Viscous theory for three-layer Rayleigh-Taylor instability.

Antoine Simon, Jorge-César Brändle de Motta, Christophe Dumouchel & Marie-Charlotte Renoult

CNRS UMR 6614 - CORIA
University of Rouen Normandy

HILIMBA: 2 labs (CORIA/LOMC), 2 PhD students, 320 k€

ICMF, Kobe, Japan, 2-7 April 2023

1 - Rayleigh-Taylor instability (RTI)

Hydrodynamic instability

- at a **fluid-fluid** interface
- when **accelerated** in the direction of the density gradient

$$\frac{(2) : \rho_2 > \rho_1}{(1) : \rho_1} \quad \uparrow \vec{\nabla}\rho \uparrow \vec{a} \qquad \frac{(2) : \rho_2 > \rho_1}{(1) : \rho_1} \quad \uparrow \vec{\nabla}\rho \downarrow \vec{g}$$

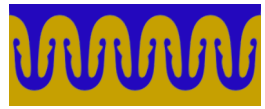
Digitations of **various** shapes

- $(1) \equiv$ oil / $(2) \equiv$ water $\rightarrow \rho_2 - \rho_1 \sim 10^2$ kg/m^3


$t = 0,04s$


$t = 0,10s$

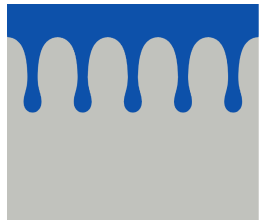

$t = 0,16s$

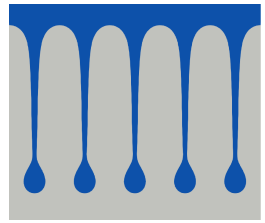

$t = 0,24s$

- $(1) \equiv$ air / $(2) \equiv$ water $\rightarrow \rho_2 - \rho_1 \sim 10^3$ kg/m^3


$t = 0,04s$


$t = 0,09s$


$t = 0,16s$

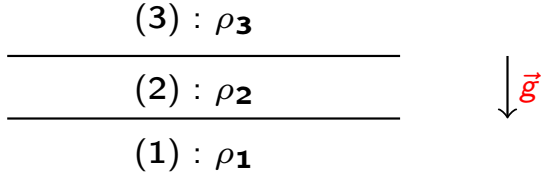

$t = 0,21s$

1 - 2-layer fluid system $\rightarrow$ 3-layer fluid system

Reference: S Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, 1961, Chapter X.

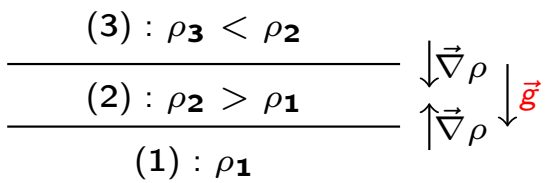
Fields of observation: geophysics, astrophysics and nuclear safety $\rightarrow$ **multi-layer** fluid systems

Case of study: a **3-layer** fluid system immersed in a gravity field



$\rightarrow$ 2 unstable interfaces if $\rho_{i+1} > \rho_i$ for $i \in \{1; 2\}$

Focus: RTI at the **lower** interface



1 - Review of the literature

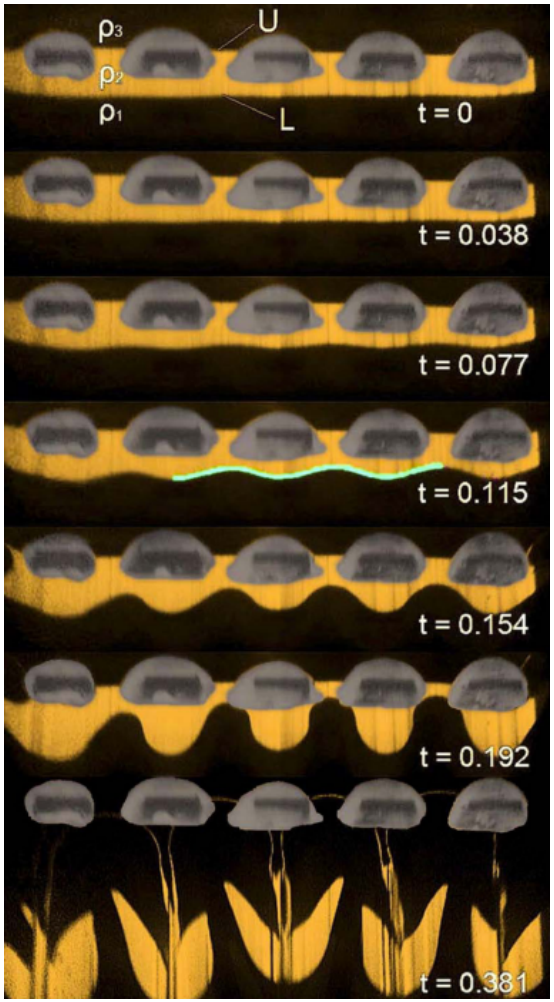
Theoretical studies

1. KO Mikaelian Phys. Rev. A 1983
   → inviscid model, surface tension ignored, analytical solutions.
2. S Parhi & G Nath Int. J. Engng Sci. 1991
   → viscous model, solving method, stability criterion.
3. KO Mikaelian Phys. Rev. E 1996
   → viscous model, solving method.

