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EXCURSION DECOMPOSITION OF THE 2D CONTINUUM GFF

JUHAN ARU, TITUS LUPU, AND AVELIO SEPÚLVEDA

ABSTRACT. In this note we show that the 2D continuum Gaussian free field (GFF) admits an excursion decomposition similar to the classical excursion decomposition of the Brownian motion. In particular, 2D continuum GFF can be written as an infinite sum of disjoint positive and negative sign excursions, which are given by Minkowski content measures of clusters of a critical 2D Brownian loop soup with i.i.d. signs. Although the 2D continuum GFF is not even a signed measure, we show that the decomposition to positive and negative parts is unique under natural conditions.

1. INTRODUCTION

The 2D continuum Gaussian free field (GFF) is a universal model of a continuum height function and has become a central object in the study of conformally invariant continuum random geometry. The main reason for this is its strong connections with other objects like for example Schramm-Loewner Evolution, Brownian loop soup and Liouville quantum gravity measures (see e.g. overviews [BP23, GHS19, WP21]) and several known or conjectured convergence results towards the Gaussian free field [Nad97, Ken01, RV07, BLR20].

In this note, we explain how to prove a decomposition of the 2D continuum Gaussian free field into an (infinite) sum of signed measures with disjoint supports. This decomposition is unique under natural conditions and can be obtained as a scaling limit of an honest excursion decomposition of the metric graph GFF. Thus our result says that there is a natural decomposition of the GFF into negative and positive parts, despite the fact that the field is not pointwise defined and not even a signed measure. The obtained decomposition shares many properties with the classical excursion decomposition of Brownian motion [Itô71], but also exhibits some new surprising ones.

We work in an open bounded simply-connected domain $D \subset \mathbb{C}$, and we consider Φ a zero boundary Gaussian free field on D . To fix a normalization, we consider the GFF as the field coming from the following functional integral

$$\exp\left(-\frac{1}{2}\int_D \|\nabla\varphi\|^2\right)\mathcal{D}\varphi.$$

More precisely, Φ is the centred Gaussian process with covariance given by the Dirichlet Green's function $G_D(z, w)$ function with the following divergence on the diagonal

$$G_D(z, w) \sim \frac{1}{2\pi} \log |z - w|^{-1}.$$

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With this normalization, the value of the height gap (used later and introduced in [SS09]) is $2\lambda = \sqrt{\pi/2}$.

The contribution of this paper comes in three theorems: first we state the existence and uniqueness of an excursion decomposition, second we list properties of this decomposition, that mirror strongly those of the excursion decomposition of the one dimensional Brownian motion and make connections with the 2D critical Brownian loop soup. Finally, we show that the naturally defined excursion decomposition of the metric GFF converges to the excursion decomposition of the continuum GFF. We make use of known couplings between GFF, CLE_4 and Brownian loop soup [SS09, SW12, ASW17, QW18, ALS20a, ALS20b] and build on techniques introduced in [SS13, ASW17, ALS20a, ALS20b].

The existence of the excursion decomposition is given in the following theorem.

Theorem 1 (Excursion decomposition of the 2D Gaussian free field). *Let Φ be a zero boundary GFF in D . There exists a unique collection of positive measures $(\nu_k)_{k \geq 1}$ with supports $(C_k)_{k \geq 1}$, and a collection of signs $(\sigma_k)_{k \geq 1}$, such that the following conditions hold:*

(1) *We can write*

$$\Phi = \lim_{N \rightarrow \infty} \sum_{k=1}^N \sigma_k \nu_k, \quad (1.1)$$

where the sum is ordered by decreasing size of the diameter of C_k . The sum converges almost surely in all the Sobolev spaces $H^{-1-\varepsilon}(D)$ (i.e. for the Sobolev norms) for $\varepsilon > 0$.

(2) *The decomposition satisfies the following Markov property. For any smooth simple path $\gamma \subset \bar{D}$, starting from the boundary, let γ^{exc} denote the closure of the union of all sets C_k that intersect γ . We can write almost surely $\Phi = \Phi^{\gamma^{exc}} + \Phi_{\gamma^{exc}}$, with*

$$\Phi_{\gamma^{exc}} = \sum_{k: C_k \cap \gamma \neq \emptyset} \sigma_k \nu_k,$$

where the sum is again ordered by decreasing size of diameter of C_k and converges almost surely in all the Sobolev spaces $H^{-1-\varepsilon}(D)$, for $\varepsilon > 0$. Further, conditionally on γ^{exc} , the field $\Phi^{\gamma^{exc}}$ is independent of $\Phi_{\gamma^{exc}}$ and has the law of a zero boundary GFF in the domain $D \setminus \gamma^{exc}$.

(3) *The collection $(C_k)_{k \geq 1}$ is pairwise disjoint, locally finite¹, and further each C_k is connected.*

We call $(C_k)_{k \geq 1}$ the (sign) excursion clusters, $(\nu_k)_{k \geq 1}$ the sign excursions and the triplet $(C_k, \sigma_k, \nu_k)_{k \geq 1}$ the excursion decomposition of Φ .

Further properties of the excursion decomposition are listed in the following theorem.

Theorem 2 (Properties of the excursion decomposition). *Let Φ be a zero boundary GFF in D and $((C_k, \nu_k, \sigma_k))_{k \geq 1}$ respectively the excursion clusters, the measure and their signs in the excursion decomposition of Theorem 1. Then, the following properties hold:*

(1) *The excursion decomposition $((C_k, \nu_k, \sigma_k))_{k \geq 1}$ is measurable w.r.t. Φ .*

(2) *In the joint law of $((C_k, \nu_k, \sigma_k))_{k \geq 1}$, the signs $(\sigma_k)_{k \geq 1}$ are independent of the rest and have the law of i.i.d. Rademacher random variables.*

¹Locally finite means that for any $\varepsilon > 0$ there are finitely many C_k with diameter bigger than ε .

(3) For all $k \geq 1$, the measures ν_k are given by Minkowski content measure of C_k defined by

$$\nu_k(f) := \lim_{r \rightarrow 0} \frac{1}{2} |\log r|^{1/2} \int_D f(z) 1_{d(z, C_k) \leq r} dz,$$

for all $f \in C(D)$. In particular for all $k \geq 1$, ν_k is determined by C_k .

(4) The law of $(C_k)_{k \geq 1}$ equals to that of clusters of a 2D Brownian loop soup at the critical intensity $\alpha = 1/2$ in D , ordered by decreasing diameter.

Further, one can justify the name excursion decomposition by showing a convergence result from the well defined excursion decomposition on the metric graph GFF. See Section 5 for the exact set-up.

Theorem 3 (Convergence of the excursion decomposition). *Let Φ be a zero boundary GFF on D and $\tilde{\phi}_n$ be a sequence of zero boundary metric graph GFFs on \tilde{D}_n that are coupled with a GFF Φ such that a.s. $\tilde{\phi}_n \rightarrow \Phi$ in $H^{-\varepsilon}(D)$, for some $\varepsilon > 0$. Further, take $(\tilde{C}_k^n, \tilde{\nu}_k^n, \tilde{\sigma}_k^n)_{k \geq 1}$ the excursion decomposition of $\tilde{\phi}_n$.*

We have that for every $k > 0$, $\tilde{C}_k^n \rightarrow C_k$, $\tilde{\nu}_k^n \rightarrow \nu_k$ and $\tilde{\sigma}_k^n \rightarrow \sigma$ as $n \rightarrow \infty$, where the convergence is in probability and in the Hausdorff topology for the first component, and in the weak topology of measures for the second component.

Let us elaborate on these theorems via some further remarks.

- It is known that the 2D continuum Gaussian free field is not a signed measure and in particular it cannot be written as a difference of two sigma-finite positive measures. Thus such a rewriting as a sum of disjoint signed measures is in itself already non-trivial. Previously this was known for the continuum limit of the magnetization field of the Ising model [CGN15], but interestingly in that context the decomposition is not measurable with respect to the continuum magnetization field itself.
- The existence of a decomposition into a signed sum of measures (without uniqueness, measurability and an explicit description of the structure of the decomposition) could be also obtained using subsequential convergence results from the metric graph, using results from [Lup18] but no further SLE theory.
- To prove existence and uniqueness of the decomposition we only need to use basic properties of the GFF and its local sets (including CLE_4 , SLE_4), and thus in particular we do not use isomorphism theorems. In fact also the excursion clusters have a writing in terms of only the nested CLE_4 : see Remark 17.
- We expect the existence and uniqueness of the decomposition, and all the properties to hold also in non-simply-connected domains. However, it adds some technicalities that we decided not to address in this work.
- The convergence of the sum can most likely be improved to $H^{-\varepsilon}$ for all $\varepsilon > 0$.
- For the convergence in (1.1), the compensation induced by the sign is crucial, and the total variation measure $\sum_{i \geq 1} \nu_i$ diverges in every open subset of D . There is some freedom in the specific order on the clusters $(C_i)_{i \geq 1}$. However, it is important to fix the order independently of the signs $(\sigma_i)_{i \geq 1}$. Notice that we do not a priori ask any independence properties of the signs, and obtain them as a corollary.
- One may wonder if any Markov property at all is necessary to prove the uniqueness of the decomposition. For example, in the case of Brownian motion on $[0, 1]$ it suffices for uniqueness to ask that we can decompose $[0, 1]$ into disjoint closed sets where BM

is either non-positive or non-negative (and not fully zero) together with a residual set where BM is equal to zero. Yet, observe that without the final condition on the residual set, the uniqueness is not so clear: one could further decompose the support of any positive or negative excursion to countable many connected closed sets, by giving up a set of zero measure. On that set BM would not be zero, but all other conditions would be satisfied. In particular, if one decomposes Brownian motion in the sense of L^2 to closed disjoint sets where BM is either non-positive or non-negative, this decomposition would not be unique without some extra condition like the Markov property after discovering some excursions, or independence of signs of excursions (which would be implied by a reasonable Markov property).

