote  $P_{ijk}$  too. These ten 2-planes  $P_{ijk}$ 's are pairwise distinct. In addition to them,  $\mathcal{S}$  contains five other planes, which can be described as follows: let  $\pi : \widetilde{\mathcal{S}} = \text{Bl}_p \mathbf{P}^3 \rightarrow \mathbf{P}^3$  be the blow-up at the five base-points of  $|\mathcal{L}_3|$ . Then  $\widetilde{\varphi}_3 = \varphi_3 \circ \pi : \widetilde{\mathcal{S}} \rightarrow \mathcal{S}$  is a morphism and if  $E_i$  stands for the exceptional divisor corresponding to  $p_i$  (i.e.  $E_i = \pi^{-1}(p_i)$ ) for any  $i$ , then  $P_i = \widetilde{\varphi}_3(E_i)$  is a 2-plane in Segre's cubic. In this way, we get five other planes included in  $\mathcal{S}$  which with the 15 described above form the whole set of 2-planes contained in Segre's cubic, which therefore has cardinality 16. In order to have an uniform notation, we set  $P^{ij} = P_{klm}$  and  $P^{i6} = P_i$ , here for any  $i, j$  such that  $1 \leq i \leq j \leq 5$ .

**Remark 3.6.** *There is a nice uniform interpretation of the planes  $P^{u,v}$  for any  $1 \leq u < v \leq 6$  in terms of Knudsen-Mumford modular compactification  $\overline{\mathcal{M}}_{0,6}$ . According to a well-known result of Kapranov, the blow-up of  $\widetilde{\mathcal{S}}$  along the union of the strict transforms by  $\pi$  of the lines  $L_{i,j}$  is isomorphic to  $\overline{\mathcal{M}}_{0,6}$  and any plane  $P^{uv}$  in Segre's cubic is the image by the morphism  $\overline{\mathcal{M}}_{0,6} \rightarrow \mathcal{S}$  associated to  $\varphi_3$ , of the boundary divisor  $\Delta_{uv}$  of  $\overline{\mathcal{M}}_{0,6}$  (naturally isomorphic to  $\overline{\mathcal{M}}_{0,5}$ ) formed by limits of configurations of 6 points on the projective line obtained when making the  $u$ -th and  $v$ -points of the configurations collapse. A generic point of  $\Delta^{uv}$  is a rational curve  $C_1 \cup C_2$  with two irreducible components  $C_1, C_2$  intersecting at a node, with three marked points on  $C_1$ , namely the  $u$ -th, the  $v$ -th and the node.*

One easily determines the incidence relations between the nodes of  $\mathcal{S}$  and the 2-planes included in it: for any indices  $i, j, k, l, m$  such that their set coincides with  $\{1, \dots, 5\}$ , one has

- $P^{ij} \cap N_{\mathcal{S}} = \{p_{ij}, p_{kl}, p_{km}, p_{lm}\}$  and  $P^{i6} \cap N_{\mathcal{S}} = \{p_{ij}, p_{ik}, p_{il}, p_{im}\}$ ; moreover, the four nodes in a plane included in  $\mathcal{S}$  are in general position in it.
- The set of 2-planes in  $\mathcal{S}$  adjacent to  $p_{ij}$  is  $\{P^{ij}, P^{kl}, P^{km}, P^{lm}, P^{i6}, P^{j6}\}$ .

Finally, for any distinct pairs  $(i, j)$  and  $(k, l)$  (with  $1 \leq i < j \leq 5$  and similarly for  $k$  and  $l$ ), the line  $\ell_{ij,kl} = \langle p_{ij}, p_{kl} \rangle \subset \mathbf{P}^4$  is included in  $\mathcal{S}$  as well. From the preceding point, it follows that these lines are pairwise distinct hence are 45 in number. These lines serve to describe the intersections of two distinct planes of Segre's cubic. For  $i, j, k, l$  in  $\{1, \dots, 5\}$  such that  $\{i, j\} \neq \{k, l\}$ , the intersection of  $P^{ij}$  with  $P^{kl}$  is either a point or a line:

$$(34) \quad \begin{aligned} P^{ij} \cap P^{i6} &= \{p_{ij}\}, P^{i6} \cap P^{j6} = \{p_{ij}\} \text{ and } P^{ij} \cap P^{ik} = \{p_{lm}\} && (\text{intersection of dim. } 0); \\ P^{ij} \cap P^{kl} &= \ell_{ij,kl} = \langle p_{ij}, p_{kl} \rangle, P^{ij} \cap P^{k6} = \ell_{kl,km} = \langle p_{kl}, p_{km} \rangle && (\text{intersection of dim. } 1). \end{aligned}$$

**Remark 3.7.** *It may be helpful to indicate how the lines  $\ell_{ij,kl}$ 's can be realized 'at the source' (that is, in terms of the  $\mathbf{P}^3$  serving as source of the rational parametrization  $\varphi_3$  of  $\mathcal{S}$ ). When  $\{i, j\} \cap \{k, l\}$  is empty,  $\ell_{ij,kl}$  is just the image of the line  $P^{ij} \cap P^{kl} \subset \mathbf{P}^3$  by  $\varphi_3$ . Giving an interpretation of  $\ell_{ij,kl}$  for  $i, j, k$  pairwise distinct is a bit more involved: the closure of the set of tangent directions at  $p_i$*

of the lines joining  $p_i$  to a generic point of  $\langle p_j, p_k \rangle$  is a line in  $\mathbf{P}(T_{p_i} \mathbf{P}^3)$ . We denote by  $l_{i,jk}$  the corresponding line in the exceptional divisor  $E_i = \pi^{-1}(p_i) \subset \widetilde{\mathcal{S}}$ . Then an easy computation shows that, by restriction,  $\widetilde{\varphi}_3$  induces a linear isomorphism from  $l_{i,jk}$  onto  $\ell_{ij,ik} \subset \mathcal{S}$ .

Beware that our notations above for the lines through two nodes of  $\mathcal{S}$  are lightly different from the ones, namely  $L[(ij), (kl)]$ , used in [Col2]. In Collino's text,  $i, j, k, l$  stand for indices ranging from 1 to 6 (with  $\{i, j\} \neq \{k, l\}$  of course). When all fourth are less than or equal to 5, then Collino's  $L[(ij), (kl)]$  coincides with the line we denote by  $\ell_{ij,kl}$ . But if one among these indices is equal to 6, for example  $l$ , then one has instead  $L[(ij), (k6)] = \langle p_{ik}, p_{jk} \rangle = \ell_{ik,jk}$ .

**3.3.3. Fano surface of Segre's cubic.** Everything here is taken from [Col2, §2.2.8]. For a line  $L$  in  $\mathbf{P}^4$ , we write  $[L]$  for the point of the Grassmann manifold  $G_1(\mathbf{P}^4)$  to which it corresponds. From the preceding description of the nodes and of the planes in  $\mathcal{S}$ , one can deduce a combinatorial description of its Fano surface denoted by  $\Sigma$ , that we will often identify with its image by the Plücker embedding

$$(35) \quad \Sigma = F_1(\mathcal{S}) \subset G_1(\mathbf{P}V) \hookrightarrow \mathbf{P}(\wedge^2 V) \simeq \mathbf{P}^9.$$

It is a non irreducible surface with 21 irreducible components, all of which are rational and can be described easily:

- for  $1 \leq i < j \leq 6$ , the set of lines of  $P^{ij}$  forms an irreducible component of  $\Sigma$ , denoted by  $\Sigma^{ij}$ . Any such component is isomorphic to  $\mathbf{P}^2$  and there are 15 components of this type;
- for  $i = 1, \dots, 6$ , the images by  $\varphi_3$  of the lines through  $p_i$  in  $\mathbf{P}^3$  if  $i \leq 5$ , or of the twisted cubics through all the  $p_k$ 's when  $i = 6$ , give rise to an irreducible component of  $\Sigma$ , denoted by  $\Sigma(i)$ . For  $i \leq 5$ , it identifies with the blow up at the  $[L_{ij}]$ 's (with  $j \in \{1, \dots, 5\} \setminus \{i\}$ ) of the projective plane parametrizing the lines of  $\mathbf{P}^3$  passing through  $p_i$ . Therefore  $\Sigma(i)$  is isomorphic to  $\overline{\mathcal{M}}_{0,5}$ . It can be verified that the same holds true for  $\Sigma(6)$  too;
- the images of the components of  $\Sigma$  by the Plücker embedding are not difficult to describe:
  - for any  $i, j$  distinct,  $\Sigma^{ij}$  is the set of lines of a 2-plane in  $\mathbf{P}V \simeq \mathbf{P}^4$  hence its image in  $\mathbf{P}^9$  is a 2-plane as well;
  - for each  $i$ , the restriction of (35) to  $\Sigma(i)$  coincides with the anticanonical embedding hence its image is a quintic del Pezzo surface (which spans a 5-plane in  $\mathbf{P}^9$ ).

Our notations are unfortunately not well suited to deal with the birational  $\mathfrak{S}_6$ -symmetries of  $\mathcal{S}$  and of its Fano surfaces. In order to describe the incidence relations between the 21 components of  $\Sigma$ , it is much more convenient to use Collino's notations  $L[(ij), (kl)]$  (recalled above) for the lines joining two nodes of  $\mathcal{S}$ . Then the intersections between the different components of  $\Sigma$  can be seen to be as follows where now  $i, j, k, l, m, n$  stand for elements ranging in  $\{1, \dots, 6\}$  (with the pairs  $\{i, j\}$  and  $\{k, l\}$  distinct, etc) :

- the relations (34) can be stated in equivalent terms for the  $\Sigma^{ij}$ 's: the two planes  $\Sigma^{ij}$  and  $\Sigma^{kl}$  do not meet if  $\{i, j\} \cap \{k, l\} \neq \emptyset$  whereas one has  $\Sigma^{ij} \cap \Sigma^{kl} = \{L[(ij), (kl)]\}$  otherwise;
- as for the intersection of two del Pezzo components, when  $i$  and  $j$  are distinct then  $\Sigma(i) \cap \Sigma(j)$  is the set of three points of  $\Sigma$  corresponding to the lines  $L[(kl), (mn)]$  for all  $k, l, m, n$  such that  $\{i, j, k, l, m, n\} = \{1, \dots, 6\}$ ;



- there are two possibilities for the intersection of a plane  $\Sigma^{ij}$  with a del Pezzo component  $\Sigma(k)$ : first, one obviously has  $\Sigma^{ij} \cap \Sigma(k) = \emptyset$  if  $k \in \{i, j\}$ . At the opposite, when  $i, j$  and  $k$  are pairwise distinct, then  $\Sigma(k)$  intersects  $\Sigma^{ij}$  along a line denoted by  $R(k, ij)$ , which is isomorphic to  $\overline{M}_{0,4}$ . In our non symmetric presentation, for  $k < 6$  the line  $R(ij, k)$  corresponds to the (images by  $\varphi_3$  of the) lines joining  $p_k$  to a point of  $L_{lm} = \langle p_l, p_m \rangle \subset \mathbf{P}^3$  and the three boundary points correspond to the lines passing through  $p_l, p_m$  and the point of intersection in  $\mathbf{P}^3$  of the line  $L_{ij} = \langle p_i, p_j \rangle$  with the plane  $P^{ij} = \langle p_k, p_l, p_m \rangle$ .

A noteworthy last point concerns the incidence between the lines joining two nodes of  $\mathcal{S}$  but considered as points of the Fano surface  $\Sigma$ , and the 2-dimensional components of the latter: using Collino's notation, when  $i, j, k, l, m, n$  stand for the elements of  $\{1, \dots, 6\}$  then the two del Pezzo's  $\Sigma(m), \Sigma(n)$  and the two planes  $\Sigma^{ij}$  and  $\Sigma^{kl}$  are exactly the components of  $\Sigma$  to which  $L[(ij), (kl)] \in \Sigma$  belongs. Essentially quoting Collino here, one concludes by saying that *'this shows that  $\Sigma = F(\mathcal{S})$  is not a normal crossing surface: at any point  $L[(ij), (kl)]$ , four irreducible components of  $\Sigma$  meet, and moreover these components split in two couples which intersect locally only at this point'*.

**3.3.4. Abelian relations and rank of  $\mathcal{W}_{0,6}$ .** We have seen above that  $\mathcal{W}_{0,6}$  identifies with the 6-web by lines  $\mathcal{LW}_{\mathcal{S}}$  on Segre's cubic. Our goal here is to establish that Proposition C of the Introduction, which is stated for a 'sufficiently generic' hypercubic of  $\mathbf{P}^4$ , holds true for  $\mathcal{S}$  as well:

**Proposition 3.8.** *1. The trace induces an isomorphism  $\text{Tr} : H^0(\Sigma, \omega_{\Sigma}^2) \simeq AR(\mathcal{LW}_{\mathcal{S}})$ .*

*2. Therefore  $\mathcal{W}_{0,6}$  is an algebraizable 6-web of maximal 2-rank  $\text{rk}(\mathcal{W}_{0,6}) = h^0(\Sigma, \omega_{\Sigma}^2) = 10$ .*

**An elementary and explicit proof of Proposition 3.8.** The key fact behind the elementary effective approach discussed below is point 2. in (29), namely that the dualizing sheaf  $\omega_F^2$  of the Fano surface  $F$  of a cubic threefold  $X$  with finitely many ordinary singular points coincides with the pull-back of  $\mathcal{O}_{\mathbf{P}^9}(1)$  under the Plücker embedding  $F \subset G_1(\mathbf{P}^4) \hookrightarrow \mathbf{P}^9$ . Since the dualizing sheaf  $\omega_F$  naturally identifies with the sheaf  $\omega_F^2$  of abelian differential 2-forms on  $F$  (cf. [Bar]), the aforementioned fact allows to get explicit expressions for a basis of  $H^0(F, \omega_F^2)$  as soon as the embedding  $F \hookrightarrow \mathbf{P}^9$  is explicitly known. One is reduced to verify whether the elements of this basis give rise to abelian relations for the associated linear web  $\mathcal{LW}_X$ , which can be done through direct computations. In the sequel, we apply this approach to Segre's cubic  $\mathcal{S}$  in order to describe the ARs then to compute the rank of  $\mathcal{W}_{0,6} \simeq \mathcal{LW}_{\mathcal{S}}$ .

Recall that to a rank 2 matrix  $M \in \text{Mat}_{2,5}(\mathbf{C})$ , one can associate the line in  $\mathbf{P}^4$ , denoted by  $\langle M \rangle$ , passing through the two points whose homogeneous coordinates are the two lines of  $M$ . hereafter, we will use the well-known fact that  $\text{Mat}_{2,5}(\mathbf{C}) \dashrightarrow G_1(\mathbf{P}^4)$ ,  $M \mapsto \langle M \rangle$  is a birational chart.

Let  $(x, y, z)$  stand for the coordinates of a generic point  $p$  of  $\mathbf{P}^3$ ; here are the rational first integrals for  $\mathcal{W}_{0,6}$  we will work with:  $U_1 = (y, z)$ ,  $U_2 = (x, z)$ ,  $U_3 = (x, y)$ ,  $U_4 = (x/z, y/z)$ ,  $U_5 = ((x-1)/(z-1), (y-1)/(z-1))$  and  $U_6 = (x(z-1)/(z(x-1)), y(z-1)/(z(y-1)))$ .

For each  $i = 1, \dots, 6$ , solving  $U_i(x, y, z) = (u_i, v_i)$  with  $u_i, v_i \in \mathbf{Q}(x, y, z)$ , one gets easily a parametrization  $\xi_{i,p} : \mathbf{P}^1 \rightarrow \mathbf{P}^3$  of the  $i$ -th leaf of  $\mathcal{W}_{0,6}$  through  $p$ , whose coefficients are rational in  $u_i$  and  $v_i$ . Then evaluating  $\varphi \circ \xi_{i,p} : \mathbf{P}^1 \rightarrow \mathbf{P}^4$  at two generic points, for instance 0 and  $\infty$ , one easily computes a  $2 \times 5$  matrix  $M_i \in \text{Mat}_{2,5}(\mathbf{Q}(u_i, v_i))$  such that the  $i$ -th leaf of  $\mathcal{LW}_{\mathcal{S}}$  through  $\varphi_3(p)$  is the line passing through the two points whose homogeneous coordinates are the two lines

of  $M_i$ . Of course, the  $M_i$ 's are not unique but some elementary computations give that they can be taken to be the following ones:

$$\begin{aligned}
M_1 &= \begin{bmatrix} 0 & 0 & u_1 v_1 & u_1 & v_1 \\ u_1 - v_1 & u_1 - 1 & u_1 & u_1 & u_1 \end{bmatrix} \\
M_2 &= \begin{bmatrix} u_2 v_2 & u_2 & 0 & 0 & v_2 \\ u_2 & u_2 & u_2 - v_2 & u_2 - 1 & u_2 \end{bmatrix} \\
M_3 &= \begin{bmatrix} u_3 v_3 & u_3(v_3 - 1) & u_3 v_3 & v_3(u_3 - 1) & v_3 \\ u_3 & 0 & v_3 & 0 & 1 \end{bmatrix} \\
M_4 &= \begin{bmatrix} 0 & u_4 & 0 & v_4 & 1 \\ u_4(1 - v_4) & -u_4 v_4 & v_4(1 - u_4) & -u_4 v_4 & -u_4 v_4 \end{bmatrix} \\
M_5 &= \begin{bmatrix} u_5(v_5 - 1) & u_5 v_5 & v_5(u_5 - 1) & u_5 v_5 & u_5 v_5 \\ (v_5 - 1)(1 - u_5) & v_5(1 - u_5) & (v_5 - 1)(1 - u_5) & u_5(1 - v_5) & (v_5 - 1)(1 - u_5) \end{bmatrix} \\
\text{and } M_6 &= \begin{bmatrix} 0 & u_6 & 0 & v_6 & 1 \\ u_6(v_6 - 1) & 0 & v_6(u_6 - 1) & 0 & (v_6 - 1)(u_6 - 1) \end{bmatrix}.
\end{aligned}$$

The map  $\mathbf{C}^2 \dashrightarrow \Sigma(i)$ ,  $(u_i, v_i) \mapsto \langle M_i \rangle$  is easily seen to be birational, from which it follows that  $(u_i, v_i)$  can be taken as global rational coordinates on  $\Sigma(i)$  for any  $i = 1, \dots, 6$ . It is then immediate to compute a parametrization of the image  $\widetilde{\Sigma}(i)$  of the component  $\Sigma(i) \subset \Sigma$  by the Plücker embedding  $\rho : \Sigma \subset G_1(\mathbf{P}^4) \hookrightarrow \mathbf{P}^9$ . Indeed, denoting by  $\Delta_{a,b}(M_i)$  the  $2 \times 2$  minor of  $M_i$  corresponding to its  $a$ -th and  $b$ -th columns (in this order) for any  $(a, b)$  such that  $1 \leq a < b \leq 5$ , the aforementioned parametrization is given by

$$\kappa_i : (u_i, v_i) \dashrightarrow \left[ \Delta_{a,b}(M_i) \right]_{1 \leq a < b \leq 5} \in \mathbf{P}^9.$$

Since  $\kappa_i$  corresponds to the expression, in the rational coordinates  $u_i, v_i$ , of the restriction on  $\Sigma(i)$  of the canonical map  $\kappa = \kappa_{|\omega_\Sigma^2|} : \Sigma \dashrightarrow \mathbf{P}H^0(\Sigma, \omega_\Sigma^2)^\vee$ , there exists a rational function  $\sigma_i = \sigma_i(u_i, v_i)$  (to be made explicit further below) such that

$$\left\{ \Delta_{a,b}(M_i)(\sigma_i du_i \wedge dv_i) \mid 1 \leq a < b \leq 5 \right\}$$

spans of the space of rational 2-forms  $\iota_i^*(H^0(\Sigma, \omega_\Sigma^2))$  on  $\Sigma(i)$ .<sup>31</sup> Actually, working a bit further, one verifies that

- (1) for any  $i$ , the composition  $\rho \circ \iota_i : \Sigma(i) \subset \Sigma \rightarrow \mathbf{P}^9$  is given by the complete anticanonical linear system  $|K_{\Sigma(i)}^{-1}|$ , hence  $\widetilde{\Sigma}(i) = \rho(\Sigma(i))$  is a quintic del Pezzo surface which spans a  $\mathbf{P}^5$  in the ambient  $\mathbf{P}^9$ ;

- (2) the union of the  $\widetilde{\Sigma}(i)$ 's spans the whole ambient projective space, that is  $\langle \cup_{i=1}^6 \widetilde{\Sigma}(i) \rangle = \mathbf{P}^9$ .

It follows that the linear map  $H^0(\Sigma, \omega_\Sigma^2) \rightarrow \oplus_{i=1}^6 \Omega_{\mathbf{C}(\Sigma(i))}^2$ ,  $\omega \mapsto (\omega|_{\Sigma(i)})_{i=1}^6$  is injective and consequently, the set of 6-tuples of rational 2-forms

$$(36) \quad \left\{ \left( \Delta_{a,b}(M_i)(\sigma_i du_i \wedge dv_i) \right)_{i=1}^6 \in \oplus_{i=1}^6 \Omega_{\mathbf{C}(\Sigma(i))}^2 \mid 1 \leq a < b \leq 5 \right\}$$

can be identified with a basis of the space of global abelian differential 2-forms on  $\Sigma$ .

<sup>31</sup>More prosaically,  $\iota_i^*(H^0(\Sigma, \omega_\Sigma^2))$  is the image of the restriction map  $H^0(\Sigma, \omega_\Sigma^2) \rightarrow \Omega_{\mathbf{C}(\Sigma(i))}^2$ ,  $\omega \mapsto \omega|_{\Sigma(i)}$ .

At this point, it is straightforward to give an explicit expression in the rational coordinates  $x, y, z$ , for the trace of the 2-form  $\omega_{a,b} \in \mathbf{H}^0(\Sigma, \omega_\Sigma^2)$  corresponding to the 6-tuple in (36) associated to the  $(a, b)$ -minor: for any  $a, b$  such that  $1 \leq a < b \leq 5$ , one has

$$(37) \quad \text{Tr}(\omega_{a,b}) = \sum_{i=1}^6 \Delta_{a,b}(M_i) (\sigma_i du_i \wedge dv_i)$$

where all the  $u_i$ 's and  $v_i$ 's are now viewed as rational functions of  $x, y, z$ , hence any term of the right-hand side sum has to be considered as a rational 2-form in the variables  $x, y$  and  $z$ .

An easy computation gives that the only 6-tuple of functions  $(\sigma_i)_{i=1}^6$  with  $\sigma_i = \sigma_i(u_i, v_i)$  such that the RHS of any expression (37) vanishes identically is a nonzero constant multiple of

$$\left( \frac{1}{m_5(u_i, v_i)} \right)_{i=1}^6$$

where  $m_5$  stands for the ‘ $\mathcal{M}_{0,5}$ -polynomial’ defined by  $m_5(u, v) = uv(u-1)(v-1)(u-v)$ .

We thus have obtained the following result which has to be seen as an explicit version of Proposition 3.8:

**Proposition 3.9.** *In the coordinates  $x, y, z$  and relatively to the first integrals  $U_i = (u_i, v_i)$ ,  $i = 1, \dots, 6$ , the ARs of  $\mathcal{W}_{0,6}$  are all rational and their space admits as a basis the set of 6-tuples of rational 2-forms (for all  $a, b$  such that  $1 \leq a < b \leq 5$ )*

$$(38) \quad \left( \frac{\Delta_{a,b}(M_i)}{m_5(u_i, v_i)} (du_i \wedge dv_i) \right)_{i=1}^6$$

which all satisfy the relation  $\sum_{i=1}^6 (\Delta_{a,b}(M_i)/m_5(u_i, v_i))(du_i \wedge dv_i) = 0$  identically.

Since the  $U_i$ 's as well as the  $M_i$ 's are all explicit, it is just an elementary computational matter to make the ARs (38) explicit. For instance, the one corresponding to  $(a, b) = (1, 2)$  is

$$\frac{u_2 du_2 \wedge dv_2}{v_2(u_2-1)(u_2-v_2)} - \frac{u_3 du_3 \wedge dv_3}{v_3(u_3-1)(u_3-v_3)} + \frac{u_4 du_4 \wedge dv_4}{v_4(u_4-1)(u_4-v_4)} - \frac{u_6 du_6 \wedge dv_6}{v_6(u_6-1)(u_6-v_6)} = 0$$

and that associated to  $(a, b) = (4, 5)$  is

$$(39) \quad \frac{du_1 \wedge dv_1}{v_1(u_1-1)(v_1-1)} - \frac{du_2 \wedge dv_2}{u_2(v_2-1)(u_2-v_2)} + \frac{du_3 \wedge dv_3}{u_3(v_3-1)(u_3-v_3)} \\ - \frac{du_4 \wedge dv_4}{(u_4-1)(u_4-v_4)} + \frac{du_5 \wedge dv_5}{(u_5-1)(u_5-v_5)} + \frac{du_6 \wedge dv_6}{u_6(u_6-v_6)} = 0.$$

Realizing that all the ARs of  $\mathcal{W}_{0,6}$  were rational in some natural coordinates led us to doubt about the validity of the decomposition (8) of  $\mathbf{AR}(\mathcal{W}_{0,6})$  as a  $\mathfrak{S}_6$ -module claimed in [Da2]. In order to clear this up, we have investigated the birational action of  $\mathfrak{S}_6$  on the ARs (38) to come to the conclusion that there was indeed an error on this point.

**3.3.5. The representation of  $\mathfrak{S}_6$  on  $AR_C(\mathcal{W}_{0,6})$ .** We use below notations almost identical to those used in the preceding subsection. Here  $x, y, z$  stand for the rational coordinates corresponding to the map  $\mathbf{C}^3 \dashrightarrow \mathcal{M}_{0,6}$ ,  $(x, y, z) \mapsto [\infty, 0, 1, x, y, z]$ . The first integrals we work with here for  $\mathcal{W}_{0,6}$  are the six rational first integrals  $U_i = (u_i, v_i) : \mathbf{C}^3 \dashrightarrow \mathbf{C}^2$  of the previous subsection, with the only difference that we have exchanged  $U_1$  and  $U_3$  (namely, one takes  $U_1 = (x, y)$  and  $U_3 = (y, z)$  below while the others  $U_i$ 's are left unchanged).

The associated normals  $\Omega_i = du_i \wedge dv_i$  for  $i = 1, \dots, 6$  are easy to compute: the sheaf of 2-forms on  $\mathbf{C}^2$  is globally spanned by  $\Omega_1 = dx \wedge dy$ ,  $\Omega_2 = dx \wedge dz$  and  $\Omega_3 = dy \wedge dz$  and as global rational 2-forms, the others  $\Omega_i$ 's are determined by the following matricial relation:

$$(40) \quad \begin{bmatrix} \Omega_4 \\ \Omega_5 \\ \Omega_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{z^2} & -\frac{y}{z^3} & \frac{x}{z^3} \\ \frac{1}{(z-1)^2} & -\frac{y-1}{(z-1)^3} & \frac{x-1}{(z-1)^3} \\ \frac{1}{(x-1)^2(y-1)^2z^2} & -\frac{y(z-1)}{(x-1)^2(y-1)z^3} & \frac{x(z-1)}{(x-1)(y-1)^2z^3} \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

The transformations

$$\mathcal{T} : [p_i]_{i=1}^6 \longmapsto [p_1, p_2, p_3, p_5, p_4, p_6] \quad \text{and} \quad \mathcal{C} : [p_i]_{i=1}^6 \longmapsto [p_6, p_1, \dots, p_5]$$

are two automorphisms of  $\mathcal{M}_{0,6}$  which correspond to the transposition  $\tau = (45)$  and to the cycle  $c = (1, \dots, 6)$  of  $\mathfrak{S}_6$  respectively. The corresponding rational maps in the affine coordinates  $(x, y, z)$  are given respectively by

$$(41) \quad \mathcal{T} : (x, y, z) \mapsto (y, x, z) \quad \text{and} \quad \mathcal{C} : (x, y, z) \mapsto \left( \frac{1}{x}, \frac{x-z}{(x-1)z}, \frac{y-z}{(y-1)z} \right).$$

Clearly,  $\mathcal{T}$  and  $\mathcal{C}$  generate the image of  $\text{Aut}(\mathcal{M}_{0,6}) \simeq \mathfrak{S}_6$  in  $\mathbf{Bir}_3$ . Their actions under pull-backs on the  $\Omega_i$ 's can be easily computed. One verifies that they are given by the following relations:

$$(42) \quad \begin{aligned} (\mathcal{T}^*(\Omega_i))_{i=1}^6 &= (-\Omega_1, \Omega_3, \Omega_2, -\Omega_4, -\Omega_5, -\Omega_6) \\ \text{and} \quad (\mathcal{C}^*(\Omega_i))_{i=1}^6 &= (\eta_2 \Omega_2, \eta_3 \Omega_3, \Omega_6, \tilde{\eta}_5 \Omega_5, \eta_1 \Omega_1, \tilde{\eta}_4 \Omega_4), \end{aligned}$$

with  $\eta_i = \eta(U_i) = \eta(u_i, v_i)$  for any  $i$  (and similarly for  $\tilde{\eta}_i$ ), where  $\eta$  and  $\tilde{\eta}$  are standing for the rational functions  $\eta(u, v) = (v-1)/(v^3(u-1)^2)$  and  $\tilde{\eta}(u, v) = \eta(1+u, 1-v) = v/(u^2(v-1)^3)$ .

From (42), it follows that the action on  $\mathcal{W}_{0,n+3} = (\mathcal{F}_i)_{i=1}^6$  induced by  $\mathcal{T}$  and  $\mathcal{C}$  are given respectively by the transposition  $\tau = (23)$  and the 6-cycle  $c = (123645)$ : for  $i = 1, \dots, 6$ , one has

$$(43) \quad \mathcal{T}^*(\mathcal{F}_i) = \mathcal{F}_{\tau(i)} \quad \text{and} \quad \mathcal{C}^*(\mathcal{F}_i) = \mathcal{F}_{c(i)}.$$

**An explicit combinatorial basis of  $AR(\mathcal{W}_{0,6})$ .** Some of the ARs (38) are combinatorial but not all of them (cf. (39)). But since these abelian relations form a basis of  $AR(\mathcal{W}_{0,6})$ , it is just a matter of computing to construct all the combinatorial ARs of  $\mathcal{W}_{0,6}$ . We want to do that explicitly here.

We set  $I = \{1, \dots, 6\}$  and  $I_{\neq}^2$  stands for the set of all pairs  $(i, j) \in I^2$  such that  $i < j$ . For any  $(i, j)$  in this set, one denotes by  $AR_{ij}$  the non trivial AR for the 4-subweb  $\mathcal{W}_{\overline{ij}} = \mathcal{W}(U_k \mid k \in I \setminus \{i, j\})$  of  $\mathcal{W}_{0,6}$ . Actually,  $AR_{ij}$  is only defined up to multiplication by a non zero scalar but this ambiguity will be irrelevant with regard to what we are going to discuss in the sequel.

As it can easily be verified, each row of the matrix page 45, denoted by  $M_{AR}$  in what follows, encodes an abelian relation  $AR_{ij}$  (where therefore  $i$  and  $j$  correspond to the zero coefficients): denoting by  $m_{ab}$  (for  $a$  and  $b$  ranging from 1 to 10 and 6 respectively) its coefficients, then  $m_{ab}$  is a function of  $U_b$  for any  $b$  (hence one can write  $m_{ab} = m_{ab}(U_b)$ ) and for any  $a = 1, \dots, 10$ , the following relation  $\sum_{b=1}^{10} m_{ab}(U_b) \Omega_b = 0$  holds true identically.

One denotes by  $AR(a)$  the combinatorial AR corresponding to the previous relation (hence  $AR(1) = AR_{13}, \dots, AR(10) = AR_{56}$ ). By elementary linear algebra, one verifies that the  $AR(a)$ 's for  $a = 1, \dots, 10$  are linearly independent abelian relations, from which we get the

**Proposition 3.10.** *1. The space  $AR_C(\mathcal{W}_{0,6})$  has dimension 10.*

*2. Consequently  $AR(\mathcal{W}_{0,6}) = AR_C(\mathcal{W}_{0,6})$  and  $\mathcal{W}_{0,6}$  has maximal rank.*

We remark that the five combinatorial abelian relations  $AR_{ij}$  which do not correspond to one of the lines of  $M_{AR}$  correspond to the pairs (12), (23), (36), (46) and (45) hence one deduces a more invariant way to describe the basis of  $AR(\mathcal{W}_{0,6})$  formed by the  $AR(a)$ 's for  $a = 1, \dots, 10$ :

**Fact 3.11.** *The family of  $AR_{ij}$ 's for all pairs  $(i, j) \in I_{\neq}^2$  minus the five abelian relations  $(C^\ell)^*(AR_{12})$  for  $\ell = 0, \dots, 4$ , form a basis of  $AR(\mathcal{W}_{0,6})$ .*

The interest of formulating things that way is that it naturally asks whether the following straightforward generalization holds true or not:

**Question 3.12.** *For any odd integer  $n \geq 2$ , does the family of  $AR_{ij}$ 's for all pairs  $(i, j) \in I_{\neq}^2$  minus the  $n + 2$  abelian relations  $(C^\ell)^*(AR_{12})$  for  $\ell = 0, \dots, n + 1$ , form a basis of  $AR(\mathcal{W}_{0,n+3})$ ?*

**The representation of  $\mathfrak{S}_6$  on  $AR(\mathcal{W}_{0,6})$ .** Looking at (43) and because the space of ARs of  $\mathcal{W}_{ij}$  is 1-dimensional for any  $(i, j) \in I_{\neq}^2$ , it is clear that

$$\mathcal{T}^*(AR_{ij}) = \lambda_{ij} AR_{\tau(i)\tau(j)} \quad \text{and} \quad C^*(AR_{ij}) = \gamma_{ij} AR_{c(i)c(j)}$$

for some non zero scalars  $\lambda_{ij}$  and  $\gamma_{ij}$  which are easy to make explicit using formulas (42). One gets that relatively to the basis  $(AR(a))_{a=1}^{10}$ , the actions on  $AR(\mathcal{W}_{0,6})$  induced by the pull-backs under  $\mathcal{T}$  and  $C$  (defined in (41)) are given by the two following matrices:

$$T = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Their traces are given by

$$\text{Tr}(T) = -2 \quad \text{and} \quad \text{Tr}(C) = -1.$$

The representation of  $\langle \mathcal{T}, C \rangle \simeq \mathfrak{S}_6$  on the vector space  $AR_C(\mathcal{W}_{0,6}) = AR(\mathcal{W}_{0,6})$  of dimension 10 is isomorphic to the one defined by the group morphism  $\rho : \mathfrak{S}_6 \rightarrow \text{GL}_{10}$  characterized by

$$\rho(\tau) = T \quad \text{and} \quad \rho(c) = C.$$

$$\left[ \begin{array}{cccccc}
0 & \frac{v_2-u_2}{u_2v_2(u_2-1)(v_2-1)} & 0 & \frac{u_4-1}{u_4(u_4-v_4)(v_4-1)} & \frac{1-u_5}{u_5(u_5-v_5)(v_5-1)} & \frac{1-u_6}{u_6(u_6-v_6)(v_6-1)} \\
0 & \frac{v_2-1}{(u_2-1)(u_2-v_2)v_2} & \frac{1-v_3}{(u_3-1)(u_3-v_3)v_3} & 0 & \frac{-1}{(u_5-1)(v_5-1)u_5v_5} & \frac{1}{(v_6-1)(u_6-1)} \\
0 & \frac{v_2}{u_2(u_2-v_2)(v_2-1)} & \frac{-v_3}{u_3(u_3-v_3)(v_3-1)} & \frac{-1}{u_4v_4(u_4-1)(v_4-1)} & 0 & \frac{1}{u_6v_6(u_6-1)(v_6-1)} \\
0 & \frac{-1}{v_2(u_2-v_2)(v_2-1)} & \frac{1}{v_3(u_3-v_3)(v_3-1)} & \frac{1}{(u_4-1)(v_4-1)} & \frac{-1}{(u_5-1)(v_5-1)} & 0 \\
\frac{v_1-1}{v_1(u_1-1)(u_1-v_1)} & 0 & \frac{1-u_3}{u_3(u_3-v_3)(v_3-1)} & 0 & \frac{-v_5}{u_5(u_5-v_5)(v_5-1)} & \frac{1}{v_6(u_6-v_6)(v_6-1)} \\
\frac{v_1}{u_1(v_1-1)(u_1-v_1)} & 0 & \frac{-u_3}{v_3(u_3-1)(u_3-v_3)} & \frac{-v_4}{u_4(u_4-v_4)(v_4-1)} & 0 & \frac{v_6}{u_6(u_6-v_6)(v_6-1)} \\
\frac{-1}{v_1(u_1-v_1)(v_1-1)} & 0 & \frac{1}{u_3(u_3-1)(u_3-v_3)} & \frac{1}{v_4(u_4-v_4)(v_4-1)} & \frac{-1}{v_5(u_5-v_5)(v_5-1)} & 0 \\
\frac{u_1-1}{u_1(u_1-v_1)(v_1-1)} & \frac{1-u_2}{u_2(u_2-v_2)(v_2-1)} & 0 & 0 & \frac{-u_5}{(u_5-1)(u_5-v_5)v_5} & \frac{1}{u_6(u_6-v_6)(u_6-1)} \\
\frac{u_1}{v_1(u_1-1)(u_1-v_1)} & \frac{-u_2}{v_2(u_2-1)(u_2-v_2)} & 0 & \frac{-u_4}{v_4(u_4-1)(u_4-v_4)} & 0 & \frac{u_6}{v_6(u_6-1)(u_6-v_6)} \\
\frac{-1}{u_1v_1} & \frac{1}{u_2v_2} & \frac{-1}{u_3v_3} & \frac{1}{u_4v_4} & 0 & 0
\end{array} \right]$$

From the explicit expression for  $T$  and  $C$  above and considering basic facts of the *linear representation theory* of  $\mathfrak{S}_6$ , it is not difficult to recognize to which representation  $\rho$  corresponds. First, elementary computations show that  $T$  and  $C$  do not have any eigenvector in common. Hence if the representation under scrutiny is not irreducible, then it is necessarily the sum of two irreducible  $\mathfrak{S}_6$ -modules both of dimension 5. Under this assumption, looking at the character table of the  $\mathfrak{S}_6$ -representations, one deduces from  $\text{Tr}(C) = -1$  that the representation is the direct sum of the standard representation with another representation with Young diagram either  $[3^2]$  or  $[2^3]$ .

In these cases, the trace of the matrix corresponding to a transposition would be  $3 - 1 = 2$  and  $3 + 1 = 4$  respectively, which would contradict  $\text{Tr}(T) = -2$ . This implies that the representation  $\rho$  is irreducible. From  $\text{Tr}(T) = -2$ , one deduces that it is the third exterior power of the standard representation of  $\mathfrak{S}_6$ . We have proved the

**Proposition 3.13.** *As a  $\mathfrak{S}_6$ -module, the 10-dimensional space  $\mathbf{AR}_C(\mathcal{W}_{0,6}) = \mathbf{AR}(\mathcal{W}_{0,6})$  is irreducible with associated Young diagram  $[31^3]$ .*

Proposition 3.10 and the one just above contradict some of the claims in [Da2]. The first thing one may think of for explaining this would be that there is a problem with the construction given by Damiano of Euler's abelian relation  $\mathcal{E}_3$  and that this AR actually does not exist. It turns out that this is not the case, Damiano's construction of Euler's abelian relation is valid for any  $n \geq 2$ : we will discuss this in depth in Section §5.1. According to us, the problem lies in the fact that, unaware of the dichotomy according to the parity of  $n$  in what concerns the properties of web  $\mathcal{W}_{0,n+3}$ , Damiano missed an elementary but crucial fact when he studied the  $\mathfrak{S}_{n+3}$ -module structure of the space of combinatorial abelian relations  $\mathbf{AR}(\mathcal{W}_{0,n+3})$ . It is what we are going to discuss in the next section.

#### 4. The space of combinatorial abelian relations as a $\mathfrak{S}_{n+3}$ -representation

We would like to get a better understanding of the space of combinatorial ARs of  $\mathcal{W}_{0,n+3}$ . Damiano used a powerful combinatorial and representation theoretic approach to do so. Using it, he obtained very interesting and almost complete results about the structure of  $AR(\mathcal{W}_{0,n+3})$  as a  $\mathfrak{S}_{n+3}$ -module. However, since he was not aware of the dichotomy between the cases when  $n$  is even or odd in what concerns the  $\mathfrak{S}_n$ -module structure of  $AR(\mathcal{W}_{0,n+3})$ , he did not go all the way and missed a complete description of it. On another hand, the proofs in Damiano's article [Da2] are very concise and/or a bit elliptical in some places. More details are given in his thesis [Da1] but this one is less easily available. For these reasons and especially for the sake of completeness, we thought it relevant to take up Damiano's approach with as much details as possible and it is the purpose of the current section.

Most of the material below, the statements as the proofs, are due to Damiano and are taken with essentially no change either from the fourth chapter of Damiano's thesis [Da1] or from Section §3 of the corresponding paper [Da2]. We only have added a few details in some places. Proofs have been included below mainly for the sake of completeness but also because these results are particularly important regarding the main purpose of this text and are quite nice, as well as their proofs. The unique genuine novelty in this section is Lemma 4.10 which together with Damiano's result and although it is elementary, allows us to show that actually all ARs of  $\mathcal{W}_{0,n+3}$  are combinatorial when  $n$  is odd.

The plan of the sequel is as follows: after having introduced some notations in §4.1, we state Theorem 4.2 which is the main result of this section. Proofs are given in the following subsections §4.3 and §4.4. Finally in §4.5, we discuss explicit formulas for the components of the combinatorial ARs of  $\mathcal{W}_{0,n+3}$ , which will be used later on to make explicit the components of Euler's abelian relation in Section §5 (see Proposition 5.22 more specifically).

**4.1. Some Notations.** Here,  $n$  is a fixed integer bigger than or equal to 2 and  $I_n$  stands for the set of pairs  $(i, j)$  with  $1 \leq i < j \leq n + 3$ . We will take  $\mathbf{C}$  as base field. Since we are only dealing with rational quantities, this choice actually is not relevant and we could have chosen to work over  $\mathbf{R}$  instead (at the cost of some cosmetic changes that we will not mention).

We use the coordinates system  $(x_1, \dots, x_n)$  associated to the birational map  $\psi : \mathbf{C}^n \dashrightarrow \mathcal{M}_{0,n+3}$ ,  $x = (x_i)_{i=1}^n \mapsto [0, 1, \infty, x_1, \dots, x_n]$  and we will work with the following cross-ratio

$$\text{cr}(z_1, z_2, z_3, z_4) = -\frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)}$$

(chosen because of the normalization  $\text{cr}(0, 1, \infty, z) = z$  for any  $z \in \mathbf{C} \setminus \{0, 1\}$ ).

We identify  $\mathcal{W}_{0,n+3}$  with its pull-back under  $\psi$ . As first integrals in the coordinates  $x_i$ 's for the foliations of this web, we take the following rational maps (for  $i$  ranging from 1 to  $n$ ):

$$(44) \quad \begin{aligned} U_i(x) &= \left( x_j \right)_{j \leq n, j \neq i} & U_{n+1}(x) &= \left( \frac{x_j - 1}{x_n - 1} \right)_{j \leq n-1} \\ U_{n+2}(x) &= \left( \frac{x_j}{x_n} \right)_{j \leq n-1} & U_{n+3}(x) &= \left( \frac{x_j(x_n - 1)}{x_n(x_j - 1)} \right)_{j \leq n-1} \end{aligned} .$$



We will denote by  $\mathcal{F}_i$  the foliation admitting  $U_i$  as a first integral. For  $i = 1, \dots, n+3$ , one sets

$$(45) \quad \Omega_i = dU_{i,1} \wedge \dots \wedge dU_{i,n-1}$$

where  $U_{i,s}$  stands for the  $s$ -th component of  $U_i$  for any  $s \leq n-1$ . Then one has  $\Omega_k = \wedge_{l=1, \dots, n, l \neq k} dx_l$  for  $k = 1, \dots, n$  hence these forms form a basis of the module of  $(n-1)$ -differential forms on any open domain of  $\mathbf{C}^n$ .

We will use the following automorphisms of  $\mathcal{M}_{0,n+3}$ :

- the cyclic shift  $C : [0, 1, \infty, x_1, \dots, x_n] \mapsto [1, \infty, x_1, \dots, x_n, 0]$ ;
- the transposition  $\mathcal{T} : [0, 1, \infty, x_1, \dots, x_n] \mapsto [0, \infty, 1, x_1, \dots, x_n]$ .

We denote by  $C$  and  $T$  the two birational maps corresponding to  $C$  and  $\mathcal{T}$  but expressed in the coordinates  $x_i$ 's (formally  $C = \psi^{-1} \circ C \circ \psi$  and similarly for  $T$  with respect to  $\mathcal{T}$ ). One verifies easily that  $C$  and  $T$  are given by the following formulas:

$$(46) \quad C(x) = \left( \frac{1-x_2}{x_1-x_2}, \frac{1-x_3}{x_1-x_3}, \dots, \frac{1-x_n}{x_1-x_n}, \frac{1}{x_1} \right) \quad \text{and} \quad T(x) = \left( \frac{x_1}{x_1-1}, \dots, \frac{x_n}{x_n-1} \right).$$

**Lemma 4.1.** *The maps  $C$  and  $T$  are automorphisms of  $\mathcal{W}_{0,n+3}$  and their actions on the foliations  $\mathcal{F}_i$  are as follow: for any  $i = 1, \dots, n+3$ , one has*

$$C^*(\mathcal{F}_i) = \mathcal{F}_{c(i)} \quad \text{and} \quad T^*(\mathcal{F}_i) = \mathcal{F}_{\tau(i)}$$

where  $c$  stands for the  $(n+3)$ -cycle  $(12 \cdots (n+3))$  and  $\tau$  for the transposition  $((n+2)(n+3))$ .