Experimental work

1. JW Jacobs & SB Dalziel J. Fluid Mech. 2005
   → miscible fluids, method of barrier removal, turbulent initial flow.
2. R Adkins, EM Shelton, MC Renoult, P Carles, C Rosenblatt Phys. Rev. Fluids 2017
   → non-miscible fluids, method of magnetic levitation, sinusoidal initial interface deformation.

1 - Motivation



Set of images versus time for one experimental run from Adkins et al.

Growth rate measurements at the lower interface

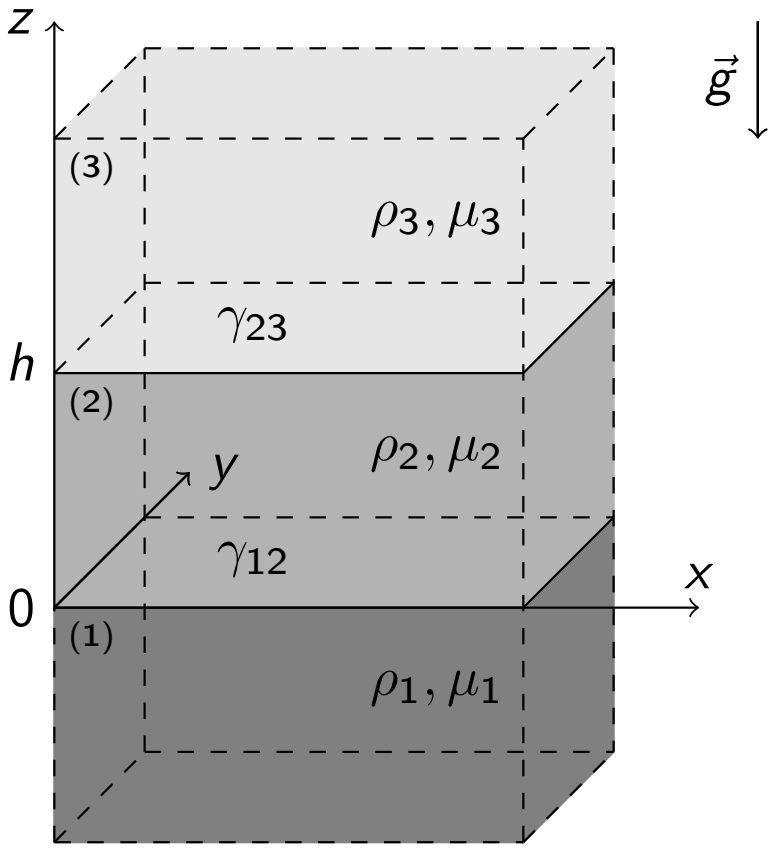
- assuming exponential growth
- **lower** values than the ones predicted by the inviscid Mikaelian theory.

Sources of discrepancy

1. Fluids **viscosity**
2. **Sides** effects
3. Interfaces **coupling**

$\rightarrow$ **viscous** theory and **comparison** to experiments and numerical simulations.

2 - Formulation



Assumptions:

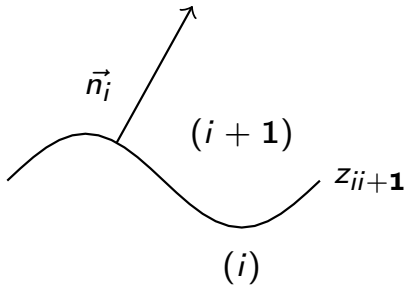
- **Incompressible** fluids.
- **Newtonian** fluids.
- **Non miscible** fluids.
- Extreme layers of **infinite** depth.
- **Isothermal** conditions.

Condition for RTI occurrence:
$\rho_3 > \rho_2$ and/or $\rho_2 > \rho_1$.

Volume equations in each phase $i \in \{1;2;3\}$:

$$\vec{\nabla} \cdot \vec{u_i} = 0 \quad \text{(mass balance)}$$
$$\rho_i \left(\vec{u}_{i,t} + (\vec{u_i} \cdot \vec{\nabla}) \vec{u_i} \right) = -\vec{\nabla} p_i + \mu_i \vec{\Delta} \vec{u_i} + \rho_i \vec{g} \quad \text{(momentum balance)}$$

Jump conditions at each interface $ii+1$ with $i \in \{1;2\}$:



$$\vec{u_i} \cdot \vec{n_i} = \vec{v}_{ii+1} \cdot \vec{n_i} = \vec{u}_{i+1} \cdot \vec{n_i} \quad \text{(mass balance)}$$
$$\left(p_{i+1} - p_i + \gamma_{ii+1} \kappa_{ii+1}\right) \vec{n_i} = (\bar{\bar{\tau}}_{i+1} - \bar{\bar{\tau}}_i) \cdot \vec{n_i} \quad \text{(momentum balance)}$$
$$\vec{u_i} \times \vec{n_i} = \vec{u}_{i+1} \times \vec{n_i} \quad \text{(no slip)}$$

with $\vec{n_i} = \left(-z_{ii+1,x}, -z_{ii+1,y}, 1\right)^T$, $\kappa_{ii+1} = \frac{z_{ii+1,x,x}}{\left(1+z^2_{ii+1,x}\right)^{\frac{3}{2}}} + \frac{z_{ii+1,y,y}}{\left(1+z^2_{ii+1,y}\right)^{\frac{3}{2}}}$ and

