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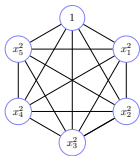
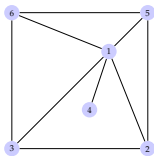
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Exploiting sparsity in polynomial optimization

Victor Magron
LAAS CNRS

<https://homepages.laas.fr/vmagron/SparsePOPJNCF23.pdf>

Journées Nationales de Calcul Formel 6-10 March 2023



What is sparse polynomial optimization?

Looks like a regular polynomial optimization problem (POP):

$$\begin{aligned} \inf \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\} \end{aligned}$$

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Correlative sparsity: few products between each variable and the others in f, g_j

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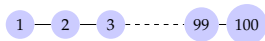
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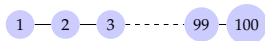
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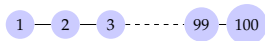
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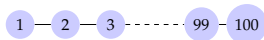
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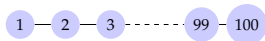
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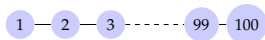
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PERFORMANCE



VS



ACCURACY

Where do we find sparse POPs?

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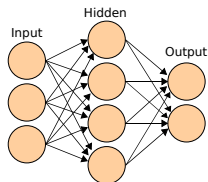
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Where do we find sparse POPs?

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Deep learning

~> robustness, computer vision

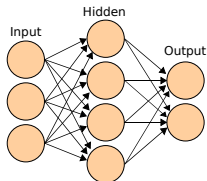


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~> AC optimal power-flow, stability

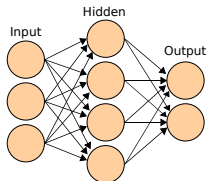


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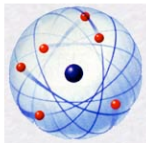
Power systems

↪ AC optimal power-flow, stability



Quantum Systems

↪ condensed matter, entanglement



The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf f(\mathbf{x})$

Theory

(Primal)		(Dual)
$\inf \int f d\mu$		$\sup \lambda$
with μ proba \Rightarrow	INFINITE LP	\Leftarrow with $f - \lambda \geq 0$

The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf f(x)$

Practice

(Primal **Relaxation**)

moments $\int x^\alpha d\mu$

finite number \Rightarrow



SDP

(Dual **Strengthening**)

$f - \lambda =$ sum of squares

\Leftarrow fixed degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$

[Lasserre '01]

degree r & n vars $\implies \binom{n+2r}{n}$ **SDP** VARIABLES



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HOW TO OVERCOME THIS **NO-FREE LUNCH** RULE?

An example

NP hard General Problem: $f_{\min} := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

Semialgebraic set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\}$

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$\mathbf{X} = [0, 1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geq 0, \quad x_2(1 - x_2) \geq 0\}$

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$$\underbrace{x_1 x_2}_f = -\frac{1}{8} + \frac{1}{2} \overbrace{\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \frac{1}{2} \overbrace{x_1(1 - x_1)}^{g_1} + \frac{1}{2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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$$\text{Quadratic module: } \mathcal{M}(\mathbf{X})_r = \left\{ \sigma_0 + \sum_j \sigma_j g_j, \deg \sigma_j g_j \leq 2r \right\}$$

The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- space $\mathcal{M}_+(\mathbf{X})$ of probability measures supported on \mathbf{X}
- quadratic module $\mathcal{Q}(\mathbf{X}) = \left\{ \sigma_0 + \sum_j \sigma_j g_j, \text{ with } \sigma_j \text{ SOS} \right\}$

Infinite-dimensional linear programs (LP)

(Primal)	=	(Dual)
$\inf \int_{\mathbf{X}} f d\mu$		$\sup \lambda$
s.t. $\mu \in \mathcal{M}_+(\mathbf{X})$		s.t. $\lambda \in \mathbb{R}$
		$f - \lambda \in \mathcal{Q}(\mathbf{X})$

The Moment-SOS Hierarchy for POP

NP-hard NON CONVEX Problem $f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Pseudo-moment sequences \mathbf{y} up to order r
- Truncated quadratic module $\mathcal{Q}(\mathbf{X})_r$

Finite-dimensional semidefinite programs (SDP)

(Moment)	=	(SOS)
$\inf \sum_{\alpha} f_{\alpha} y_{\alpha}$		$\sup \lambda$
s.t. $\mathbf{M}_{r-r_j}(g_j \mathbf{y}) \succcurlyeq 0$		s.t. $\lambda \in \mathbb{R}$
$y_0 = 1$		$f - \lambda \in \mathcal{Q}(\mathbf{X})_r$

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Primal-dual “SPARSE” variants?

Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

Tutorial session

Sparse SDP

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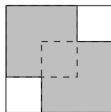
Tutorial session

Sparse matrices

Symmetric matrices indexed by graph vertices

Sparse matrices

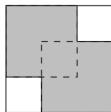
Symmetric matrices indexed by graph vertices



💡 no edge between 1 and 3 \iff 0 entry in the (1,3) entry

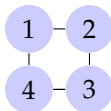
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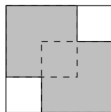
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cycle =

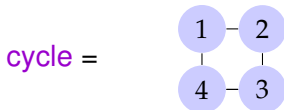


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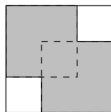
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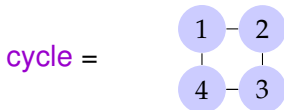
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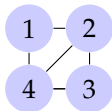


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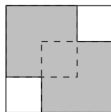
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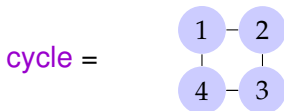


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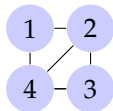
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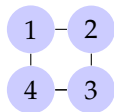
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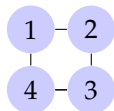
clique = a fully connected subset of vertices



Chordal extensions



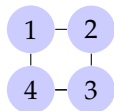
Chordal extensions



Fact

Any non-chordal graph can always be extended to a chordal graph, by adding appropriate edges

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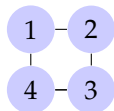


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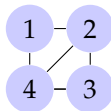
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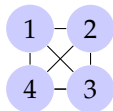
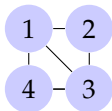
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approximately minimal



maximal

Theorem [Gavril '72, Vandenberghe & Andersen '15]

The maximal cliques of a chordal graph can be enumerated in linear time in the number of nodes and edges.

Running intersection property (RIP)

RIP Theorem for chordal graphs [Blair & Peyton '93]

For a chordal graph with maximal cliques I_1, \dots, I_p :

$$(RIP) \quad \forall k < p \quad I_{k+1} \cap \bigcup_{j \leq k} I_j \subseteq I_i \quad \text{for some } i \leq k$$

(possibly after reordering)

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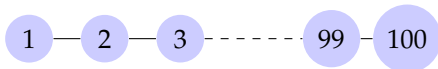
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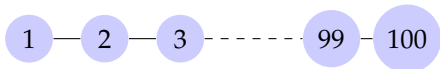
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💡 RIP holds for numerous applications!