- This theorem can be further tweaked to write the 2D continuum GFF using a Poisson point process of excursion very similarly to the classical writing of the Brownian motion by concatenating a PPP of Brownian excursions.
- A similar decomposition can be given also for the Wick powers of the GFF, although interestingly the measurability part is again an issue. This will either appear in a second version of the current paper, or in a separate note.
- It would be very interesting to see similar decompositions for other random distributions and indeed an upcoming work by Jego, Lupu and Qian manages to prove similar decompositions for random fields constructed from sub-critical Brownian loop soups [JLQ23]. Among other things, they will also give an alternative proof for existence of the decomposition in the critical case that does not rely local sets of the GFF (nor CLE_4 , SLE_4), but that does not provide uniqueness and measurability.

The rest of this note is structured as follows: we collect definitions of main objects in Section 2; in Section 3 we prove the existence part of Theorem 1 and deduce the properties of Theorem 2. In Section 4 we prove the uniqueness of the decomposition and in Section 5 the convergence.

2. DEFINITIONS AND PRELIMINARIES

For the convenience of the reader, we collect here the definitions of the 2D continuum Gaussian free field and its local sets, CLE_4 and Brownian loop soup. For more information, see e.g. preliminaries of [ALS20a, ALS20b] or the book [WP21].

The continuum Gaussian free field (GFF) is the generalisation of Brownian motion, replacing the time axis by a d -dimensional domain. More precisely, it is defined as follows.

Definition 4 (Gaussian free field). *Let $D \subseteq \mathbb{C}$ denote a finitely connected domain. The 2-dimensional zero boundary continuum GFF in D is the centred Gaussian process $(\Phi, f)_{f \in C_c^\infty(\mathbb{C})}$ whose covariance is given by*

$$\mathbb{E}[(\Phi, f)(\Phi, g)] = \int \int_{D \times D} f(z) G^D(z, w) g(w) dz dw; \quad f, g \in C_c^\infty(\mathbb{C}),$$

where G^D denotes the zero boundary Green's function for the Laplacian in D .

For any open set U that is a union of countably many finitely-connected domains, we define the zero boundary GFF on U as a disjoint union of independent zero boundary GFFs in the connected components. The GFF is almost surely in $H^{-\varepsilon}(U)$ for any $\varepsilon > 0$, but we can also consider the GFF as a random distribution on larger domains $U' \supseteq U$, extending

it outside of U by zero. The continuum GFF can be essentially characterized by its Markov property [BPR20, BPR21, AP22] and random sets coupled with the GFF that satisfy a strong Markov property are called local sets. For a more general discussion of local sets and their properties we refer to [Aru15, SS13, WP21].

Definition 5 (Local sets). *Consider a random triple (Φ, A, Φ_A) , where Φ is a GFF in D , A is a random closed subset of \overline{D} and Φ_A a random distribution that can be viewed as a harmonic function when restricted to $D \setminus A$. We say that A is a local set for Φ if conditionally on (A, Φ_A) , $\Phi^A := \Phi - \Phi_A$ is a GFF in $D \setminus A$.*

We list here some properties of local sets that we use implicitly or explicitly, see for instance [SS13, Aru15] for derivations and further properties.

Lemma 6. *The following properties hold for local sets of the GFF.*

- (1) *Any local set can be coupled in a unique way with a given GFF: Let $(\Phi, A, \Phi_A, \widehat{\Phi}_A)$ be a coupling, where (Φ, A, Φ_A) and $(\Phi, A, \widehat{\Phi}_A)$ satisfy the conditions of this definition. Then, a.s. $\Phi_A = \widehat{\Phi}_A$. Thus, being a local set is a property of the coupling (Φ, A) , as Φ_A is a measurable function of (Φ, A) .*
- (2) *If A and B are local sets coupled with the same GFF Φ , and (A, Φ_A) and (B, Φ_B) are conditionally independent given Φ , then $A \cup B$ is also a local set coupled with Φ and the boundary values of $\Phi_{A \cup B}$ agree with those of Φ_B or Φ_A at every point of the boundary of $A \cup B$ that is of positive distance of A or B respectively². Additionally, $B \setminus A$ is a local set of Φ^A with $(\Phi^A)_{B \setminus A} = \Phi_{B \cup A} - \Phi_A$.*
- (3) *Let $(\Phi, (A_n)_{n \in \mathbb{N}}, (\Phi_{A_n})_{n \in \mathbb{N}})$ a sequence of conditionally independent local sets coupled with the same GFF Φ . Furthermore, assume that A_n is increasing. Then $A_\infty = \overline{\bigcup_{n \in \mathbb{N}} A_n}$ is a local set. Furthermore, if a.s. for all $n \in \mathbb{N}$, A_n is connected to the boundary, then a.s. $\Phi_{A_n} \rightarrow \Phi_A$.*

In particular, we will use the existence and uniqueness of the following type of local sets: two-valued local sets introduced in [ASW17] and studied in [ALS20a], and first passage sets, introduced in [ALS20a, ALS20b]. For definitions of thin local sets, bounded type local sets we refer e.g. to [ASW17, ALS22].

Theorem 7 (Two-valued local sets: existence and uniqueness). *Let $a, b > 0$ be such that $a + b \geq 2\lambda$. Then one can couple a thin³ bounded type local set $\mathbb{A}_{-a,b} \neq \emptyset$ with a GFF Φ such that in each connected component O of $D \setminus \mathbb{A}_{-a,b}$ the harmonic function Φ_A is equal to either $-a$ or b . Moreover, the sets $\mathbb{A}_{-a,b}$ are*

- *unique in the sense that if A' is another BTLS coupled with the same Φ , such that a.s. it satisfies the conditions above, then $A' = \mathbb{A}_{-a,b}$ almost surely;*
- *measurable functions of the GFF Φ that they are coupled with;*
- *monotone in the following sense: if $[-a, b] \subset [-a', b']$ with $b + a \geq 2\lambda$, then almost surely, $\mathbb{A}_{-a,b} \subset \mathbb{A}_{-a',b'}$.*

It comes out [ASW17] that the Minkowski dimension of all of any two-valued set is a.s. strictly smaller than 2 (it was precisely calculated in [SSV22]); we will make use of this fact

²We say that $\Phi_{A \cup B}$ agrees with Φ_A at a point $x \in \partial(A \cup B) \cap \partial A$ if for any sequence of $x_n \notin A \cup B$ converging to x , $\Phi_{A \cup B}(x_n) - \Phi_A(x_n) \rightarrow 0$ as $n \rightarrow \infty$.

³Thin means that Φ_A is a.s. equal to a harmonic function everywhere, see [Sep19].

for this for $\mathbb{A}_{-2\lambda, 2\lambda}$. Indeed, the boundaries of sign excursions will turn out to be two-valued local sets $\mathbb{A}_{-2\lambda, 2\lambda}$. The clusters themselves are given by first passage sets.

Definition 8 (First passage set). *Let $a \in \mathbb{R}$ and Φ be a GFF in D . We define the first passage set of Φ of level $-a$ as the local set of Φ such that $\partial D \subseteq \mathbb{A}_{-a}$, with the following properties:*

- (1) *Inside each connected component O of $D \setminus \mathbb{A}_{-a}$, the harmonic function $\Phi_{\mathbb{A}_{-a}}|_{D \setminus \mathbb{A}_{-a}}$ is equal to $-a$.*
- (2) *$\Phi_{\mathbb{A}_{-a}} + a \geq 0$, i.e., for any smooth positive test function f we have $(\Phi_{\mathbb{A}_{-a}} + a, f) \geq 0$, in other words $\nu := \Phi_{\mathbb{A}_{-a}} + a$ is a positive measure with support \mathbb{A}_{-a} .*

The key result is the following.

