A dense subset of $M\_n(R)$ containing diagonalizable matrices

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A DENSE SUBSET OF $M_n(\mathbb{R})$ CONTAINING DIAGONALIZABLE MATRICES

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Abstract. In this note, we consider matrices similar to X-form matrices, which are the matrices for which only the diagonal and the anti-diagonal elements can be different from zero. First, we give a characterization of these matrices using the minimal polynomial. Then, we prove that the set of matrices similar to X-form matrices over $\mathbb{R}$ and $\mathbb{C}$ are dense and we give a characterization of the interior of this set.

Keywords : X-form matrix, minimal polynomial, Jordan normal form, companion matrix.

Declarations of interest: none

"La mémoire est aussi menteuse que l'imagination, et bien plus dangereuse avec ses petits airs studieux."
Françoise Sagan, *Derrière l'épaule*

1. Introduction

In this note, all fields considered are commutative. Let $\mathbb{K}$ be an arbitrary commutative field. The set of all square matrices of size n over $\mathbb{K}$ is denoted $M_n(\mathbb{K})$, 0_n is the zero matrix of $M_n(\mathbb{K})$. If $A \in M_n(\mathbb{K})$ we denote $\pi_A(X)$ the minimal polynomial of A (with the convention π_A monic polynomial) and $\chi_A(X) = \det(XI_n - A)$ the characteristic polynomial of A (with this definition $\chi_A(X)$ is a monic polynomial). We use the convention $\prod_{i=1}^{0} a_i = 1$. An elementary Jordan matrix is a matrix composed of zeroes everywhere except for the diagonal, which is filled with a fixed element $\lambda \in \mathbb{K}$, and for the superdiagonal, which is composed of ones. The Frobenius companion matrix of the monic polynomial $P(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ is the square matrix defined as $C(P) = \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \ddots & \vdots & -a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{d-1} \end{pmatrix}$.

A classical results states that the set of diagonalizable matrices is dense over $\mathbb{C}$ but not over $\mathbb{R}$. Here, we want to find a subset of $M_n(\mathbb{K})$ containing all diagonalizables matrices which is dense in the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and we want to find the interior of his set. For this, we will study the objects introduced in the following definitions :

Definition 1.1. *i) A X-form matrix is a square matrix for which only the diagonal and the anti-diagonal elements can be different from zero.*
ii) An endomorphism u of a finite dimensional vector space E is X-formable if there is a basis of E with respect to which the matrix of u is a X-form matrix.
iii) A X-formable matrix is a matrix similar to a X-form matrix.

First, we give some of the easy properties verifying by that kind of matrices :

- Matrices of size 1 or 2 are X-form matrices;

- X-form matrices of odd size has an eigenvalue belonging to $\mathbb{K}$;
- If A is a X-form matrix then A^t is a X-form matrix;
- If A is diagonalizable then A is X-formable;
- The set of X-form matrices of size n is a vector subspace of $M_n(\mathbb{K})$;
- If A and B are X-form matrices then AB is a X-form matrix;
- If A is an invertible X-form matrix then A^{-1} is a X-form matrix (this follows from the last two points and the equality $A^{-1} = \frac{-1}{a_0}(A^{r-1} + \sum_{i=1}^{r-1} a_i A^{i-1})$ with $\pi_A(X) = X^r + \sum_{i=0}^{r-1} a_i X^i$, $a_0 \neq 0$ since A is invertible).

We also have some other properties in the case of central-symmetric X-form matrices (see [6, 7]) and also in the case of block central-symmetric X-form matrices (see [8]).

A classical result states that a square matrix is diagonalizable if and only if its minimal polynomial is a product of distinct linear factors over $\mathbb{K}$. It is natural to find a similar characterization in the case of X-formable matrices. This seems to be not very difficult but we have been unable to locate such a result in the litterature. Hence, our first objective is to prove the following result :

Theorem 1.2. *Let $\mathbb{K}$ be a commutative field and $A \in M_n(\mathbb{K})$. A is X-formable if and only if*

$$\pi_A(X) = \prod_{i=1}^{r} P_i(X) \prod_{i=1}^{q} (X - \lambda_i)^{n_i},$$

with $r, q \geq 0$, P_i irreducible monic polynomials of degree 2, $\lambda_i \in \mathbb{K}$, $1 \leq n_i \leq 2$, $P_i \neq P_j$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.

