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# Large time behavior of finite difference schemes for the transport equation

Lucas COEURET<sup>1</sup>

## Abstract

In order to study the large time behavior of finite difference schemes for the transport equation, we need to describe the pointwise asymptotic behavior of iterated convolutions for finitely supported sequences indexed on  $\mathbb{Z}$ . In this paper, we investigate this question by presenting the main result of [Coe22] which is a generalization of the so-called local limit theorem in probability theory to complex valued sequences.

## Notations

Throughout this article, we define the unit circle in the complex plane:

$$\mathbb{S}^1 := \{z \in \mathbb{C}, \quad |z| = 1\}.$$

For  $E$  a Banach space and  $T$  a bounded operator acting on  $E$ , the notation  $\sigma(T)$  stands for the spectrum of the operator  $T$ .

For  $1 \leq q < +\infty$ , we let  $\ell^q(\mathbb{Z})$  denote the Banach space of complex valued sequences indexed by  $\mathbb{Z}$  and such that the norm:

$$\|u\|_{\ell^q} := \left( \sum_{j \in \mathbb{Z}} |u_j|^q \right)^{\frac{1}{q}}$$

is finite. We also let  $\ell^\infty(\mathbb{Z})$  denote the Banach space of bounded complex valued sequences indexed by  $\mathbb{Z}$  equipped with the norm

$$\|u\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |u_j|.$$

We define the convolution  $a * b$  of two elements  $a$  and  $b$  of  $\ell^1(\mathbb{Z})$  by

$$\forall j \in \mathbb{Z}, \quad (a * b)_j := \sum_{l \in \mathbb{Z}} a_l b_{j-l}.$$

When equipped with this product,  $\ell^1(\mathbb{Z})$  is a commutative Banach algebra. We define for a sequence  $a \in \ell^1(\mathbb{Z})$  and an integer  $n \in \mathbb{N}$

$$a^n := a * \dots * a.$$

## 1 Context

This paper is devoted to presenting the main result of [Coe22] which describes the pointwise asymptotic behavior of iterated convolutions  $a^n := a * \dots * a$  for some sequences  $a \in \ell^1(\mathbb{Z})$ . This question arises for instance when one discretizes the scalar 1D-transport equation

$$\begin{aligned} \partial_t u + v \partial_x u &= 0, \quad t \geq 0, x \in \mathbb{R}, \\ u(0, \cdot) &= u_0, \end{aligned} \tag{1.1}$$

with a velocity  $v \in \mathbb{R}$  and a Cauchy datum  $u_0$  at  $t = 0$ . If we introduce a time step  $\Delta t > 0$  and a space step  $\Delta x > 0$  and assume that the Courant number  $\lambda := \frac{\Delta t}{\Delta x}$  is kept constant, we want to approach the value  $u(n\Delta t, j\Delta x)$  where  $u$  is the solution of (1.1) by a quantity  $u_j^n$  that is determined by using a finite difference scheme. We claim that any explicit one-step finite difference scheme can be written with some well chosen finitely supported sequence  $a \in \ell^1(\mathbb{Z})$  as the following system

$$\forall n \in \mathbb{N}, \quad u^{n+1} = a * u^n \quad \text{and} \quad u^0 \in \mathbb{C}^{\mathbb{Z}}. \tag{1.2}$$

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Therefore, the precise understanding of the large time behavior of the finite difference scheme (1.2) relies on the description of the pointwise asymptotic behavior of the family  $(a^n)_n$  for the complex valued sequence  $a \in \ell^1(\mathbb{Z})$ .