Theorem 9 (Theorem 4.3 and Proposition 4.5 of [ALS20a], Proposition 5.7 of [ALS20b]). *For all $a \geq 0$, the first passage set, \mathbb{A}_{-a} , of Φ of level $-a$ exists and satisfies the following properties:*

- (1) *Uniqueness: if A' is another local set coupled with Φ and satisfying Definition 8, then a.s. $A' = \mathbb{A}_{-a}$.*
- (2) *Measurability: \mathbb{A}_{-a} is a measurable function of Φ .*
- (3) *Monotonicity: If $a \leq a'$, then $\mathbb{A}_{-a} \subseteq \mathbb{A}_{-a'}$.*
- (4) *Locally finiteness: for any $\varepsilon > 0$ there are only finitely many connected components of $D \setminus \mathbb{A}_{-a}$ of diameter larger than ε .*

2.1. Couplings between different objects. We first that from the connection between GFF and CLE_4 that was first discovered by Miller & Sheffield [MS11], based on the work of Schramm and Sheffield [SS13], and it says that CLE_4 can be coupled as a local set of the GFF. In [ASW17] this was rephrased in the language of two-valued sets - the set $\mathbb{A}_{-2\lambda, 2\lambda}$ has the law of a CLE_4 carpet.

Theorem 10 (Section 4 of [ASW17]). *Let Φ be a GFF in D and $\mathbb{A}_{-2\lambda, 2\lambda}$ be its TVS of levels -2λ and 2λ . Then $\mathbb{A}_{-2\lambda, 2\lambda}$ has the law of CLE_4 carpet. Moreover, it satisfies the following properties:*

- (1) *The loops of $\mathbb{A}_{-2\lambda, 2\lambda}$ (i.e. the boundaries of the connected components of $D \setminus \mathbb{A}_{-2\lambda, 2\lambda}$) are continuous simple loops. $\mathbb{A}_{-2\lambda, 2\lambda}$ is the closure of the union of all loops.*
- (2) *The collection of loops of $\mathbb{A}_{-2\lambda, 2\lambda}$ is locally finite, i.e. for any $\varepsilon > 0$ there are only finitely many loops that have diameter bigger than ε .*
- (3) *Almost surely no two loops of $\mathbb{A}_{-2\lambda, 2\lambda}$ intersect, nor does any loop intersect the boundary.*
- (4) *The conditional law of the labels of the loops of $\mathbb{A}_{-2\lambda, 2\lambda}$ given $\mathbb{A}_{-2\lambda, 2\lambda}$ is that of i.i.d. random variables taking values $\pm 2\lambda$ with equal probability.*

From the ground-setting work of Sheffield and Werner, we know further that in simply-connected domains CLE_4 loops can be described using the critical Brownian loop soup.

Theorem 11 (Theorem 1.6 in [SW12]). *Let D be a simply-connected domain and consider the critical Brownian loop-soup \mathcal{L} in D . Then CLE_4 loops are exactly the outer boundaries of the outermost clusters of this Brownian loop soup (BLS).*

This theorem together with Theorem 10 implies the following Markov property for $\mathbb{A}_{-2\lambda, 2\lambda}$

Proposition 12. *Let Φ be a GFF in a simply connected domain D and $\gamma : [0, 1] \mapsto \bar{D}$ be a simple continuous curve such that $\gamma(0) \in \partial D$ and $\gamma((0, 1)) \subseteq D$. Define γ^{ext} the closure of the union of all loops of a $\mathbb{A}_{-2\lambda, 2\lambda}$ that intersect γ . We have that γ^{ext} is a BTLS of Φ , where $\Phi_{\gamma^{ext}}$ can be characterised as follows. Take \mathcal{I} the union of the interior all loops ℓ of $\mathbb{A}_{-2\lambda, 2\lambda}$ that intersect γ , then $\Phi_{\gamma^{ext}}(z) = \pm 2\lambda$ for any $z \in \mathcal{I}$ and $\Phi_{\gamma^{ext}}(z) = 0$ for all $z \in D \setminus \bar{\mathcal{I}}$.*

In fact, the relation of Theorem 11 can be further strengthened. First in [QW19] the authors show that one can couple the critical Brownian loop soup, CLE_4 and the zero boundary GFF all in one coupling where both CLE_4 describes the outer boundaries of outermost BLS clusters as above, but also the Wick square of the GFF equals the renormalised occupation time of the BLS. We will not use this statement directly, however we use a certain strengthening that further identifies the Brownian loop soup clusters given their boundary with first passage sets defined and constructed in [ALS20a].

Proposition 13 (Corollary 5.4 in [ALS20b]). *Let D be a simply connected domain. Conditionally on the outer boundary ℓ of a Brownian loop-soup cluster in $\mathcal{L}_{1/2}^D$, the topological closure of the cluster itself is distributed like a first passage set $\mathbb{A}_{-2\lambda}$ inside $\text{Int}(L)$, the interior surrounded by Υ .*

Finally, one can identify the GFF restricted to a first passage set by its Minkowski content measure.

Theorem 14 (Theorem 5.1 in [ALS20a]). *Let D be simply-connected and Φ a GFF and suppose \mathbb{A}_{-a} is a first passage set of level $-a$. Writing $\Phi = \Phi_{\mathbb{A}_{-a}} + \Phi^{\mathbb{A}_{-a}}$ as in Definition 5, we obtain the following. The measure $\nu_{\mathbb{A}_{-a}} := \Phi_{\mathbb{A}_{-a}} + a$ is a measurable function of \mathbb{A}_{-a} . Moreover, it is proportional to the Minkowski content measure in the gauge $r \mapsto |\log(r)|^{1/2} r^2$. More precisely, almost surely for any continuous f compactly supported in D ,*

$$\nu_{\mathbb{A}_{-a}} = \lim_{r \rightarrow 0} \frac{1}{2} |\log(r)|^{1/2} \int_D f(z) \mathbf{1}_{d(z, \mathbb{A}_{-a}) \leq r} dz.$$

3. EXISTENCE OF THE EXCURSION DECOMPOSITION AND ITS PROPERTIES

In this section, we prove the existence of the excursion decomposition together with the properties stated in Theorem 2. The proof follows rather directly from the theory of bounded type local sets and first passage sets of the GFF, though some care is needed in collecting and combining the results and techniques. The most technical part consist in proving the Markov property of the decomposition.

We start by an elementary estimate on the H^{-1} norm of a GFF open sets that are disjoint unions of domains of small diameter. This lemma is used to show that contributions to the excursion decomposition coming from small excursions can be summed.

Lemma 15. *Suppose $\hat{D}_n \subseteq D$ is a sequence of decreasing open set (not necessarily connected) such that the maximal diameter over its connected components goes to 0 as $n \rightarrow \infty$. Consider $\Phi^{\hat{D}_n}$ a GFF in \hat{D}_n . Then $\mathbb{E} \left[\|\Phi^{\hat{D}_n}\|_{H^{-1}(D)}^2 \right] \rightarrow 0$, as $n \rightarrow \infty$*

Proof. This just follows from the dominated convergence theorem ($G_{\widehat{D}_n} \leq G_D$) and the computation

$$\mathbb{E} \left[\|\Phi^{\widehat{D}}\|_{H^{-1}(D)}^2 \right] = \iint_{\widehat{D} \times \widehat{D}} G_D(x, y) G_{\widehat{D}}(x, y) dx dy \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

□

We are now ready to prove the existence part of the main theorem.

Proof of the existence of an excursion decomposition in Theorem 1. We start by considering the coupling $(\Phi, \text{CLE}_4 = (\ell_k)_{k \geq 1}, (\sigma_k)_{k \geq 1})$ between the GFF, CLE_4 loops and the i.i.d. signs coming from Theorem 10. We can order the loops in descending order of their diameter. Note that this theorem implies the almost sure equality

$$\Phi = \sum_{k \geq 1} 2\lambda \sigma_k 1_{\text{Int}(\ell_k)} + \Phi^{\text{Int}(\ell_k)},$$

where given the CLE_4 loops $(\ell_k)_{k \geq 1}$, $\Phi^{\text{Int}(\ell_k)}$ are independent zero boundary GFFs⁴ inside $\text{Int}(\ell_k)$ and $(\sigma_k)_{k \geq 1}$ are i.i.d. Rademacher random variables.

Using Lemma 15 to control the tails, we can restrict our attention to the subset J_ε of $k \in \mathbb{N}$ such that the diameter of C_k is at least $\varepsilon > 0$. As the set of CLE_4 loops is locally finite, J_ε is finite.

Now, we take $k \in J_\varepsilon$. Conditionally on ℓ_k the law of Φ restricted to $\text{Int}(\ell_k)$ is equal to that of $2\lambda \sigma_k + \Phi^{\text{Int}(\ell_k)}$, where the conditional law of $\Phi^{\text{Int}(\ell_k)}$ is that of a GFF in $\text{Int}(\ell_k)$. We now sample $A_k := \mathbb{A}_{-2\lambda}(\sigma_k \Phi^{\text{Int}(\ell_k)})$ and define the positive measure $\nu_k := \sigma_k \Phi_{A_k}^{\text{Int}(\ell_k)} + 2\lambda$. This measure is supported in A_k thanks to Definition 8. We then have that

$$2\lambda \sigma_k + \Phi^{\text{Int}(\ell_k)} = \sigma_k \nu_{\mathbb{A}_{-2\lambda}} + \Phi^{\text{Int}(\ell_k) \setminus \mathbb{A}_{-2\lambda}}.$$

Thus we obtain the following decomposition

$$\Phi = \sum_{k \in J_\varepsilon} (\sigma_k \nu_{A_k} + \Phi^{\text{Int}(\ell_k) \setminus A_k}) + \sum_{k' \notin J_\varepsilon} (2\lambda \sigma_{k'} 1_{\text{Int}(\ell_{k'})} + \Phi^{\text{Int}(\ell_{k'})}),$$

where the outermost boundaries ℓ_k and $\ell_{k'}$ are ordered in the descending order of their diameter. We now iterate this process inside each connected component of $D \setminus \bigcup_{k \in J_\varepsilon} A_k$.