The group generated by  $C$  and  $T$ , denoted by  $S_{n+3}$ , is isomorphic to  $\mathfrak{S}_{n+3}$ . More precisely, for any  $G$  in this group, let  $\delta_G$  be the permutation of  $\{1, \dots, n+3\}$  such that  $G^*(\mathcal{F}_i) = \mathcal{F}_{\delta_G(i)}$  for  $i = 1, \dots, n+3$ . Then  $\delta : S_{n+3} \rightarrow \mathfrak{S}_{n+3}$ ,  $G \mapsto \delta_G$  is an isomorphism.<sup>32</sup> We will denote by  $G_\sigma$  the birational map of  $\mathcal{M}_{0,n+3}$  corresponding to  $\sigma \in \mathfrak{S}_{n+3}$  up to this isomorphism, that is

$$G_\sigma = \delta^{-1}(\sigma) \in \mathbf{Bir}(\mathcal{M}_{0,n+3}).$$

Note that  $S_{n+3}$  identifies with the group of birational automorphisms of  $\mathcal{W}_{0,n+3}$  (which coincides with the group of automorphisms of  $\mathcal{M}_{0,n+3}$ ).

The  $\mathfrak{S}_{n+3}$ -module structure of  $\mathbf{AR}(\mathcal{W}_{0,n+3})$  is as follows: for any permutation  $\sigma \in \mathfrak{S}_{n+3}$  and any abelian relation  $A = (A^i \cdot \Omega_i)_{i=1}^{n+3} \in \mathbf{AR}(\mathcal{W}_{0,n+3})$ , one has

$$(47) \quad \sigma \cdot A = G_\sigma^*(A) = \left( G_\sigma^*(A^i \Omega_i) \right)_{i=1}^{n+3}.$$

For any  $k = 1, \dots, n+3$  and any  $(i, j) \in \mathbf{I}_n$ , we denote by  $\mathcal{W}_{\widehat{k}}$  and  $\mathcal{W}_{\widehat{ij}}$  the subwebs of  $\mathcal{W}_{0,n+3}$  defined by the first integrals  $U_l$  for  $l \in \{1, \dots, n+3\} \setminus \{k\}$  and  $l \in \{1, \dots, n+3\} \setminus \{i, j\}$  respectively. The corresponding spaces of abelian relations will be denoted by  $\mathbf{A}_{\widehat{k}}$  and  $\mathbf{A}_{\widehat{ij}}$  respectively. These vector spaces can naturally be seen as subspaces of  $\mathbf{AR}(\mathcal{W}_{0,n+3})$ .

<sup>32</sup>For  $G, G' \in S_{n+3}$ , one has  $\delta_{G' \circ G} = \delta_{G'} \cdot \delta_G$  where  $\cdot$  denotes the product on  $\mathfrak{S}_{n+3}$  given by applying the permutations according to their order of appearance from the right to the left, i.e. one has  $(\delta_{G'} \cdot \delta_G)(k) = \delta_G(\delta_{G'}(k))$  for any  $k$ .

**4.2. The main theorem.** Here we aim to give an almost complete proof of the theorem below, which describes  $\mathbf{AR}(\mathcal{W}_{0,n+3})$  as a  $\mathfrak{S}_{n+3}$ -module. This result contradicts, when  $n$  is odd, one of the main results claimed by Damiano. However the proof, whether  $n$  is odd or not, is essentially Damiano's one! The single novelties here are the few details we have added and also a final argument, elementary but missed by Damiano, showing that all ARs of  $\mathcal{W}_{0,n+3}$  are combinatorial when  $n$  is odd.

The approach we will follow is Damiano's and can be summarised as such:

- we start by considering the  $(n + 1)$ -subwebs  $\mathcal{W}_{\widehat{ij}}$  and their abelian relations;
- then we turn to the  $(n + 1)$ -subwebs  $\mathcal{W}_{\widehat{k}}$  whose structure of the space of its abelian relations will be determined as an  $\mathfrak{S}_{n+2}$ -module;
- eventually we consider  $\mathbf{AR}(\mathcal{W}_{0,n+3})$  and establish Theorem 4.2 below.

For any pair  $(i, j)$  in  $\mathbf{I}_n$ , the space  $A_{\widehat{ij}} = \mathbf{AR}(\mathcal{W}_{\widehat{ij}})$  is a subspace of dimension at most 1 of  $\mathbf{AR}(\mathcal{W}_{0,n+3})$ . Our goal here is first to give an effective construction of a non trivial element in  $A_{\widehat{ij}}$ . We will thus get a explicit family of ARs generating  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  which will make it possible to study this  $\mathfrak{S}_{n+3}$ -module more in depth.

We first consider some explicit scalar quantities  $A_{n+2,n+3}^i$ 's for  $i = 1, \dots, n + 3$ : one has  $A_{n+2,n+3}^k = 0$  for  $k = n + 2, n + 3$  and the other  $A_{n+2,n+3}^i$ 's are given by the following formulas:

$$A_{n+2,n+3}^i = \frac{(-1)^{i-1}}{\prod_{k=1}^{n-1} (U_{i,k} - 1)} \quad \text{for } i = 1, \dots, n \quad \text{and} \quad A_{n+2,n+3}^{n+1} = \frac{(-1)^n}{\prod_{k=1}^{n-1} U_{n+1,k}}.$$

Then it can be verified that the following  $(n + 3)$ -tuple of differential forms

$$\mathbf{AR}_{n+2,n+3} = \left( A_{n+2,n+3}^i \cdot \Omega_i \right)_{i=1}^{n+3}$$

is an abelian relation for  $\mathcal{W}_{\widehat{n+2,n+3}}$  (*i.e.* one has  $\sum_{i=1}^{n+1} A_{n+2,n+3}^i \cdot \Omega_i = 0$  as a rational  $(n - 1)$ -form on  $\mathbf{C}^n$ ). Our approach to build the ARs of the other subwebs  $\mathcal{W}_{\widehat{ij}}$  consists in taking the pull-backs of  $\mathbf{AR}_{n+2,n+3}$  under some birational isomorphisms the constructions of which are not difficult to show.

One sets

- $T_{k-1,k} = (C)^{-k} \circ T \circ (C)^k$  for  $k = 2, \dots, n + 4$ ; and  $T_{1,n+3} = T_{n+3,n+4}$ ;
- $C_1 = \mathbf{Id}$  and  $C_l = T_{l-1,l} \circ T_{l-2,l-1} \cdots \circ T_{1,2}$  for  $l = 2, \dots, n + 3$ ;
- $T_{1,\ell} = (C_{\ell-1})^{-1} \circ T_{\ell-1,\ell} \circ C_{\ell-1}$  and  $T_{2,\ell} = (T_{1,2})^{-1} \circ T_{1,\ell} \circ T_{1,2}$  for  $\ell = 3, \dots, n + 3$ ;
- $T_{i,j} = T_{2,j} \circ T_{1,i}$  for any pair  $(i, j)$  such that  $3 \leq i < j \leq n + 3$ .

It follows from elementary standard computations in  $\mathfrak{S}_{n+3}$  that for any  $(i, j) \in \mathbf{I}_n$ ,  $T_{i,j}$  is an involution such that  $\delta(T_{i,j}) = (1i)(2j)$ . Consequently, for any such pair, the pull-back

$$\mathbf{AR}_{i,j} = \left( T_{i,j} \circ T_{n+2,n+3} \right)^* (\mathbf{AR}_{n+2,n+3})$$

is a non trivial abelian relation of  $\mathcal{W}_{\widehat{ij}}$ . Thus any space  $A_{\widehat{ij}}$  is spanned by  $\mathbf{AR}_{i,j}$  hence has dimension 1 and  $\{\mathbf{AR}_{i,j} \mid (i, j) \in \mathbf{I}_n\}$  is a generating family for the space of combinatorial ARs. Because the  $\mathbf{AR}_{i,j}$ 's have been constructed effectively, this gives a way to investigate  $\mathbf{AR}_C(\mathcal{W}_{0,n+3}) = \langle \mathbf{AR}_{i,j} \mid (i, j) \in \mathbf{I}_n \rangle$  as a  $\mathfrak{S}_{n+3}$ -module.

It will be useful to set  $AR_{k,l} = AR_{l,k}$  for any  $k, l = 1, \dots, n+3$  with  $n+3 \geq k > l \geq 1$ .

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For  $k \leq n+1$ , one denotes by  $I_n^k$  the set of  $(i, j)$ 's in  $I_n$  such that  $i = 1, \dots, k$  and  $j = i+1, \dots, n+2$ , and one defines

$$J_n = \begin{cases} I_n^n & \text{when } n \text{ is even;} \\ I_n^{n+1} & \text{when } n \text{ is odd.} \end{cases}$$

Remark that one has  $I_n^{n+1} = I_n^n \cup \{(n+1, n+2)\}$  and  $\text{Card}(I_n^{n+1}) = \text{Card}(I_n^n) + 1 = (n+1)(n+2)/2$ .

Our goal below is to prove the following

**Theorem 4.2.** *The following assertions hold true for any integer  $n \geq 2$ :*

1. *If  $n$  is odd, then:*

- *the abelian relations  $AR_{ij}$ 's for  $(i, j) \in J_n$  form a basis of  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  which has dimension  $(n+1)(n+2)/2$ . Consequently  $\mathbf{AR}(\mathcal{W}_{0,n+3}) = \mathbf{AR}_C(\mathcal{W}_{0,n+3})$  and the web  $\mathcal{W}_{0,n+3}$  has maximal rank;*
- *as a  $\mathfrak{S}_{n+3}$ -representation, the space  $\mathbf{AR}(\mathcal{W}_{0,n+3}) = \mathbf{AR}_C(\mathcal{W}_{0,n+3})$  is irreducible and isomorphic to the  $n$ -th wedge product of the standard  $\mathfrak{S}_{n+3}$ -representation (with associated Young symbol  $[3, 1^n]$ ).<sup>33</sup>*

2. *When  $n$  is even:*

- *the abelian relations  $AR_{ij}$ 's for  $(i, j) \in J_n$  form a basis of  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ . Thus this space has dimension  $n(n+3)/2 = (n+1)(n+2)/2 - 1$ .*
- *as a  $\mathfrak{S}_{n+3}$ -representation, the space  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  of combinatorial ARs is irreducible with associated Young symbol  $[221^{n-1}]$ .*

**4.3. The subwebs  $\mathcal{W}_i$ 's and their abelian relations.** Since  $\mathfrak{S}_{n+3}$  acts transitively on the foliations of  $\mathcal{W}_{0,n+3}$ , any  $(n+2)$ -web  $\mathcal{W}_i$  is isomorphic to  $\mathcal{W}_{\widehat{n+3}}$ . The latter is the linear web formed by the  $n+2$  linear families of lines passing through  $n+2$  points in general position in  $\mathbf{P}^n$  (which can be taken to be the points  $p_k$  of §2.1.3, cf. (15)).

For any  $i$ , the subgroup  $\mathfrak{S}_{n+2} \simeq \text{Fix}(i) \subset \mathfrak{S}_{n+3}$  acts on the space of combinatorial ARs

$$A_C(i) = \langle AR_{i,j} \mid j = 1, \dots, n+3, j \neq i \rangle \subset \mathbf{AR}(\mathcal{W}_i) = A(i).$$

From the action (47) of  $\mathfrak{S}_{n+3}$  on  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ , one gets a  $\mathfrak{S}_{n+2}$ -module structure on  $A_C(i)$ , whose structure is made clear thanks to the following

**Proposition 4.3** (Damiano).

1. *The space  $A_C(i)$  has dimension  $n+1$  and the abelian relations  $AR_{i,j}$ 's for  $j = 1, \dots, n+3$  with  $j \neq i$  are in general position in it: any set of  $n+1$  of them form a basis of this space.*
2. *Consequently, one has  $A_C(i) = A(i)$  and  $\mathcal{W}_i$  has maximal rank  $n+1$ .*
3. *As a  $\mathfrak{S}_{n+2}$ -representation,  $A(i)$  is irreducible with Young symbol  $[2, 1^n]$ .*

<sup>33</sup>Remember that for  $m \geq 3$  and any  $p \in \{1, \dots, m\}$ , the  $p$ -th wedge product of the standard  $\mathfrak{S}_m$ -representation with Young symbol  $[m-1, 1]$  is irreducible with Young symbol  $[m-p, 1^p]$  (see Proposition 3.12 and Exercise 4.6 in [FH]).

To prove this result, we will need the following lemma which will be used again at the end:

**Lemma 4.4.** *For any elements  $(i, j)$  and  $(k, l)$  of  $\mathbf{I}_n$  with  $\{i, j\} \cap \{k, l\} = \emptyset$ , one has*

$$(48) \quad (i, j) \cdot AR_{i,j} = (-1)^{n-1} AR_{i,j} \quad \text{and} \quad (k, l) \cdot AR_{i,j} = -AR_{i,j}.$$

*Proof.* Considering the action of  $\mathfrak{S}_{n+3}$  on the foliations of  $\mathcal{W}_{0,n+3}$ , one can restrict oneself to the case when  $(i, j) = (n+2, n+3)$  and  $(k, l) = (1, 2)$ .

In the rational coordinates  $x_1, \dots, x_n$  we are working with, the birational maps associated to the two corresponding transpositions are given by

$$G_{n+2,n+3}(x) = T(x) = \left( x_i / (x_i - 1) \right)_{i=1}^n \quad \text{and} \quad G_{1,2}(x) = (x_2, x_1, x_3, \dots, x_n).$$

Let  $\tau$  stand for  $(n+2, n+3)$  or  $(1, 2)$ . Since  $G_\tau$  leaves  $\mathcal{W}_{ij}$  invariant and because  $\tau$  is an involution, one has  $\tau \cdot AR_{n+2,n+3} = G_\tau^*(AR_{n+2,n+3}) = \epsilon_\tau AR_{n+2,n+3}$  for  $\epsilon_\tau \in \{\pm 1\}$  to be determined.

To do so, it suffices to compare  $G_\tau^*(A_{n+2,n+3}^3 \Omega_3)$  with  $A_{n+2,n+3}^3 \Omega_3$ , which is easy to do. Indeed, one verifies easily that  $A_{n+2,n+3}^3 \Omega_3$  is equal to the wedge product of the logarithmic 1-forms  $d\text{Log}(x_\ell - 1)$  for  $\ell = 1, 2, \dots, n$  distinct from 3:

$$A_{n+2,n+3}^3 \Omega_3 = d\text{Log}(x_1 - 1) \wedge d\text{Log}(x_2 - 1) \wedge \left( \bigwedge_{\ell=4}^n d\text{Log}(x_\ell - 1) \right).$$

Since  $G_{1,2}$  is just exchanging  $x_1$  and  $x_2$ , one gets  $(1, 2) \cdot A_{n+2,n+3}^3 \Omega_3 = -A_{n+2,n+3}^3 \Omega_3$  hence  $\epsilon_{1,2} = -1$ . Because taking the pull-back under  $T$  consists in replacing  $x_i$  by  $x_i / (x_i - 1)$ , one has

$$\text{Log} \left( \frac{x_i}{x_i - 1} - 1 \right) = \text{Log} \left( \frac{1}{x_i - 1} \right) = -\text{Log}(x_i - 1)$$

for any  $i = 1, \dots, n$ , which immediately gives us that  $T^*(A_{n+2,n+3}^3 \Omega_3) = (-1)^{n-1} A_{n+2,n+3}^3 \Omega_3$ . Hence one has  $\epsilon_{n+2,n+3} = (-1)^{n-1}$ , which finishes the proof of the lemma.  $\square$

We now turn to the proof of the preceding proposition. The one below is entirely taken from [Da1] (cf. §4.1.4 in it) and is reproduced here for the sake of completeness. Next, in §4.3.1, we will describe (without proof) a more geometric approach of Proposition 4.3 which is much more in the spirit of classical web geometry.

*Proof of Proposition 4.3.* It suffices to prove the proposition for  $i = n+3$ . To simplify the notation, we set  $\mathbf{A} = \mathbf{A}(n+3) = \mathbf{AR}(\widehat{\mathcal{W}_{n+3}})$  and  $\mathbf{A}_C = \mathbf{A}_C(n+3) = \mathbf{AR}_C(\widehat{\mathcal{W}_{n+3}})$ . For  $l = 1, \dots, n+2$ , we denote by  $q_l$  the point of  $\mathbf{P}(\mathbf{A})$  corresponding to  $AR_{l,n+3}$ . By definition,  $\mathbf{P}(\mathbf{A}_C)$  is the subspace of  $\mathbf{P}(\mathbf{A})$  spanned by all the  $q_l$ 's and from Proposition 1.1, we know that  $\dim(\mathbf{P}(\mathbf{A})) \leq n$ .

Given  $\sigma \in \mathfrak{S}_{n+2}$  and  $q = [Q] \in \mathbf{P}(\mathbf{A})$  with  $Q \in \mathbf{AR}(\mathcal{W}) \setminus \{0\}$ , we set  $\sigma \cdot q = [\sigma \cdot Q]$ . Then for any  $i, j$  such that  $1 \leq i < j \leq n+2$ , the transposition  $\sigma = (i, j)$  exchanges  $q_i$  and  $q_j$  and admits all the other  $q_l$ 's (for  $l \in \{1, \dots, n+2\}$  distinct from  $i$  and  $j$ ) as fixed points.

Let  $\kappa \in \{1, \dots, n\}$  be the largest integer satisfying the property that there exists a subset  $J = \{j_1, \dots, j_{\kappa+2}\} \subset \{1, \dots, n+2\}$  of cardinality  $\kappa + 2$  such that the points  $q_{j_1}, \dots, q_{j_{\kappa+2}}$  are in general position in their span  $\mathbf{P}(J) = \langle q_j, j \in J \rangle$  which is of dimension  $\kappa$ . If  $\kappa = n$  then 1. is proved hence assume that it is not the case. Then for any  $i \in \{1, \dots, n+2\} \setminus J$ , the transposition  $\sigma_i = (i, j_1)$  lets  $q_{j_s}$  fixed for any  $s = 2, \dots, \kappa + 2$ . Since these points form a projective frame of  $\mathbf{P}(J)$  according

to the definition of  $J$ , we have that  $\sigma_i \cdot \mathbf{P}(J) = \mathbf{P}(J)$ . On the other hand,  $\sigma_i$  exchanges  $q_i$  with  $q_{j_i}$  hence we get that  $q_i$  belongs to  $\mathbf{P}(J)$ . Because this applies to any  $i$  not in  $J$ , we obtain that  $\mathbf{P}(J) = \mathbf{P}(A_C)$ . Assume that  $\kappa < n - 1$ . Then there exist  $i, i' \in \{1, \dots, n + 2\} \setminus J$  distinct. On the one hand, the permutation  $(i, i')$  lets  $q_j$  invariant for all  $j \in J$ . Since these points are in general position in  $\mathbf{P}(J)$ , it follows that  $(i, i')$  acts as the identity on this space. On the other hand,  $(i, i')$  exchanges  $q_i$  and  $q_{i'}$ , which is a contradiction. Thus necessarily, one has  $\kappa = n - 1$  or  $\kappa = n$ .

Since we are done in the latter case, let us assume than the former holds true: one has  $\kappa = n - 1$ . We denote by  $i$  the element of  $\{1, \dots, n + 2\}$  not in  $J$ . Then  $\mathfrak{S}_{n+1} \simeq \text{Fix}_{\mathfrak{S}_{n+2}}(i)$  acts on  $\mathbf{P}(J) = \mathbf{P}(A_C) \simeq \mathbf{P}^{n-1}$  inducing the standard permutation action on the  $n + 1$  points  $q_1, \dots, \widehat{q_i}, \dots, q_{n+2}$ . It is easily verified that necessarily, the action on  $\mathbf{P}^{n-1}$  comes from a linear action on a vector space of dimension  $n$  which is irreducible. But this action also lets  $q_i$  invariant, a contradiction.

We thus have proved 1. The second point follows immediately.

Let us turn to the structure of the space  $\mathbf{A}$  as a  $\mathfrak{S}_{n+2}$ -module. Picking a projective basis for  $\mathbf{P}(\mathbf{A})$ , say the  $(n + 1)$ -tuple  $(q_1, \dots, q_{n+1})$ , we identify it with  $\mathbf{P}^n$ . From the arguments above, it follows that the image of  $\mathfrak{S}_{n+2}$  in  $\text{PGL}_{n+1}(\mathbf{C})$  corresponding to this identification can be described as the set of projective transformations leaving  $\{q_i\}_{i=1}^{n+1}$  invariant and inducing a permutation of it. From this, one gets easily that  $\widehat{A_{n+3}}$  necessarily is an irreducible  $\mathfrak{S}_{n+2}$ -representation. Since moreover it has dimension  $n + 1$ , there are only two possibilities: either it is the standard representation or its dual (with respective Young symbols  $[n + 1, 1]$  and  $[2, 1^n]$ ).

From above, we know that the abelian relations  $AR_{1,n+3}$  for  $i = 1, \dots, n + 1$  form a basis of  $\mathbf{A} = A_C$  which has dimension  $n + 1$ . In order to recognize it as a  $\mathfrak{S}_{n+2}$ -representation, we consider the matrix in this basis of the linear involution associated to the transposition  $(1, 2)$ , denoted by  $M_{(1,2)}$ . From Lemma 4.4, we have  $(1, 2) \cdot AR_{j,n+3} = -AR_{j,n+3}$  for any  $j = 3, \dots, n + 1$  on the one hand, whereas one has  $(1, 2) \cdot AR_{k,n+3} = \epsilon_k AR_{l,n+3}$  for  $k, l$  such that  $\{k, l\} = \{1, 2\}$ , with  $\epsilon_k, \epsilon_l \in \{\pm 1\}$  on the other hand. It follows that  $\text{Tr}(M_{(1,2)}) = -(n - 1)$ . Since the character map  $\chi_{[n+1,1]}$  of the standard representation is such that  $\chi_{[n+1,1]}(\sigma) = |\text{Fix}(\sigma)| - 1$  for any  $\sigma \in \mathfrak{S}_{n+2}$  where  $\text{Fix}(\sigma) \subset \{1, \dots, n + 2\}$  stands for the set of fixed points of  $\sigma$ , we deduce that the space  $\mathbf{A}(n + 3)$  is not the standard representation but its dual, which concludes the proof of the proposition.  $\square$

**4.3.1. An alternative view on Proposition 4.3.** Here we would like to explain another way to establish Proposition 4.3 much more in the spirit of classical web geometry and which suggests an interesting possible development (see §6.2 further). This approach is a rather straightforward generalization for curvilinear  $(n + 2)$ -web in dimension  $n$  of the classical linearization/algebraization of maximal rank planar 4-webs which goes back to works by Lie, Poincaré and Blaschke (for which we refer to [PP, §4.3]).

We denote by  $V$  the complex vector space  $\mathbf{C}^{n+1}$  the projectivization of which is the ambient projective space  $\mathbf{P}^n$  we are working in:  $\mathbf{P}^n = \mathbf{P}(V)$ . If  $v_1, \dots, v_{n+1} \in V$  stand for the vectors of the canonical basis of  $V$  and  $v_{n+2} = (1, \dots, 1) \in V$ , then one has  $p_i = [v_i]$  for  $i = 1, \dots, n + 2$  where  $[\cdot] : V \setminus \{0\} \rightarrow \mathbf{P}^n$  stands for the standard projectivization map. We denote by  $P$  the set of the  $p_i$ 's and by  $\mathcal{LW}_P$  the  $(n + 2)$ -linear web on  $\mathbf{P}^n$  formed by the families of lines through the  $p_i$ 's. It is nothing else but the subweb  $\widehat{\mathcal{W}_{n+3}}$  of  $\mathcal{W}_{0,n+3}$ .

Here we continue to work with the notations of §4.1. In particular, the rational coordinates  $x_i$ 's are those associated to the birational map  $\mathbf{P}^n \dashrightarrow \overline{\mathcal{M}}_{0,n+3}$ ,  $(x_1, \dots, x_n) \rightarrow [0, 1, \infty, x_1, \dots, x_n]$ . The

subgroup  $\text{Fix}(3) \subset \mathfrak{S}_{n+3}$  is isomorphic to  $\mathfrak{S}_{n+2}$  hence the latter group acts on  $\mathbf{P}^n$  by birational maps which (1) preserve the foliation by RNC of degree  $n$  of  $\mathcal{W}_{0,n+3}$  (2) are birational automorphisms of  $\mathcal{LW}_P$ . We thus have a representation

$$(49) \quad \mathfrak{S}_{n+2} \simeq \text{Fix}(3) \rightarrow \text{Bir}(\mathbf{P}^n).$$

The transpositions  $(i, j)$  with  $3 < i < j \leq n + 3$ ,  $(2, 4)$  and  $(1, 2)$  generate  $\text{Fix}(3)$  and the corresponding Cremona transformations  $G_\sigma$  for such any transpositions  $\sigma$  are easily seen to be given by

$$G_\sigma(x) = \begin{cases} (x_{\sigma(k+3)-3})_{k=1}^n & \text{for } \sigma = (i, j) \\ (1 - x_k)_{k=1}^n & \text{for } \sigma = (1, 2) \\ \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) & \text{for } \sigma = (2, 4). \end{cases}$$

These birational maps are projectivizations of linear automorphisms  $\widehat{G}_\sigma$  of  $V$  which it is straightforward to make explicit. Using their explicit form, it is not difficult to verify that (49) is induced by a linear representation which is irreducible. Since moreover one has  $\text{Tr}(\widehat{G}_\sigma) = n - 1$ , we obtain that this representation is equivalent to the standard one. We thus have established the

**Fact:** *The representation (49) is linear. More precisely, it is the projectivization of a  $\mathfrak{S}_{n+2}$ -representation on  $V$  which is irreducible and isomorphic to the standard representation.*

As in the preceding section, we denote by  $\mathbf{A}$  the space of abelian relations  $\mathbf{AR}(\mathcal{LW}_P) = \mathbf{AR}(\widehat{\mathcal{W}_{n+3}})$ . For any  $i = 1, \dots, n + 2$ , given  $x \in \mathbf{P}^n$  generic, the map  $\text{ev}_i : \mathbf{A} \rightarrow \mathbf{C}, (A_k \Omega_k)_{k=1}^{n+2} \mapsto A_i(x)$  is a non trivial linear form hence  $\ker(\text{ev}_i)$  is a hyperplane in  $\mathbf{A}$ . After projectivization, we get the  $i$ -th canonical map of the web under scrutiny, which by definition is the map

$$\begin{aligned} \kappa_i : \mathbf{P}^n &\dashrightarrow \check{\mathbf{P}}^n = \mathbf{P}(V^\vee) \\ x &\longrightarrow \kappa_i(x) = [\ker(\text{ev}_i)]. \end{aligned}$$

Clearly, each  $\kappa_i$  is constant along the leaves of the  $i$ -th foliation (which is the bundle  $\mathcal{L}_{p_i}$  of lines through  $p_i$ ) and, as a rational map, it factorizes through the map  $U_i : \mathbf{P}^n \rightarrow \mathbf{P}^{n-1}$ . The map  $\kappa_i$  has maximal rank  $n - 1$  (generically), it is a canonical first integral for  $\mathcal{L}_{p_i}$  and (the closure of) its image is the hyperplane  $H_i \subset \check{\mathbf{P}}^n$  which is dual to  $p_i$ : considering the  $v_i$ 's as the canonical coordinates on  $V^\vee$ ,  $H_i$  is cut out by  $v_i = 0$  for  $i = 1, \dots, n + 1$  and by  $v_{n+2} = \sum_{i=1}^{n+1} v_i = 0$ . The union of the  $H_i$ 's, denoted by  $H$ , is a hypersurface of  $\check{\mathbf{P}}^n$ , of degree  $n + 2$ , cut out by  $v = v_1 \cdots v_{n+2} = 0$ .

Any AR  $(A_i \Omega_i)_{i=1}^{n+2}$  is such that  $\sum_{i=1}^{n+2} A_i \Omega_i = 0$  as a rational  $(n - 1)$ -form. This relation is equivalent to  $n$  linearly scalar identities, from which it can be deduced that generically, the  $n + 2$  points  $\kappa_i(x)$ 's actually span a line, that we denote by  $\mathfrak{L}(x)$ : one has

$$\mathfrak{L}(x) = \langle \kappa_i(x) \mid i = 1, \dots, n + 2 \rangle \in G_1(\check{\mathbf{P}}^n).$$

We then obtain a map  $\mathfrak{L} : \mathbf{P}^n \dashrightarrow G_1(\check{\mathbf{P}}^n)$  which allows to get an interesting geometric model of  $\mathcal{LW}_P$ . Indeed, the closure  $Z_n = \overline{\mathfrak{L}(\mathbf{P}^n)}$  is a  $n$ -dimensional submanifold of the grassmannian variety of lines in  $\check{\mathbf{P}}^n$  and the push-forward onto it of  $\mathcal{LW}_P$  by  $\mathfrak{L}$  is defined by the incidence in  $\check{\mathbf{P}}^n$  between the lines belonging to  $Z_n$  and the points of the hyperplanes  $H_i$ 's. In other terms, the following holds true:

**Fact.** *The push-forward of  $\mathcal{LW}_P$  by  $\mathfrak{L}$  is the trace along  $Z_n$  of the algebraic web  $\mathcal{W}_H$ : one has*

$$\mathfrak{L}_*(\mathcal{LW}_P) = (\mathcal{W}_H)|_{Z_n}.$$

Now, as it classically follows from Abel's theorem and its converse, the ARs of  $\mathcal{W}_H$  can be described in terms of the abelian  $(n-1)$ -differential forms on  $H$  (cf. [Bar, HP]). The maps  $\rho_i : G_1(\check{\mathbf{P}}^n) \dashrightarrow H_i$ ,  $\ell \mapsto \rho_i(\ell)$  where  $\rho_i(\ell)$  stands for the intersection point of a generic line  $\ell$  in  $\check{\mathbf{P}}^n$  with the hyperplane  $H_i$  for each  $i$ , are natural rational first integrals of  $\mathcal{W}_H$ . Moreover, the trace gives rise to a linear isomorphism  $H^0(H, \omega_H^{n-1}) \rightarrow \mathbf{AR}(\mathcal{W}_H)$ ,  $\omega \mapsto (\rho_i^*(\omega))_{i=1}^{n+2}$ . Furthermore, it can be verified that the natural map  $\mathbf{AR}(\mathcal{W}_H) \rightarrow \mathbf{AR}((\mathcal{W}_H)|_{Z_n})$  is injective, hence up to pull-back under  $\mathfrak{L}$ , we obtain a linear isomorphism of complex vector spaces

$$(50) \quad \mathbf{AR}(\mathcal{LW}_P) \simeq \mathbf{AR}(\mathcal{W}_H) \simeq H^0(H, \omega_H^{n-1}).$$

On the other hand, we have the following short exact sequence of sheaves

$$(51) \quad 0 \rightarrow \Omega_{\check{\mathbf{P}}^n}^n \rightarrow \Omega_{\check{\mathbf{P}}^n}^n \otimes \mathcal{O}_{\check{\mathbf{P}}^n}(H) \xrightarrow{R} \omega_H^{n-1} \rightarrow 0$$

(where  $\omega_H^{n-1}$  denotes the sheaf of abelian differentials of top degree on  $H$  and  $R$  is the morphism of sheaves induced by Poincaré residue, see [HP]). Since  $\deg(H) = n+2$  and because  $\Omega_{\check{\mathbf{P}}^n}^n \simeq \mathcal{O}_{\check{\mathbf{P}}^n}(-n-1)$ , we have  $\Omega_{\check{\mathbf{P}}^n}^n \otimes \mathcal{O}_{\check{\mathbf{P}}^n}(H) \simeq \mathcal{O}_{\check{\mathbf{P}}^n}(1)$ . Since  $\check{\mathbf{P}}^n = \mathbf{P}(V^\vee)$ ,  $H^0(\check{\mathbf{P}}^n, \mathcal{O}_{\check{\mathbf{P}}^n}(1))$  naturally identifies with the dual of  $V^\vee$ . Injecting this into the first non trivial part of the long cohomology sequence associated to (51) gives rise to a sequence of natural linear isomorphisms

$$(52) \quad V = (V^\vee)^\vee = H^0(\check{\mathbf{P}}^n, \mathcal{O}_{\check{\mathbf{P}}^n}(1)) \simeq H^0(H, \omega_H^{n-1}).^{34}$$

One verifies that the preceding isomorphisms actually are isomorphisms of  $\mathfrak{S}_{n+2}$ -modules, when the action on  $H^0(H, \omega_H^{n-1})$  is induced by push-forward by the birational maps  $G_\sigma$  for  $\sigma \in \mathfrak{S}_{n+2}$ . Considering (50), and because the action (47) is induced by pull-backs under the  $G_\sigma$ 's, we get the

**Proposition 4.5.** *The action of  $\mathfrak{S}_{n+2}$  by push-forward on  $\mathbf{AR}(\mathcal{LW}_P)$  is isomorphic to the action of  $\mathfrak{S}_{n+2}$  on  $V$  (hence is isomorphic to the standard representation). Consequently, the  $\mathfrak{S}_{n+2}$ -action (47) on this space of ARs is the dual action (irreducible module with Young symbol  $[2, 1^n]$ ).*

In addition to giving a more conceptual explanation for Proposition 4.3.3, the material discussed above is interesting since it asks possibly fruitful questions, in two distinct directions at least:

- One may think that the construction of canonical maps and of a canonical geometric model considered above for  $\mathcal{LW}_P$  could be generalized to any  $(n+2)$ -web in dimension  $n$  with maximal  $(n-1)$ -rank. This suggests a possible approach for algebraizing such webs. We will say a few words about this problem further in §6.2.
- Another interesting question that emerges from the above considerations concerns the varieties  $Z_n$ . Given a generic point  $p \in \mathbf{P}^n$ , let  $C_p$  be the rational normal curve in  $\mathbf{P}^n$  passing through all the  $p_i$ 's and  $p$ . Denoting by  $\mathcal{L}_p$  the (embedded) tangent line to  $C_p$  at  $p$ , one gets a rational map  $\mathcal{L} : \mathbf{P}^n \dashrightarrow G_1(\mathbf{P}^n)$ ,  $p \mapsto \mathcal{L}_p$  whose image  $\mathcal{L}_n = \overline{\mathcal{L}(\mathbf{P}^n)}$  is a  $n$ -dimensional submanifold of the grassmannian variety of lines in  $\mathbf{P}^n$ . For  $n=3$  and up to an identification of  $\mathbf{P}^n$  with its dual  $\check{\mathbf{P}}^n$ , we have verified that  $\mathcal{L}_3$  coincides with  $Z_3$

<sup>34</sup>The map  $V \rightarrow H^0(H, \omega_H^{n-1})$  can be made explicit rather easily: the abelian differential on  $H$  associated to  $\bar{v} \in V$  is the Poincaré residue along  $H$  of the global section  $(\bar{v}/v) \sum_{i=1}^{n+1} (-1)^i v_i dv_1 \wedge \cdots \wedge \widehat{dv_i} \wedge \cdots \wedge dv_{n+1}$  of  $\Omega_{\check{\mathbf{P}}^n}^n \otimes \mathcal{O}_{\check{\mathbf{P}}^n}(H)$ .

and that it is a degree six complex of lines in  $\mathbf{P}^3$ . Contrarily to what we expected, we have not found any mention of it in the classical literature on line complexes. Is it really new? And in higher dimension, do  $Z_n$  and  $\mathcal{L}_n$  coincide? In any case, we think that they are interesting subvarieties of grassmannian varieties of lines that deserve further study.

**4.4. Proof of Theorem 4.2.** This subsection is devoted to proving Theorem 4.2. We have reproduced Damiano's arguments for completeness, with only minor modifications added in order to make them clearer. Our only significant (although elementary) contribution is Lemma 4.10 which definitively settles the case when  $n$  is odd.

We start with two technical lemmata taken from [Da1, §4]. To state the first, one considers  $n + 2$  points  $q_1, \dots, q_{n+2}$  in general position in  $\mathbf{P}^n$  together with an irreducible linear  $\mathfrak{S}_{n+2}$ -action on this projective space which induces the natural action by permutations on the  $q_i$ 's (i.e. one has  $\sigma \cdot q_i = q_{\sigma(i)}$  for every  $i$  and every  $\sigma \in \mathfrak{S}_{n+2}$ ). Let  $m \leq n + 1$  be fixed. The subgroup  $\bigcap_{s=m+2}^{n+2} \text{Fix}(s)$  is naturally isomorphic to  $\mathfrak{S}_m$  and will be denoted in the same way.

**Lemma 4.6.** *For any  $q \in \mathbf{P}^n \setminus \langle q_{m+1}, \dots, q_{n+2} \rangle$ , one has  $\langle \mathfrak{S}_m \cdot q, q_{m+1}, \dots, q_{n+2} \rangle = \mathbf{P}^n$ .*

*Proof.* By induction on  $m$ , one can restrict oneself to only deal with the case when  $m = n + 1$ .

Let  $V$  be the  $\mathfrak{S}_{n+2}$ -module of dimension  $n + 1$  whose projectivization is the considered  $\mathfrak{S}_{n+2}$ -action on  $\mathbf{P}^n$ . Since  $V$  is irreducible and has dimension  $n + 1$ , its Young symbol  $\lambda$  is either  $[n + 1, 1]$  or  $[2, 1^n]$ . It then follows immediately from the branching rule (see [Sag, Theorem 2.8.3]) that the restriction  $V \downarrow_{\mathfrak{S}_{n+1}}$  is the sum of two irreducible  $\mathfrak{S}_{n+1}$ -representations, one with Young symbol  $\lambda_-$  of dimension  $n$ , denoted by  $V_-$ , the other of dimension 1 denoted by  $W$ . Since  $q_{n+2}$  is fixed by  $\mathfrak{S}_{n+1} = \text{Fix}(n + 2)$ , this point is the projectivization  $\mathbf{P}(W)$  of  $W$ . Any  $q \in \mathbf{P}^n \setminus \{q_{n+2}\}$  has a lift in  $V$  which can be written  $\hat{q}_- + w$  with  $\hat{q}_- \in V_- \setminus \{0\}$  and  $w \in W$ . Denote by  $q_-$  the image of  $\hat{q}_-$  in  $\mathbf{P}^n$ .

Clearly, one has  $\langle \mathfrak{S}_{n+1} \cdot q, q_{m+2} \rangle = \langle \mathfrak{S}_{n+1} \cdot q_-, q_{m+2} \rangle$ . On the other hand, since  $V_-$  is an irreducible  $\mathfrak{S}_{n+1}$ -module, the orbit  $\mathfrak{S}_{n+1} \cdot q_-$  spans the projectivization  $\mathbf{P}(V_-)$  of  $V_-$ . Thus the linear span of  $\mathfrak{S}_{n+1} \cdot q$  and  $q_{m+2}$  contains both  $\mathbf{P}(V_-)$  and  $\mathbf{P}(W)$  hence the lemma follows.  $\square$

The previous lemma will be used in the proof of the second one, which is the following:

**Lemma 4.7.** *Let  $\{q_{i,j} = q_{j,i} \mid (i, j) \in I_n\}$  be a family of  $(n + 3)(n + 2)/2$  points in a projective space  $\mathbf{P}^N$ . Assume that these points satisfy the following hypotheses:*

- (H<sub>1</sub>). *For any  $i = 1, \dots, n + 3$ , the  $n + 2$  points  $q_{i,j}$ 's for  $j \neq i$  are in general position in their span, which is a linear subspace of dimension  $n$  of  $\mathbf{P}^N$ , denoted by  $\mathbf{P}_i^n$ .*
- (H<sub>2</sub>). *For any  $(i, j) \in I_n$ , the intersection of  $\mathbf{P}_i^n$  with  $\mathbf{P}_j^n$  consists of the point  $q_{i,j}$ ;*
- (H<sub>3</sub>). *There is a linear  $\mathfrak{S}_{n+3}$ -representation on  $\mathbf{P}^N$  such that :*
  - (a). *for every  $\sigma \in \mathfrak{S}_{n+3}$  and any  $(i, j) \in I_n$ , one has  $\sigma q_{i,j} = q_{\sigma i, \sigma j}$ ;*
  - (b). *the restricted action of  $\mathfrak{S}_{n+2} \simeq \text{Fix}_{\mathfrak{S}_{n+3}}(i)$  on  $\mathbf{P}_i^n$  is irreducible for any  $i \leq n + 3$ .*

*Then one necessarily has  $N \geq (n + 2)(n + 1)/2 - 2$ .*

*Proof.* There is no loss of generality by assuming that the ambient projective space  $\mathbf{P}^N$  is spanned by the  $q_{i,j}$ 's, which we do in what follows.



For  $\ell \in \{1, \dots, n+3\}$ , we set  $P_\ell = \langle \mathbf{P}_1^n, \dots, \mathbf{P}_\ell^n \rangle \subset \mathbf{P}^N$  and one denotes (a bit abusively) by  $\mathfrak{S}_{n+3-\ell}$  the subgroup  $\cap_{i=1}^\ell \text{Fix}_{\mathfrak{S}_{n+3}}(i) = \mathfrak{S}_{\{\ell+1, \dots, n+3\}}$  of  $\mathfrak{S}_{n+3}$ . The  $P_\ell$ 's are linear subspaces of  $\mathbf{P}^N = P_{n+3}$  which form an increasing sequence (for the inclusion). First remark that for any  $\ell = 1, \dots, n+2$ , since  $q_{i, \ell+1} \in \mathbf{P}_i^n \subset P_\ell$  for  $i = 1, \dots, \ell$  and because these points are in general position in  $\mathbf{P}_{\ell+1}^n$  (by  $(H_1)$ ), one has  $\dim(P_\ell \cap \mathbf{P}_{\ell+1}^n) \geq \max(\ell-1, n)$ .

One has  $\dim(P_1 \cap \mathbf{P}_2^n) = 0$  thanks to  $(H_2)$  hence the set of indices  $\ell$  such that  $\dim(P_\ell \cap \mathbf{P}_{\ell+1}^n) = \ell-1$  is not empty. Its maximal element, denoted by  $L$ , is a well defined element of  $\{1, \dots, n+1\}$ . We are going to prove that  $P_{L+1} = \mathbf{P}^N$  with  $L \geq n-1$ , from which it will be easy to get the looked sought-after minoration.

For  $\ell$  such that  $1 \leq \ell \leq L$ , the points  $q_{1, \ell+1}, \dots, q_{\ell, \ell+1}$  all belong to  $P_{\ell+1}$ . Since  $\dim(P_\ell \cap \mathbf{P}_{\ell+1}^n)$  has dimension  $\ell-1$  and because of  $(H_1)$ , we get that the points  $q_{i, \ell+1}$  for  $i = \ell+1, \dots, n+2$  are projectively linearly independent modulo  $P_\ell$ . By induction, we deduce that

1. the points  $q_{i,j}$ 's for  $i = 1, \dots, \ell$  and  $j = i+1, \dots, n+2$  form a projective frame for  $P_\ell$ ;
2. hence one has  $\dim(P_{\ell+1}) = N(\ell)$  with  $N(\ell) = [(n+1) + (n+1-1) + \dots + (n+1-\ell)] - 1$ .

It follows that in order to prove the proposition, one has to establish that  $L \geq n-1$ . Assuming the contrary and setting  $k = L+1$ , one has  $k \leq n-1$  and  $\dim(P_k \cap \mathbf{P}_{k+1}^n) > k-1$  hence there exists a point  $q \in P_k \cap \mathbf{P}_{k+1}^n \setminus \langle q_{1, k+1}, \dots, q_{k, k+1} \rangle$ . The group  $\cap_{i=1}^{k+1} \text{Fix}(i) \subset \mathfrak{S}_{n+2}$ , denoted a bit abusively by  $\mathfrak{S}_{n+3-(k+1)}$  here, acts on  $\mathbf{P}_{k+1}^n$ . Thanks to hypothesis  $(H_3.b)$ , Lemma 4.6 applies and gives us that  $\mathbf{P}_{k+1}^n$  is spanned by the union of  $\{q_{1, k+1}, \dots, q_{k, k+1}\}$  with the  $\mathfrak{S}_{n+3-(k+1)}$ -orbit of  $q$ : one has

$$(53) \quad \langle \mathfrak{S}_{n+3-(k+1)} \cdot q, q_{1, k+1}, \dots, q_{k, k+1} \rangle = \mathbf{P}_{k+1}^n.$$

Since  $q$  belongs to  $P_k$  which is invariant under the action of  $\mathfrak{S}_{n+3-(k+1)} = \cap_{i=1}^k \text{Fix}(i)$ , one has that  $\mathfrak{S}_{n+3-k} \cdot q \subset P_k$ . Combined with the fact that  $q_{i, k+1} \in \mathbf{P}_i^n \subset P_k$  for  $i = 1, \dots, k$ , it then follows from (53) that  $\mathbf{P}_{k+1}^n \subset P_k$ . On the other hand, for any  $l > k+1$ , the permutation  $(k+1, l) \in \mathfrak{S}_{n+3}$  interchanges  $\mathbf{P}_{k+1}^n$  and  $\mathbf{P}_l^n$  while leaving  $P_k$  fixed. One deduces that  $\mathbf{P}_l^n \subset P_k$  for any  $l = k+1, \dots, n+3$  hence  $P_k = P_{n+3} = \mathbf{P}^N$ .