Semidefinite Programming (SDP)

$$\begin{aligned} \min_{\mathbf{y}} \quad & \mathbf{c}^\top \mathbf{y} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i y_i \succcurlyeq \mathbf{F}_0 \end{aligned}$$

- Linear cost \mathbf{c}
- Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities “ $\mathbf{F} \succcurlyeq 0$ ”
(\mathbf{F} has nonnegative eigenvalues)



Spectrahedron

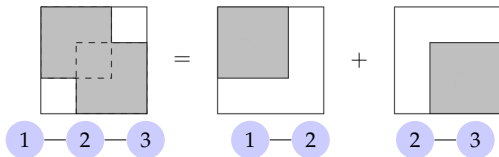
Sparse SDP matrices

Theorem [Griewank Toint '84, Agler et al. '88]

Chordal graph G with n vertices & maximal cliques I_1, I_2

$Q_G \succcurlyeq 0$ with nonzero entries corresponding to edges of G

$\implies Q_G = P_1^T Q_1 P_1 + P_2^T Q_2 P_2$ with $Q_k \succcurlyeq 0$ indexed by I_k



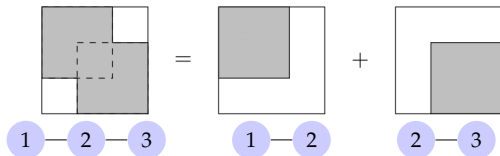
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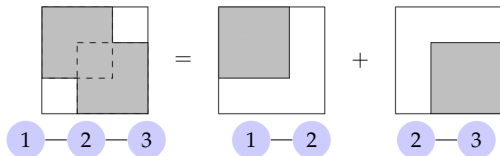
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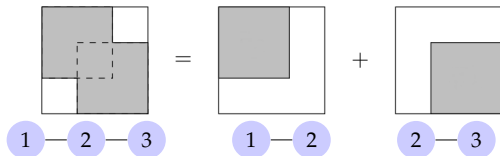
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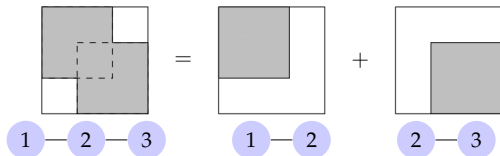
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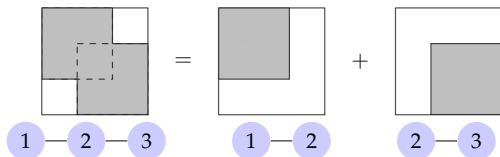
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💡 $P_1^T Q_1 P_1$ inflates a $|I_1| \times |I_1|$ matrix Q_1 into a sparse $n \times n$ matrix

Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

Tutorial session

What is correlative sparsity?

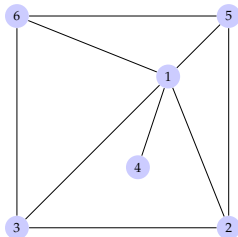
💡 Exploit few links between **variables** [Lasserre, Waki et al. '06]

$$f(\mathbf{x}) = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

Correlative sparsity pattern (csp) graph G

Vertices = $\{1, \dots, n\}$

$(i, j) \in \text{Edges} \iff x_i x_j$ appears in f



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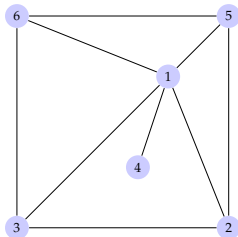
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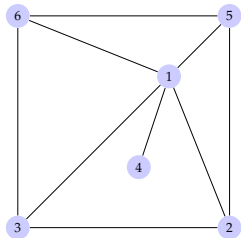


Similar construction with constraints $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\}$

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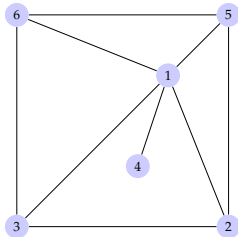
Chordal graph after adding edge (3,5)



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maximal cliques $I_1 = \{1, 4\}$ $I_2 = \{1, 2, 3, 5\}$ $I_3 = \{1, 3, 5, 6\}$

$f = f_1 + f_2 + f_3$ where f_k involves **only** variables in I_k

💡 Let us index moment matrices and SOS with the cliques I_k

A sparse variant of Putinar's Positivstellensatz

Convergence of the Moment-SOS hierarchy is based on:

Theorem [Putinar '93] Positivstellensatz

If \mathbf{X} contains a ball constraint $N - \sum_i x_i^2 \geq 0$ then

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Theorem: Sparse Putinar's representation [Lasserre '06]

$f = \sum_k f_k$, f_k depends on $\mathbf{x}(I_k)$

$f > 0$ on \mathbf{X}

Each g_j depends on some I_k

RIP holds for (I_k)

ball constraints for each $\mathbf{x}(I_k)$

$$\boxed{f = \sum_k (\sigma_{0k} + \sum_{j \in I_k} \sigma_j g_j)}$$

\implies SOS σ_{0k} "sees" vars in I_k
 σ_j "sees" vars from g_j

A first key message



SUMS OF SQUARES PRESERVE SPARSITY



A proof of sparse Putinar's Positivstellensatz

Let $\mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$ be compact and $f = \sum_k f_k$, with f_k depends on $\mathbf{x}(I_k)$, and $f > 0$ on \mathbf{X}

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💡 Then apply Putinar to each h_k

Sparse moment matrices

For each subset I_k , submatrix of $\mathbf{M}_r(\mathbf{y})$ corresponding of rows & columns indexed by monomials in $\mathbf{x}(I_k)$

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$$I_1 = \{1, 4\} \implies \text{monomials in } x_1, x_4$$

$$\mathbf{M}_1(\mathbf{y}, I_1) = \left(\begin{array}{c|cc} 1 & y_{1,0,0,0,0,0} & y_{0,0,0,1,0,0} \\ \hline y_{1,0,0,0,0,0} & y_{2,0,0,0,0,0} & y_{1,0,0,1,0,0} \\ y_{0,0,0,1,0,0} & y_{1,0,0,1,0,0} & y_{0,0,0,2,0,0} \end{array} \right)$$

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💡 same for each localizing matrix $\mathbf{M}_r(\mathbf{g}_j\mathbf{y})$

Sparse primal-dual Moment-SOS hierarchy

$$f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) \text{ with } \mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$$

Dense Moment-SOS hierarchy

(Moment)	=	(SOS)
$\inf \sum_{\alpha} f_{\alpha} y_{\alpha}$		$\sup \lambda$
s.t. $\mathbf{M}_r(\mathbf{y}) \succeq 0$		s.t. $\lambda \in \mathbb{R}$
$\mathbf{M}_{r-r_j}(g_j \mathbf{y}) \succeq 0$		$f - \lambda = \sigma_0 + \sum_j \sigma_j g_j$
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$$\text{s.t. } \mathbf{M}_r(\mathbf{y}, I_k) \succeq 0$$

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RIP holds for (I_k) + ball constraints for each $\mathbf{x}(I_k) \implies$ Primal and dual optimal value converge to f_{\min} by sparse Putinar

Computational cost

$$f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) \text{ with } \mathbf{X} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0, j \leq m\}$$

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$(m + p)$ SOS in at most τ vars of degree $\leq 2r$

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💡 $(m + p) \mathcal{O}(r^{\tau})$ SDP vars

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💡 $(m + p) \mathcal{O}(r^{\tau})$ SDP vars vs $(m + 1) \mathcal{O}(r^n)$ in the dense SDP

Sparse linear program over measures

In the dense setting:

$$\begin{aligned} f_{\min} &= \inf_{\mu} \int_{\mathbf{X}} f d\mu \\ \text{s.t. } & \mu \in \mathcal{M}_+(\mathbf{X}) \end{aligned}$$