We denote the outermost loops and clusters of the n -th iteration that themselves have diameter larger than ε by $(\ell_{n,k})_{k \in J_{n,\varepsilon}}, (C_{n,k})_{k \in J_{n,\varepsilon}}$, having ordered them decreasingly by diameter, and the corresponding signs and Minkowski measures by $(\sigma_{n,k})_{k \in J_{n,\varepsilon}}, (\nu_{n,k})_{k \in J_{n,\varepsilon}}$.

The iteration of the construction allows us to write the following almost sure equality:

$$\Phi = \sum_{k \in \bigcup_{n \leq N} J_{n,\varepsilon}} (\sigma_k \nu_{A_{n,k}} + \Phi^{\text{Int}(\ell_{n,k}) \setminus A_{n,k}}) + \sum_{\substack{(n',k'): \\ k' \notin J_{n',\varepsilon} \\ n' \leq N}} (2\lambda \sigma_{k'} 1_{\text{Int}(\ell_{k'})} + \Phi^{\text{Int}(\ell_{k'})}),$$

where again the ordering in the first finite sum is along decreasing size of the diameter. This writing allows us to apply Lemma 15 directly to obtain an error of order $o_\varepsilon(1)$ independently of the level of iteration N on the second term. The existence of the decomposition now follows from the a.s. martingale convergence theorem and the fact that for any $\varepsilon > 0$, there

⁴Note that $\Phi^{\mathbb{A}_{-2\lambda, 2\lambda}} = \sum_k \Phi^{\text{Int}(\ell_k)}$

is almost surely a finite N such that all loops of diameter larger than ε have been discovered. This follows directly from the locally finiteness of CLE_4 and FPS.

All the properties listed in the Theorem for the excursion clusters $(C_k)_{k \geq 1}$ hold by construction. The Markov property follows from the following claim combined with the argument above that again shows we can sum the sign excursions in their decreasing order of diameter.

Claim 16. *Consider γ a smooth simple path in \bar{D} starting from the boundary. Let $(C_k, \ell_k)_{k \in I}$ be the collection of outermost clusters with $C_k \cap \gamma \neq \emptyset$ and denote their outer boundaries by ℓ_k . Let I_ε denote the set of $k \in I$ for which the diameter of C_k is at least ε and define $A_\varepsilon = \bigcup_{k \in I_\varepsilon} C_k \cup \bigcup_{k \in I \setminus I_\varepsilon} \ell_k$. Then A_ε is a local set, such that $\phi_{A_\varepsilon} = \sum_{i \in I_\varepsilon} \sigma_i \nu_i + \sum_{i \in I \setminus I_\varepsilon} 2\lambda \sigma_i 1_{z \in \text{Int } L_i}$.*

Proof. The claim follows directly from iterating the Markov property of $\mathbb{A}_{-2\lambda, 2\lambda}$ in Proposition 12, together with the strong Markov property of FPS and the construction above. \square

\square

\square

We now proceed to discuss further properties of the excursion decomposition. In essence, this amounts to collecting and applying a few interesting results from the literature.

We are now ready to give a proof of Theorem 2, assuming already uniqueness.

Proof of Theorem 2. Properties (1) and (2) follow directly from the construction given above. Property (3) follows from the construction of the excursion clusters and excursions via First passage sets of height $\pm 2\lambda$ and Theorem 14.

The identification with clusters of 2D Brownian loop soup follows further from iterating Theorem 11 to identify the outer boundaries of critical BLS clusters with those of excursion clusters in the construction above, and Theorem 13 to identify the critical BLS clusters with the excursion clusters above. \square

Remark 17. *Indeed, as above, we know that conditioning on the outer boundary of an excursion cluster, the cluster itself is distributed as the FPS of height 2λ . But now by Lemma 2.5 in [APS20] and the discussion under it, we see that this FPS can be obtained by iterating the two-valued local set $\mathbb{A}_{-2\lambda, 2\lambda}$ until every component has label 2λ . As $\mathbb{A}_{-2\lambda, 2\lambda}$ has the law of CLE_4 , we indeed have a way of describing the whole sign cluster using iterated CLE_4 .*

4. UNIQUENESS OF THE EXCURSION DECOMPOSITION

In this section, we prove the uniqueness part of Theorem 1. Throughout this section $(\nu_k, C_k, \sigma_k)_{k \geq 1}$ denotes the excursion decomposition constructed in Section 3, and $(\hat{\mu}_k, \hat{C}_k, \hat{\sigma}_k)$ is another decomposition that satisfies the properties of Theorem 1 for the same GFF Φ . By conformal invariance we may assume that we work in the unit disk \mathbb{D} throughout this section.

The proof of uniqueness relies on two preliminary propositions. We first show that the excursion clusters in the construction of the previous section are in a certain sense minimal:

Proposition 18. *A.s. for any $k \in \mathbb{N}$ there exists $\hat{k}(k) \in \mathbb{N}$ such that $C_k \subseteq \hat{C}_{\hat{k}(k)}$.*

Then, we show that the signs of intersecting sign clusters of the two decompositions introduced above have to match.

Proposition 19. *Let $\hat{k}(k)$ be as in Proposition 18, we have that a.s. $\sigma_k = \hat{\sigma}_{\hat{k}(k)}$.*

Next, we argue that there is a 1-1 correspondence between the clusters.

Proposition 20. *The function $k \rightarrow \hat{k}(k)$ of Proposition 18 is also injective.*

And we conclude by saying that the clusters are almost surely equal; the equality of measures is concluded in the proof of the theorem.

Proposition 21. *For all $k \geq 1$, with $k \rightarrow \hat{k}(k)$ as above, we have $C_k = \hat{C}_{\hat{k}(k)}$. Moreover, the function $k \rightarrow \hat{k}(k)$ is also surjective.*

Throughout the proofs we will make use of the following local set, obtained by exploring the clusters around a line segment until some stopping time.

Lemma 22. *Let $\gamma : [0, 1] \mapsto \mathbb{D}$ be a simple curve. We define $\hat{\gamma}^{exc}(t)$ as the union of all \hat{C}_k that intersect $\gamma([0, t])$ and take τ a stopping time for the filtration $\mathcal{F}_t := \bigvee_{s \leq t} \sigma(\hat{\gamma}^{exc}(s))$. We have that $\hat{\gamma}^{exc}(\tau)$ is also a local set, more precisely $\Phi = \Phi^{\hat{\gamma}^{exc}(\tau)} + \Phi_{\hat{\gamma}^{exc}(\tau)}$ where conditionally on $\hat{\gamma}^{exc}(\tau)$ the law of $\Phi^{\hat{\gamma}^{exc}(\tau)}$ is a GFF in $D \setminus \hat{\gamma}^{exc}(\tau)$ and $\Phi_{\hat{\gamma}^{exc}(\tau)} = \sum_k \sigma_k \nu_k \mathbf{1}_{C_k \cap \eta \neq \emptyset}$ (where again the sum is ordered by descending diameter size of clusters).*

Proof. As by the Markov property of the excursion decomposition $(\hat{\gamma}^{exc}(t))_{t \in [0, 1]}$ is a family of increasing local sets, this strong Markov property follows from Lemma 1.3.13 in [Aru15]. \square

In fact, to circumvent some technical issues, we have to tweak this local set further to be able to also explore only a subset of the excursions intersecting the line:

Lemma 23. *Let γ be the straight line segment from $-i$ to 0 and define $\hat{\gamma}^{exc}$ to be the closure of the union of all \hat{C}_k that intersect γ . For any $\varepsilon > 0$, there exists a local set χ_ε that has the following property:*

- (1) *It is equal to the union of certain excursion of $(\hat{C}_k)_k$.*
- (2) *It is contained in $\hat{\gamma}^{exc}$, and contains any cluster \hat{C}_k that surrounds 0 with diameter bigger than or equal to ε .*
- (3) *if there is a simple loop $\ell \subseteq \chi_\varepsilon$ with diameter bigger than ε that surrounds 0, then there exists k such that $\ell \subseteq \hat{C}_k$.*

Proof of Lemma 23. We construct the set recursively. We first define the stopping time

$$\hat{\tau}^1 := \inf\{t \geq 0 : \exists \delta > 0, \exists \mathcal{O} \text{ c.c. of } B(0, 1) \setminus (\hat{\gamma}^{exc}(t) \cup \gamma|_{[-i, -\delta i]}) \text{ with } 0 \in \mathcal{O} \text{ and } \partial \mathbb{D} \cap \overline{\mathcal{O}} = \emptyset\}.$$

Note that $\gamma^{exc}(\tau^1) \setminus \bigcup_{t < \tau^1} \gamma^{exc}(t)$ is equal to a certain excursion \hat{C}_k . If this excursion surrounds 0 and has diameter smaller than or equal to ε we finish our iteration and define $\chi_\varepsilon = \gamma^{exc}(\tau^1)$; by definition it satisfies the desired conditions.