If we suppose that the sequence  $a$  has real non negative coefficients and that the sum of its coefficients equals 1 (as would be the case for a consistent monotone scheme), then the sequence  $a^n$  is the probability distribution<sup>2</sup> of the sum of  $n$  independent random variables supported on  $\mathbb{Z}$  each with the probability distribution  $a$ . A lot is known on the pointwise asymptotic behavior of the sequence  $a^n$  in this case. In particular, the local limit theorem states, under suitable hypotheses on  $a$ , that

$$a_j^n - \frac{1}{\sqrt{2\pi\sigma^2n}} \exp\left(-\frac{|j-n\alpha|^2}{2n\sigma^2}\right) \underset{n \rightarrow +\infty}{=} o\left(\frac{1}{\sqrt{n}}\right), \quad (1.3)$$

where  $\alpha = \sum_{k \in \mathbb{Z}} ka_k$  and  $\sigma^2 = \sum_{k \in \mathbb{Z}} k^2 a_k - \alpha^2 > 0$  are respectively the mean and the variance of any random variable with probability distribution  $a$  and where the error term in (1.3) is uniform on  $\mathbb{Z}$  (see [Pet75, Chapter VII] for more details). The expansion (1.3) gives a precise description of the asymptotic behavior of  $a_j^n$  in the range  $|j-n\alpha| \lesssim \sqrt{n}$  and implies that the convolution powers of  $a$  are attracted towards the heat kernel. In the partial differential equation language, this corresponds to a parabolic behavior for the numerical scheme (1.2).

Following, among other works, [DSC14, RSC15, CF22], we are interested in generalizing the local limit theorem (1.3) of the probability theory to the case where the sequence  $a$  is complex valued. Extending the works of Schoenberg [Sch53], Greville [Gre66] and Diaconis and Saloff-Coste [DSC14, Theorem 2.6], the article [RSC15] of Randles and Saloff-Coste already provides a generalization of the local limit theorem for a large class of complex valued finitely supported sequences. By doing so, the authors of [RSC15] describe an asymptotic expansion similar to (1.3) with an explicit expression of the attractors. Our goal in this paper is to present the result [Coe22, Theorem 1] which introduces a more precise description of the asymptotic result proved in [RSC15] by proving a sharp rate of convergence together with a generalized Gaussian bound for the remainder (see Theorem 1 below). In the case where  $a$  is the probability distribution of a random variable, as above, [Coe22, Theorem 1] would translate in saying that, under suitable assumptions on  $a$ , there exist two constants  $C, c > 0$  such that

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j \in \mathbb{Z}, \left| a_j^n - \frac{1}{\sqrt{2\pi\sigma^2n}} \exp\left(-\frac{|j-n\alpha|^2}{2n\sigma^2}\right) \right| \leq \frac{C}{n} \exp\left(-c\frac{|j-n\alpha|^2}{n}\right).$$

To be more precise, [Coe22, Theorem 1] actually allows us to obtain an asymptotic expansion not only of order 1 but up to any order, which was not proved in [RSC15] but was expected due to the local limit theorem in the probabilistic case (see [Pet75, Chapter VII]). The interested reader can find more details in the latest version of [Coe22].

However, [Coe22, Theorem 1] relies on a stronger hypothesis on the elements of  $\ell^1(\mathbb{Z})$  than the conditions imposed in [RSC15]. We will consider here elements  $a$  of  $\ell^1(\mathbb{Z})$  which are finitely supported and such that the sequence  $(a^n)_{n \in \mathbb{N}}$  is bounded in  $\ell^1(\mathbb{Z})$ . The fundamental contribution [Tho65] by Thomée completely characterizes such elements and is an important starting point for our work.

## 2 Main result: Generalization of the local limit theorem

We now present in more details the necessary hypothesis and the main result of [Coe22]. We consider a sequence  $a \in \ell^1(\mathbb{Z})$  associated with a one-step in time explicit finite difference scheme (1.2) for the transport equation (1.1). We recall that the sequence  $a$  is thus finitely supported. We introduce the Fourier series  $F$  associated with the sequence  $a$  defined as

$$\forall \kappa \in \mathbb{C} \setminus \{0\}, \quad F(\kappa) := \sum_{j \in \mathbb{Z}} a_j \kappa^j. \quad (2.1)$$

The function  $F$  is holomorphic on  $\mathbb{C} \setminus \{0\}$ . We now introduce a hypothesis on the Fourier series  $F$  which is based on the work of Thomée in [Tho65].

