Supplementary Material
Logarithmic Mathematical Morphology: theory and applications

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APPENDIX

PROOFS OF PROPOSITIONS 1 (P. 4) AND 2 (P. 4)

In this section, the proofs of propositions 1 and 2 are given. They were previously demonstrated in [18].

Proof of proposition 1: $\delta_b^{\triangleq}$ is a dilation (def. 1) because as the addition $\triangleq$ preserves the order $\leq$ [16], we have $\forall f, g \in \overline{\mathcal{F}}_M$, $\forall x \in D$,

$$\begin{aligned}\delta_b^{\triangleq}(f \vee g)(x) &= \vee_{h\in D}\left\{[(f \vee g)(x-h)] \triangleq b(h)\right\}\\ &= \vee_{h\in D}\left\{[f(x-h) \triangleq b(h)] \vee [g(x-h) \triangleq b(h)]\right\}\\ &= [\vee_{h\in D}\left\{f(x-h) \triangleq b(h)\right\}] \vee [\vee_{h\in D}\left\{g(x-h) \triangleq b(h)\right\}]\\ &= \delta_b^{\triangleq}(f)(x) \vee \delta_b^{\triangleq}(g)(x).\end{aligned}$$

For the least element O of $\overline{\mathcal{F}}_M$, we have $\forall x \in D$:

$$\begin{aligned}\delta_b^{\triangleq}(O)(x) = \delta_b^{\triangleq}(f_{-\infty})(x) &= \vee_{h\in D}\{(-\infty(x-h) \triangleq b(h))\}\\ &= \vee_{h\in D}\{-\infty(1-b(h)/M) + b(h)\} = -\infty\\ &= O(x).\end{aligned}$$

Similarly, $\varepsilon_b^{\triangleq}$ is an erosion because we have $\forall f, g \in \overline{\mathcal{F}}_M$, $\varepsilon_b^{\triangleq}(f \wedge g) = \varepsilon_b^{\triangleq}(f) \wedge \varepsilon_b^{\triangleq}(g)$ and $\varepsilon_b^{\triangleq}(I) = \varepsilon_b^{\triangleq}(f_M) = M = I$. Finally, the pair $(\varepsilon_b^{\triangleq}, \delta_b^{\triangleq})$ is an adjunction because, $\forall f, g \in \overline{\mathcal{F}}_M$, we have:

$$\begin{aligned}\delta_b^{\triangleq}(f) \leq g &\Leftrightarrow \forall x \in D,\ \vee_{h\in D}\{f(x-h) \triangleq b(h)\} \leq g(x)\\ &\Leftrightarrow \forall x, h \in D,\ f(x-h) \triangleq b(h) \leq g(x)\\ &\Leftrightarrow \forall y, h \in D,\ f(y) \leq g(y+h) \triangleq b(h)\\ &\Leftrightarrow \forall y \in D,\ f(y) \leq \wedge_{h\in D}\{g(y+h) \triangleq b(h)\}\\ &\Leftrightarrow f \leq \varepsilon_b^{\triangleq}(g).\end{aligned}$$

□

Proof of proposition 2: The logarithmic-erosion $\varepsilon_b^{\triangleq}$ and dilation $\delta_b^{\triangleq}$ are dual by their *negative function* because for $b \in \overline{\mathcal{F}}_M$ we have, $\forall f, g \in \overline{\mathcal{F}}_M$, $\forall x \in D$,

$$\begin{aligned}(\delta_b^{\triangleq}(f^*))^*(x) &= \triangleq(\vee_{h\in D}\{\triangleq f(x-h) \triangleq b(h)\})\\ &= \wedge_{h\in D}\{f(x-h) \triangleq b(h)\}\\ &= \wedge_{h\in D}\{f(x+h) \triangleq \overline{b}(h)\}\\ &= \varepsilon_{\overline{b}}^{\triangleq}(f)(x).\end{aligned}$$

Similarly, we have $(\varepsilon_b^{\triangleq}(f^*))^* = \delta_{\overline{b}}^{\triangleq}(f)$. □

PROOF OF PROPOSITION 3 (P. 5)

In this section, the proof of proposition 3 is given. It was previously demonstrated in [18]. The dilation δ_b and the erosion ε_b are mappings of the lattice $\overline{\mathbb{R}}^D$, whereas the logarithmic-dilation $\delta_b^{\triangleq}$ and erosion $\varepsilon_b^{\triangleq}$ are mappings of the lattice $\overline{\mathcal{F}}_M$. In order to link these operations, a bijective mapping (i.e. an isomorphism) is needed between these two lattices. Such an isomorphism $\xi : \overline{\mathcal{F}}_M \to \overline{\mathbb{R}}^D$ and its inverse $\xi^{-1} : \overline{\mathbb{R}}^D \to \overline{\mathcal{F}}_M$ were both defined in [44] by $\xi(f) = -M\ln(1-f/M)$ and $\xi^{-1}(f) = M(1-\exp(-f/M))$.

We also need to demonstrate the following lemma.