To simplify the proof, we will use the following notation : a polynomial verifies the property $(\mathcal{P})$ if it has the same factorization as in the previous theorem.

Then, we will consider some topological properties of X-formable matrices over $\mathbb{R}$ and $\mathbb{C}$ related to our initial goal. The main results of this text gathered in the following theorem give an answer to this problem :

Theorem 1.3. *$n \in \mathbb{N}^*$. Let F be the set of X-formable matrices of size n over $\mathbb{C}$ and G the set of X-formable matrices of size n over $\mathbb{R}$.*
i) F is dense in $M_n(\mathbb{C})$.
ii) The interior of F is $\{A \in M_n(\mathbb{C}),\ \chi_A$ verifies $(\mathcal{P})\} = \{A \in M_n(\mathbb{C}),\ \chi_A$ has only simple or double roots$\}$.
iii) G is dense in $M_n(\mathbb{R})$.
iv) The interior of G is $\{A \in M_n(\mathbb{R}),\ \chi_A$ verifies $(\mathcal{P})\}$.

2. X-formable matrices

In this subpart, $\mathbb{K}$ is an arbitrary commutative field. We begin by an easy lemma.

Lemma 2.1. *A matrix A is X-formable if and only if it is similar to a block-diagonal matrix in which each block has a size less than 2.*

Proof. First, we consider the case $n = 2m$. Let A be a X-formable matrix of size n. Let u be an endomorphism of $\mathbb{K}^n$ whose matrix in the canonical basis is A. It exists $\mathcal{B} = (e_1, \ldots, e_{2m})$ a basis of $\mathbb{K}^n$ such that the matrix of u in $\mathcal{B}$ is a X-form matrix. The matrix of u in the basis $(e_1, e_{2m}, e_2, e_{2m-1}, \ldots, e_m, e_{m+1})$ is a block-diagonal matrix in which each block has a size less than 2. Let A be a matrix similar to a block-diagonal matrix in which each block has a size less than 2 and u be an endomorphism of $\mathbb{K}^n$ whose matrix in the canonical basis is A. It exists $\mathcal{B}' = (f_1, \ldots, f_{2m})$ a basis of $\mathbb{K}^n$ such that the matrix of u

in $\mathcal{B}'$ is a block-diagonal matrix in which each block has a size less than 2. The matrix of u in the basis $(f_1, f_3, \ldots, f_4, f_2)$ is a X-form matrix. The proof of the case $n = 2m + 1$ is similar.

□

Note that this result, combined with the normal matrices reduction theorem (see [3] Theorem 4, p 271), allows us to see that normal matrices on $\mathbb{R}$ are X-formable.

2.1. **Proof of theorem 1.2.**

Proof. Let A be a X-formable matrix of size n. By the previous lemma, it exists a basis of $\mathbb{K}^n$ such that the matrix of u in this basis is a block-diagonal matrix C in which each block has a size less than 2. Thus, π_A is the least common multiple of the minimal polynomials of the diagonal blocks of C. The minimal polynomials of matrix of size 1 or 2 are one of the following type :

- irreducible monic polynomial of degree 2;
- $(X - \lambda)^2$;
- $(X - \lambda)$;
- $(X - \lambda_1)(X - \lambda_2)$, $\lambda_1 \neq \lambda_2$.

Hence, they verify $(\mathcal{P})$. Thus, π_A verifies $(\mathcal{P})$.

Now, we want to prove the remaining implication of the theorem.