Let  $F = \{p \in \mathbf{P}^N \mid \tau \cdot p = p\} \subset \mathbf{P}^N$  be the fixed point set of  $\tau = (n+2, n+3) \in \mathfrak{S}_{n+3}$ . From  $(H_1)$ , it comes that the set of  $q_{i,j}$ 's with  $i = 1, \dots, k$  and  $j = i+1, \dots, n+2$  spans  $P_k$ . But since any such point  $q_{i,j}$  belongs to  $F$  if  $j \leq n+1$ , it comes that

$$(54) \quad \dim(P_k) \leq \dim(F) + \text{Card}(\{q_{1, n+2}, \dots, q_{k, n+2}\}) = \dim(F) + k.$$

On the other hand, since  $\tau(F) = F$  and  $\tau(\mathbf{P}_{n+3}^n) = \mathbf{P}_{n+2}^n$ , it follows that  $F \cap \mathbf{P}_{n+3}^n = \{q_{n+2, n+3}\}$ . Hence  $\dim(F \cap \mathbf{P}_{n+3}^n) = 0$  so  $\text{codim}_{\mathbf{P}^N}(F) \geq n$ , which can also be written

$$(55) \quad \dim(F) + n \leq \dim(P_k).$$

From (54) and (55) together, one gets  $n \leq k$  contradicting the assumption  $L < n-1$ . Thus necessarily  $L \geq n$  which gives us the proposition.  $\square$

This lemma has an interesting consequences when applied to  $\mathbf{P}^N = \mathbf{P}(\text{AR}_C(\mathcal{W}_{0, n+3}))$  with  $q_{i,j} = \mathbf{P}(\langle \text{AR}_{i,j} \rangle)$  for any  $i, j = 1, \dots, n+3$  with  $i \neq j$ . For any subset  $J \subset \{1, \dots, n+3\}$ , one sets

$A_J = \langle A_{\bar{j}} \mid j \in J \rangle \subset \mathbf{AR}(\mathcal{W}_{0,n+3})$  and

$$(56) \quad \eta = \min_{J \subset \{1, \dots, n+3\}} \left\{ |J| \mid A_J = \mathbf{AR}_C(\mathcal{W}_{0,n+3}) \right\}.$$

Then, from the previous lemma as well as from its proof, one deduces the

**Corollary 4.8.** 1. *Either  $\eta = n$  or  $\eta = n + 1$ .*

2. *In both cases, one has  $A_J = \mathbf{AR}_C(\mathcal{W}_{0,n+3})$  for any subset  $J$  of cardinality  $\eta$  and the  $AR_{i,j}$ 's for  $i = 1, \dots, \eta$  and  $j = i + 1, \dots, n + 2$  form a basis of  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ .*
3. *Consequently,  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  has dimension  $(n + 1)(n + 2) - 1$  if  $\eta = n$  and dimension one more when  $\eta = n + 1$ . In the latter case,  $\mathcal{W}_{0,n+3}$  has maximal rank hence necessarily  $\mathbf{AR}_C(\mathcal{W}_{0,n+3}) = \mathbf{AR}(\mathcal{W}_{0,n+3})$ .*

Another nice feature of Lemma 4.7 is that it is rather easy to deduce from it a description of  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  as a  $\mathfrak{S}_{n+3}$ -module:

**Corollary 4.9.** 1. *As a  $\mathfrak{S}_{n+3}$ -representation,  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  is irreducible.*

2. *Its Young symbol is  $[221^{n-1}]$  if  $\eta = n$  and  $[31^n]$  when  $\eta = n + 1$ .*

*Proof.* Assume that  $A_C = \mathbf{AR}_C(\mathcal{W}_{0,n+3})$  is not irreducible. Pick  $F \subset A_C$ , a non trivial and proper stable sub- $\mathfrak{S}_{n+3}$ -module of maximal dimension. Since the  $\mathfrak{S}_{n+3}$ -orbit of any  $A_{\bar{i}}$  coincides with the whole space  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ , we have that  $F \cap A_{\bar{i}} = \{0\}$  for any  $i = 1, \dots, n + 3$ , hence  $\text{codim}(F) \geq n + 1$ .

Let  $\pi : \mathbf{PA}_C \setminus \mathbf{PF} \rightarrow \mathbf{P}(A_C/F) = \mathbf{P}^M$  with  $M = \dim(A_C) - \dim(F) - 1$ . For any  $i = 1, \dots, n + 3$ , since  $\mathbf{PF} \cap \mathbf{PA}_{\bar{i}} = \emptyset$ , it follows that  $\pi(\mathbf{PA}_{\bar{i}})$  has dimension  $n$ , which justifies denoting this subspace by  $\mathbf{P}_i^n$ . In particular, one has  $M \geq n$ .

For any  $i, j$  distinct, let us prove that  $\mathbf{P}_i^n \cap \mathbf{P}_j^n = q_{i,j} = \pi(p_{i,j})$  with  $p_{i,j} = [AR_{i,j}] \in \mathbf{PA}_C$ . Otherwise, there would exist another point  $q$  in this intersection, distinct from  $q_{i,j}$ . We denote here by  $\mathfrak{S}_{n+2}$  (resp.  $\mathfrak{S}_{n+1}$ ) the subgroup of  $\mathfrak{S}_{n+3}$  formed by permutations fixing  $i$  (resp. fixing both  $i$  and  $j$ ): one has  $\mathfrak{S}_{n+1} = \text{Fix}(i) \cap \text{Fix}(j) < \mathfrak{S}_{n+2} = \text{Fix}(i) < \mathfrak{S}_{n+3}$ . Note that the action of  $\mathfrak{S}_{n+2}$  on  $\mathbf{P}_i^n$  is irreducible with Young symbol  $[2, 1^n]$  according to Proposition 4.3.3. On the other hand

$$[2, 1^n] \Big|_{\mathfrak{S}_{n+1}} = [1^{n+1}] \oplus [2, 1^{n-1}]$$

as it follows immediately from the classical branching rules for the representations of symmetric groups (see [Sag, §2.8]). This means that the representation of  $\mathfrak{S}_{n+1}$  on  $\mathbf{P}_i^n$  induced by restriction admits a unique fixed point, which is  $q_{i,j}$  of course, and acts irreducibly on a supplementary hyperplane. It follows that the orbit  $\mathfrak{S}_{n+1} \cdot q$  spans a hyperplane in  $\mathbf{P}_i^n$  which does not contain  $q_{i,j}$  hence one has  $\langle q_{i,j}, \mathfrak{S}_{n+1} \cdot q \rangle = \mathbf{P}_i^n$  (cf. also Lemma 4.6). Since all these arguments also apply when exchanging  $i$  and  $j$ , one has  $\langle q_{i,j}, \mathfrak{S}_{n+1} \cdot q \rangle = \mathbf{P}_j^n$  as well, from which it comes that  $\mathbf{P}_i^n = \mathbf{P}_j^n$ . Since all  $\mathbf{P}_i^n$ 's coincide and because their union spans  $\mathbf{P}^M$ , we get that  $M = n$ , hence that  $\text{codim}(F) = n + 1$ . But this is impossible: indeed, from our assumption of the maximality of  $F$  (as a proper submodule of  $A_C$ ), it comes that  $A_C/F$  is an irreducible  $\mathfrak{S}_{n+3}$ -representation of dimension  $n + 1$  and such a thing does not exist when  $n \geq 2$ .

From the above, it comes that  $\mathbf{P}^M$  and the points  $q_{i,j}$  with  $(i, j) \in I_n$  satisfy all the assumptions of Lemma 4.7 which therefore applies and gives us that  $M \geq (n + 1)(n + 2) - 2$ . Since  $F$  has

been assumed to be non trivial and proper, the single possibility is that  $A_C$  and  $F$  have dimension  $(n+1)(n+2)/2$  and 1 respectively. But since any  $\mathfrak{S}_{n+3}$ -representation is completely reducible,  $F$  must admit a complementary invariant subspace of dimension  $(n+1)(n+2) - 1 > 1 = \dim(F)$ , contradicting the maximality of  $\dim(F)$  among the non trivial proper sub-representations of  $A_C$ . Thus there is no such a sub-representation, which proves the first point.

Let  $[\lambda]$  be the Young diagram of  $AR_C(\mathcal{W}_{0,n+3})$  as a  $\mathfrak{S}_{n+3}$ -module, for a partition  $\lambda$  of  $n+3$ . Since  $\mathfrak{S}_{n+2} = \text{Fix}(n+3)$  acts irreducibly on  $A_{\widehat{n+3}}$  with Young symbol  $[2, 1^n]$  according to the third point of Proposition 4.3, it comes that  $[\lambda] \Big|_{\mathfrak{S}_{n+2}}$  admits  $[2, 1^n]$  as one of its irreducible factors. Since  $[\lambda]$  is obtained by adding one box to  $[2, 1^n]$ , we deduce that there are only two possibilities for the former Young symbol: it is either  $[3, 1^n]$  which has dimension  $(n+1)(n+2)/2$ , or  $[2, 2, 1^n]$ , which is of dimension one less. This finishes the proof of the proposition.  $\square$

**Lemma 4.10.** *When  $n$  is odd,  $AR_{n+2,n+3}$  does not belong to  $\langle AR_{i,j} \mid \begin{smallmatrix} i = 1, \dots, n \\ j = i+1, \dots, n+2 \end{smallmatrix} \rangle$ .*

*Proof.* Assume that there exist scalars  $c_{i,j}$  for  $(i,j) \in I_n^n$  such that

$$(57) \quad A_{n+2,n+3} = \sum_{(i,j) \in J_n} c_{i,j} AR_{i,j}.$$

From the second point of Corollary 4.8, it follows that this relation is necessarily unique.

On the other hand, for any  $(k,l)$  with  $k = 1, \dots, n$  and  $l = k+1, \dots, n+1$ , it follows from Lemma 4.4 that  $(k,l) \cdot A_{n+2,n+3} = -A_{n+2,n+3}$  and  $(k,l) \cdot A_{k,l} = (-1)^{n-1} A_{k,l} = A_{k,l}$  (since  $n-1$  is even). Applying  $(k,l)$  to (57) would give another such identity if  $c_{k,l}$  were not zero, and this is not possible by the argument given just above. We thus get that all the constants  $c_{i,j}$  are zero for  $i = 1, \dots, n$  and  $j = 1, \dots, n+1$ .

Hence (57) actually is written

$$A_{n+2,n+3} = \sum_{i=1}^n c_{i,n+2} AR_{i,n+2}.$$

and such a relation does not exist since, as it follows from the first point of Proposition 4.3, the combinatorial ARs appearing in it form a free family. Thus there is no identity of the form (57), which gives us the lemma.  $\square$

With the previous results at hand, it is then easy to establish Theorem 4.2:

*Proof of Theorem 4.2.* When  $n$  is odd, it follows from Lemma 4.10 that the integer  $\eta$  defined in (56) is equal to  $n+1$ . Hence  $AR_C(\mathcal{W}_{0,n+3})$  has dimension  $(n+1)(n+2)/2$  from which it follows that  $\mathcal{W}_{0,n+3}$  has maximal rank with all its ARs combinatorial. Then Corollary 4.9.2 allows us to conclude.

To prove the theorem in case when  $n$  is even, it suffices to prove that  $\eta = n$ , which would follow from exhibiting a non trivial abelian relation which is not combinatorial. We will show that the so-called Euler's abelian relation does the job, hence the theorem follows.  $\square$

4.5. Leaving aside the question of the structure of  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  as a  $\mathfrak{S}_{n+3}$ -module, it is not difficult to describe another rather efficient computational approach for building the combinatorial abelian relations. To this end, we first need to set some notation which will be also be used later on.

Given  $x = (x_1, \dots, x_n)$ , we use the shorthand  $x' = (x_2, \dots, x_n) = U_1(x)$ ; we introduce two new but fixed variables  $x_{n+1} = 0$  and  $x_{n+2} = 1$ ; we set  $\tilde{x} = (x_1, \dots, x_n, x_{n+1}, x_{n+2})$ ; and for  $i, j$  such that  $i = 1, \dots, n+1$  and  $j = i+1, \dots, n+2$ , we set  $\tilde{x}'_{ij} = (x_s)_{s=1, s \neq i, j}^{n+2}$ . We allow ourselves to combine all these notation: for instance, for  $i, j$  as above  $\tilde{x}'_{ij}$  stands for the  $(n-1)$ -tuple obtained by removing the first, the  $i$ th and the  $j$ -th entries to  $\tilde{x}$ . Since we will always deal with pairs of indices  $(i, j)$  with  $2 \leq i \leq n$ , there will be no ambiguity with this notation in what follows. We denote by  $B_n$  the set of pairs we will work with:

$$B_n = \left\{ (i, j) \mid i = 2, \dots, n, j = i+1, \dots, n+2 \right\}.$$

For any  $N > 0$ , we consider the following homogeneous polynomial of  $z \in \mathbf{C}^N$ :

$$M_{0,N}(z) = \prod_{1 \leq p < q \leq N} (z_p - z_q).$$

**Remark 4.11.** For  $N = n+3$ , one has  $M_{0,N}(\tilde{x}) = -\prod_{1 \leq i \leq n} x_i(x_i - 1) \prod_{1 \leq i < j \leq n} (x_i - x_j)$  for any  $x \in \mathbf{C}^n$  hence the equation  $M_{0,N}(\tilde{x}) = 0$  cuts out the braid arrangement  $A_{n+3}$  in  $\mathbf{C}^n$ , whose complement  $\mathbf{C}^n \setminus A_{n+3}$  is isomorphic to the moduli space  $\mathcal{M}_{0,n+3}$  (see (131)). This explains the notation  $M_{0,N} = M_{0,n+3}$  for the polynomial above.

We define rational functions  $F_0$  and  $F_{ij}$  for  $(i, j) \in B_n$ , of  $x'$ , by setting

$$(58) \quad F_0(x') = \frac{1}{x_2 \dots x_n} \quad \text{and} \quad F_{ij}(x') = (\tilde{x}_i - \tilde{x}_j)^{n-1} \frac{M_{0,n-1}(\tilde{x}'_{ij})}{M_{0,n+1}(\tilde{x}')} \quad \text{for any } (i, j) \in B_n.$$

Then we set  $B_n(x') = \{F_{ij}\}_{(i,j) \in B_n}$  and

$$(59) \quad \mathfrak{B}_n(x') = \begin{cases} B_n(x') & \text{if } n \text{ is even;} \\ B_n(x') \cup \{F_0\} & \text{if } n \text{ is odd.} \end{cases}$$

The cardinal of  $\mathfrak{B}_n(x')$  is  $(n-1)(n+2)/2$  for  $n$  even and  $n(n+1)/2 = (n-1)(n+2)/2 + 1$  otherwise. In view of describing the components of the ARs of  $\mathcal{W}_{0,n+3}$ , we set

$$(60) \quad \begin{aligned} & \bullet V_1(x) = U_1(x) = x' = (x_2, \dots, x_n) \text{ and } \Pi_1 = \Omega_1 = dx_2 \wedge \dots \wedge dx_n; \\ & \bullet V_i(x) = (C^{\circ(i-1)})^*(V_1) \text{ and } \Pi_i = (C^{\circ(i-1)})^*(\Pi_1) \text{ for } i = 2, \dots, n+3; \\ & \bullet \mathfrak{B}_n(V_i) = (C^{\circ(i-1)})^*(\mathfrak{B}_n(x')) = \{F \circ V_i\}_{F \in \mathfrak{B}_n(x')} \text{ for } i = 1, \dots, n+3. \end{aligned}$$

By considering their respective poles, one gets easily that the elements of  $\mathfrak{B}_n(x')$  are linearly independent. Then, by direct elementary but computational checks in what concerns the second point, we prove the

**Proposition 4.12.** 1. For any  $n \geq 2$ , the elements of  $\mathfrak{B}_n(x')$  are linearly independent over  $\mathbf{C}$ ;

2. The following holds true for any  $n \leq 12$ : for any  $i = 1, \dots, n+3$ , the space  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})[i]$  of  $V_i$ -th components of combinatorial abelian relations admits  $\{F_i \Pi_i \mid F_i \in \mathfrak{B}_n(V_i)\}$  as a basis.

The interest of this result is threefold: first and of course, we conjecture that it holds true in full generality. Second, it indicates that  $AR_C(\mathcal{W}_{0,n+3})$  can be determined by pure linear algebra in the finite dimensional direct sum  $\oplus_{i=1}^{n+3} \mathfrak{B}_n(V_i)$ . This elementary fact opens the door for the possibility to arrive at an explicit closed formula for (a multiple of) the non trivial abelian relation  $AR_{i,j}$  of  $AR_{\bar{i}}$  discussed just above, this for any  $(i, j) \in I_n$ . Finally, as we will see in the next section, the proposition above and more generally the notation and formulas of the current subsection are useful to explicitly describe Euler's abelian relation.

**Remark 4.13.** *One can verify that, if taking instead of the map  $C$  in (46) the cyclic birational automorphism of  $\mathcal{W}_{0,n+3}$  given in the coordinates  $x_i$ 's by*

$$\tilde{C}(x) = \left( \frac{x_n}{x_n - x_{i-1}} \right)_{i=1}^n$$

(with the convention that  $x_0 = 1$ ), then all the preceding definitions make sense and all the subsequent results (especially the preceding proposition) hold true *mutatis mutandis*. This will be used later on in §5.1.7.

## 5. On Euler's abelian relation.

In this section, we discuss Euler's AR. Since every AR of  $\mathcal{W}_{0,n+3}$  is combinatorial hence rational when  $n$  is odd, it is certain that there is something wrong in Damiano's claims (8) about this particular abelian relation for this parity of  $n$ . In order to make everything as clear as possible, in this section we present a thorough study of this AR and before that, of its construction.

Following Damiano but giving more details, we first describe the construction of a new abelian relation  $\mathcal{E}_n$  for  $\mathcal{W}_{0,n+3}$  using Gelfand-MacPherson's theory applied to a characteristic class on the *oriented* Grassmannian variety  $G_2^{or}(\mathbf{R}^{n+3})$  of oriented 2-planes in  $\mathbf{R}^{n+3}$ . In §5.1.4, we give a rather concise integral formula for the components of  $\mathcal{E}_n$  which we use to determine the fourth-order jet of  $\mathcal{E}_3$  at a specific point. Then in §5.1.5, we discuss some invariance properties satisfied by  $\mathcal{E}_n$  for a certain dihedral action preserving a particular connected component of  $G_2^{or}(\mathbf{R}^{n+3})$  (its *positive part*  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ ) and explain how two non trivial functional identities satisfied by the components of  $\mathcal{E}_n$  can be deduced from them.

After having completely explicated  $\mathcal{E}_3$  in §5.1.6, we turn to the invariant properties of  $\mathcal{E}_n$  with respect to the birational action of the whole symmetric group  $\mathfrak{S}_{n+3}$  on  $\mathcal{M}_{0,n+3}$  in §5.1.7. We show that things actually are more subtle than as described by Damiano, the main and crucial fact being that, contrarily to the case when  $n$  is even, when  $n$  is odd Euler's abelian relation is actually not canonically defined on each connected component of  $\mathcal{M}_{0,n+3}$ , but only up to sign. We explain that this fact, that does not seem really significant at first glance, actually has the consequence that Damiano's construction of a 1-dimensional  $\mathfrak{S}_{n+3}$ -representation associated to Euler's abelian relation is irremediably flawed when  $n$  is odd.

Finally, in §5.1.8 and §5.1.9, which concern the cases when  $n$  is odd and even respectively, we give two closed formulas for the components of Euler's abelian relation. Both formulas are conjectural in full generality but are proven to hold true indeed for  $n$  less than or equal to 12 (by means of formal computations).

**5.1. On Gelfand-MacPherson's theory of generalized dilogarithm forms.** Euler's abelian relation is discussed in [Da2, §5] (see also [Da1, Chap. 6]). Its construction relies on the general theory of 'generalized dilogarithm forms' exposed in [GM]. Gelfand and MacPherson worked with usual grassmannian varieties  $G_N(\mathbf{R}^{N+M})$  whereas Damiano's chose to do so with oriented grassmannians  $G_N^{or}(\mathbf{R}^{N+M})$ , moreover in the specific case when  $N = 2$  and  $M = n + 1$ . In order to show more clearly where the problem in Damiano's construction of a polylogarithmic AR for  $\mathcal{W}_{0,n+3}$  lies, we give below an overview of the corresponding constructions. For proofs, we refer the reader to the two previously cited articles.

5.1.1. In order to consider both cases together, namely the standard and the oriented grassmannians and associated configuration spaces, we agree that  $*$  either stands for nothing or for 'or', the former notation referring of course to the standard case and the latter to the oriented one. Considering this, we set  $O_m^{or}(\mathbf{R}) = SO_m(\mathbf{R})$  for any  $m \geq 1$ .

Below, '*plane*' means a subvector space of dimension 2, and when  $*$  = or, a '*\*-plane*' is a 2-plane provided with an orientation. We will denote a *\*-plane* by  $\xi^*$  whereas  $\xi$  will stand for the associated (non oriented) plane.

- $G_2^*(\mathbf{R}^{n+3}) = O_{n+3}^*(\mathbf{R})/(O_2^*(\mathbf{R}) \times O_{n+1}^*(\mathbf{R}))$  is the grassmannian of *\*-planes* in  $\mathbf{R}^{n+3}$ ;

- $\mathrm{GL}_{n+3}(\mathbf{R})$  acts on  $\mathbf{R}^{n+3}$  hence on  $G_2^*(\mathbf{R}^{n+3})$  as well. This action is not effective: since the center  $\mathbf{R}^*\mathrm{Id}$  acts trivially, this action factors through  $\mathrm{PGL}_{n+3}(\mathbf{R}) = \mathrm{GL}_{n+3}(\mathbf{R})/(\mathbf{R}^*\mathrm{Id})$ ;
- $\widehat{G}_2^*(\mathbf{R}^{n+3})$  stands for the dense open subset of *generic*  $*$ -planes in  $\mathbf{R}^{n+3}$ , which are those which do not contain any coordinate line and which are not contained in any coordinate hyperplane;
- $\widetilde{H} \subset \mathrm{GL}_{n+3}(\mathbf{R})$  denotes the abelian subgroup of diagonal matrices and  $H = \widetilde{H}/(\mathbf{R}^*\mathrm{Id})$  stands for its image in  $\mathrm{PGL}_{n+3}(\mathbf{R})$ . The latter group is a maximal torus isomorphic to  $(\mathbf{R}^*)^{n+2}$  (in natural but non unique ways);
- $\widetilde{H}_0$  and  $H_0$  are the connected components of the identity of  $\widetilde{H}$  and  $H$  respectively (thus  $H_0 \simeq (\mathbf{R}_{>0})^{n+2}$  but again not in a canonical way);
- $\widetilde{K}$  is the subgroup of  $\widetilde{H}$  formed by the matrices with diagonal coefficients in  $\{\pm 1\} \simeq \mathbf{Z}_2$  and  $K$  stands for its image in  $H$ . One has  $K \simeq (\mathbf{Z}_2)^{n+2}$  and since  $H$  is commutative, we have that  $H \simeq H_0 \times K \simeq (\mathbf{R}_{>0})^{n+2} \times (\mathbf{Z}_2)^{n+2}$ ;
- we define  $C_2^*(n+3) = \widehat{G}_2^*(\mathbf{R}^{n+3})/H$  (resp.  $EC_2^*(n+3) = \widehat{G}_2^*(\mathbf{R}^{n+3})/H_0$ ) as the space of  $*$ -configurations (resp. *enhanced*  $*$ -configurations);
- given an oriented 2-plane  $\xi^{or}$ , we denote by  $\check{\xi}^{or}$  the same plane but endowed with the opposite orientation, and by  $\xi$  the standard plane (disregarding the orientations);
- The map  $D : \xi^{or} \rightarrow D(\xi^{or}) = \check{\xi}^{or}$  is an involutive isomorphism which is the deck transformation of the 2-1 (universal) covering  $\nu : G_2^{or}(\mathbf{R}^{n+3}) \rightarrow G_2(\mathbf{R}^{n+3})$ ,  $\xi^{or} \mapsto \xi$ .

The spaces, the quotients maps by  $H$  and  $H_0$  and the coverings induced by  $\nu$  discussed above all fit into the following commutative diagram :

$$(61) \quad \begin{array}{ccc} \widehat{G}_2^{or}(\mathbf{R}^{n+3}) & \xrightarrow{\nu} & \widehat{G}_2(\mathbf{R}^{n+2}) \\ \downarrow \pi_0 & & \downarrow \pi_0 \\ EC_2^{or}(n+3) & \xrightarrow{\nu} & EC_2(n+3) \\ \downarrow \kappa & & \downarrow \kappa \\ C_2^{or}(n+3) & \xrightarrow{\nu} & C_2(n+3) \end{array} \quad \begin{array}{l} \swarrow \pi_H \\ \searrow \pi_H \end{array}$$

In this diagram, the horizontal maps all are non ramified 2-1 coverings induced by  $\nu$  (hence are denoted by the same notation), the maps  $\pi_0$  and  $\kappa$  correspond to quotienting by  $H_0$  and  $K$  respectively, hence the compositions  $\pi_H = \kappa \circ \pi_0$  are the maps induced by quotienting by  $H$ .

As is well-known,  $C_2^*(n+3)$  naturally identifies with the space  $(\mathbf{P}_{\mathbf{R}}^1)^{n+3} - \Delta$  of  $(n+3)$ -tuple of pairwise distinct points on the real projective line<sup>35</sup>, quotiented by  $\mathrm{GL}_2^*(\mathbf{R})/(\mathbf{R}_{>0}\mathbf{I}_2)$  where we agree that  $\mathrm{GL}_2^{or}(\mathbf{R}) = \mathrm{GL}_2^+(\mathbf{R}) = \{g \in \mathrm{GL}_2(\mathbf{R}) \mid \det(g) > 0\}$ . In the standard case when  $*$  is empty, one recovers  $\mathcal{M}_{0,n+3}(\mathbf{R})$ . When  $*$  = *or*, the quotient  $C_2^{or}(n+3) = \widehat{G}_2^{or}(\mathbf{R}^{n+3})/H$  can be seen as the set of  $(n+3)$ -tuples of pairwise distinct points on  $\mathbf{P}_{\mathbf{R}}^1 \simeq \mathbf{S}^1 = \{z \in \mathbf{C}, |z|=1\} \subset \mathbf{C}$  modulo the action of the subgroup  $\mathrm{Mob}^+(\mathbf{S}^1)$  of Moëbius transformations preserving the standard orientation

<sup>35</sup>Here  $\Delta$  stands for the union of all the small diagonals in  $(\mathbf{P}_{\mathbf{R}}^1)^{n+3}$ .

of  $\mathbf{S}^1$ . Hence it appears natural to denote this space of oriented configurations by  $\mathcal{M}_{0,n+3}^{or}(\mathbf{R})$ . The map

$$\nu : \mathcal{M}_{0,n+3}^{or}(\mathbf{R}) \rightarrow \mathcal{M}_{0,n+3}(\mathbf{R})$$

is simply given by identifying configurations when reversing the orientation of  $\mathbf{P}_{\mathbf{R}}^1 \simeq \mathbf{S}^1$ .

To explain what an enhanced configuration is, we denote by  $\mathbb{S}^1 \subset \mathbf{R}^2$  the circle from which  $\mathbf{P}_{\mathbf{R}}^1 = \mathbf{S}^1$  is obtained by identifying diametrically opposite points. Then an ‘enhanced configuration’ (that is, an element of  $EC_2(n+3)$ ) is a projective equivalence class of a  $(n+3)$ -tuple  $(c_i)_{i=1}^{n+3} \in \mathbf{P}_{\mathbf{R}}^1 - \Delta$  enhanced by a choice of a lift  $\tilde{c}_i \in \mathbb{S}^1$  of  $c_i$  for each  $i = 1, \dots, n+3$ . Of course, there is a similar description of what an ‘enhanced oriented configuration’ is, we leave it to the interested reader to elaborate on this.

Recall (see §2.1.5 above) that  $\mathcal{M}_{0,n+3}(\mathbf{R})$  is not connected and that its connected components  $\mathcal{M}(\sigma)$  are first, biunivocally labeled by classes  $\sigma \in \mathfrak{R}_{n+3} = \mathfrak{S}_{n+3}/D_{0,n+3}$  and second, all isomorphic to  $\mathcal{M}(\mathbf{1}) = \mathcal{M}_{0,n+3}^{>0}(\mathbf{R})$ . Our purpose below is to explain Damiano’s construction of an AR on any component  $\mathcal{M}(\sigma)$  and to describe some of its properties. We will denote by  $\mathcal{E}_n(\sigma)$  this AR which is an element of  $\mathbf{AR}(\sigma)$ . The question of the nature of the  $\mathcal{E}_n(\sigma)$ ’s and how these ARs are related is crucial and will be discussed. But in a first step, we will only consider the case of the positive part  $\mathcal{M}_{0,n+3}^{>0}(\mathbf{R})$  and of the AR  $\mathcal{E}_n(\mathbf{1})$  it carries, that we will denote by  $\mathcal{E}_n^{>0}$  to simplify.

**5.1.2. Euler’s differential form.** When  $n$  is even, one could work with usual grassmannians of 2-planes, which are precisely the manifolds used by Gelfand and MacPherson to develop their theory of higher polylogarithmic forms and of the differential relations they satisfy. However this is not possible when  $n$  is odd (see Remark 5.19 below for an explanation of this) and it is then necessary to work at the level of the oriented grassmannian. In order to give a presentation independent of the parity of  $n$ , we will place ourselves within this ‘oriented framework’.

We assume below that  $\mathbf{R}^{n+3}$  is endowed with its standard euclidean structure:  $e_1, \dots, e_{n+3}$  are the elements of the canonical basis, that is  $e_i = (\delta_{ij})_{j=1}^{n+3}$  for  $i = 1, \dots, n+3$ . One denotes by  $x_1, \dots, x_{n+3}$  the corresponding standard coordinates, and one has

$$(x, y) = \sum_{k=1}^{n+3} x_k y_k \quad \text{and} \quad |x| = \sqrt{(x, x)} = \sqrt{(x_1)^2 + \dots + (x_{n+3})^2}$$

for any two elements  $x = (x_k)_{k=1}^{n+3}$  and  $y = (y_k)_{k=1}^{n+3}$  of  $\mathbf{R}^{n+3}$ . Finally, we will denote by  $\mathbf{R}_i^{n+2}$  the coordinate hyperplane in  $\mathbf{R}^{n+3}$  cut out by the equation  $x_i = 0$ .

Let  $\mathcal{T}^{or}$  be the tautological bundle over  $G_2^{or}(\mathbf{R}^{n+3})$ . Its Euler’s class  $E_n^{or} = E(\mathcal{T}^{or})$  is a non trivial element in  $H^2(G_2^{or}(\mathbf{R}^{n+3}), \mathbf{R})$ . Since the oriented grassmannian is a symmetric space, there exists a unique  $SO_{n+2}(\mathbf{R})$ -invariant global differential 2-form  $E_n^{or}$  on it such that  $[E_n^{or}] = E(\mathcal{T}^{or})$ . To simplify, we will no longer write the superscript  $or$  and just write  $E_n$  in what follows.

We consider the Stiefel manifold  $\mathbb{S}_2^{or}(\mathbf{R}^{n+3})$  of oriented 2-frame in  $\mathbf{R}^{n+3}$ : it is the manifold of dimension  $2n+3$  formed by orthogonal 2-frames in  $\mathbf{R}^{n+3}$ :

$$\mathbb{S}_2^{or}(\mathbf{R}^{n+3}) = \left\{ (e_1, e_2) \in (\mathbf{R}^{n+3})^2 \mid |e_1| = |e_2| = 1 \text{ and } \langle e_1, e_2 \rangle = 0 \right\}.$$



It comes with a natural map  $\mathcal{S} : \mathbb{S}_2^{or}(\mathbf{R}^{n+3}) \rightarrow G_2^{or}(\mathbf{R}^{n+3})$ ,  $(e_1, e_2) \mapsto e_1 \wedge e_2$  which makes of the Stiefel variety a  $\text{SO}_2(\mathbf{R})$ -bundle over the oriented grassmannian. The interest of considering the Stiefel manifold is that there is a simple explicit formula for the pull-back of  $E_n$  under  $\mathcal{S}$ .

Indeed, for  $i = 1, 2$ , let  $de_i$  be the  $\mathbf{R}^{n+3}$ -valued 1-form whose components are the exterior derivatives of the scalar components of  $e_i$ : if  $e_i = (e_i^k)_{k=1}^{n+3}$ , then  $de_i = (de_i^k)_{k=1}^{n+3}$ . Then the componentwise wedge product of  $de_1$  with  $de_2$  is a multiple of the pull-back of  $E_n$  under  $\mathcal{S}$  since

$$(62) \quad \mathcal{E}_n = de_1 \wedge de_2 = \sum_{k=1}^{n+3} de_1^k \wedge de_2^k = \frac{-1}{2\pi} \mathcal{S}^*(E_n)$$

(cf. [Da1, §6.2] or [Da2, §6] for more details and references). Since we are only interested in  $E_n$  up to a non zero multiple, it will be more convenient to work with  $\mathcal{E}_n \in \Gamma(\Omega^2(\mathbb{S}_2^{or}(\mathbf{R}^{n+3})))$  in what follows.

Formula (62) above gives us an effective way to compute  $\mathcal{E}_n$ . Indeed, let  $\gamma : U \rightarrow G_2^{or}(\mathbf{R}^{n+3})$  be a smooth map where  $U$  stands for an open domain in an affine space of dimension  $N$ . Given affine coordinates  $u_1, \dots, u_N$  on  $U$ , the question is to give an explicit formula for (a multiple of)  $\gamma^*(E_n^{or})$  in the  $u_i$ 's. We assume that  $\gamma$  is given by  $u \mapsto \gamma_1(u) \wedge \gamma_2(u)$  where both  $\gamma_1$  and  $\gamma_2$  are smooth maps from  $U$  to  $\mathbf{R}^{n+3}$  such that  $\gamma_1(u)$  and  $\gamma_2(u)$  are not colinear for any  $u \in U$ . Denote by  $GS$  the Gram-Schmidt orthogonalization process, defined here as the map

$$GS : \Omega^{n+3} \longrightarrow \mathbb{S}_2^{or}(\mathbf{R}^{n+3}) \\ (x, y) \longmapsto \left( \frac{x}{|x|}, \frac{\tilde{y}}{|\tilde{y}|} \right)$$

where  $\Omega^{n+3}$  stands for the set of pairs of two linearly independent vectors of  $\mathbf{R}^{n+3}$  and where for such a pair  $(x, y)$ , we have set  $\tilde{y} = y - ((x, y)/|x|^2)x$ .

Then  $\Gamma = GS \circ \gamma$  is a lift of  $\gamma$  to the Stiefel variety, *i.e.* the following diagram is commutative

$$\begin{array}{ccc} & & \mathbb{S}_2^{or}(\mathbf{R}^{n+3}) \\ & \nearrow \Gamma & \downarrow \mathcal{S} \\ U & \xrightarrow{\gamma} & G_2^{or}(\mathbf{R}^{n+3}) \end{array}$$

hence we obtain that  $\Gamma^*(\mathcal{E}_n)$  coincides with  $-1/(2\pi)$  times the pull-back of  $E_n$  under  $\gamma$ : one has

$$(63) \quad \Gamma^*(\mathcal{E}_n) = \frac{-1}{2\pi} \gamma^*(E_n).$$

We use the material of the previous paragraph in the somehow ‘most general map  $\gamma$ ’ in order to deduce some formulas for (pull-backs of) Euler’s form which will be used later on. The map we are referring to here, denoted by  $\xi$ , is just the map associating to a  $2 \times (n+3)$  matrix  $M$  the oriented 2-plane  $\xi(M)$  spanned by the first and the second rows of  $M$ , denoted by  $x(M)$  and  $y(M)$  respectively: one has  $\xi(M) = x(M) \wedge y(M)$ . For  $x, y \in \mathbf{R}^{n+3}$  we denote by  $M_{x,y}$  the matrix such that  $x(M_{x,y}) = x$  and  $y(M_{x,y}) = y$  and we set  $\xi(x, y) = \xi(M_{x,y}) \in G_2^{or}(\mathbf{R}^{n+3})$ . The map  $\xi$  under

scrutiny here is defined on the subset  $\Omega_2 \subset M_{2 \times (n+3)}(\mathbf{R})$  of rank 2 matrices:

$$(64) \quad \begin{aligned} \xi : \Omega_2 &\longrightarrow G_2^{or}(\mathbf{R}^{n+3}) \\ M = \begin{bmatrix} x \\ y \end{bmatrix} &\longmapsto \xi(M) = \xi(x, y) = x \wedge y. \end{aligned}$$

From  $\gamma : \Omega_2 \rightarrow G_2^{or}(\mathbf{R}^{n+3})$ , we build other such maps by setting for any  $M_{x,y} \in \Omega_2$

$$\check{\gamma}(M_{x,y}) = \gamma(M_{x,y}) \quad \text{and} \quad \gamma_G(M_{x,y}) = \gamma(M_{x,y} \cdot G)$$

where  $G$  stands for a previously given constant element of the linear group  $GL_{n+3}(\mathbf{R})$ . For  $\epsilon = (\epsilon_k)_{k=1}^{n+3} \in \{\pm 1\}$ , we denote by  $D_\epsilon$  the diagonal matrix with  $\epsilon_k$  for its  $k$ -th coefficient, that is  $D_\epsilon = \text{diag}(\epsilon_1, \dots, \epsilon_{n+3}) \in GL_{n+3}(\mathbf{R})$  and we set  $\gamma_\epsilon = \gamma_{D_\epsilon}$ . With the preceding notation, one can state some transformation formulas for (the pull-backs of) Euler's form that will prove to be important in the sequel.

**Lemma 5.1.** *The 2-form  $E_n$  satisfies the following transformation formulas:*

1.  $\check{\gamma}^*(E_n) = -\gamma^*(E_n)$ ;
2.  $\gamma_\epsilon^*(E_n) = \gamma^*(E_n)$  for any  $\epsilon \in \{\pm 1\}^{n+3}$ ;
3.  $\gamma_G^*(E_n) = \gamma^*(E_n)$  for any  $G \in SO_{n+3}(\mathbf{R})$ .

*Proof.* As the notation suggests, one has  $\check{\gamma} = D \circ \gamma$  where  $D$  is the change of orientation. Since  $D$  has for  $\mathcal{S}$ -equivariant lift the exchange map  $(e_1, e_2) \rightarrow (e_2, e_1)$  on the Stiefel manifold, 1. follows from  $de_2 \wedge de_1 = -de_1 \wedge de_2$  together with (62). The second point is a direct consequence of this formula as well. As for 3., it follows immediatly from the  $SO_{n+3}(\mathbf{R})$ -invariance of  $E_n$ .  $\square$

For  $i \in \{1, \dots, n+3\}$ , the natural inclusion  $\mathbf{R}_i^{n+2} \subset \mathbf{R}^{n+3}$  induces a natural embedding  $\iota_i : G_2^{or}(\mathbf{R}_i^{n+2}) \hookrightarrow G_2^{or}(\mathbf{R}^{n+3})$  which is such that the following lemma holds true:

**Lemma 5.2.** *For any  $k \geq 1$ , denote by  $E_n^k$  the  $k$ -th wedge product of  $E_n$ .*

1. *The pull-back  $\iota_i^*(E_n)$  coincides with the Euler form of the tautological bundle of  $G_2^{or}(\mathbf{R}_i^{n+2})$ .*
2. *The pull-back  $\iota_i^*(E_n^n)$  is a  $SO_{n+2}$ -invariant volume form on  $G_2^{or}(\mathbf{R}_i^{n+2})$ .*

*Proof.* Since  $\iota_i^*(\mathcal{T}^{or})$  coincides with the tautological bundle on  $G_2^{or}(\mathbf{R}_i^{n+2})$ , the first point follows from the naturality of the Euler's class and of the  $SO_{n+3}$ -invariance of the representative  $E_n$ .

The second point follows from the first combined with all the following facts: *i.*  $G_2^{or}(\mathbf{R}_i^{n+2})$  is orientable and compact hence admits a necessarily non exact volume form; and *ii.* the top degree cohomology space  $H^{2n}(G_2^{or}(\mathbf{R}_i^{n+2}), \mathbf{R}) \simeq \mathbf{R}$  is generated over  $\mathbf{R}$  by (the class of) the  $SO_{n+2}$ -invariant form  $E_{n-1}^n$ .  $\square$

With the constructions and results discussed above at hand, one can now follow Damiano's construction of an AR from a power of the Euler class  $E_n$ . It is the subject of the next subsection.

**5.1.3. The positive Eulerian abelian relation.** Here, we review Damiano's construction (which itself heavily relies on Gelfand-MacPherson theory [GM]) of the Euler abelian relation for  $\mathcal{W}_{0,n+3}$  on the positive part  $\mathcal{M}_{0,n+3}^{>0}$ , which is the privileged component of the moduli space  $\mathcal{M}_{0,n+3}$  we will work on. We will discuss later the Euler's abelian relations on the other components  $\mathcal{M}(\sigma)$ .

Let  $U \subset \mathbf{R}^n$  be the set of  $n$ -tuples  $u = (u_i)_{i=1}^n \in \mathbf{R}^n$  such that  $1 < u_1 < u_2 < \dots < u_n$ . The map

$$(65) \quad \begin{aligned} \varphi : U &\longrightarrow \mathcal{M}_{0,n+3}^{or,>0} \\ u &\longmapsto [\infty, 0, -1, -u_1, -u_2, \dots, -u_n] \end{aligned}$$

can be seen as a global isomorphism hence the  $u_i$ 's form a global coordinate system on  $\mathcal{M}_{0,n+3}^{>0}$ , which is the one we will work with.

Recall that a matrix  $M \in M_{2,n+3}(\mathbf{R})$  is said 'positive' if all its  $2 \times 2$  minors are positive. We then define the 'positive oriented grassmannian'  $G_2^{or}(\mathbf{R}^{n+2})^{>0}$  as the image by the map (64) of the open subset  $\Omega_2^{>0} \subset \Omega_2$  of positive matrices: one has

$$(66) \quad G_2^{or}(\mathbf{R}^{n+2})^{>0} = \xi(\Omega_2^{>0}).$$

One verifies that the positive grassmannian  $G_2^{or}(\mathbf{R}^{n+2})^{>0}$  is

- included in  $\widehat{G}_2^{or}(\mathbf{R}^{n+2})$ ;
- stable under the action of the 'positive part'  $H_0$  of the Cartan torus  $H \subset \mathrm{SL}_{n+3}(\mathbf{R})$ ;
- such that the quotient of it by  $H$  is  $\mathcal{M}_{0,n+3}^{>0}$ , i.e. one has (cf. diagram (61)):

$$\pi_H(G_2^{or}(\mathbf{R}^{n+2})^{>0}) = \mathcal{M}_{0,n+3}^{>0}.$$

We consider the following map

$$(67) \quad \begin{aligned} M : U &\longrightarrow \Omega_2^{>0} \\ u &\longmapsto M_u = \begin{bmatrix} 1 & 0 & -1 & -u_1 & -u_2 & \dots & -u_n \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 \end{bmatrix} \end{aligned}$$

which is such that (1) the composition  $\gamma = \xi \circ M : U \rightarrow G_2^{or}(\mathbf{R}^{n+2})$  takes values into the positive grassmannian; and (2) makes the following diagram commutative:

$$(68) \quad \begin{array}{ccc} & & G_2^{or}(\mathbf{R}^{n+3})^{>0} \\ & \nearrow \gamma = \xi \circ M & \downarrow \pi_H \\ U & \xrightarrow{\varphi} & \mathcal{M}_{0,n+3}^{>0} \end{array}$$

It is not difficult to give an explicit formula for  $\pi = \varphi^{-1} \circ \pi_H : G_2^{or}(\mathbf{R}^{n+3})^{>0} \rightarrow U$ . Given  $\xi^{or} \in G_2^{or}(\mathbf{R}^{n+3})^{>0}$ , one has  $\xi^{or} = \xi_{x,y} = \xi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$  for  $x, y \in \mathbf{R}^{n+3}$  such that  $x_i y_j - y_i x_j > 0$  for any  $i, j$  such that  $1 \leq i < j \leq n+3$ . In particular, the submatrix  $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$  is invertible, hence one has  $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & \tilde{x}_3 & \dots & \tilde{x}_{n+3} \\ 0 & 1 & \tilde{y}_3 & \dots & \tilde{y}_{n+3} \end{bmatrix}$  for some  $\tilde{x}_k$ 's and  $\tilde{y}_k$ 's with  $k = 3, \dots, n+3$ , which are explicit rational expressions in the original  $x_i$ 's and  $y_i$ 's. Notice that, since all  $2 \times 2$  minors are positive,

one necessarily has  $\tilde{x}_k < 0$  and  $\tilde{y}_k > 0$  for any  $k = 3, \dots, n+3$ . Then setting  $h_1 = -\tilde{x}_3$ ,  $h_2 = \tilde{y}_3$  and  $h_k = \tilde{y}_3/\tilde{y}_k$  for all such  $k$ , one has

$$\begin{bmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \tilde{x}_3 & \cdots & \tilde{x}_{n+3} \\ 0 & 1 & \tilde{y}_3 & \cdots & \tilde{y}_{n+3} \end{bmatrix} \cdot \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_{n+3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -\frac{\tilde{x}_4 \tilde{y}_3}{\tilde{y}_4 \tilde{x}_3} & \cdots & -\frac{\tilde{x}_{n+3} \tilde{y}_3}{\tilde{y}_{n+3} \tilde{x}_3} \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

from which it follows that the map  $\pi$  under scrutiny is given in coordinates by

$$(69) \quad \pi : G_2^{or}(\mathbf{R}^{n+3})^{>0} \longrightarrow U$$

$$\xi_{x,y} \longmapsto \left( \frac{\tilde{x}_4 \tilde{y}_3}{\tilde{y}_4 \tilde{x}_3}, \dots, \frac{\tilde{x}_{n+3} \tilde{y}_3}{\tilde{y}_{n+3} \tilde{x}_3} \right)$$

(it is easy and left to the reader to show that this is well-defined and is the right formula).

★

Recall the way of parametrizing the  $H_0$ -orbit of any generic oriented 2-plane  $\xi^{or}$  by means of the hypersimplex  $\Delta_2^{n+3} = \{(t_i)_{i=1}^{n+3} \in ]0, 1[^{n+3} \mid \sum_{i=1}^{n+3} t_i = 2\}$  (see [GM]): the map

$$h : \Delta_2^{n+3} \longrightarrow (\mathbf{R}_{>0})^{n+3}, \quad t = (t_i)_{i=1}^{n+3} \longmapsto h(t) = (t_i/(1-t_i))_{i=1}^{n+3}$$

is such that the image of

$$Dh = \text{Diag} \circ h : \Delta_2^{n+3} \longrightarrow \mathbf{PSL}_{n+3}(\mathbf{R})$$

$$t \longmapsto \text{Diag}(h(t)) = \begin{bmatrix} \frac{t_1}{1-t_1} & & 0 \\ & \ddots & \\ 0 & & \frac{t_{n+3}}{1-t_{n+3}} \end{bmatrix}$$

is equal to the positive component  $H_0$  of the Cartan torus  $H \subset \mathbf{PSL}_{n+3}(\mathbf{R})$ .

