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# Matroid Power Series

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## Abstract

Inspired by set functions and the greedy algorithm, we introduce matroid power series to provide short formulations and proofs for many classical and new results in the theory of graphs, matroids and oriented matroids, emphasizing the interest to study a matroid together with its minors in a single object.

## 1 Introduction

Let  $G = (V, E)$  be a finite undirected multigraph, permitted to have both multiple edges and loops. The main idea of [29, 30, 32] was to study not just  $G$ , but to consider at the same time all of its  $2^{|V|}$  subgraphs induced by subsets  $V'$  of  $V$ . In other words, for every vertex  $v \in V$  we have the choice to delete it (together with all its incident edges) or to conserve it. The main advantage of this approach is the possibility to use the commutative algebra  $A[V]$  of set functions  $f : 2^V \rightarrow A$ , where  $A$  is a commutative ring.

It may, however, be preferable to study  $G$  from its set of edges  $E$ . In this situation, we have three possibilities for every edge  $e \in E$  : we can *delete* it, *contract* it (i.e. delete it and identify its endpoint nodes) or *conserve* it. Each of those  $3^{|E|}$  choices gives us a different *minor* of  $G$  (all those minors are considered to be different although some of them can be isomorphic).

In this situation, the most important numerical values associated to  $G$  are the *cardinality*  $|E|$  and the *rank*  $r(G) := |V| - c(G)$ , where  $c(G)$  denotes the number of connected components of  $G$ . In other words,  $r(G)$  is the cardinality of each spanning forest of  $G$ , called *base* of  $G$ .

This approach can be generalized in matroid theory. A *matroid*  $M = (E, r)$ ,  $|M| := |E|$ , on a finite set  $E$  (in the special case of multigraphs,  $E$  is the set of edges) with rank function  $r$  can be defined as follows. For every partition  $E = E' \uplus D' \uplus C'$ , we have the minor  $M \setminus D' / C'$  which is a matroid on  $E'$  of cardinality  $|M \setminus D' / C'| := |E'|$ , where (in the case of multigraphs) all  $d' \in D'$  were deleted and all  $c' \in C'$  were contracted. We denote by  $3^M$  the set of minors of  $M$ ,  $|3^M| = 3^{|M|}$ . Then our rank function  $r : 3^M \rightarrow \mathbb{R}$  must satisfy the following three axioms :

- a) For every minor  $M' := M \setminus D' / C'$  on  $E'$  ( $E = E' \uplus D' \uplus C'$ ) and for every partition  $E' = E'' \uplus E'''$ , we have the fundamental relations  $(M' / E''' = M \setminus D' / (C' \cup E'''))$  and  $M' \setminus E'' = M \setminus (D' \cup E'') / C'$  :

$$|M'| = |M' / E'''| + |M' \setminus E''| \quad \text{and} \quad \mathbf{r}(M') = \mathbf{r}(M' / E''') + \mathbf{r}(M' \setminus E''). \quad (1.1)$$

In particular, the rank of every minor of zero cardinality is equal to zero.

- b) For every minor  $M' \in 3^M$  of cardinality one, we have  $r(M') \in \{0, 1\}$ .
- c) For every minor  $M' \in 3^M$  of cardinality two on the set  $\{e, f\}$ , we have

$$\begin{aligned} r(M') \leq r(M' \setminus e) + r(M' \setminus f) &\Leftrightarrow r(M') \geq r(M'/e) + r(M'/f) \\ \Leftrightarrow r(M'/e) \leq r(M' \setminus e) &\Leftrightarrow r(M'/f) \leq r(M' \setminus f). \end{aligned} \quad (1.2)$$

In other words, contractions reduce the rank more than deletions.

It is not difficult to prove that those axioms are satisfied for graphs, and everybody familiar with matroid theory can easily see their equivalence with any classical system of axioms. Of course, most questions of exact enumeration in matroid theory make use only of our axiom a). Therefore, one might want to call structures satisfying only this axiom *praepolymatroids* or *polypraematroids*. If, moreover, our axiom b) is satisfied, we call the structure *praematroid*.

Up to isomorphism, the only praematroids of cardinality one are indeed the matroids  $E_0$  of rank zero (a loop) and  $E_1$  of rank one (an isthmus also called bridge). There are only five praematroids of cardinality two, namely  $E_0 \oplus E_0 = U_{0,2}$  (two loops),  $E_1 \oplus E_1 = U_{2,2}$  (two bridges),  $E_0 \oplus E_1$  (a loop and a bridge), as well as the two connected praematroids  $U_{1,2}$  (a cycle of length two) and  $U_{1,2}^{\sim}$ , which can be defined as follows on the set  $\{e, f\}$ :  $r(U_{1,2}) = r(U_{1,2} \setminus e) = r(U_{1,2} \setminus f) = 1$ ; and  $r(U_{1,2}^{\sim}) = 1$ ,  $r(U_{1,2}^{\sim} \setminus e) = r(U_{1,2}^{\sim} \setminus f) = 0$ . Therefore a matroid is a praematroid without any minor (of cardinality two) isomorphic to  $U_{1,2}^{\sim}$ .

The aim of this article, however, is not a generalization of matroid theory [41, 55]. On the contrary, we often follow the example of the last chapter of Bollobás [6] and usually restrict ourselves to graphical matroids (i.e. to multigraphs) leaving it to interested readers to formulate, according to their needs, the evident generalizations to more general structures such as regular, representable or oriented matroids, etc. In some cases, however, it is even easier to consider matroids or oriented matroids, in particular for results motivated by the fact that oriented matroids do not only generalize oriented graphs but also oriented hyperplane arrangements. In all cases, we try to provide the easiest approach to our results.

It is classical that the structure of matroids is somehow equivalent to the fact that the greedy algorithm provides a maximal basis, in particular a spanning tree of maximal weight for a connected graph. This situation is easiest if all edges of the graph have a different weight. For our purposes, however, it is most interesting to study carefully the case in which edges can have the same weight. In that case, the optimal bases are actually the bases of a new decomposed matroid, as we will show in section 2. The decompositions appearing there are essential to understand the algebraic operations in all other sections. They will also be used directly in section 6.

In section 3 we give a systematic introduction to our algebraic tools of matroid power series. Those tools can be considered as classical, namely as the incidence algebra of the poset of all subsets of the set of edges, and therefore as a subalgebra of upper triangular matrices. Other motivations to introduce such algebraic tools came from Hopf algebras, coalgebras and mathematical physics [12, 13, 25, 46, 47]. Our main motivation was to mimic the method of formal power series and set functions [29, 30, 31, 32] in matroid theory. In particular, we think that their most important ingredients of substitutions and derivations are used systematically for the first time in this article (and in [29, 31]). Our approach allows us to formulate many classical and new results in a very short way. The proofs become then almost automatic. In particular,

they often rely only on a fundamental lemma already mentioned at the end of section 3. In other words, everything follows from the one element minors of our matroid.

In section 4 we give a short introduction to flows and tensions on graphs. In particular, we use our matroid power series in order to formulate a new duality theorem. This theorem implies that flow and tension polynomials can be calculated easily.

In section 5 we show that many classical and new results of algebraic or enumerative matroid theory are just corollaries of the associativity and partial commutativity of the multiplication of matroid power series. This concerns in particular the Tutte and Whitney polynomials and their weighted generalizations such as the Potts model in physics or the weight enumerator in the theory of error correcting codes. Proofs of convolution formulas [13, 26] become very short, and simple substitutions allow us to reduce proofs of several pages to one line, for example Matiyasevich's attempt to prove the four-color theorem [39].

Our section 6 is devoted to consequences of the fact that there are just two non isomorphic minors of cardinality 1 : one of rank 0 and one of rank 1. This is useful for counting forests and trees, in particular of maximal weight. It also shows that the incidence algebra of the lattice of flats of a matroid is a subalgebra of matroid power series. We use in this section already some results on orientations established in [30, 32], although we provide a special section on orientations (and oriented matroids) at the end of this article. The Tutte and Whitney polynomials or the partition function of the random cluster model cover products of two factors, but we show that more than two factors naturally occur in the theory of random minors and also in the work of [42, 14] showing that the Tutte and Whitney polynomials only depend on the lattice of cyclic flats.

In section 7 we introduce and study derivations and their many applications. First of all, they allow us to get all known recurrence relations for Tutte polynomials and their weighted generalizations. Moreover, we obtain easy formulations and proofs for results from algebraic geometry [35, 36, 37, 38]. Last but not least, we get differential equations for flows and tensions (first established in [29, 30]) with the help of Crapo's beta invariants. These equations can be solved with the help of a non commutative exponential function answering questions asked by Gioan at the Dagstuhl Seminar «Comparative Theory for Graph Polynomials» (2019) on possible algebraic proofs of his nice enumerative results established in [20]. At the AMS-EMS-SMF Joint International Meeting 2022, Brändén asked for a combinatorial treatment of a variant of the Tutte polynomial introduced in algebraic geometry [4]. This is possible in almost the same way and done at the end of section 7.

Since the algebra of matroid power series is not commutative, it is interesting to calculate Lie brackets, to which our section 8 is devoted. In particular, we get a better understanding of Crapo's beta invariants in the light of the classical Baker-Hausdorff formula. Moreover, we see that the axioms of matroids are closely related to Lie algebras.

In section 9 we finally study orientations for graphs and oriented matroids. It is particularly surprising that we do not only get identities for the number of acyclic or strongly connected (totally cyclic) orientations, but also for the indicator functions of acyclic minors or strongly connected minors, answering questions in [2]. Once again, some identities, are closely related to the axioms of oriented matroids. In particular, the famous 3-painting axiom [5] becomes a simple identity for matroid power series.

## 2 Decompositions

One of the most important ideas of our approach consists in exploiting in an automatic way the decompositions appearing in the context of the greedy algorithm providing a maximal basis of our matroid  $M = (E, r)$  with respect to a weight function  $w : E \rightarrow \{w_1, w_2, w_3, \dots, w_k\}$ ,  $w_1 > w_2 > w_3 > \dots > w_k$ , see [31]. Indeed, let  $B$  be an arbitrary basis of  $M$  having  $e_1$  elements of weight  $w_1$ ,  $e_2$  elements of weight  $w_2, \dots$ ,  $e_k$  elements of weight  $w_k$ , then summation by parts (also called Abel transformation) gives :

$$w(B) = e_1 w_1 + e_2 w_2 + e_3 w_3 + \dots + e_k w_k \quad (2.1)$$

$$\begin{aligned} &= e_1(w_1 - w_2) + \\ &\quad (e_1 + e_2)(w_2 - w_3) + \\ &\quad (e_1 + e_2 + e_3)(w_3 - w_4) + \dots + \\ &\quad (e_1 + e_2 + e_3 + \dots + e_k)w_k \end{aligned} \quad (2.2)$$

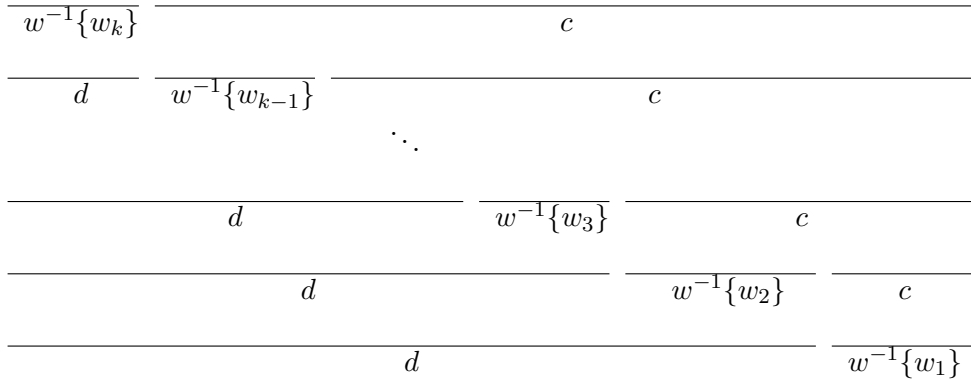
$$\begin{aligned} &\leq r(M \setminus w^{-1}\{w_2, w_3, \dots, w_k\})(w_1 - w_2) + \\ &\quad r(M \setminus w^{-1}\{w_3, \dots, w_k\})(w_2 - w_3) + \\ &\quad r(M \setminus w^{-1}\{w_4, \dots, w_k\})(w_3 - w_4) + \dots + \\ &\quad r(M)w_k \end{aligned} \quad (2.3)$$

$$\begin{aligned} &= r(M \setminus w^{-1}\{w_2, w_3, \dots, w_k\})w_1 + \\ &\quad r(M \setminus w^{-1}\{w_3, \dots, w_k\}/w^{-1}\{w_1\})w_2 + \\ &\quad r(M \setminus w^{-1}\{w_4, \dots, w_k\}/w^{-1}\{w_1, w_2\})w_3 + \dots + \\ &\quad r(M/w^{-1}\{w_1, w_2, w_3, \dots, w_{k-1}\})w_k. \end{aligned} \quad (2.4)$$

In other words,  $B$  is a maximal basis of  $M$  if and only if  $B$  is a basis of the matroid

$$M_1 \oplus M_2 \oplus \dots \oplus M_k, \quad M_i = M \setminus w^{-1}\{w_{i+1}, w_{i+2}, \dots, w_k\} / w^{-1}\{w_{i-1}, w_{i-2}, \dots, w_1\}$$

for every  $i = 1, 2, 3, \dots, k$ , see the following figure (c = contract, d = delete) :



The sequences of  $k$  matroids  $M_1, M_2, M_3, \dots, M_k$  will appear in the products of  $k$  factors in the following section.

### 3 Algebraic tools

The main idea of this article is to study not just one matroid  $M$ , but to look at its  $3^{|M|}$  minors simultaneously. The advantage of this approach is the possibility to use systematically the associative algebra  $A[M]$  formed by matroid functions  $f : 3^M \rightarrow A$ , where  $A$  is a commutative ring with 1 (in most cases, the associativity of our ring is sufficient). Let us identify  $f$  with its generating function called *matroid power series*

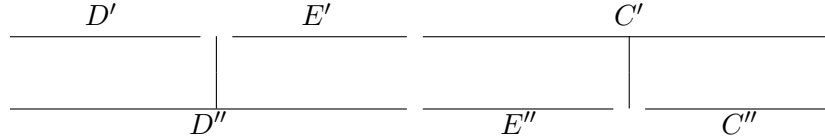
$$M_f(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{E' \uplus D' \uplus C' = E} f(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (3.1)$$

where the sum is taken over all ordered partitions of  $E$  (empty sets are allowed everywhere). The  $A$ -module structure of  $A[M]$  is evident, and the following definition for the multiplication seems most natural, inspired by our axiom a) mentioned in the introduction and by the preceding section.

**Definition.** For pairwise disjoint sets  $E', D', C'$  as well as  $E'', D'', C''$  let us define

$$(\mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}) \cdot (\mathbf{e}^{E''} \mathbf{d}^{D''} \mathbf{c}^{C''}) := \mathbf{e}^{E' \cup E''} \mathbf{d}^{(D' \cup D'') \setminus (E' \cup E'')} \mathbf{c}^{(C' \cup C'') \setminus (E' \cup E'')} \quad (3.2)$$

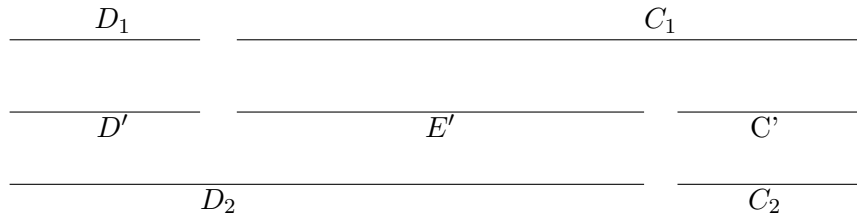
if  $E' \cap E'' = E' \cap C'' = D' \cap E'' = D' \cap C'' = C' \cap D'' = \emptyset$ ; otherwise this product equals zero. In particular, if  $E' \uplus D' \uplus C' = E'' \uplus D'' \uplus C'' = E$ , then the product is equal to zero or equal to  $\mathbf{e}^{E' \cup E''} \mathbf{d}^{D' \cap D''} \mathbf{c}^{C' \cap C''}$ , see the following figure.



It is evident that our multiplication is associative and that its unity is given by

$$1 = \sum_{D \uplus C = E} \mathbf{d}^D \mathbf{c}^C. \quad (3.3)$$

Our algebra  $A[M]$  of matroid power series is by no means new. Indeed, let  $P$  be the poset of all subsets of  $E$ , et let us associate bijectively to each pair  $D_1, D_2 \subseteq E$  such that  $D_1 \subseteq D_2$  a partition  $E = E' \uplus D' \uplus C'$  by defining  $D' := D_1$ ,  $C' := E \setminus D_2$  and  $E' := D_2 \setminus D_1$  (as well as  $C_1 := E \setminus D_1$  and  $C_2 := E \setminus D_2$ , see the following figure).



In this way, the incidence algebra of  $P$  becomes isomorphic to our algebra  $A[M]$  (see [1], chapter IV.1.A, or [50], chapter 3.6). Related algebraic approaches can be found in [12, 13, 25, 46, 47]. We have

$$(\mathbf{d}^{D_1} \mathbf{c}^{C_1}) M_f(\mathbf{e}, \mathbf{d}) (\mathbf{d}^{D_2} \mathbf{c}^{C_2}) = f(M \setminus D' / C') \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (3.4)$$

and if  $E'$  is not empty,

$$\begin{aligned} (\mathbf{d}^{D_1} \mathbf{c}^{C_1})(\mathbf{d}^{D_2} \mathbf{c}^{C_2}) M_f(\mathbf{e}, \mathbf{d}) &= 0, \\ M_f(\mathbf{e}, \mathbf{d})(\mathbf{d}^{D_1} \mathbf{c}^{C_1})(\mathbf{d}^{D_2} \mathbf{c}^{C_2}) &= 0, \\ (\mathbf{d}^{D_2} \mathbf{c}^{C_2}) M_f(\mathbf{e}, \mathbf{d})(\mathbf{d}^{D_1} \mathbf{c}^{C_1}) &= 0. \end{aligned} \quad (3.5)$$

Therefore  $f : 3^M \rightarrow A$  must vanish on all minors with nonempty support (i.e.  $M_f(\mathbf{e}, \mathbf{d}) = M_f(0, \mathbf{d})$ ), if  $M_f(\mathbf{e}, \mathbf{d})$  shall commute with all matroid power series (clearly, the algebra of matroid power series satisfying  $M_f(\mathbf{e}, \mathbf{d}) = M_f(0, \mathbf{d})$  is isomorphic to  $A^{2^{|M|}}$ ). In this case (i.e.  $M_f(\mathbf{e}, \mathbf{d}) = M_f(0, \mathbf{d})$ ) we further have

$$\begin{aligned} M_f(\mathbf{e}, \mathbf{d})(\mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}) &= f(M \setminus D_1 / C_1) \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \\ (\mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}) M_f(\mathbf{e}, \mathbf{d}) &= f(M \setminus D_2 / C_2) \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}. \end{aligned} \quad (3.6)$$

Therefore a matroid power series commuting with all elements of  $A[M]$  has to be a multiple of 1.

The product  $fg$  of two matroid functions  $f, g : 3^M \rightarrow A$  can also be defined, for every minor  $M \setminus D / C$ , by the formula

$$(fg)(M \setminus D / C) := \sum_{E' \uplus E'' = E \setminus (D \cup C)} f(M \setminus D / (C \cup E'')) \cdot g(M \setminus (D \cup E') / C). \quad (3.7)$$

It follows

$$M_{fg}(\mathbf{e}, \mathbf{d}) = M_f(\mathbf{e}, \mathbf{d}) \cdot M_g(\mathbf{e}, \mathbf{d}). \quad (3.8)$$

It is well-known from the theory of incidence algebras that  $M_f(\mathbf{e}, \mathbf{d})$  is invertible if and only if  $f(M \setminus D / C)$  is invertible for every partition  $D \uplus C = E$ . In this cas,

$$M_f(\mathbf{e}, \mathbf{d})^{-1} = M_{f^{-1}}(\mathbf{e}, \mathbf{d}), \quad (3.9)$$

where  $f^{-1} : 3^M \rightarrow A$  can be calculated recursively by using the identity  $M_f(\mathbf{e}, \mathbf{d}) \cdot M_f(\mathbf{e}, \mathbf{d})^{-1} = 1$  or  $M_f(\mathbf{e}, \mathbf{d})^{-1} \cdot M_f(\mathbf{e}, \mathbf{d}) = 1$  :  $f^{-1}(M \setminus D / C) = f(M \setminus D / C)^{-1}$  for all  $D \uplus C = E$ , and otherwise, for all  $\emptyset \subset E^* \uplus D \uplus C = E$  (i.e.  $E^*$  is not empty) :

$$\begin{aligned} f^{-1}(M \setminus D / C) &= f(M \setminus D / (C \cup E^*))^{-1} \cdot \\ &\quad \left( - \sum_{E' \cup E'' = E^*, E'' \neq \emptyset} f(M \setminus D / (C \cup E'')) f^{-1}(M \setminus (D \cup E') / C) \right) \\ &= \left( - \sum_{E' \cup E'' = E^*, E'' \neq \emptyset} f^{-1}(M \setminus D / (C \cup E'')) f(M \setminus (D \cup E') / C) \right) \cdot \\ &\quad f(M \setminus (D \cup E^*) / C)^{-1}. \end{aligned} \quad (3.10)$$

Let us associate to every  $e \in E$  a variable  $x_e$  (without loss of generality, we can suppose  $x_e \in A$ ), and let us define, for every  $z \in A$ ,

$$M_f(x_e \mathbf{e}, z \mathbf{d}) := \sum_{E' \uplus D' \uplus C' = E} f(M \setminus D' / C') \cdot \left( \prod_{e \in E'} x_e \right) \cdot z^{r(M \setminus D' / C')} \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}. \quad (3.11)$$

The most important particular case is  $x_e = x$  for every  $e \in E$ . It gives the identity

$$M_f(x\mathbf{e}, z\mathbf{d}) := \sum_{E' \uplus D' \uplus C' = E} f(M \setminus D' / C') \cdot x^{|M \setminus D' / C'|} \cdot z^{r(M \setminus D' / C')} \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}. \quad (3.12)$$

It is evident that those definitions are compatible with the addition and multiplication of matroid power series and that their behavior with respect to  $\mathbf{e}$  (or  $x$ ) does not differ from well known properties of set functions (see [29, 30, 32]). In particular,  $M_f(\mathbf{e}, \mathbf{d})$  is nilpotent if  $M_f(0, \mathbf{d}) = M_f(0\mathbf{e}, \mathbf{d})$  is equal to zero (in what follows, this condition will almost always be satisfied). In that case,

$$[1 + M_f(\mathbf{e}, \mathbf{d})]^{-1} = \sum_{k=0}^{\infty} (-1)^k M_f(\mathbf{e}, \mathbf{d})^k, \quad (3.13)$$

where the last sum is finite.