Firt, we consider a matrix A of size n such that π_A verifies $(\mathcal{P})$ and such that π_A has only one irreducible factor (over $\mathbb{K}$). We have three cases :

- $\pi_A = (X - \lambda)$ then A is diagonalizable, and so X-formable.
- $\pi_A = (X-\lambda)^2$. By the decomposition theorem of Jordan, A is similar to a block diagonal matrix $H = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_l \end{pmatrix}$ in which J_i is an elementary Jordan matrix. Besides, each J_i is a square matrix of size $n_i \in \{1, 2\}$. Indeed, $\pi_A = \pi_H = \mathrm{lcm}(\pi_{J_i}, 1 \leq i \leq l)$ and $\pi_{J_i} = (X - \lambda)^{n_i}$. Hence, A is X-formable, by lemma 2.1.
- π_A is an irreducible monic polynomial of degree 2. By the decomposition theorem of Frobenius, A is similar to a block diagonal matrix
$$H = \begin{pmatrix} C(R_1) & & \\ & \ddots & \\ & & C(R_t) \end{pmatrix},$$
with R_i monic polynomials verifying R_i divides R_{i+1}. Since $\pi_A = \pi_H = \mathrm{lcm}(R_i, 1 \leq i \leq t)$ and π_A irreducible, $R_i = \pi_A$. Thus, H is block-diagonal matrix in which each block has size 2 and A is X-formable, by lemma 2.1.

Now, we proceed by induction on the size n of the matrix. If $n = 1$, then the result is true. Suppose it exists $n \geq 1$ such that all matrices of size less than n whose minimal polynomial verifies $(\mathcal{P})$ are X-formable. Let A be a matrix of size $n + 1$ whose minimal polynomial verifies $(\mathcal{P})$ and u the endomorphism canoniquely associated to A. If π_A has only one irreducible factor, then the result is true by the previous discussion. Suppose π_A has several irreducible factors. It exists a monic polynomial P such that $\pi_A = P\frac{\pi_A}{P}$, P has degree greater or equal to 1, P and $\frac{\pi_A}{P}$ are relatively prime polynomials. P and $\frac{\pi_A}{P}$ verify $(\mathcal{P})$. Consider $F = \mathrm{Ker}(P(u))$ and $G = \mathrm{Ker}(\frac{\pi_A}{P}(u))$. Hence, by the kernel lemma[1], $\mathbb{K}^{n+1} = F \oplus G$

[1]This translation of the French name "lemme des noyaux" seems to be the most used name for this result.

and F, G are invariant subspaces of u.

By induction assumption, $u_{|G}$ (restriction of u to G) and $u_{|F}$ are X-formable. Hence, u is X-formable and theorem 1.2 is proved.

□

2.2. **Some additional elements about theorem 1.2.**

In the case of an algebraically closed field we have the following result :

Corollary 2.2. *Let $\mathbb{K}$ be an algebraically closed field and $A \in M_n(\mathbb{K})$. The following assertions are equivalent :*
i) A is X-formable.
ii) $\pi_A(X) = \prod_{i=1}^{q}(X-\lambda_i)^{n_i}$ with $q \geq 1$, $\lambda_i \in \mathbb{K}$, $1 \leq n_i \leq 2$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.
iii) All the Jordan blocks appearing in the Jordan normal form of A have their size equal to 1 or 2.

If A is a X-formable matrix then there is not a unique X-form matrix similar to A. For example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ is a X-form matrix but A is also similar to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ which is also a X-form matrix.

We now give the two following examples :

- We consider $\mathbb{K} = \mathbb{R}$ and $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. $\pi_B(X) = (X-1)^3$. Hence, by the theorem 1.2, B is not X-formable.

- We consider $\mathbb{K} = \mathbb{R}$ and $C = \begin{pmatrix} -22 & 47 & -19 & 18 \\ 1 & 3 & -3 & -5 \\ 14 & -23 & 7 & -16 \\ -15 & 27 & -9 & 17 \end{pmatrix}$. $\pi_C(X) = (X-2)(X-3)(X^2+2)$. Hence, by the theorem 1.2, C is X-formable. For instance,
$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -2 & 1 & -1 \\ 4 & -7 & 2 & -4 \\ -3 & 6 & -2 & 3 \end{pmatrix}.$$

We conclude this part by giving some elements about the product of X-formable matrices. We have the following result :

Theorem 2.3 (Botha, [1] Theorem 2.1)**.** *Let $\mathbb{K}$ be any field such that the characteristic of $\mathbb{K}$ is different from 2 and such that $\mathbb{K} \neq \mathbb{F}_3$ (the field with 3 elements). Then every matrix over $\mathbb{K}$ is a product of two diagonalizable matrices.*