If the above is not the case, we note that $\hat{\gamma}^{exc}(\tau^1)$ is a local set and that the set of excursions $(\hat{C}_k : C_k \cap \hat{\gamma}^{exc}(\tau^1) = \emptyset)_{k \geq 1}$ generate an excursion decomposition of the GFF $\Phi^{\hat{\gamma}^{exc}(\tau^1)}$. We define

$$\hat{x} = \sup\{\Im(x) : x \in \gamma^{exc}(\tau^1) \cap \gamma\}$$

where $\Im(x)$ denotes the imaginary part of x and consider the line γ_1 that goes from $-i\hat{x}$ to $-\varepsilon i$. In this case we define $\hat{\tau}^2$ in the analogue way with γ_1 instead of γ and iterate the procedure above. This procedure finishes at a finite step j , because the set of clusters is

locally finite and at some point $\hat{\tau}^j$ discovers a loop that surrounds 0 whose diameter is less than or equal to ε .

We now need to prove that this sets satisfies the claimed properties. (1) and (2) are clear from construction. Moreover, if a cluster \hat{C}_k intersects γ and separates 0 from the boundary, then it has to appear at one of the times $\hat{\tau}^j$.

To see the point (3), we note that by definition there can be no simple loops ℓ surrounding 0 contained in $\bigcup_{t < \tau_{j+1}} \hat{\gamma}_j^{exc}(t) \setminus \hat{\gamma}_{j-1}^{exc}(\tau_j)$ and furthermore $\bigcup_{t < \tau_{j+1}} \hat{\gamma}_j^{exc}(t) \cap \hat{\gamma}_{j-1}^{exc}(\tau_j)$ and $\bigcup_{t < \tau_{j+1}} \hat{\gamma}_j^{exc}(t) \cap \hat{\gamma}_j^{exc}(\tau_{j+1})$ both have exactly one point and thus there can not be a simple loop going between $\bigcup_{t < \tau_{j+1}} \hat{\gamma}_j^{exc}(t)$ and either $\hat{\gamma}_j^{exc}(\tau_{j+1})$ or $\hat{\gamma}_{j-1}^{exc}(\tau_j)$. \square

Let us now start proving the propositions.

Proof of Proposition 18. It suffices to prove the proposition for any cluster C_k that surrounds 0. Let γ be the straight line segment from $-i$ to 0 and define $\hat{\gamma}^{exc}$ to be the closure of the union of all \hat{C}_k that intersect γ as before and consider the local set χ_ε from Lemma 23.

We now work recursively: we start by building $\mathbb{A}_{-2\lambda, 2\lambda}$, say via $\text{SLE}_4(-2)$ like in [ASW17], and we consider $k \in \mathbb{N}$ such that the outer boundary of C_k is in $\mathbb{A}_{-2\lambda, 2\lambda}$ and surrounds 0 (i.e. C_k is the outer-most excursion surrounding 0); we let $L_k \subseteq \mathbb{A}_{-2\lambda, 2\lambda}$ denote its outer boundary. We aim to show that L_k is contained in some $\hat{C}_{\hat{k}(k)}$.

By Claim 17 of [ASW17] we see that a priori L_k can touch at most two-prime ends of χ_ε (at the start and at the end of the loop). Furthermore, using the same proof as that of Claim 17 of [ASW17] one can see that we can touch only one prime end. This is because the loop is simple, and thus at the beginning of the curve it can intersect only one prime end. Then, on the event that C_k intersect $\gamma|_{[-i, -\varepsilon i]}$ the construction of the loop implies that the SLE_4 -type curve that makes the loop is a generalized level line of the GFF in $\mathbb{D} \setminus \chi_\varepsilon$ and by Lemma 16 of [ASW17] it can only hit at the starting prime end. Given that $\mathcal{L}_k \subseteq \gamma^{exc}$ is a simple loop, then either it has diameter smaller than ε or it has to be contained in one $\hat{C}_{\hat{k}(k)}$ by Lemma 23. By taking $\varepsilon \rightarrow 0$, we show that there always has to be on such $\hat{C}_{\hat{k}(k)}$.

We now show that the whole cluster C_k is contained in $\hat{C}_{\hat{k}(k)}$. To do this, recall that given the outer boundary, C_k is constructed by taking $\mathbb{A}_{-2\lambda}$ inside the connected component of $\mathbb{D} \setminus L_k$ containing 0. We can use the following iterative construction: to obtain A^k , we sample $\mathbb{A}_{-\lambda, \lambda}$ inside the connected component of $\mathbb{D} \setminus A^{k-1}$ that contains 0, unless the boundary value is already equal to 0. We can now use the same argument as under *Uniqueness* in Section 6 of [ASW17] to see that each A^k is contained in $\hat{\gamma}^{exc}$. Indeed, the fact that level lines used to construct $\mathbb{A}_{-\lambda, \lambda}$ do not self-intersect and the fact that if they enter any connected component of $\mathbb{D} \setminus \hat{\gamma}^{exc}$ they cannot touch the boundary γ^{exc} imply that they cannot enter any connected component of $\mathbb{D} \setminus \gamma^{exc}$ at all.

To show that the next excursion $C_{k'}$ surrounding 0 is also contained in some excursion $\hat{C}_{\hat{k}(k')}$ it suffices to note that, the law of Φ restricted to the connected component \mathcal{O} containing 0 of $\mathbb{D} \setminus C_k$ is that of a GFF in $\mathbb{D} \setminus C_k$. This implies that the restriction of $(\hat{C}_k, \hat{\mu}_k, \hat{\sigma}_k)_{k \geq 1}$ to \mathcal{O} is also that of an excursion decomposition, so we can repeat the above procedure. \square

We now prove Proposition 19. To do this we show that we can recover the sign of the cluster by integrating positive function whose support are close to it.

Proof of Proposition 19. Similarly to above, it suffices to prove the claim for the outermost cluster C_k surrounding 0. Let $\gamma : [0, 1] \mapsto \overline{B(0, 1)}$ be a straight line from $-i$ to 0 and \hat{C}_k (or C_k) be the outermost cluster surrounding 0. Define

$$\tau := \inf\{t \in [0, 1] : C_k \in \gamma^{exc}(t)\}. \quad (4.1)$$

By Proposition 18, we know that for all $t \in [0, 1]$, it holds that $\gamma^{exc}(t) \subseteq \hat{\gamma}^{exc}(t)$. Thus $\hat{\gamma}^{exc}(\tau)$ is also a local set, and as it contains the cluster $\hat{C}_{\hat{k}(k)}$. Observe that by construction the closure of $\bigcup_{t < \tau} \gamma^{exc}(t)$ intersects C_k only at $\hat{x} = \gamma(\tau)$ and that the same holds for \hat{C} .

Claim 24. *Let $A \subseteq \hat{C}_k \setminus \{\hat{x}\}$ be a closed set of positive diameter that is measurable w.r.t. $\hat{\gamma}^{exc}(\tau)$. Consider $(f_n : \mathbb{D} \mapsto [0, 1])_{n \geq 1}$, a family of smooth functions all taking value 1 in A and equal to 0 for all points of distance at least 2^{-n} of \hat{C}_k . Further, assume that conditionally on $\hat{\gamma}^{exc}(\tau)$, f_n are independent from the GFF $\Phi^{\hat{\gamma}^{exc}(\tau)}$.*

Then, almost surely if $\hat{\nu}_k(A) > 0$, $\liminf(\Phi, f_n) \geq \hat{\nu}_k(A)$ and if $\hat{\nu}_k(A) < 0$, $\liminf(\Phi, f_n) \leq -\hat{\nu}_k(A)$.

As this holds for clusters of any excursion decomposition, it holds in particular also for the one constructed in the previous section, i.e. if we omit all the hats in the claim.

Further, by Proposition 18, we know that $\gamma^{exc}(\tau)$ is a local set contained in $\hat{\gamma}^{exc}(\tau)$. In particular, it is conditionally independent of $\Phi^{\hat{\gamma}(\tau)}$. We can now apply the claim with a closed set $A \subseteq C_k \setminus \{\hat{x}\}$, and functions f_n chosen depending only on $\hat{\gamma}(\tau)$ and $\Phi_{\hat{\gamma}(\tau)}$ twice (once for C_k , once for $\hat{C}_{\hat{k}(k)}$) to obtain the proposition.

It remains to argue the claim.