Using  $Dh$  and the maps  $\xi$  and  $M$  defined in 64 and (67), one constructs

$$(70) \quad \Phi_U = \xi \circ (M \cdot Dh) : U \times \Delta_2^{n+3} \longrightarrow G_2^{or}(\mathbf{R}^{n+3})^{>0}$$

$$(u, t) \longrightarrow \xi(M_u \cdot Dh(t))$$

which can easily be proven to be an isomorphism (from the direct product  $U \times \Delta_2^{n+3}$  onto the whole positive oriented grassmannian  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ ). Let

$$(71) \quad \delta : U \times \Delta_2^{n+3} \longrightarrow U$$

be the projection onto the first factor. Since  $E_n$  is a global smooth 2-form on the oriented grassmannian, for any  $k \geq 1$ , its  $k$ -th wedge product  $E_n^k$  is integrable along the fibers of the  $H_0$ -action. Consequently, for any such  $k$  the corresponding  $2k$ -form  $\widetilde{E}_n^k = (\Phi_U)^*(E_n^k)$  on  $U \times \Delta_2^{n+3}$  admits a push-forward by  $\delta$  which we will denote by

$$\mathcal{E}_n^k = \delta_*(\widetilde{E}_n^k).$$

It is a smooth  $(2k - n + 2)$ -form on  $U$ , where we use the convention that a  $\ell$ -differential form is 0 when  $\ell$  is negative. Two important properties of these forms are given in the following:

**Proposition 5.3** ([Da2]). 1. The  $(n-2)$ -form  $\mathcal{E}_n^n$  is trivial: one has  $\mathcal{E}_n^n = 0$ ;

2. The  $n$ -form  $\mathcal{E}_n^{n+1}$  is non zero and can be written

$$(72) \quad \mathcal{E}_n^{n+1}(u) = e_n(u) du_1 \wedge \dots \wedge du_n$$

for a non vanishing global analytic function  $e_n$  on  $U$ .

*Proof.* The first point is proved through a direct computation by Damiano (see [Da1, §6.3]). The proof of 2. relies on the very same arguments used to establish the second point of Lemma 5.2.  $\square$

In order to build the Eulerian AR discussed here, it is necessary to extend  $\Phi_U$  to the product  $U \times \overline{\Delta}_2^{n+3}$  where  $\overline{\Delta}_2^{n+3}$  stands for the closure of the hypersimplex:

$$\overline{\Delta}_2^{n+3} = \left\{ (t_1, \dots, t_{n+3}) \in [0, 1]^{n+3} \mid t_1 + \dots + t_{n+3} = 2 \right\}.$$

In the classical case, that is when the target is the usual grassmannian, the extension property of the corresponding map  $\nu \circ \Phi_U : U \times \Delta_2^{n+3} \rightarrow G_2(\mathbf{R}^{n+3})$  is well-known (cf. [GM, Prop. 2.3.2]) but since the map takes values into the oriented grassmannian in the case under scrutiny, we believe that it has to be justified, which is easy to do.

**Proposition 5.4.** 1. The map  $\Phi_U$  in (70) admits a unique continuous extension

$$(73) \quad \overline{\Phi}_U : U \times \overline{\Delta}_2^{n+3} \longrightarrow G_2^{or}(\mathbf{R}^{n+3}).$$

2. Moreover, this extension  $\overline{\Phi}_U$  takes values into the totally non negative oriented grassmannian  $G_2^{or}(\mathbf{R}^{n+3})^{\geq 0}$  and is actually smooth as a map between manifolds with corners.

This proposition is rather direct consequence of the following lemma:

**Lemma 5.5.** 1. The preimage of  $G_2(\mathbf{R}^{n+3})^{>0}$  by the covering  $\nu : G_2^{or}(\mathbf{R}^{n+3}) \rightarrow G_2(\mathbf{R}^{n+3})$  is the disjoint union of the positive and the negative oriented grassmannians: one has

$$\nu^{-1}(G_2(\mathbf{R}^{n+3})^{>0}) = G_2^{or}(\mathbf{R}^{n+3})^{>0} \sqcup G_2^{or}(\mathbf{R}^{n+3})^{<0}.$$

2. The map (70) is a smooth diffeomorphism (in particular, it is surjective);

3. The isomorphism between  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$  and its non oriented counterpart  $G_2(\mathbf{R}^{n+3})^{>0}$  induced by the restriction of  $\nu$  (again denoted by  $\nu$  to simplify) extends to an isomorphism of smooth varieties with corner

$$\nu : G_2^{or}(\mathbf{R}^{n+3})^{\geq 0} \longrightarrow G_2(\mathbf{R}^{n+3})^{\geq 0}.$$

*Proof.* The first point is obvious and the second is hardly more difficult to prove. The third point follows from the first two together with [GM, Prop. 2.3.2].  $\square$

**Remark 5.6.** That  $\Phi_U$  is an isomorphism has the following interesting corollary: since  $U \simeq \mathcal{M}_{0,n+3}^{>0}$  is known to be isomorphic to (the interior of) a polyhedron, the so-called  $n$ -th associahedron  $\text{Ass}_n$ , (70) provides a description of the positive grassmannian of 2-planes in  $\mathbf{R}^{n+3}$  as the product of (the interiors of) two polyhedra: one has

$$G_2(\mathbf{R}^{n+3})^{>0} \simeq \text{Ass}_n \times \Delta_2^{n+3}.$$

As far as we are aware of, such a polyhedral description of  $G_2(\mathbf{R}^{n+3})^{>0}$  is not mentioned in the literature yet, although it is quite likely that it is known by specialists.

From the second point of Proposition 5.4, one deduces that  $\widetilde{E}_n^k$  admits a unique smooth extension  $\overline{E}_n^k$  to  $U \times \overline{\Delta}_2^{n+3}$ , which is such that

$$\overline{E}_n^k = (\overline{\Phi}_U)^*(E_n^k).$$

Considering the boundary  $\partial\Delta_2^{n+3} = \overline{\Delta}_2^{n+3} \setminus \Delta_2^{n+3}$ , the natural inclusion  $\partial\Delta_2^{n+3} \subset \overline{\Delta}_2^{n+3}$  gives rise to an injective morphism of bundles over  $U$  denoted by  $\tau : U \times \partial\Delta_2^{n+3} \hookrightarrow U \times \overline{\Delta}_2^{n+3}$ . The map  $\delta^\partial = \delta \circ \tau$  is nothing else than the projection map  $U \times \partial\Delta_2^{n+3} \rightarrow U$ . Then, from the Stokes-type formula for integration along fibers with boundary, we get that the following identity (which is an equality between differential  $(n-1)$ -forms on  $U$ ) is satisfied:

$$(74) \quad (-1)^{n+2} d\mathcal{E}_n^n = (\delta^\partial)_*(\tau^*(\mathcal{E}_n^n)).$$

Let us now explain how this relation gives rise to an AR for the web  $\mathcal{W}_{0,n+3}$  on  $\mathcal{M}_{0,n+3}^{>0} \simeq U$ . First, it follows from Proposition 5.3.1 that the RHS in (74) is 0. Secondly, for  $i = 1, \dots, n+3$  and  $\nu = 0, 1$ , let  $d_i^\nu : \mathbf{R}^{n+2} \hookrightarrow \mathbf{R}^{n+3}$  be the affine map associating  $(t_1, \dots, t_{i-1}, \nu, t_i, \dots, t_{n+2})$  to any  $(n+2)$ -tuple  $(t_s)_{s=1}^{n+2}$  (' $\nu$  is inserted at the  $i$ -th position'). Then the restriction of this map to

$$\overline{\Delta}_{2-\nu}^{n+2} = \left\{ (s_k)_{k=1}^{n+2} \in [0, 1]^{n+2} \mid \sum_k s_k = 2 - \nu \right\}$$

induces a linear inclusion whose image, denoted by  $\overline{\Delta}_{2-\nu}^{n+2}(i)$ , is the facet of  $\overline{\Delta}_2^{n+3}$  obtained by intersecting it with the affine hyperplane cut out by  $x_i = \nu$ : one has

$$\overline{\Delta}_{2-\nu}^{n+2} \xrightarrow{\sim} \overline{\Delta}_{2-\nu}^{n+2}(i) = d_i^\nu(\overline{\Delta}_{2-\nu}^{n+2}) = \overline{\Delta}_2^{n+3} \cap \{x_i = \nu\} \subset \overline{\Delta}_2^{n+3} \subset \mathbf{R}^{n+3}.$$

The codimension 1 boundary of  $\overline{\Delta}_2^{n+3}$  is exactly the disjoint union of the interiors  $\Delta_{2-\nu}^{n+2}(i)$  of the (hyper)simplices  $\overline{\Delta}_{2-\nu}^{n+2}(i)$  for  $i = 1, \dots, n+3$  and  $\nu = 0, 1$ .

An important fact from which a crucial property satisfied by  $\mathcal{E}_n^n$  will be deduced below is the following homological relation

$$\partial \left[ \overline{\Delta}_2^{n+3} \right] = \sum_{i=1}^{n+3} (-1)^i (d_i^1)_* \left[ \overline{\Delta}_1^{n+2} \right] - \sum_{i=1}^{n+3} (-1)^i (d_i^0)_* \left[ \overline{\Delta}_2^{n+2} \right]$$

which holds true in the relative homology group  $\mathbf{H}_{n+3}(\overline{\Delta}_2^{n+3}, \partial\Delta_2^{n+3}, \mathbf{Z})$  (cf. [GM, Prop. 2.1.3]).

Since the facets  $\overline{\Delta}_1^{n+2}(i)$  for  $i = 1, \dots, n+3$  (all isomorphic to the  $(n+1)$ -dimensional simplex  $\Delta_1^{n+2}$ ) do not play any role in building Euler's abelian relation, we will no longer consider them in what follows and we will write  $d_i = d_i^0$  from now on.

Combining the above homological relation with (74) and Proposition 5.3.1, we obtain that

$$(75) \quad 0 = \sum_{i=1}^{n+3} (-1)^i (\delta_i)_* (\tau_i^*(\mathcal{E}_n^n))$$

where  $\tau_i : U \times \Delta_2^{n+2}(i) \subset U \times \partial\Delta_2^{n+3}$  is the morphism of trivial bundles over  $U$  induced by the inclusion  $\Delta_2^{n+2}(i) \subset \partial\Delta_2^{n+3}$  and where  $\delta_i = \delta \circ \tau_i : U \times \Delta_2^{n+2}(i) \rightarrow U$  is the natural projection.

In order of interpreting (75) as an abelian relation, one has now to express each term  $(\delta_i)_*(\tau_i^*(\mathcal{E}_n^n))$  in a different way. The key point to do this is Lemma 5.2.1, which says that up to the natural identification of  $G_2^{or}(\mathbf{R}_i^{n+2})$  with  $G_2^{or}(\mathbf{R}^{n+2})$ , the restriction of  $E_n$  along  $G_2^{or}(\mathbf{R}_i^{n+2})$  coincides with the invariant representative  $E_{n-1}$  of the Euler class of the tautological bundle on  $G_2^{or}(\mathbf{R}^{n+2})$ .

To build Euler's AR, we then consider the following diagram

$$(76) \quad \begin{array}{ccccc} U \times \overline{\Delta}_2^{n+3} & \xrightarrow{\overline{\Phi}_U} & G_2^{or}(\mathbf{R}^{n+3})^{\geq 0} & & \\ \uparrow \tau_i & & \parallel & & \\ U \times \overline{\Delta}_2^{n+2} & \xrightarrow{\overline{\Phi}_U} & G_2^{or}(\mathbf{R}_i^{n+2})^{\geq 0} & \xleftarrow[\sim]{\mu_i} & G_2^{or}(\mathbf{R}^{n+2})^{\geq 0} & \xleftarrow{\overline{\Phi}_{U'}} & U' \times \overline{\Delta}_2^{n+2} \\ \delta_i \downarrow & & \psi_i & & \pi' \downarrow & \swarrow \gamma' & \downarrow \delta' \\ U & \xrightarrow{\psi_i} & U' & \xlongequal{\quad} & U' & & U' \end{array}$$

where

- $U'$  stands for the set of  $(n-1)$ -tuples  $u' = (u'_k)_{k=1}^n \in \mathbf{R}^{n-1}$  such that  $1 < u'_1 < \dots < u'_{n-1}$ ;
- $\gamma', \pi', \overline{\Phi}_{U'}$  and  $\delta'$  correspond respectively to the maps (68), (69), (73) and (71) above but in the case when the dimension has been taken to be  $n-1$  instead of  $n$ ;
- the map  $\mu_i$  is the restriction of the natural identification between  $G_2^{or}(\mathbf{R}^{n+2})$  and  $G_2^{or}(\mathbf{R}_i^{n+2})$  induced by the linear map  $d_i : \mathbf{R}^{n+2} \xrightarrow{\sim} \text{Im}(d_i) = \mathbf{R}_i^{n+2} \subset \mathbf{R}^{n+3}$ ;
- the maps  $\psi_i : U \rightarrow U'$  are given by

$$\psi_1(u) = \left( \frac{u_1(u_j - 1)}{u_j(u_1 - 1)} \right)_{j=2}^n, \quad \psi_2(u) = \left( \frac{u_j - 1}{u_1 - 1} \right)_{j=2}^n, \quad \psi_3(u) = \left( \frac{u_j}{u_1} \right)_{j=2}^n$$

and  $\psi_{3+i}(u) = (u_1, \dots, \widehat{u}_i, \dots, u_n)$  for  $i = 1, \dots, n$ .

- for  $i = 1, \dots, n+3$  and  $u' \in U'$ ,  $\gamma_i(u')$  is the oriented 2-plane in  $\mathbf{R}_i^{n+2} \subset \mathbf{R}^{n+3}$  associated to the  $2 \times (n+3)$  matrix obtained from  $M_{u'} \in \text{Mat}_{2, n+2}(\mathbf{R})$  (see (67)) by inserting in it the zero column at the  $i$ -th place.

By straightforward computations (left to the reader), it can be verified that (76) is commutative. Combining this with the fact recalled in the paragraph just before this diagram, one deduces the

**Proposition 5.7.** 1. One has  $(\delta_i)_*(\tau_i^*(\mathcal{E}_n^n)) = \psi_i^*(\mathcal{E}_{n-1}^n)$  for  $i = 1, \dots, n+3$ .

2. Consequently (75) can be written  $0 = \sum_{i=1}^{n+3} (-1)^i \psi_i^*(\mathcal{E}_{n-1}^n)$ .

3. Since  $\mathcal{E}_{n-1}^n$  never vanishes (according to Proposition 5.3.2) it follows that

$$(77) \quad \mathcal{E}_n^{>0} = \left( (-1)^{i-1} \psi_i^*(\mathcal{E}_{n-1}^n) \right)_{i=1}^{n+3}$$

is a complete hence in particular non-trivial abelian relation for  $\mathcal{W}_{0, n+3}^{>0}$ .

The construction above gives indeed an AR on  $U \simeq \mathcal{M}_{0,n+3}^{>0}$  but several questions about it remain to be answered:

- (78) *i. Does the construction above of  $\mathcal{E}_n^{>0}$  depend on some choices? If yes, on which ones and to what extent?*
- ii. Following a completely similar approach, one can construct an Eulerian abelian relation  $\mathcal{E}_n(\sigma)$  for  $\mathcal{W}_{0,n+3}$  on the component  $\mathcal{M}(\sigma)$  of  $\mathcal{M}_{0,n+3}(\mathbf{R})$  for any  $\sigma \in \mathcal{R}_{n+3}$  (cf. §2.1.5). How are the  $\mathcal{E}_n(\sigma)$ 's related, especially with respect to the birational action of  $\mathfrak{S}_{n+3}$  on the real moduli space  $\mathcal{M}_{0,n+3}(\mathbf{R})$ ?*

We will come back to these important points further on but we will first discuss how to make  $\mathcal{E}_n^{>0}$  explicit by giving formulas for its components. Since all are pull-backs of the form  $\mathcal{E}_{n-1}^n$  on  $U' \simeq \mathcal{M}_{0,n+2}^{>0}$ , this amounts to studying the function  $e_{n-1}$  defined in Proposition 5.3.2.

**5.1.4. An integral formula for  $e_n$ .** A multivariable integration scheme for computing  $e_{n-1}$  has already been described by Damiano in [Da2, §6] (more details are given in [Da1, §6]). But it is not quite explicit and the author was able to use it only in the case when  $n = 2$ , recovering the well-known *Rogers' dilogarithm*. Our goal below is to go a bit further by describing a slight simplification of Damiano's integration scheme, which gives rise to a more explicit integral formula for  $e_{n-1}$  that we will use to get effective informations about  $e_{n-1}$  for higher but still small  $n$  (namely  $n = 2, 3$  and  $4$ ).

In what follows,  $m$  is an integer bigger than or equal to 1. We do not make any assumption about the parity of  $m$ . The integer  $m$  has to be thought of as  $n$  shifted by 1, namely  $m = n - 1$ , this just in order to simplify the writing. We use below the same notation as above, but in which  $n$  has been replaced by  $m$ . For instance,  $U$  here stands for the set of  $m$ -tuples  $u = (u_1, \dots, u_m) \in \mathbf{R}^m$  such that  $1 < u_1 < \dots < u_{m-1} < u_m$ , etc.

Our main goal here is to give a nice closed formula for the function  $e_m(u)$  defined by equality (72). The main facts we use for this purpose are first that  $E_m^{m+1}$  coincides (up to sign) with an invariant volume form on  $G_2^{or}(\mathbf{R}^{m+3})$  (cf. Lemma 5.2.2) and secondly that there is a simple formula for such a volume form when working with a natural parametrization.

We use the notation introduced circa (64). For  $M \in \Omega_2$  (that is,  $M$  is a  $2 \times (m+3)$ -matrix of rank 2), we write  $M = [M_{12}, N]$  where  $M_{12}$  is the square  $2 \times 2$  matrix obtained by considering the first 2 columns of  $M$  while  $N$  is the one formed by the  $n+1$  others. Then  $\Omega'_2 = \{M \mid \det(M_{12}) > 0\}$  is an open subset of  $\Omega_2$  and the map  $A : \Omega'_2 \rightarrow M_{2,m+1}(\mathbf{R})$ ,  $M \mapsto M_{12}^{-1} \cdot N$  is a surjective  $\text{GL}_2^{>0}(\mathbf{R})$ -fibration. The injection  $B : M_{2,m+1}(\mathbf{R}) \hookrightarrow \Omega'_2$ ,  $N \mapsto [\text{Id}_2, N]$  is such that  $A \circ B$  is the identity. Denoting again by  $\xi$  the restriction of the map (64) to  $\Omega'_2$ , we define a map  $\Xi : M_{2,m+1}(\mathbf{R}) \rightarrow G_2^{or}(\mathbf{R}^{m+3})$  by requiring that the following diagram commutes:

$$(79) \quad \begin{array}{ccc} & \Omega'_2 & \xrightarrow{\xi} & G_2^{or}(\mathbf{R}^{m+3}) \\ & \downarrow A & & \nearrow \Xi \\ & M_{2,m+1}(\mathbf{R}) & & \end{array}$$

(Note: A curved arrow labeled  $B$  points from  $M_{2,m+1}(\mathbf{R})$  back to  $\Omega'_2$ .)



More prosaically,  $\Xi$  is nothing else but the map that associates to the  $2 \times (m+1)$  matrix whose two lines correspond to the vectors  $(x_s)_{s=0}^m, (y_s)_{s=0}^m \in \mathbf{R}^{m+1}$ , the oriented 2-plane of  $\mathbf{R}^{m+3}$  directly spanned by  $(1, 0, x_0, \dots, x_m)$  and  $(0, 1, y_0, \dots, y_m)$ , taken in this order. It is well-known that  $\Xi$  induces a coordinates chart on  $G_2^{or}(\mathbf{R}^{m+3})$ , the corresponding coordinates being the  $x_s$  and  $y_s$  for  $s = 0, \dots, m$ . We will denote by  $d\text{Vol}(N)$  the standard Euclidean volume in these coordinates, *i.e.*

$$d\text{Vol}(N) = (dx_0 \wedge dy_0) \wedge (dx_1 \wedge dy_1) \cdots \wedge (dx_m \wedge dy_m).$$

The formula in the coordinates  $x_s, y_s$  for the pull-back under  $\Xi$  of an invariant volume form on  $G_2^{or}(\mathbf{R}^{m+3})$  is well known:<sup>36</sup>

**Lemma 5.8.** *There exists a non zero constant  $C_m$  such that*<sup>37</sup>

$$(80) \quad \Xi^*(E_m^{m+1}) = C_m \cdot \frac{d\text{Vol}(N)}{\det(\text{Id}_2 + N^t N)^{\frac{m+3}{2}}}.$$

It is easy to get a nice integral formula for  $e_m$  from (80). We list below the facts and notation we need to establish the sought-after formula:

- one denotes by  $k = (k_1, k_2)$  (resp. by  $h = (h_1, \dots, h_m)$ ) an element of  $(\mathbf{R}_{>0})^2$  (resp. of  $(\mathbf{R}_{>0})^m$ ). By means of the map  $(k, h) \mapsto (k_1, k_2, 1, h_1, \dots, h_m)$ , we naturally identify  $(\mathbf{R}_{>0})^2 \times (\mathbf{R}_{>0})^m = (\mathbf{R}_{>0})^{2+m}$  with the open subset of  $(m+3)$ -tuples in  $\mathbf{R}^{m+3}$  with positive coordinates and such that the third coordinate is 1,<sup>38</sup>
- we will consider the following map which is an isomorphism:

$$(81) \quad \Delta_2^{m+3} \longrightarrow (\mathbf{R}_{>0})^{2+m}, (t_i)_{i=1}^{m+3} \longmapsto \left( \frac{t_s(t_3 - 1)}{t_3(t_s - 1)} \right)_{s=1}^{m+3};$$

- we set  $dk = dk_1 \wedge dk_2$ ,  $dh = dh_1 \wedge \cdots \wedge dh_m$  and  $du = du_1 \wedge \cdots \wedge du_m$ ;
- the notation  $k, h > 0$  will be used to mean that the pair  $(k, h)$  varies in  $(\mathbf{R}_{>0})^{2+m}$ ;
- the following map is an isomorphism onto its image which is an open subset in  $\Omega_2$ :

$$(82) \quad F : U \times (\mathbf{R}_{>0})^{2+m} \longrightarrow M_{2, n+3}(\mathbf{R})$$

$$(u, k, h) \longmapsto \begin{bmatrix} \sqrt{k_1} & 0 & -1 & -u_1 \sqrt{h_1} & \cdots & -u_m \sqrt{h_m} \\ 0 & \sqrt{k_2} & 1 & \sqrt{h_1} & \cdots & \sqrt{h_m} \end{bmatrix}$$

- $X$  and  $Y$  stand for the line vectors of the matrix  $F(u, k, h)$ ; one has:

$$X = (\sqrt{k_1}, 0, -1, -u_1 \sqrt{h_1}, \dots, -u_m \sqrt{h_m})$$

$$\text{and } Y = (0, \sqrt{k_2}, 1, \sqrt{h_1}, \dots, \sqrt{h_m});$$

- for  $(u, k, h) \in U \times (\mathbf{R}_{>0})^{2+m}$ , we denote by  $S_{u,k,h}$  the matrix  $F(u, k, h) \cdot F(u, k, h)^t$ ;

<sup>36</sup>This formula is a particular case of a more general one holding true for some affine parametrizations (in terms of Jordan structures) of a vast class of homogeneous varieties, see Proposition X.6.3 in [Ber]. We have not been able to locate a classical reference for this formula in the case of real grassmannians.

<sup>37</sup>One has  $C_m = (m+1)!$  but knowing the exact value of this constant is actually not relevant for our purpose hence we will not give a proof of this.

<sup>38</sup>The fact that we have chosen to normalize the third coordinate is not important at all. We could have decided to normalize any other coordinate with essentially no change.

- one has

$$S_{u,k,h} = \begin{bmatrix} X^2 & XY \\ XY & Y^2 \end{bmatrix}$$

with

$$(83) \quad X^2 = 1 + k_1 + \sum_{i=1}^m h_i u_i^2, \quad XY = -1 - \sum_{i=1}^m h_i u_i \quad \text{and} \quad Y^2 = 1 + k_2 + \sum_{i=1}^m h_i.$$

**Proposition 5.9.** 1. Up to the isomorphism (81) between  $\Delta_2^{m+3}$  and  $(\mathbf{R}_{>0})^{2+m}$ , the map

$$\xi \circ F : U \times (\mathbf{R}_{>0})^{2+m} \rightarrow G_2^{or}(\mathbf{R}^{m+3})$$

identifies with the parametrization (70) of  $G_2^{or}(\mathbf{R}^{m+3})^{>0}$ .

2. In particular, the projection  $(u, k, h) \rightarrow u$  corresponds to quotienting by the torus action.
3. There exists a non zero scalar  $\lambda_m$  such that the pull-back of  $E_m^{m+1}$  under  $\xi \circ F$  is given by

$$(\xi \circ F)^*(E_m^{m+1}) = \lambda_m \det(S_{u,k,h})^{-\frac{m+3}{2}} dk \wedge dh \wedge du.$$

4. Consequently, up to multiplication by a non zero constant, one has

$$(84) \quad e_m(u) = \int_{k,h>0} (X^2 \cdot Y^2 - (XY)^2)^{-\frac{m+3}{2}} dk \wedge dh$$

for any  $u \in U$ , where  $X^2, Y^2$  and  $XY$  are the expressions given in (83) above.

*Proof.* Both the two first points are easy to establish (this is left to the reader). Since 4. follows immediatly from 3., we will only discuss the latter.

For  $(u, k, h) \in U \times (\mathbf{R}_{>0})^{2+m}$ , we denote by  $\widetilde{F}(u, k, h)$  the  $2 \times (m+1)$  submatrix of  $F(u, k, h)$  given by its last  $m+1$  columns and we consider the map

$$f : U \times (\mathbf{R}_{>0})^{2+m} \rightarrow M_{2,m+1}(\mathbf{R})$$

$$(u, k, h) \longmapsto \begin{bmatrix} 1/\sqrt{k_1} & & & & \\ & 1/\sqrt{k_2} & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \cdot \widetilde{F}(u, k, h) = \begin{bmatrix} -1/\sqrt{k_1} & -u_1 \sqrt{h_1/k_1} & \cdots & -u_m \sqrt{h_m/k_1} \\ 1/\sqrt{k_2} & \sqrt{h_1/k_2} & \cdots & \sqrt{h_m/k_2} \end{bmatrix}.$$

The two following identities hold true:

- $f^*(\det(I_2 + N^t N)) = \det(S_{u,k,h})/(k_1 k_2)$ ;
- $f^*(d\text{Vol}(N)) = \lambda'_m \cdot (k_1 k_2)^{-(m+3)/2} dk \wedge dh \wedge du$  for a certain non zero constant  $\lambda'_m$ .

(The first identity is elementary and the second can be obtained by an easy recurrence on  $m$ ).

Since  $\xi \circ F = \Xi \circ f$ , point 3. follows from (80) combined with the two preceding identities.  $\square$

**Remark 5.10.** There is no point in justifying the convergence of the integral in (84): it follows immediatly from the fact that  $E_m^{m+1}$  is a global smooth form on  $G_2^{or}(\mathbf{R}^{m+3})$  which is compact.

The interest of (84) lies in the fact that it is an explicit and closed formula. This offers a way to study Euler's abelian relation more concretely. We use this formula below to describe explicitly Gelfand-MacPherson's computation leading to Rogers dilogarithm in the case  $n = 2$ . Then we discuss the case when  $n = 3$  and give the first terms of the Taylor series of  $e_2$  that can be computed from (84).

**Computation of  $e_1$ : Rogers dilogarithm.** In this case, the preceding proposition gives us that for any  $u \in ]1, +\infty[$ , one has

$$e_1(u) = \iiint_{k_1, k_2, h_1=0}^{+\infty} \frac{dk_1 dk_2 dh_1}{(\alpha_0 - 2\alpha_1 u + \alpha_2 u^2)^2}$$

with  $\alpha_0 = k_1 k_2 + h_1(k_1 + 1) + k_2$ ,  $\alpha_1 = h_1$  and  $\alpha_2 = h_1(1 + k_2)$ .

By successive direct computations, we get that

$$\begin{aligned} e_1(u) &= \iint_{k_1, k_2=0}^{+\infty} \frac{dk_1 dk_2}{(k_1 k_2 + k_1 + k_2)((k_2 + 1)u^2 - 2u + (k_1 + 1))} \\ &= \iint_{k_1, k_2=1}^{+\infty} \frac{dk_1 dk_2}{(k_1 k_2 - 1)(k_2 u^2 - 2u + k_1)} \\ &= \int_{k_2=1}^{+\infty} \frac{\log(k_2/(k_2 - 1)) + \log(k_2 u^2 - 2u + 1)}{(u k_2 - 1)^2} dk_2 \\ &= 2 \left( \frac{\log(u)}{u - 1} - \frac{\log(u - 1)}{u} \right) = 4 \mathcal{R}'(u) \end{aligned}$$

with  $\mathcal{R}(u) = R\left(\frac{u-1}{u}\right)$  for any  $u > 1$ , where  $R$  is Rogers dilogarithm (3).

It would be interesting to study (84) further and in particular to see whether the explicit computation above can be generalized for  $n > 2$ . Our few preliminary attempts in this direction have not been successful and it might be the case that this is something difficult. However, again by means of explicit computations (performed on a computer algebra system), we have been able to extract interesting information about  $e_{n-1}$  for  $n = 3, 4$ . We present some of the results we have obtained in the following two paragraphs.

**On the Taylor expansion of  $e_2$ .** We consider here the case  $n = 3$  which corresponds to the function  $e_2$ . This function takes as arguments elements of  $U \subset ]1, +\infty[^2$ , that we will denote here by  $(u_2, u_3)$  (hence  $u_2$  and  $u_3$  are such that  $1 < u_2 < u_3$ ).

For such a pair  $(u_2, u_3)$ , modulo an elementary and unimportant change of the integration scheme, (84) reads in explicit form

$$(85) \quad e_2(u_1, u_2) = \int_{h_2, \dots, h_5 > 0} B(u, h)^{-5/2} dh$$

with  $dh = dh_2 \wedge \dots \wedge dh_5$  and where  $B(u, h)$  is the following polynomial of degree 2 in  $u_1$  and  $u_2$

$$B(u, h) = (\beta_{11} u_1^2 - 2h_4 h_5 u_1 u_2 + \beta_{22} u_2^2) - 2h_3(h_4 u_1 + h_5 u_2) + \beta_0$$

with  $\beta_{11} = h_4(h_2 + h_3 + h_5)$ ,  $\beta_{22} = h_5(h_2 + h_3 + h_4)$  and  $\beta_0 = (1 + h_3)(1 + h_2 + h_4 + h_5) - 1$ .

Even with the help of a symbolic integration software, we have not been able to compute the quadruple multiple integral above for  $u_1, u_2$  arbitrary in  $U$ . However, it is not the same if working with jets of finite order at a suitable base-point in  $U$ , such as

$$u^* = (2, 3).$$

Let  $\varepsilon_2(u, h)$  be the integrand in (85), which is obviously analytic at  $u^*$ . We denote by  $\varepsilon_2(u, h) = \sum_{w=0}^{+\infty} \sum_{|\ell|=w} \varepsilon_2^\ell(h)(u - u^*)^\ell$  its Taylor expansion at  $u^*$ , where we use the notation  $(u - u^*)^\ell = (u_1 - 2)^{\ell_1}(u_2 - 3)^{\ell_2}$  for any  $\ell = (\ell_1, \ell_2) \in \mathbf{N}^2$ . Setting  $B(h) = B(u^*, h)$ , one can verify that there are polynomials  $P^\ell(h)$  for any  $\ell^{39}$  such that  $\varepsilon_2^\ell(h) = P^\ell(h)/B(h)^{\frac{5}{2}+w}$  for any  $w \geq 0$  and any  $\ell$  such that  $|\ell| = \ell_1 + \ell_2 = w$ . With some work (left to the reader), one can justify the exchange of the integration and the summation below

$$e_2(u_1, u_2) = \int_{h>0} \left[ \sum_{\substack{w \geq 0 \\ |\ell|=w}} \varepsilon_2^\ell(h)(u - u^*)^\ell \right] dh = \sum_{\substack{w \geq 0 \\ |\ell|=w}} \left[ \int_{h>0} \varepsilon_2^\ell(h) dh \right] (u - u^*)^\ell.$$

The interest of doing this lies in the fact that we have been able to compute in closed form the integrals  $\int_{h>0} \varepsilon_2^\ell(h) dh$ , at least for the pairs  $\ell$  of weight  $|\ell|$  small enough and we have been able to get that way the first terms of the Taylor expansion of  $e_2$  at  $u^*$ :

$$\begin{aligned} \frac{4}{\pi} e_2(u) &= \frac{2}{9} \left( \frac{(u_1 - 2)}{9} + \frac{5(u_2 - 3)}{27} \right) + \frac{(u_1 - 2)^2}{18} + \frac{5(u_1 - 2)(u_2 - 3)}{54} + \frac{19(u_2 - 3)^2}{162} \\ &\quad - \left( \frac{(u_1 - 2)^3}{36} + \frac{5(u_1 - 2)^2(u_2 - 3)}{108} + \frac{19(u_1 - 2)(u_2 - 3)^2}{324} + \frac{65(u_2 - 3)^3}{972} \right) \\ &\quad + \frac{(u_1 - 2)^4}{72} + \frac{5(u_1 - 2)^3(u_2 - 3)}{216} + \frac{19(u_1 - 2)^2(u_2 - 3)^2}{648} + \frac{65(u_1 - 2)(u_2 - 3)^3}{1944} \\ &\quad + \frac{211(u_2 - 3)^4}{5832} + O((u - u^*)^5). \end{aligned}$$

We observe that, up to the fourth order, the RHS of this equality coincides with the Taylor series at  $u^*$  of

$$\frac{8}{3u_1u_2(u_2 - 1)} = \frac{2}{9} \sum_{w=0}^{+\infty} \sum_{l=0}^w (-1)^w \frac{(3^{l+1} - 2^{l+1})}{2^w 3^l} (u_1 - 2)^{w-l} (u_2 - 3)^l.$$

This coincidence could leads us to think that we have identically

$$(86) \quad e_2(u_1, u_2) = \frac{(2\pi/3)}{u_1u_2(u_2 - 1)}$$

and we will see a bit further that this is indeed the case.

**5.1.5. Invariance properties of the function  $e_n$ : some transformation formulas.** The positive grassmannian  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$  admits dihedral symmetries. Our goal here is to explain how one can deduce from them some quite strong invariance properties for the function  $e_{n-1}$ , this for any  $n \geq 2$ .

Euler's abelian relation  $\mathcal{E}_n^{>0}$  (see (77)) is defined on the positive part  $\mathcal{M}_{0,n+3}^{>0}$ . As explained in §2.1.5, this space admits a dihedral group of symmetries and a natural question is about the behavior of  $\mathcal{E}_n^{>0}$  with respect to them and what can be deduced from that for  $e_{n-1}$ .

The subgroup  $D_{0,n+3}$  of  $\mathfrak{S}_{n+3}$  letting  $\mathcal{M}_{0,n+3}^{>0}$  invariant is generated by the maps  $C$  and  $R$ , respectively induced by

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<sup>39</sup>One can give an explicit formula for  $P^\ell(k, h)$  but since it is a bit involved and not relevant for the discussion here we will not elaborate on this.

- the shift  $\tilde{C} : (\mathbf{P}^1)^{n+3} \rightarrow (\mathbf{P}^1)^{n+3}$ ,  $(z_i)_{i=1}^{n+3} \mapsto (z_{n+3}, z_1, \dots, z_{n+2})$  which is of order  $n+3$ ;
- the involutive transformation  $\tilde{R} : (\mathbf{P}^1)^{n+3} \rightarrow (\mathbf{P}^1)^{n+3}$ ,  $(z_i)_{i=1}^{n+3} \mapsto (z_1, z_{n+3}, z_{n+2}, \dots, z_3, z_2)$  which, as an automorphism of a regular  $(n+3)$ -gon, corresponds to the reflection with respect to the dihedral axis joining the center on the  $(n+3)$ -gon to that of its vertices labeled by 1.

We set  $n' = \lceil (n+3)/2 \rceil$ . The two permutations of  $\{1, \dots, n+3\}$  corresponding to  $C$  and  $R$  are the  $(n+3)$ -cycle  $c = (1 \dots n+3)$  and the product of  $n'-1$  transpositions given by  $r = (2, n+3)(3, n+2) \cdots (n', n'+1)$  when  $n$  is even and  $r = (2, n+3)(3, n+2) \cdots (n', n'+2)$  otherwise (see Figure 3 below). In particular, if  $\mathbf{sgn} : \mathfrak{S}_{n+3} \rightarrow \{\pm 1\}$  denotes the signature, then one has

$$(87) \quad \mathbf{sgn}(c) = (-1)^n \quad \text{and} \quad \mathbf{sgn}(r) = (-1)^{n'-1}.$$

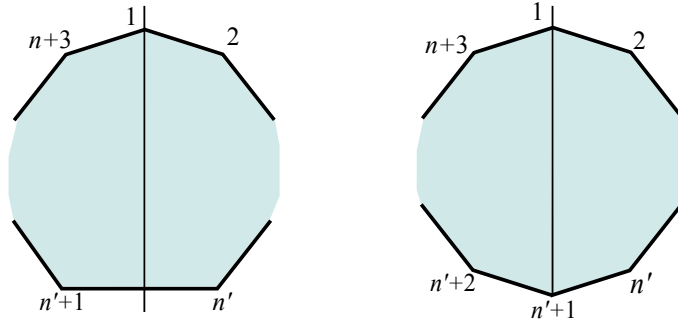


FIGURE 3. Axial symmetry of the  $(n+3)$ -gon when  $n$  is even (left) and odd (right).

**Warning:** The transformation denoted by  $C$  here corresponds to the inverse of the one unfortunately denoted the same in §4.1 which moreover is expressed there with respect to another coordinates system on  $\mathcal{M}_{0,n+3}$  than the one used here.<sup>40</sup> We apologize to the reader for this inconvenience.

The map (65) gives rise to an isomorphism  $U \rightarrow \mathcal{M}_{0,n+3}^{>0}$  again denoted by  $\varphi$ . It is straightforward (and left to the reader) to express  $C$  and  $R$  in the affine coordinates  $u_1, \dots, u_n$  on  $U \subset ]1, +\infty[^n$  explicitly. Denoting (a bit abusively) by the same notation the automorphisms of  $U$  corresponding to  $R$  and  $C$  (given by conjugating them by  $\varphi$ ) and agreeing that  $u_0 = 1$  and  $u_{-1} = 0$ , one has

$$C(u) = \left( \frac{u_n}{u_n - u_{i-1}} \right)_{i=1}^n \quad \text{and} \quad R(u) = \left( \frac{u_n - u_{n-i-1}}{u_n - u_{n-1}} \right)_{i=1}^n.$$

Actually, for our purpose here, we do not have to only consider  $C$  and  $R$ , which are automorphisms of  $\mathcal{M}_{0,n+3}^{>0}$ , but some lifts of these to the oriented grassmannian  $G_2^{or}(\mathbf{R}^{n+3})$  as well, and there is a subtlety about this. This subtle point has not been considered in [Da2] although it is quite relevant regarding the invariant properties of  $\mathcal{E}_n^{>0}$  with respect to the action of  $\mathfrak{S}_{n+3}$ .

Actually the lifts to the oriented grassmannian of automorphisms of  $\mathcal{M}_{0,n+3}$  considered by Damiano (either in [Da2] or in [Da1]) are the simplest/most naive ones: given a permutation

<sup>40</sup>How the  $x_i$ 's of §4.1 and the  $u_k$ 's of this section are related can be summarized by the equality  $[0, 1, \infty, x_1, \dots, x_n] = [\infty, 0, -1, -u_1, \dots, -u_n]$  (in  $\mathcal{M}_{0,n+3}$ ) which has to be thought of as identically satisfied.

$\sigma \in \mathfrak{S}_{n+3} \simeq \text{Aut}(\mathcal{M}_{0,n+3})$ , the lift over it that Damiano considers is the automorphism  $\tilde{\sigma} \in \text{Aut}(G_2^{or}(\mathbf{R}^{n+3}))$  induced by the natural linear action of  $\sigma$  on  $\mathbf{R}^{n+3}$ , namely

$$\sigma \cdot x = \sigma \cdot (x_i)_{i=1}^{n+3} = (x_{\sigma(i)})_{i=1}^{n+3}.$$

The nice feature of the naive lifting map  $\sigma \mapsto \tilde{\sigma}$  considered by Damiano is that it gives rise to a group monomorphism  $\mathfrak{S}_{n+3} = \text{Aut}(\mathcal{M}_{0,n+3}) \rightarrow \text{Aut}(G_2^{or}(\mathbf{R}^{n+3}))$ . But the problem with it is clearly shown when considering the case of  $C$ : using the notations (67) and (68), the associated permutation  $c = (1 \dots n+3)$  is such that Damiano's naive lift  $\tilde{c}$  satisfies

$$\tilde{c} \cdot \gamma(u) = \xi(\tilde{c} \cdot M_u) = \xi\left(\begin{bmatrix} -u_n & 1 & 0 & -1 & -u_1 & \cdots & -u_{n-1} \\ 1 & 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}\right)$$

for any  $u \in U$ . Clearly, some of the  $2 \times 2$  minors of  $\tilde{c} \cdot M_u$  are positive, whereas others (namely all those involving the first column of  $\tilde{c} \cdot M_u$ ) are negative. This has the consequence that the matrix  $\tilde{c} \cdot M_u$  is not merely positive, which translates into the fact that with Damiano's choice for the lift  $\tilde{c}$ , the image of  $\tilde{c} \cdot \gamma$  into the oriented grassmannian of 2-planes in  $\mathbf{R}^{n+3}$  does not coincide with the positive part  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ . Actually, the image  $\tilde{c} \cdot \gamma(U)$ , which is  $\tilde{c} \cdot G_2^{or}(\mathbf{R}^{n+3})^{>0}$ , is disjoint from the positive part of the oriented grassmannian: one has

$$\tilde{c}(G_2^{or}(\mathbf{R}^{n+3})^{>0}) \cap G_2^{or}(\mathbf{R}^{n+3})^{>0} = \emptyset.$$

In order to study the invariant properties of  $\mathcal{E}_n^{>0}$  with respect to the action of  $C$ , one has to look for a lift  $\hat{c}$  making the following diagram commutative:

$$(88) \quad \begin{array}{ccc} G_2^{or}(\mathbf{R}^{n+3})^{>0} & \xrightarrow{\hat{c}} & G_2^{or}(\mathbf{R}^{n+3})^{>0} \\ \gamma \uparrow & & \uparrow \gamma \\ U & \xrightarrow{c} & U \end{array}$$

In particular, the lift  $\hat{c}$  must let invariant the positive grassmannian, which precisely Damiano's lift  $\tilde{c}$  does not do.

**Invariance properties of Euler's abelian relation  $\mathcal{E}_n^{>0}$  with respect to pull-back under  $C$ .** The relevant lift of  $c$  to deal when considering the notion of positivity on the oriented grassmannian is well-known: it is the cyclic automorphism  $\hat{c}$  of  $G_2^{or}(\mathbf{R}^{n+3})$  induced by the linear isomorphism of  $\mathbf{R}^{n+3}$  given by

$$(89) \quad (x_1, \dots, x_{n+3}) \mapsto (-x_{n+3}, x_1, \dots, x_{n+2})$$

whose matrix (written by blocs) with respect to the standard basis of  $\mathbf{R}^{n+3}$  is

$$\begin{bmatrix} 0 & I_{n+2} \\ -1 & 0 \end{bmatrix}.$$

This matrix belongs to  $O_{n+3}(\mathbf{R})$  and has determinant equal to  $(-1)^{n-1}$ . Hence  $\hat{c}$  is not always in the special orthogonal group but it is easily verified (using (62) for instance) that one has  $\hat{c}^*(E_n) = E_n$  in any case. It follows that  $E_n^n$  is left invariant by the lift  $\hat{c}$  as well. Consequently, it follows that the pull-back of Euler's abelian relation  $\mathcal{E}_n^{>0}$  under  $C$  coincides with  $\mathcal{E}_n^{>0}$  or with its opposite, according whether  $\hat{c}$  preserves the orientation of a generic  $H_0$ -orbit in  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ .<sup>41</sup>

<sup>41</sup>We recall that  $H_0 \simeq (\mathbf{R}_{>0})^{n+2}$  stands for the positive part of the diagonal Cartan torus  $H \subset \text{SL}(\mathbf{R}^{n+3})$ .