Since diagonalizable matrices are X-formable, this result covers a lot of cases. Here, we consider the case of $\mathbb{F}_3$. By the decomposition theorem of Frobenius, it is sufficient to consider companion matrices. We have the following equality :

$$\begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \ddots & \vdots & a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & a_{d-1} \end{pmatrix} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \begin{pmatrix} & & 1 & a_{d-1} \\ & \iddots & & \vdots \\ 1 & & & a_1 \\ 0 & \cdots & 0 & a_0 \end{pmatrix}.$$

The first matrix on the right is diagonalizable since its minimal polynomial is $(X-1)(X+1)$ and $1 \neq -1$ in $\mathbb{F}_3$. If $a_0 = \pm 1$ then the second matrix on the right is X-formable since its minimal polynomial divides $(X-1)^2(X+1)^2$ (theorem 1.2). If $a_0 = 0$ then the second matrix on the right is X-formable since its minimal polynomial divides $X^2(X-1)^2(X+1)^2$ (theorem 1.2). Hence, every matrix over $\mathbb{F}_3$ is a product of two X-formable matrices.

3. Some topological aspects of the set of X-formable matrices

The aim of this section is to prove theorem 1.3. Here, we suppose $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The set of polynomials of degree less than n over $\mathbb{K}$ is denoted $\mathbb{K}_n[X]$. We use the following norm, if $A = (a_{i,j})_{1 \leq i,j \leq n} \in M_n(\mathbb{K})$ we denote $\|A\|_\infty = \max(|a_{i,j}|, 1 \leq i,j \leq n)$ (we recall that all norms are equivalent on $M_n(\mathbb{K})$). We start by some preliminary results.

3.1. **Prelimirary lemmas.**

In this subsection, we give some results concerning polynomials over $\mathbb{K}$. If $P(X) = \sum_{k=0}^{n} a_k X^k$, we denote $\|P\|_\infty = \max(|a_k|,\ 0 \leq k \leq n)$ and $D(z,r) = \{w \in \mathbb{C},\ |z-w| < r\}$. We begin by the following well-known result :

Theorem 3.1 (Continuity of the roots of a polynomial, [4])**.** *Let P be a polynomial, z_i its distinct roots with $i = 1, \ldots, p$, m_i the multiplicity of the root z_i ($m_1 + ... + m_p = deg(P)$). Then for any $\epsilon > 0$ such that $D(z_i, \epsilon) \cap D(z_j, \epsilon) = \emptyset$ for any $i \neq j$ there exists $\eta > 0$ such that any polynomial Q whose coefficients differ from those of P only by less than η has exactly m_i roots (distinct or not) in $D(z_i, \epsilon)$.*

Now we can prove the following result :

Proposition 3.2. *$n \in \mathbb{N}^*$. The set $\hat{F}$ of polynomials over $\mathbb{C}$ of degree n whose roots are simple or double is an open subset of $\mathbb{C}_n[X]$.*

Proof. Let $P \in \mathbb{C}_n[X]$ with $\deg(P) = n$. We assume that P has only simple and double roots. We denote z_i its distinct roots with $i = 1, \ldots, p$ and m_i the multiplicity of the root z_i. We have $m_i \leq 2$ and $m_1 + ... + m_p = n$.

Since $\mathbb{C}$ is a separated nomed space, it exists $\epsilon > 0$ such that $D(z_i, \epsilon) \cap D(z_j, \epsilon) = \emptyset$ for any $i \neq j$. Hence, by the previous theorem it exists $\eta > 0$ such that any polynomial Q verifying $\|P - Q\|_\infty < \eta$ has exactly m_i roots (distinct or not) in $D(z_i, \epsilon)$. So, in each $D(z_i, \epsilon)$, Q has exactly one root of multiplicity less than two or two roots of multiplicity one. Besides, Q has n roots (with multiplicity) in $\bigcup_{i=1}^{p} D(z_i, \epsilon)$ and $\deg(Q) \geq n$.

Hence, a polynomial T belonging to $\mathbb{C}_n[X]$ and verifying $\|P - T\|_\infty < \eta$ has degree n and all its roots are simple or double. Thus, $\hat{F}$ is an open subset of $\mathbb{C}_n[X]$.