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Sparse moment SDPs relax the sparse LP over measures:

$$f_{\text{CS}} = \inf_{\mu_k} \sum_k \int_{\mathbf{X}_k} f_k d\mu_k$$

s.t. $\pi_{jk}\mu_j = \pi_{kj}\mu_k, \quad \mu_k \in \mathcal{M}_+(\mathbf{X}_k)$

The dual of sparse Putinar's Positivstellensatz

Theorem [Lasserre '06]

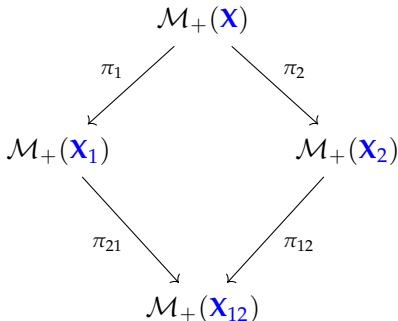
$$\begin{aligned} \text{RIP holds for } (I_k) \implies f_{\min} = f_{\text{CS}} = \inf_{\mu_k} & \sum_k \int_{\mathbf{X}_k} f_k d\mu_k \\ \text{s.t.} & \pi_{jk} \mu_j = \pi_{kj} \mu_k \\ & \mu_k \in \mathcal{M}_+(\mathbf{X}_k) \end{aligned}$$

The dual of sparse Putinar's Positivstellensatz

Theorem [Lasserre '06]

$$\begin{aligned} \text{RIP holds for } (I_k) \implies f_{\min} = f_{\text{CS}} = \inf_{\mu_k} \sum_k \int_{\mathbf{X}_k} f_k d\mu_k \\ \text{s.t. } \pi_{jk}\mu_j = \pi_{kj}\mu_k \\ \mu_k \in \mathcal{M}_+(\mathbf{X}_k) \end{aligned}$$

💡 Proof: there exists $\mu \in \mathcal{M}_+(\mathbf{X})$ with marginal μ_k on \mathbf{X}_k



A first (dual) key message



THE MEASURE LP PRESERVES SPARSITY



Extracting minimizers: the dense case

Let r_{\min} be the minimal relaxation order.

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Theorem: dense extraction [Lasserre & Henrion '05]

Assume that the moment SDP has an optimal solution \mathbf{y} with cost f^r and

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Then $f^r = f_{\min}$ and the LP over measures has an optimal solution $\mu \in \mathcal{M}_+(\mathbf{X})$ supported on $t = \text{rank } \mathbf{M}_{r'}(\mathbf{y})$ points of \mathbf{X} .

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Extraction possible with the Gloptipoly software

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$a_k = \max_{j \in J_k} r_j = \max$ half degree of g_j depending on I_k .

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💡 RIP is not required!

💡 Extract $\mathbf{x}(k)$ from $\mathbf{M}_r(\mathbf{y}, I_k) \implies$ minimizer \mathbf{x} with $(x_i)_{i \in I_k} = \mathbf{x}(k)$

Application to rational functions

$$f_{\min} = \inf_{\mathbf{x} \in \mathbf{X}} \sum_i \frac{p_i(\mathbf{x})}{q_i(\mathbf{x})}, \quad q_i > 0 \text{ on } \mathbf{X}, \quad p_i, q_i \text{ depends only on } I_i$$

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Application to roundoff errors

[Magron Constantinides Donaldson '17]

Exact $f(\mathbf{x}) = x_1x_2 + x_3x_4$

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3: Bound $\ell(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

💡 $I_k \rightarrow \{\mathbf{x}, e_k\} \implies \boxed{m r^{n+1} \text{ instead of } r^{n+m}}$ SDP vars

Application to roundoff errors

$$f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

$$\mathbf{x} \in [4.00, 6.36]^6, \quad \mathbf{e} \in [-\epsilon, \epsilon]^{15}, \quad \epsilon = 2^{-53}$$

Dense SDP: $\binom{6+15+4}{6+15} = 12650$ variables \leadsto **Out of memory**

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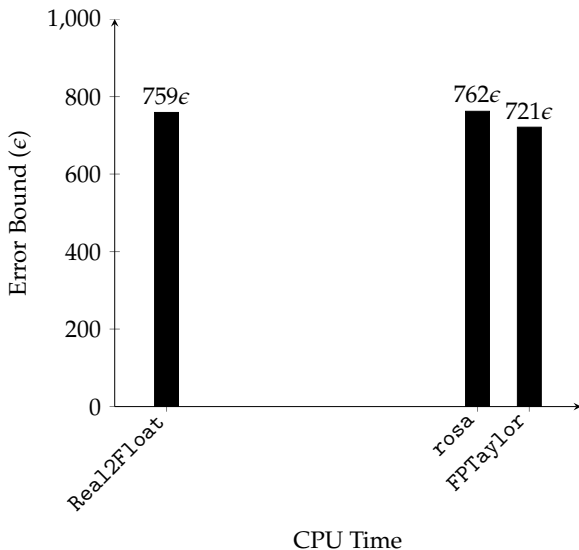
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SMT-based rosa tool: 762ϵ (19 \times more CPU)

Application to roundoff errors



Extension to noncommutative optimization

Self-adjoint noncommutative variables $a_i, b_j \in \mathcal{B}(\mathcal{H})$

$$f = a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 - b_3) + a_3(b_1 - b_2) - b_1 - 2b_2 - b_3$$

with $a_1a_2 \neq a_2a_1$, **involution** $(a_1b_3)^* = b_3a_1$

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MINIMAL EIGENVALUE OPTIMIZATION

$$\lambda_{\min} = \inf \{ \langle f(a, b) \mathbf{v}, \mathbf{v} \rangle : (a, b) \in \mathbf{X}, \|\mathbf{v}\| = 1 \}$$

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Ball constraint $N - \sum_i x_i^2 \succcurlyeq 0$ in \mathbf{X}

Theorem: NC Putinar's representation [Helton & McCullough '02]

$$f \succcurlyeq 0 \text{ on } \mathbf{X} \implies f = \sum_i s_i^* s_i + \sum_j \sum_i t_{ji}^* g_j t_{ji} \text{ with } s_i, t_{ji} \in \mathbb{R}\langle \underline{x} \rangle$$

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NC variant of Lasserre's Hierarchy for λ_{\min} :

💡 replace " $f - \lambda \mathbf{I} \succcurlyeq 0$ on \mathbf{X} " by $f - \lambda \mathbf{I} = \sum_i s_i^* s_i + \sum_j \sum_i t_{ji}^* g_j t_{ji}$
with s_i, t_{ji} of **bounded** degrees = SDP 🎯

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Theorem [Helton & McCullough '02]

$f \succcurlyeq 0 \Leftrightarrow f$ SOS (all positive polynomials are sums of squares)

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[Klep Magron Povh '21]

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Theorem: Sparse NC Positivstellensatz [Klep Magron Povh '21]

$f = \sum_k f_k$, f_k depends on $x(I_k)$

$f > 0$ on $\{x : g_j(x) \geq 0\}$

chordal graph with cliques $I_k \implies$

ball constraints for each $x(I_k)$

$$f = \sum_{k,i} (s_{ki}^* s_{ki} + \sum_{j \in J_k} t_{ji}^* g_j t_{ji})$$

s_{ki} "sees" vars in I_k

t_{ji} "sees" vars from g_j

Application to violation of Bell inequalities

I₃₃₂₂ Bell inequality (entanglement in quantum information)

$$f = a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 - b_3) + a_3(b_1 - b_2) - a_1 - 2b_1 - b_2$$