Lemma 1. *Let $f, g \in \overline{\mathcal{F}}_M$ be two functions. The isomorphism ξ transforms the LIP-sum $\triangleq$ and the LIP-difference $\triangleq$ between functions into the usual sum $+$ and difference $-$, respectively. We have*

$$\xi(f \triangleq g) = \xi(f) + \xi(g), \tag{A.1}$$

$$\xi(f \triangleq g) = \xi(f) - \xi(g). \tag{A.2}$$

Proof of lemma 1: Let $f, g \in \overline{\mathcal{F}}_M$ be two functions. There is:

$$\begin{aligned}
\xi(f \mathbin{\overline{\triangle}} g) &= -M \ln\left(1 - (f \mathbin{\overline{\triangle}} g)/M\right) \\
&= -M \ln\left((1 - f/M)(1 - g/M)\right) \\
&= -M \ln\left(1 - f/M\right) - M \ln\left(1 - g/M\right) \\
&= \xi(f) + \xi(g), \\
\xi(\overline{\triangle} g) &= -M \ln\left(1 + g/(M - g)\right) = -M \ln\left(M/(M - g)\right) \\
&= M \ln\left(1 - g/M\right) = -\xi(g), \\
\xi(f \mathbin{\underline{\triangle}} g) &= \xi(f \mathbin{\overline{\triangle}} (\overline{\triangle} g)) = \xi(f) + \xi(\overline{\triangle} g) = \xi(f) - \xi(g).
\end{aligned}$$

□

We can now establish the proof of proposition 3.

Proof of proposition 3: Let f be a function and b a structuring function of $\overline{\mathcal{F}}_M$. Let $\acute{f} : D \to \overline{\mathbb{R}}$ be a function equal to $\acute{f} = -\ln\left(1 - f/M\right)$. As increasing bijections, the isomorphisms ξ and ξ^{-1} distribute over infima and suprema. The dilation $\delta_b^{\overline{\triangle}}$ can therefore be expressed for all $x \in D$ by:

$$\begin{aligned}
\delta_b^{\overline{\triangle}}(f)(x) &= \xi^{-1} \circ \xi[\vee_{h \in D} \{f(x - h) \mathbin{\overline{\triangle}} b(h)\}] \\
&= \xi^{-1}\left[\vee_{h \in D} \{\xi[f(x - h) \mathbin{\overline{\triangle}} b(h)]\}\right] \\
&= \xi^{-1}\left[\vee_{h \in D} \{\xi[f(x - h)] + \xi[b(h)]\}\right] \\
&= \xi^{-1}\left[\delta_{\xi(b)}[\xi(f)]\right]
\end{aligned} \tag{A.3}$$

From (A.3), we have:

$$\begin{aligned}
\delta_b^{\overline{\triangle}}(f)(x) &= M\left[1 - e^{\frac{-1}{M} \vee_{h \in D}\left\{-M \ln\left[1 - \frac{f(x-h)}{M}\right] - M \ln\left[1 - \frac{b(h)}{M}\right]\right\}}\right] \\
&= M\left[1 - e^{\frac{-1}{M} \vee_{h \in D}\left\{M\left(-\ln\left[1 - \frac{f(x-h)}{M}\right] - \ln\left[1 - \frac{b(h)}{M}\right]\right)\right\}}\right] \\
&= M\left[1 - e^{\frac{-M}{M} \vee_{h \in D}\left\{-\ln\left[1 - \frac{f(x-h)}{M}\right] - \ln\left[1 - \frac{b(h)}{M}\right]\right\}}\right] \\
&= M\left[1 - e^{-\vee_{h \in D}\left\{\acute{f}(x-h) + \acute{b}(h)\right\}}\right] \\
&= M\left[1 - e^{-\delta_{\acute{b}}(\acute{f})(x)}\right].
\end{aligned}$$

Similarly, we have

$$\varepsilon_b^{\overline{\triangle}}(f) = \xi^{-1}\left[\varepsilon_{\xi(b)}[\xi(f)]\right] = M[1 - e^{-\varepsilon_{\acute{b}}(\acute{f})}].$$

This ends the proof. □

PROOF OF PROPOSITION 4 (P. 6)

Proof of proposition 4: From (15), we have $\forall x \in D$:

$$\begin{aligned}
c_{1_b}(f)(x) &= \vee \{f(x + h) \mathbin{\underline{\triangle}} b(h), h \in D_b\} \\
&= \vee \{f(x - h) \mathbin{\underline{\triangle}} b(-h), -h \in D_b\} \\
&= \vee \left\{f(x - h) \mathbin{\overline{\triangle}} (\overline{\triangle}\overline{b}(h)), h \in D_{\overline{b}}\right\} \\
&= \delta_{\overline{\triangle}\overline{b}}^{\overline{\triangle}}(f)(x).
\end{aligned}$$

By comparing (8) with (16), we have for all $x \in D$:

$$c_{2_b}(f)(x) = \varepsilon_b^{\overline{\triangle}}(f)(x).$$

From (14), we deduce that $Asp_b^{\overline{\triangle}}(f) = \delta_{\overline{\triangle}\overline{b}}^{\overline{\triangle}}(f) \mathbin{\underline{\triangle}} \varepsilon_b^{\overline{\triangle}}(f)$. □

MAP OF ASPLUND DISTANCES: LINK BETWEEN DOUBLE PROBING AND LOGARITHMIC-OPENING AND CLOSING

Let us study the location of the supremum or infimum of the translated probes $b \in \overline{\mathcal{F}}_M$ such that they are in contact with the function f from below or from above, respectively. In a similar way to that used by Heijmans et al. in [1], the mapping $\tau_{h,v}$ of horizontal and vertical translations on $\overline{\mathcal{F}}_M$ is defined by $(\tau_{h,v}(f))(x) = f(x - h) \mathbin{\overline{\triangle}} v$, where $h \in D$ and $v \in]-\infty, M[$. For the lower probes, the supremum of the translated probes b which are in contact with the function f from below is the supremum of the translated probes $\tau_{h,v}(b)$ which are less or equal to the function f. It is equal to $\Gamma_b^{\overline{\triangle}}(f) = \sup\{\tau_{h,v}(b) \mid h \in D, v \in]-\infty, M[, \tau_{h,v}(b) \leq f\}$. As demonstrated in [2], $\Gamma_b^{\overline{\triangle}}(f)$ is an opening.