Let

$$E_M(\mathbf{e}, \mathbf{d}) := \sum_{e \in E} \sum_{D \uplus C = E \setminus e} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C \quad (3.14)$$

be the indicator function of minors of  $M$  of cardinality one, such that

$$\exp[E_M(\mathbf{e}, \mathbf{d})] = \sum_{E' \uplus D' \uplus C' = E} \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \quad (3.15)$$

is the indicator function of all minors of  $M$ . Therefore,

$$\exp[E_M(x_e \mathbf{e}, z\mathbf{d})] = \sum_{E' \uplus D' \uplus C' = E} \left( \prod_{e \in E'} x_e \right) \cdot z^{r(M \setminus D' / C')} \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \quad (3.16)$$

counts the weight (i.e. the product of the variables of the support  $E'$ ) and the rank for all minors of  $M$ .

**Remark.** For every matroid  $M = (E, r)$ , there exists the *dual* matroid  $M^* = (E, r^*)$ , which can be defined with the help of the identities  $r(M) + r^*(M^*) = |E|$  and  $M^* \setminus D' / C' = (M \setminus C' / D')^*$  for every partition  $E = E' \uplus D' \uplus C'$ . In other words,  $r^*(M^* \setminus D' / C') := |E'| - r(M \setminus C' / D')$ . This implies the following relation for the indicator functions of the minors of cardinality one (see [29]) :

$$\begin{aligned} E_{M^*}(\mathbf{e}, z\mathbf{d}) &= \sum_{e \in E} \sum_{D \uplus C = E \setminus e} z^{r(M^* \setminus D / C)} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C \\ &= \sum_{e \in E} \sum_{D \uplus C = E \setminus e} z^{r^*(M / D \setminus C)} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C \\ &= \sum_{e \in E} \sum_{D \uplus C = E \setminus e} z^{|M / D \setminus C|} \left( \frac{1}{z} \right)^{r(M / D \setminus C)} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C \\ &= E_M(z\mathbf{e}, \frac{1}{z}\mathbf{d}). \end{aligned} \quad (3.17)$$

We could have chosen notations that emphasize more the duality aspects of matroid theory. However, nobody writes neither  $(k, l)$  nor  $\binom{k+l}{k, l}$  for  $\binom{k+l}{k} = \binom{k+l}{l}$ ; and in this perspective we have preferred a notation which relies only on the matroid  $M$  itself. It allows us very well



to express all aspects of  $M^*$  while offering the additional advantage of uniqueness and of its proximity to set functions.

Let us associate to each  $e \in E$  two variables  $x_e$  and  $y_e$ . It is evident that  $E_M(x_e \mathbf{e}, \mathbf{d}_c)$  and  $E_M(y_e \mathbf{e}, \mathbf{d}_c)$  commute, because they are in reality just two set functions (see [29, 30, 32]). The identity  $E_M(x_e \mathbf{e}, \mathbf{d}_c) \cdot E_M(y_e \mathbf{e}, \mathbf{d}_c) = E_M(y_e \mathbf{e}, \mathbf{d}_c) \cdot E_M(x_e \mathbf{e}, \mathbf{d}_c)$ , however, remains valid if we replace  $\mathbf{d}_c$  by  $z_c^{\mathbf{d}}$ , where  $z$  is an additional variable. We have proved the following lemma.

**Lemma 1. Fundamental Lemma.** *For every  $x_e, y_e, z \in A$  or for variables  $x_e, y_e, z$ , the matrix power series  $E_M(x_e \mathbf{e}, z_c^{\mathbf{d}})$  and  $E_M(y_e \mathbf{e}, z_c^{\mathbf{d}})$  commute, and their sum equals  $E_M(x_e \mathbf{e}, z_c^{\mathbf{d}}) + E_M(y_e \mathbf{e}, z_c^{\mathbf{d}}) = E_M((x_e + y_e) \mathbf{e}, z_c^{\mathbf{d}})$ . Therefore,*

$$\begin{aligned} \exp[E_M((x_e + y_e) \mathbf{e}, z_c^{\mathbf{d}})] &= \exp[E_M(x_e \mathbf{e}, z_c^{\mathbf{d}})] \cdot \exp[E_M(y_e \mathbf{e}, z_c^{\mathbf{d}})] \\ &= \exp[E_M(y_e \mathbf{e}, z_c^{\mathbf{d}})] \cdot \exp[E_M(x_e \mathbf{e}, z_c^{\mathbf{d}})]. \end{aligned} \quad (3.18)$$

In particular,  $\exp[E_M(x_e \mathbf{e}, z_c^{\mathbf{d}})]$  and  $\exp[E_M(y_e \mathbf{e}, z_c^{\mathbf{d}})]$  are inverse to each other if and only if  $x_e + y_e = 0$  for every  $e \in E$ .  $\square$

## 4 Flows and Tensions

Let us choose an orientation for every edge of our multigraph  $G = (V, E)$  in an arbitrary way such that, for every edge  $e \in E$ , we are able to distinguish its head  $h(e)$  and its tail  $t(e)$  ( $h(e) = t(e)$  if and only if  $e$  is a loop). For every positive integer  $n$ , let  $C_n$  be a commutative group of cardinality  $n$  equipped with a bilinear form  $\langle \cdot, \cdot \rangle : C_n \times C_n \rightarrow C$ , where  $C$  is a commutative group. Let us denote by  $C_n(E)$  (resp.  $C_n(V)$ ) the commutative group generated by the functions  $c : E \rightarrow C_n$  (resp.  $c : V \rightarrow C_n$ ) and let us extend our bilinear form to  $C_n(E)$  and to  $C_n(V)$  in a canonical way. Let  $\partial$  be the incidence matrix of our multigraph  $G$  defined for every pair  $(v, e) \in V \times E$  by  $\partial(v, e) = 0$  if  $v$  and  $e$  are not incident (or  $e$  is a loop),  $\partial(v, e) = -1$  if  $t(e) = v$ , and  $\partial(v, e) = 1$  if  $h(e) = v$ . We have the following two complexes well known from topology :

$$0 \longrightarrow C_n(E) \xrightarrow{\partial} C_n(V) \longrightarrow 0, \quad (4.1)$$

$$0 \longleftarrow C_n(E) \xleftarrow{\partial^T} C_n(V) \longleftarrow 0. \quad (4.2)$$

It is evident that  $\text{Im } \partial$  and  $\text{Ker } \partial^T$  as well as  $\text{Im } \partial^T$  and  $\text{Ker } \partial$  are orthogonal subspaces with respect to our bilinear form. Moreover, the rank of  $\partial$  equals  $|V| - c(G) = r(G)$  if  $G$  has  $c(G)$  connected components (see the introduction), and we have  $|\text{Ker } \partial^T| = n^{c(G)}$ ,  $|\text{Im } \partial^T| = |\text{Im } \partial| = n^{r(G)}$ , and  $|\text{Ker } \partial| = n^{|E| - r(G)}$ .

The elements of  $\text{Ker } \partial$  (resp.  $\text{Im } \partial^T$ ) are called  $n$ -flows (resp.  $n$ -tensions or sometimes *potential differences*), see[3]. For every  $n$ -tension  $c \in \text{Im } \partial^T$ , the number of corresponding potentials (i.e. the cardinality of  $(\partial^T)^{-1}(c)$ ) equals  $n^{c(G)}$  since a potential can be chosen arbitrarily on a vertex of every connected component of  $G$ .

Let us look at a maximal forest (i.e. a base) of  $G$  having  $r(G)$  edges. We can define an  $n$ -tension (resp. an  $n$ -flow) of  $G$  arbitrarily on the  $r(G)$  edges of this forest (resp. on the  $|E| - r(G)$  edges not belonging to our forest), but then its value is determined uniquely everywhere. Therefore we see once again that the number of different  $n$ -tensions (resp.  $n$ -flows) of our multigraph  $G$  equals  $n^{r(G)}$  (resp.  $n^{|E| - r(G)}$ ).

Our aim, however, is to study the matroid  $M$  corresponding to our multigraph  $G$  (see the introduction). For every minor  $M' \in 3^M$  defined on the set of edges  $E' \subseteq E$ , let us denote the space of  $n$ -flows by  $F_n(M')$  and the space of  $n$ -tensions by  $T_n(M')$ . We have the following lemma.

**Lemma 2.** *For every partition  $E' = E'' \uplus E'''$ , we have two canonical split exact sequences :*

$$0 \longrightarrow T_n(M'/E''') \longrightarrow T_n(M') \xrightarrow{\quad} T_n(M' \setminus E'') \longrightarrow 0, \quad (4.3)$$

$$0 \longleftarrow F_n(M'/E''') \xleftarrow{\quad} F_n(M') \longleftarrow F_n(M' \setminus E'') \longleftarrow 0. \quad (4.4)$$

*Proof.* An  $n$ -tension of  $M'/E'''$  is also an  $n$ -tension of  $M'$ ; and if we delete the edges of  $E''$  of an  $n$ -tension of  $M'$ , we get an  $n$ -tension of  $M' \setminus E''$ . On the other hand, an  $n$ -tension of  $M' \setminus E''$  can be extended in  $n^{r(M'/E''')}$  different ways to get an  $n$ -tension of  $M'$ .

Dually, an  $n$ -flow of  $M' \setminus E''$  is also an  $n$ -flow of  $M'$ ; and if we contract the edges of  $E'''$  of an  $n$ -flow of  $M'$ , we get an  $n$ -flow of  $M'/E'''$ . On the other hand, an  $n$ -flow of  $M'/E'''$  can be extended in  $n^{|M' \setminus E''| - r(M' \setminus E'')}$  different ways to get an  $n$ -flow of  $M'$ .  $\square$

Let us attach to every  $e \in E$  a subset (not necessarily a subgroup)  $C_n(e)$  of the commutative group  $C_n$  and let us denote  $\overline{C_n(e)} := C_n \setminus C_n(e)$ . For every minor  $M' \in 3^M$  defined on  $E' \subseteq E$ , we denote  $\theta_{M'}[C_n(e)]$  (resp.  $\phi_{M'}[C_n(e)]$ ) the number of  $n$ -tensions (resp.  $n$ -flows)  $c : E' \rightarrow C_n$  satisfying the condition  $c(e) \in C_n(e)$  for every edge  $e \in E'$ . Let us put

$$\theta_{M, C_n(e)}(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} \theta_{M \setminus D' / C'}[C_n(e)] \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (4.5)$$

$$\phi_{M, C_n(e)}(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} \phi_{M \setminus D' / C'}[C_n(e)] \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (4.6)$$

where the sums are taken over all partitions such that  $E' \neq \emptyset$  ( $D' = \emptyset$  or  $C' = \emptyset$  is not excluded).

If  $C_n(e) = C_n$  for every  $e \in E$ , then  $E_M(\mathbf{e}, n\mathbf{d}, \mathbf{c})$  (resp.  $E_M(n\mathbf{e}, \frac{1}{n}\mathbf{d}, \mathbf{c})$ ) counts the number of  $n$ -tensions (resp.  $n$ -flows) for all minors of cardinality one of our matroid  $M$ . Therefore the preceding lemma allows us to conclude that

$$1 + \theta_{M, C_n}(\mathbf{e}, \mathbf{d}, \mathbf{c}) = \exp[E_M(\mathbf{e}, n\mathbf{d}, \mathbf{c})] \quad \text{and} \quad 1 + \phi_{M, C_n}(\mathbf{e}, \mathbf{d}, \mathbf{c}) = \exp[E_M(n\mathbf{e}, \frac{1}{n}\mathbf{d}, \mathbf{c})]. \quad (4.7)$$

In particular, we have a third proof of the fact that  $\theta_M[C_n] = n^{r(M)}$  and  $\phi_M[C_n] = n^{|M| - r(M)}$ . Actually, if  $C_n(e) = \{0\}$  for every  $e \in E$ , then  $\theta_M[\{0\}] = \phi_M[\{0\}] = 1$  and globally  $1 + \theta_{M, \{0\}}(\mathbf{e}, \mathbf{d}, \mathbf{c}) = 1 + \phi_{M, \{0\}}(\mathbf{e}, \mathbf{d}, \mathbf{c}) = \exp[E_M(\mathbf{e}, \mathbf{d}, \mathbf{c})]$ .

Our preceding lemma (or its proof) as well as the principle of inclusion-exclusion imply the identities

$$\exp[E_M(\mathbf{e}, n\mathbf{d}, \mathbf{c})] \cdot [1 + \theta_{M, C_n(e)}(-\mathbf{e}, \mathbf{d}, \mathbf{c})] = 1 + \theta_{M, \overline{C_n(e)}}(\mathbf{e}, \mathbf{d}, \mathbf{c}), \quad (4.8)$$

$$[1 + \phi_{M, C_n(e)}(-\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M(n\mathbf{e}, \frac{1}{n}\mathbf{d}, \mathbf{c})] = 1 + \phi_{M, \overline{C_n(e)}}(\mathbf{e}, \mathbf{d}, \mathbf{c}), \quad (4.9)$$

proving the following theorem.

**Theorem 1. Duality theorem for flows and tensions.**

$$\begin{aligned}\exp[E_M(\mathbf{e}, n\mathbf{d})] &= [1 + \theta_{M, \overline{C_n(e)}}(\mathbf{e}, \mathbf{d})] \cdot [1 + \theta_{M, C_n(e)}(-\mathbf{e}, \mathbf{d})]^{-1} \\ &= [1 + \theta_{M, C_n(e)}(\mathbf{e}, \mathbf{d})] \cdot [1 + \theta_{M, \overline{C_n(e)}}(-\mathbf{e}, \mathbf{d})]^{-1},\end{aligned}\quad (4.10)$$

$$\begin{aligned}\exp[E_M(n\mathbf{e}, \frac{1}{n}\mathbf{d})] &= [1 + \phi_{M, C_n(e)}(-\mathbf{e}, \mathbf{d})]^{-1} \cdot [1 + \phi_{M, \overline{C_n(e)}}(\mathbf{e}, \mathbf{d})] \\ &= [1 + \phi_{M, \overline{C_n(e)}}(-\mathbf{e}, \mathbf{d})]^{-1} \cdot [1 + \phi_{M, C_n(e)}(\mathbf{e}, \mathbf{d})]. \quad \square\end{aligned}\quad (4.11)$$

Let  $\lambda$  and  $\mu$  be two positive integers. In order to count *true*  $\lambda$ -tensions and  $\mu$ -flows of  $M$  which are not equal to zero on any edge, let us define  $\theta_M(\lambda) := \theta_M[C_\lambda \setminus \{0\}]$  and  $\phi_M(\mu) := \phi_M[C_\mu \setminus \{0\}]$  as well as  $\theta_{M, \lambda}(\mathbf{e}, \mathbf{d}) := \theta_{M, C_\lambda \setminus \{0\}}(\mathbf{e}, \mathbf{d})$  and  $\phi_{M, \mu}(\mathbf{e}, \mathbf{d}) := \phi_{M, C_\mu \setminus \{0\}}(\mathbf{e}, \mathbf{d})$ . Our duality theorem now implies the following corollary.

**Corollary 1.** *We have*

$$1 + \theta_{M, \lambda}(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, \lambda\mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, \mathbf{d})], \quad (4.12)$$

$$1 + \phi_{M, \mu}(\mathbf{e}, \mathbf{d}) = \exp[E_M(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d})]. \quad (4.13)$$

*In particular,  $\theta_M(\lambda)$  and  $\phi_M(\mu)$  are two polynomials : the tension polynomial (also called characteristic polynomial of the matroid  $M$ ) and the flow polynomial (also called characteristic polynomial of the dual matroid  $M^*$ ), respectively.  $\square$*

The identities of our corollary are equivalent to

$$[1 + \theta_{M, \lambda}(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(\mathbf{e}, \lambda\mathbf{d})], \quad (4.14)$$

$$\exp[E_M(\mathbf{e}, \mathbf{d})] \cdot [1 + \phi_{M, \mu}(\mathbf{e}, \mathbf{d})] = \exp[E_M(\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d})], \quad (4.15)$$

which have the following additional proofs : Every  $\lambda$ -tension vanishes on some (possibly empty) subset of the edges, and is in bijection with a true  $\lambda$ -tension of the graph obtained after contracting the vanishing edges. Dually, every  $\mu$ -flow vanishes on some (possibly empty) subset of the edges, and is in bijection with a true  $\mu$ -flow of the graph obtained after deleting the vanishing edges. Our Fundamental Lemma also allows us to calculate the following inverses.

**Corollary 2.** *We have*

$$[1 + \theta_{M, \lambda}(\mathbf{e}, \mathbf{d})]^{-1} = 1 + \theta_{M, \frac{1}{\lambda}}(\mathbf{e}, \lambda\mathbf{d}), \quad (4.16)$$

$$[1 + \phi_{M, \mu}(\mathbf{e}, \mathbf{d})]^{-1} = 1 + \phi_{M, \frac{1}{\mu}}(\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d}). \quad \square \quad (4.17)$$

**Remark.** If  $M$  is the matroid corresponding to our multigraph  $G = (V, E)$ , let us define  $\theta_G(\lambda) := \theta_M(\lambda)$  and  $\phi_G(\mu) := \phi_M(\mu)$ .

The *chromatic polynomial*  $\chi_G(\lambda)$  counts the potentials of  $G$  providing nowhere zero  $\lambda$ -tensions. According to our remarks at the beginning of this section, we have the fundamental identity

$$\chi_G(\lambda) = \lambda^{c(G)} \cdot \theta_G(\lambda), \quad (4.18)$$

where  $c(G)$  denotes the number of connected components of  $G$ .

## 5 The associativity and partial commutativity of multiplication

This section relies only on the axiom a) of our introduction and on the first corollary of our preceding section, which we could consider as an additional axiom a').

Following an implicit suggestion made by Etienne [16], let us define for the two families of variables  $(x_1)_e, (x_2)_e$  as well as for  $\lambda_1, \lambda_2, \mu_1, \mu_2$  :

$$\begin{aligned} & 1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d}) \\ &= 1 + \sum_{\emptyset \subset E' \cup D' \cup C' = E} R_{M \setminus D' / C'}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \\ &:= \exp[E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \cdot \exp[E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})], \end{aligned} \quad (5.1)$$

and let us call  $R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e)$  the *Etienne polynomial* of our matroid  $M$ .

**Remark.** We have  $R_{M^*}(\mu_2, \mu_1, \lambda_2, \lambda_1, (x_2)_e, (x_1)_e) = R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e)$  for the dual matroid (see [29]) :

$$\begin{aligned} & 1 + R_{M^*, \mu_2, \mu_1, \lambda_2, \lambda_1, (x_2)_e, (x_1)_e}(\mathbf{e}, \mathbf{d}) \\ &= \exp[E_{M^*}((x_2)_e \lambda_2 \mathbf{e}, \frac{\mu_2 \mathbf{d}}{\lambda_2 \mathbf{c}})] \cdot \exp[E_{M^*}((x_1)_e \lambda_1 \mathbf{e}, \frac{\mu_1 \mathbf{d}}{\lambda_1 \mathbf{c}})] \\ &= \exp[E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{c}}{\mu_2 \mathbf{d}})] \cdot \exp[E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{c}}{\mu_1 \mathbf{d}})] \\ &= 1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{c}), \end{aligned} \quad (5.2)$$

because exchanging  $c \leftrightarrow d$  also exchanges the order in all multiplications.