□

This result is no longer true if we replace $\mathbb{C}$ by $\mathbb{R}$. For instance, we can consider the polynomial sequence $P_n = X^2 + \frac{1}{n}X + \frac{1}{n}$. For all $n \in \mathbb{N}^*$, P_n is irreducible, since its discriminant is $\frac{1}{n}(\frac{1}{n} - 4) < 0$. However, this sequence converges to X^2. Hence, the complementary of the set of polynomials over $\mathbb{R}$ of degree 2 whose roots are simple or double is not closed. So, this set is not an open set. However, we can still have a similar result :

Proposition 3.3. *$n \in \mathbb{N}^*$. The set $\hat{G}$ of polynomials over $\mathbb{R}$ of degree n which verify $(\mathcal{P})$ is an open subset of $\mathbb{R}_n[X]$.*

Proof. Let $P \in \mathbb{R}_n[X]$ with $\deg(P) = n$. We assume that P verifies $(\mathcal{P})$. We denote z_i its distinct roots in $\mathbb{C}$ with $i = 1, \ldots, p$ and m_i the multiplicity of the root z_i. Since P verify $(\mathcal{P})$, we have $m_i \leq 2$ and $m_1 + ... + m_p = n$. Besides, if $z_i \in \mathbb{C} - \mathbb{R}$ then $m_i = 1$ and $\overline{z_i}$ is also a root of P of multiplicity 1.

Since $\mathbb{C}$ is a separated nomed space, it exists $\epsilon > 0$ such that $D(z_i, \epsilon) \cap D(z_j, \epsilon) = \emptyset$ for any $i \neq j$. Hence, by the previous theorem it exists $\eta > 0$ such that any polynomial Q in $\mathbb{R}[X] \subset \mathbb{C}[X]$ verifying $\|P - Q\|_\infty < \eta$ has exactly m_i roots (distinct or not) in $D(z_i, \epsilon)$. In particular, Q has n roots in $\mathbb{C}$ (with multiplicity) belonging to $\bigcup_{i=1}^{p} D(z_i, \epsilon)$ and $\deg(Q) \geq n$. We have several possible cases for the roots of Q belonging to $D(z_i, \epsilon)$:

- $D(z_i, \epsilon)$ contains exactly one real root of multiplicity less than two or two real roots of multiplicity one.

- $D(z_i, \epsilon)$ contains exactly one root λ of multiplicity one belonging to $\mathbb{C} - \mathbb{R}$. Then, $D(\overline{z_i}, \epsilon)$ contains exactly one root $\overline{\lambda}$ of multiplicity one. Thus, the polynomial $(X - \lambda)(X - \overline{\lambda})$ is an irreducible real factor of Q and this factor appears only one time in Q.

- $D(z_i, \epsilon)$ contains exactly two roots of multiplicity one, λ and μ, belonging to $\mathbb{C} - \mathbb{R}$. In this case, $z_i \in \mathbb{R}$ (since the non-real roots of P has multiplicity one). Besides, $\overline{\lambda}$ and $\overline{\mu}$ are roots of Q of multiplicity one and they also belong to $D(z_i, \epsilon)$ (since $z_i \in \mathbb{R}$). Hence, $\mu = \overline{\lambda}$. Thus, the polynomial $(X - \lambda)(X - \overline{\lambda})$ is an irreducible real factors of Q and this factor appears only one time in Q.

- $D(z_i, \epsilon)$ contains exactly one real root of multiplicity one x and one root of multiplicity one λ belonging to $\mathbb{C} - \mathbb{R}$. In this case, $z_i \in \mathbb{R}$ (since the non-real roots of P has multiplicity one). Besides, $\overline{\lambda}$ is a root of Q of multiplicity one which belongs to $D(z_i, \epsilon)$ (since $z_i \in \mathbb{R}$). Hence, this case is not possible.

Hence, a polynomial T belonging to $\mathbb{R}_n[X]$ and verifying $\|P - T\|_\infty < \eta$ has degree n and verifies $(\mathcal{P})$. Thus, $\hat{G}$ is an open subset of $\mathbb{R}_n[X]$.