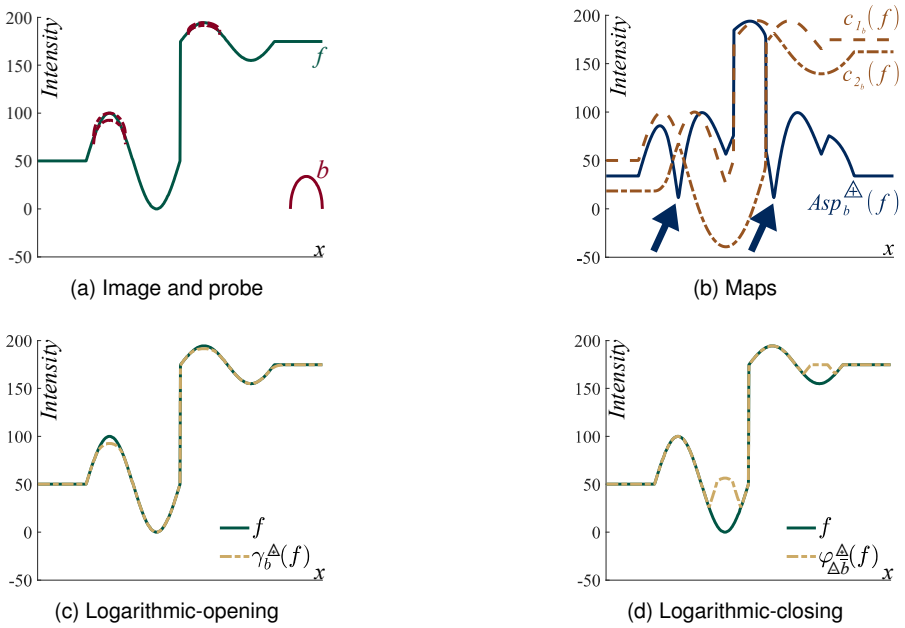

Fig. 1. (a) An image f is analysed by a probe b from above and below. (b) The *mlub* $c_{1_b}(f)$, the *mglb* $c_{2_b}(f)$ and the map of Asplund distances $Asp_b^{\triangleq}(f)$ between the image and the probe. (c) The supremum of the probe b such that the probe is in contact with the function f from below is the logarithmic-opening $\gamma_b^{\triangleq}$ of the function f. (d) The infimum of the probe b such that the probe is in contact with the function f from above is the logarithmic-closing $\varphi_{\triangle\bar{b}}^{\triangleq}(f)$ of the function f by the structuring function $\triangle\bar{b}$.

Similarly, for the upper probes, the infimum of the translated probes b which are in contact with the function f from above is equal to $\Phi_b^{\triangleq}(f) = \inf\{\tau_{h,v}(b) \mid h \in D, v \in]-\infty, M[, \tau_{h,v}(b) \geq f\}$. As demonstrated in [2], $\Phi_b^{\triangleq}(f)$ is a closing.

Proposition A.1. *The supremum of the translated probes b which are in contact with the function f from below, $\Gamma_b^{\triangleq}(f)$, is equal to the logarithmic-opening of f, $\gamma_b^{\triangleq}(f)$:*

$$\Gamma_b^{\triangleq}(f)(x) = \gamma_b^{\triangleq}(f)(x). \tag{A.4}$$

The infimum of the translated probes b which are in contact with the function f from above, $\Phi_b^{\triangleq}(f)$, is equal to the logarithmic-closing of f, $\varphi_{\triangle\bar{b}}^{\triangleq}(f)$:

$$\Phi_b^{\triangleq}(f)(x) = \varphi_{\triangle\bar{b}}^{\triangleq}(f)(x). \tag{A.5}$$

Fig. 1c and 1d illustrate the supremum or infimum of the translated probes b which are in contact with the function f from below or from above, respectively. In Fig. 1c, one can notice that the opening $\gamma_b^{\triangleq}(f)$ of f is very close the image f. In Fig. 1d, the closing $\varphi_{\triangle\bar{b}}^{\triangleq}(f)$ of f is similar to the image f with the exception of both valleys of f, where the structuring function f is very dissimilar to f.

Proof of proposition A.1:

Let f and b be a function and a probe of $\overline{\mathcal{F}}_M$, respectively. For $h \in D$ and $v \in -]\infty, M[$, the mapping $\tau_{h,v}$ of horizontal and vertical translations on $\overline{\mathcal{F}}_M$ is defined by $(\tau_{h,v}(f))(x) = f(x-h) \triangleq v$. The supremum of the translated probes b which are in contact with the function f from below is equal to:

$$\begin{aligned}
\Gamma_b^{\triangleq}(f)(x) &= \sup\{\tau_{h,v}(b)(x) \mid h \in D, v \in]-\infty, M[, \tau_{h,v}(b) \leq f\} \\
&= \sup\{b(x-h) \triangleq v \mid h \in D, v \in]-\infty, M[, x' \in D,\ b(x'-h) \triangleq v \leq f(x')\} \\
&= \vee_{h\in D}\{b(x-h) \triangleq v(h) \mid v(h) = \sup_{x'\in D}\{v, b(x'-h) \triangleq v \leq f(x')\}\} \\
&= \vee_{h\in D}\{b(x-h) \triangleq v(h) \mid v(h) = \sup_{x'\in D}\{v, b(x') \triangleq v \leq f(h+x')\}\} \\
&= \vee_{h\in D}\{b(x-h) \triangleq c_{2_b}f(h)\} \quad \text{(from (16))} \\
&= \vee_{h\in D}\{b(x-h) \triangleq \varepsilon_b^{\triangleq}(f)(h)\} \quad \text{(from Prop. 4)} \\
&= \vee_{h\in D}\{\varepsilon_b^{\triangleq}(f)(h) \triangleq b(x-h)\} \\
&= \vee_{h\in D}\{\varepsilon_b^{\triangleq}(f)(x-h) \triangleq b(h)\} \\
&= \delta_b^{\triangleq}\varepsilon_b^{\triangleq}(f)(x) \quad \text{(from (7))} \\
&= \gamma_b^{\triangleq}(f)(x). \quad \text{(from (9))}
\end{aligned}$$

Such a relation is an opening one.

The infimum of the translated probes b which are in contact with the function f from above is equal to:

$$\begin{aligned}
\Phi_b^{\triangleq}(f)(x) &= \inf\{\tau_{h,v}(b)(x) \mid h \in D, v \in]-\infty, M[\,, \tau_{h,v}(b) \geq f\} \\
&= \inf\{b(x-h) \triangleq v \mid h \in D, v \in]-\infty, M[\,, x' \in D,\ b(x'-h) \triangleq v \geq f(x')\} \\
&= \wedge_{h \in D_b}\{b(x-h) \triangleq v(h) \mid v(h) = \inf_{x' \in D_b}\{v, b(x'-h) \triangleq v \geq f(x')\}\} \\
&= \wedge_{h \in D_b}\{b(x-h) \triangleq v(h) \mid v(h) = \inf_{x' \in D_b}\{v, b(h) \triangleq v \geq f(x'+h)\}\} \\
&= \wedge_{h \in D_b}\{b(x-h) \triangleq c_{1_b} f(h)\} \quad \text{(from (15))} \\
&= \wedge_{h \in D_b}\{b(x-h) \triangleq \delta^{\triangleq}_{\triangle \bar{b}}(f)(h)\} \quad \text{(from Prop. 4)} \\
&= \wedge_{h \in D_b}\{\delta^{\triangleq}_{\triangle \bar{b}}(f)(h) \triangleq b(x-h)\} \\
&= \wedge_{h \in D_b}\{\delta^{\triangleq}_{\triangle \bar{b}}(f)(x-h) \triangleq b(h)\} \\
&= \wedge_{-h \in D_b}\{\delta^{\triangleq}_{\triangle \bar{b}}(f)(x+h) \triangleq b(-h)\} \\
&= \wedge_{h \in D_{\bar{b}}}\{\delta^{\triangleq}_{\triangle \bar{b}}(f)(x+h) \triangle (\triangle \bar{b}(h))\} \\
&= \varepsilon^{\triangleq}_{\triangle \bar{b}} \delta^{\triangleq}_{\triangle \bar{b}}(f)(x) \quad \text{(from (8))} \\
&= \varphi^{\triangleq}_{\triangle \bar{b}}(f)(x) \quad \text{(from (10))}.
\end{aligned}$$

Such a relation is a closing one. □

PROOFS OF EQUATIONS (20) AND (21) (P. 7)

Proof of equation 20: Let B_0 a flat structuring element with the same support D_{b_0} as the symmetric and constant structuring element b_0. $b_0 : D \to \overline{\mathbb{R}}$ is defined for all $x \in D_{b_0}$, where $D_{b_0} \subset D$, by $b_0(x) = b_0$ and $b_0(-x) = b_0(x)$. For all $x \in D$, we have:

$$\begin{aligned}
Asp_{b_0}^{\triangleq}(f)(x) &= \delta^{\triangleq}_{\triangle \bar{b}_0}(f)(x) \triangle \varepsilon^{\triangleq}_{b_0}(f)(x) \\
&= \vee_{h \in D_{\bar{b}_0}}\left\{f(x-h) \triangleq (\triangle \bar{b}_0)\right\} \triangle \wedge_{h \in D_{b_0}}\left\{f(x+h) \triangle b_0\right\} \\
&= \left(\vee_{h \in D_{b_0}}\left\{f(x-h)\right\} \triangle b_0\right) \triangle \left(\wedge_{h \in D_{b_0}}\left\{f(x+h)\right\} \triangle b_0\right) \\
&= \vee_{h \in D_{b_0}}\left\{f(x-h)\right\} \triangle \wedge_{h \in D_{b_0}}\left\{f(x+h)\right\} \\
&= \delta_{B_0}(f)(x) \triangle \varepsilon_{B_0}(f)(x) \\
&= \varrho^{LIP}_{B_0}(f)(x).
\end{aligned}$$