**Hypothesis 1.** *We assume that*

$$F(1) = 1, \quad F'(1) = v\lambda, \quad (\text{Consistency condition})$$

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<sup>2</sup>We say that a sequence  $a$  is the probability distribution of a random variable  $X$  with values in  $\mathbb{Z}$  when  $\mathbb{P}(X = j) = a_j$  for all  $j \in \mathbb{Z}$ .

where  $v$  is the velocity in the transport equation (1.1). Moreover, we suppose that

$$\forall t \in [-\pi, \pi] \setminus \{0\}, \quad |F(e^{it})| < 1 \quad (\text{Dissipativity condition})$$

and that there exist an integer  $\mu \in \mathbb{N} \setminus \{0\}$  and a complex number  $\beta \in \mathbb{C}$  with  $\Re(\beta) > 0$  such that

$$F(e^{it}) \underset{t \rightarrow 0}{=} \exp(iv\lambda t - \beta t^{2\mu} + o(t^{2\mu})). \quad (\text{Diffusivity condition}) \quad (2.2)$$

The consistency condition on the Fourier series  $F$  implies that the numerical scheme (1.2) is consistent with the transport equation (1.1). The dissipativity condition is a stronger version of the Von Neumann condition which is necessary and sufficient so that the numerical scheme (1.2) is also  $\ell^2$ -stable. The main contribution of Thomée in [Tho65] is the introduction of the diffusivity condition (2.2) which assures us that the numerical scheme (1.2) is actually  $\ell^p$ -stable for all  $p \in [1, +\infty]$ . For instance, the Lax-Wendroff scheme does not satisfy Hypothesis 1 and is not  $\ell^\infty$ -stable. To be more precise, it is shown in [Tho65] that Hypothesis 1 is one of two conditions which characterize the sequences  $a \in \ell^1(\mathbb{Z})$  so that the family  $(a^n)_n$  is bounded in  $\ell^1(\mathbb{Z})$ .

As discussed in the previous section, the leading order of the pointwise asymptotic behavior of  $a^n$  is already known and justified in [RSC15]. For  $\mu \in \mathbb{N} \setminus \{0\}$  and  $\beta \in \mathbb{C}$  with positive real part, we let  $H_{2\mu}^\beta : \mathbb{R} \rightarrow \mathbb{C}$  be the function defined by

$$\forall x \in \mathbb{R}, \quad H_{2\mu}^\beta(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} e^{-\beta u^{2\mu}} du.$$

We call those functions generalized Gaussians since for  $\mu = 1$ , we have

$$\forall x \in \mathbb{R}, \quad H_2^\beta(x) = \frac{1}{\sqrt{4\pi\beta}} \exp\left(-\frac{x^2}{4\beta}\right).$$

We can now define the so-called "attractors" for the numerical scheme (1.2) satisfying Hypothesis 1:

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j \in \mathbb{Z}, \quad \mathcal{H}_j^n := \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}^\beta\left(\frac{j - nv\lambda}{n^{\frac{1}{2\mu}}}\right).$$

The result [RSC15, Theorem 1.2] claims that if the sequence  $a \in \ell^1(\mathbb{Z})$  satisfies Hypothesis 1, then

$$a_j^n - \mathcal{H}_j^n \underset{n \rightarrow +\infty}{=} o\left(\frac{1}{n^{\frac{1}{2\mu}}}\right) \quad (2.3)$$

where the error term is uniform on  $\mathbb{Z}$ . This result is obviously a generalization of (1.3) to complex valued sequences. We now state the main result of [Coe22].