Proof of Claim 24. We can use the local set property of $\hat{\gamma}^{exc}(\tau)$ to write

$$(\Phi, f_n) = (\Phi_{\hat{\gamma}^{exc}(\tau)}, f_n) + (\Phi_{\hat{\gamma}^{exc}(\tau)}, f_n).$$

By conditioning on $\hat{\gamma}^{exc}(\tau)$ we can see that the variance of $(\Phi_{\hat{\gamma}^{exc}(\tau)}, f_n)$ goes to 0. We conclude by noting that there exists an n such that the support of f_n does not intersect the closure of $\bigcup_{t < \hat{\tau}} \hat{\gamma}^{exc}(t)$. □

□

Before showing Proposition 20 let us state the following lemma.

Lemma 25. *For any connected component \mathcal{O} of $\mathbb{D} \setminus C_k$ that does not contain $\partial\mathbb{D}$ we have that $\mathcal{O} \cap \hat{C}_{\hat{k}(k)} = \emptyset$. In particular, if ℓ_k is the outer boundary of C_k and \mathcal{O}' is the interior, then $\overline{\hat{C}_{\hat{k}(k)}} \cap \mathcal{O}' = C_k$.*

Proof. We show it for the connected component containing 0. Define τ as in the proof of Proposition 19. Note that \mathcal{O} is also the connected component of $\mathbb{D} \setminus \gamma^{exc}(\tau)$ that contains 0. Because $C_k \subseteq \hat{C}_{\hat{k}(k)}$ we have that $\mathcal{O} \cap \hat{\gamma}^{exc}(\tau)$ is equal to $\mathcal{O} \cap \hat{C}_{\hat{k}(k)}$. The theory of local sets implies then that $\mathcal{O} \cap \hat{C}_{\hat{k}(k)}$ is a local set of $\Phi^{\hat{\gamma}^{exc}(\tau)}$ restricted to \mathcal{O} . Conditionally on γ^{exc} and the sign of the cluster C_k (WLOG we assume it +1), we see that restricted to \mathcal{O} , $(\Phi^{\hat{\gamma}^{exc}(\tau)})_{\hat{C}_{\hat{k}(k)}} \geq 0$. The first part of Proposition 4.5 of [ALS20a] implies that this set has to be empty.

The second statement follows directly. □

Proposition 20 needs again different arguments.

Proof of Proposition 20. By Proposition 18 we know that for each C_k , there is some \hat{k} with $C_k \subseteq \hat{C}_{\hat{k}}$. We start by showing that the signs $\sigma_k, \sigma_{k'}$ are independent even when we further condition on the event that they do not belong to the same cluster of $(\hat{C}_k)_{k \geq 1}$, more precisely,

$$E_{k,k'} = \{\hat{k}(k) \neq \hat{k}(k')\}.$$

This is done through the following claim

Claim 26. *We have that*

$$\mathbb{E} [(\sigma_k \nu_k, 1)(\sigma_{k'} \nu_{k'}, 1) 1_{E_{k,k'}}] = 0.$$

It follows from this claim that the function $k \rightarrow \hat{k}(k)$ is injective: indeed, for any points z_1, \dots, z_n and any nesting levels j_1, \dots, j_n , we can first write.

$$\mathbb{E} \left[\left(\sum_{k=1}^n (\sigma_k \nu_k, 1) \right)^2 \right] = \sum_{k=1}^n \mathbb{E} [(\nu_k, 1)^2]. \quad (4.2)$$

Using the claim we can alternatively write the LHS as:

$$\mathbb{E} \left[\left(\sum_{k=1}^n (\sigma_k \nu_k, 1) \right)^2 \right] = \sum_{k=1}^n \mathbb{E} [(\nu_k, 1)^2] + \sum_{k \neq k'} \mathbb{E} [(\sigma_k \nu_k, 1)(\sigma_{k'} \nu_{k'}, 1) 1_{E_{k,k'}^c}].$$

But on the event $E_{(z,j),(w,h)}^c$, we have that $\sigma_{w,h} = \sigma_{z,j}$ and thus all the terms in the second sum are non-negative. But then they have to actually be equal to zero by (4.2). As this holds for any collection of z_1, \dots, z_n and any nesting heights, and all clusters can be listed this way, we obtain that $k \rightarrow \hat{k}(k)$ is injective.

Proof of Claim 26. For this proof, it is easier to denote clusters using the point they surround and their level of nesting as follows: for $z \in \mathbb{D}$ and $j \in \mathbb{N}$, we denote $C_{z,j}$ denote respectively the j -th outermost cluster that surrounds z . So we now fix $z, z' \in \mathbb{D}$ and $j, j' \in \mathbb{N}$ and denote $k = k(z, j)$ and $k' = k(z', j')$ the k and k' such that $C_k = C_{z,j}$ and $C_{k'} = C_{z',j'}$ respectively.

We first prove the result when both the cluster are in the first layer, i.e., $j = j' = 1$. Consider the local set $\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$ along a line segment γ from the boundary to z and then to w , stopped at time $\tau_z \wedge \tau_w$, when either the cluster of \hat{C} around z or w appears. Assume, WLOG that $\hat{C}_{z,1}$. In that case, and on the event $E_{k,k'}$, we have that $C_{z',1} \cap \hat{C}_{z,1} = \emptyset$, we call \mathcal{O}' the connected component of $\mathbb{D} \setminus \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$ that contains z' . Furthermore we have two options

- (1) $C_{z,1} \subseteq \hat{C}_{z,1}$, in which case we define $\mathcal{O} = \emptyset$
- (2) or $C_{z,1} \cap \hat{C}_{z,1} = \emptyset$, in which case we define \mathcal{O} as the connected component of $\mathbb{D} \setminus \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$ that contains z . Note that $\mathcal{O} \cap \mathcal{O}' = \emptyset$.

In both cases we claim the following. Take the A closure of the union of the outer-most boundary of the outer-most clusters in $(C_k)_{k \geq 1}$ that are contained in either \mathcal{O} or \mathcal{O}' . We have that a.s. A is equal to the union of $\mathbb{A}_{-2\lambda, 2\lambda}$ of $\Phi^{\hat{\gamma}^{exc}}(\tau_z \wedge \tau_{z'})$ restricted to \mathcal{O} and to \mathcal{O}' .

To show this, we use Proposition 18 to see that none of the loops of C_k that make part of A intersect $\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$. Furthermore, the set $A \cup \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$ is a local set that, restricted to $\mathcal{O} \cup \mathcal{O}'$ is a.s. equal to $\mathbb{A}_{-2\lambda, 2\lambda} \cup \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$. This is because the outer-most boundary

of the outermost loop of C_k is equal to $\mathbb{A}_{-2\lambda, 2\lambda}$. Now, we use that again Proposition 18 and basic properties of local sets to see that for any $x \in D \setminus \mathbb{A}_{-2\lambda, 2\lambda} \cup \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$ is equal to either $\pm 2\lambda$ if x is not surrounded by a cluster in $\gamma^{exc}(\tau_z \wedge \tau_{z'})$ and 0 if not. Thus by (2) Lemma 6, we see that A is a local set of $\Phi^{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})}$ that inside \mathcal{O}' and \mathcal{O} (when it is not empty) is thin (because its Minkowski dimension is smaller than 2) and its harmonic function is equal to $\pm 2\lambda$. We conclude using the uniqueness of TVS.

We now notice that A can be obtain in a measurable function with respect to $\Phi^{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})}$, and that conditionally on $\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$, $\Phi^{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})}$ is independent of $\Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})}$. So now, if we are on the case (1)

$$\begin{aligned} & \mathbb{E} \left[\sigma_k \sigma_{k'}(\nu_k, 1)(\nu_{k'}, 1) \mathbf{1}_{E_{k,k'}} \mathbf{1}_{(1)} \right] \\ &= \mathbb{E} \left[\sigma_k(\nu_k, 1) \mathbb{E} \left[\sigma_{k'}(\nu_{k'}, 1) \mid \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'}), \Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})} \right] \mathbf{1}_{E_{k,k'}} \mathbf{1}_{(1)} \right] = 0, \end{aligned}$$

where we use that (1) and $E_{k,k'}$ are measurable with respect to $\gamma^{exc}(\tau_z \wedge \tau_{z'}), \Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})}$, thanks to the fact that $\gamma^{exc}(\tau_z \wedge \tau_{z'}) \subseteq \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})$, and that the law of $(\sigma_{z'}, \nu_{z'})$ conditionally on $(\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'}), \Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})})$ is that of the measure of the excursion decomposition of a GFF in the domain \mathcal{O}' .

For the case (2), a similar computation is needed

$$\begin{aligned} & \mathbb{E} \left[\sigma_k \sigma_{k'}(\nu_k, 1)(\nu_{k'}, 1) \mathbf{1}_{E_{k,k'}} \mathbf{1}_{(2)} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sigma_{k'}(\nu_{k'}, 1) \mid \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'}), \Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})} \right] \mathbb{E} \left[\sigma_k(\nu_k, 1) \mid \hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'}), \Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})} \right] \mathbf{1}_{E_{k,k'}} \mathbf{1}_{(2)} \right] \\ &= 0, \end{aligned}$$

where we used that conditionally on $(\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'}), \Phi_{\hat{\gamma}^{exc}(\tau_z \wedge \tau_{z'})})$ the law of $(\sigma_{z'}, \nu_{z'}, \sigma_z, \nu_z)$ is that of two independent excursion decompositions of one GFF in \mathcal{O}' and another in \mathcal{O} , respectively.