Our fundamental lemma directly implies the following identity for variables  $\lambda_1, \lambda_2, \dots, \lambda_n, \mu_1, \mu_2, \dots, \mu_n$  and families of variables  $(x_1)_e, (x_2)_e, \dots, (x_n)_e$ .

**Lemma 3.** *We have*

$$\begin{aligned} & \exp[E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \cdot \exp[E_M(-(x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})] \cdot \\ & \exp[E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})] \cdot \exp[E_M(-(x_3)_e \mu_3 \mathbf{e}, \frac{\lambda_3 \mathbf{d}}{\mu_3 \mathbf{c}})] \cdot \\ & \dots \\ & \exp[E_M((x_{n-1})_e \mu_{n-1} \mathbf{e}, \frac{\lambda_{n-1} \mathbf{d}}{\mu_{n-1} \mathbf{c}})] \cdot \exp[E_M(-(x_n)_e \mu_n \mathbf{e}, \frac{\lambda_n \mathbf{d}}{\mu_n \mathbf{c}})] \\ &= \exp[E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \cdot \exp[E_M(-(x_n)_e \mu_n \mathbf{e}, \frac{\lambda_n \mathbf{d}}{\mu_n \mathbf{c}})]. \quad \square \end{aligned} \quad (5.3)$$

This proves the following theorem for the Etienne polynomial.

**Theorem 2.** *We have*

$$\begin{aligned} & [1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, -(x_2)_e}(\mathbf{e}, \mathbf{d})] \cdot [1 + R_{M, \lambda_2, \lambda_3, \mu_2, \mu_3, (x_2)_e, -(x_3)_e}(\mathbf{e}, \mathbf{d})] \cdot \dots \\ & [1 + R_{M, \lambda_{n-1}, \lambda_n, \mu_{n-1}, \mu_n, (x_{n-1})_e, -(x_n)_e}(\mathbf{e}, \mathbf{d})] = 1 + R_{M, \lambda_1, \lambda_n, \mu_1, \mu_n, (x_1)_e, -(x_n)_e}(\mathbf{e}, \mathbf{d}). \quad \square \end{aligned} \quad (5.4)$$

If  $(x_1)_e = \dots = (x_n)_e = 1$  for every  $e \in E$ , then the preceding theorem is equivalent to proposition 5.10 of [13]. It is easy to prove the following relations for the Etienne polynomial.

**Proposition 1.** *We have*

$$[1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d})]^{-1} = 1 + R_{M, \lambda_2, \lambda_1, \mu_2, \mu_1, -(x_2)_e, -(x_1)_e}(\mathbf{e}, \mathbf{d}), \quad (5.5)$$

$$1 + R_{M, \lambda_1, c \lambda_1, \mu_1, c \mu_1, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d}) = \exp[E_M((x_1) \mu_1 + x_2 c \mu_1) \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})], \quad (5.6)$$

$$1 + R_{M, c_1 \lambda_1, c_2 \lambda_2, c_1 \mu_1, c_2 \mu_2, (x_1/c_1)_e, (x_2/c_2)_e}(\mathbf{e}, \mathbf{d}) = 1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d}) \quad (5.7)$$

for all nonzero  $c, c_1, c_2$ . □

The multivariate Tutte polynomial (alias Potts model) studied in [48] can be defined with the help of matroid power series :

$$1 + P_{M,q,w_e}(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(w_e \mathbf{e}, \frac{1}{q} \mathbf{d})]. \quad (5.8)$$

This definition immediately implies the following proposition.

**Proposition 2.** *We have*

$$1 + P_{M,q,w_e}(\mathbf{e}, \mathbf{d}) = 1 + R_{M,1,1/q,1,1,1,w_e}(\mathbf{e}, \mathbf{d}), \quad (5.9)$$

$$1 + R_{M,\lambda_1,\lambda_2,\mu_1,\mu_2,(x_1)_e,(x_2)_e}(\mathbf{e}, \mathbf{d}) = 1 + P_{M,\frac{\lambda_1\mu_2}{\mu_1\lambda_2},\frac{(x_2)_e\mu_2}{(x_1)_e\mu_1}}((x_1)_e\mu_1\mathbf{e}, \frac{\lambda_1}{\mu_1}\mathbf{d}). \quad \square \quad (5.10)$$

In other words, the Potts model and the Etienne polynomial can be calculated from each other by easy transformations of variables. Kung's subset-corank polynomial [26] can be defined with the help of matroid power series in the following way :

$$1 + SC_{M,\lambda,x_e}(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M(x_e \mathbf{e}, \mathbf{d})] \quad (5.11)$$

$$= 1 + P_{M,\lambda,x_e}(\mathbf{e}, \lambda \mathbf{d}) \quad (5.12)$$

$$= 1 + R_{M,\lambda,1,1,1,1,x_e}(\mathbf{e}, \mathbf{d}). \quad (5.13)$$

If we put in our preceding theorem or lemma or (5.3)  $n = 3$ ,  $\mu_1 = \mu_2 = \mu_3 = 1$ ,  $\lambda_3 = 1$ ,  $\lambda_2 = \mu$ ,  $\lambda_1 = \lambda\mu$ ,  $(x_1)_e = 1$ ,  $(x_2)_e = x_e$  and  $(x_3)_e = x_e y_e$  for all  $e \in E$ , then we get the main theorem of [26] (identity 1).

**Corollary 3.** *(Kung) We have*

$$[1 + SC_{M,\lambda,-x_e}(\mathbf{e}, \mu \mathbf{d})] \cdot [1 + SC_{M,\mu,-y_e}(x_e \mathbf{e}, \mathbf{d})] = [1 + SC_{M,\lambda\mu,-x_e y_e}(\mathbf{e}, \mathbf{d})]. \quad \square \quad (5.14)$$

If  $(x_1)_e = (x_2)_e = 1$  for every  $e \in E$  and  $\lambda_2 = \mu_1 = 1$ , then we get, by definition, the *Whitney polynomial*  $R_M(\lambda, \mu) := R_M(\lambda, 1, 1, \mu, 1, 1)$  (see [55], chapter 15.4). For its matroid power series, we have the following proposition (see [51], chapter IX.6).

**Proposition 3.** *We have*

$$1 + R_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M(\mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})], \quad (5.15)$$

$$[1 + R_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d})]^{-1} = 1 + R_{M,\frac{1}{\lambda},\frac{1}{\mu}}(-\mu \mathbf{e}, \frac{\lambda}{\mu} \mathbf{d}), \quad (5.16)$$

$$1 + R_{M,-\lambda,-1}(\mathbf{e}, -\mathbf{d}) = 1 + \theta_{M,\lambda}(\mathbf{e}, \mathbf{d}), \quad (5.17)$$

$$1 + R_{M,-1,-\mu}(-\mathbf{e}, -\mathbf{d}) = 1 + \phi_{M,\mu}(\mathbf{e}, \mathbf{d}). \quad \square \quad (5.18)$$

For everybody familiar with the *Tutte polynomial*  $T_M(\lambda, \mu)$  (see [9]), it may be useful to recall that  $T_M(\lambda, \mu) = R_M(\lambda - 1, \mu - 1)$ . We have preferred to work with the Whitney polynomial making all formulas slightly easier. The usefulness of the Whitney polynomial is precisely that it allows the calculation of products of the following form (for  $z' \neq z''$ ).

**Lemma 4.** *We have*

$$\exp[E_M(x\mathbf{e}, z' \mathbf{d})] \cdot \exp[E_M(y\mathbf{e}, z'' \mathbf{d})] = 1 + R_{M,\frac{z'x}{z''y},\frac{y}{x}}(x\mathbf{e}, \frac{z''y}{x} \mathbf{d}). \quad \square \quad (5.19)$$

If  $x_e = y_e = 1$  for every  $e \in E$ , then our identity (5.3) can be rewritten using only the Whitney polynomial. In this way, we get the following analogue of our preceding theorem.

**Theorem 3.** *We have*

$$\begin{aligned} & [1 + R_{M, -\frac{\lambda_1}{\lambda_2}, -\frac{\mu_2}{\mu_1}}(\mu_1 \mathbf{e}, -\frac{\lambda_2}{\mu_1} \mathbf{d})] \cdot [1 + R_{M, -\frac{\lambda_2}{\lambda_3}, -\frac{\mu_3}{\mu_2}}(\mu_2 \mathbf{e}, -\frac{\lambda_3}{\mu_2} \mathbf{d})] \cdots \\ & [1 + R_{M, -\frac{\lambda_{n-1}}{\lambda_n}, -\frac{\mu_n}{\mu_{n-1}}}(\mu_{n-1} \mathbf{e}, -\frac{\lambda_n}{\mu_{n-1}} \mathbf{d})] = [1 + R_{M, -\frac{\lambda_1}{\lambda_n}, -\frac{\mu_n}{\mu_1}}(\mu_1 \mathbf{e}, -\frac{\lambda_n}{\mu_1} \mathbf{d})]. \quad \square \end{aligned} \quad (5.20)$$

For two factors, the identity of our preceding theorem can be rewritten in the following way :

$$[1 + R_{M, -\frac{\lambda_1}{\lambda_2}, -\frac{\mu_2}{\mu_1}}(\mathbf{e}, \frac{\lambda_2}{\lambda_3} \mathbf{d})] \cdot [1 + R_{M, -\frac{\lambda_2}{\lambda_3}, -\frac{\mu_3}{\mu_2}}(\frac{\mu_2}{\mu_1} \mathbf{e}, \frac{\mu_1}{\mu_2} \mathbf{d})] = 1 + R_{M, -\frac{\lambda_1}{\lambda_3}, -\frac{\mu_3}{\mu_1}}(\mathbf{e}, \mathbf{d}). \quad (5.21)$$

Putting  $\frac{\lambda_1}{\lambda_2} = a$ ,  $\frac{\lambda_2}{\lambda_3} = b$ ,  $\frac{\mu_2}{\mu_1} = c$ ,  $\frac{\mu_3}{\mu_2} = d$ , we get the following corollary, proved in [26] (identity 3), [53] (theorem 5.3), [13] (proposition 5.11).

**Corollary 4.** *(Kung) We have*

$$[1 + R_{M, -a, -c}(\mathbf{e}, b \mathbf{d})] \cdot [1 + R_{M, -b, -d}(c \mathbf{e}, \frac{1}{c} \mathbf{d})] = 1 + R_{M, -ab, -cd}(\mathbf{e}, \mathbf{d}). \quad \square \quad (5.22)$$

If  $a = 1$  or  $b = 1$ , then the corresponding Whitney polynomial can be replaced by a flow polynomial, and if  $c = 1$  or  $d = 1$ , then the corresponding Whitney polynomial can be replaced by a tension polynomial, as shown in our preceding proposition.

For every identity of this section, we can find a direct one line proof, if we replace all terms by products of type  $\exp[E_M(x_e \mathbf{e}, z_e \mathbf{d})]$  and use our fundamental lemma as well as the associativity of multiplication. Nevertheless, we propose another quite general theorem, such that all other results can be obtained by different specializations.

**Theorem 4.** *For all  $t_e, f_e, t'_e, f'_e$  as well as for  $\lambda$  and  $\mu$ , we have*

$$\begin{aligned} & \exp[E_M(t'_e \mathbf{e}, \lambda \mathbf{d})] \cdot [1 + \theta_{M, \lambda}(t_e \mathbf{e}, \mathbf{d})] \cdot \\ & \exp[E_M((t_e + f_e) \mathbf{e}, \mathbf{d})] \\ & \cdot [1 + \phi_{M, \mu}(f_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(f'_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \\ & = \exp[E_M((t_e + t'_e) \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M((f_e + f'_e) \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \\ & = 1 + R_{M, \lambda, 1, 1, \mu, t_e + t'_e, f_e + f'_e}(\mathbf{e}, \mathbf{d}). \end{aligned} \quad (5.23)$$

*If there exists  $z$  such that  $(f_e + f'_e)/(t_e + t'_e) = z$  for all  $e \in E$ , the the result can be expressed with the help of the Whitney (or Tutte) polynomial :*

$$1 + R_{M, \lambda, 1, 1, \mu, t_e + t'_e, f_e + f'_e}(\mathbf{e}, \mathbf{d}) = 1 + R_{M, \lambda, \frac{1}{z}, \mu \cdot z}((t_e + t'_e) \cdot \mathbf{e}, z \cdot \mathbf{d}), \quad (5.24)$$

*where we can replace  $R$  by  $\theta$  (resp.  $\phi$ ) if and only if  $\mu \cdot z$  (resp.  $\lambda \cdot \frac{1}{z}$ ) equals  $-1$ .*

*Proof.* Using our formulae (4.12), (4.13) and the fundamental lemma, it is evident that all terms of our theorem are equal to

$$\begin{aligned} & \exp[E_M(t'_e \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M(t_e \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M(-t_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(t_e \mathbf{e}, \mathbf{d})] \cdot \\ & \exp[E_M(f_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(-f_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(f_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \cdot \exp[E_M(f'_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})]. \end{aligned} \quad (5.25)$$

□

Let us start by looking at the specialization  $t'_e = f'_e = 0$  for every  $e \in E$ .

**Corollary 5.** *We have*

$$[1 + \theta_{M,\lambda}(t_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M((t_e + f_e) \mathbf{e}, \mathbf{d})] \cdot [1 + \phi_{M,\mu}(f_e \mathbf{e}, \mathbf{d})] = 1 + R_{M,\lambda,1,1,\mu,t_e,f_e}(\mathbf{e}, \mathbf{d}). \quad \square \quad (5.26)$$

If, moreover, we specialize  $t_e + f_e = 1$  for every  $e \in E$ , then we get the main theorem of Etienne's article [16], proved on almost an entire page with the help of the Möbius algebra.

In order to obtain the Whitney (or Tutte) polynomial, we can specialize  $t_e = t$  and  $f_e = f$  for every  $e \in E$  (see [29], chapter 5.4).

**Corollary 6.** *We have*

$$[1 + \theta_{M,\lambda}(t \mathbf{e}, \mathbf{d})] \cdot \exp[E_M((t + f) \mathbf{e}, \mathbf{d})] \cdot [1 + \phi_{M,\mu}(f \mathbf{e}, \mathbf{d})] = 1 + R_{M,\lambda \frac{t}{f}, \mu \frac{f}{t}}(t \mathbf{e}, \frac{f}{t} \mathbf{d}). \quad \square \quad (5.27)$$

If we specialize  $t + f = 1$ , then we get the main theorem of Reiner's article [45]. If, moreover, we take  $\lambda = \mu =: q$ , then we get the main theorem of Jaeger's article [23]. Both proved their results by verification of the recurrence relation on more than one page.

Particularly beautiful formulas can be obtained by putting  $t = f = 1$  or  $t = -f = 1$  (see [29], chapter 5.4).

**Corollary 7.** *We have*

$$[1 + \theta_{M,\lambda}(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(2\mathbf{e}, \mathbf{d})] \cdot [1 + \phi_{M,\mu}(\mathbf{e}, \mathbf{d})] = 1 + R_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d}), \quad (5.28)$$

$$[1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{d})] \cdot [1 + \phi_{M,-\mu}(-\mathbf{e}, -\mathbf{d})] = 1 + R_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d}). \quad \square \quad (5.29)$$

The second formula (5.29) is the main theorem of Kook, Reiner and Stanton's article [24], proved on an entire page by a method closely related to ours. This result can also be found in the chapters 4.5 and 5.4 of [29] and is also a consequence of [17].

If we take  $t = f =: s$  in (5.27) and  $t'_e = t'$  and  $f'_e = f'$  for every  $e \in E$ , then the preceding theorem implies the following corollary.

**Corollary 8.** *We have*

$$\begin{aligned} & \exp[E_M(t' \mathbf{e}, \lambda \mathbf{d})] \cdot [1 + R_{M,\lambda,\mu}(s \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(f' \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \\ &= \exp[E_M((s + t') \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M((s + f') \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \\ &= 1 + R_{M,\lambda \frac{s+t'}{s+f'}, \mu \frac{s+f'}{s+t'}}((s + t') \mathbf{e}, \frac{s+f'}{s+t'} \mathbf{d}). \quad \square \end{aligned} \quad (5.30)$$

The special case  $f' = 0$  and  $t' = 1$  is the main technical lemma in [54], pointed out to Welsh by Vertigan.

On the one hand, let us consider the specializations  $\mu = 1$  and  $f_e = 0$  for every  $e \in E$  of our preceding autodual theorem, and on the other hand, let us consider the dual specializations  $\lambda = 1$  and  $t_e = 0$  for every  $e \in E$ .

**Corollary 9.** *We have*

$$\begin{aligned}
& \exp[E_M(t'_e \mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot [1 + \theta_{M,\lambda}(t_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((t_e + f'_e) \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\
&= \exp[E_M((t_e + t'_e) \mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(f'_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\
&= 1 + R_{M,\lambda,1,1,1,t_e+t'_e,f'_e}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}), \tag{5.31}
\end{aligned}$$

$$\begin{aligned}
& \exp[E_M((f_e + t'_e) \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot [1 + \phi_{M,\mu}(f_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(f'_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{c}^{\mathbf{d}})] \\
&= \exp[E_M(t'_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((f_e + f'_e) \mu \mathbf{e}, \frac{1}{\mu} \mathbf{c}^{\mathbf{d}})] \\
&= 1 + R_{M,1,1,1,\mu,t'_e,f_e+f'_e}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}). \quad \square \tag{5.32}
\end{aligned}$$

Let  $t_e^*$  be a family of additional variables. In order to count true  $\lambda$ -tensions according to their support, let us follow [33], chapter 5.6, and introduce the homogenous *weight enumerator*

$$1 + \theta_{M,\lambda;t_e,t_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}) := [1 + \theta_{M,\lambda}(t_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(t_e^* \mathbf{e}, \mathbf{c}^{\mathbf{d}})], \tag{5.33}$$

where  $[1 + \theta_{M,\lambda}(t_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  counts the  $e \in E$  on which our tension does not vanish (the support), whereas  $\exp[E_M(t_e^* \mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  counts the  $e \in E$  on which our tension equals zero (the monochromatic edges, where the potential is the same for the head and the tail). In other words, the matroid power series  $1 + \theta_{M,\lambda;t_e,t_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}})$  counts  $\lambda$ -tensions in such a way that every edge  $e \in E$  contributes a multiplication by  $t_e^*$  if it is monochromatic, and a multiplication by  $t_e$  otherwise.

Dually, in order to count true  $\mu$ -flows according to their support, let us introduce the homogenous *weight enumerator*

$$1 + \phi_{M,\mu;f_e,f_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}) := \exp[E_M(f_e^* \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot [1 + \phi_{M,\mu}(f_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})], \tag{5.34}$$

where  $[1 + \phi_{M,\mu}(f_e \mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  counts the  $e \in E$  on which our flow does not vanish (the support) whereas  $\exp[E_M(f_e^* \mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  counts the  $e \in E$  on which the flow is equal to zero.

In the first identity of our preceding corollary, let us put  $f'_e := t_e^* - t_e$ , and in its second identity, let us put  $t'_e := f_e^* - f_e$ .

**Corollary 10.** *We have*

$$\begin{aligned}
& \exp[E_M(t'_e \mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot [1 + \theta_{M,\lambda;t_e,t_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\
&= \exp[E_M((t'_e + t_e) \mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((t_e^* - t_e) \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\
&= 1 + R_{M,\lambda,1,1,1,t'_e+t_e,t_e^*-t_e}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}), \tag{5.35}
\end{aligned}$$

$$\begin{aligned}
& [1 + \phi_{M,\mu;f_e,f_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(f'_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{c}^{\mathbf{d}})] \\
&= \exp[E_M((f_e^* - f_e) \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((f'_e + f_e) \mu \mathbf{e}, \frac{1}{\mu} \mathbf{c}^{\mathbf{d}})] \\
&= 1 + R_{M,1,1,1,\mu,f_e^*-f_e,f'_e+f_e}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}), \tag{5.36}
\end{aligned}$$

and in particular our weight enumerators can be expressed with the help of the Etienne polynomial :

$$\begin{aligned}
1 + \theta_{M,\lambda;t_e,t_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}) &= \exp[E_M(t_e \mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((t_e^* - t_e) \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\
&= 1 + R_{M,\lambda,1,1,1,t_e,t_e^*-t_e}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}), \tag{5.37}
\end{aligned}$$

$$\begin{aligned}
1 + \phi_{M,\mu;f_e,f_e^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}) &= \exp[E_M((f_e^* - f_e) \mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(f_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{c}^{\mathbf{d}})] \\
&= 1 + R_{M,1,1,1,\mu,f_e^*-f_e,f_e}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}). \quad \square \tag{5.38}
\end{aligned}$$



If  $\lambda = \mu =: q$ , then the last two identities of our preceding corollary immediately imply the following duality relations.