In order to determine the action of  $\hat{c}$  on the orientation of the  $H_0$ -orbits of elements in  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ , let us consider the following parametrization of the positive part of this oriented grassmannian

$$(90) \quad U \times (\mathbf{R}_{>0})^2 \times (\mathbf{R}_{>0})^n \longrightarrow G_2^{or}(\mathbf{R}^{n+3})^{>0}$$

$$(u, k, h) = \left( (u_i)_{i=1}^n, (k_1, k_2), (h_i)_{i=1}^n \right) \longmapsto \begin{bmatrix} k_1 & 0 & -1 & -u_1 h_1 & \dots & -u_n h_n \\ 0 & k_2 & 1 & h_1 & \dots & h_n \end{bmatrix}$$

(where  $U$  is the set of  $n$ -tuples  $(u_i)_{i=1}^n \in \mathbf{R}^n$  such that  $1 < u_1 < \dots < u_{n-1} < u_n$ ). This map is an isomorphism hence the  $u_i$ ,  $k_s$  and  $h_i$  for  $i = 1, \dots, n$  and  $s = 1, 2$  form a system of global coordinates on  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ , which is particularly useful for our purpose since relatively to these coordinates, quotienting by the  $H_0$ -action corresponds exactly to the standard projection  $(u, k, h) \mapsto u$  (cf. Proposition 5.9 above). Let  $\hat{C}$  be the automorphism of  $U \times (\mathbf{R}_{>0})^2 \times (\mathbf{R}_{>0})^n$  obtained from the automorphism  $\hat{c}$  of  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$  by means of the identification induced by (90). It is not difficult to give an explicit formula for  $\hat{C}$ . Indeed setting  $u_0 = 1$  and denoting by  $c_i(u) = u_n / (u_n - u_{i-1})$  the  $i$ -th component of  $C(u)$  for  $i = 1, \dots, n$ , straightforward computations (the details of which are left to the reader) give us that

$$\hat{C}(u, k, h) = (\hat{u}, \hat{k}, \hat{h})$$

with

$$\hat{u} = C(u) = (c_i(u))_{i=1}^n = \left( \frac{u_n}{u_n - u_{i-1}} \right)_{i=1}^n,$$

$$\hat{k} = (\hat{k}_1, \hat{k}_2) = \left( \frac{h_n}{k_2}, \frac{k_1}{k_2 u_n} \right)$$

and  $\hat{h} = (\hat{h}_i)_{i=1}^n = \left( \frac{1}{k_2 c_1(u)}, \frac{h_1}{k_2 c_2(u)}, \dots, \frac{h_{n-1}}{k_2 c_n(u)} \right).$

The implicitly chosen (because natural) orientation on the  $H_0$ -orbits of elements of  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$  corresponds to the  $(n+2)$ -form  $dk \wedge dh = dk_1 \wedge dk_2 \wedge dh_1 \wedge \dots \wedge dh_n$  in the coordinates  $u_i$ ,  $k_s$  and  $h_i$  ( $i = 1, \dots, n$ ,  $s = 1, 2$ ). Using the expressions above for the components of  $\hat{k}$  and  $\hat{h}$ , one easily computes what is the  $dk \wedge dh$ -component of the pull-back of this form under  $\hat{C}$ : setting  $\partial k \wedge \partial h = \partial_{k_1} \wedge \partial_{k_2} \wedge \partial_{h_1} \wedge \dots \wedge \partial_{h_n}$ , one obtains that

$$(\partial k \wedge \partial h) \lrcorner \hat{C}^*(dk \wedge dh) = (-1)^n \frac{\prod_{i=0}^{n-1} (u_n - u_i)}{k_2^{n+3} u_n^{n+1}}.$$

Since  $u \in U$ , one has  $u_n - u_i > 0$  for any  $i = 0, \dots, n-1$  hence the sign of the above expression is  $(-1)^n$ . On the other hand,  $\hat{c}$  lets  $E_n^n$  invariant as we have seen above, hence we deduce the

**Proposition 5.11.** *The positive Euler's abelian relation is stable up to multiplication by  $(-1)^n$  under pull-back by  $C$ , i.e. identically on  $U$ , one has*

$$C^*(\mathcal{E}_n^{>0}) = (-1)^n \mathcal{E}_n^{>0}.$$

When  $n$  is even, we will state a more general result whose proof is much more conceptual and which admits the above as a particular case (see Proposition 5.18 further below).

In any case, the preceding proposition has two interesting outcomes. The first is that one recovers the fact that  $\mathcal{E}_n^{>0}$  is entirely known as soon as one, say the first, of its components is

known. Indeed, the map  $U_{n+3}(u) = (u_1, \dots, u_{n-1})$  is a first integral of the  $(n+3)$ -th foliation of  $\mathcal{W}_{0,n+3}^{>0}$  on  $U$  (that foliation corresponding to forgetting the  $(n+3)$ -th point on  $\mathcal{M}_{0,n+3}^{>0}$ ).

For  $i = 0, \dots, n+2$ , the rational map

$$U_{n+3-i} = (C^{oi})^*(U_{n+3})$$

(which is just given by taking the first  $n-1$  components of the map obtained by composing  $C$   $i$ -th times) is a rational first integral of the  $(n+3-i)$ -th foliation of  $\mathcal{W}_{0,n+3}^{>0}$  (corresponding to forgetting the  $(n+3-i)$ -th point on  $\mathcal{M}_{0,n+3}^{>0}$ ). For any  $i$ , denote by  $U_i^k$  for  $k = 1, \dots, n-1$  the components of  $U_i$  and set  $dU_i = dU_i^1 \wedge \dots \wedge dU_i^{n-1}$ . Since the  $U_{n+3}$ -component of  $\mathcal{E}_n^{>0}$  is

$$e_{n-1}(U_{n+3}) dU_{n+3} = e_{n-1}(u_1, \dots, u_{n-1}) du_1 \wedge \dots \wedge du_{n-1},$$

it follows from the above proposition that  $\mathcal{E}_n^{>0}$  is equivalent to the fact that the differential relation

$$(91) \quad 0 = \sum_{i=1}^{n+3} (-1)^{in} e_{n-1}(U_i) dU_i^1 \wedge \dots \wedge dU_i^{n-1}$$

holds true identically for any  $n$ -tuple of real numbers  $(u_1, \dots, u_n) \in U$ .

**Example 4.** When  $n = 2$ , the maps  $U_i$  are given by

$$U_5 = u_1, \quad U_4 = \frac{u_2}{u_2 - 1}, \quad U_3 = \frac{u_2}{u_1}, \quad U_2 = \frac{u_2 - 1}{u_2 - u_1} \quad \text{and} \quad U_1 = \frac{u_1(u_2 - 1)}{u_2(u_1 - 1)}.$$

Up to integration, (91) is equivalent to the fact that the relation

$$(92) \quad R(u_1) + R\left(\frac{u_2}{u_2 - 1}\right) + R\left(\frac{u_2}{u_1}\right) + R\left(\frac{u_2 - 1}{u_2 - u_1}\right) + R\left(\frac{u_1(u_2 - 1)}{u_2(u_1 - 1)}\right) = 0$$

holds true for any  $u_1, u_2$  such that  $1 < u_1 < u_2$ , where  $R$  stands for the function defined by

$$R(u) = \frac{1}{2} \int_u^{+\infty} \left( \frac{\log(u-1)}{u} - \frac{\log(u)}{u-1} \right) du + \frac{\pi^2}{10} \quad (u > 1).^{42}$$

It can be verified that given  $u > 1$ , one has  $R(u) = R_1((u-1)/u)$  where  $R_1$  stands for the following version of Rogers dilogarithm:  $R_1(x) = \mathbf{Li}_2(x) + \log(x) \log(1-x)/2 - \pi^2/15$  for  $x \in ]0, 1[$ .

One checks easily that (92) is indeed left invariant by the 5-cyclic birational map

$$C(u_1, u_2) = \left( \frac{u_2}{u_2 - 1}, \frac{u_2}{u_2 - u_1} \right).$$

The second interesting consequence of Proposition 5.11 is a nice transformation formula for the function  $e_{n-1}$  appearing in the components of  $\mathcal{E}_n^{>0}$ . Recall the rational functions  $\psi_i$  introduced just before Proposition 5.7, which are first integrals for  $\mathcal{W}_{0,n+3}^{>0}$  as well. With respect to these, the last two components of  $\mathcal{E}_n^{>0}$  in (75) are written respectively

$$(93) \quad \begin{aligned} & f_{n+2}(u_1, \dots, u_{n-2}, u_n) \cdot du_1 \wedge \dots \wedge du_{n-2} \wedge du_n \\ & \text{and} \quad f_{n+3}(u_1, \dots, u_{n-1}) \cdot du_1 \wedge \dots \wedge du_{n-2} \wedge du_{n-1} \end{aligned}$$

for two functions  $f_{n+2}$  and  $f_{n+3}$  globally defined (and analytic) on  $U$ .

<sup>42</sup>The constant  $\pi^2/10$  in the definition of  $R$  is just a normalization added in order that the RHS of (92) be zero.



On the other hand, for any  $i = 1, \dots, n+3$ , the  $i$ -th component  $(-1)^{i-1} \psi_i^*(\mathcal{E}_{n-1}^n)$  of  $\mathcal{E}_n^{>0}$  can also be written  $(\delta_i)_*(\tau_i^*(\mathcal{E}_n^n))$ . Then following carefully the computations of §5.1.4 for these expressions when  $i$  is  $n+2$  or  $n+3$  (with  $m = n-1$ ), we obtain that the following holds true:

**Fact 5.12.** *As functions of  $n-1$  variables, one has (up to multiplication by a non zero constant)*

$$(94) \quad f_{n+2} = f_{n+3} = e_{n-1}$$

where  $e_{n-1}$  stands of course for the function defined in (84) (in the case when  $m = n-1$ ).

On the other hand, one has:

$$(95) \quad \begin{aligned} C^*(du_1 \wedge \dots \wedge du_{n-1}) &= d\left(\frac{u_n}{u_n-1}\right) \wedge d\left(\frac{u_n}{u_n-u_1}\right) \wedge \dots \wedge d\left(\frac{u_n}{u_n-u_{n-2}}\right) \\ &= (-1)^{n-1} \frac{(u_n)^{n-2}}{\prod_{i=0}^{n-2} (u_n - u_i)^2} \cdot (du_1 \wedge \dots \wedge du_{n-2} \wedge du_n). \end{aligned}$$

Combining (93), (94) and (95), we deduce the

**Proposition 5.13.** *The function  $e_{n-1}$  satisfies the following relation:*

$$(96) \quad e_{n-1}\left(\frac{u_n}{u_n-u_0}, \frac{u_n}{u_n-u_1}, \dots, \frac{u_n}{u_n-u_{n-2}}\right) = \frac{\prod_{i=0}^{n-2} (u_n - u_i)^2}{(u_n)^{n-2}} \cdot e_{n-1}(u_1, \dots, u_{n-2}, u_n)$$

for any real numbers  $u_0, u_1, \dots, u_{n-2}, u_n$  such that  $1 = u_0 < u_1 < \dots < u_{n-2} < u_n$ .

**Example 4 (continued).** *Up to multiplication by a non zero constant, one has  $e_1(u) = \log(u-1)/u - \log(u)/(u-1)$  for any  $u > 1$ . Given such a  $u$ , one has*

$$(97) \quad \begin{aligned} e_1\left(\frac{u}{u-1}\right) &= -\frac{(u-1)\log(u-1)}{u} - (u-1)(\log(u) - \log(u-1)) \\ &= \left(-\frac{u-1}{u} + (u-1)\right) \log(u-1) - (u-1)\log(u) \\ &= (u-1)^2 \frac{\log(u-1)}{u} - (u-1)^2 \frac{\log(u)}{u-1} = (u-1)^2 e_1(u). \end{aligned}$$

**Invariance properties of Euler's abelian relation  $\mathcal{E}_n^{>0}$  with respect to pull-back under  $R$ .** The case of the automorphism  $R$  of  $\mathcal{M}_{0,n+3}^{>0}$  is handled by similar but somewhat more subtle arguments that we are going to discuss below. The lift  $\hat{r}$  of  $R$  that we consider here is the automorphism of the oriented grassmannian induced by the linear map  $(x_i)_{i=1}^{n+3} \mapsto (-x_1, x_{n+3}, x_{n+2}, \dots, x_2, x_1)$  whose matrix (written by blocs) with respect to the standard basis of  $\mathbf{R}^{n+3}$  is

$$N_n = \begin{bmatrix} -1 & 0 \\ 0 & J_{n+2} \end{bmatrix}$$

where  $J_{n+2}$  stands for the anti-diagonal  $(n+2) \times (n+2)$  matrix. The matrix  $N_n$  is orthogonal with determinant  $\delta_n = (-1)^{\lfloor n/2 \rfloor}$  thus does not always belong to  $\mathrm{SO}_{n+3}(\mathbf{R})$ . Nevertheless, in any case (using for instance (62)) it can be proved that  $\hat{r}$  lets the Euler form  $E_n$  invariant: one has  $\hat{r}^*(E_n) = E_n$ . But contrarily to  $\hat{c}$ , the map  $\hat{r}$  does not give rise to an automorphism of the positive part of the oriented grassmannian since, as it can be verified straightforwardly, one has

$$\hat{r}(G_2^{or}(\mathbf{R}^{n+3})^{>0}) = G_2^{or}(\mathbf{R}^{n+3})^{<0}.$$

In order to land into  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ , it is necessary to post compose  $\hat{r}$  with the change of orientation map  $D$  defined in §5.1.1. One then obtains a commutative diagram

$$(98) \quad \begin{array}{ccc} G_2^{or}(\mathbf{R}^{n+3})^{>0} & \xrightarrow{D \circ \hat{r}} & G_2^{or}(\mathbf{R}^{n+3})^{>0} \\ \uparrow \gamma & & \uparrow \gamma \\ U & \xrightarrow{R} & U \end{array}$$

Because  $D^*(E_n) = -E_n$  (by Lemma 5.1.1), it follows that one has

$$(99) \quad (D \circ \hat{r})^*(E_n^n) = (-1)^n E_n^n.$$

Now one has to determine the action of  $D \circ \hat{r}$  on the orientation of the  $H_0$ -orbits in  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ . We will proceed as in the case of  $C$ , by considering the map  $\tilde{R}$  from  $U \times (\mathbf{R}_{>0})^2 \times (\mathbf{R}_{>0})^n$  into itself corresponding to  $D \circ \hat{r}$ . In the coordinates  $u, k$  and  $h$  on  $U \times (\mathbf{R}_{>0})^2 \times (\mathbf{R}_{>0})^n$  considered above, one has

$$\tilde{R}(u, k, h) = (\tilde{u}, \tilde{k}, \tilde{h})$$

with

$$\tilde{u} = R(u) = \left( \frac{u_n - u_{n-1-i}}{u_n - u_{n-1}} \right)_{i=1}^n, \quad \tilde{k} = \left( \frac{k_1}{h_{n-1}(u_n - u_{n-1})}, \frac{h_n}{h_{n-1}} \right) \quad \text{and} \quad \tilde{h} = \left( \frac{h_{n-1-i}}{h_{n-1}} \right)_{i=1}^n$$

where we use the following notation:  $u_{-1} = 0, u_0 = 1, h_0 = 1$  and  $h_{-1} = k_2$ .

Using the expressions above for the components of  $\tilde{k}$  and  $\tilde{h}$ , one easily computes what is the  $dk \wedge dh$ -component of the pull-back of this form under  $\tilde{R}$ : it is given by

$$(\partial k \wedge \partial h) \lrcorner \tilde{R}^*(dk \wedge dh) = \frac{(-1)^{\lfloor n/2 \rfloor + 1}}{(h_{n-1})^{n+3}(u_n - u_{n-1})}.$$

The sign of the above expression on  $U \times (\mathbf{R}_{>0})^{n+2}$  is that of  $(-1)^{\lfloor n/2 \rfloor + 1}$ . Together with (99) and because  $n + \lfloor n/2 \rfloor + 1 - \lfloor (n-1)/2 \rfloor = 2\lfloor n/2 \rfloor + 2$  is even for any  $n \geq 2$ , this gives us the

**Proposition 5.14.** *Identically on  $U$ , one has*

$$(100) \quad R^*(\mathcal{E}_n^{>0}) = (-1)^{\lfloor (n-1)/2 \rfloor} \mathcal{E}_n^{>0}.$$

**Example 4 (finished).** *When  $n = 2$ , the birational involution  $R$  is given by*

$$R(u_1, u_2) = \left( \frac{u_2 - 1}{u_2 - u_1}, \frac{u_2}{u_2 - u_1} \right).$$

*A straightforward verification gives that it lets the relation (92) entirely invariant.*

The ingredients needed in order to get a transformation formula for  $e_{n-1}$  from (100) are similar but not completely the same according to the parity of  $n$ . We now set  $m = \lfloor n/2 \rfloor$ .

Let us first discuss the case when  $n$  is odd. In this situation,  $R$  lets invariant the  $((n+5)/2)$ -th foliation of  $\mathcal{W}_{0,n+3}^{>0}$ . This foliation admits the map  $\psi_{(n+5)/2}(u)$  as first integral whose components are the  $u_i$ 's for  $i = 1, \dots, n$  distinct from  $(n-1)/2$ . The component of  $\mathcal{E}_n^{>0}$  with respect to this first integral can be seen to be a multiple of

$$(101) \quad e_{n-1}(u_1, \dots, u_{(n-3)/2}, \widehat{u_{(n-1)/2}}, u_{(n+1)/2}, \dots, u_n) \Omega_{(n+5)/2}$$

with  $\Omega_{(n+5)/2} = \wedge_{i \neq (n-1)/2} du_i = du_1 \wedge \dots \wedge du_{(n-3)/2} \wedge du_{(n+1)/2} \wedge \dots \wedge du_n$ .

By a direct computation, one gets that

$$(102) \quad R^*(\Omega_{(n+5)/2}) = \frac{(-1)^{\lfloor n/2 \rfloor}}{(u_n - u_{n-1})^n} \Omega_{(n+5)/2}.$$

Let us now deal with the case when  $n$  is even. Setting  $m = n/2$ , one has  $n' = \lceil (n+3)/2 \rceil = m+2$  and the birational involution  $R$  exchanges the  $n'$ -th and the  $(n' + 1)$ -th foliations of  $\mathcal{W}_{0,n+3}^{>0}$  (see Figure 3). The two corresponding first integrals  $\psi_{n'}$  and  $\psi_{n'+1}$  have for components the  $u_i$ 's for  $i = 1, \dots, n$  distinct from  $m - 1$  and  $m$  respectively. Accordingly, we set  $\Omega_{n'} = \wedge_{i \neq m-1} du_i$  and  $\Omega_{n'+1} = \wedge_{i \neq m} du_i$ . Then, up to a common multiple, the components of  $\mathcal{E}_n^{>0}$  with respect to these first integrals can be seen to be

$$(103) \quad \begin{aligned} & e_{n-1}(u_1, \dots, \widehat{u_{m-1}}, \dots, u_n) \Omega_{n'} \\ \text{and} \quad & -e_{n-1}(u_1, \dots, \widehat{u_m}, \dots, u_n) \Omega_{n'+1}. \end{aligned}$$

On the other hand, a direct computation gives us that when  $n$  is (even and) strictly bigger than 2, one has

$$(104) \quad R^*(\Omega_{n'}) = \frac{(-1)^{\lfloor n/2 \rfloor}}{(u_n - u_{n-1})^n} \Omega_{n'+1}$$

whereas when  $n = 2$ , one has  $n' = 3$  and the following relation holds true:

$$(105) \quad R^*(\Omega_3) = \frac{-du_2}{(u_2 - 1)^2} = \frac{-1}{(u_2 - 1)^2} \Omega_4.$$

We recall that  $m = \lfloor n/2 \rfloor \geq 1$ . We now have everything at hand to get another transformation formula satisfied by  $e_{n-1}$ . Indeed, from (100) together with (101) and (102) when  $n$  is odd, and with (103) and (104) (or (105) if  $n = 2$ ) when  $n$  is even, we deduce the

**Proposition 5.15.** 1. *The function  $e_1$  satisfies the following relation for any  $x_2 > 1$*

$$(106) \quad \frac{e_1\left(\frac{x_2}{x_2-1}\right)}{(x_2 - 1)^2} = e_1(x_2).$$

2. *For any  $n$  strictly bigger than 2, the function  $e_{n-1}$  satisfies the relation:*

$$(107) \quad \frac{e_{n-1}\left(\frac{u_n - u_{n-2}}{u_n - u_{n-1}}, \dots, \frac{\widehat{u_n - u_m}}{u_n - u_{n-1}}, \dots, \frac{u_n - 1}{u_n - u_{n-1}}, \frac{u_n}{u_n - u_{n-1}}\right)}{(u_n - u_{n-1})^n} = e_{n-1}(u_1, \dots, \widehat{u_m}, \dots, u_n)$$

for any  $u_1, \dots, u_{m-1}, \widehat{u_m}, u_{m+1}, \dots, u_n$  such that  $1 < u_1 < \dots < u_{m-1} < u_{m+1} < \dots < u_n$ .

Remark that in the case when  $n = 2$ , the two functional relations given by Proposition 5.13 and Proposition 5.15 that  $e_1$  satisfies actually coincide (equations (97) and (106) are the same!).

**5.1.6. Euler's abelian relation  $\mathcal{E}_3^{>0}$ .** Let us write down (96) and (107) explicitly when  $n = 3$ : the two corresponding formulas are respectively

$$(108) \quad \frac{e_2\left(\frac{u_3}{u_3-1}, \frac{u_3}{u_3-u_1}\right) u_3}{(u_3 - 1)^2 (u_3 - u_1)^2} = e_2(u_1, u_3) \quad \text{and} \quad \frac{e_2\left(\frac{u_3-1}{u_3-u_2}, \frac{u_3}{u_3-u_2}\right)}{(u_3 - u_2)^3} = e_2(u_2, u_3),$$

and these two equalities are satisfied for any  $u_1, u_2, u_3$  such that  $1 < u_1 < u_2 < u_3$ .

We have determined above the first terms of the Taylor expansion of  $e_2$  at a given base point  $u^*$  and noticed that, up to multiplication by a non zero constant, these terms were the same as those of the Taylor expansion at  $u^*$  of the rational function

$$\tilde{e}_2 : (u_1, u_2) \mapsto \frac{1}{u_1 u_2 (u_2 - 1)}.$$

As elementary verifications show, the function  $\tilde{e}_2$  satisfies both identities (108) as well. That is not a coincidence.

For  $i = 1, \dots, n+3$ , denote by  $\mathbf{AR}(\mathcal{W}_{0,n+3}^{>0})[i]$  the subspace of  $\psi_i^* \Omega^{n-1}$  spanned by the  $i$ -th components of the ARs of  $\mathcal{W}_{0,n+3}^{>0}$ , with respect to the first integrals  $\psi_i$ . From the results of §4.5, one gets that  $\mathbf{AR}(\mathcal{W}_{0,n+3}^{>0})[n+3]$  is exactly the space of 2-forms  $F(u_1, u_2) du_1 \wedge du_2$  where  $F$  ranges in the vector space of rational functions spanned by the set

$$\mathfrak{B}_3(u_1, u_2) = \left\{ F_0(u_1, u_2) = \frac{1}{u_1 u_2}, F_{ij}(u_1, u_2) \mid i = 2, 3, j = i+1, \dots, 5 \right\} \subset \mathbf{Q}(u_1, u_2)$$

defined in (59). It is straightforward to verify that the function  $\tilde{e}_2$  defined above first belongs to  $\langle \mathfrak{B}_3(u_1, u_2) \rangle$ ; and secondly and most importantly, is the unique element of this space (up to multiplication by a non zero constant) satisfying the same identities as (108). Then taking into account the determination of the order three jet of  $e_2$  at  $u^*$  computed above, we deduce that equality (86) is indeed indentially satisfied on  $U \subset ]1, +\infty[^2$ .

It is just a computational matter to write down (91) in explicit form: Euler's abelian relation  $\mathcal{E}_3^{>0}$  corresponds to the following identity

$$(109) \quad \sum_{i=1}^6 (-1)^i \mathcal{E}_{3,i} = 0$$

where the six 2-forms  $\mathcal{E}_{3,i} = (C^{o6-i})^*(\mathcal{E}_{3,6})$ 's are given by

$$\mathcal{E}_{3,6} = \frac{du_1 \wedge du_2}{u_1 u_2 (u_2 - 1)}, \quad \mathcal{E}_{3,5} = \frac{du_1 \wedge du_3}{u_1 u_3 (u_3 - 1)}, \quad \mathcal{E}_{3,4} = \frac{du_2 \wedge du_3}{u_2 u_3 (u_3 - 1)}$$

and

$$\begin{aligned} \mathcal{E}_{3,3} &= \frac{-du_1 \wedge du_2}{u_2(u_3 + u_1)} + \frac{du_1 \wedge du_3}{u_3(u_3 - u_1)} - \frac{du_2 \wedge du_3}{u_2 u_3 (u_3 - u_1)} \\ \mathcal{E}_{3,2} &= \frac{du_1 \wedge du_2}{(u_2 - 1)(u_3 u_1)} - \frac{du_1 \wedge du_3}{(u_3 - 1)(u_3 - u_1)} + \frac{(u_1 - 1) du_2 \wedge du_3}{(u_2 - 1)(u_3 - 1)(u_3 - u_1)} \\ \mathcal{E}_{3,1} &= \frac{-u_3 du_1 \wedge du_2}{u_1 u_2 (u_2 - 1)(u_3 - u_1)} + \frac{du_2 \wedge du_3}{u_1 (u_3 - 1)(u_3 - u_1)} - \frac{(u_1 - 1) du_2 \wedge du_3}{u_2 (u_2 - 1)(u_3 - 1)(u_3 - u_1)}. \end{aligned}$$

Obviously,  $\mathcal{E}_3^{>0}$  is rational. Actually, since all the ARs of  $\mathcal{W}_{0,6}$  are combinatorial, one can state the following striking result:

**Proposition 5.16.** *Euler's abelian relation  $\mathcal{E}_3^{>0}$  is combinatorial: one has*

$$\mathcal{E}_3^{>0} \in \mathbf{AR}_C(\mathcal{W}_{0,n+3}).$$

**5.1.7. Euler's abelian relations and their behavior with respect to the action of  $\mathfrak{S}_{n+3}$ .** We now want discuss the questions (78) raised above.

Let us start by considering the first one: ‘Does the construction given above of  $\mathcal{E}_n^{>0}$  depend on some choices? If yes, on which ones?’ The main object we used to construct  $\mathcal{E}_n^{>0}$  on  $\mathcal{M}_{0,n+3}^{>0}$  in section §5.1.3 is the map  $\gamma = \xi \circ M : U \rightarrow G_2^{or}(\mathbf{R}^{n+3})^{>0}$  defined in diagram (68). Up to the identification (65) between  $U$  and  $\mathcal{M}_{0,n+3}^{>0}$ ,  $\gamma$  corresponds to a section over  $\mathcal{M}_{0,n+3}^{>0}$  of the natural map  $\widehat{G}_2^{or}(\mathbf{R}^{n+3}) \rightarrow \mathcal{M}_{0,n+3}$  obtained by composing the arrows from the top left corner to the bottom right corner in diagram (61). Considering this diagram is helpful to better understand  $\mathcal{E}_n^{>0}$ : it shows that from an intrinsic point of view,  $\mathcal{E}_n^{>0}$  is not defined on  $\mathcal{M}_{0,n+3}^{>0}$  but rather on one of the two components over  $\mathcal{M}_{0,n+3}^{>0}$  in  $\mathcal{M}_{0,n+3}^{or}(\mathbf{R})$ .

Let us write

$$\nu^{-1}(\mathcal{M}_{0,n+3}^{>0}) = \mathcal{M}_{0,n+3}^{or,>0} \sqcup \mathcal{M}_{0,n+3}^{or,<0}$$

this notation being justified by the fact that one has  $\mathcal{M}_{0,n+3}^{or,>0}(\mathbf{R}) = G_2^{or}(\mathbf{R}^{n+3})^{>0}/H \subset \mathcal{M}_{0,n+3}^{or}(\mathbf{R})$  and similarly for the negative setting (corresponding to formally replacing  $>$  by  $<$ ). Actually (65) is an identification between  $U$  and  $\mathcal{M}_{0,n+3}^{or,>0}$  and  $\gamma$  corresponds to a section of

$$\pi_H : \widehat{G}_2^{or}(\mathbf{R}^{n+3}) \longrightarrow \mathcal{M}_{0,n+3}^{or}(\mathbf{R})$$

over the latter component. Since  $H \simeq (\mathbf{R}_{>0})^{n+2} \times (\mathbf{Z}_2)^{n+2}$ , there is no canonical choice for such a section: the section  $\gamma$  we have considered is the one landing in  $G_2^{or}(\mathbf{R}^{n+3})^{>0}$ . Any other section is of the form  $\gamma_{\underline{\varepsilon}} = \xi \circ M_{\underline{\varepsilon}}$  for  $\underline{\varepsilon} = (\varepsilon_i)_{i=1}^{n+2} \in \{\pm 1\}^{n+2}$  where  $M_{\underline{\varepsilon}} : U \rightarrow \Omega_2$  is the map  $u \mapsto M_u \cdot \text{Diag}(\varepsilon_1, \dots, \varepsilon_{n+3})$ , i.e. for  $u \in U$ ,  $M_{\underline{\varepsilon}}(u)$  is the matrix whose  $i$ -th column is  $\varepsilon_i$  times the corresponding column of  $M_u$  (cf. (67)). For each section  $\gamma_{\underline{\varepsilon}}$ , similar arguments to those in §5.1.3 apply. But in view of Lemma 5.1.2, the Eulerian AR one constructs using  $\gamma_{\underline{\varepsilon}}$  actually coincides with the one obtained by means of  $\gamma$ , which is  $\mathcal{E}_n^{>0}$ . This shows that this AR is well-defined on  $\mathcal{M}_{0,n+3}^{or,>0}$ .

But a totally similar construction can be made on the other component  $\mathcal{M}_{0,n+3}^{or,<0}$  of the inverse image of  $\mathcal{M}_{0,n+3}^{>0}$  by  $\nu$ . One obtains a well-defined Eulerian AR on this component, that we will denote accordingly by  $\mathcal{E}_n^{<0}$ . In order to construct it in coordinates, that is on  $U$ , one just has to consider the map  $\tilde{\gamma} = D \circ \gamma : U \rightarrow G_2^{or}(\mathbf{R}^{n+3})^{<0}$ . Since the pull-back under  $D$  of Euler's 2-form  $E_n$  is  $-E_n$ , we obtain that  $D^*(E_n) = (-1)^n E_n$ , which gives us the following

**Lemma 5.17.** *Let  $\nu^{>0}$  and  $\nu^{<0}$  be the restrictions of  $\nu$  to  $\mathcal{M}_{0,n+3}^{or,>0}$  and  $\mathcal{M}_{0,n+3}^{or,<0}$  respectively.*

1. *One has  $(\nu^{>0})_*(\mathcal{E}_n^{>0}) = (-1)^n (\nu^{<0})_*(\mathcal{E}_n^{<0})$ .*
2. *Consequently, Euler's abelian relation  $\mathcal{E}_n^{>0}$  constructed on  $\mathcal{M}_{0,n+3}^{>0}$  in §5.1.3 is*
  - a. *well-defined when  $n$  is even;*
  - b. *only defined up to sign if  $n$  is odd.*

Albeit rather elementary, this result is of crucial importance regarding the birational action of  $\mathfrak{S}_{n+3}$  on the abelian relations of  $\mathcal{W}_{0,n+3}$ .

A first remark, is that when  $n$  is odd, the notation  $\mathcal{E}_n^{>0}$  is a bit misleading since it may let the reader think that there is a more natural/canonical Eulerian AR among  $\mathcal{E}_n^{>0}$  and  $-\mathcal{E}_n^{>0}$ . As it can

be verified easily (using one of the two formulas (5.11) or (5.14) for instance) this is not the case: there is no non arbitrary way to distinguish one from the other. As we will see below, this has non trivial consequences regarding the invariance properties satisfied (or not) by Euler's abelian relation on  $\mathcal{M}_{0,n+3}^{or,>}(\mathbf{R})$ . Note however that in spite of this, it does not make null and void some invariance results stated in §5.1.5 (namely Proposition 5.11 and Proposition 5.15) when  $n$  is odd: it would just be necessary to make the statements of these propositions more precise in this case, which is easy to do and is left to the reader.

Secondly, the construction above is in no way specific to the positive component of  $\mathcal{M}_{0,n+3}(\mathbf{R})$ . It can be generalized to any component of it: for any  $\sigma \in \mathfrak{R}_{n+3}$ , we construct an Eulerian AR on  $\mathcal{M}(\sigma)$  (of the web  $\mathcal{W}_{0,n+3}$ ) which is well-defined, but only up to sign when  $n$  is odd. In any case, we denote by  $\mathcal{E}_n^\sigma$  this (when  $n$  is even) or one of these two (when  $n$  is odd) abelian relations, the choice of  $\mathcal{E}_n^\sigma$  instead of its opposite being arbitrary when  $n$  is odd. The span  $\langle \mathcal{E}_n^\sigma \rangle$  is a well-defined 1-dimensional vector subspace of the space  $AR(\sigma)$  of abelian relations of the web  $\mathcal{W}_{0,n+3}^\sigma$ . Since the permutations in  $\mathfrak{S}_{n+3}$  induce isomorphisms between the  $AR(\sigma)$ 's and because these spaces are pairwise non-identical, it is necessary to have a fixed way to identify them all to one of them, say  $AR(\mathbf{1}) = AR(\mathcal{W}_{0,n+3}^{>0})$ , by means of fixed isomorphisms  $\Xi^\sigma : AR(\sigma) \rightarrow AR(\mathbf{1})$ , one for each  $\sigma$ .

Of course, it is natural to assume that these isomorphisms satisfy the following properties: first, one should have  $\Xi^{\mathbf{1}} = \text{Id}$ . Secondly, since the space  $AR_R(\mathcal{W}_{0,n+3})$  of rational ARs of  $\mathcal{W}_{0,n+3}$  (on the whole moduli space  $\mathcal{M}_{0,n+3}(\mathbf{R})$  or even on its complexification  $\mathcal{M}_{0,n+3}$ ) naturally embeds into  $AR(\sigma)$  for any  $\sigma$ , a natural property that the isomorphisms  $\Xi^\sigma$  must all satisfy is that they coincide with the identity when taking their restriction along  $AR_R(\mathcal{W}_{0,n+3})$ . Finally, if one expects the Euler ARs to give rise to a 1-dimensional representation, then each  $\Xi^\sigma(\mathcal{E}_n^\sigma)$  must be a non trivial multiple of  $\mathcal{E}_n^{>0}$  for any  $\sigma$  (which makes sense even in the case when  $n$  is odd).

To summarize, in order that the action of  $\mathfrak{S}_{n+3}$  on the space(s) of abelian relations of the web(s) under scrutiny give rise to a representation with the expected properties, it is necessary that the  $\Xi^\sigma$ 's satisfy the following properties:

- (110) (i). one has  $\Xi^{\mathbf{1}} = \text{Id}_{AR(\mathbf{1})}$ ;  
(ii). for any  $\sigma \in \mathfrak{R}_{n+3}$ , one has  $\Xi^\sigma|_{AR_R(\mathcal{W}_{0,n+3})} = \text{Id}_{AR_R(\mathcal{W}_{0,n+3})}$ ;  
(iii). for any  $\sigma \in \mathfrak{R}_{n+3}$ , there exists  $\lambda^\sigma \neq 0$  such that  $\Xi^\sigma(\mathcal{E}_n^\sigma) = \lambda^\sigma \mathcal{E}_n^{>0}$ .

We are going to consider the two cases according to the parity of  $n$  separately, this because there exist isomorphisms satisfying (110) when  $n$  is even, whereas this does not seem to be the case when  $n$  is odd (which we prove explicitly when  $n$  is equal to 3)

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**Case when  $n$  is even.** When  $n$  is even, essentially all the results claimed in [Da2] are indeed satisfied. Actually in this case, it is not necessary to work with oriented grassmannians but everything can be obtained within the (let say) 'classical' Gelfand-MacPherson theory developed in [GM]. For this reason, the treatment below of the case when  $n$  is even is rather concise.

We then assume that  $n \geq 2$  is even: hence  $m = n/2$  is a positive integer. In this case, one has  $E_n^n = (E_n^2)^m$  and Euler's AR can be constructed from  $P_1 = E_n^2$ . This invariant 4-form represents the first Pontryagin class of the oriented tautological bundle  $\mathcal{T}^{or}$  hence is the pull-back under the 2-to-1 projection  $G_2^{or}(\mathbf{R}^{n+3}) \rightarrow G_2(\mathbf{R}^{n+3})$  of the invariant 4-form on  $G_2(\mathbf{R}^{n+3})$ , denoted by

the same notation, and which represents the first Pontryagin class of  $\mathcal{T}$ . Then the  $2n$ -form  $P_1^m$  on  $G_2(\mathbf{R}^{n+3})$  is ‘leading’ according to the terminology of Gelfand-MacPherson’s paper (*cf.* §1.3.4 therein). Then from [GM, §1.3.2], one constructs the abelian relation  $\mathcal{E}_n^\sigma$  for any  $\sigma$ , which is well-defined on any component  $\mathcal{M}(\sigma)$  of  $\mathcal{M}_{0,n+3}(\mathbf{R}^{n+3})$ . This explains more conceptually the point 2.a. of Lemma 5.17.

But the results of [GM] can also be used to investigate the invariant properties of the Euler ARs. Contrarily to when dealing with oriented grassmannians, in which case the choices of lifts of permutations can be a bit subtle, here we will only consider the naive lifts  $\tilde{\sigma} \in \text{Aut}(G_2(\mathbf{R}^{n+3}))$  of permutations  $\sigma \in \mathfrak{S}_{n+3} \simeq \text{Aut}(\mathcal{M}_{0,n+3})$ , whose definition we recall:  $\tilde{\sigma}$  is the automorphism of the grassmannian induced by the linear map  $\mathbf{R}^{n+3} \rightarrow \mathbf{R}^{n+3}$ ,  $(x_i)_{i=1}^{n+3} \mapsto (x_{\sigma(i)})_{i=1}^{n+3}$ , also denoted by  $\tilde{\sigma}$ . It is well-known that the latter is an orthogonal transformation. On the other hand, it is proved in [GM, Corollary 3.25] that the 4-form  $P_1$  on  $G_2(\mathbf{R}^{n+3})$  not only is left invariant under the action of  $\text{SO}_{n+3}(\mathbf{R})$ , but also under that of the full orthogonal group  $\text{O}_{n+3}(\mathbf{R})$ . Thus one has  $\tilde{\sigma}_*(P_1) = P_1$  hence  $\tilde{\sigma}_*(P_1^m) = P_1^m$  for any permutation  $\sigma$ . On the other hand, in the case when  $\sigma$  is a transposition, one verifies easily that  $\tilde{\sigma}$  reverses the natural orientation of the  $H_0$ -orbits.

From all the preceding considerations, one deduces the

**Proposition 5.18.** *For any  $\sigma \in \mathfrak{S}_{n+3}$ :*

1. *one has  $\tilde{\sigma}^*(\mathcal{E}_n^\sigma) = \text{sgn}(\sigma) \mathcal{E}_n^{>0}$ ;*
2. *Euler’s abelian relation  $\mathcal{E}_n^\sigma$  does not belong to  $\text{AR}_C(\mathcal{W}_{0,n+3})$  hence*

$$(111) \quad \text{AR}(\sigma) = \text{AR}_C(\mathcal{W}_{0,n+3}) \oplus \langle \mathcal{E}_n^\sigma \rangle.$$

*Proof.* The first point follows from the arguments given just before the proposition so let us deal with the second. If  $\mathcal{E}_n^\sigma$  belongs to  $\text{AR}_C(\mathcal{W}_{0,n+3})$  for one permutation  $\sigma$ , this holds true for all according to 1. We thus have a line  $\langle \mathcal{E}_n^{>0} \rangle = \langle \mathcal{E}_n^\sigma \rangle$  in the space of combinatorial ARs, which is invariant by  $\mathfrak{S}_{n+3}$ . But this is impossible since, as proved by Damiano (see Corollary 4.9.1 here),  $\text{AR}_C(\mathcal{W}_{0,n+3})$  is an irreducible  $\mathfrak{S}_{n+3}$ -representation. Thus for any  $\sigma$ , one has  $\mathcal{E}_n^\sigma \notin \text{AR}_C(\mathcal{W}_{0,n+3})$  and the decomposition in direct sum (111) follows from dimensional considerations.  $\square$

The first point of this proposition answers question (78).ii. when  $n$  is even.

**Remark 5.19.** 1. *When  $n$  is odd, one cannot do as above and get an  $(n-1)$ -abelian relation for  $\mathcal{W}_{0,n+3}$  using the standard Gelfand-MacPherson theory applied to an invariant  $2n$ -form on the usual (i.e. non oriented) grassmannian  $G_2(\mathbf{R}^{n+3})$ . As explained above, when  $n$  is even one can construct Euler’s AR by considering the  $(n/2)$ -wedge power of an invariant representative of the first Pontryagin class. This is not possible when  $n$  is odd: in this case,  $2n$  is not a multiple of 4 hence one really has to work with an invariant 2-form at some point. And if  $n = 2m + 1$ , the thing is that there is no such invariant 2-form on  $G_2(\mathbf{R}^{n+3})$  as it follows from the well known description*

$$H^*(G_2(\mathbf{R}^{n+3}), \mathbf{R}) \simeq \frac{\mathbf{R}[p_1, \bar{p}_1, \dots, \bar{p}_{m+1}]}{(p\bar{p} - 1)}$$

of its cohomology ring over  $\mathbf{R}$ , where  $p_1$  stands for the first Pontryagin class of the tautological bundle  $\mathcal{T}$  on  $G_2(\mathbf{R}^{n+3})$ , the  $\bar{p}_k$ 's are the Pontryagin classes of the rank  $n+1$  cotautological bundle  $\bar{\mathcal{T}}$ , and where  $p$  and  $\bar{p}$  denote the corresponding Pontryagin characters.<sup>43</sup>

At this point, we believe it is interesting to recall how the cohomology ring of the oriented grassmannian  $G_2^{or}(\mathbf{R}^{n+3})$  is related to that of the standard one: via the injection of the former cohomology ring into the latter induced by the 2-1 covering  $G_2^{or}(\mathbf{R}^{n+3}) \rightarrow G_2(\mathbf{R}^{n+3})$ , one has

$$(112) \quad H^*(G_2^{or}(\mathbf{R}^{n+3}), \mathbf{R}) \simeq \frac{H^*(G_2(\mathbf{R}^{n+3}), \mathbf{R})[e]}{(p_1 - e^2)} = \frac{\mathbf{R}[e, p_1, \bar{p}_1, \dots, \bar{p}_{n/2}]}{(p\bar{p} - 1, p_1 - e^2)}$$

when  $n$  is even, and

$$(113) \quad H^*(G_2^{or}(\mathbf{R}^{n+3}), \mathbf{R}) \simeq \frac{H^*(G_2(\mathbf{R}^{n+3}), \mathbf{R})[e, \bar{e}]}{(e\bar{e}, p_1 - e^2, \bar{p}_{m+1} - \bar{e}^2)} = \frac{\mathbf{R}[e, p_1, \bar{e}, \bar{p}_1, \dots, \bar{p}_{m+1}]}{(p\bar{p} - 1, e\bar{e}, p_1 - e^2, \bar{p}_{m+1} - \bar{e}^2)}$$

when  $n$  is odd and is written  $n = 2m + 1$  for a positive integer  $m$ , where in both cases  $e$  (resp.  $\bar{e}$ ) stands here for the Euler class of the oriented tautological bundle  $\mathcal{T}^{or}$  (resp. the cotautological bundle  $\bar{\mathcal{T}}^{or}$ ) on  $G_2^{or}(\mathbf{R}^{n+3})$ , with  $p_1$  and the  $\bar{p}_k$ 's being the associated Pontryagin classes.<sup>44</sup>

When working with oriented grassmannians, one can consider (an invariant representative of) the Euler class independently of the parity of  $n$ , which has the convenience to allow to build Euler's abelian relation  $\mathcal{E}_n^{>0}$  in an uniform manner.

2. From (112) and (113), it follows that as an  $\mathbf{R}$ -algebra, the cohomology ring of  $G_2^{or}(\mathbf{R}^{n+3})$  is generated by  $e$  when  $n$  is odd, and by  $e$  and  $\bar{e}$  but not by  $e$  alone when  $n$  is even (cf. the remark in [GMZ, p. 267]). This shows that the claim that 'the real cohomology ring of  $G_2^{or}(\mathbf{R}^{n+3})$  is generated by the Euler class' p. 1341 of [Da2] is incorrect when  $n$  is odd. However, this is irrelevant in what concerns the validity of the proof of [Da2, Theorem 5.1] given in the sixth section of [Da2]: since its degree is  $n+1$ , the class  $\bar{e}$  does not belong to the set of characteristic classes considered in the statement of this theorem and because its square is non zero and has degree  $2n+2 = \dim(G_2^{or}(\mathbf{R}^{n+3}))$ , it is a non trivial multiple of the  $(n+1)$ -th wedge power of  $e$ . This remark, which is missing in [Da2, §6] from our point of view, shows that no case has been forgotten and that the proof given there is complete.

For each class  $\sigma \in \mathfrak{S}_{n+3} = \mathfrak{S}_{n+3}/D_{0,n+3}$ , thanks to the decomposition in direct sum (111), one constructs an isomorphism  $\Xi^\sigma : \mathbf{AR}(\sigma) \rightarrow \mathbf{AR}(\mathcal{W}_{0,n+3}^{>0})$  uniquely characterized by the properties (110) with  $\lambda^\sigma = 1$  in (iii). We then consider the following map:

$$(114) \quad \begin{aligned} \mathfrak{S}_{n+3} &\longrightarrow \text{Aut}(\mathbf{AR}(\mathcal{W}_{0,n+3}^{>0})) \\ \sigma &\longmapsto \Xi^\sigma \circ \sigma^* : \Psi \longmapsto \Xi^\sigma(\sigma^*(\Psi)). \end{aligned}$$

**Corollary 5.20.** *The above map makes of*

$$(115) \quad \mathbf{AR}(\mathcal{W}_{0,n+3}^{>0}) = \mathbf{AR}_C(\mathcal{W}_{0,n+3}) \oplus \langle \mathcal{E}_n^{>0} \rangle.$$

<sup>43</sup>We recall that  $\mathcal{T}$  and  $\bar{\mathcal{T}}$  fit into the following short exact sequence  $0 \rightarrow \mathcal{T} \rightarrow \mathbf{R}^{n+3} \rightarrow \bar{\mathcal{T}} \rightarrow 0$  of fiber bundles over  $G_2(\mathbf{R}^{n+3})$ , where  $\mathbf{R}^{n+3}$  stands for the trivial bundle of rank  $n+3$ . Also, the associated Pontryagin characters are the elements  $p = 1 + p_1$  and  $\bar{p} = 1 + \bar{p}_1 + \dots + \bar{p}_{\lfloor (n+3)/2 \rfloor - 1}$  of the cohomology ring with real coefficients of  $G_2(\mathbf{R}^{n+3})$ .

<sup>44</sup>The descriptions of the cohomology rings above are very classical, see [Sad] or [He] and the references therein.



a  $\mathfrak{S}_{n+3}$ -module with  $AR_C(\mathcal{W}_{0,n+3})$  and  $\langle \mathcal{E}_n^{>0} \rangle$  as irreducible components, with Young symbols  $[221^{n-1}]$  and  $[1^{n+3}]$  respectively. In particular,  $\mathfrak{S}_{n+3}$  acts as the signature representation on the 1-dimensional component  $\langle \mathcal{E}_n^{>0} \rangle$ .