□

Proof of equation 21: Let $Y \subset D$ be a constant (i.e. flat) zone of a function f and let $X = Y \ominus D_b$ be the eroded flat zone by the domain D_b of the structuring function b. $\ominus$ represents the binary erosion [3], [4]. We have, for all $x \in X$:

$$\begin{aligned}
Asp_b^{\triangleq}(f)(x) &= c_{1_b} f(x) \triangle c_{2_b} f(x) \\
&= \vee_{h \in D_b}\left\{f(x+h) \triangle b(h)\right\} \triangle \wedge_{h \in D_b}\left\{f(x+h) \triangle b(h)\right\} \\
&= (f(x) \triangle \vee_{h \in D_b}\left\{b(h)\right\}) \triangle (f(x) \triangle \wedge_{h \in D_b}\left\{b(h)\right\}) \\
&= \triangle \vee_{h \in D_b}\left\{b(h)\right\} \triangle \wedge_{h \in D_b}\left\{b(h)\right\} \\
&= \wedge_{h \in D_b}\left\{b(h)\right\} \triangle \vee_{h \in D_b}\left\{b(h)\right\} \\
&= b_{sup} \triangle b_{inf}.
\end{aligned}$$

b_{sup} and b_{inf} are the supremum and the infimum, respectively, of the structuring element b. In the eroded flat zone X, the Asplund metric is therefore equal to the constant $b_{sup} \triangle b_{inf}$. □

PROOFS OF PROPERTIES 1 (P. 9) AND 2 (P. 9)

Proof of property 1: Let $f \in \overline{\mathcal{F}}_M$ be a function and b^l, $b^r \in \overline{\mathcal{F}}_M$ be structuring functions. From (8), we have $\forall x \in D$, $\wedge_{h \in D_b}\{f(x+h) \triangle b^l(h)\} = \varepsilon^{\triangleq}_{b^l}(f)(x)$ and from (19), $c_{2_b}(f) = \varepsilon^{\triangleq}_b(f)$. From (25), we deduce that:

$$\begin{aligned}
E(b^l, f)(x) &= \wedge_{h \in D_b}\{f(x+h) \triangle b^l(h)\} \triangle c_{2_b}(f)(x) \\
&= \varepsilon^{\triangleq}_{b^l}(f)(x) \triangle \varepsilon^{\triangleq}_b(f)(x).
\end{aligned}$$

In a similar way, we have $E(b^r, f)(x) = \varepsilon^{\triangleq}_{b^r}(f)(x) \triangle \varepsilon^{\triangleq}_b(f)(x)$. □

Proof of property 2: From (8), we have $\forall c \in]-\infty, M[$ and $\forall x \in D$:

$$\begin{aligned}\varepsilon_b^{\triangleq}(f \triangleq c)(x) &= \wedge_{h\in D_b}\{(f \triangleq c)(x+h) \triangleq b(h)\}\\ &= \wedge_{h\in D_b}\{f(x+h) \triangleq c \triangleq b(h)\}\\ &= \wedge_{h\in D_b}\{f(x+h) \triangleq b(h)\} \triangleq c\\ &= \varepsilon_b^{\triangleq}(f)(x) \triangleq c. \end{aligned} \tag{A.6}$$

From (28), one deduces that:

$$\begin{aligned}E(b^l, f \triangleq c) &= \varepsilon_{b^l}^{\triangleq}(f \triangleq c) \triangleq \varepsilon_b^{\triangleq}(f \triangleq c)\\ &= (\varepsilon_{b^l}^{\triangleq}(f) \triangleq c) \triangleq (\varepsilon_b^{\triangleq}(f) \triangleq c)\\ &= \varepsilon_{b^l}^{\triangleq}(f) \triangleq \varepsilon_b^{\triangleq}(f) \triangleq c \triangleq c\\ &= \varepsilon_{b^l}^{\triangleq}(f) \triangleq \varepsilon_b^{\triangleq}(f)\\ &= E(b^l, f).\end{aligned}$$

In a similar way, we obtained $E(b^r, f \triangleq c) = E(b^r, f)$. As a result, from (27), we have $E(b, f \triangleq c) = E(b, f)$. □

PROOF OF PROPERTY 3 (P. 10)

Proof of property 3: Let $f \in \overline{\mathcal{F}}_M$ be a function and b, $b_r \in \overline{\mathcal{F}}_M$ be structuring functions. Similarly as in (A.6), we have $\forall c \in]-\infty, M[$ and $\forall x \in D$:

$$\begin{aligned}\delta_b^{\triangleq}(f \triangleq c)(x) &= \vee_{h\in D_b}\{(f \triangleq c)(x-h) \triangleq b(h)\}\\ &= \vee_{h\in D_b}\{f(x-h) \triangleq c \triangleq b(h)\}\\ &= \vee_{h\in D_b}\{f(x-h) \triangleq b(h)\} \triangleq c\\ &= \delta_b^{\triangleq}(f)(x) \triangleq c. \end{aligned} \tag{A.7}$$

From (9), (A.6) and (A.7), we have $\forall c \in]-\infty, M[$:

$$\begin{aligned}\gamma_b^{\triangleq}(f \triangleq c) &= \delta_b^{\triangleq}\left[\varepsilon_b^{\triangleq}(f \triangleq c)\right]\\ &= \delta_b^{\triangleq}\left[\varepsilon_b^{\triangleq}(f) \triangleq c\right]\\ &= \delta_b^{\triangleq}\left[\varepsilon_b^{\triangleq}(f)\right] \triangleq c\\ &= \gamma_b^{\triangleq}(f) \triangleq c. \end{aligned} \tag{A.8}$$