**Corollary 11.** *If  $(t_e^* - t_e)(f_e^* - f_e) = t_e f_e q$  for every  $e \in E$ , then*

$$1 + \theta_{M,q;t_e,t_e^*}((f_e^* - f_e)\mathbf{e}, \mathbf{c}^{\mathbf{d}}) = 1 + \phi_{M,q;f_e,f_e^*}(t_e \mathbf{e}, q\mathbf{c}^{\mathbf{d}}), \quad (5.39)$$

$$1 + \phi_{M,q;f_e,f_e^*}((t_e^* - t_e)\mathbf{e}, \mathbf{c}^{\mathbf{d}}) = 1 + \theta_{M,q;t_e,t_e^*}(f_e q\mathbf{e}, \frac{1}{q}\mathbf{c}^{\mathbf{d}}). \quad \square \quad (5.40)$$

If  $t_e = f_e = 1$  for every  $e \in E$ , then those duality relations become theorem 3.1 of [58], proven by Woodall on almost two pages with the help of the deletion contraction method. He has written : « There may be a direct way of doing this, but I do not know of one. » This clearly shows the usefulness of substitutions in matroid power series.

If we want to replace the Etienne polynomial in our equations (5.35) and (5.36) by the Whitney polynomial (or the Tutte polynomial), we can specialize  $f_e = f$ ,  $f'_e = f'$ ,  $f_e^* = f^*$ ,  $n_e = n$ ,  $n'_e = n'$  and  $n_e^* = n^*$  for every  $e \in E$ .

**Corollary 12.** *We have*

$$\begin{aligned} & \exp[E_M(t' \mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot [1 + \theta_{M,\lambda;t,t^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\ &= \exp[E_M((t' + t)\mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((t^* - t)\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\ &= 1 + R_{M,\lambda} \frac{t'+t}{t^*-t}, \frac{t^*-t}{t'+t} ((t' + t)\mathbf{e}, \frac{t^*-t}{t'+t}\mathbf{c}^{\mathbf{d}}), \end{aligned} \quad (5.41)$$

$$\begin{aligned} & [1 + \phi_{M,\mu;f,f^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(f' \mu \mathbf{e}, \frac{1}{\mu}\mathbf{c}^{\mathbf{d}})] \\ &= \exp[E_M((f^* - f)\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((f' + f)\mu \mathbf{e}, \frac{1}{\mu}\mathbf{c}^{\mathbf{d}})] \\ &= 1 + R_{M,\frac{f^*-f}{f'+f}, \mu} \frac{f'+f}{f^*-f} ((f^* - f)\mathbf{e}, \frac{f'+f}{f^*-f}\mathbf{c}^{\mathbf{d}}), \end{aligned} \quad (5.42)$$

and in particular our weight enumerators can be expressed with the help of the Whitney polynomial :

$$\begin{aligned} 1 + \theta_{M,\lambda;t,t^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}) &= \exp[E_M(t\mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M((t^* - t)\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \\ &= 1 + R_{M,\lambda} \frac{t}{t^*-t}, \frac{t^*-t}{t} (t\mathbf{e}, \frac{t^*-t}{t}\mathbf{c}^{\mathbf{d}}), \end{aligned} \quad (5.43)$$

$$\begin{aligned} 1 + \phi_{M,\mu;f,f^*}(\mathbf{e}, \mathbf{c}^{\mathbf{d}}) &= \exp[E_M((f^* - f)\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(f\mu \mathbf{e}, \frac{1}{\mu}\mathbf{c}^{\mathbf{d}})] \\ &= 1 + R_{M,\frac{f^*-f}{f}, \mu} \frac{f}{f^*-f} ((f^* - f)\mathbf{e}, \frac{f}{f^*-f}\mathbf{c}^{\mathbf{d}}). \quad \square \end{aligned} \quad (5.44)$$

If we take  $t^* = 1$  in (5.43), then we get Greene's classical result [21], proved by Brylawski and Oxley in [9] on pages 182-184 in two different ways. If, however, we take  $t = 1$  in (5.43), then we get the main result of the paragraph « Two-variable Colouring » of Brylawski and Oxley's article [9], proved on the pages 156-158 in three different ways, one of which is closely related to ours.

We can simplify (5.41) by defining  $\mu' := (t^* - t)/(t' + t) \Leftrightarrow \mu' t' = t^* - t - \mu' t$  and by replacing  $\lambda$  by  $\lambda \mu'$  and  $\mathbf{e}$  by  $\mu' \mathbf{e}$ . Dually, we can simplify (5.42) by defining  $\lambda' := (f^* - f)/(f' + f) \Leftrightarrow \lambda' f' = f^* - f - \lambda' f$  and by replacing  $\mu$  by  $\lambda' \mu$  and  $\mathbf{e}$  by  $\lambda' \mathbf{e}$ .

**Corollary 13.** *We have*

$$\begin{aligned} & \exp[E_M((t^* - t - \mu't)\mathbf{e}, \lambda\mu'\mathbf{c})] \cdot [1 + \theta_{M,\lambda\mu';t,t^*}(\mu'\mathbf{e}, \mathbf{d})] \\ &= 1 + R_{M,\lambda,\mu'}((t^* - t)\mathbf{e}, \mu'\mathbf{c}), \end{aligned} \quad (5.45)$$

$$\begin{aligned} & [1 + \phi_{M,\lambda'\mu;f,f^*}(\lambda'\mathbf{e}, \mathbf{d})] \cdot \exp[E_M((f^* - f - \lambda'f)\lambda'\mu\mathbf{e}, \frac{1}{\lambda'\mu}\mathbf{d})] \\ &= 1 + R_{M,\lambda',\mu}((f^* - f)\lambda'\mathbf{e}, \frac{1}{\lambda'}\mathbf{d}). \quad \square \end{aligned} \quad (5.46)$$

If we want to replace the Whitney polynomial in (5.45) by the flow polynomial, we have to put  $\lambda := -1$  and replace  $\mu'$  by  $-\mu'$  and  $\mathbf{e}$  by  $-\mathbf{e}$ . If we want to replace it by the tension polynomial, however, we have to put  $\mu' := -1$  and replace  $\lambda$  by  $-\lambda$ .

Dually, if we want to replace the Whitney polynomial in (5.46) by the tension polynomial, we have to put  $\mu := -1$  and replace  $\lambda'$  by  $-\lambda'$  and  $\mathbf{e}$  by  $-\mathbf{e}$ . If we want to replace it by the flow polynomial, however, we have to put  $\lambda' := -1$  and replace  $\mu$  by  $-\mu$ .

**Corollary 14.** *We have*

$$\begin{aligned} & \exp[E_M((t - t^* - \mu't)\mathbf{e}, \mu'\mathbf{c})] \cdot [1 + \theta_{M,\mu';t,t^*}(\mu'\mathbf{e}, \mathbf{d})] \\ &= 1 + \phi_{M,\mu'}((t^* - t)\mathbf{e}, \mu'\mathbf{c}), \end{aligned} \quad (5.47)$$

$$\begin{aligned} & \exp[E_M(t^*\mathbf{e}, \lambda\mathbf{c})] \cdot [1 + \theta_{M,\lambda;t,t^*}(-\mathbf{e}, \mathbf{d})] \\ &= 1 + \theta_{M,\lambda}((t^* - t)\mathbf{e}, \mathbf{d}), \end{aligned} \quad (5.48)$$

$$\begin{aligned} & [1 + \phi_{M,\lambda';f,f^*}(\lambda'\mathbf{e}, \mathbf{d})] \cdot \exp[E_M((f - f^* - \lambda'f)\lambda'\mathbf{e}, \frac{1}{\lambda'}\mathbf{d})] \\ &= 1 + \theta_{M,\lambda'}((f^* - f)\lambda'\mathbf{e}, \frac{1}{\lambda'}\mathbf{d}), \end{aligned} \quad (5.49)$$

$$\begin{aligned} & [1 + \phi_{M,\mu;f,f^*}(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(f^*\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d})] \\ &= 1 + \phi_{M,\mu}((f^* - f)\mathbf{e}, \mathbf{d}). \quad \square \end{aligned} \quad (5.50)$$

If we take  $t := -1$  and  $t^* := 1$  in our identity (5.47), then we get a result proved by Galluccio and Loebli in [18] (theorem 4.9).

In order to simplify our equation (5.45) even further, let us put  $t^* - t = \mu't \Leftrightarrow t^* = (\mu' + 1)t$  and let us replace  $\mu'\mathbf{e}$  by  $\mathbf{e}$ . Dually, in order to simplify (5.46), we take  $f^* - f = \lambda'f \Leftrightarrow f^* = (\lambda' + 1)f$  and replace  $\lambda'\mathbf{e}$  by  $\mathbf{e}$ .

**Corollary 15.** *We have*

$$1 + \theta_{M,\lambda\mu';t,(\mu'+1)t}(\mathbf{e}, \mathbf{d}) = 1 + R_{M,\lambda,\mu'}(t\mathbf{e}, \mu'\mathbf{c}), \quad (5.51)$$

$$1 + \phi_{M,\lambda'\mu;f,(\lambda'+1)f}(\mathbf{e}, \mathbf{d}) = 1 + R_{M,\lambda',\mu}(f\lambda'\mathbf{e}, \frac{1}{\lambda'}\mathbf{d}). \quad \square \quad (5.52)$$

If, moreover, we take  $t = f = 1$ , we get results allowing us to express the Whitney polynomial by true  $\lambda\mu'$ -tensions or by true  $\lambda'\mu$ -flows. These results can also be found in the chapters 4.5 and 5.3 of [29] and in [58] (theorem 2.1).

**Corollary 16.** *We have*

$$[1 + \theta_{M,\lambda\mu'}(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M((\mu' + 1)\mathbf{e}, \mathbf{d})] = 1 + R_{M,\lambda,\mu'}(\mathbf{e}, \mu'\mathbf{c}), \quad (5.53)$$

$$\exp[E_M((\lambda' + 1)\mathbf{e}, \mathbf{d})] \cdot [1 + \phi_{M,\lambda'\mu}(\mathbf{e}, \mathbf{d})] = 1 + R_{M,\lambda',\mu}(\lambda'\mathbf{e}, \frac{1}{\lambda'}\mathbf{d}). \quad \square \quad (5.54)$$

Let us replace, in the preceding corollary,  $\lambda$  by  $-1$ ,  $\mu'$  by  $-\mu'$ ,  $\mu$  by  $-1$  and  $\lambda'$  by  $-\lambda'$ . Then we get results allowing us to express the flow polynomial by true  $\mu'$ -tensions and the tension polynomial by true  $\lambda'$ -flows.

**Corollary 17.** *We have*

$$[1 + \theta_{M,\mu'}(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M((1 - \mu')\mathbf{e}, \mathbf{d})] = 1 + \phi_{M,\mu'}(-\mathbf{e}, \mu' \mathbf{d}), \quad (5.55)$$

$$\exp[E_M((1 - \lambda')\mathbf{e}, \mathbf{d})] \cdot [1 + \phi_{M,\lambda'}(\mathbf{e}, \mathbf{d})] = 1 + \theta_{M,\lambda'}(-\lambda' \mathbf{e}, \frac{1}{\lambda'} \mathbf{d}). \quad \square \quad (5.56)$$

For graphs, an expansion of the form (5.56) for its chromatic or tension polynomial was given by Nagle [40]. In an attempt to understand the four-color theorem, Matiyasevich proved that  $\phi$  is, in fact, the flow polynomial : this is the main result of his article [39]. His proof, by number theoretic methods, is longer than two pages. Other proofs can be found in the chapters 4.4 and 5.3 of [29], in [44] and in [58].

## 6 Minors of cardinality one

Since the rank of every one element minor of  $M$  is either 0 (loop) or 1 (coloop), it is natural to define

$$E_M^0(\mathbf{e}, \mathbf{d}) := \sum_{e \in E} \sum_{\substack{D \uplus C = E \setminus e, \\ r(M \setminus D/C) = 0}} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C, \quad E_M^1(\mathbf{e}, \mathbf{d}) := \sum_{e \in E} \sum_{\substack{D \uplus C = E \setminus e, \\ r(M \setminus D/C) = 1}} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C. \quad (6.1)$$

Of course,  $\exp[E_M^0(\mathbf{e}, \mathbf{d})]$  is the indicator function of all minors which are collections of loops whereas  $\exp[E_M^1(\mathbf{e}, \mathbf{d})]$  is the indicator function of all minors which are collections of coloops. Moreover,

$$E_M((x)_e \mu \mathbf{e}, \frac{\lambda}{\mu} \mathbf{d}) = \mu \cdot E_M^0((x_e) \mathbf{e}, \mathbf{d}) + \lambda \cdot E_M^1((x_e) \mathbf{e}, \mathbf{d}). \quad (6.2)$$

The following lemma is an immediate consequence of the definition for the multiplication of matroid power series.

**Lemma 5.** *For all minors of  $M$ ,*

$$1 + R_{M,0,0}(\mathbf{e}, \mathbf{d}) = \exp[E_M^0(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d})] \quad (6.3)$$

*counts the number of bases (or spanning forests),*

$$1 + R_{M,1,0}(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d})] \quad (6.4)$$

*counts the number of independent sets (or forests), and*

$$1 + R_{M,0,1}(\mathbf{e}, \mathbf{d}) = \exp[E_M^0(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] \quad (6.5)$$

*counts the number of coindependent sets (or spanning subgraphs).*

Let  $w : E \rightarrow \{w_1, w_2, w_3, \dots, w_k\}$  be a fixed weight function with  $w_1 > w_2 > w_3 > \dots > w_k$ . For every  $i \in \{1, \dots, k\}$  let us define

$$E_{M,w_i}^0(\mathbf{e}, \mathbf{d}) := \sum_{\substack{e \in E, \\ w(e) = w_i}} \sum_{\substack{D \uplus C = E \setminus e, \\ r(M \setminus D/C) = 0}} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C, \quad (6.6)$$

$$E_{M,w_i}^1(\mathbf{e}, \mathbf{d}) := \sum_{\substack{e \in E, \\ w(e) = w_i}} \sum_{\substack{D \uplus C = E \setminus e, \\ r(M \setminus D/C) = 1}} e^{w_i} \mathbf{e}^{\{e\}} \mathbf{d}^D \mathbf{c}^C, \quad (6.7)$$

where  $e$  is Euler's number. Our section on decompositions immediately implies the following theorem.

**Theorem 5.** *The matroid power series*

$$\exp \left[ \sum_{i=1}^k E_{M,w_i}^0(\mathbf{e}, \mathbf{d}) \right] \cdot \exp \left[ \sum_{i=1}^k E_{M,w_i}^1(\mathbf{e}, \mathbf{d}) \right] \quad (6.8)$$

counts every basis  $B$  of our matroid  $M$  with its weight

$$\prod_{b \in B} e^{w(b)} = e^{\sum_{b \in B} w(b)}, \quad (6.9)$$

whereas

$$\exp[E_{M,w_k}^0(\mathbf{e}, \mathbf{d})] \exp[E_{M,w_k}^1(\mathbf{e}, \mathbf{d})] \cdots \exp[E_{M,w_1}^0(\mathbf{e}, \mathbf{d})] \exp[E_{M,w_1}^1(\mathbf{e}, \mathbf{d})] \quad (6.10)$$

counts only every maximal basis of our matroid with its weight.

For every  $M' \in \mathfrak{Z}^M$  let  $L(M')$  be the set of its loops and  $C(M')$  be the set of its coloops. Then we have the following lemma.

**Lemma 6.** *We have*

$$\begin{aligned} & \exp[E_M(x_e \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M^0(y_e \mathbf{e}, \mathbf{d})] = \\ & 1 + \sum_{\substack{\emptyset \subset E' \uplus D' \uplus C' = E, \\ M' = M \setminus D' / C'}} \lambda^{r(M')} \prod_{e \in E' \setminus L(M')} x_e \prod_{e \in L(M')} (x_e + y_e) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \end{aligned} \quad (6.11)$$

$$\begin{aligned} & \exp[E_M^1(y_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(x_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] = \\ & 1 + \sum_{\substack{\emptyset \subset E' \uplus D' \uplus C' = E, \\ M' = M \setminus D' / C'}} \mu^{|M'| - r(M')} \prod_{e \in E' \setminus C(M')} x_e \prod_{e \in C(M')} (x_e + y_e) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}. \end{aligned} \quad (6.12)$$

In particular,  $\exp[E_M(\mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M^0(-\mathbf{e}, \mathbf{d})]$  is the rank generating function for loop free minors whereas  $\exp[E_M^1(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})]$  is the corank generating function for coloop free minors.  $\square$

Our preceding lemma and the identity

$$\begin{aligned} & \left( \exp[E_M(x_e \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M^0(-x_e \mathbf{e}, \mathbf{d})] \right) \cdot \left( \exp[E_M^0(x_e \mathbf{e}, \mathbf{d})] \cdot \exp[E_M(y_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \right) \\ & = \exp[E_M(x_e \mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M(y_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})] \Leftrightarrow \\ & [1 + R_{M,\lambda,0,1,1,x_e,-x_e}(\mathbf{e}, \mathbf{d})] \cdot [1 + R_{M,0,1,1,\mu,x_e,y_e}(\mathbf{e}, \mathbf{d})] \\ & = 1 + R_{M,\lambda,1,1,\mu,x_e,y_e}(\mathbf{e}, \mathbf{d}) \end{aligned} \quad (6.13)$$

for the Etienne polynomial imply the following proposition generalizing the proposition 5.12 of [13].

**Proposition 4.** For any matroid  $M = (E, r)$  we have

$$R_M(\lambda, 1, 1, \mu, x_e, y_e) = \sum_{F \text{ flat of } M} R_{M|F}(0, 1, 1, \mu, x_e, y_e) \cdot \lambda^{r(M)-r(F)} \cdot \prod_{e \in E \setminus F} x_e, \quad (6.14)$$

where  $M|F$  denotes the restriction of  $M$  to the flat  $F$ , i.e. the deletion of its complement. In particular,

$$\left( \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \exp[-E_M^0(\mathbf{e}, \mathbf{d})] \right) \cdot \left( \exp[E_M^0(\mathbf{e}, \mathbf{d})] \cdot \exp[-E_M(\mathbf{e}, \mathbf{d})] \right) = 1 \quad (6.15)$$

reflects the fact that the zeta function of the lattice of flats of  $M$  (first factor) and its Möbius function (second factor)  $1 + \theta_{M,0}(\mathbf{e}, \mathbf{d})$  are inverse to each other.  $\square$

More generally, we have the following theorem.

**Theorem 6.** If the matroid function  $f : 3^M \rightarrow A$  does not depend on loops, then the matroid power series  $M_f(\mathbf{e}, \mathbf{d}) \cdot \exp[-E_M^0(\mathbf{e}, \mathbf{d})]$  counts  $f$  only for minors without loops; and the incidence algebra of the lattice of flats of our matroid  $M$  is isomorphic to the algebra of matroid power series vanishing on minors with loops.  $\square$

Let us define the absolute value of the Möbius function of the lattice of flats of our matroid  $M$  by

$$\begin{aligned} 1 + A_M^*(\mathbf{e}, \mathbf{d}) &= 1 + \theta_{M,0}(\mathbf{e}, -\mathbf{d}) = 1 + R_{M,0,-1}(\mathbf{e}, \mathbf{d}) \\ &= \exp[E_M^0(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d}) - E_M^0(\mathbf{e}, \mathbf{d})]. \end{aligned} \quad (6.16)$$

For graphs, it also counts the number of acyclic orientations with a fixed source (or sink) (see [30]), and for oriented matroids, it counts the number of equivalence classes of acyclic orientations, where acyclic orientations are considered to be equivalent if they can be obtained from each other by reorientations of oriented cuts. Dually, we can see with the help of the deletion contraction method or other methods that

$$\begin{aligned} 1 + S_M^*(\mathbf{e}, \mathbf{d}) &= 1 + \phi_{M,0}(-\mathbf{e}, -\mathbf{d}) = 1 + R_{M,-1,0}(\mathbf{e}, \mathbf{d}) \\ &= \exp[E_M^0(\mathbf{e}, \mathbf{d}) - E_M^1(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d})] \end{aligned} \quad (6.17)$$

counts the number of equivalence classes of strongly connected orientations, where strongly connected orientations are considered to be equivalent if they can be obtained from each other by reorientations of oriented cycles. In particular,

$$[1 + A_M^*(\mathbf{e}, \mathbf{d})] \cdot [1 + S_M^*(\mathbf{e}, \mathbf{d})] = \exp[E_M^0(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d})] \quad (6.18)$$

counts the number of bases or spanning forests.

Moreover,  $\exp[E_M(\mathbf{e}, \lambda \mathbf{d})] \cdot \exp[E_M^0(-\mathbf{e}, \mathbf{d})]$  (the rank generating function for loop free minors) and  $\exp[E_M^1(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mu \mathbf{e}, \frac{1}{\mu} \mathbf{d})]$  (the corank generating function for coloop free minors) are nonnegative. We have proved the following proposition corresponding to theorem 5.1 of [30] (see also [22, 52]), where we have shown that  $1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{d})$  counts all acyclic orientations according to their numbers of components.