Maximal violation levels \rightarrow **upper bounds** on λ_{\max} of f on $\{a, b : a_i^2 = a_i \quad b_i^2 = b_i \quad a_i b_j = b_j a_i\}$

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level	sparse	dense [Pál & Vértesi '18]
2	0.2550008	0.2509397

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3	0.2511592	0.2508756
3'		0.2508754 (1 day)
4	0.2508917	
5	0.2508763	

Application to violation of Bell inequalities

I₃₃₂₂ Bell inequality (entanglement in quantum information)

$$f = a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 - b_3) + a_3(b_1 - b_2) - a_1 - 2b_1 - b_2$$

Maximal violation levels \rightarrow **upper bounds** on λ_{\max} of f on

$$\{a, b : a_i^2 = a_i \quad b_i^2 = b_i \quad a_i b_j = b_j a_i\}$$

💡 $I_k \rightarrow \{a_k, b_1, b_2, b_3\}$

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6	0.2508753977180	(1 hour)

PERFORMANCE



VS



ACCURACY

Application to SOS of bounded degrees

Theorem: sparse BSOS representation [Weisser et al. '18]

If $0 \leq g_j \leq 1$ on \mathbf{X} , $f > 0$ on \mathbf{X} & RIP holds for (I_k) then

$$f = \sum_k \left(\sigma_k + \sum_{\alpha, \beta} c_{k, \alpha \beta} \prod_{j \in I_k} g_j^{\alpha_j} (1 - g_j)^{\beta_j} \right),$$

with σ_k SOS of degree $\leq 2r$

Application to sparse positive definite forms

Theorem: [Reznick '95] Positivstellensatz

$$\text{pd form } f \implies f = \frac{\sigma}{\|\mathbf{x}\|_2^{2r}} \text{ with } \sigma \text{ SOS, } r \in \mathbb{N}$$

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Sparse $f \sum_k f_k$, with f_k only depends on I_k

RUNNING INTERSECTION PROPERTY (RIP)

$$\forall k \quad \underbrace{I_k \cap \bigcup_{j < k} I_j}_{\hat{I}_k} \subseteq I_{s_k} \quad \text{for some } s_k < k$$

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Theorem: sparse Reznick [Mai Lasserre Magron '20]

$$\text{RIP} \implies f = \sum_k \frac{\sigma_k}{H_k^r} \text{ with } \sigma_k \text{ SOS only depends on } I_k$$

Uniform H_k involve products $\|\mathbf{x}(I)\|_2^2$ for $I \in \{I_k, \hat{I}_k, \hat{I}_i : s_i = k\}$

More and more applications!

Robust Geometric Perception [Yang & Carlone '20]

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Robustness of implicit deep networks [Chen et al. '21]

Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

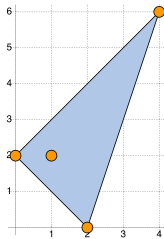
Tutorial session

Term sparsity via Newton polytope

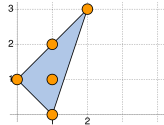
$$f = 4x_1^4x_2^6 + x_1^2 - x_1x_2^2 + x_2^2$$

$$\text{spt}(f) = \{(4,6), (2,0), (1,2), (0,2)\}$$

Newton polytope $\mathcal{B} = \text{conv}(\text{spt}(f))$



Squares in SOS decomposition $\subseteq \frac{\mathcal{B}}{2} \cap \mathbb{N}^n$
 [Reznick '78]



$$f = \left(x_1 \quad x_2 \quad x_1x_2 \quad x_1x_2^2 \quad x_1^2x_2^3 \right) \underbrace{Q}_{\succeq 0} \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1x_2^2 \\ x_1^2x_2^3 \end{pmatrix}$$

Term sparsity: the unconstrained case

$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 \\ + 6x_3^2 + 18x_2^2x_3 - 54x_2x_3^2 + 142x_2^2x_3^2$$

[Reznick '78] $\rightarrow f = (1 \quad x_1 \quad x_2 \quad x_3 \quad x_1x_2 \quad x_2x_3)$

$\rightsquigarrow \frac{6 \times 7}{2} = 21$ “unknown” entries in \mathcal{Q}

$$\underbrace{\mathcal{Q}}_{\succeq 0} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}$$

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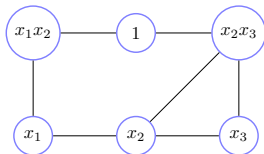
$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 18x_2^2x_3 - 54x_2x_3^2 + 142x_2^2x_3^2$$

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💡 **Term sparsity pattern graph G**



Term sparsity: the unconstrained case

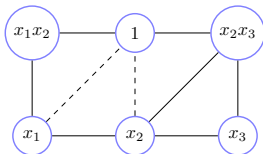
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💡 **Term sparsity pattern graph G**
+ chordal extension G'



Term sparsity: the unconstrained case

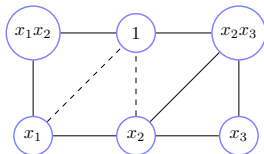
$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 18x_2^2x_3 - 54x_2x_3^2 + 142x_2^2x_3^2$$

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$$\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}$$

💡 **Term sparsity pattern graph G**
+ chordal extension G'



Replace Q by $Q_{G'}$ with nonzero entries at edges of G'

$\rightsquigarrow 6 + 9 = 15$ "unknown" entries in $Q_{G'}$

Term sparsity: the constrained case

At step r of the hierarchy, tsp graph G has

Nodes $V =$ monomials of degree $\leq r$

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Edges E with

$$\{\alpha, \beta\} \in E \Leftrightarrow \alpha + \beta \in \text{supp } f \cup \bigcup_{|\alpha| \leq r} \text{supp } g_j \cup 2\alpha$$

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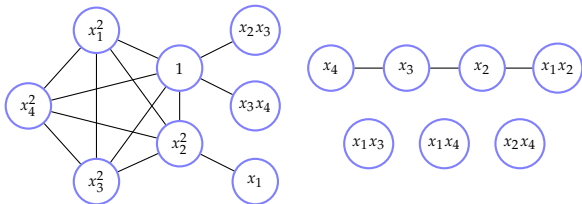
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An example with $r = 2$

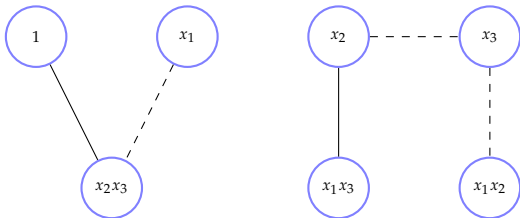
$$f = x_1^4 + x_1x_2^2 + x_2x_3 + x_3^2x_4^2$$

$$g_1 = 1 - x_1^2 - x_2^2 - x_3^2 \quad g_2 = 1 - x_3x_4$$



Term sparsity: support extension

$$\alpha' + \beta' = \alpha + \beta \text{ and } (\alpha, \beta) \in E \Rightarrow (\alpha', \beta') \in E$$



Term sparsity: the constrained case

At step r of the hierarchy, tsp graph G has

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Term sparsity: the constrained case

At step r of the hierarchy, **tsp** graph G has

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\rightsquigarrow **support extension**