From (30), one deduces $\forall c \in]-\infty, M[$ that:

$$\begin{aligned}G_b^{\triangleq}(f \triangleq c) &= \gamma_b^{\triangleq}(f \triangleq c) \triangleq \gamma_{b_r}^{\triangleq}(f \triangleq c)\\ &= \gamma_b^{\triangleq}(f) \triangleq c \triangleq \left[\gamma_{b_r}^{\triangleq}(f) \triangleq c\right]\\ &= \gamma_b^{\triangleq}(f) \triangleq c \triangleq \gamma_{b_r}^{\triangleq}(f) \triangleq c\\ &= \gamma_b^{\triangleq}(f) \triangleq \gamma_{b_r}^{\triangleq}(f)\\ &= G_b^{\triangleq}(f).\end{aligned}$$

□

PROOF OF PROPERTY 4 (P. 11)

Proof of property 4: Let $f \in \overline{\mathcal{F}}_M$ be a function and b, $b_r \in \overline{\mathcal{F}}_M$ be structuring functions. From (32) and (A.8), we have $\forall c \in]-\infty, M[$:

$$\begin{aligned}R_b^{\triangleq}(f \triangleq c) &= [f \triangleq c] \triangleq \gamma_b^{\triangleq}(f \triangleq c)\\ &= f \triangleq c \triangleq \left[\gamma_b^{\triangleq}(f) \triangleq c\right]\\ &= f \triangleq c \triangleq \gamma_b^{\triangleq}(f) \triangleq c\\ &= f \triangleq \gamma_b^{\triangleq}(f)\\ &= R_b^{\triangleq}(f).\end{aligned}$$

□

PROOF OF PROPOSITION 5 (P. 11)

Proof of proposition 5: First of all, let us remind the definition of the LIP-additive Asplund metric with tolerance which was introduced in [5].

Definition A.1 (LIP-additive Asplund metric with tolerance)**.** *Let $(1-p)$ be a percentage of points of D to be discarded and D' the set of these discarded points. The LIP-additive Asplund metric with tolerance between two functions f and $g \in \mathcal{F}_M$ is defined by:*

$$d^{\triangleq}_{asp,p}(f,g) = c_{1,p} \triangleq c_{2,p}. \tag{A.9}$$

The constants $c_{1,p}$ and $c_{2,p}$ are equal to:

$$c_{1,p} = \inf\{c, \forall x \in D, \gamma^{\triangleq}_{(f_{|D\setminus D'}, g_{|D\setminus D'})}(x) \leq c\},$$
$$c_{2,p} = \sup\{c, \forall x \in D, c \leq \gamma^{\triangleq}_{(f_{|D\setminus D'}, g_{|D\setminus D'})}(x)\}.$$

A percentage $(1-p)/2$ of the points $x \in D$ with the greatest, respectively lowest contrast values $\gamma^{\triangleq}_{(f,g)}(x) = f(x) \triangleq g(x)$ are discarded.

Remark A.1. In practice, for a function $f : D \to]\infty, M[$ defined on a discrete grid D, e.g. $D \subset \mathbb{Z}^n$, the number of points to be suppressed are selected as follows. Let $\#D$ be the cardinal of D. The number of points n_{supr} to be suppressed is equal to $n_{suppr} = round[(1-p)\#D]$, where $round$ is the rounding operator of any real number to its nearest integer. For the constant $c_{1,p}$ and $c_{2,p}$, the number of points to be suppressed are respectively equal to:

$$n_1 = round(n_{suppr}/2), \tag{A.10}$$
$$n_2 = n_{suppr} - n_1, \tag{A.11}$$

Let us now recall the rank filter definitions. Let $\zeta_{b,k} : \overline{\mathbb{R}}^D \to \overline{\mathbb{R}}^D$ be the erosion-rank filter of rank k by the structuring function b. Let $\vartheta_{b,k}$ be the dilation-rank filter of rank k by the structuring function b. Both operations are defined by:

$$\zeta_{b,k}(f)(x) = \bigwedge^{k} \{f(x+h) - b(h), h \in D_b\}, \tag{A.12}$$
$$\vartheta_{b,k}(f)(x) = \bigvee^{k} \{f(x-h) + b(h), h \in D_b\}. \tag{A.13}$$

The LIP-erosion-rank filter $\zeta^{\triangleq}_{b,k} : \overline{\mathcal{F}}^D_M \to \overline{\mathcal{F}}^D_M$ and the LIP-dilation-rank filter $\vartheta^{\triangleq}_{b,k}$ are defined as follows:

$$\zeta^{\triangleq}_{b,k}(f)(x) = \bigwedge^{k} \{f(x+h) \triangleq b(h), h \in D_b\}, \tag{A.14}$$
$$\vartheta^{\triangleq}_{b,k}(f)(x) = \bigvee^{k} \{f(x-h) \triangleq b(h), h \in D_b\}. \tag{A.15}$$

The map of LIP-additive Asplund distances with tolerance (to noise extrema) is defined as follows.

Definition A.2 (Map of LIP-additive Asplund distances with tolerance [5])**.** *Let $f \in \mathcal{F}_M$ be a grey-level image and $b : D_b \to]-\infty, M[$ a probe. Let $(1-p)$ be a percentage of points of D_b to be discarded. The map of Asplund distances $Asp^{\triangleq}_b : \mathcal{F}_M \to \mathcal{F}_M$ with a tolerance (to extrema) p is defined by:*

$$Asp^{\triangleq}_{b,p}(f)(x) = d^{\triangleq}_{asp,p}(f_{|D_b(x)}, b). \tag{A.16}$$

For each point $x \in D$, the distance $d^{\triangleq}_{asp}(f_{|D_b(x)}, b)$ is computed in the neighbourhood $D_b(x)$ centred in x and the template b is acting like a structuring function. $f_{|D_b(x)}$ is the restriction of f to $D_b(x)$. $Asp^{\triangleq}_{b,p}(f) : D \to \mathcal{F}_M$ is the map of Asplund distances between the image f and the probe b.