**Theorem 7.** *We have*

$$\begin{aligned} 1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{d}) &= \exp[E_M(\mathbf{e}, \lambda\mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{d})] \\ &= \left( \exp[E_M(\mathbf{e}, \lambda\mathbf{d})] \cdot \exp[E_M^0(-\mathbf{e}, \mathbf{d})] \right) \cdot [1 + A_M^*(\mathbf{e}, \mathbf{d})], \end{aligned} \quad (6.19)$$

$$\begin{aligned} 1 + \phi_{M,-\mu}(-\mathbf{e}, -\mathbf{d}) &= \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[E_M(\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d})] \\ &= [1 + S_M^*(\mathbf{e}, \mathbf{d})] \cdot \left( \exp[E_M^1(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d})] \right). \end{aligned} \quad (6.20)$$

In particular, it follows that  $(-1)^{r(M)}\theta_M(-\lambda)$  is a polynomial with nonnegative coefficients in  $\lambda$  whereas  $(-1)^{|M|-r(M)}\phi_M(-\mu)$  is a polynomial with nonnegative coefficients in  $\mu$ .  $\square$

If we put  $\lambda = 1$  and  $\mu = 1$  in the preceding theorem, then we count all acyclic and strongly connected orientations, respectively (see our section on orientations).

**Corollary 18.** *The matroid power series*

$$\begin{aligned} 1 + A_M(\mathbf{e}, \mathbf{d}) &= 1 + \theta_{M,-1}(\mathbf{e}, -\mathbf{d}) = 1 + R_{M,1,-1}(\mathbf{e}, \mathbf{d}) \\ &= \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{d})] \\ &= \left( \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M^0(-\mathbf{e}, \mathbf{d})] \right) \cdot [1 + A_M^*(\mathbf{e}, \mathbf{d})] \end{aligned} \quad (6.21)$$

*counts acyclic orientation whereas*

$$\begin{aligned} 1 + S_M(\mathbf{e}, \mathbf{d}) &= 1 + \phi_{M,-1}(-\mathbf{e}, -\mathbf{d}) = 1 + R_{M,-1,1}(\mathbf{e}, \mathbf{d}) \\ &= \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] \\ &= [1 + S_M^*(\mathbf{e}, \mathbf{d})] \cdot \left( \exp[E_M^1(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] \right) \end{aligned} \quad (6.22)$$

*counts strongly connected orientations. In particular, we have the identities*

$$[1 + A_M(\mathbf{e}, \mathbf{d})] \cdot [1 + S_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(2\mathbf{e}, \mathbf{d})], \quad (6.23)$$

$$[1 + \theta_{M,\lambda}(\mathbf{e}, \mathbf{d})][1 + A_M(\mathbf{e}, \mathbf{d})] \cdot [1 + S_M(\mathbf{e}, \mathbf{d})][1 + \phi_{M,\mu}(\mathbf{e}, \mathbf{d})] = 1 + R_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d}). \quad (6.24)$$

$\square$

Gessel and Sagan [19] have counted acyclic orientations not only according to their numbers of components with respect to a variable  $x$ , but also according to their numbers of edges with respect to a variable  $y$ . Moreover, they were interested in acyclic suborientations, i.e. edges could also be deleted instead of being oriented. We can formulate their theorem 3.4 in the following way.

**Corollary 19.** *We have*

$$\begin{aligned} \exp[E_M(\mathbf{e}, x\mathbf{d})] \cdot [1 + \theta_{M,-x}(y\mathbf{e}, -\mathbf{d})] &= \exp[E_M((1+y)\mathbf{e}, x\mathbf{d})] \cdot \exp[E_M(-y\mathbf{e}, -\mathbf{d})] \\ &= 1 + R_{M, \frac{x(1+y)}{y}, \frac{-y}{1+y}}((1+y)\mathbf{e}, \frac{y}{1+y}\mathbf{d}). \quad \square \end{aligned} \quad (6.25)$$

Gessel and Sagan [19] have also counted subdigraphs according to their numbers of components with respect to a variable  $x$ , and according to their numbers of edges with respect to a variable  $y$ . Once again, edges can be deleted, but now, they can also be contracted if and only if they were oriented in both directions contributing a factor of  $y^2$ . Moreover, all edges belonging to oriented cycles can be contracted without any influence on the number of components. Therefore we can formulate theorem 4.4 of [19] in the following way.

**Corollary 20.** *We have*

$$\begin{aligned}
& \exp[E_M(\mathbf{e}, x_{\mathbf{c}}^{\mathbf{d}})] \cdot [1 + \theta_{M,-x}(\mathbf{y}\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + S_M(\mathbf{y}\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot \exp[E_M(y^2\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \\
&= \exp[E_M((1+y)\mathbf{e}, x_{\mathbf{c}}^{\mathbf{d}})] \cdot \exp[E_M((1+y)\mathbf{y}\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \\
&= 1 + R_{M, \frac{x}{y}, y}((1+y)\mathbf{e}, y_{\mathbf{c}}^{\mathbf{d}}). \quad \square
\end{aligned} \tag{6.26}$$

**Remark.** According to the preceding theorem, we know that

$$1 + \theta_{M,-\lambda}(\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}}) = \exp[E_M(\mathbf{e}, \lambda_{\mathbf{c}}^{\mathbf{d}})] \cdot \exp[E_M^1(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}}) - E_M^0(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})], \tag{6.27}$$

$$1 + \phi_{M,-\mu}(-\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}}) = \exp[E_M^0(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}}) - E_M^1(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot \exp[E_M(\mu\mathbf{e}, \frac{1}{\mu}\frac{\mathbf{d}}{\mathbf{c}})]. \tag{6.28}$$

Since  $1 + \theta_{M,-\lambda}(\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})$  vanishes on minors with loops and  $1 + \phi_{M,-\mu}(-\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})$  vanishes on minors with coloops, the products

$$1 + R_{M,\lambda,\mu}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}}) = [1 + \theta_{M,-\lambda}(\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + \phi_{M,-\mu}(-\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})], \tag{6.29}$$

$$\exp[E_M((1 + \frac{1}{\lambda})\mathbf{e}, \lambda_{\mathbf{c}}^{\mathbf{d}})] = [1 + \theta_{M,-\lambda}(\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + \phi_{M,-\frac{1}{\lambda}}(-\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})], \tag{6.30}$$

$$\exp[E_M((1 + \mu)\mathbf{e}, \frac{1}{\mu}\frac{\mathbf{d}}{\mathbf{c}})] = [1 + \theta_{M,-\frac{1}{\mu}}(\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + \phi_{M,-\mu}(-\mathbf{e}, -\frac{\mathbf{d}}{\mathbf{c}})] \tag{6.31}$$

are just sums over cyclic flats (i.e. flats without coloops).

In a similar way, Plesken, Bächler and Eberhardt [42, 14] have introduced and studied what they called the *cloud polynomial*  $C_M(\lambda)$  and *flock polynomial*  $F_M(\mu)$  of a matroid  $M$ . We can easily define them with the help of matroid power series :

$$\begin{aligned}
1 + C_{M,\lambda}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}}) &= 1 + \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} C_{M \setminus D' / C'}(\lambda) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \\
&:= \exp[E_M(\mathbf{e}, \lambda_{\mathbf{c}}^{\mathbf{d}})] \cdot \exp[-E_M^0(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot \exp[E_M^1(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})],
\end{aligned} \tag{6.32}$$

$$\begin{aligned}
1 + F_{M,\mu}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}}) &= 1 + \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} F_{M \setminus D' / C'}(\mu) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \\
&:= \exp[-E_M^1(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot \exp[E_M^0(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot \exp[E_M(\mu\mathbf{e}, \frac{1}{\mu}\frac{\mathbf{d}}{\mathbf{c}})].
\end{aligned} \tag{6.33}$$

Since those two polynomials are products of three factors, they cannot be evaluated with the help of the Whitney (or Tutte) polynomial. Nevertheless, the cloud polynomial vanishes on minors with loops and the flock polynomial vanishes on minors with coloops. Therefore the products

$$1 + R_{M,\lambda,\mu}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}}) = [1 + C_{M,\lambda}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + F_{M,\mu}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})], \tag{6.34}$$

$$\exp[E_M((1 + \frac{1}{\lambda})\mathbf{e}, \lambda_{\mathbf{c}}^{\mathbf{d}})] = [1 + C_{M,\lambda}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + F_{M,\frac{1}{\lambda}}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})], \tag{6.35}$$

$$\exp[E_M((1 + \mu)\mathbf{e}, \frac{1}{\mu}\frac{\mathbf{d}}{\mathbf{c}})] = [1 + C_{M,\frac{1}{\mu}}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \cdot [1 + F_{M,\mu}(\mathbf{e}, \frac{\mathbf{d}}{\mathbf{c}})] \tag{6.36}$$

are just sums over cyclic flats and were used by Eberhardt [14] to show that the cloud, flock, Whitney and Tutte polynomials all depend only on the lattice of cyclic flats. All those formulae

can be dualized :

$$1 + F_{M,\lambda}^*(\mathbf{e}, \mathbf{d}, \mathbf{c}) = \exp[E_M(\mathbf{e}, \lambda \mathbf{d}, \mathbf{c})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[-E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c})], \quad (6.37)$$

$$1 + C_{M,\mu}^*(\mathbf{e}, \mathbf{d}, \mathbf{c}) = \exp[E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[-E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M(\mu \mathbf{e}, \frac{1}{\mu} \mathbf{d}, \mathbf{c})], \quad (6.38)$$

$$1 + R_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d}, \mathbf{c}) = [1 + F_{M,\lambda}^*(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot [1 + C_{M,\mu}^*(\mathbf{e}, \mathbf{d}, \mathbf{c})], \quad (6.39)$$

$$\exp[E_M((1 + \frac{1}{\lambda})\mathbf{e}, \lambda \mathbf{d}, \mathbf{c})] = [1 + F_{M,\lambda}^*(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot [1 + C_{M,\frac{1}{\lambda}}^*(\mathbf{e}, \mathbf{d}, \mathbf{c})], \quad (6.40)$$

$$\exp[E_M((1 + \mu)\mathbf{e}, \frac{1}{\mu} \mathbf{d}, \mathbf{c})] = [1 + F_{M,\frac{1}{\mu}}^*(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot [1 + C_{M,\mu}^*(\mathbf{e}, \mathbf{d}, \mathbf{c})]. \quad (6.41)$$

In particular, the multiplication of  $\exp[E_M(\mathbf{e}, \lambda \mathbf{d}, \mathbf{c})]$  with

$$\exp[E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c}) - E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c})], \quad \exp[-E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})], \quad \exp[E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[-E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c})]$$

gives three different results, but all of them vanish on minors with loops.

Products of more than two factors also appear naturally in the context of random minors. For each  $e \in E$  of our matroid  $M$ , let us delete  $e$  with probability  $p_e$  and let us contract  $e$  with probability  $q_e$ . Then  $e$  remains in our random minor of  $M$  with probability  $r_e$  such that  $p_e + q_e + r_e = 1$  for every  $e \in E$ . Let  $R_{M[(p)_e, (q)_e, (r)_e]}(\lambda, \mu)$  be the expected value of the Whitney polynomial of our random minor of  $M$  and let

$$1 + R_{M[(p)_e, (q)_e, (r)_e], \lambda, \mu}(\mathbf{e}, \mathbf{d}, \mathbf{c}) := 1 + \sum_{\emptyset \subset E' \cup D' \cup C' = E} R_{M \setminus D' / C' [(p)_e, (q)_e, (r)_e]}(\lambda, \mu) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \quad (6.42)$$

be the matroid power series of those expected values for all minors of  $M$ .

**Proposition 5.** *We have*

$$\begin{aligned} 1 + R_{M[(p)_e, (q)_e, (r)_e], \lambda, \mu}(\mathbf{e}, \mathbf{d}, \mathbf{c}) &= \exp[E_M(p_e \mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \\ &\exp[E_M(r_e \mathbf{e}, \lambda \mathbf{d}, \mathbf{c})] \cdot \exp[E_M(r_e \mu \mathbf{e}, \frac{1}{\mu} \mathbf{d}, \mathbf{c})] \cdot \\ &\exp[E_M(q_e \mathbf{e}, \mathbf{d}, \mathbf{c})]. \quad \square \end{aligned} \quad (6.43)$$

Let us suppose that  $p_e = p$ ,  $q_e = q$  and  $r_e = r$  for all  $e \in E$ . Then we can evaluate those expected values with the help of the classical Whitney (or Tutte) polynomial in some special cases allowing to reduce our product of four factors to two factors, see our equation (5.19).

**Corollary 21.** *For  $q = 0$  the expected value of the flow polynomial is given by*

$$\begin{aligned} \exp[E_M(p\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot [1 + \phi_{M,\mu}(r\mathbf{e}, \mathbf{d}, \mathbf{c})] &= \exp[E_M((p-r)\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M(r\mu\mathbf{e}, \frac{1}{\mu}\mathbf{d}, \mathbf{c})] \\ &= 1 + R_{M, \frac{p-r}{r}, \frac{r\mu}{p-r}}((p-r)\mathbf{e}, \frac{r}{p-r}\mathbf{d}, \mathbf{c}), \end{aligned} \quad (6.44)$$

the expected number of acyclic orientations is given by

$$\begin{aligned} \exp[E_M(p\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot [1 + A_M(r\mathbf{e}, \mathbf{d}, \mathbf{c})] &= \exp[E_M(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M(-r\mathbf{e}, -\mathbf{d})] \\ &= 1 + R_{M, \frac{1}{r}, -r}(\mathbf{e}, r\mathbf{d}, \mathbf{c}) \end{aligned} \quad (6.45)$$

and the expected number of independent sets is given by

$$\begin{aligned} \exp[E_M(p\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M(r\mathbf{e}, \mathbf{d}, \mathbf{c})] \exp[E_M^1(r\mathbf{e}, \mathbf{d}, \mathbf{c})] &= \exp[E_M(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot \exp[E_M^1(\mathbf{e}, r\mathbf{d}, \mathbf{c})] \\ &= 1 + R_{M, \frac{1}{r}, 0}(\mathbf{e}, r\mathbf{d}, \mathbf{c}). \quad \square \end{aligned} \quad (6.46)$$



In the case of graphs, the results of the preceding corollary can be found in [54]. For matroids, however, it is natural to dualize them as well.

**Corollary 22.** *For  $p = 0$  the expected value of the tension polynomial is given by*

$$\begin{aligned} [1 + \theta_{M,\lambda}(r\mathbf{e}, \mathbf{d}_c)] \cdot \exp[E_M(q\mathbf{e}, \mathbf{d}_c)] &= \exp[E_M(r\mathbf{e}, \lambda\mathbf{d}_c)] \cdot \exp[E_M((q-r)\mathbf{e}, \mathbf{d}_c)] \\ &= 1 + R_{M, \frac{r\lambda}{q-r}, \frac{q-r}{r}}(r\mathbf{e}, \frac{q-r}{r}\mathbf{d}_c), \end{aligned} \quad (6.47)$$

the expected number of strongly connected orientations is given by

$$\begin{aligned} [1 + S_M(r\mathbf{e}, \mathbf{d}_c)] \cdot \exp[E_M(q\mathbf{e}, \mathbf{d}_c)] &= \exp[E_M(r\mathbf{e}, -\mathbf{d}_c)] \cdot \exp[E_M(\mathbf{e}, \mathbf{d}_c)] \\ &= 1 + R_{M, -r, \frac{1}{r}}(r\mathbf{e}, \frac{1}{r}\mathbf{d}_c) \end{aligned} \quad (6.48)$$

and the expected number of coindependent sets is given by

$$\begin{aligned} \exp[E_M^0(r\mathbf{e}, \mathbf{d}_c)] \exp[E_M(r\mathbf{e}, \mathbf{d}_c)] \cdot \exp[E_M(q\mathbf{e}, \mathbf{d}_c)] &= \exp[E_M^0(r\mathbf{e}, \frac{1}{r}\mathbf{d}_c)] \cdot \exp[E_M(\mathbf{e}, \mathbf{d}_c)] \\ &= 1 + R_{M, 0, \frac{1}{r}}(r\mathbf{e}, \frac{1}{r}\mathbf{d}_c). \quad \square \end{aligned} \quad (6.49)$$

**Remark.** For  $q = 0$  we cannot simplify

$$\exp[E_M(p\mathbf{e}, \mathbf{d}_c)] \cdot [1 + \theta_{M,\lambda}(r\mathbf{e}, \mathbf{d}_c)] \quad (6.50)$$

but we can simplify

$$\begin{aligned} \exp[E_M(p\mathbf{e}, \lambda\mathbf{d}_c)] \cdot [1 + \theta_{M,\lambda}(r\mathbf{e}, \mathbf{d}_c)] &= \exp[E_M(\mathbf{e}, \lambda\mathbf{d}_c)] \cdot \exp[E_M(-r\mathbf{e}, \mathbf{d}_c)] \\ &= 1 + R_{M, \frac{\lambda}{-r}, -r}(\mathbf{e}, -r\mathbf{d}_c), \end{aligned} \quad (6.51)$$

see [54]. Moreover, for  $q = 0$

$$\exp[E_M^0(p\mathbf{e}, \mathbf{d}_c)] \cdot \exp[E_M(r\mathbf{e}, \mathbf{d}_c)] = 1 + R_{M, 0, \frac{r}{p}}(p\mathbf{e}, \frac{r}{p}\mathbf{d}_c) \quad (6.52)$$

denotes the probability that our random minor is coindependent (it is the all terminal reliability, see [56]) whereas

$$\exp[E_M(p\mathbf{e}, P\mathbf{d}_c)] \cdot \exp[E_M(r\mathbf{e}, \mathbf{d}_c)] = 1 + R_{M, \frac{pP}{r}, \frac{r}{p}}(p\mathbf{e}, \frac{r}{p}\mathbf{d}_c) \quad (6.53)$$

is essentially the partition function of the random cluster model, see [56].

## 7 Derivations

Let  $f : 3^M \rightarrow A$  be a matroid function and let

$$M_f(\mathbf{e}, \mathbf{d}_c) = \sum_{E' \uplus D' \uplus C' = E} f(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'} \quad (7.1)$$

be its matroid power series, where the sum is taken over all ordered partitions of  $E$  (empty sets are allowed everywhere). For every  $e \in E$ , we can define the partial derivative  $\partial^e$  by

$$\partial^e M_f(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{e \in E' \uplus D' \uplus C' = E} f(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}. \quad (7.2)$$

In other words,  $\partial^e$  leaves the value of  $f(M \setminus D' / C')$  unchanged if  $e \in E'$  where  $E' = E \setminus (D' \cup C')$ , but it replaces the value of  $f(M \setminus D' / C')$  by 0 if  $e \notin E'$ .

**Remark.** More generally, for any  $E'' \subseteq E$  we can define  $\partial^{E''}$ : it leaves the value of  $f(M \setminus D' / C')$  unchanged if  $E'' \subseteq E'$ , but it replaces the value of  $f(M \setminus D' / C')$  by 0 otherwise.

Of course,  $\partial^e$  is a linear operator acting on matroid functions and matroid power series. For deriving a product of matroid power series, we have the following lemma, well known from set functions [29, 30, 32].

**Lemma 7. Product Rule.** *For any matroid functions  $f, g : 3^M \rightarrow A$  we have*

$$\partial^e [M_f(\mathbf{e}, \mathbf{d}, \mathbf{c}) \cdot M_g(\mathbf{e}, \mathbf{d}, \mathbf{c})] = [\partial^e M_f(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot M_g(\mathbf{e}, \mathbf{d}, \mathbf{c}) + M_f(\mathbf{e}, \mathbf{d}, \mathbf{c}) \cdot [\partial^e M_g(\mathbf{e}, \mathbf{d}, \mathbf{c})]. \quad (7.3)$$

*Proof.* The lemma immediately follows from the fact that for any disjoint subsets  $E', E'' \subseteq E$  we have

$$e \in E' \uplus E'' \quad \Leftrightarrow \quad e \in E' \quad \text{or} \quad e \in E''. \quad (7.4)$$

□

**Remark.** The multiplication of matroid power series is not commutative, and in general, it is not true that

$$[\partial^e M_f(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot M_f(\mathbf{e}, \mathbf{d}, \mathbf{c}) = M_f(\mathbf{e}, \mathbf{d}, \mathbf{c}) \cdot [\partial^e M_f(\mathbf{e}, \mathbf{d}, \mathbf{c})]. \quad (7.5)$$

However, if our matroid function  $f : 3^M \rightarrow A$  depends only on its support, that is  $f(M \setminus D' / C') = f(M \setminus D'' / C'')$  for all  $D' \uplus C' = D'' \uplus C'' = E \setminus E'$ , then (7.5) is obviously true, and remains true if  $(\mathbf{e}, \mathbf{d}, \mathbf{c})$  is replaced by  $(x_e \mathbf{e}, z_e \mathbf{d})$  everywhere. In particular, we have

$$\begin{aligned} \partial^e \exp[E_M(x_e \mathbf{e}, z_e \mathbf{d})] &= [\partial^e E_M(x_e \mathbf{e}, z_e \mathbf{d})] \cdot \exp[E_M(x_e \mathbf{e}, z_e \mathbf{d})] \\ &= \exp[E_M(x_e \mathbf{e}, z_e \mathbf{d})] \cdot [\partial^e E_M(x_e \mathbf{e}, z_e \mathbf{d})]. \end{aligned} \quad (7.6)$$

First of all, our element derivation  $\partial^e$  allows us to prove the following recurrence relations for the Etienne polynomial.