Term sparsity: the constrained case

At step r of the hierarchy, **tsp** graph G has

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Edges E with

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\rightsquigarrow support extension \rightsquigarrow chordal extension G'

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\rightsquigarrow **support extension** \rightsquigarrow **chordal extension** G'

By iteratively performing **support extension** & **chordal extension**

$$G^{(1)} = G' \subseteq \dots \subseteq G^{(s)} \subseteq G^{(s+1)} \subseteq \dots$$

💡 Two-level hierarchy of lower bounds for f_{\min} , indexed by sparse order s and relaxation order r

Term sparsity: primal moment relaxations

Let G' be a chordal extension of G with maximal cliques (C_i)

$$C_i \longmapsto \mathbf{M}_{C_i}(\mathbf{y})$$

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💡 Each constraint $G_j \rightsquigarrow G_j^{(s)} \rightsquigarrow \text{cliques } C_{j,i}^{(s)}$

Term sparsity: primal moment relaxations

Let $C_{j,i}^{(s)}$ be the maximal cliques of $G_j^{(s)}$. For each $s \geq 1$

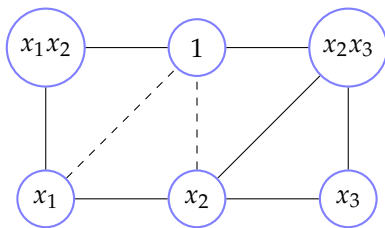
$$\begin{aligned} f_{\text{ts}}^{r,s} &= \inf \sum_{\alpha} f_{\alpha} y_{\alpha} \\ \text{s.t. } & \mathbf{M}_{C_{0,i}^{(s)}}(\mathbf{y}) \succeq 0 \\ & \mathbf{M}_{C_{j,i}^{(s)}}(g_j \mathbf{y}) \succeq 0 \\ & y_0 = 1 \end{aligned}$$

💡 dual yields the TSSOS hierarchy

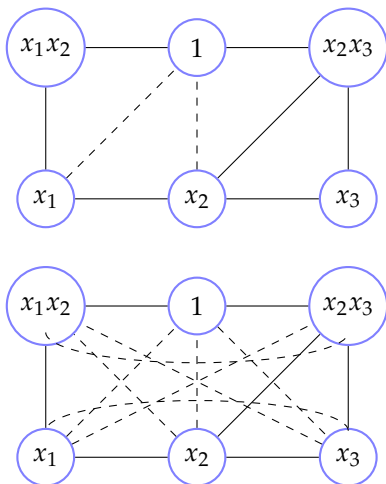
A two-level hierarchy of lower bounds

$$\begin{array}{cccc} f_{\text{ts}}^{r_{\min},1} \leq & f_{\text{ts}}^{r_{\min},2} \leq & \dots \leq & f^{r_{\min}} \\ \wedge | & \wedge | & & \wedge | \\ f_{\text{ts}}^{r_{\min}+1,1} \leq & f_{\text{ts}}^{r_{\min}+1,2} \leq & \dots \leq & f^{r_{\min}+1} \\ \wedge | & \wedge | & & \wedge | \\ \vdots & \vdots & \vdots & \vdots \\ \wedge | & \wedge | & & \wedge | \\ f_{\text{ts}}^{r,1} \leq & f_{\text{ts}}^{r,2} \leq & \dots \leq & f^r \\ \wedge | & \wedge | & & \wedge | \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Different choices of chordal extensions



Different choices of chordal extensions



Term sparsity: convergence guarantees

Theorem [Lasserre Magron Wang '21]

Fixing a sparse order s , the sequence $(f_{ts}^{r,s})_{r \geq r_{\min}}$ is monotonically non-decreasing.

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$$x_2 \mapsto -x_2$$



Sign-symmetries blocks

$$(1 \quad x_1 x_2^2 \quad x_1^2 x_2^2) \quad (x_1 x_2 \quad x_1^2 x_2)$$

TSSOS blocks

$$(1 \quad x_1 x_2^2 \quad x_1^2 x_2^2) \quad (x_1 x_2) \quad (x_1^2 x_2)$$

A second key message

 **TSSOS preserves the block structure related to sign-symmetries** 

Comparison with (S)DSOS

Let f be a nonnegative polynomial of degree $2d$

f is SOS $\Leftrightarrow f = \mathbf{v}^T \mathbf{Q} \mathbf{v}$ with $\mathbf{Q} \succeq 0 \rightsquigarrow$ semidefinite program

where \mathbf{v} contains $1, x_1, \dots, x_n, x_1^2, \dots, x_n^d$

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To reduce the number of “unknown” entries in \mathbf{Q} , one can force:

[Ahmadi & Majumdar '14]

- 1 \mathbf{Q} diagonally dominant: $Q_{ii} \geq \sum_{j \neq i} Q_{ij} \rightsquigarrow$ linear program
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Theorem [Lasserre Magron Wang '21]

The first TSSOS relaxation is always more accurate than the SDSOS relaxation

Combining correlative & term sparsity

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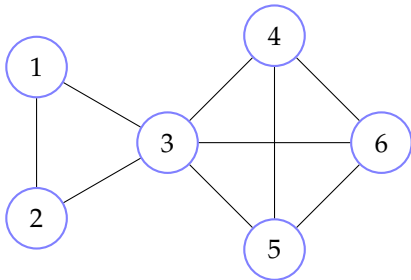
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- 💡 a two-level CS-TSSOS hierarchy of lower bounds for f_{\min}

Combining correlative & term sparsity

$$f = 1 + \sum_{i=1}^6 x_i^4 + x_1x_2x_3 + x_3x_4x_5 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6$$

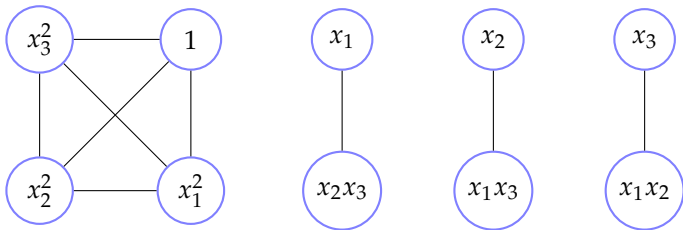
csp graph



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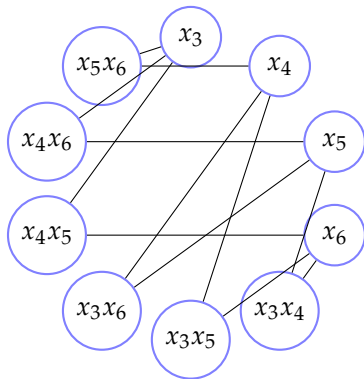
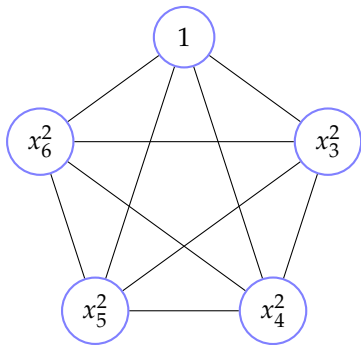
tsp graph for the first clique