$\bar{\bar{\tau}}_i = \mu_i (\vec{\nabla} \vec{u_i} + (\vec{\nabla} \vec{u_i})^T)$

2 - Basic state

Definition:

- Fluids at **rest**: $\vec{u_i^*} = \vec{0}$
- Interface 12 **plane**: $z_{12}^* = 0$
- Interface 23 **plane**: $z_{23}^* = h$

Solution for the pressure:

$$
\begin{aligned}
p_1^*(z) &= -\rho_1 g z + p_c \\
p_2^*(z) &= -\rho_2 g z + p_c \\
p_3^*(z) &= -\rho_3 g z + g h(\rho_3 - \rho_2) + p_c
\end{aligned}
$$

with $p_c = p_1^*(z = 0)$ a constant.

2 - Perturbed state

Definition:

$$\vec{u}_i(x,y,z,t) = \vec{u}^*_i + \vec{u}'_i(x,y,z,t) = \vec{u}'_i \text{ with } u'_i \ll 1$$
$$p_i(x,y,z,t) = p^*_i(z) + p'_i(x,y,z,t) \text{ with } p'_i \ll p^*_i$$
$$z_{ii+1}(x,y,t) = z^*_{ii+1} + z'_{ii+1}(x,y,t) \text{ with } z'_{ii+1} \ll z^*_{ii+1}$$

Volume equations in each phase $i \in \{1;2;3\}$:
Mass balance: $u_{ij,j} = 0$
Linearized momentum balance: $\rho_i u_{ij,t} = -p'_{i,j} + \mu_i \Delta u_{ij} \quad j \in \{x;y;z\}$

Linearized jump conditions at $z = (i-1)h$, $i \in \{1;2\}$:

$$u_{iz} = z'_{ii+1,t} = u_{i+1z}$$
$$u_{ix} = u_{i+1x}$$
$$u_{iy} = u_{i+1y}$$
$$p'_{i+1} - p'_i - (\rho_{i+1} - \rho_i) g z'_{ii+1} - \gamma_{ii+1}(z'_{ii+1,x,x} + z'_{ii+1,y,y}) = 2(\mu_{i+1} u_{i+1z,z} - \mu_i u_{iz,z})$$
$$\mu_{i+1}(u_{i+1z,x} + u_{i+1x,z}) = \mu_i(u_{iz,x} + u_{ix,z})$$
$$\mu_{i+1}(u_{i+1z,y} + u_{i+1y,z}) = \mu_i(u_{iz,y} + u_{iy,z})$$

2 - Dispersion relation

- Solutions searched under the form: $e^{(ik_x x + ik_y y + \omega t)}$ with $k^2 = k_x^2 + k_y^2$ the wavenumber of the perturbation and ω the time coefficient (complex number).
- Volume equations $\rightarrow u_{iz}$:

$$u_{1z} = (C_1 e^{kz} + D_1 e^{q_1 z}) e^{(ik_x x + ik_y y + \omega t)}$$
$$u_{2z} = (A_2 e^{-kz} + B_2 e^{-q_2 z} + C_2 e^{kz} + D_2 e^{q_2 z}) e^{(ik_x x + ik_y y + \omega t)}$$
$$u_{3z} = (A_3 e^{-kz} + B_3 e^{-q_3 z}) e^{(ik_x x + ik_y y + \omega t)}$$

  with C_1, D_1, A_2, B_2, C_2, D_2, A_3 and B_3 **8 unknown** coefficients and $q_i = \sqrt{k^2 + \omega \rho_i / \mu_i}$ the **modified** wavenumber in phase i.
- 4 equations at each interface $\rightarrow$ **8** equations $\rightarrow$ **closed** problem

Dispersion relation: solution of $\boxed{M \cdot (C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3)^T = 0}$

where M is a 8×8 matrix dependant of 9 dimensionless numbers (Π theorem: 12 physical parameters - 3 fundamental units).

2 - Solutions

Non-trivial solutions are solutions of

$$|M| = 0$$

There is one **analytical** solution, stable $\forall k$:

$$\Omega_1 = \frac{\omega_1}{\sqrt{gk}} = \sqrt{\frac{g}{k^3}}\frac{\rho_i}{\mu_i} = \boxed{-Re_2^{-1}}$$

And 3 or 4 other solutions, depending on k, determined **numerically**.