To conclude this discussion of the case when  $n$  is even, we recall that when  $n = 2$ , considering the local system (with non trivial monodromy) of complex abelian relations of  $\mathcal{W}_{0,n+3}$  on the complex moduli space  $\mathcal{M}_{0,n+3}(\mathbf{C})$ , we have constructed in Example 3 (via a method different from that discussed just above) a complex  $\mathfrak{S}_5$ -representation on the complexification of  $AR(\mathcal{W}_{0,5}^{>0})$ . The latter has two irreducible components: one is the complexification of the natural  $\mathfrak{S}_5$ -representation on  $AR_C(\mathcal{W}_{0,5})$  and the other, denoted by  $\langle \mathcal{A}b \rangle$  in Example 3, is the trivial representation.

We believe that this generalizes to any  $n$  even: considering the general form for the components of  $\mathcal{E}_n^{>0}$  we conjecture and have proved this conjecture to be correct for  $n$  even and less than 12 (cf. (125) and Proposition 5.28 further), that the complexification  $\mathcal{E}_n^{\mathbf{C}}$  of  $\mathcal{E}_n^{>0}$  extends to a global multivalued AR on the whole complexified moduli space  $\mathcal{M}_{0,n+3}(\mathbf{C})$ , with logarithmic hence unipotent monodromy. Using the same elementary arguments as in Example 3, this would give a 1-step filtration on  $AR(\mathcal{W}_{0,n+3}^{\mathbf{C}})$  inducing a decomposition into irreducible  $\mathfrak{S}_{n+3}$ -modules

$$(116) \quad \mathbf{Gr}AR^*(\mathcal{W}_{0,n+3}^{\mathbf{C}}) = AR_C(\mathcal{W}_{0,n+3}^{\mathbf{C}}) \oplus \langle \mathcal{E}_n^{\mathbf{C}} \rangle$$

with the 1-dimensional component  $\langle \mathcal{E}_n^{\mathbf{C}} \rangle$  being the trivial complex  $\mathfrak{S}_{n+3}$ -representation.

This might seem a bit surprising at first sight, a natural (but naive) guess would be that (116) coincides with the complexification of the real  $\mathfrak{S}_{n+3}$ -representation (115). But this cannot be the case since the 1-dimensional component of the former would be the trivial representation, whereas that of the latter is the signature representation. Here is a (not fully rigorous) explanation of this apparent inconsistency: focusing only on the components of dimension 1, one can say that (115) is obtained by comparing the Euler's abelian relations  $\mathcal{E}_n^{>0}$  and  $\mathcal{E}_n^\sigma$  at two fixed points  $u^{>0}$  and  $u^\sigma$  of two connected components  $\mathcal{M}_{0,n+3}^{>0}$  and  $\mathcal{M}(\sigma)$  for any  $\sigma \in \mathfrak{S}_{0,n+3}$ .

It is sufficient to only consider the case when  $\sigma$  is a transposition in order to understand the complex action on  $\langle \mathcal{E}_n^{\mathbf{C}} \rangle$ . One identifies  $\mathcal{M}_{0,n+3}$  with  $\mathbf{R}^n \setminus A_n$  and accordingly  $\mathcal{M}_{0,n+3}(\mathbf{C})$  with  $\mathbf{C}^n \setminus A_n$  and for  $i, j$  such that  $1 \leq i < j \leq n+3$ , one denotes by  $H_{i,j}$  the hyperplane of the braid arrangement corresponding to the limits obtained by making coinciding the  $i$ -th and  $j$ -th components of configurations of  $n+3$  points on the Riemann sphere. Then the action of  $\sigma = (i, j)$  on  $\mathcal{E}_n^{\mathbf{C}}$  can be understood as follows: one can find a smooth loop  $\gamma : [0, 1] \rightarrow \mathbf{C}^n \setminus A_n \simeq \mathcal{M}_{0,n+3}(\mathbf{C})$ , based at  $\gamma(0) = \gamma(1) = u^{>0}$  and such that  $\gamma(1/2) = u^\sigma$ , which is of trivial index with respect to all the hyperplanes of the braid arrangement  $A_n$  at the exception of the hyperplane  $H_{i,j}$ , with respect to which it is of index 1. For  $\epsilon = 0, 1$ , one sets  $\gamma_\epsilon : [0, 1] \rightarrow \mathbf{C}^n \setminus A_n$ ,  $t \mapsto \gamma(t/2 + \epsilon/2)$ . For any path  $\bar{\gamma} : [0, 1] \rightarrow \mathbf{C}^n \setminus A_n$  and any germ of holomorphic object  $F$  at  $\bar{\gamma}(0)$  one denotes by  $\bar{\gamma} \cdot F$  the germ at  $\bar{\gamma}(1)$  obtained after analytic continuation of  $F$  along  $\bar{\gamma}$  (in case its exists of course).

Since the loop  $\gamma$  is the concatenation  $\gamma_1 \gamma_0$ , one has

$$\sigma \mathcal{E}_n^{\mathbf{C}} = (i, j) \mathcal{E}_n^{\mathbf{C}} = \gamma \cdot \mathcal{E}_n^{\mathbf{C}} = \gamma_1 \cdot (\gamma_0 \cdot \mathcal{E}_n^{\mathbf{C}})$$

as germs of holomorphic ARs at  $u^{>0}$ . Up to sign, the complex analytic germ of abelian relation  $\gamma_0 \cdot \mathcal{E}_n^{\mathbf{C}}$  coincides with (the germ at  $u^\sigma$  of) the complexification of  $\mathcal{E}_n^\sigma$ , denoted by  $\mathcal{E}_n^{\sigma, \mathbf{C}}$ . Since the

orientation of  $H_0$ -orbits does not change along continuous deformations, one has  $\gamma_0 \cdot \mathcal{E}_n^{\mathbf{C}} = -\mathcal{E}_n^{\sigma, \mathbf{C}}$ . Applying the same reasoning to  $\gamma_1$ , one gets that the following fact holds true:

*As  $\mathfrak{S}_{n+3}$ -representations, the irreducible components of dimension 1 in (115) and in (116) are related in the following manner: the latter  $\langle \mathcal{E}_n^{\mathbf{C}} \rangle$  identifies with the square of the complexification  $\langle \mathcal{E}_n^{>0} \rangle^{\mathbf{C}}$  of the former. Mathematically, one has*

$$\langle \mathcal{E}_n^{\mathbf{C}} \rangle \simeq \left( \langle \mathcal{E}_n^{>0} \rangle^{\mathbf{C}} \right)^{\otimes 2}.$$

Since it has not yet been proved that Euler' AR  $\mathcal{E}_n^{\mathbf{C}}$  has unipotent monodromy for all  $n \geq 2$  even, the existence of a decomposition (116) with a trivial irreducible component of dimension 1 is not known in full generality, so the previous statement is only conjectural for the moment.

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**Case when  $n$  is odd.** When  $n$  is odd, the situation is quite different from that in the former (even) case. Indeed, since the Eulerian abelian relations  $\mathcal{E}_n^{\sigma}$ 's for  $\sigma \in \mathfrak{R}_{n+3}$  are only well-defined up to sign in this case, looking for isomorphisms  $\Xi^{\sigma}$  satisfying (110) actually requires to have beforehand suitably chosen one of the two representatives  $\pm \mathcal{E}_n^{\sigma}$  of Euler's AR on  $\mathcal{M}(\sigma)$  for any  $\sigma$ .

Since  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  is of maximal dimension  $(n+1)(n+2)/2$ ,  $\mathcal{E}_n^{\sigma}$  is combinatorial for any  $\sigma \in \mathfrak{S}_{n+3}$ . Since  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$  is irreducible as a  $\mathfrak{S}_{n+3}$ -representation when  $n$  is odd (according to Theorem 4.2), one immediately gets the following negative result:

**Proposition 5.21.** *There is no set  $\{ \star \mathcal{E}_n^{\sigma} \mid \sigma \in \mathfrak{R}_{n+3} \}$  of representatives of Euler's abelian relations such that for all permutations  $\sigma \in \mathfrak{S}_{n+3}$ ,  $\sigma_*(\star \mathcal{E}_n^1)$  is colinear with  $\star \mathcal{E}_n^{\sigma}$  (as elements of  $\mathbf{AR}(\sigma)$ ).*

This result implies in particular that there is no way to build a 1-dimensional  $\mathfrak{S}_{n+3}$ -representation from the Euler's abelian relations following the lines considered by Damiano in [Da2].

It is worth considering the simplest case when this occurs, namely when  $n = 3$ . Let us discuss a concrete example of how the  $\mathcal{E}_3^{\sigma}$ 's are related with the abelian relation  $\mathcal{E}_3^{>0}$  considered in §5.1.6. Let  $S$  be the involutive automorphism of  $\mathcal{W}_{0,6}$  given by  $S(u_1, u_2, u_3) = (u_2, u_1, u_3)$  in the coordinates  $u_1, u_2, u_3$ . This map can be seen to be the realization as a birational automorphism of the transposition  $s = (45) \in \mathfrak{S}_6$ . It induces an isomorphism between  $\mathcal{M}_{0,6}^{>0}$  and  $\mathcal{M}(s) \simeq \{(u_i)_{i=1}^3 \mid 1 < u_2 < u_1 < u_3\}$ . Using the explicit expression (109) for  $\mathcal{E}_3^{>0}$ , one gets that its pull-back under  $S$  is an AR on  $\mathcal{M}(s)$  whose sixth component is

$$S^*(\mathcal{E}_{3,6}) = S^* \left( \frac{du_1 \wedge du_2}{u_1 u_2 (u_2 - 1)} \right) = - \frac{du_1 \wedge du_2}{u_1 u_2 (u_1 - 1)}.$$

On the other hand, by direct computations, one can compute the first terms of the Taylor expansion of the Eulerian AR  $\mathcal{E}_3^s$  and verify that one has  $S^*(\mathcal{E}_3^{>0}) = -\mathcal{E}_3^s$ . However, as rational abelian relations of  $\mathcal{W}_{0,6}$ ,  $\mathcal{E}_3^{>0}$  and  $\mathcal{E}_3^s$  are easily seen to be linearly independent.

By direct computation, one verifies that  $\mathbf{AR}_C(\mathcal{W}_{0,6}) = \mathbf{AR}(\mathcal{W}_{0,6})$  is spanned by the  $\mathfrak{S}_6$ -orbit of  $\mathcal{E}_3^{>0}$  which is consistent with the fact that the space of ARs of  $\mathcal{W}_{0,6}$  is an irreducible  $\mathfrak{S}_6$ -representation.

**5.1.8. An explicit formula for Euler's abelian relation when  $n$  is odd.** By means of direct computations, we have been able, for  $n$  odd and less than or equal to 9, to give a closed formula for Euler's abelian relation. Indeed, using Proposition 4.12 (together with Remark 4.13), it is just a matter of linear algebra to determine the functions  $F$  depending on  $u' = U_1(u) = (u_1, \dots, u_{n-1})$  satisfying the following properties on the whole domain  $U$ :

- (117) • the function  $F$  satisfies the same functional identities as (96) and (107);  
 • the identity  $0 = \sum_{i=1}^{n+3} (-1)^i \left( C^{\circ(i-1)} \right)^* \left( F(u') du_1 \wedge \dots \wedge du_{n-1} \right)$  holds true identically.

We obtained that the set of such functions is a vector space of dimension one, spanned by an explicit rational function. In particular, it gives us an explicit closed rational formula for the function  $e_{n-1}(u')$  of Proposition 5.9.

**Proposition 5.22.** *Let  $n$  be an odd integer.*

1. *Euler's abelian relation  $\mathcal{E}_n^{>0}$  is combinatorial (hence rational): one has*

$$\mathcal{E}_n^{>0} \in \mathbf{AR}_C(\mathcal{W}_{0,n+3}).$$

2. *If  $n$  is moreover assumed to be less than or equal to 11 then as a function of  $u'$ , the function corresponding to the  $U_1$ -component of Euler's AR  $\mathcal{E}_n^{>0}$  is rational and given by*

$$(118) \quad \tilde{e}_{n-1} = F_0 + \sum_{2 \leq i < j \leq n} (-1)^{i+j} F_{ij} + \sum_{i=2}^n (-1)^i F_{i,n+2}.$$

*Consequently, up to a non zero multiplicative constant, the function  $e_{n-1}$  defined via the integral formula (84) coincides with the above rational function  $\tilde{e}_{n-1}$ .*

We conjecture that the previous statement actually is satisfied for any odd integer  $n \geq 3$ .

**Remark 5.23.** 1. *Since it is irrelevant regarding its content, the preceding proposition has been stated without mentioning the subtlety that Euler's abelian relation  $\mathcal{E}_n^{>0}$  on  $\mathcal{M}_{0,n+3}^{>0}$  actually is only defined up to sign (cf. Lemma 5.17).*

2. *It turns out that both conditions in (117) have to be assumed in order to get a space of solutions of dimension 1. Indeed, for  $n$  odd less than 11, we have verified that the elements of the space of functions spanned by  $\mathfrak{B}_n(u')$  (see (59)) and satisfying only one of these two conditions has dimension  $(n+1)/2$ . We have also verified (by means of direct computations as well) that for an element  $F \in \langle \mathfrak{B}_n(u') \rangle$ , satisfying the functional identity (107) is automatic if one assumes that  $F$  satisfies (96). We conjecture that these facts hold true in full generality.*

**5.1.9. An explicit formula for Euler's abelian relation when  $n$  is even.** If the results of §5.1.7 (especially Corollary 5.20) give a clear picture of  $\mathbf{AR}(\mathcal{W}_{0,n+3}^{>0})$  as a vector space (and even as a  $\mathfrak{S}_{n+3}$ -module) when  $n$  is even, they have the defect of not being explicit. However, as in the case when  $n$  is odd (cf. the proposition just above), it is possible to make (the components of) Euler's abelian relation explicit when  $n$  is odd as well, at least for small values of  $n$  (we give a conjectural closed formula in the general case).

We assume that  $n$  is even in what follows. Since  $\mathcal{E}_n^{>0}$  does not belong to  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ , one cannot use a similar approach to that used just above to get Proposition 5.22. The path we follow below is the following. We first deal with the  $n = 4$  case: applying a variation of Abel's method for solving functional equations, we determine in explicit form (one of) the components of  $\mathcal{E}_4^{>0}$ .

Then we deduce from it a conjectural closed formula for the function  $e_{n-1}$  in (84) for  $n \geq 2$  arbitrary. Then, by means of explicit computations, we verify that this formula is indeed correct for all even integers less than or equal to 12.

**The case  $n = 4$  via Abel's method.** It is now well-known that Abel's method for solving functional equations, by means of successive differentiations and eliminations in order to get a (partial) differential equation eventually, gives rise to a powerful and efficient method to determine the abelian relations of 1-codimensional webs.<sup>45</sup> Facing the problem of making the components of  $\mathcal{E}_4^{>0}$  explicit, we have realised that it could be handled by adapting Abel's method again. Since it could be useful in dealing with other (curvilinear) webs, we say below a few words on how Abel's method works in the case under scrutiny.

We work on  $\mathbf{C}^4$  and  $x_1, \dots, x_4$  stand for the standard variables on it. We denote by  $\partial_{x_1}, \dots, \partial_{x_4}$  the associated constant vector fields. The first integrals we are going to work with are the  $U_1, \dots, U_7$  given in (44), and  $\Omega_1, \dots, \Omega_7$  stand for the corresponding normals (see (45)). Our goal is to apply an appropriate adaptation of Abel's method to determine, for all  $i$ , the general form of the functions  $e_i$  of three variables appearing in the following identity:

$$(119) \quad \sum_{i=1}^7 e_i(U_i) \Omega_i = 0.$$

Thanks to the action of the cyclic map  $C$  defined in (46), just dealing with  $e_1$  is enough.

For  $s = 1, 2, 3$  and any  $i$ , we denote by  $\delta_s(e_i)$  the partial derivative of  $e_i$  with respect to the  $s$ -th variable and we also consider the following rational vector fields on  $\mathbf{C}^4$ :

$$X_i = \partial_{x_i} \quad (i = 1, \dots, 4), \quad X_5 = \sum_{i=1}^4 (x_i - 1) \partial_{x_i}, \quad X_6 = \sum_{i=1}^4 x_i \partial_{x_i} \quad \text{and} \quad X_7 = \sum_{i=1}^4 x_i(x_i - 1) \partial_{x_i}.$$

Each  $X_i$  defines the foliation which admits  $U_i$  as a primitive first integral. Consequently one has identically  $X_i(U_i) = 0$  whereas for  $i \neq j$ ,  $X_i(U_j) \neq 0$  holds true (generically) thanks to the transversality of the leaves of the foliations of  $\mathcal{W}_{0,7}$ .

One has  $\Omega_i = \wedge_{k \neq i} dx_k$  for  $i = 1, \dots, 4$  hence  $(\Omega_i)_{i=1}^4$  is a (essentially the canonical) basis of  $\Omega^4(\mathbf{C}^4)$ . Setting  $\omega_i = \Omega_i(\partial_{x_2}, \partial_{x_3}, \partial_{x_4})$  for  $i = 1, \dots, 7$ , and considering only the  $\Omega_1$  components of each term of the sum in (119), we deduce that the following functional identity with scalar coefficients holds true

$$(120) \quad e_1(U_1) + e_5(U_5) \omega_5 + e_6(U_6) \omega_6 + e_7(U_7) \omega_7 = 0.$$

Dividing (120) by  $\omega_7^1$  and applying  $X_7$  makes  $e_7(U_7)$  disappear and gives us the following 'partial functional-differential equation':

$$e_1(U_1) X_7 \left( \frac{1}{\omega_7} \right) + \sum_{s=1}^3 \delta_s(e_1)(U_1) X_7(U_{1,s}) + \sum_{k=5,6} \left( e_k(U_k) X_7 \left( \frac{\omega_k}{\omega_7} \right) + \sum_{s=1}^3 \delta_s(e_k)(U_k) \frac{\omega_k X_7(U_{k,s})}{\omega_7} \right) = 0.$$

Dividing the LHS by  $X_7(U_{k,3}) \omega_6 / \omega_7$  and applying  $X_6$  makes the term involving  $\delta_3(e_6)$  disappear. One can then continue this process of division-derivation-elimination following an algorithmic process quite similar to that described in [P2, §2.2.2], to eventually get that any function

<sup>45</sup>For instance, see [P4, §1.5] and the references there.

$e_1 = e_1(x_2, x_3, x_4)$  appearing in an identity of the form (119) necessarily also satisfies

$$\left( x_{34}x_{24}x_{23} D_{234} - x_{23}(x_2 - 2x_4 + x_3) D_{23} + x_{24}(x_4 + 2x_3 + x_2) D_{24} \right. \\ \left. - x_{34}(x_3 - 2x_2 + x_4) D_{34} + 2x_{34} D_2 - 2x_{24} D_3 + 2x_{23} D_4 \right) \cdot e_1 = 0$$

with  $x_{ij} = x_i - x_j$  for any  $i, j = 1, \dots, 4$ , where we use the following notations:  $D_{234}$  stands for the order three partial derivative  $\partial_{x_2}\partial_{x_3}\partial_{x_4}$ ,  $D_{23}$  for  $\partial_{x_2}\partial_{x_3}$ , etc.

Actually, changing the order in which the terms involving the  $e_i$ 's for  $i \neq 1$  (and their partial derivatives) are eliminated gives rise to other partial differential equations. What we get at the end is an explicit system of partial differential equations, which we denote  $(Se_1)$  (it is formed by several PDE's similar to the one above and there is no point in making it explicit here). Contrarily to so many similar computations we did in the past to determine the ARs of given planar webs, we have not been able to solve  $(Se_1)$  using a computer algebra system without making additional assumptions. Looking for solutions of a certain kind<sup>46</sup>, we have been able to find one of these in explicit form.

We consider the following function, defined on the set  $X = \{(x_i)_{i=2}^4 \in \mathbf{R}^3 \mid 1 < x_2 < x_3 < x_4\}$ :

$$\begin{aligned} \varepsilon_3(x_2, x_3, x_4) = & \frac{x_2^2 \ln|x_2|}{x_3x_4(x_2-1)(x_2-x_3)(x_2-x_4)} - \frac{x_3^2 \ln|x_3|}{x_2x_4(x_2-x_3)(x_3-1)(x_3-x_4)} \\ & + \frac{x_4^2 \ln|x_4|}{x_2x_3(x_4-1)(x_2-x_4)(x_3-x_4)} - \frac{(x_2-1)^2 \ln|x_2-1|}{x_2(x_2-x_3)(x_2-x_4)(x_3-1)(x_4-1)} \\ (121) \quad & - \frac{(x_2-x_3)^2 \ln|x_2-x_3|}{x_2x_3(x_2-1)(x_2-x_4)(x_3-1)(x_3-x_4)} + \frac{(x_2-x_4)^2 \ln|x_2-x_4|}{x_2x_4(x_2-1)(x_4-1)(x_2-x_3)(x_3-x_4)} \\ & + \frac{(x_3-1)^2 \ln|x_3-1|}{x_3(x_2-1)(x_4-1)(x_2-x_3)(x_3-x_4)} - \frac{(x_3-x_4)^2 \ln|x_3-x_4|}{x_3x_4(x_4-1)(x_3-1)(x_2-x_3)(x_2-x_4)} \\ & - \frac{(x_4-1)^2 \ln|x_4-1|}{x_4(x_2-1)(x_3-1)(x_2-x_4)(x_3-x_4)}. \end{aligned}$$

**Remark 5.24.** *If one restricts to  $X \simeq \mathcal{M}_{0,6}^{>0}(\mathbf{R})$ , one can drop all the absolute values in (121). However the presence of absolute values in the above definition is useful since it allows to extend  $\varepsilon_3$  straightforwardly to a function defined on the whole complement  $\mathbf{R}^3 \setminus A_3 \simeq \mathcal{M}_{0,6}(\mathbf{R})$  where the affine space has  $x_2, x_3, x_4$  as coordinates with  $A_3$  being the braid arrangement cut out by*

$$x_2x_3x_4(x_2-1)(x_3-1)(x_4-1)(x_2-x_3)(x_2-x_4)(x_3-x_4) = 0.$$

By direct computations, we then get the following result:

**Proposition 5.25.** *The function  $\varepsilon_3$  defined just above:*

1. *is a solution of the system of PDEs  $(Se_1)$ ;*

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<sup>46</sup>We were looking for solutions of  $(Se_1)$  which are Laurent polynomials in the expressions  $x_i, x_i - 1, x_{ij}$  ( $i, j = 2, 3, 4$ ) and their logarithms. It is not relevant to elaborate more on this here.

2. satisfies the functional relations corresponding to (96) and (107): identically on  $X$ , one has

$$\varepsilon_3 \left( \frac{x_4}{x_4 - 1}, \frac{x_4}{x_4 - x_2}, \frac{x_4}{x_4 - x_3} \right) = \varepsilon_3(x_2, x_3, x_4) (x_4 - 1)^2 (x_4 - x_2)^2 (x_4 - x_3)^2 x_4^{-2}$$

and

$$\varepsilon_3 \left( \frac{x_4 - x_2}{x_4 - x_3}, \frac{x_4 - 1}{x_4 - x_3}, \frac{x_4}{x_4 - x_3} \right) = \varepsilon_3(x_2, x_3, x_4) (x_4 - x_3)^4 ;$$

3. is such that the following identity holds true on  $X$ :

$$(122) \quad 0 = \sum_{i=1}^7 \left( C^{\circ(i-1)} \right)^* \left( \varepsilon_3(x_2, x_3, x_4) dx_2 \wedge dx_3 \wedge dx_4 \right)$$

where  $C$  stands for the cyclic birational map defined in (46).<sup>47</sup>

The relation (122) means that the 7-tuple

$$(123) \quad \left( \left( C^{\circ(i-1)} \right)^* \left( \varepsilon_3(x_2, x_3, x_4) dx_2 \wedge dx_3 \wedge dx_4 \right) \right)_{i=1}^7$$

is an AR for  $\mathcal{W}_{0,7}^{>0}$ . Since its components involve logarithms, it is clear that this AR does not belong to  $\mathbf{AR}_C(\mathcal{W}_{0,7})$  and is colinear to  $(\mathcal{E}_4^{>0})$  modulo the combinatorial ARs. Because we know an explicit basis for  $\mathbf{AR}_C(\mathcal{W}_{0,7})$  (cf. Theorem 4.2), one can verify that the properties 2. and 3. of the preceding proposition characterize a subspace of dimension 1 of  $\mathbf{AR}(\mathcal{W}_{0,7})$ . We thus get the

**Corollary 5.26.** *Up to multiplication by a non zero constant:*

1. the identity (122) corresponds to Euler's abelian relation  $\mathcal{E}_4^{>0}$ ;
2. the function  $\varepsilon_3$  defined above coincides with the function  $e_3$  of Proposition 5.9.

An interesting feature of  $\varepsilon_3$ , which follows from the fact that the arguments of all the logarithms involved in its definition are absolute values, is that  $\varepsilon_3$  extends to  $\mathbf{R}^3 \setminus A_4$  (see Remark 5.24 above) and one can verify that (122) actually holds true identically on the whole complement  $\mathbf{R}^4 \setminus A_4$  where  $A_4$  stands for the braid arrangement of type  $A_4$ .<sup>48</sup> Thus up to the standard identification  $\mathbf{R}^4 \setminus A_4 \simeq \mathcal{M}_{0,7}(\mathbf{R})$ , for any  $\sigma \in \mathfrak{R}_7$  the restriction of (123) on any connected component  $\mathcal{M}(\sigma)$  of  $\mathcal{M}_{0,7}(\mathbf{R})$  is an AR which can be proved to coincide with Euler's one  $\mathcal{E}_4^\sigma$ .

But  $\varepsilon_3$  has another characteristic that is most important for our purpose, which relies on the fact that it is a function of several variables. Indeed, in clear contrast with the single-variable formula (3) for Rogers' dilogarithm, it is rather easy to guess from the quite specific type of formula (121), what might be its generalization for  $n \geq 2$  even arbitrary. Indeed, if one sets

$$x_{i0} = x_i, \quad x_{i1} = x_i - 1 \quad \text{and} \quad x_{ij} = x_i - x_j$$

for all  $i, j$  such that  $2 \leq i < j \leq 4$  and if  $M_{0,6}(x_2, x_3, x_4)$  denotes the polynomial which cuts out  $A_3$  in Remark 5.24, then  $\varepsilon_3$  can be written as follows

$$\varepsilon_3(x_2, x_3, x_4) = \frac{\sum_{i=2}^4 p_{i0}(x_i)^3 \text{Log } |x_{i0}| + \sum_{i=2}^4 p_{i1}(x_{i1})^3 \text{Log } |x_{i1}| + \sum_{2 \leq i < j \leq 4} p_{ij}(x_{ij})^3 \text{Log } |x_{ij}|}{M_{0,6}(x_2, x_3, x_4)}$$

<sup>47</sup>In the case under consideration, one has  $C(x) = ((x_2 - 1)/(x_2 - x_1), (x_3 - 1)/(x_3 - x_1), (x_4 - 1)/(x_4 - x_1), 1/x_1)$ .

<sup>48</sup> $A_4$  is cut out by the polynomial equation  $\prod_{i=1}^4 x_i(x_i - 1) \cdot \prod_{1 \leq i < j \leq 4} (x_i - x_j) = 0$ .

for some polynomials  $p_{i0}$ ,  $p_{i1}$  and  $p_{ij}$  which it is not difficult to make explicit (see below). And the nice feature of the preceding expression is that it can be generalized to any even integer  $n \geq 2$  quite straightforwardly.

**Remark 5.27.** (1) *Of course, being closed, the 3-form  $E_4$  admits a (local) not unique primitive  $\int E_4$  which can be explicitly computed and seen to carry many dilogarithmic terms. Since the expression of  $\int E_4$  is quite involved and because we will not use it here, we will not elaborate on this.*

(2) *A direct computation gives  $C^*(\varepsilon_3(U_1)\Omega_1) = -\varepsilon_3(U_2)\Omega_2$  hence the first two components of  $(\mathcal{E}_4)$  are  $\varepsilon_3(x_2, x_3, x_4)dx_2 \wedge dx_3 \wedge dx_4$  and  $-\varepsilon_3(x_1, x_3, x_4)dx_1 \wedge dx_3 \wedge dx_4$ . It follows that the involution consisting in exchanging the variables  $x_1$  and  $x_2$  transforms  $(\mathcal{E}_4)$  into its opposite. This is coherent with the first point of Proposition 5.18.*

**An explicit formula for  $e_{n-1}$  for any  $n \geq 2$  even.** Here  $n \geq 2$  stands for a fixed even integer. First we set some notation that we will use to define a function  $\varepsilon_{n-1}$  through an explicit formula. Then we will discuss how this function is meaningful regarding the Euler's abelian relation of  $\mathcal{W}_{0,n+3}^{>0}$ .

We denote here by  $X$  the open domain in  $\mathbf{R}^n$  whose elements are  $n$ -tuples  $x = (x_i)_{i=1}^n$  such that  $1 < x_1 < \dots < x_n$  (recall that there is a natural identification  $X \simeq \mathcal{M}_{0,n+3}^{>0}(\mathbf{R})$ ).

Here  $C$  stands for the cyclic birational map defined in (46) again. It is useful to work with the following first integrals for  $\mathcal{W}_{0,n+3}^{>0}$ , which behave well regarding the cyclic symmetry induced by  $C$ : one sets  $V_1 = U_1 : \mathbf{R}^n \rightarrow \mathbf{R}^{n-1}$ ,  $x \mapsto (x_2, \dots, x_n)$  and

$$V_i = (V_i^2, \dots, V_i^n) = V_1 \circ (C^{\circ(i-1)})$$

for  $i = 1, \dots, n+3$ . The associated normals are denoted by  $\Gamma_i$ : one has

$$\Gamma_1 = \Omega_1 = dx_2 \wedge \dots \wedge dx_n \quad \text{and} \quad \Gamma_i = (C^{\circ(i-1)})^*(\Omega_1) = dV_i^2 \wedge \dots \wedge dV_i^n \quad (i = 2, \dots, n+3).$$

Looking for an explicit function  $\varepsilon_{n-1}(V_1) = \varepsilon_{n-1}(x_2, \dots, x_n)$  such that the following relation

$$(124) \quad \sum_{i=1}^{n+3} (C^{\circ(i-1)})^*(\varepsilon_{n-1}(V_1) dx_2 \wedge \dots \wedge dx_n) = \sum_{i=1}^{n+3} \varepsilon_{n-1}(V_i) \Gamma_i = 0$$

holds true identically on  $X$ , one sets for any  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ :

- $x' = V_1(x) = (x_2, \dots, x_n) \in \mathbf{R}^{n-1}$ ;
- $x_{n+1} = 0$  and  $x_{n+2} = 1$ ;
- $\tilde{x} = (x_1, \dots, x_n, x_{n+1}, x_{n+2}) \in \mathbf{R}^{n+2}$ ;
- $\tilde{x}' = (x_2, \dots, x_n, x_{n+1}, x_{n+2}) \in \mathbf{R}^{n+1}$ ;
- $M_n(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ ;
- $M_{0,n+3}(x) = M_{n+2}(\tilde{x}) = \prod_{i=1}^n x_i (x_i - 1) \prod_{1 \leq k < \ell \leq n} (x_k - x_\ell)$ ;
- for  $i, j = 1, \dots, n$ , one sets  $x_{ij} = x_i - x_j$  (thus  $\tilde{x}_{i,n+1} = x_i$  and  $\tilde{x}_{i,n+2} = x_i - 1$  for any  $i \leq n$ );
- for  $i = 2, \dots, n$  and  $j = i+1, \dots, n+2$ , one denotes by  $x_{\bar{ij}}$  the  $(n-1)$ -tuple obtained from  $\tilde{x}$  by removing from it, in the same step: its first,  $i$ -th and  $j$ -th coefficients.

The notation  $M_{0,n+3}$  is not absolutely needed but we use it because it is enlightening: indeed, for  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{C}$ , the equation  $M_{0,n+3}(z) = 0$  for  $z \in \mathbf{K}^n$  cuts out the braid arrangement  $A_n$  in  $\mathbf{K}^n$  which is such that there is a natural identification between  $M_{0,n+3}(\mathbf{K})$  and  $\mathbf{K}^n \setminus A_n$ .

Then, with these notations at hand, for any  $x' = (x_2, \dots, x_n) \in \mathbf{R}^{n-1} \setminus A_{n-1}$ , one sets:

$$(125) \quad \varepsilon_{n-1}(x') = \frac{1}{M_{0,n+2}(x')} \sum_{\substack{i=2,\dots,n \\ j=i+1,\dots,n+2}} (-1)^{j-i} M_{n-1}(x_{\widehat{i}}) (\tilde{x}_{ij})^{n-1} \text{Log} |\tilde{x}_{ij}|.$$

For  $n = 4$ , one recovers exactly the function defined in (121).

By direct computations, we get the following generalization of Proposition 5.25:

**Proposition 5.28.** *Let  $n \geq 2$  be an even integer less than or equal to 12.*

1. *The function  $\varepsilon_{n-1}$  satisfies the functional relations corresponding to (96) and (107): identically on  $X$ , one has*

$$\varepsilon_{n-1} \left( \frac{x_n}{x_n - 1}, \frac{x_n}{x_n - x_2}, \dots, \frac{x_n}{x_n - x_{n-1}} \right) = \varepsilon_{n-1}(x') M_{n+1}(\tilde{x}')^2 (x_n)^{-n}$$

and 
$$\varepsilon_{n-1} \left( \frac{x_n - x_{n-2}}{x_n - x_{n-1}}, \dots, \frac{x_n - x_2}{x_n - x_{n-1}}, \frac{x_n - 1}{x_n - x_{n-1}}, \frac{x_n}{x_n - x_{n-1}} \right) = \varepsilon_{n-1}(x') (x_n - x_{n-1})^n;$$

2. *Identity (124) holds true identically and consequently it corresponds to Euler's abelian relation  $\mathcal{E}_n^{>0}$  and  $\varepsilon_{n-1}$  coincides with the function  $e_{n-1}$  of Proposition 5.9 (up to multiplication by a non zero constant).*

Of course, we conjecture that this proposition is actually satisfied for any even integer  $n \geq 2$ .

**An Eulerian basis of the space of combinatorial abelian relations for any  $n \geq 2$  even?** The identity (124) has an interesting feature which comes from the fact that there are logarithmic terms in the definition of  $\varepsilon_{n-1}$ : these being transcendent over any field of rational functions, one can construct many rational ARs from Euler's one. In this paragraph, we formalise this in precise terms and we show that, at least for the first values (namely 2, 4, 6, 8) of  $n$ , this approach is relevant regarding the question of constructing a basis of  $\mathbf{AR}_C(\mathcal{W}_{0,n+3})$ .

We continue to use the notation introduced in the preceding paragraph. We also set

$$\Pi'_n = \left\{ (i, j) \in \mathbf{N}^2 \mid \begin{array}{l} 2 \leq i \leq n \\ i+1 \leq j \leq n+2 \end{array} \right\} \quad \text{and} \quad \Pi_n = \left\{ (i, j) \in \mathbf{N}^2 \mid \begin{array}{l} 1 \leq i \leq n \\ i+1 \leq j \leq n+2 \end{array} \right\}.$$

(Note that  $\Pi'_n$  is precisely the set of  $(i, j)$ 's appearing in the sum in the definition (125) of  $\varepsilon_{n-1}$ ).

Setting also

$$R_\gamma(V_k) = (-1)^{j-i} (V_{k,ij})^{n-1} M_{n-1}(V_{k,\widehat{i}}) / M_{0,n+2}(V_k)$$

and 
$$\text{Log} |\tilde{V}_{k,\gamma}| = V_k^*(\text{Log} |\tilde{x}_\gamma|)$$

for any  $k = 1, \dots, n+3$  and any  $\gamma = (i, j) \in \Pi'_n$ , one can write

$$(126) \quad \varepsilon_{n-1}(V_k) = \sum_{\gamma \in \Pi'_n} R_\gamma(V_k) \text{Log} |\tilde{V}_{k,\gamma}|.$$

On the other hand, one has  $V_k = (C^{\circ(k-1)})^*(V_1)$ . From the fact that  $C$  is birational and induces an automorphism of  $\mathbf{C}^n \setminus A_n$ , one deduces the



**Lemma 5.29.** 1. For any  $k \leq n+3$  and  $\gamma \in \Pi'_n$ , there exist constants  $c_{k,\gamma}^v$ 's with  $v \in \Pi_n$  such that

$$(127) \quad \text{Log} |\tilde{V}_{k,\gamma}| = \sum_{v \in \Pi_n} c_{k,\gamma}^v \text{Log} |\tilde{x}_v|.$$

2. Moreover for  $v = (i, j) \in \Pi_n$ , one has  $c_{k,\gamma}^v = 0$  if  $k \in \{i, j\}$ .

3. The family of functions  $\{\text{Log} |\tilde{x}_v|\}_{v \in \Pi_n}$  is free over the field of rational functions in the  $x_i$ 's.

Using (125) and (127), it is straightforward to develop Euler's identity (124) in order to get a linear expression in the  $\text{Log} |\tilde{x}_v|$ 's:

$$0 = \sum_{k=1}^{n+3} \varepsilon_{n-1}(V_k) \cdot \Gamma_k = \sum_{k=1}^{n+3} \sum_{\gamma \in \Pi'_n} R_\gamma(V_k) \text{Log} |\tilde{V}_{k,\gamma}| \Gamma_k = \sum_{v \in \Pi_n} \left[ \sum_{k=1}^{n+3} \sum_{\gamma \in \Pi'_n} c_{k,\gamma}^v R_\gamma(V_k) \Gamma_k \right] \text{Log} |\tilde{x}_v|.$$

From the third point of Lemma 5.29, it follows that Euler's identity (124) holds true if and only if all the following relations are identically satisfied:

$$(128) \quad \sum_{k=1}^{n+3} \sum_{\gamma \in \Pi'_n} c_{k,\gamma}^v R_\gamma(V_k) \Gamma_k = 0 \quad v \in \Pi_n.$$

For any  $v = (i, j) \in \Pi_n$ , the identity above gives rise to an abelian relation for  $\mathcal{W}_{0,n+3}$  and since  $c_{i,\gamma}^{(i,j)} = c_{j,\gamma}^{(i,j)} = 0$  according to the second point of Lemma 5.29, it follows that (128) is an AR of the quadrilateral  $(n+1)$ -web  $\mathcal{W}_{\bar{i}\bar{j}}$  which has rank 1. We will denote this AR by  $AR_v(\mathcal{E}_n)$ .

**Proposition 5.30.** For  $n = 2, 4, 6, 8$ , both assertions below hold true.

1. For any  $v = (i, j) \in \Pi_n$ ,  $AR_v(\mathcal{E}_n)$  is non trivial hence spans the vector space  $AR(\mathcal{W}_{\bar{i}\bar{j}})$ .
2. The abelian relations  $AR_v(\mathcal{E}_n)$ 's for  $v \in \Pi_n$  form a basis of  $AR_C(\mathcal{W}_{0,n+3})$ .

This result has been proved by explicit formal computations. Considering this, it is quite natural to wonder about the statements of Proposition 5.30: do they hold true as well for any even integer  $n \geq 2$ ? We conjecture that it is indeed the case:

**Conjecture 5.31.** The statements of Proposition 5.30 hold true for any even integer  $n \geq 2$ .

It is worth noticing that the results of this paragraph share great similarities with some in [P4, §6.2.3]. Finally, let us mention that another approach to build a basis of the space of rational ARs of  $\mathcal{W}_{0,n+3}$  from Euler's abelian relation might be to use the 'monodromy method'<sup>49</sup> which relies on the fact that the complexification of  $\mathcal{E}_n^{>0}$  has unipotent monodromy on the complex moduli space  $\mathcal{M}_{0,n+3}(\mathbb{C})$ . We shall not pursue on this here.

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<sup>49</sup>We are referring here to the generalization for curvilinear webs of the method discussed in [P4, §6.2.3.2].

## 6. Additional problems

Many problems and conjectures have been stated previously in this text, most of them in direct relation with the webs under scrutiny here, namely the curvilinear webs  $\mathcal{W}_{0,n+3}$ . Generally speaking, webs by curves have not been studied as much as webs by hypersurfaces, which is a pity since several of the most interesting results of web geometry precisely concern curvilinear webs.

In this last section, we state a few questions/problems about webs by curves we find interesting.

**6.1. Octahedral webs.** The notion of ‘*octahedral web*’ considered in [Da2] is interesting since it appears as a generalization to curvilinear webs in arbitrary dimension of the basic notion of ‘*hexagonal*’ (aka ‘*flat*’) planar 3-web. However, some quite basic questions about this notion remain to be answered, such as the following ones:

- Is a curvilinear  $(n + 1)$ -web of maximal rank (*i.e.* of rank 1) necessarily octahedral?
- What is the moduli space of equivalence classes of curvilinear  $(n + 1)$ -webs of rank 1?
- For planar 3-webs, being hexagonal is equivalent to being of rank 1, which is in turn equivalent to being ‘*flat*’, that is of having zero curvature. In dimension  $n \geq 2$  arbitrary, is there a similar characterization of octahedral curvilinear  $(n + 1)$ -webs in terms of the vanishing of certain differential invariants attached to such webs? Same question with ‘maximal rank’ instead of ‘octahedral’ (in case these two notions do not coincide when  $n \geq 3$ ).

**6.2. Algebraization of curvilinear  $(n + 2)$ -webs with maximal rank in dimension  $n$ ?** It is classically known by web-geometers that Lie-Poincaré’s approach to the linearization and algebraization of planar 4-webs with maximal rank generalizes to  $2n$ -webs by hypersurfaces with rank  $n + 1$  on  $n$ -dimensional manifolds.<sup>50</sup> The discussion in §4.3.1, aiming to give a representation-theoretic description of the ARs of the curvilinear web formed by the bundles of lines passing through  $n + 2$  points in general position in  $\mathbf{P}^n$  (for any  $n \geq 2$ ), is of geometric nature and can be seen as well as a generalization for any  $n \geq 2$ , but now for webs by curves, of Lie-Poincaré’s approach just mentioned. It is natural to wonder whether the arguments presented in §4.3.1 actually could apply to any curvilinear  $(n + 2)$ -web in  $\mathbf{C}^n$  of rank  $n + 1$ .

More precisely, for  $n \geq 2$ , let  $\mathcal{W} = (\mathcal{F}_i)_{i=1}^{n+2}$  be a curvilinear of rank  $n + 1$  on a domain  $\Omega \subset \mathbf{C}^n$ . For any  $i$  and for  $\omega$  generic in  $\Omega$ , the  $i$ -th evaluation map associating to an AR for  $\mathcal{W}$  the value at  $\omega$  of its  $i$ -th component, is a non trivial linear map the kernel of which defines an hyperplane  $\kappa_i(\omega)$  in  $\mathbf{P}(\mathbf{AR}(\mathcal{W})) \simeq \mathbf{P}^n$ . Letting  $\omega$  vary in  $\Omega$  (or possibly in a subdomain of it), we construct that way the ‘ *$i$ -th canonical map*’  $\kappa_i : \Omega \rightarrow \check{\mathbf{P}}^n$  of  $\mathcal{W}$ . Then, inspired by the (unproven) claims of §4.3.1, we consider the five statements below (where for each of them the open domain  $\Omega$  of  $\mathbf{C}^n$  we work on is allowed to be shrunk as much as needed):

1. for any  $i$ , the map  $\kappa_i$  has rank  $n - 1$  hence is a canonical first integral for the  $i$ -th foliation  $\mathcal{F}_i$  of  $\mathcal{W}$  and the image  $V_i = \text{Im}(\kappa_i)$  is then an analytic hypersurface in  $\check{\mathbf{P}}^n$  which is a canonical space of leaves for  $\mathcal{F}_i$ ;

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<sup>50</sup>This generalization follows from works by Lie and Wirtinger on the classification of double-translation hyper-surfaces; see §4.4 and especially Theorem 4.4.3 in [PP].

2. for  $\omega \in \Omega$ , the  $\kappa_i(\omega)$ 's all are on a line  $\mathfrak{L}(\omega) \subset \check{\mathbf{P}}^n$  which intersects any  $V_i$  transversely;
3. the map  $\mathfrak{L} : \Omega \rightarrow G_1(\check{\mathbf{P}}^n)$ ,  $\omega \mapsto \mathfrak{L}(\omega)$  has rank  $n$  hence  $Z = \text{Im}(\mathfrak{L})$  is a  $n$ -dimensional subvariety of  $G_1(\check{\mathbf{P}}^n)$  and the push-forward web  $\mathfrak{L}_*(\mathcal{W})$  is a canonical model of  $\mathcal{W}$  which can be described geometrically as the trace along  $Z$  of the incidence web (locally) defined on  $G_1(\check{\mathbf{P}}^n)$  by means of the incidences between the lines in  $\check{\mathbf{P}}^n$  and the points of the  $V_i$ 's.

If the above facts indeed occur then we get the following Abel's type description of the ARs of the canonical model  $\mathcal{W}^{\text{can}} = \mathfrak{L}_*(\mathcal{W})$  of  $\mathcal{W}$ :

4. any AR of  $\mathcal{W}^{\text{can}}$  corresponds to a tuple  $(\eta_i)_{i=1}^{n+2}$  of differential  $(n-1)$ -forms  $\eta_i \in \Omega^{n-1}(V_i)$  for every  $i$  such that the trace of them with respect to the intersection of the  $V_i$ 's with the lines belonging to  $Z$  vanishes identically.