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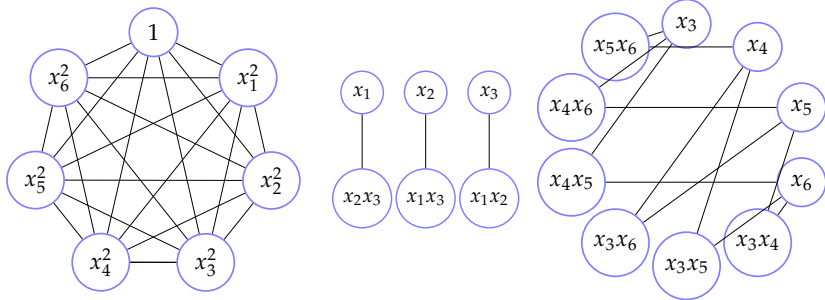
tsp graph for the second clique



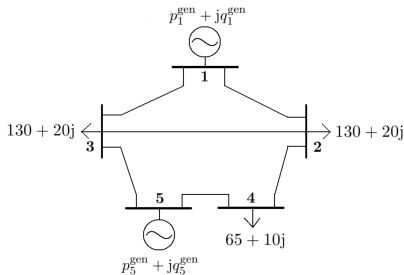
Combining correlative & term sparsity

$$f = 1 + \sum_{i=1}^6 x_i^4 + x_1x_2x_3 + x_3x_4x_5 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6$$

tsp graph without correlative sparsity



Application to optimal power-flow



Optimal Powerflow $n \simeq 10^3$
[Josz et al. '18]

$$\left\{ \begin{array}{l} \inf_{V_i, S_s^g, S_{ij}} \quad \sum_{s \in G} (\mathbf{c}_{2s} (\Re(S_s^g))^2 + \mathbf{c}_{1s} \Re(S_s^g) + \mathbf{c}_{0s}) \\ \text{s.t.} \quad \angle V_{\text{ref}} = 0, \\ \mathbf{S}_s^{gl} \leq S_s^g \leq \mathbf{S}_s^{gu} \quad \forall s \in G, \quad \mathbf{v}_i^l \leq |V_i| \leq \mathbf{v}_i^u \quad \forall i \in N \\ \sum_{s \in G_i} S_s^g - \mathbf{S}_i^d - \mathbf{Y}_i^s |V_i|^2 = \sum_{(i,j) \in E_i \cup E_i^R} S_{ij}, \quad \forall i \in N \\ S_{ij} = (\mathbf{Y}_{ij}^* - \mathbf{i} \frac{\mathbf{b}_{ij}^c}{2}) \frac{|V_i|^2}{|\mathbf{T}_{ij}|^2} - \mathbf{Y}_{ij}^* \frac{V_i V_j^*}{\mathbf{T}_{ij}}, \quad S_{ji} = \dots \\ |S_{ij}| \leq \mathbf{s}_{ij}^u, \quad \theta_{ij}^{\Delta l} \leq \angle(V_i V_j^*) \leq \theta_{ij}^{\Delta u}, \quad \forall (i,j) \in E \end{array} \right.$$

Application to optimal power-flow

mb = the maximal size of blocks

m = number of constraints

n	m	CS ($r = 2$)			CS+TS ($r = 2, s = 1$)		
		mb	time (s)	gap	mb	time (s)	gap
114	315	66	5.59	0.39%	31	2.01	0.73%
348	1809	253	—	—	34	278	0.05%
766	3322	153	585	0.68%	44	33.9	0.77%
1112	4613	496	—	—	31	410	0.25%
4356	18257	378	—	—	27	934	0.51%
6698	29283	1326	—	—	76	1886	0.47%

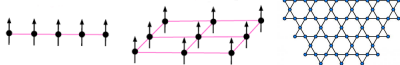
Application to noncommutative optimization

Ground-state energy \Leftrightarrow minimal eigenvalue of an Hamiltonian

$$H = \sum_{\langle i,j \rangle} (x_i x_j + y_i y_j + z_i z_j)$$

spin states (x_i, y_i, z_i) , constraints

Lattices: 1D 2D Kagome



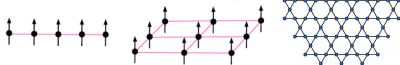
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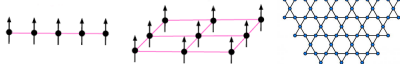
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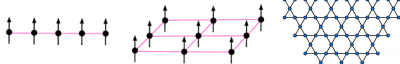
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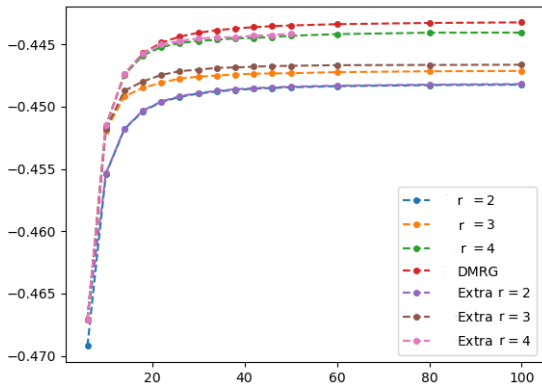
periodic boundary conditions \Rightarrow $n + 1 = 1$

Existing \pm efficient techniques: quantum Monte Carlo & variational algorithms \Rightarrow **upper bounds** on minimal energy

Application to noncommutative optimization

Lower bounds of the energy

1D lattice

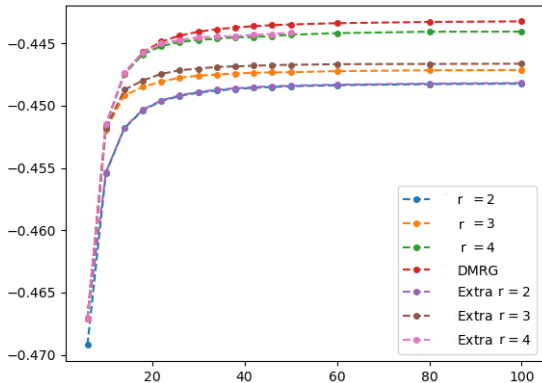


Dense $r = 4, n = 10^2 \Rightarrow 10^{11}$ variables (solvers handle $\simeq 10^4$)

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Sparse solved within 1 hour on PFCALCUL at LAAS

Application to noncommutative optimization

CLASSICAL WORLD

$$\psi^*(A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2)\psi \leq 2$$

for separable states $\psi \in \mathbb{C}^k \otimes \mathbb{C}^k$ and matrices A_j, B_j satisfying $A_j^* = A_j, A_j^2 = I, B_j^* = B_j, B_j^2 = I$

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Application to noncommutative optimization

COVARIANCES OF QUANTUM CORRELATIONS

$$\text{cov}_\psi(A, B) = \psi^*(A \otimes B)\psi - \psi^*(A \otimes I)\psi \cdot \psi^*(I \otimes B)\psi$$

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💡 2nd sparse SDP gives also 5 ... **10 times faster**

Application to networked systems stability

Lyapunov function

$$f = \sum_{i=1}^N a_i (x_i^2 + x_i^4) - \sum_{i,k=1}^N b_{ik} x_i^2 x_k^2 \quad a_i \in [1, 2] \quad b_{ik} \in \left[\frac{0.5}{N}, \frac{1.5}{N}\right]$$

$\rightsquigarrow \binom{N+2}{2} (\binom{N+2}{2} + 1) / 2$ “unknown” entries in $Q = 231$ for $N = 5$

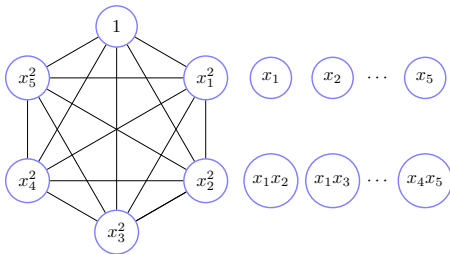
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💡 **tsp** graph G



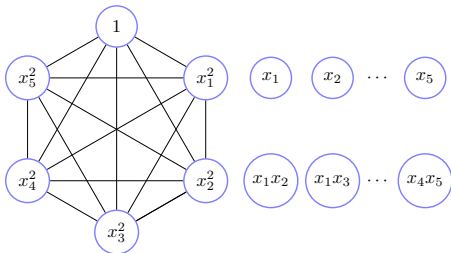
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$\rightsquigarrow (N+1)^2$ “unknown” entries in $Q_G = 36$ for $N = 5$

Proof that $f \geq 0$ for $N = 80$ in ~ 10 seconds!