□

Remark 3.4. In the first case considered in the proof, $z_i \in \mathbb{R}$. Indeed, suppose z_i is a non-real root of P. $D(z_i, \epsilon)$ contains only one root of multiplicity one (since the non-real roots of P has multiplicity one). $\overline{z_i}$ is a root of P of multiplicity one. Hence, $D(\overline{z_i}, \epsilon)$ contains exactly one root of Q. We set x this root. x is necessarily real since otherwise $D(z_i, \epsilon)$ would contain a non-real root which would be $\overline{x}$. Besides, $|\overline{z_i} - x| = |z_i - x| \leq \epsilon$. Thus, $x \in D(z_i, \epsilon) \cap D(\overline{z_i}, \epsilon)$. This is not possible since $D(z_i, \epsilon) \cap D(\overline{z_i}, \epsilon) = \emptyset$. Thus, $z_i \in \mathbb{R}$.

3.2. **Proof of theorem 1.3.**

Proof. We consider the following continued map $\begin{array}{ccccc} \varphi & : & M_n(\mathbb{K}) & \longrightarrow & \mathbb{K}_n[X] \\ & & A & \longmapsto & \chi_A \end{array}$.

i) The set of diagonalizable matrices $D_n(\mathbb{C})$ is included in F. Moreover, $D_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$. Hence, F is dense in $M_n(\mathbb{C})$.

ii) $\hat{F}$ is an open subset of $\mathbb{C}_n[X]$ (proposition 3.2). Hence,

$$\varphi^{-1}(\hat{F}) = \{A \in M_n(\mathbb{C}),\ \chi_A \text{ verifies } (\mathcal{P})\} = \{A \in M_n(\mathbb{C}),\ \chi_A \text{ has only simple or double roots}\}$$

is an open subset of $M_n(\mathbb{C})$. Besides, $\varphi^{-1}(\hat{F})$ is included in F (by the theorem 1.2 and the theorem of Cayley-Hamilton). Hence, $\varphi^{-1}(\hat{F})$ is included in the interior of F.

Let $A \in F$ such that χ_A doesn't verify $(\mathcal{P})$. By corollary 2.2, all the Jordan blocks appearing in the

Jordan normal form of A have their size equal to 1 or 2. Since, χ_A doesn't verify $(\mathcal{P})$, it exists λ such that $(X-\lambda)^3$ divides χ_A. Thus, one of the following occurs :

- a) the Jordan normal form of A contains two blocks $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$;
- b) the Jordan normal form of A contains three blocks (λ);
- c) the Jordan normal form of A contains one block $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and one block (λ).

We consider each case separately :

- If a) occurs. It exists $P \in GL_n(\mathbb{C})$ such that $A = P \begin{pmatrix} J_\lambda & & \\ & J_\lambda & \\ & & B \end{pmatrix} P^{-1}$. For all $n \in \mathbb{N}^*$, we define $A_n = P \begin{pmatrix} \lambda & 1 & \frac{1}{n} & 0 & \\ 0 & \lambda & 0 & \frac{1}{n} & \\ 0 & 0 & \lambda & 1 & \\ 0 & 0 & 0 & \lambda & \\ & & & & B \end{pmatrix} P^{-1}$. (A_n) converges to A. However, for all $n \in \mathbb{N}^*$, $(X-\lambda)^3$ divides π_{A_n}. Hence, $A_n \notin F$ (theorem 1.2). Thus, A doesn't belong to the interior of F.
- If b) occurs. It exists $P \in GL_n(\mathbb{C})$ such that $A = P \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & B \end{pmatrix} P^{-1}$. For all $n \in \mathbb{N}^*$, we define $A_n = P \begin{pmatrix} \lambda & \frac{1}{n} & 0 & \\ 0 & \lambda & \frac{1}{n} & \\ 0 & 0 & \lambda & \\ & & & B \end{pmatrix} P^{-1}$. (A_n) converges to A. However, for all $n \in \mathbb{N}^*$, $(X-\lambda)^3$ divides π_{A_n}. Hence, $A_n \notin F$ (theorem 1.2). Thus, A doesn't belong to the interior of F.
- If c) occurs. It exists $P \in GL_n(\mathbb{C})$ such that $A = P \begin{pmatrix} J_\lambda & & \\ & \lambda & \\ & & B \end{pmatrix} P^{-1}$. For all $n \in \mathbb{N}^*$, we define $A_n = P \begin{pmatrix} \lambda & 1 & 0 & \\ 0 & \lambda & \frac{1}{n} & \\ 0 & 0 & \lambda & \\ & & & B \end{pmatrix} P^{-1}$. (A_n) converges to A. However, for all $n \in \mathbb{N}^*$, $(X-\lambda)^3$ divides π_{A_n}. Hence, $A_n \notin F$ (theorem 1.2). Thus, A doesn't belong to the interior of F.