In order to establish the link with MM, we will express the map of Asplund distances with neighbourhood operations. From Eq. A.9, for each $x \in D$, the map expression becomes

$$d^{\triangleq}_{asp,p}(f_{|D_b(x)}, b) = c_{1_b,p}(f)(x) \triangleq c_{2_b,p}(f)(x),$$

where

$$c_{1_b,p}(f)(x) = \inf\{c, \forall h \in D_b, \quad \gamma^{\triangleq}_{(f_{|D_b(x)\setminus D'_b(x)}, b_{|D_b\setminus D'_b(x)})}(h) \leq c\}$$
$$c_{2_b,p}(f)(x) = \sup\{c, \forall h \in D_b, \quad c \leq \gamma^{\triangleq}_{(f_{|D_b(x)\setminus D'_b(x)}, b_{|D_b\setminus D'_b(x)})}(h)\}.$$

This leads to the following definition.

Definition A.3 (LIP-additive maps of the least upper and of the greatest lower bounds with tolerance)**.** *Let $f \in \overline{\mathcal{F}}_M$ be a function and $b : D_b \to]\infty, M[$ a probe. For any $x \in D$, let $(1-p)$ be a percentage of points of the neighbourhood $D_b(x)$ to be*

discarded and $D'_b(x)$ the set of these discarded points. The map of the least upper bounds (mlub) $c_{1_b,p} : \overline{\mathcal{F}}_M \to \overline{\mathcal{F}}_M$ and the map of the greatest lower bounds (mglb) $c_{2_{b,p}} : \overline{\mathcal{F}}_M \to \overline{\mathcal{F}}_M$, of f by b are defined, for any $x \in D$, by:

$$c_{1_b,p}(f)(x) = \inf\left\{c, \forall h \in D_b, \quad \gamma^{\triangleq}_{(f_{|D_b(x)\setminus D'_b(x)}, b_{|D_b\setminus D'_b(x)})}(h) \leq c\right\}, \tag{A.17}$$

$$c_{2_b,p}(f)(x) = \sup\left\{c, \forall h \in D_b, \quad c \leq \gamma^{\triangleq}_{(f_{|D_b(x)\setminus D'_b(x)}, b_{|D_b\setminus D'_b(x)})}(h)\right\}. \tag{A.18}$$

The number of points to be suppressed n_1 and n_2 for the mlub $c_{1_b,p}$ and for the mglb $c_{2_b,p}$ are equal to $n_1 = round(n_{suppr}/2)$, and $n_2 = n_{suppr} - n_1$, respectively, where $n_{suppr} = round[(1-p)\#D_b]$ and D_b is the cardinal of $\#D_b$.

From (A.17), one deduces, for any $x \in D$, that:

$$\begin{aligned}
c_{1_b,p}(f)(x) &= \bigvee\left\{\gamma^{\triangleq}_{(f_{|D_b(x)\setminus D'_b(x)}, b_{|D_b\setminus D'_b(x)})}(h), h \in D_b\right\} \\
&= \bigvee^{n_1}\left\{\gamma^{\triangleq}_{(f_{|D_b(x)}, b)}(h), h \in D_b\right\} \\
&= \bigvee^{n_1}\{f(x+h) \triangleq b(h), h \in D_b\} \\
&= \bigvee^{n_1}\{f(x-h) \triangleq b(-h), -h \in D_b\} \\
&= \bigvee^{n_1}\{f(x-h) \triangleq (\triangleq\overline{b}(h)), h \in D_{\overline{b}}\} \\
&= \vartheta^{\triangleq}_{\triangleq\overline{b},n_1}(f)(x) \qquad \text{(from (A.13))}.
\end{aligned}$$

Similarly, from (A.18), one deduces, for any $x \in D$, that:

$$\begin{aligned}
c_{2_b,p}(f)(x) &= \bigwedge\left\{\gamma^{\triangleq}_{(f_{|D_b(x)\setminus D'_b(x)}, b_{|D_b\setminus D'_b(x)})}(h), h \in D_b\right\} \\
&= \bigwedge^{n_2}\{f(x+h) \triangleq b(h), h \in D_b\} \\
&= \zeta^{\triangleq}_{b,n_2}(f)(x) \qquad \text{(from (A.12))}.
\end{aligned}$$

As a consequence, the following proposition holds.