Application to networked systems stability

Duffing oscillator Hamiltonian $V = \sum_{i=1}^N a_i \left(\frac{x_i^2}{2} - \frac{x_i^4}{4} \right) + \frac{1}{8} \sum_{i,k=1}^N b_{ik} (x_i - x_k)^4$

On which domain $V > 0$? $f = V - \sum_{i=1}^N \underbrace{\lambda_i}_{>0} x_i^2 (g - x_i^2) \geq 0$

$$\implies V > 0 \text{ when } x_i^2 < g$$

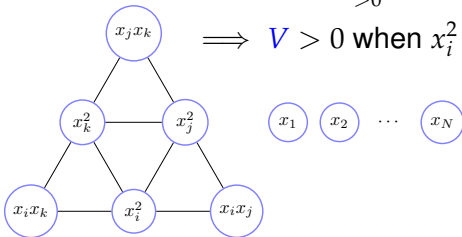
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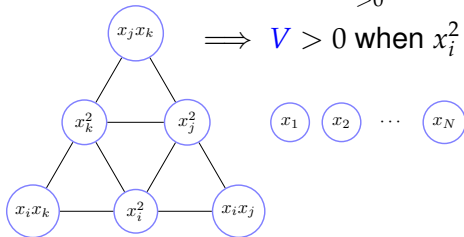
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$\rightsquigarrow \frac{N(N+1)}{2} + 6\binom{N}{2} + N$ “unknown” entries in $Q_G = 80$ for $N = 5$

Proof that $f \geq 0$ for $N = 50$ in ~ 1 second!

Application to joint spectral radius (JSR)

Given $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathbb{R}^{n \times n}$, the JSR is

$$\rho(\mathcal{A}) := \lim_{k \rightarrow \infty} \max_{\sigma \in \{1, \dots, m\}^k} \|A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_k}\|^{1/k}$$

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Theorem [Parrilo & Jadbabaie '08]

Given $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathbb{R}^{n \times n}$, if a positive definite form f of degree $2r$ satisfies

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Bisection on γ + SDP

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Theorem: Sparse JSR [Maggio Magron Wang '21]

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Closed-loop system evolves according to either a **completed** or a **missed** computation (A_H or A_M): $\mathcal{A} = \{A_H A_M^i \mid i < m\}$

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💡 takes less than 10 seconds with the Sparse JSR!

Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

Tutorial session

Ideal sparsity

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Theorem [Korda-Laurent-Magron-Steenkamp '22]

Ideal-sparse hierarchies provide better bounds than the dense ones



ACCURACY

Application to matrix ranks

Given a symmetric nonnegative matrix A , find the smallest r_+ s.t.

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✗ hard to compute

✓ Relax/convexify with a linear program over measures

$$r_+ \geq \inf_{\mu} \left\{ \int_{K_A} 1 d\mu : \int_{K_A} x_i x_j d\mu = A_{ij} \ (i, j \in V), \ \text{supp}(\mu) \subseteq K_A \right\}$$

Application to matrix ranks

Given a symmetric nonnegative matrix A , find the smallest r_+ s.t.

$$A = \sum_{\ell=1}^{r_+} a_\ell a_\ell^T \quad \text{for } a_\ell \geq 0$$

r_+ is called the completely positive rank

✗ hard to compute

✓ Relax/convexify with a linear program over measures

$$r_+ \geq \inf_{\mu} \left\{ \int_{K_A} 1 d\mu : \int_{K_A} x_i x_j d\mu = A_{ij} \ (i, j \in V), \quad \text{supp}(\mu) \subseteq K_A \right\}$$

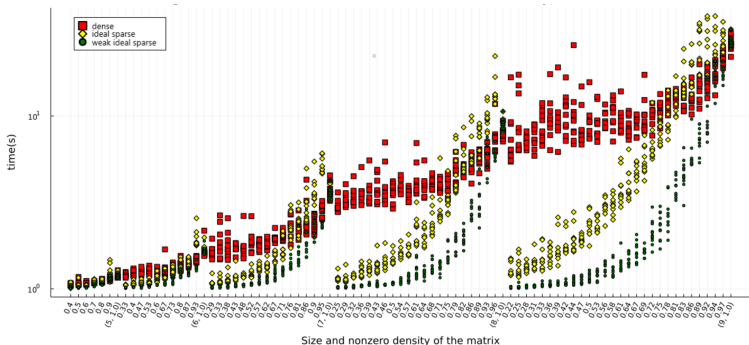
$$K_A = \{ \mathbf{x} : \sqrt{A_{ii}} x_i - x_i \geq 0, \quad A_{ij} - x_i x_j \geq 0 \ (i, j) \in E_A, \\ x_i x_j = 0 \ (i, j) \in \bar{E}_A, \quad A - \mathbf{x}\mathbf{x}^T \succeq 0 \}$$

Application to matrix ranks

Random instances, order 2

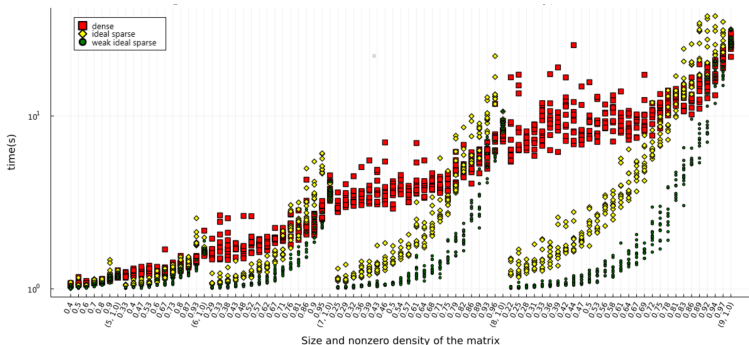
Application to matrix ranks

Random instances, order 2



Application to matrix ranks

Random instances, order 2



PERFORMANCE



AND



ACCURACY

Conclusion

SPARSITY EXPLOITING **CONVERGING HIERARCHIES** to minimize polynomials, eigenvalue/trace, joint spectral radii

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FAST IMPLEMENTATION IN JULIA: **TSSOS**, **NCTSSOS**, **SparseJSR**

Conclusion

SPARSITY EXPLOITING CONVERGING HIERARCHIES to minimize polynomials, eigenvalue/trace, joint spectral radii

FAST IMPLEMENTATION IN JULIA: TSSOS, NCTSSOS, SparseJSR

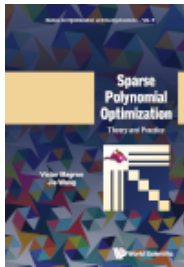
💡 Combine correlative & term sparsity for problems with $n = 10^3$

Conclusion

SPARSITY EXPLOITING **CONVERGING HIERARCHIES** to minimize polynomials, eigenvalue/trace, joint spectral radii

FAST IMPLEMENTATION IN JULIA: **TSSOS**, **NCTSSOS**, **SparseJSR**

💡 **Combine** correlative & term sparsity for problems with $n = 10^3$



Further topics

Correlative sparsity: Convergence rate?