Hence, the interior of F is equal to $\varphi^{-1}(\hat{F})$.

iii) Let $A \in M_n(\mathbb{R})$. If A is triangularizable then it exists a sequence (A_n) of diagonalizable matrices such that (A_n) converges to A.

Suppose now A is not triangularizable. Thus, $\pi_A = \prod_{i=1}^{r} P_i^{n_i} \prod_{i=1}^{l} (X-\lambda_i)^{m_i}$ with $r \geq 1$, $l \geq 0$, $\lambda_i \in \mathbb{R}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, $n_i, m_i \geq 1$ and P_i irreducible monic polynomials of degree 2, $P_i \neq P_j$ for $i \neq j$,. Therefore, A is similar to a block-diagonal matrix in which each block has its minimal polynomial equals to $P_i^{n_i}$ or $(X-\lambda_i)^{m_i}$ (by the kernel lemma). Hence, it is sufficient to consider square matrices whose minimal polynomial is the power of an irreducible monic polynomial of degree 2 (since the case of triangularizable matrices has already been considered).

Let $B \in M_n(\mathbb{R})$ such that $\pi_B = P^m$ with P an irreducible monic polynomial of degree 2 and $m \leq \frac{n}{2}$. It exists $\lambda \in \mathbb{C} - \mathbb{R}$ such that $P(X) = (X - \lambda)(X - \overline{\lambda})$. By the decomposition theorem of Frobenius, B is similar to a block diagonal matrix

$$H = \begin{pmatrix} C(R_1) & & \\ & \ddots & \\ & & C(R_t) \end{pmatrix},$$

with R_i real polynomials verifying R_i divides R_{i+1}. Since $\pi_B = \pi_H = \text{lcm}(R_i, 1 \leq i \leq t)$ and $\pi_B = P^m$ with P irreducible, R_i is a power of P. Thus, it exists $1 \leq r_i \leq m$ such that $R_i = P^{r_i} = (X-\lambda)^{r_i}(X-\overline{\lambda})^{r_i}$. We set $R_{i,n} = \prod_{j=1}^{r_i}(X-\lambda+\frac{j}{n^2})(X-\overline{\lambda}+\frac{j}{n^2})$. $(R_{i,n})$ converges to R_i. Hence, $C(R_{i,n})$ converges to $C(R_i)$. Besides, $R_{i,n}$ verifies $(\mathcal{P})$ for all $n \in \mathbb{N}^*$ and $\pi_{C(R_{i,n})} = R_{i,n}$. Thus, $C(R_{i,n})$ is X-formable (theorem 1.2).

So, it exists a sequence of X-formable matrices which converges to B.

Hence, G is dense in $M_n(\mathbb{R})$.

iv) $\hat{G}$ is an open subset of $\mathbb{R}_n[X]$ (proposition 3.3). Hence, $\varphi^{-1}(\hat{G}) = \{A \in M_n(\mathbb{R}),\ \chi_A \text{ verifies } (\mathcal{P})\}$ is an open subset of $M_n(\mathbb{R})$. Besides, $\varphi^{-1}(\hat{G})$ is included in G (by the theorem 1.2 and the theorem of Cayley-Hamilton). Hence, $\varphi^{-1}(\hat{G})$ is included in the interior of G.