At this point, any reader even slightly familiar with web geometry reading this line will surely think that the preceding points necessarily imply the following one:

5. the vanishing of the traces of the tuples of differential forms corresponding to the ARs of  $\mathcal{W}^{\text{can}}$  (cf. 4. just above) necessarily implies that everything, the  $V_i$ 's and the ARs of  $\mathcal{W}^{\text{can}}$  actually are algebraic in the following (expected since classical) sense:
  - there exists a hypersurface  $V \subset \check{\mathbf{P}}^n$  of degree  $n+2$  such that  $V_i \subset V$  for every  $i$ ;
  - the map induced by taking their restrictions along the  $V_i$ 's induces a linear isomorphism between the space of global abelian differential forms of top degree on  $V$  and the space of abelian relations of  $\mathcal{W}^{\text{can}}$ :

$$\begin{aligned} H^0(V, \omega_V^{n-1}) &\xrightarrow{\sim} \text{AR}(\mathcal{W}^{\text{can}}) \\ \omega &\longmapsto (\omega|_{V_i})_{i=1}^{n+2}. \end{aligned}$$

All the points above are indeed satisfied for any linear web  $\mathcal{L}\mathcal{W}_{p_1, \dots, p_{n+1}}$  defined by  $n+2$  points in general position in  $\mathbf{P}^n$ . One expects that this actually holds true in full generality and that the following statement holds true for any  $n \geq 2$ :

*any curvilinear  $(n+2)$ -web of maximal  $(n-1)$ -rank in dimension  $n$  can be obtained by taking the restriction of an algebraic web  $\mathcal{W}_V$  associated to a reduced hypersurface  $V \subset \mathbf{P}^n$  of degree  $n+2$  along a (non necessarily algebraic)  $n$ -dimensional subvariety of the grassmannian of lines in  $\check{\mathbf{P}}^n$ .*

That the five points above are always satisfied seems very plausible to us. We believe that the one which would require the most work to be established is the fifth: an approach to prove it would be to use the classical Abel-inverse theorem but this would require to show that in the situation under scrutiny, the vanishing of the trace of the tuples  $(\eta_i)_{i=1}^{n+2} \in \prod_{i=1}^{n+2} \Omega^{n-1}(V_i)$  corresponding to the ARs of  $\mathcal{W}^{\text{can}}$  holds true not only on the subvariety  $Z$  but on a whole open neighborhood of it in  $G_1(\check{\mathbf{P}}^n)$ . Since the former is of dimension  $n$  and the latter of dimension  $2n-2$ , proving this when  $n \geq 3$  requires a new approach we have no idea of when writing these lines.

**6.3. A conjectural more conceptual description of the  $\mathfrak{S}_{n+3}$ -module  $\text{AR}_C(\mathcal{W}_{0, n+3})$  when  $n$  is odd.** It seems to us that one of the most surprising results of this text is the fact that the ARs of  $\mathcal{W}_{0,6}$  actually are all combinatorial and come from the canonical forms on the Fano surface  $\Sigma = F_1(\mathcal{S})$  of Segre's cubic  $\mathcal{S} \subset \mathbf{P}^4$ . Since all the ARs of  $\mathcal{W}_{0, n+3}$  are combinatorial too when  $n$

is odd, one can wonder whether a similar picture to the one described above might hold true for  $n \geq 5$ .

6.3.1. Let us be a bit more precise about what we have in mind here. Assume that  $n$  is odd: one writes  $n = 2m - 1$  with  $m = (n + 1)/2$ . Hence  $n + 3 = 2m + 2$ . As we have seen in §2.2.1 and §2.2.2,  $\mathcal{W}_{0,n+3}$  admits a birational model  $W_{0,n+3}$  which is a web by lines on a certain  $n$ -dimensional projective variety  $V_n \subset \mathbf{P}^N$ . These varieties have been studied by several authors, in particular in the recent paper [BM] where the authors establish many interesting properties of them. Those which are interesting for our purpose here are the following ones, which appear as generalizations of some basic properties of Segre's cubic (which corresponds to the case  $m = 2$ ):

- The variety  $V_n$  has isolated singularities and one has  $\text{Card}(\text{Sing}(V_n)) = \binom{2m+2}{m}$ . Moreover  $V_n$  contains  $\binom{2m+1}{m+1} + \binom{2m+1}{m-1}$  linear subspaces of dimension  $m$ .
- The Fano variety of lines  $\Sigma_n = F_1(V_n) \subset G_1(\mathbf{P}^N)$  has dimension  $n - 1 = 2m - 2$ . Among its components,  $n + 3$  are isomorphic to  $\overline{\mathcal{M}}_{0,n+2}$  and are covering families. The others are formed by the lines contained in one of the  $m$ -planes included in  $V_n$  hence all are isomorphic to  $G_1(\mathbf{P}^m)$ .

Considering the case  $n = 3$ , it is more than tantalizing to hope for a description of the ARs of  $W_{0,n+3}$  by means of global  $(n - 1)$ -forms on the Fano variety  $\Sigma_n$ . Let  $\omega_{\Sigma_n} = \omega_{\Sigma_n}^{n-1}$  stand for the dualizing sheaf of  $\Sigma_n$ . We find the following questions interesting:

- Does the trace induce a well-defined map  $H^0(\Sigma_n, \omega_{\Sigma_n}) \rightarrow AR(\mathcal{W}_{0,n+3})$ ? In the affirmative, is this map an isomorphism?
- Does the  $\mathfrak{S}_{n+3}$ -action on  $V_n$  induce a birational action on  $\Sigma_n$  and a linear one on  $H^0(\Sigma_n, \omega_{\Sigma_n})$ ? If yes, what is the latter as a  $\mathfrak{S}_{n+3}$ -module? Is it irreducible with Young symbol [31<sup>n</sup>]?
- Is there a coherent subsheaf  $\omega_{\Sigma_n}^{m-1}$  of the sheaf of meromorphic  $(m - 1)$ -differential forms on  $\Sigma_n$  satisfying the following properties:
  - the space  $H^0(\Sigma_n, \omega_{\Sigma_n}^{m-1})$  of its global sections has dimension  $n + 2$ ;
  - the wedge product gives rise to an isomorphism  $\wedge^2 H^0(\Sigma_n, \omega_{\Sigma_n}^{m-1}) \simeq H^0(\Sigma_n, \omega_{\Sigma_n})$ ;
  - the action of  $\mathfrak{S}_{n+3}$  by pull-backs makes of  $H^0(\Sigma_n, \omega_{\Sigma_n}^{m-1})$  an irreducible  $\mathfrak{S}_{n+3}$ -module.
- Do the elements of  $H^0(\Sigma_n, \omega_{\Sigma_n}^{m-1})$  give rise to  $(m - 1)$ -abelian relations for  $W_{0,n+3}$ ?

In another direction, the fact that Segre's cubic  $\mathcal{S}$  is just a particular example of a whole family of varieties (cubic hypersurfaces in  $\mathbf{P}^4$ ) all carrying a web by lines of maximal rank naturally leads to ask the following questions for any odd integer  $n \geq 3$ :

- As a subvariety of  $\mathbf{P}^N$ , can  $V_n$  be deformed in such a way that its deformations carry  $(n+3)$ -webs by projective lines? Does it exist deformations of this kind which are smooth?
- If the answer to the previous question is affirmative, what can be said about the Fano variety of lines of such a deformation of  $V_n$ ? Do some rational differential forms on this Fano variety give rise to abelian relations for the  $(n + 3)$ -web by lines on the deformation under scrutiny?

We believe that giving answers in full generality to the questions asked above is difficult. The next case beyond the classical one of Segre's cubic is already interesting and deserves to be discussed on its own.

**6.3.2. Case  $n = 5$ .** The case when  $n = 5$  is interesting since  $V_5$  has already been studied by several authors: see for instance **[R1]** (in particular §3 and §11 therein) for a classical reference and **[FSM]** or **[HMSV]** for recent ones.

Let  $p_1, \dots, p_7$  be seven points in general position in  $\mathbf{P}^5$ . According to §2.2.1,  $V_5$  is the birational image of  $\mathbf{P}^5$  by the rational map associated to the linear system of cubic hypersurfaces with a double point at each  $p_i$ . It can be verified that  $V_5$  lives in  $\mathbf{P}^{13}$ , has 35 isolated double points, is of degree 40 and carries 8 covering families of lines. Many other interesting geometric properties of  $V_5$  are listed in **[R1, §11]**.

Another construction of  $V_5$  has been given by modern authors: from the recent references indicated above, it can be deduced that  $V_5$  is the singular locus of an explicit cubic hypersurface  $C \subset \mathbf{P}^{13}$ , namely the hypersurface cut out by the equation  $C = 0$  where  $C$  stands for the following cubic form (in homogeneous coordinates  $X_1, X_2, Y_a$  with  $a = 1, \dots, 4$  and  $Z_b$  for  $b = 1, \dots, 8$ ):

$$\begin{aligned} C = & X_1X_2(X_1 + X_2) + X_1X_2(Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8) - (X_1Y_2Y_4 + X_2Y_3Y_1) \\ & + (X_1Z_2Z_6 + X_2Z_3Z_7 + X_1Z_4Z_8 + X_2Z_5Z_1) + (Y_1Z_2Z_6 + Y_2Z_3Z_7 + Y_3Z_4Z_8 + Y_4Z_5Z_1) \\ & - (Z_1Z_2Z_3 + Z_2Z_3Z_4 + Z_3Z_4Z_5 + Z_4Z_5Z_6 + Z_5Z_6Z_7 + Z_6Z_7Z_8 + Z_7Z_8Z_1 + Z_8Z_1Z_2). \end{aligned}$$

The homogeneous ideal  $I(V_5)$  of  $V_5 \subset \mathbf{P}^{13}$  is then generated by the quadrics given by the partial derivatives  $\partial C / \partial U$  where  $U$  stands for any one of the fourteen homogeneous coordinates  $X_1, X_2, Y_a$  or  $Z_b$ .

Knowing an explicit generating set for  $I(V_5)$  makes it possible to attack the questions raised in the preceding subsection by means of effective computations. We have tried to do so using the software Macaulay2 but our attempt failed: even when working over a finite field of small cardinal to make everything simpler, the computations to get the ideal of the Fano variety of lines  $\Sigma_5 = F_1(V_5)$  were too memory-consuming so that we were not able to get to the end. A brute force approach to study  $\Sigma_5$  does not seem very efficient, understanding this Fano variety may require a more conceptual approach.

Finally, let us mention a possibly naive idea about the varieties  $V_n$  for  $n$  odd bigger than 1. Both  $V_3$  and  $V_5$  can be defined by means of an invariant cubic:  $V_3$  itself is a cubic (namely Segre's cubic  $\mathcal{S}$ ) and  $V_5 = \text{Sing}(C)$  where  $C \subset \mathbf{P}^{13}$  is the  $\mathfrak{S}_8$ -invariant cubic hypersurface cut out by  $C = 0$ . These two remarks lead to wonder about the general case and to ask the following questions:

- Can  $V_n$  be defined by means of a particular  $\mathfrak{S}_{n+3}$ -invariant cubic hypersurface  $C_n \subset \mathbf{P}^N$ ?
- If the answer to the previous question is affirmative, as an algebraic subset of  $\mathbf{P}^N$  and up to projective equivalence, does  $V_n$  coincide with the  $(n - 3)$ -th higher singular locus  $\text{Sing}^{(n-3)}(C_n)$  of the cubic hypersurface  $C_n$ ?<sup>51</sup>

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<sup>51</sup>For any projective variety  $X \subset \mathbf{P}^N$ , the higher order singular loci  $\text{Sing}^{(k)}(X)$  ( $k \in \mathbf{N}$ ) are the schemes inductively defined  $\text{Sing}^{(0)}(X) = X$  and  $\text{Sing}^{(k)}(X) = \text{Sing}(\text{Sing}^{(k-1)}(X))$  for any  $k \geq 1$ .

We confess not having any argument to support positive answers to these questions. To tell the truth, we would be amazed if they can be answered in a positive way. In any case, these questions were too beautiful and intriguing for not to be asked!

**6.4. About the characterization of the webs by lines on cubic threefolds.** Here we would like to discuss briefly some interesting questions indicated to us by Prof. J.-M. Hwang a few years ago. The problem under scrutiny here can be considered in the bigger realm of Fano manifolds of dimension 3 but for the sake of simplicity, we will only consider the case of cubic threefolds below.

Let  $X \subset \mathbf{P}^4$  be a generic (in particular smooth) cubic hypersurface and denote by  $F = F_1(X)$  its Fano surface. It follows from a more general result by Hwang (namely [Hw, Theorem 1.2]) that the local analytic class of the 6-web by lines  $\mathcal{LW}_X$  characterizes  $X$  as a projective variety:

**Theorem 6.1** (Hwang). *Let  $X' \subset \mathbf{P}^4$  be another smooth cubic hypersurface. Assume that there exists a local biholomorphism  $\Phi : (X, x) \rightarrow (X', x')$  such that  $\mathcal{LW}_{X,x} = \Phi^*(\mathcal{LW}_{X',x'})$ . Then  $\Phi$  is the germ at  $x$  of a global isomorphism between  $X$  and  $X'$ , which necessarily coincides with the restriction along  $X$  of a global projective automorphism of  $\mathbf{P}^4$ .*

This nice result is very much in the original spirit of web geometry, namely that webs are quite rigid objects for which local analytic equivalence may imply (under some additional assumptions of course) algebraic/global equivalence hence algebraization.

We find it interesting to consider the previous result in the light of the remarkable algebraization theorem of Blaschke and Walberer (about the algebraization of some maximal rank curvilinear 3-webs in dimension 3, see Appendix B for more details). This suggests to wonder about a possible algebraization result for the 6-webs by lines in dimension 3. In view of stating it, let us draw up a list of some nice properties enjoyed by any curvilinear 6-web  $\mathcal{W}$  equivalent to a web  $\mathcal{LW}_X$  associated to a generic cubic threefold  $X$ : from the results of §3.1, it is clear that such a web  $\mathcal{W}$

1. is linearizable;
2. is skew;
3. has maximal 2-rank (equal to 10);
4. has its 1-rank bigger than or equal to 5;
5. is such that there exists a subspace  $A^1(\mathcal{W}) \subset AR^{(1)}(\mathcal{W})$  of dimension 5 such that the wedge map  $\wedge^2 A^1(\mathcal{W}) \rightarrow AR^{(2)}(\mathcal{W})$  is a well-defined isomorphism.

Note that all these properties are stated in invariant form. Clearly, 5. implies 3. and 4. Remark also that 2. is not satisfied by  $\mathcal{W}_{0,6} \simeq \mathcal{LW}_S$ . However, for any smooth cubic  $X \subset \mathbf{P}^4$ , the web  $\mathcal{LW}_X$  is skew indeed. A first question that comes to mind for such  $X$  is the following:

**Question 6.2.** *Assume that  $X$  is smooth and denote by  $F = F_1(X)$  its Fano surface. Is  $\text{rk}^{(1)}(\mathcal{LW}_X)$  equal to 5? More precisely, is the trace map  $H^0(F, \Omega_F^1) \rightarrow AR^{(1)}(\mathcal{LW}_X)$  surjective or not?*

Each of the five conditions listed above is a strong property and it is natural to expect that any 6-web satisfying all of them must be of a very particular type. Considering this as well as Blaschke-Walberer's algebraization theorem, it is natural to make the following

**Conjecture 6.3.** *Any 6-web  $\mathcal{W}$  satisfying the properties 1. to 5. above is equivalent to an algebraic web  $\mathcal{LW}_X$  associated to a cubic hypersurface  $X \subset \mathbf{P}^4$ .*

Actually, we believe that the same conclusion holds true under the weaker assumption that only the first three of these conditions are satisfied. It would be interesting to even drop the linearizability assumption and to know whether a skew curvilinear 6-web in dimension 3 with maximal 2-rank is necessarily linearizable. However, answering this may be quite difficult.

**6.5. The 6-web by planes on Perazzo's cubic fourfold.** In [Pera], Perazzo considers a certain cubic hypersurface in  $\mathbf{P}^5$ , namely the one cut out by

$$X_1X_2X_3 - Y_1Y_2Y_3 = 0$$

with respect to some homogeneous coordinates  $X_1, X_2, X_3, Y_1, Y_2, Y_3$  on  $\mathbf{P}^5$ , named after him and denoted by  $\mathcal{P}$  here. This cubic enjoys several interesting properties and has been considered in several papers such as [Bak] or the more recent one [Loo], to which we refer the interested reader.

Here are some of the properties of  $\mathcal{P}$  which are going to be relevant for our purpose:

- Perazzo's cubic has 9 double lines  $D_1, \dots, D_9$  as singularities;
- for  $p \in \mathcal{P}$  generic (namely  $p \in \mathcal{P}_{\text{reg}} = \mathcal{P} \setminus (\cup_{i=1}^6 D_i)$ ), the intersection  $S_p = \mathcal{P} \cap T_p\mathcal{P}$  of Perazzo's cubic with its tangent hyperplane  $T_p\mathcal{P} \simeq \mathbf{P}^4$  at  $p$  is a cubic threefold with 10 nodal points:  $p$  and the nine intersection points  $T_p\mathcal{P} \cap D_i$  for  $i = 1, \dots, 9$ . Hence  $S_p$  is projectively equivalent to Segre's cubic threefold;
- the Fano scheme  $F_2(\mathcal{P})$  of 2-planes included in  $\mathcal{P}$  is of pure dimension 2 and through a general point  $p \in \mathcal{P}$  pass six pairwise distinct 2-planes  $\Pi_1(p), \dots, \Pi_6(p)$  included in  $\mathcal{P}$ . Consequently, Perazzo's cubic fourfold carries a linear 6-web of codimension 2, that we will denote by  $\mathcal{LW}_{\mathcal{P}}$ . One can define six regular germs of map  $\Pi_i : (\mathcal{P}, p) \rightarrow G_2(\mathbf{P}^5)$  ( $i = 1, \dots, 6$ ) such that  $\Pi_i(p'), \dots, \Pi_6(p')$  are the six planes included in  $\mathcal{P}$  through  $p'$  for any  $p'$  sufficiently close to  $p$ . Then  $\Pi_1, \dots, \Pi_6$  are local first integrals for  $\mathcal{LW}_{\mathcal{P}}$  at  $p$ : locally at this point, one has

$$\mathcal{LW}_{\mathcal{P}} = \mathcal{W}(\Pi_1, \dots, \Pi_6);$$

- given  $p \in \mathcal{P}_{\text{reg}}$ , for  $\tilde{p}$  generic the restrictions along  $S_p$  of the (germs of) maps

$$\begin{aligned} \Pi_{i,p} : (\mathcal{P}, \tilde{p}) &\longrightarrow F_1(S_p) \subset G_1(T_p\mathcal{P}) \simeq G_1(\mathbf{P}^4) \\ q &\longmapsto \Pi_i(q) \cap T_p\mathcal{P} \end{aligned}$$

give rise to local first integrals for the linear web  $\mathcal{LW}_{S_p}$  carried by Segre's cubic  $S_p$ . Consequently: one has  $(\mathcal{LW}_{\mathcal{P}})|_{S_p} = \mathcal{LW}_{S_p}$  locally at  $p$  on  $S_p$ ; and the (germs at the points  $\Pi_{i,p}(\tilde{p})$ 's of the) Fano surface  $F_1(S_p)$  can be taken as 'the' space(s) of leaves of the foliations of  $\mathcal{LW}_{\mathcal{P}}$  locally at  $\tilde{p}$  on  $\mathcal{P}$ .

We set  $x_5 = x_1x_2x_3/x_4$  and  $x_6 = x_4$  and see both  $x_5$  and  $x_6$  as rational functions in the  $x_k$ 's with  $k = 1, \dots, 4$ . Then the following map is a birational affine parametrization of Perazzo's cubic

$$\mathcal{P} : \mathbf{C}^4 \dashrightarrow \mathcal{P} \subset \mathbf{P}^5, \quad (x_i)_{i=1}^4 \longmapsto [x_s]_{s=1}^6$$

and the pull-back  $\mathcal{P}^*(\mathcal{LW}_{\mathcal{P}})$  (that we will still denote by  $\mathcal{LW}_{\mathcal{P}}$ , at bit abusively), can be described quite explicitly. Indeed one verifies immediately that the six 2-planes included in  $\mathcal{P}$  passing through a generic point of  $\mathcal{P}$  are cut out by the systems of equations

$$(129) \quad X_1/Y_{v(1)} = \lambda_1, \quad X_2/Y_{v(2)} = \lambda_2 \quad \text{and} \quad X_3/Y_{v(3)} = \lambda_1\lambda_2$$

for some generic parameters  $\lambda_1, \lambda_2 \in \mathbf{C}$ , where  $\nu$  ranges in the set  $\mathfrak{S}_3$  of permutations of  $\{1, 2, 3\}$ .

For each  $\nu \in \mathfrak{S}_3$ , the pull-back under  $\mathcal{P}$  of the foliations on  $\mathcal{P}$  by the 2-planes with equations (129), denoted by  $\mathcal{F}_\nu$ , admits the rational map  $x \mapsto (x_1/x_{3+\nu(1)}, x_2/x_{3+\nu(2)})$  as first integral. After some elementary simplifications, we obtain that the following very simple rational functions can be taken as first integrals for  $\mathcal{LW}_\mathcal{P}$  on  $\mathbf{C}^4$ :

$$\begin{aligned} \Pi_{\mathbf{1}} &= \left( x_3, \frac{x_1}{x_4} \right) & \Pi_{(23)} &= \left( x_2, \frac{x_1}{x_4} \right) & \Pi_{(1,2)} &= \left( x_3, \frac{x_2}{x_4} \right) \\ \Pi_{(13)} &= \left( x_1, \frac{x_3}{x_4} \right) & \Pi_{(123)} &= \left( x_2, \frac{x_3}{x_4} \right) & \Pi_{(132)} &= \left( x_1, \frac{x_2}{x_4} \right). \end{aligned}$$

Remark that  $\Pi_{\mathbf{1}}$  and  $\Pi_{(23)}$  both have  $x_1/x_4$  as a component. Thus the tangent distribution  $\langle T_{\mathcal{F}_{\mathbf{1}}}, T_{\mathcal{F}_{(23)}} \rangle$  is not of dimension 4 but is the integrable 3-dimensional one defined by  $d(x_1/x_4) = 0$ . This shows that  $\mathcal{LW}_\mathcal{P}$ , which is a 6-web of codimension 2 on a space of dimension 4, does not satisfy the general position hypothesis usually assumed when working classically with webs of this kind. Consequently the classical results about the ranks of webs of this type (*cf.* the material of Chapter 8 of [Go]) do not apply to  $\mathcal{LW}_\mathcal{P}$ . Considering the remarkable properties enjoyed by  $W_{0,6} = \mathcal{LW}_S$ , one can wonder about the status of their direct generalizations to the web by planes associated to Perazzo's cubic fourfold. Here is a sample of questions we find interesting:

**Questions 6.4.** Let  $p$  be a regular point of  $\mathcal{P}$ , set  $F = F_1(\mathcal{S}_p)$  and let  $k$  stand for 1 or 2.

1. What precisely is the Fano scheme of planes  $F_2(\mathcal{P}) \subset G_2(\mathbf{P}^5)$ ? How is it related to the Fano surface of lines  $F_1(\mathcal{S}_p) \subset G_1(\mathbf{P}^4)$  of Segre's cubic?
2. What is the  $k$ -rank of  $\mathcal{LW}_\mathcal{P}$ ?
3. Does the following assertion hold true: 'for any  $\eta \in \mathbf{H}^0(F, \omega_F^k)$ , the trace  $\text{Tr}(\eta)$ , locally defined as  $\text{Tr}(\eta) = \sum_{i=1}^6 \Pi_{i,p}^*(\eta)$ , vanishes identically'?
4. If the answer to the previous question is affirmative, what can be said about the then well-defined associated linear map  $\mathbf{H}^0(F, \omega_F^k) \rightarrow \mathbf{AR}^{(k)}(\mathcal{LW}_\mathcal{P})$ ,  $\eta \mapsto (\Pi_{i,p}^*(\eta))_{i=1}^6$ ? Is it surjective? Is it an isomorphism?



### Appendix A: the abelian 1-ARs of $\mathcal{W}_{0,6}$ .

In §3 (see Proposition 3.1 more specifically), we have translated into terms of webs some classical results about smooth cubic threefolds and their Fano surface. We then specialized some of these results to the case of Segre’s cubic  $\mathcal{S}$  to obtain a conceptual way to describe the 2-abelian relations of  $\mathcal{LW}_{\mathcal{S}} \simeq \mathcal{W}_{0,6}$ : the suitable version of ‘Abel’s theorem’ holds true for Segre’s cubic and its Fano surface  $\Sigma = F_1(\mathcal{S})$  as well and gives us a natural linear isomorphism  $H^0(\Sigma, \omega_{\Sigma}^2) \simeq \mathbf{AR}^{(2)}(\mathcal{LW}_X)$ . Then in §3.3.4, by means of direct computations, we gave another much more down to earth and elementary proof of this result .

For a smooth cubic  $X \subset \mathbf{P}^4$ , the 1-ARs of  $\mathcal{LW}_S$  all come from holomorphic 1-forms on the corresponding Fano surface  $F_1(X)$ : see the content of Proposition 3.1 corresponding to  $k = 1$ . It is natural to wonder whether this can be specialized to the pair  $(\mathcal{S}, \Sigma)$  as well. This question has already be discussed before (page 34) in the more general context of cubic threefolds with isolated singularities. A strategy relying on deformations is evokated page 36 where it is mentioned that it applies to Segre’s cubic. This approach relies on some unpublished results due to Collino.

In this Appendix, we are going to explain the main points of the abstract approach via deformations in order to construct 1-ARs for  $\mathcal{LW}_S$  (identified with  $\mathcal{W}_{0,6}$ ) from the global *abelian* differential 1-forms on a certain modification  $\tilde{\Sigma}$  of  $\Sigma$ . Because we find it justified and funny, the ARs obtained that way will be called ‘*Abelian 1-ARs*’. Although conceptually interesting, the lack of explicitness of the first construction of these ARs is not fully satisfying. We will remedy to this in a second stage. Proceeding as in §3.3.4, using explicit computations and some basic facts of the theory of representations of  $\mathfrak{S}_6$ , we will construct an explicit basis of a 5-dimensional subspace of  $\mathbf{AR}^{(1)}(\mathcal{W}_{0,6})$  which naturally corresponds to the space of abelian 1-forms on  $\tilde{\Sigma}$ .

★

Before entering into the considerations below, let us recall the following elementary fact about the space of 1-ARs of  $\mathcal{W}_{0,6}$ . We know that this web is totally non skew: two distinct of its foliations are tangent to a foliation of codimension 1 (admitting for a first integral a forgetful map  $\mathcal{M}_{0,6} \rightarrow \mathcal{M}_{0,4} \simeq \mathbf{P}^1 \setminus \{0, 1, \infty\}$ ). Its follows that any of its 3-subwebs, and even more so  $\mathcal{W}_{0,6}$  itself, has a space of 1-ARs of infinite dimension. The abelian relations we are interested in in this Appendix are those coming from particular global rational 1-forms on the Fano surface of Segre’s cubic. Thus these are 1-ARs of a very specific kind. In particular, they span a space of finite dimension which makes undertaking their study more reasonable.

**A.1. An abstract construction of the Abelian 1-ARs of  $\mathcal{W}_{0,6}$ .** We freely use below the results of [Col2] (especially from §2.3 and §3 therein) to which we refer the reader for more details.<sup>52</sup> We explain how the approach discussed page 34 above can be applied to get some 1-ARs for  $\mathcal{W}_{0,6}$ .

Let  $S$  and  $T$  be two homogeneous cubic forms in five variables such that  $S = 0$  cuts out  $\mathcal{S}$  and where  $T$  is sufficiently generic so that  $X_t = \{S + tT = 0\} \subset \mathbf{P}^4$  be an equation of a smooth cubic hypersurface for all  $t \in \mathbf{A}$  such that  $0 < |t| \ll 1$ . For any  $t$ , one denotes by  $F_t$  the Fano surface of lines in  $X_t$  and  $\mathcal{F} \subset G_1(\mathbf{P}^4) \times \mathbf{A}^1$  stands for the total space of the associated family of Fano surfaces. We denote by  $f$  the restriction along  $\mathcal{F}$  of the second projection  $G_1(\mathbf{P}^4) \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$ .

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<sup>52</sup>Note however that some of the results we use are not fully proved in [Col2].

Our goal is to establish a property (namely, the vanishing of the trace) of some global rational 1-forms on  $F_0 = \Sigma$  by using the fact that it can be obtained as a degeneration of the smooth Fano surfaces  $F_t$  (with  $0 < |t| \ll 1$ ). For this reason, we are only interested in the geometry of the affine pencil  $\mathcal{F}$  on a vertical (analytic) neighbourhood of the fiber  $\Sigma = f^{-1}(0)$ . According to [Col2, Prop. 2.4], near  $t = 0$  the pencil  $\mathcal{F}$  is singular exactly at the points of  $\Sigma$  denoted by  $\ell_{ij,kl}$  above (and  $L[(ij), (kl)]$  in Collino's paper). Moreover, these points, which are 45 in number, all are ordinary quadratic singularities. Then Collino explains that when performing a particular small blow-up at each of these isolated singular points, one gets a modified family

$$\tilde{f} = f \circ \nu : \widetilde{\mathcal{F}} \xrightarrow{\nu} \mathcal{F} \xrightarrow{f} \mathbf{A}$$

enjoying nice properties. Indeed, locally near the origin of the base,  $\widetilde{\mathcal{F}} \rightarrow \mathbf{A}$  is a semi-stable degeneration of surfaces: the total space  $\widetilde{\mathcal{F}}$  is smooth and the central fiber  $\widetilde{\Sigma} = \tilde{f}^{-1}(0) = \nu^{-1}(\Sigma)$  is a union of smooth (rational) surfaces which have normal crossings. The surface  $\widetilde{\Sigma}$  is a birational modification of  $\Sigma$  whose abelian forms can be studied via deformations from those of the smooth fibers  $F_t$  for  $t \neq 0$  sufficiently close to  $0 \in \mathbf{A}$ , which was a priori not doable for  $\Sigma$ , this surface not having normal crossings (see §3.3.3).

As mentioned in §3.3.3, the covering families of lines on  $\mathcal{S}$  give rise to six components of  $\Sigma$  isomorphic to  $\overline{\mathcal{M}}_{0,5}$ , denoted by  $\Sigma(i)$  for  $i = 1, \dots, 6$ . If  $\widetilde{\Sigma}(i)$  stands for the strict transform of  $\Sigma(i)$  in  $\widetilde{\mathcal{F}}$  under  $\nu$ , then it can be verified that  $\nu : \widetilde{\Sigma}(i) \rightarrow \Sigma(i) \simeq \overline{\mathcal{M}}_{0,5}$  is an isomorphism. Let  $\psi_i$  be the rational map associating to a generic point of  $\mathcal{S}$  the line of  $\Sigma(i)$  passing through it. Up to the natural identification of  $\mathcal{S}$  and  $\Sigma(i)$  with  $\overline{\mathcal{M}}_{0,6}$  and  $\overline{\mathcal{M}}_{0,5}$  respectively,  $\psi_i : \mathcal{S} \dashrightarrow \Sigma(i)$  corresponds to the map  $\overline{\mathcal{M}}_{0,6} \dashrightarrow \overline{\mathcal{M}}_{0,5}$  consisting in forgetting the  $i$ -th point.

The family  $\widetilde{\mathcal{F}} \rightarrow \mathbf{A}$  being a semi-stable deformation of  $\widetilde{\Sigma}$ , the general theory of [Fr] applies locally on the pointed germ of curve  $\Delta = (\mathbf{A}, 0)$ : denoting by  $\Omega_{\widetilde{\mathcal{F}}}^1(\text{Log}(\widetilde{\Sigma}))$  the associated relative log complex, one sets  $\Lambda_{\widetilde{\Sigma}}^1 = \Omega_{\widetilde{\mathcal{F}}}^1(\text{Log}(\widetilde{\Sigma}))|_{\widetilde{\Sigma}}$ . This sheaf of meromorphic 1-forms on  $\widetilde{\Sigma}$  enjoys nice properties which we will now describe. We denote by  $\partial\widetilde{\Sigma}$  the 1-boundary of  $\widetilde{\Sigma}$ , defined as the union of the irreducible curves lying in the intersection of two distinct components of  $\widetilde{\Sigma}$ . For any irreducible component  $\widetilde{F}$ , one sets  $\partial\widetilde{F} = \widetilde{F} \cap \partial\widetilde{\Sigma}$  and  $\Omega_{\widetilde{F}}^1(\text{Log}(\partial\widetilde{F}))$  stands for the sheaf of meromorphic forms on  $\widetilde{F}$ , holomorphic on  $\widetilde{F} \setminus \partial\widetilde{F}$  and with logarithmic poles along  $\partial\widetilde{F}$ . For any such  $\widetilde{F}$ , we denote the natural inclusion by  $a : \widetilde{F} \hookrightarrow \widetilde{\Sigma}$ . Then from the general theory of Friedman, one gets that  $\Lambda_{\widetilde{\Sigma}}^1$ :

- (i). is a locally free sheaf of  $\mathcal{O}_{\widetilde{\Sigma}}$ -modules;
- (ii). coincides with the sheaf of abelian 1-forms on  $\widetilde{\Sigma}$ : one has  $\Lambda_{\widetilde{\Sigma}}^1 = \omega_{\widetilde{\Sigma}}^1$ ;
- (iii). is such that there is an embedding of sheaves  $\Lambda_{\widetilde{\Sigma}}^1 \hookrightarrow \bigoplus_{\widetilde{F}} a_* (\Omega_{\widetilde{F}}^1(\text{Log}(\partial\widetilde{F})))$  where the sum is taken over all the irreducible components  $\widetilde{F}$  of  $\widetilde{\Sigma}$  in the target sheaf.

But much more can be said in the case under scrutiny. Indeed, from [Col2, §3], one gets:

- (iv). For any  $i$ , the pair  $(\widetilde{\Sigma}(i), \partial\widetilde{\Sigma}(i))$  is isomorphic to  $(\overline{\mathcal{M}}_{0,5}, \partial\overline{\mathcal{M}}_{0,5})$  hence through  $a$ , one gets a map  $H^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1) \rightarrow H^0(\overline{\mathcal{M}}_{0,5}, \Omega_{\overline{\mathcal{M}}_{0,5}}^1(\text{Log}(\partial\overline{\mathcal{M}}_{0,5})))$  which is an isomorphism;
- (v). Consequently, one has  $h^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1) = 5$  (cf. the proof of [Col2, Prop. 3.1])

- (vi). Hence  $\nu_*\left(\Omega_{\mathcal{F}}^1(\text{Log}(\widetilde{\Sigma}))\right)$  is a locally free sheaf of rank 5 on  $(\mathbf{A}, 0)$ .
- (vii). The  $\mathfrak{S}_6$ -action lifts to  $\widetilde{\Sigma}$  and acts preserving  $\omega_{\widetilde{\Sigma}}^1$ . Consequently  $\mathbf{H}^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1)$  naturally carries a structure of  $\mathfrak{S}_6$ -module.

We obtain that the points **2.** when  $k = 1$  and **3.b** of Proposition 3.1 hold true in the case when  $X$  is specialized to Segre's hypercubic  $\mathbf{S}$ .

**Proposition 6.5.** 1. For any  $\eta \in \mathbf{H}^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1)$ , its trace  $\text{Tr}(\eta) = \sum_{i=1}^5 \psi_i^*(\eta)$  vanishes identically.

2. Hence there is an injective well-defined map

$$(130) \quad \text{Tr}^{(1)} : \mathbf{H}^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1) \rightarrow \mathbf{AR}^{(1)}(\mathcal{W}_{0,6}), \quad \eta \longmapsto \left(\psi_i^*(\eta)\right)_{i=1}^6.$$

- 3. The image  $\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) = \text{Im}(\text{Tr}^{(1)})$  is a subspace of dimension 5 of  $\mathbf{AR}^{(1)}(\mathcal{W}_{0,6})$  such that the wedge map  $\wedge^2 \mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) \rightarrow \mathbf{AR}^{(2)}(\mathcal{W}_{0,6})$  is a well-defined isomorphism.
- 4. For any  $i$ , the  $i$ -th projection  $\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) \rightarrow \mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6})[i]$  is an isomorphism where  $\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6})[i]$  stands for the space of  $i$ -th components of the abelian 1-ARs of  $\mathcal{W}_{0,6}$ .
- 5. The map  $\text{Tr}^{(1)}$  induces an isomorphism of  $\mathfrak{S}_6$ -representations  $\mathbf{H}^0(\omega_{\widetilde{\Sigma}}^1) \simeq \mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6})$ . Hence  $\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6})$  is an irreducible  $\mathfrak{S}_6$ -module with associated Young symbol [33].

*Sketch of a proof.* Since the sheaf in (v). above is locally free, one can argue using a simple deformation argument, similarly as in the proof of Proposition 3.3 page 32. The points 1., 2. and the first part of 3. (namely that  $\wedge^2 \mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) \rightarrow \mathbf{AR}^{(2)}(\mathcal{W}_{0,6})$  is indeed well-defined) of the Proposition follow rather directly from this. The point 4. is a direct translation into terms of  $\mathcal{W}_{0,6}$  of the point (iv) above.

As for 5., the first part is easy and left to the reader. To get the second part, for any  $i$  we consider the induced representation of  $\mathfrak{S}_5 = \text{Fix}(i) \subset \mathfrak{S}_6$  on  $\mathbf{H}^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1)$ . Then it can be verified that the map in (iv). is an isomorphism of  $\mathfrak{S}_5$ -representation. According to Lemma 2.1 of [DFL],  $\mathbf{H}^0(\overline{\mathcal{M}}_{0,5}, \Omega_{\overline{\mathcal{M}}_{0,5}}^1(\text{Log}(\partial\overline{\mathcal{M}}_{0,5})))$  is an irreducible  $\mathfrak{S}_5$ -module, with associated Young symbol [32]. Considering the branching rules for representations of the symmetric groups, one deduces that  $\mathbf{H}^0(\omega_{\widetilde{\Sigma}}^1)$  is an irreducible  $\mathfrak{S}_6$ -module with Young symbol [33], which gives us the fifth point.

The single point remaining to be proved is that the wedge map of 3. is an isomorphism. Unfortunately, we do not have a conceptual argument to offer for this. This will follow from our explicit computations in the next part of this appendix.  $\square$

Although interesting, this is the expected result, which somewhat reduces its relevance. Below we will recover most of its content by means of an elementary explicit approach. However, taking an abstract approach naturally leads to ask several questions, that we find interesting.

**Questions 6.6.** 1. We have given a description of some 1-ARs of  $\mathcal{W}_{0,6}$  by means of the abelian 1-forms on  $\widetilde{\Sigma}$ . But this birational model of  $\Sigma$  relies on some choices (namely, the way the singular points of  $\mathcal{F}$  near  $t = 0$  are blown up) hence is non canonical. It would be interesting to have a more intrinsic description of the 1-ARs of  $\mathcal{W}_{0,6}$  under scrutiny.

For any element  $\tilde{\omega} \in \mathbf{H}^0(\tilde{\Sigma}, \omega_{\tilde{\Sigma}}^1)$ , the push-forward  $\omega = \nu_*(\tilde{\omega})$  is a rational 1-form on  $\Sigma$  whose restriction on each  $\Sigma(i)$  is holomorphic on  $\Sigma(i) \setminus \partial\Sigma(i) \simeq \mathcal{M}_{0,5}$  with logarithmic singularities along  $\partial\Sigma(i) = \overline{\partial\mathcal{M}_{0,5}}$ . Is such an  $\omega$  an abelian differential on  $\Sigma$ ? If so, does the trace of any abelian 1-form on  $\Sigma$  vanish? Or in other terms, does  $\nu_*$  give rise to an isomorphism  $\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) \simeq \mathbf{H}^0(\tilde{\Sigma}, \omega_{\tilde{\Sigma}}^1) \rightarrow \mathbf{H}^0(\Sigma, \omega_{\Sigma}^1)$ ?

2. According to [DFL] (see the paragraph just before Lemma 2.1 therein), the pull-back under the composition  $\Sigma(i) \subset \Sigma \subset G_1(\mathbf{P}^4)$  of the dual of the tautological bundle  $\mathcal{T}$  on the grassmannian  $G_1(\mathbf{P}^4)$  identifies with  $\Omega_{\Sigma(i)}^1(\text{Log}(\partial\Sigma(i)))$  which coincides with the restriction of  $\omega_{\Sigma}^1$  on  $\Sigma(i)$ . It is then natural to wonder whether the isomorphism  $(\mathcal{T}^\vee)|_{\Sigma_{reg}} \simeq \Omega_{\Sigma_{reg}}^1$  of [AK, Theorem 1.10.ii] extends to an identification  $(\mathcal{T}^\vee)|_{\Sigma} \simeq \omega_{\Sigma}^1$  on the whole  $\Sigma$ , although this Fano surface is singular.

**A.2. An explicit description of the abelian 1-ARs of  $\mathcal{W}_{0,6}$ .** We now turn to another approach to describe the abelian 1-ARs of  $\mathcal{W}_{0,6}$  which is much more elementary and explicit than the previous one.

The birational model of  $\mathcal{W}_{0,6}$  we are going to work with is the following: relatively to some rational coordinates  $x_1, x_2, x_3$ , we identify  $\mathcal{W}_{0,6}$  with the web defined by the six following rational first integrals:  $U_1 = (x_2, x_3)$ ,  $U_2 = (x_1, x_3)$ ,  $U_3 = (x_1, x_2)$  and

$$U_4 = \left( \frac{x_1 - 1}{x_3 - 1}, \frac{x_2 - 1}{x_3 - 1} \right), \quad U_5 = \left( \frac{x_1}{x_3}, \frac{x_2}{x_3} \right) \quad \text{and} \quad U_6 = \left( \frac{x_1(x_3 - 1)}{x_3(x_1 - 1)}, \frac{x_2(x_3 - 1)}{x_3(x_2 - 1)} \right).$$

We denote by  $U_{i,1}$  and  $U_{i,2}$  the components of  $U_i$ . Recall that for any  $n \geq 2$ ,  $A_n$  stands for the braid arrangement in  $\mathbf{C}^n$  which is such that there is a natural identification

$$(131) \quad \mathbf{C}^n \setminus A_n \simeq \mathcal{M}_{0,n+3}$$

Then each  $U_i$  induces a regular map  $\mathbf{C}^3 \setminus A_3 \rightarrow \mathbf{C}^2 \setminus A_2$  which corresponds to the forgetful mapping  $\mathcal{M}_{0,6} \rightarrow \mathcal{M}_{0,5}$  modulo the preceding identifications.

Up to the identification (131) when  $n = 2$ , the 5-tuple of rational 1-forms in two variables

$$(\alpha_s)_{s=1}^5 = \left( \frac{du}{u}, \frac{dv}{v}, \frac{du}{u-1}, \frac{dv}{v-1}, \frac{du-dv}{u-v} \right)$$

is a basis of a complex vector space denoted by  $\mathbf{Log}\Omega_{0,5}^1$ , which is naturally isomorphic to the space of global holomorphic 1-forms on  $\mathcal{M}_{0,5}$  with logarithmic poles along  $\partial\overline{\mathcal{M}_{0,5}}$ .

Consequently, for any  $i = 1, \dots, 6$ , the 5-tuple of rational 1-forms in  $x_1, x_2$  and  $x_3$

$$(\omega_{i,s})_{s=1}^5 = (U_i^*(\alpha_s))_{s=1}^5 = \left( \frac{dU_{i,1}}{U_{i,1}}, \frac{dU_{i,2}}{U_{i,2}}, \frac{dU_{i,1}}{U_{i,1}-1}, \frac{dU_{i,2}}{U_{i,2}-1}, \frac{dU_{i,1}-dU_{i,2}}{U_{i,1}-U_{i,2}} \right)$$

is a basis of a vector space denoted by  $U_i^*\mathbf{Log}\Omega_{0,5}^1$  identifying with  $\psi_i^*(\mathbf{H}^0(\Sigma(i), \Omega_{\Sigma(i)}^1(\text{Log}(\partial\Sigma(i))))$ . It follows that the abelian 1-ARs of  $\mathcal{W}_{0,6}$  have to be looked for inside the space

$$\mathbf{Log}\mathcal{AR}^{(1)} = \left\{ (c_{i,s})_{(i,s) \in J} \in (\mathbf{C}^5)^6 \mid \sum_{i=1}^6 \sum_{s=1}^5 c_{i,s} \omega_{i,s} = 0 \right\}$$

where  $J$  stands for the set of indices  $J = \{(i, s) \mid i = 1, \dots, 6, s = 1, \dots, 5\}$ .

- Proposition 6.7.**
1. The space  $\mathbf{LogAR}^{(1)}$  has dimension 21.
  2. As a  $\mathfrak{S}_6$ -module, its decomposition into irreducibles is  $\mathbf{LogAR}^{(1)} = V_{(3,2,1)} \oplus V_{(3,3)}$ .
  3. The 5-dimensional component  $V_{(3,3)}$  of  $\mathbf{LogAR}^{(1)}$  is precisely the image by the trace map  $\mathrm{Tr}^{(1)}$  of the space of abelian differentials on  $\widetilde{\Sigma}$ : one has  $V_{(3,3)} = \mathrm{Tr}^{(1)}(\mathbf{H}^0(\widetilde{\Sigma}, \omega_{\widetilde{\Sigma}}^1))$ .
  4. A basis for  $V_{(3,3)} = \mathbf{AR}_{ab}^{(1)}(\mathbf{W}_{0,6})$  can be explicitated: for instance, such a basis is provided by the five 1-ARs associated to the explicit functional relations of Table 2 below.

Let us discuss how this proposition can be established. Our proof relies on some explicit computations (performed on Maple) that we will not give here<sup>53</sup> but only describe.