**Proposition 6.** *For any  $e \in E$ , we have*

$$R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) = [(x_1)_e \mu_1 + (x_2)_e \mu_2] R_{M \setminus e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \quad (7.7)$$

*if  $e$  is a loop,*

$$R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) = [(x_1)_e \lambda_1 + (x_2)_e \lambda_2] R_{M/e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \quad (7.8)$$

*if  $e$  is a coloop (also called isthmus or bridge) and*

$$\begin{aligned} R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) &= (x_1)_e \mu_1 R_{M \setminus e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) + \\ &\quad (x_2)_e \lambda_2 R_{M/e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \end{aligned} \quad (7.9)$$

*otherwise.*

*Proof.* We have

$$\begin{aligned}
& \partial^e [1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d})] \\
&= [\partial^e \exp[E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})]] \cdot \exp[E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})] + \\
& \quad \exp[E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \cdot [\partial^e \exp[E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})]] \\
&= [\partial^e E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \cdot [1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d})] + \\
& \quad [1 + R_{M, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d})] \cdot [\partial^e E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})]. \tag{7.10}
\end{aligned}$$

The factor  $[\partial^e E_M((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})]$  is nonzero only on one element minors with support  $e$ . If we contract all other elements, then it has rank 0 providing a multiplication by  $(x_1)_e \mu_1$ , unless  $e$  is a coloop of rank 1 providing a multiplication by  $(x_1)_e \lambda_1$ . In any case,  $e$  must be deleted for the second factor, but if  $e$  is a coloop, it can equivalently be contracted.

The factor  $[\partial^e E_M((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})]$  is nonzero only on one element minors with support  $e$ . If we delete all other elements, then it has rank 1 providing a multiplication by  $(x_2)_e \lambda_2$ , unless  $e$  is a loop of rank 0 providing a multiplication by  $(x_2)_e \mu_2$ . In any case,  $e$  must be contracted for the first factor, but if  $e$  is a loop, it can equivalently be deleted.  $\square$

**Remark.** For every  $e \in E$ , let us define the coefficients of the preceding proposition by

$$x_e := (x_2)_e \lambda_2, \quad y_e := (x_1)_e \mu_1, \tag{7.11}$$

$$X_e := (x_1)_e \lambda_1 + (x_2)_e \lambda_2, \quad Y_e := (x_1)_e \mu_1 + (x_2)_e \mu_2. \tag{7.12}$$

According to Bollobás and Riordan [7] the following determinants have to be equal for all  $e, f \in E$ :

$$\begin{vmatrix} x_e & y_e \\ x_f & y_f \end{vmatrix} = \begin{vmatrix} x_e & Y_e \\ x_f & Y_f \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} x_e & y_e \\ x_f & y_f \end{vmatrix} = \begin{vmatrix} X_e & y_e \\ X_f & y_f \end{vmatrix}. \tag{7.13}$$

The first of those two equations is valid for all  $e, f \in E$  if and only if either the vector  $x_e$  ( $e \in E$ ) is zero (this happens for  $\lambda_2 = 0$ ) or the difference of the vectors  $Y_e$  ( $e \in E$ ) and  $y_e$  ( $e \in E$ ) is a multiple of the vector  $x_e$  ( $e \in E$ ) (this happens for  $\lambda_2 \neq 0$ ). The second of those two equations is valid for all  $e, f \in E$  if and only if either the vector  $y_e$  ( $e \in E$ ) is zero (this happens for  $\mu_1 = 0$ ) or the difference of the vectors  $X_e$  ( $e \in E$ ) and  $x_e$  ( $e \in E$ ) is a multiple of the vector  $y_e$  ( $e \in E$ ) (this happens for  $\mu_1 \neq 0$ ).

On the other hand, our coefficients  $x_e, y_e, X_e, Y_e$  are of the most general form satisfying the equations (7.13) even if  $\lambda_2 = \mu_1 = 1$  (Zaslavsky's normal functions [62]) or  $\lambda_2 = 0$  et  $\mu_1 = 1$  (Zaslavsky's primal elementary functions [62]) or  $\lambda_2 = 1$  et  $\mu_1 = 0$  (Zaslavsky's dual elementary functions [62]) or  $\lambda_2 = \mu_1 = 0$  (Zaslavsky's paranil functions [62]).

Therefore our Etienne polynomial provides a natural common generalization of all those families of solutions.

**Corollary 23.** For any  $e \in E$ , we have

$$\begin{aligned} & \frac{\partial}{\partial(x_1)_e} R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \\ &= \begin{cases} \lambda_1 R_{M \setminus e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) & \text{if } e \text{ is a coloop,} \\ \mu_1 R_{M \setminus e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) & \text{otherwise,} \end{cases} \end{aligned} \quad (7.14)$$

$$\begin{aligned} & \frac{\partial}{\partial(x_2)_e} R_M(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \\ &= \begin{cases} \mu_2 R_{M/e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) & \text{if } e \text{ is a loop,} \\ \lambda_2 R_{M/e}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) & \text{otherwise.} \quad \square \end{cases} \end{aligned} \quad (7.15)$$

**Remark.** It is also possible to use matroid power series with different supports. For example, if  $M = M_1 + M_2$  is a direct sum, then

$$\begin{aligned} & 1 + R_{M_1+M_2, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d}) \\ &= \exp[E_{M_1}((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}}) + E_{M_2}((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \cdot \\ & \quad \exp[E_{M_1}((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}}) + E_{M_2}((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})] \\ &= \exp[E_{M_1}((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \exp[E_{M_1}((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})] \cdot \\ & \quad \exp[E_{M_2}((x_1)_e \mu_1 \mathbf{e}, \frac{\lambda_1 \mathbf{d}}{\mu_1 \mathbf{c}})] \exp[E_{M_2}((x_2)_e \mu_2 \mathbf{e}, \frac{\lambda_2 \mathbf{d}}{\mu_2 \mathbf{c}})] \\ &= [1 + R_{M_1, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d})] \cdot [1 + R_{M_2, \lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e}(\mathbf{e}, \mathbf{d})]. \end{aligned} \quad (7.16)$$

Therefore

$$\begin{aligned} & R_{M_1+M_2}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \\ &= R_{M_1}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e) \cdot R_{M_2}(\lambda_1, \lambda_2, \mu_1, \mu_2, (x_1)_e, (x_2)_e). \end{aligned} \quad (7.17)$$

If  $x_e = y_e = 1$  for every  $e \in E$  and  $\lambda_2 = \mu_1 = 1$ , then our preceding proposition simplifies to the classical recurrence relations for the Whitney polynomial (see [55], chapter 15.4).

**Corollary 24.** For any  $e \in E$ , we have

$$R_M(\lambda, \mu) = [1 + \mu] R_{M \setminus e}(\lambda, \mu) \quad \text{if } e \text{ is a loop,} \quad (7.18)$$

$$R_M(\lambda, \mu) = [\lambda + 1] R_{M/e}(\lambda, \mu) \quad \text{if } e \text{ is a coloop,} \quad (7.19)$$

$$R_M(\lambda, \mu) = R_{M \setminus e}(\lambda, \mu) + R_{M/e}(\lambda, \mu) \quad \text{otherwise.} \quad (7.20)$$

In particular,  $R_M(\lambda, \mu)$  is a polynomial with nonnegative coefficients in  $\lambda + 1$  and  $\mu + 1$ ,  $(-1)^{r(M)} \theta_M(-\lambda) = R_M(\lambda, -1)$  is a polynomial with nonnegative coefficients in  $\lambda + 1$  and  $(-1)^{|M|-r(M)} \phi_M(-\mu) = R_M(-1, \mu)$  is a polynomial with nonnegative coefficients in  $\mu + 1$ .  $\square$

**Remark.** For all  $e, f \in E$ ,  $\partial^e E_M(\mathbf{e}, \mathbf{d})$  and  $\partial^f E_M(\mathbf{e}, \mathbf{d})$  commute because both are set functions. By substitution, this implies the commutativity of  $\partial^e E_M((x)_e \mu \mathbf{e}, \frac{\lambda \mathbf{d}}{\mu \mathbf{c}})$  and  $\partial^f E_M((x)_e \mu \mathbf{e}, \frac{\lambda \mathbf{d}}{\mu \mathbf{c}})$ . In particular,  $\partial^e E_M^0(\mathbf{e}, \mathbf{d})$  and  $\partial^f E_M^0(\mathbf{e}, \mathbf{d})$  as well as  $\partial^e E_M^1(\mathbf{e}, \mathbf{d})$  and  $\partial^f E_M^1(\mathbf{e}, \mathbf{d})$  commute with each other, as can also be seen by direct verification on all two element minors.  $\partial^e E_M^0(\mathbf{e}, \mathbf{d})$

and  $\partial^f E_M^1(\mathbf{e}, \mathbf{d})$ , however, do not commute, as can be seen on the only connected two element matroid, a circle of length two.

With the help of the deleting contraction method we have already seen that the coefficients of the *Tutte polynomial*  $T_M(\lambda, \mu) = R_M(\lambda - 1, \mu - 1)$  are nonnegative. In particular, this is true for *Crapo's beta invariants* (see [11] and [55], chapter 15.4)

$$\beta^1(M) := \frac{\partial}{\partial \lambda} T_M(\lambda, \mu) \Big|_{\lambda=\mu=0}, \quad \beta^0(M) := \frac{\partial}{\partial \mu} T_M(\lambda, \mu) \Big|_{\lambda=\mu=0}. \quad (7.21)$$

We will study them in detail in the next section (in [20],  $\beta^1(M) = \beta(M)$  and  $\beta^0(M) = \beta^*(M)$ ). In this section, we show that their nonnegativity implies the nonnegativity of the coefficients of the Tutte polynomial. For this purpose, we use the matroid power series

$$T_{M, \lambda, \mu}(\mathbf{e}, \mathbf{d}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} T_{M \setminus D' / C'}(\lambda, \mu) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (7.22)$$

$$\beta_M^1(\mathbf{e}, \mathbf{d}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} \beta^1(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (7.23)$$

$$\beta_M^0(\mathbf{e}, \mathbf{d}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} \beta^0(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (7.24)$$

and get the following theorem proved in [29].

**Theorem 8.** *We have*

$$1 + T_{M, \lambda, \mu}(\mathbf{e}, \mathbf{d}) = \exp[\lambda E_M^1(\mathbf{e}, \mathbf{d}) + E_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[\mu E_M^0(\mathbf{e}, \mathbf{d}) - E_M(\mathbf{e}, -\mathbf{d})]. \quad (7.25)$$

Moreover, for every  $e \in E$  we have

$$\begin{aligned} \partial^e T_{M, \lambda, \mu}(\mathbf{e}, \mathbf{d}) &= \exp[\lambda E_M^1(\mathbf{e}, \mathbf{d}) + E_M(\mathbf{e}, -\mathbf{d})] \cdot \\ &\quad [\lambda \partial^e E_M^1(\mathbf{e}, \mathbf{d}) + \mu \partial^e E_M^0(\mathbf{e}, \mathbf{d})] \cdot \\ &\quad \exp[\mu E_M^0(\mathbf{e}, \mathbf{d}) - E_M(\mathbf{e}, -\mathbf{d})], \end{aligned} \quad (7.26)$$

$$\partial^e \beta_M^1(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot [\partial^e E_M^1(\mathbf{e}, \mathbf{d})] \cdot \exp[-E_M(\mathbf{e}, -\mathbf{d})], \quad (7.27)$$

$$\partial^e \beta_M^0(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot [\partial^e E_M^0(\mathbf{e}, \mathbf{d})] \cdot \exp[-E_M(\mathbf{e}, -\mathbf{d})]. \quad \square \quad (7.28)$$

**Remark.** We have already remarked that for all  $e, f \in E$ ,  $\partial^e E_M^0(\mathbf{e}, \mathbf{d})$  and  $\partial^f E_M^0(\mathbf{e}, \mathbf{d})$  as well as  $\partial^e E_M^1(\mathbf{e}, \mathbf{d})$  and  $\partial^f E_M^1(\mathbf{e}, \mathbf{d})$  commute with each other. Therefore, this is also true for  $\partial^e \beta_M^0(\mathbf{e}, \mathbf{d})$  and  $\partial^f \beta_M^0(\mathbf{e}, \mathbf{d})$  as well as for  $\partial^e \beta_M^1(\mathbf{e}, \mathbf{d})$  and  $\partial^f \beta_M^1(\mathbf{e}, \mathbf{d})$ .

**Corollary 25.** *For every  $e \in E$  we have*

$$\exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \partial^e \beta_M^1(-\mathbf{e}, -\mathbf{d}) = \partial^e E_M^1(\mathbf{e}, \mathbf{d}) \cdot \exp[E_M(\mathbf{e}, \mathbf{d})], \quad (7.29)$$

$$-\partial^e \beta_M^1(\mathbf{e}, -\mathbf{d}) \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \partial^e E_M^1(\mathbf{e}, \mathbf{d}), \quad (7.30)$$

$$-\exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \partial^e \beta_M^0(-\mathbf{e}, -\mathbf{d}) = \partial^e E_M^0(\mathbf{e}, \mathbf{d}) \cdot \exp[E_M(\mathbf{e}, \mathbf{d})], \quad (7.31)$$

$$\partial^e \beta_M^0(\mathbf{e}, -\mathbf{d}) \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \partial^e E_M^0(\mathbf{e}, \mathbf{d}). \quad \square \quad (7.32)$$

Let  $w : E \rightarrow A$  be a fixed weight function and let

$$\partial^w := \sum_{e \in E} w(e) \partial^e \quad (7.33)$$

be a linear combination of our partial derivatives  $\partial^e$ . We have

$$\begin{aligned} \partial^w \exp[\mu E_M^0(\mathbf{e}, \mathbf{d}) - E_M(\mathbf{e}, -\mathbf{d})] &= [\mu \partial^w E_M^0(\mathbf{e}, \mathbf{d}) + \partial^w E_M^1(\mathbf{e}, \mathbf{d}) - \partial^w E_M^0(\mathbf{e}, \mathbf{d})] \cdot \\ &\quad \exp[\mu E_M^0(\mathbf{e}, \mathbf{d}) - E_M(\mathbf{e}, -\mathbf{d})] \end{aligned} \quad (7.34)$$

proving the following corollary.

**Corollary 26.** *We have*

$$\begin{aligned} \partial^w T_{M, \lambda, \mu}(\mathbf{e}, \mathbf{d}) &= \exp[\lambda E_M^1(\mathbf{e}, \mathbf{d}) + E_M(\mathbf{e}, -\mathbf{d})] \cdot \\ &\quad [(\lambda + \mu - \lambda \mu) \partial^w E_M^0(\mathbf{e}, \mathbf{d}) + \lambda \partial^w] \cdot \\ &\quad \exp[\mu E_M^0(\mathbf{e}, \mathbf{d}) - E_M(\mathbf{e}, -\mathbf{d})], \end{aligned} \quad (7.35)$$

$$\begin{aligned} \partial^w \theta_{M, -\lambda}(\mathbf{e}, -\mathbf{d}) &= \exp[E_M(\mathbf{e}, \lambda \mathbf{d})] \cdot \\ &\quad [(\lambda + 1)(\partial^w E_M^0(\mathbf{e}, \mathbf{d}) + \partial^w)] \cdot \\ &\quad \exp[E_M(-\mathbf{e}, -\mathbf{d})]. \end{aligned} \quad (7.36)$$

The first identity of our preceding corollary can be written down without using matroid power series in the following way

$$w(E) T_M(\lambda, \mu) = \sum_{E' \subseteq E} (\lambda - 1)^{r(E) - r(E')} (\mu - 1)^{|E'| - r(E')} [(\lambda + \mu - \lambda \mu) w(\overline{E'} \setminus E') + \lambda w(E')], \quad (7.37)$$

where  $\overline{E'}$  is the closure of  $E'$  and

$$w(E') = \sum_{e \in E'} w(e) \quad (7.38)$$

its weight. This identity (generalizing results from [35, 36, 37]) is the most general formula which Massey, Simion, Stanley, Vertigan, Welsh and Ziegler obtained in [38] (and proved on more than two pages).

The Möbius inversion identities

$$\left( \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[-E_M^0(\mathbf{e}, \mathbf{d})] \right) \cdot [1 + A_M^*(\mathbf{e}, \mathbf{d})] = 1, \quad (7.39)$$

$$[1 + S_M^*(\mathbf{e}, \mathbf{d})] \cdot \left( \exp[-E_M^1(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{d})] \right) = 1 \quad (7.40)$$

inspire the following corollary.

**Corollary 27.** *For every  $e \in E$  we have*

$$\left( \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[-E_M^0(\mathbf{e}, \mathbf{d})] \right) \cdot \partial^e A_M^*(\mathbf{e}, \mathbf{d}) = \partial^e \beta_M^1(\mathbf{e}, \mathbf{d}), \quad (7.41)$$

$$\partial^e S_M^*(\mathbf{e}, \mathbf{d}) \cdot \left( \exp[-E_M^1(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{d})] \right) = \partial^e \beta_M^0(\mathbf{e}, \mathbf{d}). \quad \square \quad (7.42)$$

We can also derive our tension polynomial (or characteristic polynomial)

$$\begin{aligned}
\partial^e[1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{c})] &= \exp[E_M(\mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \partial^e[E_M(\mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}}) + E_M(-\mathbf{e}, -\mathbf{c})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{c})] \\
&= \exp[E_M(\mathbf{e}, \lambda \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{c})] \cdot \\
&\quad \exp[E_M(\mathbf{e}, -\mathbf{c})] \cdot (\lambda + 1) \partial^e E_M^1(\mathbf{e}, \mathbf{c}) \cdot \exp[E_M(-\mathbf{e}, -\mathbf{c})] \\
&= [1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{c})] \cdot (\lambda + 1) \partial^e \beta_M^1(\mathbf{e}, \mathbf{c})
\end{aligned} \tag{7.43}$$

proving the following theorem established in [29] and [30], theorem 5.2.

**Theorem 9. Differential Equations for Tensions and Flows.** *For every  $e \in E$  we have*

$$\partial^e[1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{c})] = [1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{c})] \cdot (\lambda + 1) \partial^e \beta_M^1(\mathbf{e}, \mathbf{c}), \tag{7.44}$$

$$\partial^e[1 + \phi_{M,-\mu}(-\mathbf{e}, -\mathbf{c})] = (\mu + 1) \partial^e \beta_M^0(\mathbf{e}, \mathbf{c}) \cdot [1 + \phi_{M,-\mu}(-\mathbf{e}, -\mathbf{c})]. \tag{7.45}$$

*In particular, for  $\lambda = 0$  and  $\mu = 0$  we have*

$$\partial^e[1 + A_M^*(\mathbf{e}, \mathbf{c})] = [1 + A_M^*(\mathbf{e}, \mathbf{c})] \cdot \partial^e \beta_M^1(\mathbf{e}, \mathbf{c}), \tag{7.46}$$

$$\partial^e[1 + S_M^*(\mathbf{e}, \mathbf{c})] = \partial^e \beta_M^0(\mathbf{e}, \mathbf{c}) \cdot [1 + S_M^*(\mathbf{e}, \mathbf{c})]. \quad \square \tag{7.47}$$

Let  $f : 3^M \rightarrow A$  be a matroid function and let  $M_f(\mathbf{e}, \mathbf{c})$  be its matroid power series. If  $M_f(0\mathbf{e}, \mathbf{c}) = M_f(0, \mathbf{c})$  is equal to 0, that is  $f$  vanishes on all minors with empty support, then  $[M_f(\mathbf{e}, \mathbf{c})]^k$  is a sum over set partitions into  $k$  nonempty blocks of support. There are  $k!$  ways to choose the order of multiplication among them, and if, for reasons of commutativity, they all give the same result, then  $[M_f(\mathbf{e}, \mathbf{c})]^k/k!$  is well defined for any commutative ring  $A$ , and instead of calculating this average, we can also calculate one specific product of the  $k!$  possible ones. If all elements of  $E$  are totally ordered, let us denote by  $\leftarrow$  (respectively  $\rightarrow$ ) the choice where the minimal elements of the  $k$  blocks of support are increasing from right to left (respectively left to right). If we sum over  $k$ , we get the operators

$$\underleftarrow{\exp}[M_f(\mathbf{e}, \mathbf{c})] \quad \text{and} \quad \underrightarrow{\exp}[M_f(\mathbf{e}, \mathbf{c})] \tag{7.48}$$

(which coincide with  $\exp$  in the case of commutativity, see [29, 30, 32]). They allow us to solve the differential equations of the preceding theorem (see [29] and [30], theorem 5.2).