Let $A \in G$ such that χ_A doesn't verify $(\mathcal{P})$. A is similar to a block-diagonal matrix in which each block has a size less than 2 (lemme 2.1). Besides, each triangularizable block has a Jordan normal form. Hence, A is similar to a block-diagonal matrix C in which each block is a Jordan block of size 1 or 2 or a square matrix of size 2 whose charateristic polynomial is an irreducible monic polynomial of degree 2. Since, χ_A doesn't verify $(\mathcal{P})$, it exists λ such that $(X - \lambda)^3$ divides χ_A or it exists an irreducible monic polynomial P of degree 2 such that P^2 divides χ_A. Thus, one of the following occurs :

- a) C contains two blocks $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$;
- b) C contains three blocks (λ);
- c) C contains one block $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and one block (λ);
- d) C contains two blocks $T = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ and $O = \begin{pmatrix} u & v \\ w & h \end{pmatrix}$ which have the same irreducible characteristic polynomial $S(X) = X^2 + \alpha X + \beta$. In particular, $y, z, v, w \neq 0$.

We can study the first three cases in the same way as that used previously.

If d) occurs. It exists $P \in GL_n(\mathbb{R})$ such that $A = P \begin{pmatrix} T & & \\ & O & \\ & & B \end{pmatrix} P^{-1}$. For all $n \in \mathbb{N}^*$, we define $A_n = P \begin{pmatrix} x & y & 0 & \frac{1}{n} & \\ z & t & 0 & 0 & \\ & & u & v & \\ & & w & h & \\ & & & & B \end{pmatrix} P^{-1}$. The minimal poynomial of $Z_n = \begin{pmatrix} x & y & 0 & \frac{1}{n} \\ z & t & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & w & h \end{pmatrix}$ is equal to the minimal

polynomial of A_n. Besides, S divides the minimal polynomial of Z_n. $S(Z_n) = \begin{pmatrix} 0 & 0 & \frac{w}{n} & \frac{x+h+\alpha}{n} \\ 0 & 0 & 0 & \frac{z}{n} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0_4$ (since $w \neq 0$) and $S^2(Z_n) = 0_4$. Hence, $\pi_{Z_n} \neq S$ and π_{Z_n} divides S^2. Thus, $\pi_{Z_n} = S^2$, since S is irreducible. Hence, $\pi_{A_n} = S^2$. So, π_{A_n} doesn't verify $(\mathcal{P})$ and $A_n \notin G$ (theorem 1.2). However, (A_n) converges to A. Thus, A doesn't belong to the interior of G.

Hence, the interior of G is equal to $\varphi^{-1}(\hat{G})$.

□

We have therefore, as announced, a dense subset of $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ and a complete characterization of the interior and of this set. We conclude by noticing a curious similarity between the diagonalizable matrices and the X-formable matrices. For this, we will use the following notation : a polynomial verifies the property $(\mathcal{Q})$ if it is a product of distinct linear factors over $\mathbb{K}$. Let $A \in M_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we have :

- A is diagonalizable if and only if π_A verifies $(\mathcal{Q})$; A belongs to the interior of the set of diagonalizable matrices if and only if χ_A verifies $(\mathcal{Q})$;
- A is X-formable if and only if π_A verifies $(\mathcal{P})$; A belongs to the interior of the set of X-formable matrices if and only if χ_A verifies $(\mathcal{P})$.

4. Some open problems

We collect here some open problems related to X-formable matrices. The firt problem is related to theorem 1.3. Indeed, we know some topological properties of the set of X-formable matrices. Therefore, it is natural to want to look for other.

Problem 4.1. *Find other topological properties of the set of X-formable matrices over $\mathbb{R}$ and $\mathbb{C}$.*

We have a certain number of results concerning the maximum dimension of a vector subspace all of whose elements verify a given property (such as Gerstenhaber's theorem, see [2]). This naturally leads to the following problem :

Problem 4.2. *What is the maximal dimension of a vector subspace of $M_n(\mathbb{K})$ containing only X-formable elements ?*

In the case of diagonalizable matrices we have the following result :

Theorem 4.3 (Klarès's criterion, [5] p 125)**.** *Let $B \in M_n(\mathbb{K})$ with $\mathbb{K}$ an algebraically closed field. We set $Ad_B : M \in M_n(\mathbb{K}) \longmapsto BM - MB$. B is diagonalizable if and only if $Ker(Ad_B) = Ker(Ad_B^2)$.*

This leads to the formulation of the problem below :

Problem 4.4. *Can we find a "similar" criterion for X-formable matrices ?*

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