Limit case: disregarding the viscosity and the surface tension
$\rightarrow$ Mikaelian relation

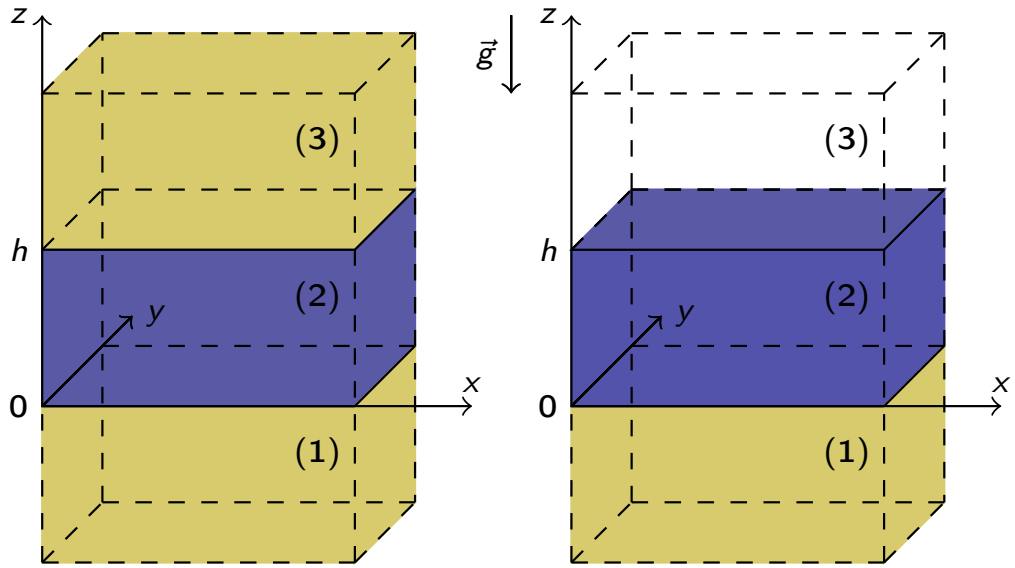
$$a(At_{12}, At_{23}, H)\,\Omega^4 + b(At_{12}, At_{23}, H)\,\Omega^2 + c(At_{12}, At_{23}, H) = 0$$

with $At_{ii+1} = (\rho_{i+1} - \rho_i)/(\rho_{i+1} + \rho_i)$ the Atwood numbers and $H = kh$ the dimensionless thickness $\rightarrow$ 4 **analytical** solutions $\forall k$.

3 - Operating conditions

Let us consider the 2 cases studied in Adkins et al.

- (1) ≡ Oil / (2) ≡ Water / (3) ≡ Oil
- (1) ≡ Oil / (2) ≡ Water / (3) ≡ Air



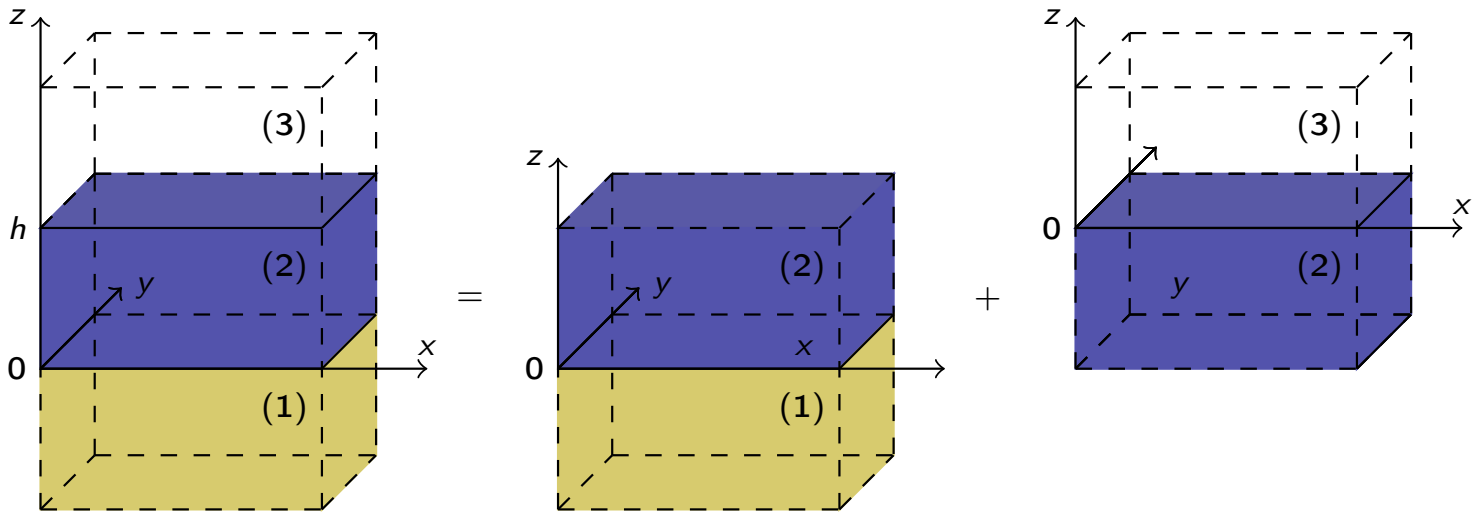
Variable quantities: h and $k \rightarrow$ various H.