**Theorem 10.** *We have*

$$1 + \theta_{M,-\lambda}(\mathbf{e}, -\mathbf{c}) = \underleftarrow{\exp}[(\lambda + 1) \beta_M^1(\mathbf{e}, \mathbf{c})], \tag{7.49}$$

$$1 + \phi_{M,-\mu}(-\mathbf{e}, -\mathbf{c}) = \underrightarrow{\exp}[(\mu + 1) \beta_M^0(\mathbf{e}, \mathbf{c})], \tag{7.50}$$

$$1 + A_M^*(\mathbf{e}, \mathbf{c}) = \underleftarrow{\exp}[\beta_M^1(\mathbf{e}, \mathbf{c})], \tag{7.51}$$

$$1 + S_M^*(\mathbf{e}, \mathbf{c}) = \underrightarrow{\exp}[\beta_M^0(\mathbf{e}, \mathbf{c})]. \tag{7.52}$$

*In particular, it follows that  $(-1)^{r(M)} \theta_M(-\lambda)$  is a polynomial with nonnegative coefficients in  $\lambda + 1$  whereas  $(-1)^{|M|-r(M)} \phi_M(-\mu)$  is a polynomial with nonnegative coefficients in  $\mu + 1$ .  $\square$*

Using the identity (5.29) we get theorem 3.13 of [20] proved there in beautiful and interesting bijective ways.

**Theorem 11.** (*Gioan-Las Vergnas*) For the Tutte polynomial we have

$$1 + T_{M,\lambda,\mu}(\mathbf{e}, \mathbf{d}, \mathbf{c}) = \underbrace{\exp[\lambda \cdot \beta_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})]}_{\leftarrow} \cdot \underbrace{\exp[\mu \cdot \beta_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c})]}_{\rightarrow}. \quad \square \quad (7.53)$$

Motivated by geometry and inequalities, the following 4-variable transformation of the Tutte polynomial was introduced in [4] :

$$t_M(x, y, z, w) := \frac{(y+z)^{r(M)}(x+w)^{|M|-r(M)}}{x+y} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right). \quad (7.54)$$

If  $M$  is the empty matroid, however,  $t_M(x, y, z, w) = \frac{1}{x+y}$ . Therefore, it is natural to introduce

$$\begin{aligned} T_M(x, y, z, w) &:= (y+z)^{r(M)}(x+w)^{|M|-r(M)} T_M\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right) \\ &= (x+y)t_M(x, y, z, w) \end{aligned} \quad (7.55)$$

as well as their matroid power series

$$t_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} t_{M \setminus D'/C'}(x, y, z, w) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (7.56)$$

$$T_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} T_{M \setminus D'/C'}(x, y, z, w) \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}. \quad (7.57)$$

Since  $T_M(\lambda, \mu) = R_M(\lambda - 1, \mu - 1)$ , we have

$$\begin{aligned} 1 + T_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}, \mathbf{c}) &= 1 + T_{M, \frac{x+y}{y+z}, \frac{x+y}{x+w}}((x+w)\mathbf{e}, \frac{y+z}{x+w}\mathbf{d}) \\ &= \exp[E_M((x+w)\mathbf{e}, \frac{x-z}{x+w}\mathbf{d})] \cdot \exp[E_M((y-w)\mathbf{e}, \frac{y+z}{y-w}\mathbf{d})] \\ &= \exp[(x+w)E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c}) + (x-z)E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})] \\ &\quad \exp[(y-w)E_M^0(\mathbf{e}, \mathbf{d}, \mathbf{c}) + (y+z)E_M^1(\mathbf{e}, \mathbf{d}, \mathbf{c})] \end{aligned} \quad (7.58)$$

implying the following propositions (in the same way as the first proposition of this section was proved).

**Proposition 7.** For any  $e \in E$ , we have

$$T_M(x, y, z, w) = (x+y)T_{M \setminus e}(x, y, z, w) \quad (7.59)$$

if  $e$  is a loop or a coloop and

$$T_M(x, y, z, w) = (x+w)T_{M \setminus e}(x, y, z, w) + (y+z)T_{M/e}(x, y, z, w) \quad (7.60)$$

otherwise. The same recurrence relations hold for  $t_M(x, y, z, w)$ , but the initialization is different :  $T_M(x, y, z, w) = 1$  for the empty matroid whereas  $t_M(x, y, z, w) = 1$  for all one element matroids.  $\square$



**Proposition 8.** *We have*

$$1 + T_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}) = \frac{\exp[xE_M(\mathbf{e}, \mathbf{d}) + wE_M^0(\mathbf{e}, \mathbf{d}) - zE_M^1(\mathbf{e}, \mathbf{d})]}{\exp[yE_M(\mathbf{e}, \mathbf{d}) - wE_M^0(\mathbf{e}, \mathbf{d}) + zE_M^1(\mathbf{e}, \mathbf{d})]}, \quad (7.61)$$

and for any  $e \in E$ , we have

$$\begin{aligned} \partial^e [1 + T_{M,x,y,z,w}(\mathbf{e}, \mathbf{d})] &= \exp[xE_M(\mathbf{e}, \mathbf{d}) + wE_M^0(\mathbf{e}, \mathbf{d}) - zE_M^1(\mathbf{e}, \mathbf{d})] \cdot (x+y)\partial^e E_M(\mathbf{e}, \mathbf{d}) \cdot \\ &\quad \exp[yE_M(\mathbf{e}, \mathbf{d}) - wE_M^0(\mathbf{e}, \mathbf{d}) + zE_M^1(\mathbf{e}, \mathbf{d})], \end{aligned} \quad (7.62)$$

$$\begin{aligned} \partial^e t_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}) &= \exp[xE_M(\mathbf{e}, \mathbf{d}) + wE_M^0(\mathbf{e}, \mathbf{d}) - zE_M^1(\mathbf{e}, \mathbf{d})] \cdot \partial^e E_M(\mathbf{e}, \mathbf{d}) \cdot \\ &\quad \exp[yE_M(\mathbf{e}, \mathbf{d}) - wE_M^0(\mathbf{e}, \mathbf{d}) + zE_M^1(\mathbf{e}, \mathbf{d})]. \quad \square \end{aligned} \quad (7.63)$$

Our last proposition as well as the fundamental lemma directly imply the following theorem.

**Theorem 12.** *We have*

$$1 + T_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}) = [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] \cdot [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})], \quad (7.64)$$

and for every  $e \in E$ , we have

$$\begin{aligned} \partial^e t_{M,x,y,z,w}(\mathbf{e}, \mathbf{d}) &= [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] \cdot \partial^e t_{M,0,0,z,w}(\mathbf{e}, \mathbf{d}) \cdot [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})] \end{aligned} \quad (7.65)$$

$$= [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] \cdot \partial^e t_{M,0,y,z,w}(\mathbf{e}, \mathbf{d}) \quad (7.66)$$

$$= \partial^e t_{M,x,0,z,w}(\mathbf{e}, \mathbf{d}) \cdot [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})], \quad (7.67)$$

$$\begin{aligned} \partial^e [1 + T_{M,x,y,z,w}(\mathbf{e}, \mathbf{d})] &= [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] \cdot (x+y)\partial^e t_{M,0,0,z,w}(\mathbf{e}, \mathbf{d}) \cdot [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})] \end{aligned} \quad (7.68)$$

$$= [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] \cdot (x+y)\partial^e t_{M,0,y,z,w}(\mathbf{e}, \mathbf{d}) \quad (7.69)$$

$$= (x+y)\partial^e t_{M,x,0,z,w}(\mathbf{e}, \mathbf{d}) \cdot [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})], \quad (7.70)$$

and

$$\partial^e [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] = [1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{d})] \cdot x\partial^e t_{M,0,0,z,w}(\mathbf{e}, \mathbf{d}), \quad (7.71)$$

$$\partial^e [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})] = y\partial^e t_{M,0,0,z,w}(\mathbf{e}, \mathbf{d}) \cdot [1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{d})]. \quad \square \quad (7.72)$$

Our identities (7.66) and (7.67) provide very short proofs of the lemma I.2 in [4]. Moreover, we have

$$\begin{aligned} \partial^e t_{M,0,0,z,w}(\mathbf{e}, \mathbf{d}) &= \exp[E_M(w\mathbf{e}, -\frac{z}{w}\mathbf{d})] \cdot \partial^e [\frac{1}{z}E_M^1(w\mathbf{e}, \frac{z}{w}\mathbf{d}) + \frac{1}{w}E_M^0(w\mathbf{e}, \frac{z}{w}\mathbf{d})] \cdot \exp[-E_M(w\mathbf{e}, -\frac{z}{w}\mathbf{d})] \\ &= \partial^e [\frac{1}{z}\beta_M^1(w\mathbf{e}, \frac{z}{w}\mathbf{d}) + \frac{1}{w}\beta_M^0(w\mathbf{e}, \frac{z}{w}\mathbf{d})] \\ &=: \partial^e \beta_{M,z,w}(\mathbf{e}, \mathbf{d}), \end{aligned} \quad (7.73)$$

where the vary last equality is a definition. Our identities (7.71), (7.72), (7.64) and (7.65) prove the following theorem.

**Theorem 13.** *We have*

$$1 + T_{M,x,0,z,w}(\mathbf{e}, \mathbf{c}) = \overleftarrow{\exp}[x\beta_{M,z,w}(\mathbf{e}, \mathbf{c})], \quad (7.74)$$

$$1 + T_{M,0,y,z,w}(\mathbf{e}, \mathbf{c}) = \overrightarrow{\exp}[y\beta_{M,z,w}(\mathbf{e}, \mathbf{c})], \quad (7.75)$$

$$1 + T_{M,x,y,z,w}(\mathbf{e}, \mathbf{c}) = \overleftarrow{\exp}[x\beta_{M,z,w}(\mathbf{e}, \mathbf{c})] \cdot \overrightarrow{\exp}[y\beta_{M,z,w}(\mathbf{e}, \mathbf{c})], \quad (7.76)$$

$$\partial^e t_{M,x,y,z,w}(\mathbf{e}, \mathbf{c}) = \overleftarrow{\exp}[x\beta_{M,z,w}(\mathbf{e}, \mathbf{c})] \cdot \partial^e \beta_{M,z,w}(\mathbf{e}, \mathbf{c}) \cdot \overrightarrow{\exp}[y\beta_{M,z,w}(\mathbf{e}, \mathbf{c})]. \quad \square \quad (7.77)$$

## 8 Lie Algebra

Since the algebra of matroid power series  $A[M]$  is not commutative, it is interesting to calculate Lie brackets. The operator

$$\text{ad} \partial^e E_M^0(\mathbf{e}, \mathbf{c}) : A[M] \rightarrow A[M] \quad (8.1)$$

is classically defined by

$$\begin{aligned} \text{ad} \partial^e E_M^0(\mathbf{e}, \mathbf{c}) \left( M_f(\mathbf{e}, \mathbf{c}) \right) &:= \left[ \partial^e E_M^0(\mathbf{e}, \mathbf{c}), M_f(\mathbf{e}, \mathbf{c}) \right] \\ &:= \partial^e E_M^0(\mathbf{e}, \mathbf{c}) \cdot M_f(\mathbf{e}, \mathbf{c}) - M_f(\mathbf{e}, \mathbf{c}) \cdot \partial^e E_M^0(\mathbf{e}, \mathbf{c}) \end{aligned} \quad (8.2)$$

for every  $f : 3^M \rightarrow A$ . We use the notation

$$M_{f_e^0}(\mathbf{e}, \mathbf{c}) := \text{ad} \partial^e E_M^0(\mathbf{e}, \mathbf{c}) \left( M_f(\mathbf{e}, \mathbf{c}) \right). \quad (8.3)$$

For all  $E' \uplus D' \uplus C' = E$  we have

$$f_e^0(M \setminus D' / C') = f(M \setminus (D' \cup e) / C'), \quad (8.4)$$

if  $e \in E'$  and  $e$  is neither a loop nor a coloop in  $M \setminus D' / C'$  ;

$$f_e^0(M \setminus D' / C') = f(M \setminus (D' \cup e) / C') - f(M \setminus D' / (C' \cup e)), \quad (8.5)$$

if  $e \in E'$  is a loop in  $M \setminus D' / C'$  (this value equals 0 if  $f : 3^M \rightarrow A$  depends only on the isomorphism classes of matroids) ; and

$$f_e^0(M \setminus D' / C') = 0 \quad (8.6)$$

in all other cases. Similarly, we define

$$M_{f_e^1}(\mathbf{e}, \mathbf{c}) := -\text{ad} \partial^e E_M^1(\mathbf{e}, \mathbf{c}) \left( M_f(\mathbf{e}, \mathbf{c}) \right) \quad (8.7)$$

and have

$$f_e^1(M \setminus D' / C') = f(M \setminus D' / (C' \cup e)), \quad (8.8)$$

if  $e \in E'$  and  $e$  is neither a loop nor a coloop in  $M \setminus D' / C'$  ;

$$f_e^1(M \setminus D' / C') = f(M \setminus D' / (C' \cup e)) - f(M \setminus (D' \cup e) / C'), \quad (8.9)$$

if  $e \in E'$  is a coloop in  $M \setminus D' / C'$  (this value equals 0 if  $f : 3^M \rightarrow A$  depends only on the isomorphism classes of matroids) ; and

$$f_e^1(M \setminus D' / C') = 0 \quad (8.10)$$

in all other cases. We have proved the following proposition for Lie-monomials.

**Proposition 9.** If  $E = \{e_1, \dots, e_n\}$ ,  $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$ ,  $M_0 := M$  and for all  $i = 1, \dots, n$

$$M_i := \begin{cases} M_{i-1} \setminus e_i & \text{if } \varepsilon_i = 0, \\ M_{i-1} / e_i & \text{if } \varepsilon_i = 1, \end{cases}$$

then we have

$$(-1)^{\varepsilon_1} \text{ad} \partial^{e_1} E_M^{\varepsilon_1}(\mathbf{e}, \mathbf{c}) \dots (-1)^{\varepsilon_{n-1}} \text{ad} \partial^{e_{n-1}} E_M^{\varepsilon_{n-1}}(\mathbf{e}, \mathbf{c}) \left( \partial^{e_n} E_M^{\varepsilon_n}(\mathbf{e}, \mathbf{c}) \right) = \varepsilon \cdot \mathbf{e}^E, \quad (8.11)$$

$\varepsilon \in \{0, 1\}$ , where  $\varepsilon = 1$  if and only if  $M_0, M_1, \dots, M_{n-1}$  and  $M_n$  are connected matroids and  $\varepsilon_{n-1} \neq \varepsilon_n$  (in the last step, a coloop must be contracted and a loop must be deleted, earlier they cannot appear anyway if  $\varepsilon \neq 0$ ).  $\square$

We call a reduction of our matroid  $M$  as in the preceding proposition with  $\varepsilon = 1$  a *connected  $e_n^{\varepsilon_n}$ -reduction*. By linearity,

$$\text{ad} \partial^{e_1} E_M(\mathbf{e}, -\mathbf{c}) \text{ad} \partial^{e_2} E_M(\mathbf{e}, -\mathbf{c}) \dots \text{ad} \partial^{e_{n-1}} E_M(\mathbf{e}, -\mathbf{c}) \left( \partial^{e_n} E_M^{\varepsilon_n}(\mathbf{e}, \mathbf{c}) \right) \quad (8.12)$$

counts the number of connected  $e_n^{\varepsilon_n}$ -reductions respecting a fixed order among the elements  $e_1, \dots, e_{n-1}$ . For any  $i, j \in \{1, \dots, n\}$ , however,  $\partial^{e_i} E_M(\mathbf{e}, \mathbf{c})$  and  $\partial^{e_j} E_M(\mathbf{e}, \mathbf{c})$  commute with each other, because both are just set functions. By substitution,  $\partial^{e_i} E_M(\mathbf{e}, -\mathbf{c})$  and  $\partial^{e_j} E_M(\mathbf{e}, -\mathbf{c})$  commute as well, and by the Jacobi identity, also  $\text{ad} \partial^{e_i} E_M(\mathbf{e}, -\mathbf{c})$  and  $\text{ad} \partial^{e_j} E_M(\mathbf{e}, -\mathbf{c})$  commute with each other. Therefore the number of connected  $e_n^{\varepsilon_n}$ -reductions is the same for any fixed order among the elements  $e_1, \dots, e_{n-1}$  and, by linearity, it is also counted by

$$\frac{1}{(n-1)!} \left( \text{ad} E_M(\mathbf{e}, -\mathbf{c}) \right)^{n-1} \left( \partial^{e_n} E_M^{\varepsilon_n}(\mathbf{e}, \mathbf{c}) \right). \quad (8.13)$$

We can now use [34], chapter 5.9, exercise 7 (see also chapter 5.10, exercise 1, as well as [8], chapter II.6, exercise 1, and [43], lectures 5-6) in order to prove the following theorem.

**Theorem 14.** Let us fix  $e \in E$ . Then, for all minors of  $M$  and for any fixed order among the elements of  $E \setminus \{e\}$

$$\begin{aligned} \partial^e \beta_M^1(\mathbf{e}, \mathbf{c}) &= \exp[E_M(\mathbf{e}, -\mathbf{c})] \cdot [\partial^e E_M^1(\mathbf{e}, \mathbf{c})] \cdot \exp[-E_M(\mathbf{e}, -\mathbf{c})] \\ &= \exp[\text{ad} E_M(\mathbf{e}, -\mathbf{c})] [\partial^e E_M^1(\mathbf{e}, \mathbf{c})] \end{aligned} \quad (8.14)$$

counts the number of connected  $e^1$ -reductions whereas

$$\begin{aligned} \partial^e \beta_M^0(\mathbf{e}, \mathbf{c}) &= \exp[E_M(\mathbf{e}, -\mathbf{c})] \cdot [\partial^e E_M^0(\mathbf{e}, \mathbf{c})] \cdot \exp[-E_M(\mathbf{e}, -\mathbf{c})] \\ &= \exp[\text{ad} E_M(\mathbf{e}, -\mathbf{c})] [\partial^e E_M^0(\mathbf{e}, \mathbf{c})] \end{aligned} \quad (8.15)$$

counts the number of connected  $e^0$ -reductions. Moreover,

$$\partial^e [\beta_M^0(\mathbf{e}, \mathbf{c}) + \beta_M^1(\mathbf{e}, \mathbf{c})] = \exp[\text{ad} E_M(\mathbf{e}, -\mathbf{c})] [\partial^e E_M(\mathbf{e}, \mathbf{c})], \quad (8.16)$$

$$\partial^e [\beta_M^0(\mathbf{e}, \mathbf{c}) - \beta_M^1(\mathbf{e}, \mathbf{c})] = \partial^e E_M(\mathbf{e}, -\mathbf{c}). \quad \square \quad (8.17)$$

For any partition  $E = E' \uplus D' \uplus C'$  and any matroid function  $f : 3^M \rightarrow A$ , the differential operator

$$\partial := \sum_{e \in E} \partial^e \quad (8.18)$$

multiplies the value of  $f(M \setminus D' / C')$  by  $|E'|$ . In particular,

$$\partial E_M^1(\mathbf{e}, \mathbf{d}) = E_M^1(\mathbf{e}, \mathbf{d}) \quad \text{and} \quad \partial E_M^0(\mathbf{e}, \mathbf{d}) = E_M^0(\mathbf{e}, \mathbf{d}). \quad (8.19)$$

It gives us the following corollary.

**Corollary 28.** *We have*

$$\begin{aligned} \partial \beta_M^1(\mathbf{e}, \mathbf{d}) &= \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot E_M^1(\mathbf{e}, \mathbf{d}) \cdot \exp[-E_M(\mathbf{e}, -\mathbf{d})] \\ &= \exp[\text{ad}E_M(\mathbf{e}, -\mathbf{d})] E_M^1(\mathbf{e}, \mathbf{d}), \end{aligned} \quad (8.20)$$

$$\begin{aligned} \partial \beta_M^0(\mathbf{e}, \mathbf{d}) &= \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot E_M^0(\mathbf{e}, \mathbf{d}) \cdot \exp[-E_M(\mathbf{e}, -\mathbf{d})] \\ &= \exp[\text{ad}E_M(\mathbf{e}, -\mathbf{d})] E_M^0(\mathbf{e}, \mathbf{d}). \quad \square \end{aligned} \quad (8.21)$$

Of course, it is always possible to make substitutions such as  $(\mathbf{e}, \mathbf{d}) \leftrightarrow (\mathbf{e}, -\mathbf{d})$ . If we replace  $(\mathbf{e}, \mathbf{d})$  by  $(d'\mathbf{e}, \frac{c'}{d'}\mathbf{d})$  and  $(\mathbf{e}, -\mathbf{d})$  by  $(d'\mathbf{e}, -\frac{c'}{d'}\mathbf{d})$  then we see that every connected  $e^\varepsilon$ -reduction has to contract  $r(M)$  elements of  $E$  and to delete  $|M| - r(M)$  ones.