*Proof.* We set  $\zeta_i = x_i$  for  $i = 1, 2, 3$  and  $\zeta_4 = 0, \zeta_5 = 1$  and  $\zeta_6 = \infty$ . For  $k$  ranging from 1 to 3 and  $l$  from  $k+1$  to 5, we set  $\zeta_{kl} = \zeta_k - \zeta_l$  and  $\eta_{k,l} = d\mathrm{Log}\zeta_{kl} = (d\zeta_k - d\zeta_l)/(\zeta_k - \zeta_l)$ . The  $\eta_{k,l}$ 's are rational logarithmic 1-forms on  $\mathbf{C}^3$  which are linearly independent over  $\mathbf{C}$ . Their span is a complex vector space of dimension 9, denoted by  $\mathbf{Log}\Omega_{0,6}^1$ , which is naturally isomorphic to

$$\mathbf{H}^0\left(\overline{\mathcal{M}}_{0,6}, \Omega_{\overline{\mathcal{M}}_{0,6}}^1\left(\mathrm{Log}(\partial\overline{\mathcal{M}}_{0,6})\right)\right).$$

By definition of  $\mathbf{LogAR}^{(1)}$ , one has a short complex of vector spaces

$$(132) \quad 0 \rightarrow \mathbf{LogAR}^{(1)} \rightarrow \bigoplus_{i=1}^6 U_i^*\left(\mathbf{Log}\Omega_{\mathcal{M}_{0,5}}^1\right) \xrightarrow{\tau} \mathbf{Log}\Omega_{\mathcal{M}_{0,6}}^1 \rightarrow 0$$

where  $\tau$  stands for the map given by  $(\omega_i)_{i=1}^6 \mapsto \sum_{i=1}^6 \omega_i$ . By plain linear algebra computations, it can be verified that the sequence (132) is exact (*i.e.*  $\tau$  is surjective) hence  $\dim_{\mathbf{C}} \mathbf{LogAR}^{(1)} = 21$ . From now on, we will work with a fixed basis of  $\mathbf{LogAR}^{(1)}$ , denoted by  $\mathfrak{B}$ .

In order to prove the second point of the proposition, one starts by determining the character of  $\mathbf{LogAR}^{(1)}$  as a representation of  $\mathfrak{S}_6$ . We describe how we have proceeded. the following is a complete set of representatives of the non trivial conjugacy classes of  $\mathfrak{S}_6$ :

$$\mathcal{S} = \left\{ (12), (12)(34), (12)(34)(56), (123), (123)(45), \right. \\ \left. (123)(456), (1234), (1234)(56), (12345), (123456) \right\}.$$

We will use the following composition of group morphism

$$(133) \quad \mathfrak{S}_6 \rightarrow \mathbf{Bir}(\mathbf{C}^3) \rightarrow \mathrm{Aut}(\mathbf{LogAR}^{(1)}) \rightarrow \mathrm{GL}_{21}(\mathbf{C}) \\ \sigma \longrightarrow G_\sigma \longrightarrow G_\sigma^* \longrightarrow M_\sigma$$

with  $G_\sigma$  being the birational map defined in (12) and where  $M_\sigma$  stands for the matrix of  $G_\sigma^*$  acting on  $\mathbf{LogAR}^{(1)}$  expressed in the basis  $\mathfrak{B}$  (*i.e.*  $M_\sigma = \mathrm{Mat}_{\mathfrak{B}}(G_\sigma^*)$ ).

For any permutation  $\sigma$ , it is just a computational matter (that we have handled using Maple) to get an explicit expression for the birational map  $G_\sigma$ , hence for the matrix  $M_\sigma$ , from which one can take the trace  $\mathrm{Tr}(M_\sigma)$ . For each  $\sigma$  in the set  $\mathcal{S}$ ,  $G_\sigma$  and  $\mathrm{Tr}(M_\sigma)$  are given in Table 1 below.

Table 1 characterizes entirely the character of  $\mathbf{LogAR}^{(1)}$  as a  $\mathfrak{S}_6$ -module, denoted by  $\chi^{(1)}$ . We write  $\chi^{(1)} = \sum_{\lambda \vdash 6} n_\lambda \chi_\lambda$  for some non negative integers  $n_\lambda$ , where  $\chi_\lambda$  stands for the character of

<sup>53</sup>We can make the corresponding Maple worksheets available to anyone requesting them.

$\sigma$	(12)	(12)(34)	(12)(34)(56)	(123)	(123)(45)
$G_\sigma$	$\left(\frac{x_i}{x_i-1}\right)_{i=1}^3$	$\left(x_1, \frac{x_1(x_2-1)}{x_2-x_1}, \frac{x_1(x_3-1)}{x_3-x_1}\right)$	$\left(x_1, \frac{x_1(x_3-1)}{x_3-x_1}, \frac{x_1(x_2-1)}{x_2-x_1}\right)$	$\left(\frac{x_i-1}{x_i}\right)_{i=1}^3$	$\left(\frac{x_1-1}{x_1}, \frac{x_3-1}{x_3}, \frac{x_2-1}{x_2}\right)$
$\text{Tr}(M_\sigma)$	1	1	-3	-1	1
$\sigma$	(123)(456)	(1234)	(1234)(56)	(12345)	(123456)
$G_\sigma$	$\left(\frac{x_2-1}{x_2}, \frac{x_3-1}{x_3}, \frac{x_1-1}{x_1}\right)$	$\left(\frac{1}{x_1}, \frac{x_2-1}{x_2-x_1}, \frac{x_3-1}{x_3-x_1}\right)$	$\left(\frac{1}{x_1}, \frac{x_3-1}{x_3-x_1}, \frac{x_2-1}{x_2-x_1}\right)$	$\left(\frac{1}{x_2}, \frac{x_1-x_2}{x_2(x_1-1)}, \frac{x_3-x_2}{x_2(x_3-1)}\right)$	$\left(\frac{1-x_2}{x_1-x_2}, \frac{1-x_3}{x_1-x_3}, \frac{1}{x_1}\right)$
$\text{Tr}(M_\sigma)$	2	-1	-1	0	0

TABLE 1.

the irreducible  $\mathfrak{S}_6$ -representation  $V_\lambda$  associated to the partition  $\lambda$ . Using the character table of  $\mathfrak{S}_6$ -representations, it is not difficult to conclude that necessarily  $\chi^{(1)} = \chi_{(3,3)} \oplus \chi_{(3,2,1)}$ . The second point of the proposition follows immediately from that. Since  $V_{(3,2,1)}$  has dimension 16 and because  $\text{Tr}^{(1)}(\mathbf{H}^0(\tilde{\Sigma}, \omega_{\tilde{\Sigma}}^1))$  is a  $\mathfrak{S}_6$ -submodule of  $\mathbf{LogAR}^{(1)}$  of dimension 5, we get point 3.

The preceding arguments used to show that  $\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) = V_{(3,3)}$  are fully non constructive and it is interesting and natural to seek for a basis of this 5-dimensional space. Such a basis is provided by the 1-ARs associated to the five functional identities of Table 2 below. We were able to determine these explicit functional relations using the following approach: first, thanks to formula (12), the group morphism  $\mathfrak{S}_6 \rightarrow \mathbf{Bir}(\mathbf{C}^3)$ ,  $\sigma \mapsto G_\sigma$  can be made explicit. Then using Maple's 'DifferentialGeometry' package, one can compute the action of the pull-back by  $G_\sigma$  on  $\mathbf{LogAR}^{(1)}$  with respect to the basis  $\mathfrak{B}$ . Eventually we computed (on a computer) the  $21 \times 21$  square matrix  $M_\sigma$  for any of the  $6! = 720$  permutations  $\sigma$  element of  $\mathfrak{S}_6$ .

Next, for  $\chi = \chi_{(3,3)}$  or  $\chi = \chi_{(3,2,1)}$ , we have considered the matrix

$$\Pi_\chi = \frac{\chi(\mathbf{1})}{|\mathfrak{S}_6|} \sum_{\sigma \in \mathfrak{S}_6} \chi(\sigma^{-1}) M_\sigma \in \text{Mat}_{21}(\mathbf{C})$$

(where  $\mathbf{1}$  stands for the identity element of  $\mathfrak{S}_6$ ). From a classical result of the general theory of representations of finite group (see [Lan, Chap. XVIII, §4] for instance), we know that  $\Pi_\chi$  is the matrix in the basis  $\mathfrak{B}$  of the projection map  $\mathbf{LogAR}^{(1)} = V_{(3,2,1)} \oplus V_{(3,3)} \rightarrow V_\chi$ . Since  $\chi_{(3,3)}$  was known and because all the  $M_\sigma$  were previously computed, we have been able to construct  $\Pi_{\chi_{(3,3)}}$  explicitly. Considering suitable linear combinations of its columns led us to the 1-ARs associated to the five functional relations of Table 2.  $\square$

**Remark 6.8.** 1. Using the same computational approach as in the preceding proof, one can show that the character  $\chi_9$  of the 9-dimensional  $\mathfrak{S}_6$ -representation  $\mathbf{Log}\Omega_{0,6}^1$  is given by the following table, where the symbols in the first line stand for the possible decompositions into cycles with pairwise distinct supports of the conjugacy classes in  $\mathfrak{S}_6$  (that is  $\langle 2 \rangle$  means a cycle of length 2,  $\langle 23 \rangle$  stands for a product of a 2-cycle with a 3-cycle, etc.):

$\sigma$	$\langle 2 \rangle$	$\langle 22 \rangle$	$\langle 222 \rangle$	$\langle 3 \rangle$	$\langle 23 \rangle$	$\langle 33 \rangle$	$\langle 4 \rangle$	$\langle 24 \rangle$	$\langle 5 \rangle$	$\langle 6 \rangle$
$\chi_9(\sigma)$	3	1	3	0	0	0	-1	1	-1	0

We see that  $\chi_9$  coincides with the character associated to the partition  $(4, 2)$ , from which it follows that  $\mathbf{Log}\Omega_{0,6}^1$  is isomorphic to the corresponding Specht module  $V_{(4,2)}$ .

$$\begin{aligned}
0 &= \ln(U_{1,1}) + \ln\left(\frac{U_{2,1}}{U_{2,2}}\right) - \ln(U_{3,2}) + \ln\left(\frac{U_{4,1}-U_{4,2}}{U_{4,1}(U_{4,2}-1)}\right) - \ln(U_{5,1}) + \ln\left(\frac{U_{6,2}-1}{U_{6,1}-U_{6,2}}\right) \\
0 &= \ln(U_{1,2}) - \ln(U_{2,2}) + \ln\left(\frac{U_{3,1}}{U_{3,2}}\right) + \ln\left(\frac{U_{4,2}(U_{4,1}-1)}{U_{4,1}(U_{4,2}-1)}\right) + \ln\left(\frac{U_{5,2}}{U_{5,1}}\right) + \ln\left(\frac{U_{6,2}-1}{U_{6,1}-1}\right) \\
0 &= \ln(U_{1,1}-1) + \ln\left(\frac{U_{2,1}-1}{U_{2,2}-1}\right) - \ln(U_{3,2}-1) - \ln(U_{4,1}) + \ln\left(\frac{U_{5,1}-U_{5,2}}{U_{5,1}(U_{5,2}-1)}\right) + \ln\left(\frac{U_{6,1}(U_{6,2}-1)}{U_{6,1}-U_{6,2}}\right) \\
0 &= \ln(U_{1,2}-1) - \ln(U_{2,2}-1) + \ln\left(\frac{U_{3,1}-1}{U_{3,2}-1}\right) + \ln\left(\frac{U_{4,2}}{U_{4,1}}\right) + \ln\left(\frac{U_{5,2}(U_{5,1}-1)}{U_{5,1}(U_{5,2}-1)}\right) + \ln\left(\frac{U_{6,1}(U_{6,2}-1)}{U_{6,2}(U_{6,1}-1)}\right) \\
-\mathcal{I}\pi &= \ln(U_{1,1}-U_{1,2}) + \ln\left(\frac{U_{2,1}-U_{2,2}}{U_{2,2}(U_{2,2}-1)}\right) + \ln\left(\frac{U_{3,1}-U_{3,2}}{U_{3,2}(U_{3,2}-1)}\right) + \ln\left(\frac{U_{4,2}}{U_{4,1}(U_{4,2}-1)}\right) + \ln\left(\frac{U_{5,2}}{U_{5,1}(U_{5,2}-1)}\right) \\
&\quad + \ln\left(\frac{U_{6,1}(U_{6,2}-1)}{(U_{6,1}-1)(U_{6,1}-U_{6,2})}\right).
\end{aligned}$$

TABLE 2. A basis (in functional form) of the space of 1-abelian relations of  $\mathcal{W}_{0,6}$  coming from abelian 1-differentials on the birational model  $\tilde{\Sigma}$  of the the Fano surface  $\Sigma$  of Segre's cubic  $\mathcal{S}$ .

2. For any  $s = 1, \dots, 5$ , let  $\mathcal{AR}_s \in \mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6})$  be the 1-AR of  $\mathcal{W}_{0,6}$  corresponding to the functional relation of the  $s$ -th line in Table 2. Each  $\mathcal{AR}_s$  is a 6-tuple  $(\mathbf{ar}_{s,i}(U_{i,1}, U_{i,2}))_{i=1}^6$  of functions of two variables, such that the sum  $\sum_{i=1}^6 \mathbf{ar}_{s,i}(U_{i,1}, U_{i,2})$  is (locally) constant. For any  $i$ , let  $\Psi_i : \mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) \rightarrow \mathbf{Log}\Omega_{0,5}^1$  be the linear map obtained as the following composition

$$\begin{aligned}
\mathbf{AR}_{ab}^{(1)}(\mathcal{W}_{0,6}) &\longrightarrow \mathbf{H}^0(\tilde{\Sigma}, \omega_{\tilde{\Sigma}}^1) \longrightarrow \mathbf{H}^0(\Sigma(i), \Omega_{\Sigma(i)}^1(\mathbf{Log}(\partial\Sigma(i)))) \\
&\quad \downarrow \mathfrak{R} \\
&\quad \mathbf{H}^0(\overline{\mathcal{M}}_{0,5}, \Omega_{\overline{\mathcal{M}}_{0,5}}^1(\mathbf{Log}(\partial\overline{\mathcal{M}}_{0,5}))) \longrightarrow \mathbf{Log}\Omega_{0,5}^1
\end{aligned}$$

where the first map is the inverse of (130), the second is that of point (iv) page 105, the third (vertical) and the last maps being the natural ones respectively induced by the isomorphisms of pairs  $(\Sigma(i), \partial\Sigma(i)) \simeq (\overline{\mathcal{M}}_{0,5}, \partial\overline{\mathcal{M}}_{0,5}) \simeq (\mathbf{C}^2, A_2)$ . Then  $\Psi_i$  is an isomorphism which is fully characterized by the relations  $\Psi_i(\mathcal{AR}_s) = d(\mathbf{ar}_{s,i}(u, v)) \in \mathbf{Log}\Omega_{0,5}^1$  for any  $s$ . In particular, one has  $\Psi_1(\mathcal{AR}_s) = \alpha_s$  for any  $s = 1, \dots, 5$ .

### Appendix B: the 1-ARs of the 3-web of lines on the chordal cubic.

In this appendix, we study the 1-ARs of the web of lines on the chordal cubic  $\mathcal{C}$  (already discussed in §3.2.1 above) and explain how this web can be seen as a particular case of a nice family of curvilinear 3-webs studied by Blaschke and Walberer. We will take advantage of this to recall some of their results which are among the most striking ones in web geometry although not very well known by modern geometers.

**B.1. the 1-ARs of the 3-web of lines on the chordal cubic.** As explained in §3.2.1, the web by lines on  $\mathcal{C}$  is degenerated:  $\mathcal{LW}_{\mathcal{C}}$  is a skew 3-web, and not a 6-web as for most of the cubic threefolds. However, one can ask about the ARs of this web. It turns out that concerning 1-ARs, this web is not less interesting than those associated to smooth cubic hypersurfaces. We use below the notations introduced in §3.2.1:  $U'_+$ ,  $U''_-$  and  $U'$  stand for the explicit rational first integrals of (a birational model of)  $\mathcal{LW}_{\mathcal{C}}$  given in (27) and  $X'_+$ ,  $X''_-$  and  $X'$  denote the associated vector fields defined just after.

According to a classical bound due to Khäler,<sup>54</sup> one has  $\text{rk}^{(1)}(\mathcal{LW}_{\mathcal{C}}) \leq 5$  and the question is whether this is actually an equality. This can be answered by determining an explicit basis for  $\text{AR}^{(1)}(\mathcal{LW}_{\mathcal{C}})$ . Since 1-ARs can be integrated into functional identities, one has to find a basis of non trivial 3-tuples of holomorphic functions of two variables  $(M, N, R)$  such that the functional relation

$$(134) \quad M(U'_+) + N(U''_-) + R(U') = 0$$

is identically satisfied. In §5.1.9, we briefly explained how Abel's method described in [P2] for determining in an effective way the ARs of planar webs can be generalized to the determination of the 2-ARs of curvilinear webs (in dimension 3). It turns out that Abel's method generalizes quite straightforwardly to the case of the 1-ARs of such webs as well, and we will apply this quite effectively to the web under scrutiny.

We denote the components of  $U''_-$  by  $U''_{-i}$  with  $i = 1, 2$  and  $N_i$  stands for the partial derivative of  $N$  with respect to the  $i$ -th variable. Then one proceeds as follows for determining the solutions  $(M, N, R)$  of (134): one first applies  $X'$  to the LHS of (134) in order to eliminate  $R(U')$ . Next one divides by  $X'(U''_{-2})$  and applies  $X''_-$ : this kills the remaining  $N_2(U''_-)$ . Similarly, one divides the expression just obtained by the coefficient of  $N_1(U''_-)$  in it and then one applies  $X''_-$  again: we have eliminated all the terms in  $N$  and  $R$  (and in their partial derivatives) thus what remains can be seen as a PDE in  $M(U'_+)$  with variable coefficients. This gives us a system of linear PDEs of the third order in  $M$  which is not difficult to solve with the help of a computer algebra system.

One gets that, up to the addition of a complex constant, any function  $M$  appearing in an 1-AR (134) is necessarily a linear combination of the five following functions  $M_i = M_i(u_1, u_2)$ :

$$M_1 = \frac{1}{u_1 - u_2}, \quad M_2 = \frac{2u_1 + u_2}{u_1 - u_2}, \quad M_3 = \frac{1}{u_1(u_1 - u_2)}, \quad M_4 = \frac{u_1 u_2}{u_1 - u_2} \quad \text{and} \quad M_5 = \frac{u_1 u_2^2}{u_1 - u_2}.$$

One verifies that the same holds true for any function  $N$  appearing in (134): up to a constant, it is a linear combination of the five previous functions.

<sup>54</sup>That this bound is due to Khäler is mentioned in [Bla3, Satz S<sub>3</sub>]. A published proof is given in [Bla2, §2].



Similarly, one obtains that, up to a constant, any function  $R = R(t_1, t_2)$  appearing in an identity (134) is a linear combination of the following five rational functions  $R_i = R_i(t_1, t_2)$ :

$$R_1 = \frac{2(t_1 + t_2)}{(t_1 - t_2)^2}, \quad R_2 = \frac{4(t_1^2 + t_1 t_2 + t_2^2)}{(t_1 - t_2)^2}, \quad R_3 = \frac{4}{(t_1 - t_2)^2}, \quad R_4 = \frac{2t_1 t_2 (t_1 + t_2)}{(t_1 - t_2)^2}, \quad R_5 = \frac{4t_1^2 t_2^2}{(t_1 - t_2)^2}.$$

Then from elementary computations, one obtains that the functional identity

$$(135) \quad M_i(U''_+) + M_i(U''_-) + R_i(U') = 0$$

holds true identically for any  $i = 1, \dots, 5$ . One then verifies that the five tuples  $(M_i, M_i, R_i)$  are linearly independent. Considering Kähler's bound above, this implies that  $\text{rk}^{(1)}(\mathcal{LW}_\mathcal{C}) = 5$  hence  $\mathcal{LW}_\mathcal{C}$  has maximal 1-rank.

Considering Proposition A.1, it is natural to ask whether the 1-ARs of this web can be described by means of some abelian differential 1-forms of the Fano surface  $F_1(\mathcal{C})$  (or possibly of a suitable birational model of it) or not. But to do so one has to face some difficulties:

- first, since  $F_1(\mathcal{C})$  is not reduced as a scheme, what could be an abelian differential on it is not clear, at least to the author;<sup>55</sup>
- assuming that the first point might be settled, one expects an abelian 1-form on  $F_1(\mathcal{C})$  to deform into a holomorphic 1-form on the Fano surfaces  $F_1(X_t)$  of any regular analytic family of smooth hypercubics  $X_t \subset \mathbf{P}^4$  (with  $t \in (\mathbf{C}^*, 0)$ ) degenerating onto  $X_0 = \mathcal{C}$ ;
- finally, for a smoothing family as just above, one has to understand how the 6-webs  $\mathcal{LW}_{X_t}$  for  $t \in (\mathbf{C}^*, 0)$  degenerate to the 3-web  $\mathcal{LW}_\mathcal{C}$  when  $t \rightarrow 0$ .

Addressing these points here would require too much space. Instead, we will discuss in the next sub-appendix below another way to understand the ARs of  $\mathcal{LW}_\mathcal{C}$ , by considering this web from another point of view.

Finally, note that since the 2-rank of any skew curvilinear 3-web in dimension 3 is zero according to [Bla3, S<sub>23</sub>], there is no point in considering the 2-ARs of  $\mathcal{LW}_\mathcal{C}$  hence we will not talk more about it.

**B.2. Blaschke-Walberer theory and its application to  $\mathcal{LW}_\mathcal{C}$ .** We briefly discuss the beautiful theory established in [BW] before explaining in B.2.2 how  $\mathcal{LW}_\mathcal{C}$  can be studied using it. In the last and short subsection B.2.3, we will finish by describing another approach to study this web, which is similar to that considered in A.1 and relies on results due to Collino as well.

**B.2.1. Blaschke-Walberer theory.** In [BW], Blaschke and Walberer associate a curvilinear 3-web to a sufficiently generic cubic hypersurface and study its properties, in particular its 1-ARs.

Their construction is as follows: let  $X \subset \mathbf{P}^4$  be a given irreducible cubic hypersurface and denote by  $F = F_1(X) \subset G_1(\mathbf{P}^4)$  its Fano surface. In the following we do not consider  $F$  with its structure of scheme: we actually work with the underlying reduced surface  $F_{red}$ , that we will still denote abusively by  $F$  to simplify the writing. By definition, the associated (labeled) ‘**triangle variety**’  $\overline{\Delta} = \overline{\Delta(X)}$  is the closed algebraic subvariety of  $F^3$  formed by ‘triangles included in  $X$ ’, namely triples  $T = (\ell_1, \ell_2, \ell_3)$  of lines included in  $X$  such that there exists a 2-plane  $\pi \in G_2(\mathbf{P}^4)$  satisfying  $X \cdot \pi = \ell_1 + \ell_2 + \ell_3$  (equality between 1-cycles on  $X$ ). Such a triangle  $T = (\ell_i)_{i=1}^3$  is

<sup>55</sup>However, the contents [AL] may allow this point to be answered.

‘**non-degenerate**’ if the  $\ell_i$ ’s are pairwise distinct with their union in  $\mathbf{P}^4$  abstractly isomorphic to the model triangle formed by three non concurrent lines in  $\mathbf{P}^2$  (see the picture below).

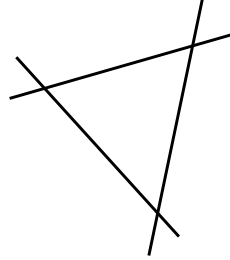


FIGURE 4. A non degenerate triangle in the plane.

One verifies that  $\overline{\Delta} \subset F^3$  is of dimension 3 and that the non-degenerate triangles form a Zariski open subset, denoted by  $\Delta = \Delta(X)$ , that we will also call abusively the ‘triangle variety’ of  $X$  when it is not empty. By restriction, the natural projections  $F^3 \rightarrow F$  onto the three distinct factors give rise to three dominant rational maps  $P_i : \overline{\Delta} \dashrightarrow F$  which define three foliations by algebraic curves on  $\Delta$ . These foliations form what we call **Blaschke-Walberer triangle web** on  $\Delta$ , denoted by

$$(136) \quad \Delta \mathbf{W}_X = \mathcal{W}(P_1, P_2, P_3).$$

Let us say that  $X$  is **W-admissible** if for  $T \in \overline{\Delta}$  generic: (1)  $T$  is a non-degenerate triangle, *i.e.*  $\Delta$  is non empty (hence it is dense in  $\overline{\Delta}$ ); (2) the tangent spaces at  $T$  of the level subsets of the  $p_i$ ’s through this point are in general position, *i.e.*  $\Delta \mathbf{W}_X$  is a genuine curvilinear web on  $\Delta$  at  $T$ ; and (3) this web is (generically) skew.

**Example 5.** 1. A smooth cubic threefold is **W-admissible** (see the proposition just below).

2. On the contrary, Segre’s cubic  $\mathbf{S}$  is not **W-admissible**. This can be verified easily using its model in  $\mathbf{P}^3$  described in §2.1.3 (see especially (17)). Let us describe the triangles in  $\mathbf{S} \simeq \mathbf{P}^3$  passing through a generic point  $q$ . Such a triangle is determined by two distinct lines passing through  $q$ . In the model we are working with, two such lines correspond to the lines  $\ell_s(q) = \langle q, p_s \rangle \subset \mathbf{P}^3$  for  $s = 1, 2$  say. The third edge of the triangle determined by  $(\ell_1(q), \ell_2(q))$  is the line  $l_3(q) = \langle q, p_1, p_2 \rangle \cap \Pi_{123}$  with  $\Pi_{123} = \langle p_3, p_4, p_5 \rangle \simeq \mathbf{P}^2$ . The map  $q \mapsto l_3(q)$  is the map  $P_3$  appearing in (136). It takes values into the dual projective plane  $\check{\Pi}_{123}$  of lines contained in  $\Pi_{123}$ .

Noticing that any line  $l_3(q)$  passes through the intersection point of  $\langle p_1, p_2 \rangle$  with  $\Pi_{123}$ , we get that actually  $\text{Im}(P_3)$  is a line in  $\check{\Pi}_{123}$ . This map is then of rank 1 and its generic fiber is a surface (actually, the fiber of  $P_3$  through  $q$  is clearly the 2-plane  $\langle q, p_1, p_2 \rangle$ ). Thus  $P_3$  defines a foliation by 2-planes, hence  $\Delta \mathbf{W}_S$  is not a curvilinear web, which shows that  $\mathbf{S}$  is not **W-admissible**.

3. But the property to be **W-admissible** for a cubic threefold is not directly related to the fact that it is singular or not. For instance, the chordal cubic  $\mathcal{C}$  is **W-admissible** although having a singular locus of dimension 1.

Remark that one could have worked without labeling the edges of the triangles included in  $X$ . This would have given a more intrinsic construction of a variety of ‘unlabeled triangles’ and of a non-ordered version of  $\Delta \mathbf{W}_X$ . Since we are interested in local analytic properties of this web, it

is pointless and actually much more convenient to work on  $\Delta_X \subset F^3$  since on it, the web we are interested in is defined by global rational first integrals.

**Proposition 6.9.** *Let  $X$  be a smooth cubic threefold in  $\mathbf{P}^4$ .*

1. *The hypersurface  $X$  is  $\mathcal{W}$ -admissible. Moreover, the skew 3-web  $\Delta\mathcal{W}_X$  is linearizable.*
2. *The first trace map  $\mathrm{Tr}^{(1)} : \omega \mapsto \sum_{i=1}^3 P_i^*(\omega)$  induces a well-defined isomorphism*

$$\mathrm{Tr}^{(1)} : \mathbf{H}^0(F, \Omega_F^1) \longrightarrow \mathbf{AR}^{(1)}(\Delta\mathcal{W}_X).$$

3. *Consequently  $\mathrm{rk}^{(1)}(\Delta\mathcal{W}_X) = h^0(F, \Omega_F^1) = 5$  hence the 3-web  $\Delta\mathcal{W}_X$  has maximal 1-rank.*

*Proof (sketched).* Assuming that  $X$  is smooth, one verifies easily that  $\Delta_X$  is non empty and generically reduced and the fact that  $\Delta\mathcal{W}_X$  is a genuine web can be obtained using the same arguments as in the proof of Lemma 3.4. As for the skewness of  $\Delta\mathcal{W}_X$ , it will be established later (see Lemma 6.11.2). And a nice geometric argument gives its linearizability. For  $T \in \Delta_X$ , one denotes by  $\ell_i^T \in F$  its edges (with  $i = 1, 2, 3$ ) and one sets  $\langle T \rangle$  for the 2-plane in  $\mathbf{P}^4$  spanned by the  $\ell_i^T$ 's. We fix a generic linear projection  $\pi : \mathbf{P}^4 \dashrightarrow \mathbf{P}^3$ . Then for  $T_0 \in \Delta_X$  generic, one considers the germ of analytic map  $\Xi : (\Delta_X, T_0) \rightarrow \check{\mathbf{P}}^3 : T \mapsto \pi(\langle T \rangle)$ . One verifies that it is a local biholomorphism. Moreover, for a generic line  $\ell \in G_1(\mathbf{P}^4)$ ,  $\pi(\ell)$  is a line in  $\mathbf{P}^3$  whose projective dual, denoted by  $\pi(\ell)^\vee$ , is a line in  $\check{\mathbf{P}}^3$  as well. For any  $T = (\ell_i)_{i=1}^3 \in (\Delta_X, T_0)$ , the three lines  $\pi(\ell_i)^\vee$  for  $i = 1, 2, 3$  are concurrent at  $\Xi(T)$ , and are easily seen to be the leaves of the push-forward of  $\Delta\mathcal{W}_X$  by  $\Xi$ . Thus  $\Xi_*(\Delta\mathcal{W}_X)$  is a germ of linear 3-web on  $\check{\mathbf{P}}^3$  at  $\Xi(T_0)$  hence, in particular,  $\Delta\mathcal{W}_X$  is linearizable.

The third and last points of the proposition follow immediately from the second which itself is a simple rephrasing of the last statement of (3.1.1) in terms of webs.  $\square$

A remarkable result obtained by Blaschke and Walberer in their paper is that the web-theoretic content of the preceding proposition actually holds true for any cubic threefold as soon as its web of triangles is  $\mathcal{W}$ -admissible:

**Proposition 6.10** (Blaschke-Walberer). *Let  $X$  be an irreducible  $\mathcal{W}$ -admissible cubic in  $\mathbf{P}^4$ .*

1. *There exists a certain 5-dimensional space  $\mathbf{H}_F$  of rational 1-forms on  $F$  whose traces vanish. This gives rise to an isomorphism  $\mathrm{Tr}^{(1)} : \mathbf{H}_F \longrightarrow \mathbf{AR}^{(1)}(\Delta\mathcal{W}_X)$ .*
2. *Consequently  $\mathrm{rk}^{(1)}(\Delta\mathcal{W}_X) = \dim(\mathbf{H}_F) = 5$  hence the 3-web  $\Delta\mathcal{W}_X$  has maximal 1-rank.*

We think that this result deserves some comments. It is proved in the sixth and last section of [BW] by means of a rather explicit approach. Very roughly, from a cubic equation defining a  $\mathcal{W}$ -admissible cubic  $X \subset \mathbf{P}^4$  and after six pages of explicit algebraic computations, the authors build rational 1-forms  $u_i$  for  $i$  ranging from 1 to 5 (see formula (68) in [BW]) which they show to give rise to five linearly independent 1-abelian relations for  $\Delta\mathcal{W}_X$ . They do not establish that the corresponding integrals  $\int u_i$  are precisely the ‘Abelsche Integrale erster Gattung’<sup>56</sup> of the Fano surface of lines included in  $X$ , but this is strongly suggested at the very end. Thanks to Proposition 6.9, we know that it is indeed the case, at least when  $X$  is smooth.

Although computational, Blaschke and Walberer’s approach is very interesting according to us, essentially for two reasons: first, being algebraic in nature, it allows a priori to handle the case of singular cubics as well; the proof of the second point of Proposition 6.9 relies on point **xvi.** of

<sup>56</sup>Translated by ‘Abelian integrals of the first kind’ into English.

§3.1.1 (with  $k = 1$ ), the only modern proof of which we are aware of (p. 332 of [CIG]) relies on a basic but non-explicit principle of complex analysis;<sup>57</sup> this leads us to the second reason, which is that the approach in [BW, §6] gives rise to explicit algebraic formulas for the abelian 1-forms on  $F_1(X)$  which is interesting and has nothing comparable in modern literature to our knowledge. Since one can take great benefit from explicit formulas for the holomorphic or abelian differentials on a projective variety<sup>58</sup> it would be very interesting to revisit the results of Blaschke and Walberer by taking a more modern (and therefore perhaps more accurate and rigorous) approach.

But the most striking result in [BW] is actually not the preceding proposition but rather its following converse which can be considered as one of the most interesting results in what concerns the algebraization of webs<sup>59</sup>

**Blaschke-Walberer’s algebraization theorem.** *Let  $\mathcal{W}$  be a skew curvilinear 3-web in  $\mathbf{C}^3$  with maximal 1-rank. Then  $\mathcal{W}$  is equivalent to the triangle web  $\Delta\mathcal{W}_X$  of a cubic hypersurface  $X \subset \mathbf{P}^4$ .*

The proof of this theorem consists in many pages of normalisations and computations that we do not have considered in detail yet. Our impression is that this is a genuine computational *tour de force*. Still, as for the previous result, we believe it would be interesting and even necessary to go over Blaschke-Walberer’s proof again, with the modern standard of mathematical rigour in order to certify that any of their arguments or computations is indeed correct.

Our opinion that the content of [BW] should be revisited should not lead one to believe that we doubt the validity of the above results: this is not the case. In the next subsection we use them to get another and better understanding of  $\mathcal{LW}_\ell$  and of its 1-ARs.

Ending this sub-section, it is interesting to point out that a notion of ‘triangle varieties’ generalizing the one discussed here has been recently considered by algebraic geometers for hyper-Kähler manifolds (see [Baz, Voi]) and first results show that it is a relevant notion in what regards the study of these varieties. It would be interesting to figure out whether this notion might give relevant outputs in web geometry as well.

**B.2.2. Another point of view on  $\mathcal{LW}_\ell$ .** The key idea here is rather simple and relies on the fact that  $\mathcal{LW}_\ell$  can be interpreted as  $\Delta\mathcal{W}_\ell$ . Blaschke-Walberer theory will then apply and Proposition 6.10 will provide an interpretation of the 1-ARs of this web.

Let  $X$  be an irreducible cubic threefold in  $\mathbf{P}^4$  containing a non degenerate triangle. By definition, for  $k = 1, 2, 3$ , its  **$k$ -th corner map** is the rational map  $c_k : \Delta_X \dashrightarrow X$  such that for any generic triangle  $T = (\ell_1, \ell_2, \ell_3) \in \Delta_X$ ,  $c_k(T)$  is equal to the intersection point of  $\ell_i$  with  $\ell_j$  if  $i$  and  $j$  are such that  $\{i, j, k\} = \{1, 2, 3\}$ . From the obvious fact that a generic triangle is determined by two of its edges, one easily gets that each  $c_k$  is dominant onto  $X$  and generically finite. Consequently, given a generic base point  $T^* = (\ell_1^*, \ell_2^*, \ell_3^*)$  and setting  $x^* = c_k(T^*)$ , each corner map  $c_k$  induces a local biholomorphism  $c_k : (\Delta_X, T^*) \rightarrow (X, x^*)$  which can be used to locally push-forward  $\Delta\mathcal{W}_X$  onto  $X$ .

<sup>57</sup>Namely, the maximal modulus principle in complex analysis which admits as a direct corollary that any holomorphic map from a simply connected compact manifold into a complex torus is necessarily constant.

<sup>58</sup>As a example of such formulas, one can think to the description of the abelian differentials on a complete intersection  $V \subset \mathbf{P}^N$  by means of successive Poincaré residues of some rational forms on  $\mathbf{P}^N$ .

<sup>59</sup>In [Ch], Chern writes that [BW] ‘is perhaps Blaschke’s deepest paper’ in web geometry.

Assume that  $i, j, k$  are as above. For a triangle  $T = (\ell_s)_{s=1}^3$  sufficiently close to  $T^*$  in  $\Delta_X$  (and in particular non degenerate), the leaves of  $\Delta\mathcal{W}_X$  passing through it, denoted by  $\mathcal{L}_m(T)$ , are the sets of triangles  $T' = (\ell'_s)_{s=1}^3$  such that  $\ell'_m = \ell_m$  for  $m = 1, 2, 3$ . Since  $c_k(T') \in \ell_i \subset X$ , we deduce that  $c_k(\mathcal{L}_i(T)) \subset \ell_i$  and similarly  $c_k(\mathcal{L}_j(T)) \subset \ell_j$ . It follows that  $(c_k)_*(\Delta\mathcal{W}_X)$  is a germ of 3-web on  $X$  at  $x^*$ , two of the local foliations of which are formed by lines included in  $X$  hence form a 2-subweb of  $\mathcal{L}\mathcal{W}_X$ . This gives us the

**Lemma 6.11.** *If  $\mathcal{L}\mathcal{W}_X$  is skew then so is  $\Delta\mathcal{W}_X$ . In particular, this applies when  $X$  is smooth.*

Note that the first statement in this lemma holds true even if the lines trough a general point of  $X$  come with multiplicities that is, even if  $\mathcal{L}\mathcal{W}_X$  is a  $k$ -web with  $k < 6$ . This remark applies in the case of the chordal cubic. Although  $\mathcal{L}\mathcal{W}_{\mathcal{C}}$  is only a 3-web, it has been verified that this web is skew in §3.2.1 (more precisely, see (28)). From the preceding lemma, we get that the triangle web  $\Delta\mathcal{W}_{\mathcal{C}}$  is skew, hence Proposition 6.10 applies and gives us a description of the 1-ARs in terms of rational 1-differentials on the Fano surface  $F_1(\mathcal{C})$ .

At this point, since both  $\mathcal{L}\mathcal{W}_{\mathcal{C}}$  and  $\Delta\mathcal{W}_{\mathcal{C}}$  are skew curvilinear 3-web with maximal 1-rank constructed from the same cubic hypersurface, one can wonder how these two webs are related. It turns out that they are actually the same as an easy computational verification shows:

**Lemma 6.12.** *In the case of the chordal cubic  $\mathcal{C}$ , any corner map  $c_k$  gives rise to an equivalence*

$$\Delta\mathcal{W}_{\mathcal{C}} \xrightarrow{\sim} (c_k)_*(\Delta\mathcal{W}_{\mathcal{C}}) = \mathcal{L}\mathcal{W}_{\mathcal{C}}.$$

The proof is left to the reader. Note that this result is specific to the case of  $\mathcal{C}$ . In general, the push-forward by a corner map  $c_k$  of the foliation induced by  $P_k$  is not linear on the considered cubic.

Taking the pull-backs under a corner map of the total derivatives of the functional identities (134), one gets a rational basis of  $\mathbf{AR}^{(1)}(\mathcal{L}\mathcal{W}_{\mathcal{C}})$  which enjoys the interesting feature of being completely explicit. On the other hand, the first point of Proposition 6.10 offers another view on the 1-ARs of  $\Delta\mathcal{W}_{\mathcal{C}}$ , more abstract and conceptual but not really explicit. It would be interesting to relate more concretely these two descriptions of the abelian relations under scrutiny and to get a description of them which would be intrinsic and conceptual as well as explicit.

In the next subsection, we describe another approach to describe the 1-ARs of  $\Delta\mathcal{W}_{\mathcal{C}} \simeq \mathcal{L}\mathcal{W}_{\mathcal{C}}$  which is conceptual and can be made explicit but lacks being intrinsic.

**B.2.3. Deforming  $\mathcal{C}$  following Collino to better understand the 1-ARs of  $\mathcal{L}\mathcal{W}_{\mathcal{C}}$ .** The same approach as the one described in Appendix A.1 in the case of Segre's cubic can be implemented in the case of the chordal cubic as well. And again, all the results needed to justify it can be found in a paper by Collino, namely in [Col1]. Since this is quite similar to what has been done in A.1, we will be rather quick and will not give any details.

Let  $C$  be the cubic form appearing in the equation (25) of  $\mathcal{C}$ . Given another generic cubic form  $G$ , one considers the pencils of cubic hypersurfaces  $g : \mathcal{G} \subset \mathbf{P}^4 \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$  whose fiber at  $s \in \mathbf{A}$  is  $\mathcal{G}_s = \{C + s^2G = 0\} \subset \mathbf{P}^4$ .<sup>60</sup> Let  $h : \mathcal{H} \subset G_1(\mathbf{P}^4) \times \mathbf{A}^1 \rightarrow \mathbf{A}^1$  be the associated family of Fano surfaces. For  $t$  with a non zero but very small modulus, the cubic threefold  $G_s = g^{-1}(s)$  is smooth hence  $F_s = F(G_s) = h^{-1}(s)$  is a smooth irreducible surface with  $h^0(F_s, \Omega_{F_s}^1) = 5$ . The

<sup>60</sup>See around (1.4) in [Col1] for an explanation of the choice of  $s^2$  as a deformation parameter

cohomology spaces  $\mathbf{H}^1(F_s, \mathbf{C})$ 's for  $s \in (\mathbf{C}^*, 0)$  form a variation of pure Hodge structures and the basic question here is how its degeneration as the deformation parameter  $s$  goes to the origin. This is nicely answered in [Col1], to which we refer the reader for details.

Recall that the singular set of  $\mathcal{C}$  is a rational quartic curve that we denote by  $\Gamma$ . For  $s \neq 0$  sufficiently close to the origin, the cubic  $G_s$  intersects this curve in 12 points which, as  $t \rightarrow 0$ , converge onto 12 pairwise distinct limit points on  $\Gamma$ . Let  $K \rightarrow \Gamma$  be the 2-to-1 covering of  $\Gamma$  ramified at the 12 limit points on  $\Gamma$ :  $K$  is a hyperelliptic curve of genus 5. Recall that  $F_0 = F(\mathcal{C})$  has two irreducible components, the symmetric product  $F' = \Gamma^{[2]}$  and another one denoted by  $F''$  (described in §3.2.1). Let  $\tilde{\mathcal{H}}$  be the proper transform of the family of Fano surfaces  $\mathcal{H} \subset G_1(\mathbf{P}^4) \times \mathbf{A}^1$  by the blow-up of  $G_1(\mathbf{P}^4) \times \mathbf{A}^1$  along  $F' \times \{0\}$  and denote by  $\tilde{h} : \tilde{\mathcal{H}} \rightarrow \mathbf{A}^1$  the natural projection. Then according to [Col1, Prop. 2.1],  $\tilde{\mathcal{H}} \rightarrow \mathbf{A}^1$  resolves the singularities of  $h : \mathcal{H} \rightarrow \mathbf{A}^1$  along the fiber over 0; and the central fiber  $\tilde{F}_0 = \tilde{h}^{-1}(0)$  is a reduced divisor with normal crossing with two irreducible components: one has  $\tilde{F}_0 = F'' \cup K^{[2]}$  and the intersection curve  $F'' \cap K^{[2]}$  corresponds to a conic in  $F'' \simeq \mathbf{P}^2$  and to the  $g_2^1$  in the symmetric product  $K^{[2]}$  associated to the covering  $K \rightarrow \Gamma \simeq \mathbf{P}^1$ .

Using the resolution  $\tilde{\mathcal{H}} \rightarrow \mathbf{A}^1$ , Collino proved the following

**Proposition 6.13** (Proposition 2.2.1 in [Col1]).

1. *The limiting Hodge structure at the origin of the  $\mathbf{H}^1(F_s, \mathbf{C})$ 's for  $s \in (\mathbf{C}^*, 0)$  exists and is equal to the one of  $\mathbf{H}^1(K^{[2]}, \mathbf{C})$ . In particular, this limit is pure and of dimension 5.*
2. *The family of associated Albanese varieties has good reduction at 0 and the special fiber  $\tilde{A}_0$  at this point is the Albanese variety of  $K^{[2]}$ , namely the Jacobian variety  $J(K)$  of  $K$ .*

For  $s \in \mathbf{C}$ , denote by  $P_{i,s} : \Delta(G_s) \dashrightarrow F_s$  (for  $i = 1, 2, 3$ ) the three rational maps defining  $\Delta\mathbf{W}_{G_s}$  (cf. (136)). Each  $\{P_{i,s}\}_{s \in \mathbf{C}}$  is a smooth family of maps from which we get that the triangles webs  $\Delta\mathbf{W}_{G_s}$ 's form a family of skew curvilinear 3-webs which is smooth at the origin (verification left to the reader). For any  $s$  such that  $G_s$  is smooth, the Albanese map  $\text{alb}_{F_s} : F_s \rightarrow \text{Alb}(F_s)$  is such that  $\Psi_s = \sum_{i=1}^3 \text{alb}_{F_s}(P_{i,s}) : \Delta(G_s) \rightarrow \text{Alb}(F_s)$  is constant. Up to some choice of a smooth family of base points that we will not detail here, one deduces from the second point of the proposition above that the  $\Psi_s$ 's extend at the origin to a smooth family of maps, with

$$\Psi_0 = \sum_{i=1}^3 \text{alb}_{\tilde{F}_0}(P_{i,0}) : \Delta(\mathcal{C}) \longrightarrow \text{Alb}(\tilde{F}_0) = \tilde{A}_0 \simeq J(K)$$

being constant as well. In terms of webs, this gives us the

**Corollary 6.14.** *The trace induces an isomorphism between the space of global 1-forms on  $\tilde{A}_0 \simeq J(K)$ , which is isomorphic to  $\mathbf{H}^0(K, \Omega_K^1)$ , and the space of 1-ARs of  $\Delta\mathbf{W}_{\mathcal{C}} \simeq \mathcal{L}\mathbf{W}_{\mathcal{C}}$ :*

$$(137) \quad \mathbf{H}^0(K, \Omega_K^1) \simeq \mathbf{H}^0(\tilde{A}_0, \Omega_{\tilde{A}_0}^1) \xrightarrow{\sim} \mathbf{AR}^{(1)}(\Delta\mathbf{W}_{\mathcal{C}}).$$

Since  $\Delta\mathbf{W}_{\mathcal{C}} \simeq \mathcal{L}\mathbf{W}_{\mathcal{C}}$ , this corollary provides a conceptual description of the 1-ARs of  $\mathcal{L}\mathbf{W}_{\mathcal{C}}$ . However, although interesting this description has the disadvantage of not being canonical: getting a smoothing of  $C$  depends of the pencil of cubics considered. Indeed, as explained in [Col1, §1], another choice for this pencil would give another limit configuration of 12 points on  $\Gamma$  hence another hyperelliptic genus 5 ramified covering  $K' \rightarrow \Gamma$ , in general non isomorphic to  $K$ . This

would give another isomorphism  $H^0(K', \Omega_{K'}^1) \simeq AR^{(1)}(\Delta\mathcal{W}_{\mathcal{C}})$ , which shows to what extent (137) is not canonical.

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