3 - 2 limit cases for comparison

1. 2 separate **2-layer** fluid systems (considering the fluid viscosities)

   (3)/(2)/(1) is compared to (2)/(1) and (3)/(2)

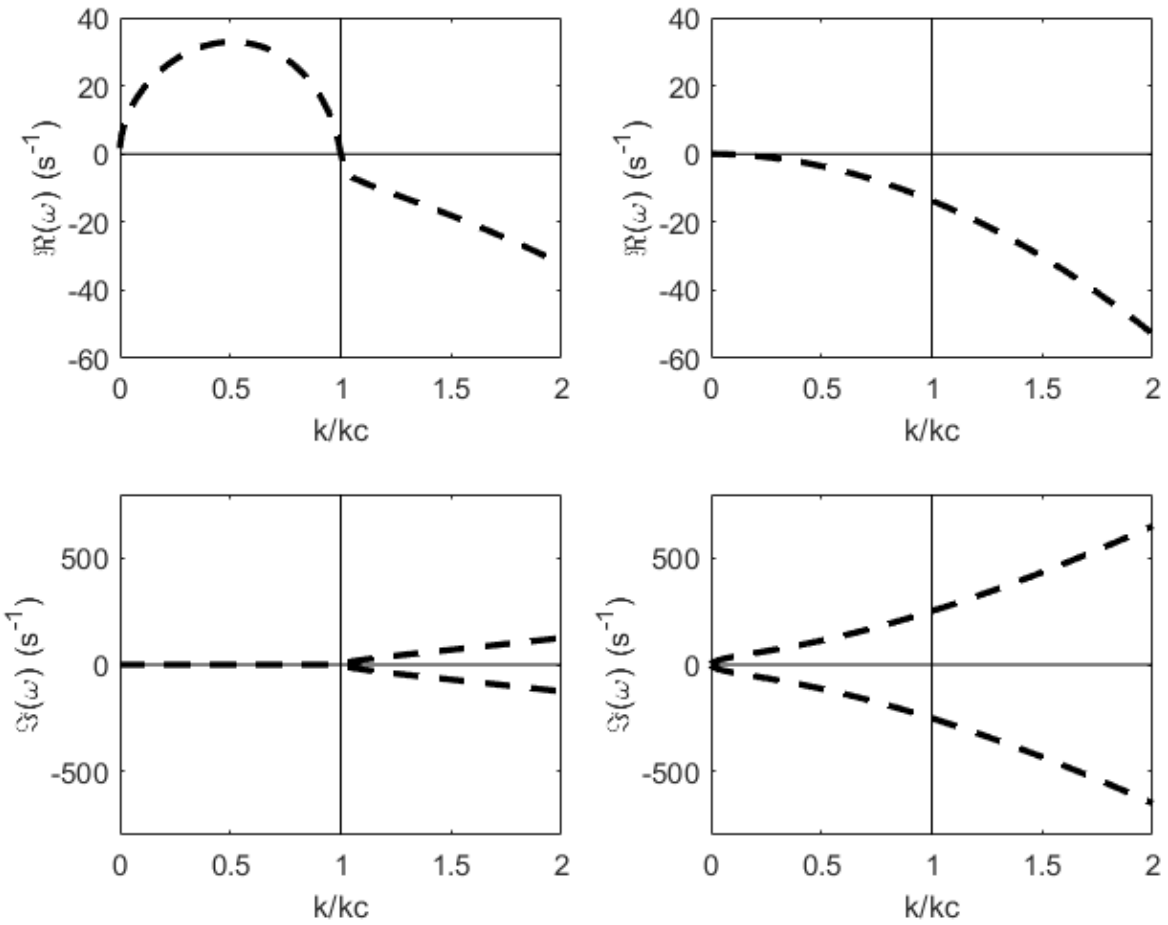
   - (2)/(1) is gravitationally **unstable**.
   - (3)/(2) is gravitationally **stable**.