The uniform matroid  $U_{k,n}$  is defined over a set of  $n$  elements. A subset of the elements is independent if and only if it contains at most  $k$  elements. A subset is a basis if it has exactly  $k$  elements, and it is a circuit if it has exactly  $k + 1$  elements. The smallest connected matroids (with at most four elements) are  $U_{1,2}$  (a circuit of length two or, equivalently, two parallel edges),  $U_{1,3}$  (three parallel edges),  $U_{2,3}$  (a circuit of length three),  $U_{1,4}$  (four parallel edges),  $U_{2,4}$  (a matroid not graphic and not even binary),  $U_{3,4}$  (a circuit of length four) and  $W_2$  (a circuit of length three with one edge doubled). Let  $U_M^{1,2}(\mathbf{e}, \mathbf{d})$ ,  $U_M^{1,3}(\mathbf{e}, \mathbf{d})$ ,  $U_M^{2,3}(\mathbf{e}, \mathbf{d})$ ,  $U_M^{1,4}(\mathbf{e}, \mathbf{d})$ ,  $U_M^{2,4}(\mathbf{e}, \mathbf{d})$ ,  $U_M^{3,4}(\mathbf{e}, \mathbf{d})$  and  $W_M^2(\mathbf{e}, \mathbf{d})$  be the indicator functions of  $U_{1,2}$ ,  $U_{1,3}$ ,  $U_{2,3}$ ,  $U_{1,4}$ ,  $U_{2,4}$ ,  $U_{3,4}$  and  $W_2$ , respectively (the matroid power series  $U_M^{1,2}(\mathbf{e}, \mathbf{d})$  takes the value 1 on all minors of  $M$  isomorphic to  $U_{1,2}$  and the value 0 elsewhere).

**Proposition 10.** *We have*

$$[E_M^0(\mathbf{e}, \mathbf{d}), E_M^1(\mathbf{e}, \mathbf{d})] = 2 \cdot U_M^{1,2}(\mathbf{e}, \mathbf{d}), \quad (8.22)$$

$$[E_M^0(\mathbf{e}, \mathbf{d}), U_M^{1,2}(\mathbf{e}, \mathbf{d})] = 3 \cdot U_M^{1,3}(\mathbf{e}, \mathbf{d}), \quad (8.23)$$

$$[U_M^{1,2}(\mathbf{e}, \mathbf{d}), E_M^1(\mathbf{e}, \mathbf{d})] = 3 \cdot U_M^{2,3}(\mathbf{e}, \mathbf{d}), \quad (8.24)$$

$$[E_M^0(\mathbf{e}, \mathbf{d}), U_M^{1,3}(\mathbf{e}, \mathbf{d})] = 4 \cdot U_M^{1,4}(\mathbf{e}, \mathbf{d}), \quad (8.25)$$

$$[U_M^{2,3}(\mathbf{e}, \mathbf{d}), E_M^1(\mathbf{e}, \mathbf{d})] = 4 \cdot U_M^{3,4}(\mathbf{e}, \mathbf{d}), \quad (8.26)$$

$$\begin{aligned} [E_M^0(\mathbf{e}, \mathbf{d}), U_M^{2,3}(\mathbf{e}, \mathbf{d})] &= [U_M^{1,3}(\mathbf{e}, \mathbf{d}), E_M^1(\mathbf{e}, \mathbf{d})] \\ &= 4 \cdot U_M^{2,4}(\mathbf{e}, \mathbf{d}) + 2 \cdot W_M^2(\mathbf{e}, \mathbf{d}), \end{aligned} \quad (8.27)$$

and for all  $e, f, g \in E$  we have

$$[\partial^{\{e\}} E_M^0(\mathbf{e}, \mathbf{d}), \partial^{\{f\}} E_M^1(\mathbf{e}, \mathbf{d})] = \partial^{\{e,f\}} U_M^{1,2}(\mathbf{e}, \mathbf{d}), \quad (8.28)$$

$$[\partial^{\{e\}} E_M^0(\mathbf{e}, \mathbf{d}), \partial^{\{f,g\}} U_M^{1,2}(\mathbf{e}, \mathbf{d})] = \partial^{\{e,f,g\}} U_M^{1,3}(\mathbf{e}, \mathbf{d}), \quad (8.29)$$

$$[\partial^{\{f,g\}} U_M^{1,2}(\mathbf{e}, \mathbf{d}), \partial^{\{e\}} E_M^1(\mathbf{e}, \mathbf{d})] = \partial^{\{e,f,g\}} U_M^{2,3}(\mathbf{e}, \mathbf{d}). \quad \square \quad (8.30)$$

Last but not least, we can use those concepts to define matroids axiomatically. Praematroids can be defined by the properties

$$\partial^e E_M^0(\mathbf{e}, \mathbf{d}), \partial^e E_M^1(\mathbf{e}, \mathbf{d}) \geq 0, \quad (8.31)$$

$$\partial^e E_M^0(\mathbf{e}, \mathbf{d}) + \partial^e E_M^1(\mathbf{e}, \mathbf{d}) = \partial^e E_M(\mathbf{e}, \mathbf{d}), \quad (8.32)$$

$$[\partial^e E_M^0(\mathbf{e}, \mathbf{d}), \partial^f E_M^0(\mathbf{e}, \mathbf{d})] = [\partial^e E_M^1(\mathbf{e}, \mathbf{d}), \partial^f E_M^1(\mathbf{e}, \mathbf{d})] = 0, \quad (8.33)$$

and for matroids we need the additional axiom

$$[\partial^e E_M^0(\mathbf{e}, \mathbf{d}), \partial^f E_M^1(\mathbf{e}, \mathbf{d})] \geq 0. \quad (8.34)$$

## 9 Orientations

Let us choose an orientation for every edge of our multigraph  $G = (V, E)$  (of course, there are  $2^{|E|}$  possible orientations of  $G$ ). This orientation also defines an orientation for every minor  $M' \in 3^M$ , where  $M$  is the matroid corresponding to our multigraph  $G$ . An orientation is called *acyclic* if and only if it does not contain an oriented cycle, and it is called *totally cyclic* or *strongly connected* if and only if it does not contain an oriented cut. For every minor  $M' \in 3^M$ , let us define  $ia(M') = 1$  if our orientation of  $M'$  is acyclic and  $ia(M') = 0$  if it contains an oriented cycle. Dually,  $is(M') = 1$  if our orientation of  $M'$  is strongly connected and  $is(M') = 0$  if it contains an oriented cut. Then the indicator matroid power series of acyclic minors and strongly connected minors are

$$IA_M(\mathbf{e}, \mathbf{d}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} ia(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (9.1)$$

$$IS_M(\mathbf{e}, \mathbf{d}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} is(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (9.2)$$

where the sums are taken over all partitions such that  $E' \neq \emptyset$  ( $D' = \emptyset$  or  $C' = \emptyset$  is not excluded).

If we delete edges of an acyclic orientation, then the orientation remains acyclic (this is not necessarily true for contractions). Dually, if we contract edges of a strongly connected orientation, then the orientation remains strongly connected (this is not necessarily true for deletions). By Minty's lemma (see [3], chapter 2.1), every edge either belongs to an oriented cycle or to an oriented cut, but not to both. If we want to get an acyclic orientation by contractions, then we have to contract at least all edges belonging to oriented cycles. If we want to get a strongly connected orientation by deletions, then we have to delete at least all edges belonging to oriented cuts. This observation is also true for all minors  $M' \in 3^M$  and equivalent to an axiomatic characterization of oriented matroids (generalizing oriented graphs) via the 3-painting axiom, where the three colors correspond to edges deleted, contracted or conserved ([5], theorem 3.4.4, Bland and Las Vergnas). It is equivalent to the following theorem.

**Theorem 15. 3-painting axiom.** *For any oriented matroid  $M$ , we have*

$$[1 + IA_M(\mathbf{e}, \mathbf{d})] \cdot [1 + IS_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(\mathbf{e}, \mathbf{d})]. \quad \square \quad (9.3)$$

**Remark.** For oriented matroids [5], acyclic orientations are usually called *topes* or *maximal covectors*, whereas *covectors* are acyclic orientations of minors obtained only by contractions and *minimal covectors* also called *cocircuits* are oriented cuts.

Dually, totally cyclic or strongly connected orientations are called *maximal vectors*, whereas *vectors* are totally cyclic orientations of minors obtained only by deletions and *minimal vectors* also called *circuits* are oriented cycles.

For every minor  $M' \in 3^M$ , let  $a(M')$  be the number of acyclic orientations of  $M'$  and let  $s(M')$  be the number of strongly connected orientations of  $M'$ . Their matroid power series are defined by

$$A_M(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} a(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (9.4)$$

$$S_M(\mathbf{e}, \mathbf{d}, \mathbf{c}) := \sum_{\emptyset \subset E' \uplus D' \uplus C' = E} s(M \setminus D' / C') \cdot \mathbf{e}^{E'} \mathbf{d}^{D'} \mathbf{c}^{C'}, \quad (9.5)$$

where the sums are taken over all partitions such that  $E' \neq \emptyset$  ( $D' = \emptyset$  or  $C' = \emptyset$  is not excluded). If we sum up the identity of the 3-painting axiom over all orientations of our graph  $G$  (or all reorientations of our oriented matroid  $M$ ), then we get the following corollary of [29], chapter 5.4.

**Corollary 29.** *For any oriented matroid  $M$ , we have*

$$[1 + A_M(\mathbf{e}, \mathbf{d}, \mathbf{c})] \cdot [1 + S_M(\mathbf{e}, \mathbf{d}, \mathbf{c})] = \exp[E_M(2\mathbf{e}, \mathbf{d}, \mathbf{c})]. \quad \square \quad (9.6)$$

Let us suppose that the vertex set of our graph  $G = (V, E)$  is  $V = \{1, \dots, n\}$ , and let us associate a variable  $x_i$  to every vertex  $i \in \{1, \dots, n\}$ . It is natural to represent an edge between the vertices  $i$  and  $j$  by the equation  $x_i = x_j$  of a hyperplane in  $\mathbb{R}^n$ . An orientation of this edge from  $i$  to  $j$  is represented by the inequality  $x_i < x_j$ , and an orientation of  $G$  is acyclic if and only if the system of inequalities has solutions. If we consider the inequalities  $x_i \leq x_j$ , then we get a convex cone in  $\mathbb{R}^n$ . The boundary of this cone is defined by equalities in some of our inequalities, and those equalities can be identified with contractions of the corresponding edges of  $G$ . In other words, we can look at an acyclic orientation of  $G$  together with all graphs obtained from  $G$  by contracting some edges such that our acyclic orientation remains acyclic. Those graphs form a ranked partially ordered set corresponding to our cone. It is called the Edmonds-Mandel or Las Vergnas face lattice ([5], definition 4.1.2). For topological reasons it is evident that the Möbius function of this partially ordered set is given by  $-1$  to the power of rank difference. These observations naturally generalize to oriented matroids ([5], corollary 4.3.8) and allow us to prove the following theorem.

**Theorem 16.** *For any oriented matroid  $M$ , we have*

$$[1 + IA_M(\mathbf{e}, -\mathbf{d}, \mathbf{c})] \cdot [1 + IA_M(\mathbf{e}, \mathbf{d}, \mathbf{c})] = 1, \quad (9.7)$$

$$[1 + IS_M(-\mathbf{e}, -\mathbf{d}, \mathbf{c})] \cdot [1 + IS_M(\mathbf{e}, \mathbf{d}, \mathbf{c})] = 1. \quad (9.8)$$

*Proof.* If we apply the first identity to the dual oriented matroid, then we get the second identity. Therefore it is sufficient to prove the first one.

If our orientation of  $M$  is not acyclic, then the second factor of our product must take care of all edges belonging to directed cycles, because all those edges have to be contracted in order

to make the orientation acyclic for the first factor. In this case, however, the first factor deletes only edges belonging to directed cuts, but this cannot make the orientation acyclic for the second factor. In other words, the contribution is 0 if our orientation of  $M$  is not acyclic.

If our orientation of  $M$  is acyclic, then it remains acyclic after all deletions of edges. Therefore the second factor always contributes a multiplication by 1. The first factor, however, contributes a multiplication by  $-1$  to the power of rank difference if the contraction of some edges provides an acyclic orientation. Otherwise, it contributes a multiplication by 0. Therefore the first identity of our theorem reflects exactly the multiplication of the Möbius function with the zeta function of the Edmonds-Mandel or Las Vergnas face lattice. This concludes our proof.  $\square$

If we sum up the identities of the preceding theorem over all orientations of our graph  $G$  (or all reorientations of our oriented matroid  $M$ ), then we get the following corollary.

**Corollary 30.** *For any oriented matroid  $M$ , we have*

$$[1 + A_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] \cdot [1 + A_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = 1, \quad (9.9)$$

$$[1 + S_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] \cdot [1 + S_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = 1. \quad \square \quad (9.10)$$

Since  $[1 + IA_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  and  $[1 + IA_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})]$ ,  $[1 + IS_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  and  $[1 + IS_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})]$  as well as  $\exp[E_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  and  $\exp[E_M(-\mathbf{e}, \mathbf{c}^{\mathbf{d}})]$  are inverse to each other, our preceding theorems also imply the following one.

**Theorem 17.** *For any oriented matroid  $M$ , we have*

$$[1 + IS_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] \cdot [1 + IA_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] = \exp[E_M(-\mathbf{e}, \mathbf{c}^{\mathbf{d}})], \quad (9.11)$$

$$[1 + IA_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = [1 + IS_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})], \quad (9.12)$$

$$[1 + IS_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(-\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = [1 + IA_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})], \quad (9.13)$$

$$\exp[E_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot [1 + IS_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] = [1 + IA_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})], \quad (9.14)$$

$$\exp[E_M(-\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot [1 + IA_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = [1 + IS_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})]. \quad \square \quad (9.15)$$

For the special case of graphs, our last two identities can easily be seen to be equivalent to the identities (53) and (54) of [2], where the authors asked for additional proofs.

If we sum up the identities of the preceding theorem over all orientations of our graph  $G$  (or all reorientations of our oriented matroid  $M$ ), then we get the following corollary.

**Corollary 31.** *For any oriented matroid  $M$ , we have*

$$[1 + S_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] \cdot [1 + A_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] = \exp[E_M(-2\mathbf{e}, \mathbf{c}^{\mathbf{d}})], \quad (9.16)$$

$$[1 + A_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(2\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = [1 + S_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})], \quad (9.17)$$

$$[1 + S_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot \exp[E_M(-2\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = [1 + A_M(\mathbf{e}, -\mathbf{c}^{\mathbf{d}})], \quad (9.18)$$

$$\exp[E_M(2\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot [1 + S_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})] = [1 + A_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})], \quad (9.19)$$

$$\exp[E_M(-2\mathbf{e}, \mathbf{c}^{\mathbf{d}})] \cdot [1 + A_M(\mathbf{e}, \mathbf{c}^{\mathbf{d}})] = [1 + S_M(-\mathbf{e}, -\mathbf{c}^{\mathbf{d}})]. \quad \square \quad (9.20)$$

For the special case of graphs, our last identity can easily be seen to be equivalent to the identity (71) of [2], where the authors asked for additional proofs.

We have already considered a fixed acyclic orientation of our graph  $G = (V, E)$  in a geometric way, but in the same way, we can consider all acyclic orientations of  $G$  together geometrically,

as well as acyclic orientations of graphs obtained from  $G$  by contracting some edges. In this way, we get a partially ordered set called the big face lattice, if we adjoin a top element 1. Once again, for topological reasons it is evident that the Möbius function of this partially ordered set is given by  $-1$  to the power of rank difference. These observations naturally generalize to oriented matroids ([5], corollary 4.3.8) and allow us to prove the following identity :

$$[1 + A_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] - \exp[E_M(\mathbf{e}, -\mathbf{d})] = 0. \quad (9.21)$$

This identity reflects the fact that the product of the Möbius function and the zeta function of the big face lattice gives the value 0 for the maximal interval from 0 to 1. Here  $-\exp[E_M(\mathbf{e}, -\mathbf{d})]$  is the contribution of the Möbius function of this interval itself, whereas  $[1 + A_M(\mathbf{e}, -\mathbf{d})]$  gives the Möbius function for all graphs or matroids obtained after some contractions.

Our preceding identity can also be considered for the dual matroid and proves the following theorem.

**Theorem 18.** *(Las Vergnas-Zaslavsky-Winder-Stanley-Cartier-Foata-Gessel-Viennot) For any oriented matroid  $M$ , we have*

$$1 + A_M(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{d})] = 1 + \theta_{M,-1}(\mathbf{e}, -\mathbf{d}), \quad (9.22)$$

$$1 + S_M(\mathbf{e}, \mathbf{d}) = \exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = 1 + \phi_{M,-1}(-\mathbf{e}, -\mathbf{d}). \quad \square \quad (9.23)$$

Several versions of the preceding theorem can be found for matroids in [27, 28, 59, 60, 61] as well as [5] (theorem 4.6.1) and [29] (chapter 5.4), for hyperplane arrangements in [57], and for graphs in [49], [10] (théorème 2.4), [52] (proposition 5.1) and [30] (théorème 3.1).

Our preceding theorem implies all our preceding corollaries of this section, as well as the following one.

**Corollary 32.** *For any oriented matroid  $M$ , we have*

$$[1 + A_M(\mathbf{e}, -\mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(\mathbf{e}, -\mathbf{d})], \quad (9.24)$$

$$\exp[E_M(\mathbf{e}, \mathbf{d})] \cdot [1 + A_M(-\mathbf{e}, \mathbf{d})] = \exp[E_M(\mathbf{e}, -\mathbf{d})], \quad (9.25)$$

$$[1 + S_M(-\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = \exp[E_M(-\mathbf{e}, -\mathbf{d})], \quad (9.26)$$

$$\exp[E_M(\mathbf{e}, \mathbf{d})] \cdot [1 + S_M(-\mathbf{e}, -\mathbf{d})] = \exp[E_M(-\mathbf{e}, -\mathbf{d})], \quad (9.27)$$

$$\exp[E_M(\mathbf{e}, \mathbf{d})] \cdot [1 + A_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(-\mathbf{e}, -\mathbf{d})] = 1 + A_M(2\mathbf{e}, \mathbf{d}), \quad (9.28)$$

$$\exp[E_M(\mathbf{e}, -\mathbf{d})] \cdot [1 + S_M(\mathbf{e}, \mathbf{d})] \cdot \exp[E_M(\mathbf{e}, \mathbf{d})] = 1 + S_M(2\mathbf{e}, \mathbf{d}). \quad \square \quad (9.29)$$

We have already seen that the first identity of our corollary is equivalent to corollary 4.3.8 of [5] if  $Y$  is the the top element 1. The second identity of our corollary is equivalent to corollary 4.3.10 of [5], see [15]. For the special case of graphs, the last identity of our corollary can easily seen to be equivalent to the identity (70) of [2], where the authors asked for additional proofs.

Let us consider an arbitrary acyclic orientation and let us fix an element  $e \in E$ . If we reorient  $e$ , then our orientation remains acyclic if and only if the contraction of  $e$  leaves it acyclic. Otherwise, reorienting  $e$  provides a unique strongly connected component (maybe all  $E$ ) to which  $e$  belongs (3-painting axiom), and if we contract this component, our orientation remains acyclic. Moreover, if we reorient not just  $e$  but the whole connected component, we get another acyclic orientation in bijection with our initial one. Of course, all those considerations



can also be dualized. Let  $\partial^e I\beta_M^1(\mathbf{e}, \mathbf{d})$  be the indicator matroid power series of acyclic minors of  $M$  becoming totally cyclic by reorienting  $e$ . By definition, its value is 1 on minors of rank 1 with support  $e$ . Dually, let  $\partial^e I\beta_M^0(\mathbf{e}, \mathbf{d})$  be the indicator matroid power series of totally cyclic minors of  $M$  becoming acyclic by reorienting  $e$ . By definition, its value is 1 on minors of rank 0 with support  $e$ . We have proved the following theorem.

**Theorem 19. Differential Equations for Acyclic and Totally Cyclic Sets.** *For every  $e \in E$  we have*

$$\partial^e [1 + IA_M(\mathbf{e}, \mathbf{d})] = [1 + IA_M(\mathbf{e}, \mathbf{d})] \cdot \partial^e I\beta_M^1(\mathbf{e}, \mathbf{d}), \quad (9.30)$$

$$\partial^e [1 + IS_M(\mathbf{e}, \mathbf{d})] = \partial^e I\beta_M^0(\mathbf{e}, \mathbf{d}) \cdot [1 + IS_M(\mathbf{e}, \mathbf{d})]. \quad \square \quad (9.31)$$

If we sum the equations of the previous theorem over all reorientations, we obtain exactly the equations (7.44) for  $\lambda = 1$  and (7.45) for  $\mu = 1$ . Therefore, we get exactly Crapo's  $\beta$  invariants and the following theorem.

**Theorem 20. Differential Equations for Acyclic and Totally Cyclic Orientations.** *For every  $e \in E$  we have*

$$\partial^e [1 + A_M(\mathbf{e}, \mathbf{d})] = [1 + A_M(\mathbf{e}, \mathbf{d})] \cdot 2\partial^e \beta_M^1(\mathbf{e}, \mathbf{d}), \quad (9.32)$$

$$\partial^e [1 + S_M(\mathbf{e}, \mathbf{d})] = 2\partial^e \beta_M^0(\mathbf{e}, \mathbf{d}) \cdot [1 + S_M(\mathbf{e}, \mathbf{d})]. \quad (9.33)$$

*In particular, for oriented matroids with at least two elements, Crapo's  $\beta$  invariants  $\beta^1(M)$  and  $\beta^0(M)$  both count the number of orientations switching between acyclic and totally cyclic by reorienting a single edge  $e$ .  $\square$*

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