Proposition A.2 (i.e. proposition 5 (p. 8))**.** *Let $f \in \overline{\mathcal{F}}_M$ be a function defined on a discrete grid, e.g. $D \subset \mathbb{Z}^n$. Let $b \in \overline{\mathcal{F}}_M$ be a structuring function, where for all $x \in D_b$, $D_b \subset D$, $b(x) > -\infty$. Let $(1-p)$ be a percentage of points of D_b to be discarded. The map of LIP-additive Asplund distances with a tolerance p between the function f and the structuring function b is equal to:*

$$Asp^{\triangleq}_{b,p}(f) = \vartheta^{\triangleq}_{\triangleq\overline{b},n_1}(f) \triangleq \zeta^{\triangleq}_{b,n_2}(f). \tag{A.19}$$

The number of points to be suppressed n_1 and n_2 for the mlub $\vartheta^{\triangleq}_{\triangleq\overline{b},n_1}$ *and for the* mglb $\zeta^{\triangleq}_{b,n_2}$ *are equal to $n_1 = round(n_{suppr}/2)$, and $n_2 = n_{suppr} - n_1$, respectively, where $n_{suppr} = round[(1-p)\#D_b]$ and $\#D_b$ is the cardinal of D_b. For the* mlub *and the* mglb *of f, $c_{1_b,p}(f)$ and $c_{2_b,p}(f)$, we have:*

$$c_{1_b,p}(f) = \vartheta^{\triangleq}_{\triangleq\overline{b},n_1}(f), \tag{A.20}$$

$$c_{2_b,p}(f) = \zeta^{\triangleq}_{b,n_2}(f). \tag{A.21}$$

□

PROOF OF PROPERTY 5 (P. 16)

Proof of property 5:

Let $f \in \overline{\mathcal{F}}_M$ be a function and b, b_θ, b^l_θ, b^c_θ $b^r_\theta \in \overline{\mathcal{F}}_M$ be structuring functions.

From (A.14), we have $\forall c \in]-\infty, M[$ and $\forall x \in D$:

$$\begin{aligned}
\zeta^{\triangleq}_b(f \triangleq c)(x) &= \wedge^k_{h\in D_b}\{(f \triangleq c)(x+h) \triangleq b(h)\} \\
&= \wedge^k_{h\in D_b}\{f(x+h) \triangleq c \triangleq b(h)\} \\
&= \wedge^k_{h\in D_b}\{f(x+h) \triangleq b(h)\} \triangleq c \\
&= \zeta^{\triangleq}_b(f)(x) \triangleq c.
\end{aligned} \tag{A.22}$$

From (A.6), (A.22) and knowing that the LIP-addition of a constant $\triangleq c$ preserves the order $\leq$, one deduces that:

$$\begin{aligned}
\grave{c}_{b_\theta,k}(f \triangleq c) &= \bigwedge \left\{ \varepsilon^{\triangleq}_{b^c_\theta}(f \triangleq c), \quad \bigwedge [\zeta^{\triangleq}_{b^l_\theta,k}(f \triangleq c), \zeta^{\triangleq}_{b^r_\theta,k}(f \triangleq c)] \right\} \\
&= \bigwedge \left\{ \varepsilon^{\triangleq}_{b^c_\theta}(f) \triangleq c, \quad \bigwedge [\zeta^{\triangleq}_{b^l_\theta,k}(f) \triangleq c, \zeta^{\triangleq}_{b^r_\theta,k}(f) \triangleq c] \right\} \\
&= \bigwedge \left\{ \varepsilon^{\triangleq}_{b^c_\theta}(f) \triangleq c, \quad \bigwedge [\zeta^{\triangleq}_{b^l_\theta,k}(f), \zeta^{\triangleq}_{b^r_\theta,k}(f)] \triangleq c \right\} \\
&= \bigwedge \left\{ \varepsilon^{\triangleq}_{b^c_\theta}(f), \quad \bigwedge [\zeta^{\triangleq}_{b^l_\theta,k}(f), \zeta^{\triangleq}_{b^r_\theta,k}(f)] \right\} \triangleq c. \qquad \text{(A.23)}
\end{aligned}$$

From (34), (A.22) and (A.23), we have:

$$\begin{aligned}
E^k(b^l_\theta, f \triangleq c) &= \zeta^{\triangleq}_{b^l_\theta,k}(f \triangleq c) \triangleq \grave{c}_{b_\theta,k}(f \triangleq c) \\
&= \zeta^{\triangleq}_{b^l_\theta,k}(f) \triangleq c \triangleq [\grave{c}_{b_\theta,k}(f) \triangleq c] \\
&= \zeta^{\triangleq}_{b^l_\theta,k}(f) \triangleq \grave{c}_{b_\theta,k}(f) \\
&= E^k(b^l_\theta, f). \qquad \text{(A.24)}
\end{aligned}$$

Similarly, from (35), we have:

$$E^k(b^r_\theta, f \triangleq c) = E^k(b^r_\theta, f). \qquad \text{(A.25)}$$

From (36), (A.24) and (A.25), we have:

$$\begin{aligned}
E^k(b_\theta, f \triangleq c) &= \bigvee \{E^k(b^l_\theta, f \triangleq c), E^k(b^r_\theta, f \triangleq c)\} \\
&= \bigvee \{E^k(b^l_\theta, f), E^k(b^r_\theta, f)\}. \\
&= E^k(b_\theta, f) \qquad \text{(A.26)}
\end{aligned}$$

As a consequence from (37), we have:

$$\begin{aligned}
E^k(b, f \triangleq c) &= \bigwedge \{E^k(b_\theta, f \triangleq c) \mid \theta \in \Theta\} \\
&= \bigwedge \{E^k(b_\theta, f) \mid \theta \in \Theta\} \\
&= E^k(b, f). \qquad \text{(A.27)}
\end{aligned}$$

Finally, from (38), we have :

$$\begin{aligned}
e^k_b(f \triangleq c) &= \bigwedge \{E^k(b_i, f \triangleq c) \mid i \in [\![1 \dots I]\!]\} \\
&= \bigwedge \{E^k(b_i, f) \mid i \in [\![1 \dots I]\!]\} \\
&= e^k_b(f). \qquad \text{(A.28)}
\end{aligned}$$

□

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