**Theorem 1.** *Let  $a \in \ell^1(\mathbb{Z})$  which satisfies Hypothesis 1. Then, there exist two positive constants  $C, c$  such that*

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j \in \mathbb{Z}, \quad |a_j^n - \mathcal{H}_j^n| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c \left(\frac{|j - nc\lambda|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right). \quad (2.4)$$

A consequence is that there also exists a positive constant  $C$  such that for all  $u^0 \in \ell^2(\mathbb{Z})$ , the solution of the numerical scheme (1.2) verifies

$$\forall n \in \mathbb{N} \setminus \{0\}, \quad \|u^n - \mathcal{H}^n * u^0\|_{\ell^2(\mathbb{Z})} \leq \frac{C}{n^{\frac{1}{2\mu}}} \|u^0\|_{\ell^2(\mathbb{Z})} \quad (2.5)$$

where  $\mathcal{H}^n := \left(\mathcal{H}_j^n\right)_{j \in \mathbb{Z}}$ .

The improvement of Theorem 1 compared to (2.3) are the precise rate of convergence in  $\frac{1}{n^{\frac{1}{\mu}}}$  and the sharp generalized Gaussian bounds in (2.4), which allow to obtain inequality (2.5). Error estimates in  $\ell^p$  are also available thanks to convolution estimates.

The main result of [Coe22] actually has further uses as it allows to obtain an asymptotic expansion of  $a_j^n$  up to any order in powers of  $n^{-\frac{1}{2\mu}}$ . Also, it takes care of the case where the Fourier series  $F$  restricted to  $\mathbb{S}^1$  does not only have one point of modulus 1 (as is the case in Hypothesis 1) but any finite number of points. This case can occur for some schemes (for instance the Lax-Friedrichs scheme) and the only modification on Theorem 1 is that there are as many generalized Gaussian waves that appear that there are points where  $F$  restricted to  $\mathbb{S}^1$

is of modulus 1. However, Theorem 1 does not fully generalize [RSC15, Theorem 1.2] as the conditions in the paper of Randles and Saloff-Coste are less restrictive. To be more precise, the result (2.3) of [RSC15] does not need to consider the diffusivity condition (2.2) in Hypothesis 1. A generalization of Theorem 1 in this difficult case has not yet been found, even though the result of [Cou23] indicates that such a result might be attainable.

We now give some insight on the proof of Theorem 1. In [RSC15], the proof to obtain (2.3) relied on the use of Fourier analysis to express the coefficients  $a_j^n$  using the Fourier series  $F$ :

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad a_j^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-ijt} F(e^{it})^n dt.$$

However, the proof of [Coe22, Theorem 1] uses another representation of the coefficients  $a_j^n$  based on an approach usually referred to in partial differential equations as "spatial dynamics". We introduce the Laurent operator

$$L_a : u \in \ell^2(\mathbb{Z}) \mapsto a * u \in \ell^2(\mathbb{Z}).$$

The Wiener theorem [New75] implies that the spectrum of the operator  $L_a$  is

$$\sigma(L_a) = F(\mathbb{S}^1).$$

Then, if we introduce the "discrete" Dirac mass  $\delta \in \ell^2(\mathbb{Z})$  defined by

$$\forall j \in \mathbb{Z}, \quad \delta_j := \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{else,} \end{cases}$$

we observe that

$$\forall n \in \mathbb{N}, \quad a^n = L_a^n \delta.$$

Thus, using functional calculus (see [Con90, Chapter VII]), we can express the coefficients  $a_j^n$  using the resolvent of the operator  $L_a$ :

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad a_j^n := \frac{1}{2i\pi} \int_{\Gamma} z^n G(z, j) dz$$

where  $G(z, \cdot) = (zId - L_a)^{-1} \delta$  is called the spatial Green's function and  $\Gamma$  is a path on the complex plane  $\mathbb{C}$  which surrounds the spectrum of the operator  $L_a$ . The proof of Theorem 1 then relies on a precise analysis of the spatial Green's function  $G(z, \cdot)$  and an adequate choice of path  $\Gamma$  (which depends on  $j, n$ ). The choice of the contour  $\Gamma$  is done using the Riemann saddlepoint method and follows similar ideas as in [God03, ZH98]. We refer to [Coe22] for the complete details.