Further topics

Correlative sparsity: Convergence rate?

Term sparsity: Smart solution extraction?



Further topics

Correlative sparsity: Convergence rate?



Term sparsity: Smart solution extraction?

Ideal sparsity: tensor (nonnegative, symmetric) ranks?

Further topics

Correlative sparsity: Convergence rate?



Term sparsity: Smart solution extraction?

Ideal sparsity: tensor (nonnegative, symmetric) ranks?

Numerical conditioning of sparse SDP?

Further topics



Correlative sparsity: Convergence rate?

Term sparsity: Smart solution extraction?

Ideal sparsity: tensor (nonnegative, symmetric) ranks?

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Combination with symmetries?

Further topics



Correlative sparsity: Convergence rate?

Term sparsity: Smart solution extraction?

Ideal sparsity: tensor (nonnegative, symmetric) ranks?

Numerical conditioning of sparse SDP?

Combination with symmetries?






💡 Tons of applications!

Thank you for your attention!








`https://homepages.laas.fr/vmagron`

GITHUB:TSSOS








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Sparse SDP

Correlative sparsity

Term sparsity

Ideal sparsity

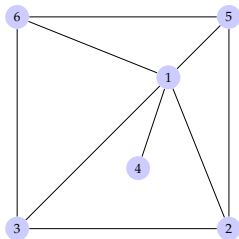
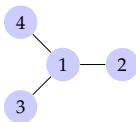
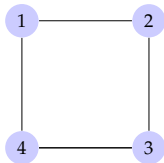
Tutorial session

Newton polytope

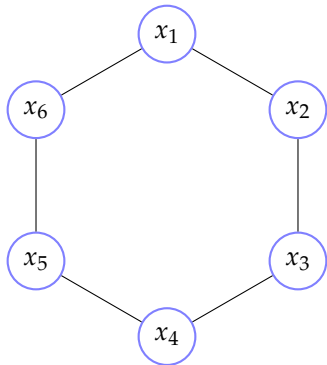
Motzkin $f = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$

- 1 Compute the Newton polytope of f
- 2 Show that f is not SOS

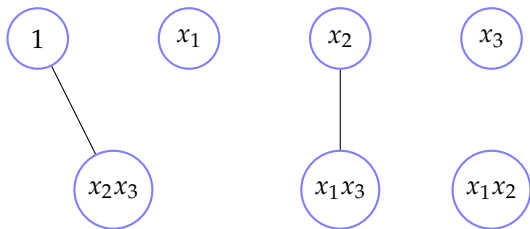
Chordal or not chordal?



Chordal extension



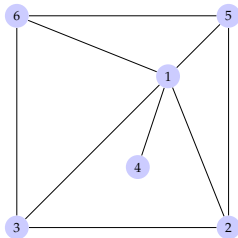
Support extension



How big is CS?

$$f(\mathbf{x}) = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

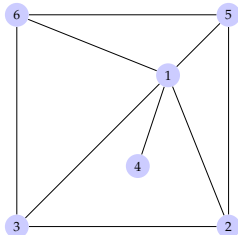
Chordal graph after adding edge (3,5)



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Chordal graph after adding edge (3,5)



How many SDP variables in the dense and sparse relaxation at order $r = 1, 2, 3$?

Moment matrix

Write the first (correlative) sparse moment relaxation of

$$\begin{aligned} \inf_x \quad & x_1x_2 + x_1x_3 + x_1x_4 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 1 \\ & x_1^2 + x_3^2 \leq 1 \\ & x_1^2 + x_4^2 \leq 1 \end{aligned}$$

Measure LP preserves sparsity

$f = f_1 + f_2$, f_k depends on I_k , \mathbf{X} compact & each g_j depends either on I_1 or I_2 .

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$$f_{\min} = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f d\mu = f_{\text{CS}} = \inf_{\mu_1, \mu_2} \int_{\mathbf{X}_1} f_1 d\mu_1 + \int_{\mathbf{X}_2} f_2 d\mu_2$$

s.t. $\pi_{12}\mu_1 = \pi_{21}\mu_2$
 $\mu_1 \in \mathcal{M}_+(\mathbf{X}_1), \quad \mu_2 \in \mathcal{M}_+(\mathbf{X}_2)$

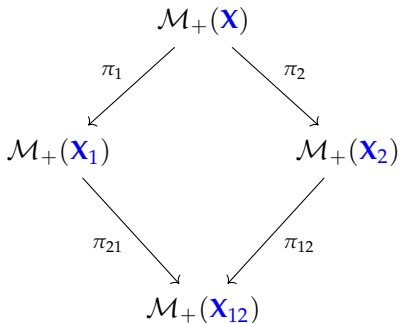
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💡 (μ_k) feasible for $f_{\text{CS}} \implies \exists \mu \in \mathcal{M}_+(\mathbf{X})$ with marginal μ_k on \mathbf{X}_k



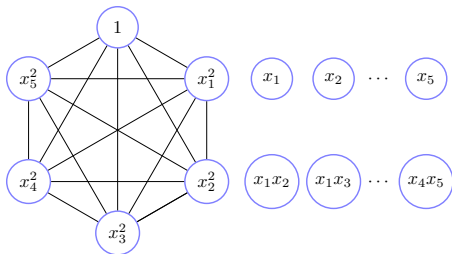
$$f = \sum_{i=1}^N (x_i^2 + x_i^4) - \sum_{i,k=1}^N x_i^2 x_k^2$$

How many entries in the dense & sparse SOS/moment matrices?

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💡 **tsp** graph G



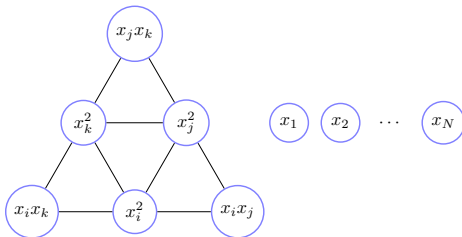
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Compute the dense relaxation f^2

Compare with the correlative sparse relaxation f_{cs}^2

Compare with the term sparse relaxation $f_{ts}^{2,s}$ for $s = 1, 2, \dots$

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using TSSOS, DynamicPolynomials
```

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f = f1+f2
dense2,sol,data=cs_tssos_first([f], x, 2,
CS=false,TS=false);
```

Download from <https://homepages.laas.fr/vmagron/ncball>:

$$f = f_1 + f_2 \quad \mathbb{B}_{nc} = \{x : 1 - x_1^2 - x_2^2 - x_3^2 \succcurlyeq 0, 1 - x_2^2 - x_3^2 - x_4^2 \succcurlyeq 0\}$$

Download from <https://homepages.laas.fr/vmagron/ncball>:

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Compute $\lambda_{\min}(f)$ on \mathbb{B}_{nc} with 2nd dense relaxation

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```
cs_nctssos_first([f;ncball],x,2,CS=false, TS=false,  
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Compare with the correlative and term sparse relaxations