2. 3-layer **inviscid** model of Mikaelian

3 - Solutions for 2 separate 2-layer fluid systems

$\vec{g}$-unstable vs $\vec{g}$-stable



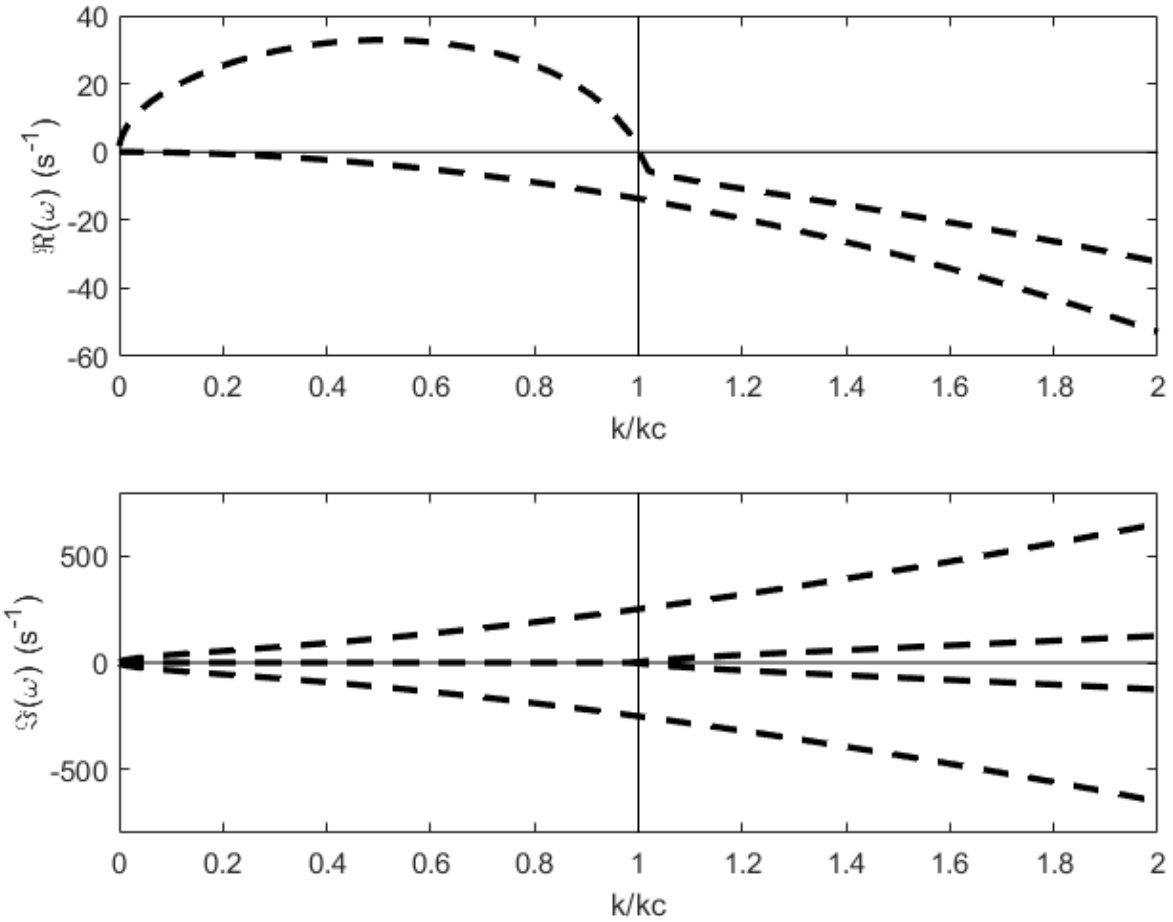
k_c: cutoff wavenumber due to surface tension.

$\mathfrak{Re}(w) > 0 \rightarrow$ unstable solution; $\mathfrak{Im}(w) \neq 0 \rightarrow$ oscillations.

3 - All solutions on the same graph!