For next levels we discover $\hat{\gamma}^{exc}(\tau_2)$, which is the stopping time where both outermost clusters appear, and then iterate inside the connected components of complement of $\hat{\gamma}^{exc}(\tau_2)$ containing either z or w similarly. \square

\square

Proof of Proposition 21. We start by showing that

$$C_k = \hat{C}_{\hat{k}(k)} \tag{4.3}$$

almost surely. By Proposition 18, we have that $C_k \subseteq \hat{C}_{\hat{k}(k)}$. Further by Lemma 25, we know that $C_{\hat{k}(k)} \setminus C_k$ only intersects the connected component \mathcal{O} of $D \setminus C_k$ that contains $\partial\mathbb{D}$ on its boundary.

Like in the proof of Claim 26, all clusters can be listed from outermost to innermost around a dyadic points z_k . Thus it suffices to prove (4.3) for outermost clusters and this in turn follows from showing the following claim: for any curve γ along the dyadics starting from $\partial\mathbb{D}$ to some z_k , we have that the complements $\mathcal{O}, \hat{\mathcal{O}}$ of $A = \gamma^{exc}(1)$ and $\hat{A} = \hat{\gamma}^{exc}(1)$, which share boundary with \mathbb{D} , agree almost surely. Both A, \hat{A} are local sets, and by Lemma 6 (2), $\hat{A} \setminus A$ is also a local set of $\phi^{\gamma^{exc}(1)}$ restricted to \mathcal{O} . On the other hand as $C_k \subseteq C_{\hat{k}(k)}$, we know that $\hat{A} \setminus A \subseteq \mathbb{A}_{-2\lambda, 2\lambda}(\mathbb{D})$ and thus it has Minkowski dimension strictly less than 2

[ASW17]. Hence, it is a thin local set of $\phi^{\mathcal{O}}$ with zero boundary values, and thus by Lemma 9 of [ASW17] it is empty.

It remains to show that the function $k \rightarrow \hat{k}$ is surjective. This follows from a very similar argument, after noticing that any cluster \hat{C}_{k^*} that is not equal to some C_k is contained in the closed union of outer boundaries of C_k in some finite iteration step of the construction of the excursion decomposition $(C_k, \nu_k, s_k)_{k \geq 1}$. But these outer boundaries are given by independent copies of $\mathbb{A}_{-2\lambda, 2\lambda}$. Thus we can repeat the argument above to obtain that such clusters of positive diameter do not exist. To see that there are no clusters whose support is just a point, we recall that almost surely the 2D GFF does not put any mass on single points. □

The uniqueness part now follows.

Proof of uniqueness in Theorem 1. Take (σ_k, ν_k, C_k) and $(\hat{\sigma}_k, \hat{\nu}_k, \hat{C}_k)$ two decompositions, where the first one is the one constructed in the previous section. From Proposition 18 we know that for every k there is $\hat{k}(k)$ such that $C_k = \hat{C}_{\hat{k}(k)}$ and by Proposition 20 this assignment is injective. In particular, for any curve γ and for any time t a.s., we have that $\gamma^{exc}(t) = \hat{\gamma}^{exc}(t)$. But now notice that by the Markov decomposition, we can conclude that almost surely for any curve γ along the dyadics and any rational time in $H^{-1-\varepsilon}(\mathbb{D})$

$$\phi_{\gamma^{exc}(t)} = \hat{\phi}_{\gamma^{exc}(t)}.$$

But the local set process $\gamma(t)^{exc}$ is right-continuous, and for any decreasing sets D_n with $\bigcap D_n = D$, Φ^{D_n} is also continuous. We conclude that in fact for all times $t \in [0, 1]$, it holds that

$$\phi_{\gamma^{exc}(t)} = \hat{\phi}_{\gamma^{exc}(t)}$$

and in particular it holds at the appearance of any cluster of diameter at least ε . This concludes that in fact for all $k \geq 1$, we have $\nu_k = \hat{\nu}_{\hat{k}(k)}$ and the theorem follows. □

5. CONVERGENCE FROM THE METRIC GRAPH

In this section, we prove the convergence of the excursion decomposition of the metric graph GFF to that of the continuum GFF. We will work in the same set-up as in Section 4.1 of [ALS20b], except that the domain D will always be simply connected.

For all $n \geq 1$, let $\tilde{\phi}_n$ be a metric GFF in a bounded graph $D_n \subseteq (2^{-n}\mathbb{Z})^2$. We define $(\tilde{C}_k^{(n)}, \tilde{\sigma}_k^{(n)}, \tilde{\nu}_k^{(n)})_{k \geq 1}$ as the sequence of sign clusters of $\tilde{\phi}_n$, the respective signs and sign excursions, ordered by decreasing size of cluster diameter. Here, by a sign excursion we mean the absolute value of the restriction of the GFF to the cluster $\tilde{C}_k^{(n)}$, i.e.

$$\tilde{\nu}_k^{(n)}(dx) = \tilde{\sigma}_k^{(n)} \tilde{\phi}_n(x) \mathbf{1}_{x \in \tilde{C}_k^{(n)}} dx.$$

We now take a sequence of (metric) graphs $\tilde{D}_n \subseteq (2^{-n}\mathbb{Z})^2$ converging to a bounded and simply connected domain $D \subseteq \mathbb{C}$ in the sense that their complement converge in the Hausdorff topology (as in Section 4.1 of [ALS20b]).

The main result of this section is the following.

Theorem 27 (Convergence of the excursion decomposition). *Let $\tilde{\phi}_n$ be a sequence of zero boundary metric graph GFFs on \tilde{D}_n that are coupled with a GFF Φ such that a.s. $\tilde{\phi}_n \rightarrow \Phi$ in $H^{-\varepsilon}$. We have that for every $k > 0$, $\tilde{C}_k^{(n)} \rightarrow C_k$, $\tilde{\nu}_k^{(n)} \rightarrow \nu_k$ and $\tilde{\sigma}_k^{(n)} \rightarrow \sigma$ as $n \rightarrow \infty$, where the convergence is in probability and in the Hausdorff topology for the first, and in the weak topology of measures for the first and second components respectively.*

In large lines, one could say that the theorem follows by patching together different convergence results for each element, all of which are already present in the literature. This patching, however, does require a bit of care. Notice that we will not use the uniqueness claim of the theorem to identify the limit, as one has to take care of the summing of excursion measures; rather we will identify excursion clusters, signs and measures one by one.

We start from a lemma that ensures the tightness of the sequences of measures $(\tilde{\nu}_k^{(n)})_{n \geq 0}$ and allows us to see that no spurious extra mass is produced in the limit by infinitesimal excursion clusters.

Lemma 28. *Let $(\tilde{C}_k^{(n)}, \tilde{\sigma}_k^{(n)}, \tilde{\nu}_k^{(n)})$ be an excursion decomposition of the metric graph GFF $\tilde{\phi}_n$ and let J be any (deterministic) index set. Then for any $q \in \mathbb{N}$*

$$\mathbb{E} \left[(\tilde{\phi}_n, f)^{2q} \right] \geq \sum_{k \in J} \mathbb{E} \left[(\tilde{\nu}_k^{(n)}, f)^{2q} \right] + \mathbb{E} \left[(\mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n, f)^{2q} \right]. \quad (5.1)$$

and

$$\mathbb{E} \left[(\tilde{\phi}_n, f)^2 \right] = \sum_{k \in J} \mathbb{E} \left[(\tilde{\nu}_k^{(n)}, f)^2 \right] + \mathbb{E} \left[(\mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n, f)^2 \right] \quad (5.2)$$

Proof. For the inequality (5.1), it suffices to prove it for any finite index set and any $q \geq 1$, then the case of infinite index sets follows by dominated convergence. We decompose

$$\tilde{\phi}_n = \sum_{k \in J} \tilde{\sigma}_k^{(n)} \tilde{\nu}_k^{(n)} + \mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n.$$

Then, we write $\mathbb{E}[(\tilde{\phi}_n, f)^{2q}]$ as the sum of three types of terms, the first being

$$\sum_{k \in J} \mathbb{E}[(\tilde{\nu}_k^{(n)}, f)^{2q}] + \mathbb{E}[(\mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n, f)^{2q}],$$

the second type of terms are a binomial coefficients times

$$\mathbb{E}[(\tilde{\sigma}_k^{(n)} \tilde{\nu}_k^{(n)}, f)^p (\mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n, f)^{2q-p}]$$

and the last type of terms are constant times

$$\mathbb{E}[(\tilde{\sigma}_k^{(n)} \tilde{\nu}_k^{(n)}, f)^p (\tilde{\sigma}_j^{(n)} \tilde{\nu}_j^{(n)}, f)^{2q-p}],$$

with $k \neq j$.