### 3 A numerical example: The O3 scheme

We will now consider a concrete example of finite difference scheme. The O3 scheme is an explicit third order accurate finite difference approximation of the transport equation (1.1). We refer to [Des08] for a detailed analysis of this scheme. It corresponds to the numerical scheme (1.2) for  $a \in \ell^1(\mathbb{Z})$  such that  $a_j = 0$  for  $j \notin \{-1, 0, 1, 2\}$  and

$$\begin{aligned} a_2 &= -\frac{v\lambda(1-(v\lambda)^2)}{6}, & a_1 &= \frac{v\lambda(1+v\lambda)(2-v\lambda)}{2}, \\ a_0 &= \frac{(1-(v\lambda)^2)(2-v\lambda)}{2}, & a_{-1} &= -\frac{v\lambda(1-v\lambda)(2-v\lambda)}{6}, \end{aligned}$$

with  $\lambda := \frac{\Delta t}{\Delta x} > 0$ . For  $v\lambda \in ]-1, 1[ \setminus \{0\}$ , we have that  $F(1) = 1$  and

$$\forall \kappa \in \mathbb{S}^1 \setminus \{1\}, \quad |F(\kappa)| < 1.$$

Also, there exists  $\beta \in ]0, +\infty[$  such that

$$F(e^{it}) \underset{t \rightarrow 0}{=} \exp(iv\lambda t - \beta t^4 + o(t^4)).$$

Hypothesis 1 is satisfied with  $\mu = 2$ . We can then apply Theorem 1. We introduce the attractors

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j \in \mathbb{Z}, \quad \mathcal{H}_j^n = \frac{1}{n^{\frac{1}{4}}} H_4^\beta \left( \frac{j - nv\lambda}{n^{\frac{1}{4}}} \right).$$

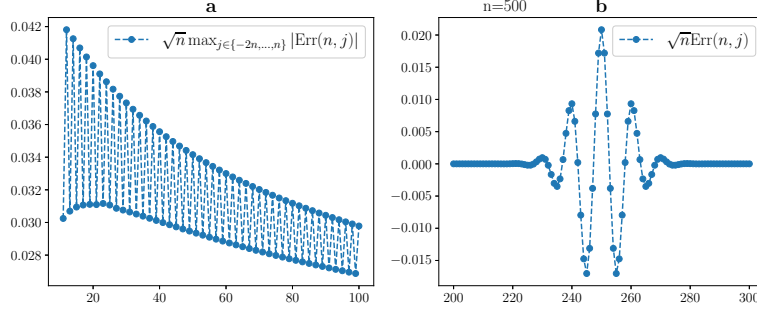


Figure 1: For these figures, we chose  $v\lambda = 1/2$ . On figure **a** : A representation of  $\sqrt{n} \max_{j \in \{-2n, \dots, n\}} |\text{Err}(n, j)|$  with  $\text{Err}(n, j) := a_j^n - \mathcal{H}_j^n$  depending on  $n$ . As expected, the function seems to be bounded. On figure **b** : We fixed  $n = 500$  and represented  $j \in \mathbb{Z} \mapsto \sqrt{n} \text{Err}(n, j)$ . We observe the exponential decay in  $j$  and a particular shape of curve that arises which can be explained by looking at the next order of the asymptotic expansion.

Theorem 1 then states that there exist two constants  $C, c > 0$  such that

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j \in \mathbb{Z}, \quad |\text{Err}(n, j)| \leq \frac{C}{\sqrt{n}} \exp\left(-c \left(\frac{|j - nv\lambda|}{n^{\frac{1}{4}}}\right)^{\frac{4}{3}}\right). \quad (3.1)$$

where  $\text{Err}(n, j) := a_j^n - \mathcal{H}_j^n$ .

This behavior is represented on Figure 1 where we even see that the remainder  $\sqrt{n}(a_j^n - \mathcal{H}_j^n)$  seems to scale like  $f\left(\frac{j - nv\lambda}{n^{\frac{1}{4}}}\right)$ . Hence, the  $\frac{1}{\sqrt{n}}$  pre-factor in (3.1) seems to be sharp. In [Coe22], we find a reliable way to compute the terms of the asymptotic expansion up to any order.

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