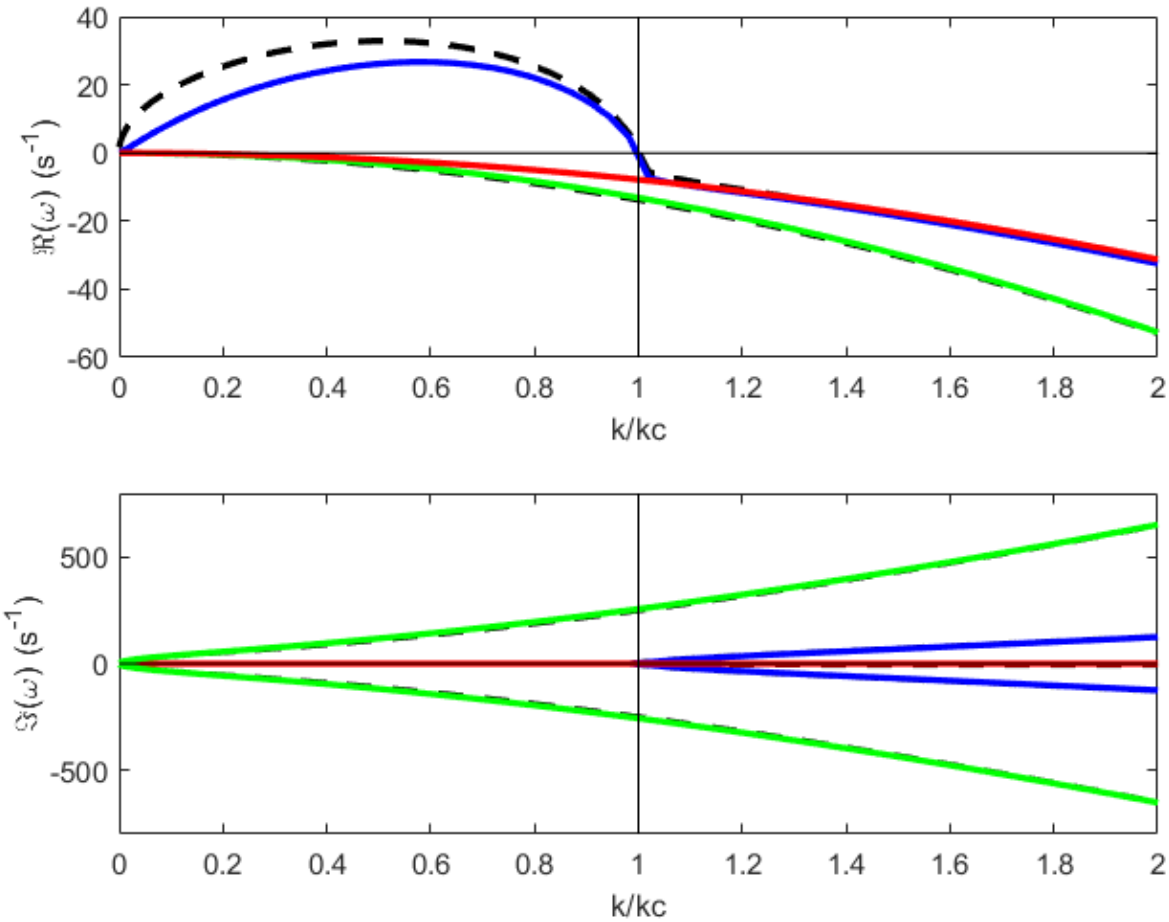
$k < k_c$: 1 unstable + 2 stables

$k > k_c$: 4 stables



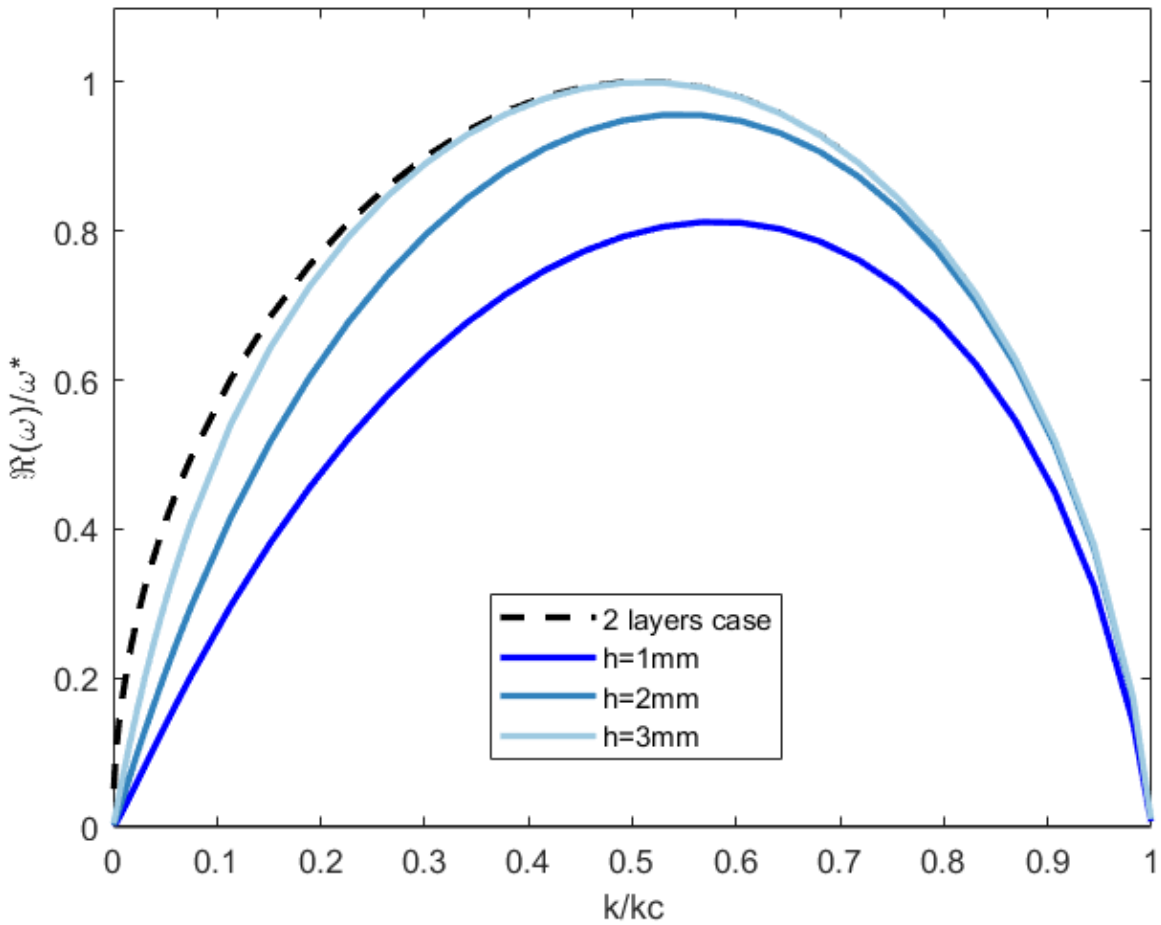
3 - Solutions for the 3-layer fluid system, h=1 mm

3 totally **stable** solutions: 1 analytically, 2 numerically determined
Other solutions: 2 **stable** for $k > k_c$, 1 **unstable** for $k < k_c$.