Now, when p is even, we can lower bound the second and third types of terms by 0. However, we claim that when p is odd, they are equally zero by sign symmetry. Indeed, conditionally on $(\tilde{C}_k^{(n)}, \tilde{\sigma}_k^{(n)}, \tilde{\nu}_k^{(n)})_{k \in J}$, the field $\mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n$ has the same distribution as its additive inverse. Thus for p odd

$$\sum_{k \in J} \mathbb{E}[(\tilde{\sigma}_k^{(n)} \tilde{\nu}_k^{(n)}, f)^p (\mathbf{1}_{\tilde{D}_n \setminus \cup_{k \in J} \tilde{C}_k^{(n)}} \tilde{\phi}_n, f)^{2q-p}] = 0.$$

But also all the signs $\tilde{\sigma}_k^{(n)}$ are i.i.d. Rademacher random variables, thus also for all $j \neq k \in J$

$$\mathbb{E}[(\tilde{\sigma}_k^{(n)} \tilde{\nu}_k^{(n)}, f)^p (\tilde{\sigma}_j^{(n)} \tilde{\nu}_j^{(n)}, f)^{2q-p}] = 0$$

and we conclude the first part. The second part for finite index sets follows from the computation above, as in the case $q = 1$, there are no cross-terms with even exponents; for the infinite sums, we can use dominated convergence, guaranteed by say the case $q = 2$ in (5.1). \square

Proof of Theorem 27. We start by noting that thanks to the uniqueness of the excursion decomposition and Lemma 4.10 of [ALS20b], we only need to prove convergence in law of $(\tilde{\phi}_n, (\tilde{C}_k^{(n)}, \tilde{\sigma}_k^{(n)}, \tilde{\nu}_k^{(n)})_k)$ as $n \rightarrow \infty$.

Now, $\tilde{\phi}_n$ are tight by assumption; for any k , $\tilde{C}_k^{(n)}$ are tight as random closed sets in a compact domain, $\tilde{\sigma}_k^{(n)}$ are tight as ± 1 valued random variables and finally $\tilde{\nu}_k^{(n)}$ are tight by the first equality in Lemma 28. Thus, using Tychonoff theorem, we see that $(\tilde{\phi}_n, (\tilde{C}_k^{(n)}, \tilde{\sigma}_k^{(n)}, \tilde{\nu}_k^{(n)})_k)$ is tight, and thus we can find a subsequence of it (we denote it the same way) and use Skorokhod's representation theorem to obtain the almost sure convergence

$$(\tilde{\phi}_n, (\tilde{C}_k^{(n)}, \tilde{\sigma}_k^{(n)}, \tilde{\nu}_k^{(n)})_k) \rightarrow (\hat{\Phi}, (\hat{C}_k, \hat{\sigma}_k, \hat{\nu}_k)_k).$$

We just need to identify $(\hat{\Phi}, (\hat{C}_k, \hat{\sigma}_k, \hat{\nu}_k)_k)$ as the elements of the excursion decomposition. First, it is clear that $(\hat{\sigma}_k)_k$ are i.i.d. Rademacher random variables and $\hat{\Phi}$ is a GFF in D .

First, note that if we only study the outer most clusters $\hat{C}_{\mathbf{k}}^{(n)} := \hat{C}_{k(\mathbf{k})}^{(n)}$ (i.e. those that are not surrounded by any other cluster), then the outer boundaries of those outermost clusters converge to the loops of CLE_4 in the sense that the outer boundaries of the m largest outermost discrete clusters converge to outer boundaries m largest continuum ones, and moreover the closed union of all outermost cluster boundaries converges to CLE_4 - these statements follow from the work in [Lup18]. More precisely, the main statement of that paper does not directly apply these claims - it does not exclude long thin filament-like clusters with limits in the interior of CLE casket; however with further work it can be deduced with the same methods; see e.g. Lemma 4.13 in [ALS20b] for a context, where similar care is needed, or proof of Lemma 6 of [QW19].

Now, notice that once we manage to identify the outermost clusters, their signs and measures, then we can recursively continue. Indeed, as the closure of the union of outermost clusters $\hat{C}_{\mathbf{k}}^{(n)}$ is a local set for all n , we conclude that in the limit, conditionally on the closure of the union of all outermost clusters $\hat{C}_{\mathbf{k}}$, the law of $\hat{\Phi}$ restricted to $D \setminus (\cup_{\mathbf{k}} \hat{C}_{\mathbf{k}})$ is that of a zero boundary GFF in $D \setminus (\cup_{\mathbf{k}} \hat{C}_{\mathbf{k}})$. Thus we see that once we can deal with outermost excursions, the convergence will also hold for excursions that are surrounded by finitely many excursions. As for any $\varepsilon > 0$ the number of excursions of diameter bigger than ε is almost surely finite, we have reduced the proposition to proving convergence for outermost clusters. This is the content of the following claim.

Claim 29. *Fix any loop ℓ of $\mathbb{A}_{-2\lambda, 2\lambda}$ and consider the sequence of clusters $\hat{C}_{\mathbf{k}}^{(n)}$ whose outer boundaries converge to ℓ . Then $\hat{C}_{\mathbf{k}}^{(n)}$ converges to the union of ℓ with the FPS $\mathbb{A}_{\pm 2\lambda}$ of the GFF $\Phi^{\mathbb{A}_{-2\lambda, 2\lambda}}$ restricted to \mathcal{O} , the interior of ℓ . Furthermore, $\hat{\nu}_{\mathbf{k}}^{(n)}$ converges to the measure $\nu_{\mathbb{A}_{\pm 2\lambda}}$ associated to this FPS.*

Finally,

Proof of Claim 29. First, we note that the union of all outermost clusters $\mathring{C}_{\mathbf{k}}^{(n)}$ is a local set of $\tilde{\phi}_n$, thus its limit $\mathring{C}_{\mathbf{k}}$ is a local set of $\hat{\Phi}$. Further, when restricted to the interior \mathcal{O} of the outermost boundary ℓ of the cluster $\mathring{C}_{\mathbf{k}}$, this limit is a local set that satisfies the properties of an FPS of level $\pm 2\lambda$ (for the GFF $\Phi^{\mathbb{A}_{\pm 2\lambda}}$ restricted to \mathcal{O}). Thus it is equal to this set by the uniqueness of FPS, Theorem 9. In particular, this means that $\mathring{C}_{\mathbf{k}}$, the limit of $\mathring{C}_{\mathbf{k}}^{(n)}$, is equal to some outermost cluster $C_{k(\mathbf{k})}$.

To identify the limiting excursion measures, we will follow a strategy similar to what was used in Section 4 to deduce the equality of excursion clusters and excursion measures by a no extra mass argument. Additional convergence issues are taken care by Lemma 28. Let us flesh it out here.

First, as no other subsequential limit of an outermost cluster $\mathring{C}_{\mathbf{k}}$ can intersect \mathcal{O} , we conclude from the Markov decomposition w.r.t. the FPS $\mathbb{A}_{\pm 2\lambda}$ in \mathcal{O} that

$$\dot{\nu}_{\mathbf{k}} := \lim_{n \rightarrow +\infty} \nu_{\mathbf{k}}^{(n)} \geq \nu_{k(\mathbf{k})} \quad (5.3)$$

in terms of positive measures.

Further, one needs to show that there is no extra mass on the subsequential limiting clusters, that is possibly compensated by some infinitesimal excursions in the limit.

To see this, recall that the closure of the union of outermost clusters $\mathring{C}_{\mathbf{k}}$ forms a local set. Also, from the argument above we see that this local set is equal to the local set obtained by taking CLE_4 and first passage sets of height $\pm 2\lambda$ inside each of the cluster, i.e. it is equal to the closed union of outermost clusters $\mathring{C}_{\mathbf{k}}$. Let us denote this set by A .

By Lemma 28, we can write the sum over outermost excursion clusters $\mathring{C}_{\mathbf{k}}$,

$$\mathbb{E} [(\Phi, 1)^2] = \sum_{\mathbf{k}} \mathbb{E} [(\nu_{k(\mathbf{k})}, 1)^2] + \mathbb{E} [(\Phi^A, 1)^2].$$

On the other hand, using the second point of Lemma 28, we see that for any \mathbf{k} , it holds that

$$\mathbb{E} [(\dot{\nu}_{\mathbf{k}}^{(n)}, 1)^2] \rightarrow \mathbb{E} [(\dot{\nu}_{\mathbf{k}}, 1)^2]$$

and moreover, also $\mathbb{E} [(\tilde{\phi}_n^{A^{(n)}}, 1)^2] \rightarrow \mathbb{E} [(\Phi^A, 1)^2]$, where $A^{(n)}$ denotes the local set given by the closed union of $\mathring{C}_{\mathbf{k}}^{(n)}$. Thus we also have

$$\mathbb{E} [(\Phi, 1)^2] = \sum_{\mathbf{k}} \mathbb{E} [(\dot{\nu}_{\mathbf{k}}, 1)^2] + \mathbb{E} [(\Phi^A, 1)^2].$$

But now recall that $\dot{\nu}_{\mathbf{k}} \geq \nu_{k(\mathbf{k})}$, from where we see that in fact we have to have a one to one correspondence between clusters with positive measure, and the equality has to hold in (5.3). □

□

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