3 - Effect of h on the unstable solution

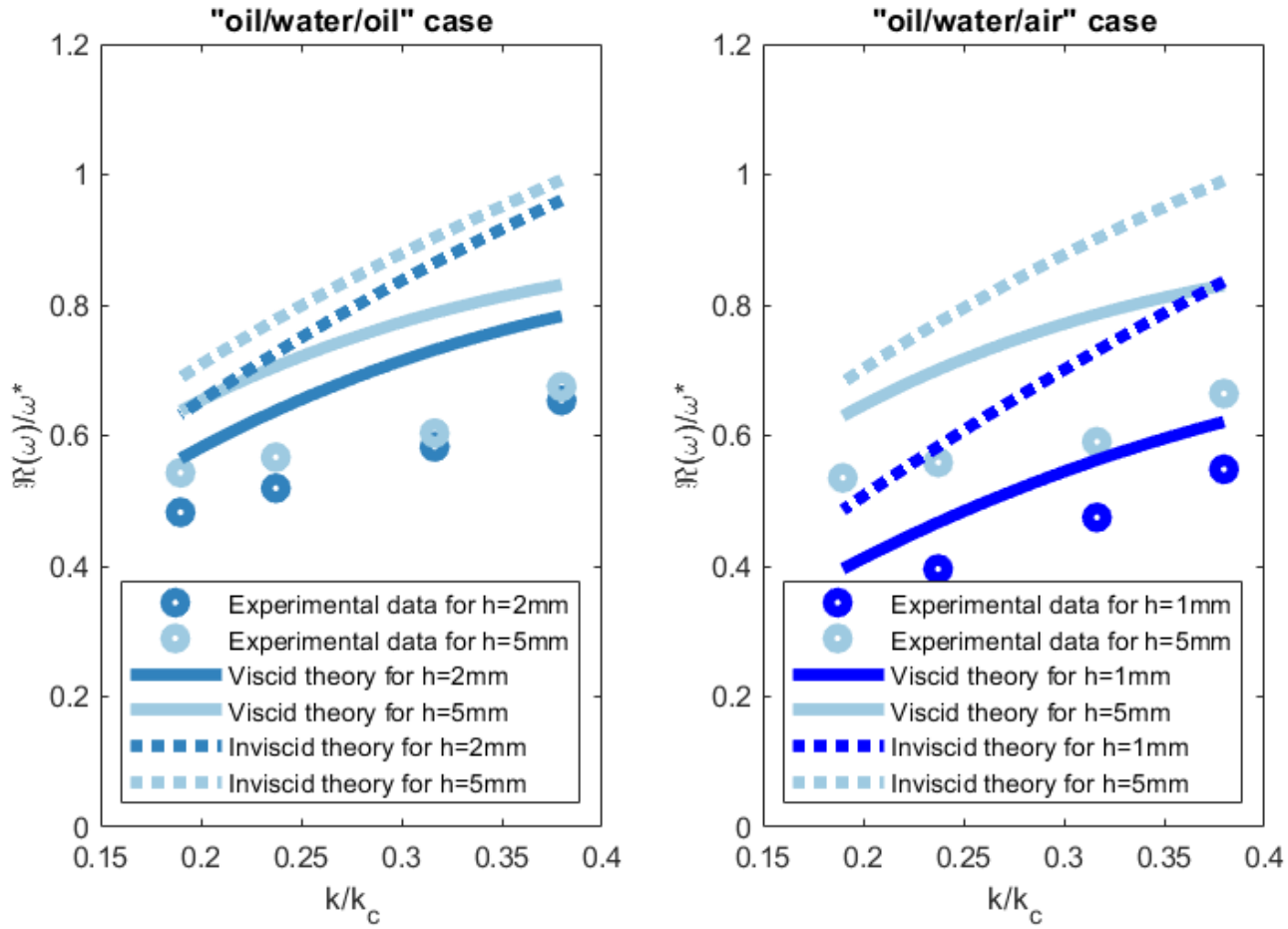
ω^*: maximum growth rate for the **viscous 2-layer** fluid model



Decreasing h (i.e. increasing interface coupling) decreases ω.

3 - Comparison to experimental values

ω^*: maximum growth rate for the **inviscid 3-layer** fluid model



The viscous model improves the prediction as h is reduced.

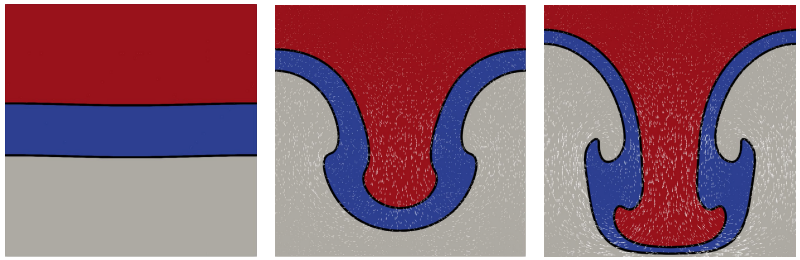
4 - Conclusion & Ongoing work

Conclusion

- Development of a theory for 3-layer RTI that considers the viscosity of the fluids and all the solutions of the dispersion relation.
- Improvement of the theoretical prediction.

Ongoing work

- Comparison of interface deformations with numerical simulations.



OPENFOAM simulations. (1) $\equiv$ Huile/(2) $\equiv$ Eau/(3) $\equiv$ Mercure. Set of images versus time.

- Inclusion of the magnetic field for a viscous model for liquid metal batteries ($\vec{g}$-stable stratification).

Future work $\rightarrow$ New experiments at Case Western Reserve University?

Thank you for your attention.

Any questions?