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Gravitational action for a massive Majorana fermion in $2d$ quantum gravity

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ABSTRACT: We compute the gravitational action of a free massive Majorana fermion coupled to two-dimensional gravity on compact Riemann surfaces of arbitrary genus. The structure is similar to the case of the massive scalar. The small-mass expansion of the gravitational yields the Liouville action at zeroth order, and we can identify the Mabuchi action at first order. While the massive Majorana action is a conformal deformation of the massless Majorana CFT, we find an action different from the one given by the David-Distler-Kawai (DDK) ansatz.

KEYWORDS: 2D Gravity, Models of Quantum Gravity

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1 Introduction

The functional integral is the most direct approach to quantum gravity coupled to matter. However, this strategy is difficult to carry out in four dimensions: first, because of the computational complexity of the functional integral, second, because gravity is non-renormalizable.

In such circumstances, a natural exercise is to study the corresponding theory in a lower number of dimensions, in particular, in two dimensions. In this case, computations can

often be carried out explicitly and one can gain intuition for approaching and understanding the higher-dimensional problem. This paper will thus be concerned about two-dimensional quantum gravity, which is particularly simple because the Einstein-Hilbert action is trivial and only the matter and the cosmological constant contribute to the dynamics of gravity.

Since the metric in two dimensions contains only one degree of freedom, the standard approach consists in splitting the metric into a conformal factor, called the Liouville field σ , and a fixed background metric $\hat{g}_{\mu\nu}$. Then, the dependence in the Liouville mode of the functional integral can be factored from the matter by introducing a Wess-Zumino-Witten term – called the gravitational action – for the Liouville field.

In his seminal paper [1], Polyakov found that the gravitational action for a scalar field is given by the famous Liouville action. In fact, this statement holds for any conformal field theory (CFT) by relating the action to the conformal anomaly on a curved manifold. The properties of this theory have been largely investigated, and the reader is referred to [2–15] for a selected set of references and to the reviews [16–20] for additional details.

For a model to be faithful to our four-dimensional world, it should contain non-conformal matter.¹ Surprisingly, this topic has been mostly ignored in the literature and only the last decade saw the beginning of an investigation from first-principles based on the heat kernel. The first terms of a development in the mass of the gravitational action for a massive scalar field (with and without non-minimal coupling to gravity) on a compact Riemann surface have been computed in [22–24]. Besides the Liouville action, two other functionals well-known from the mathematicians appear in the development: the Mabuchi² and the Aubin-Yau functionals [29, 30]. Finally, the gravitational action has later been computed completely in the case of manifolds with and without boundaries [31, 32].

The historical approach to the coupling of non-conformal matter to $2d$ gravity follows the David-Distler-Kawai (DDK) construction [6, 7]. It provides an ansatz for the gravitational action in two situations: 1) when the matter is a CFT deformed by a set of primary operators, 2) when changing to the free field measure for the Liouville mode. The second case has received strong support, both directly from an explicit derivation [33–35], and indirectly from the rigorous constructions of the Liouville CFT using the conformal bootstrap [17, 20, 36] or the Gaussian multiplicative chaos [37, 38]. On the other hand, it is known that the first case presents some problems. In particular, the β -functions associated with the matter coupling constants do not vanish, and the theory is not conformally invariant at the quantum level [39, 40]. Moreover, the functional integral of the Liouville field cannot be factorized from the matter functional integral.

The goal of this paper is to compute the gravitational action of a massive Majorana fermion on a compact Riemann surface of arbitrary genus (without boundary), developing the computations performed in [41] on the sphere (see also [42] for earlier computations and alternative methods). On flat space, a massive Majorana fermion in two dimensions is equivalent to the Ising model at finite temperature (or massive Ising model) [43, 44]. The latter is a conformal deformation of the Ising model, which is a CFT with central charge $c = 1/2$ [45].

¹Some aspects of massive matter coupled to classical two-dimensional gravity have been recently discussed in [21].

²The string susceptibility and spectrum of this action have been discussed in [25–27]. A rigorous mathematical construction based on a generalization of the Gaussian multiplicative chaos has been given in [28].

It is interesting to study the coupling of this model to gravity for two reasons. First, it is the simplest model after the massive scalar field, since it is also a free theory. Thus, it is expected that the gravitational action can be explicitly computed. This point would be helpful to provide more insights on the possible sources for the Mabuchi action. In a second time, one can compare this expression with the ansatz provided by the DDK construction. Ultimately, we find that both actions *do not agree*, which calls for a deeper study of the DDK ansatz.

Outline. In section 2, we introduce the field theory of a massive Majorana fermion on a curved space. We discuss in length how to define the functional integral of a Majorana field in a general basis. This yields an expression for the effective action as a functional determinant. Section 3 is devoted to defining the tools needed for the spectral analysis. In particular, we define the spectral functions which appear in the computations (Green functions, zeta function, heat kernel) and derive their conformal variations. We also explain how to handle zero-modes properly. The gravitational action for a Majorana fermion coupled to $2d$ gravity on a Riemann surface of arbitrary genus is derived in section 4, see (4.36), which is the main result of our paper. We also comment on the small mass expansion and show that the first correction contains the Mabuchi action. Appendix A summarizes our conventions, while appendix B gathers formulas on two-dimensional Euclidean fermions and gamma matrices. In appendix C, we recall the DDK construction, describe its possible problems and discuss them, in view of our results. Finally, in appendix D, we give some detailed computations of identities that were used in 4.

Note added. While we were finalizing our manuscript, the paper [46] appeared and obtained results similar with this paper and the earlier draft [42] (written prior to [41] and which already computed the zero-modes, projectors and finite variations) for arbitrary genus.

2 Majorana fermion field theory

In this section, we present the action and the associated functional integral of a two-dimensional massive Majorana fermion coupled to gravity in Euclidean signature. We consider a compact Riemann surface without boundaries. Conventions are given in appendix A, and general properties of gamma matrices and spinors in two dimensions are given in appendix B.

2.1 Classical action

The action for a two-dimensional Majorana fermion Ψ coupled to gravity is given by [43, 47]

$$S_m[g, \Psi] = \frac{1}{4\pi} \int d^2x \sqrt{g} \bar{\Psi} (i\cancel{\nabla} + m\gamma_*) \Psi, \tag{2.1}$$

where the Dirac adjoint reads $\bar{\Psi} = \Psi^\dagger$ and m is the mass.³The Dirac operator $\cancel{\nabla}$ is defined as (see appendix A for more details):

$$\cancel{\nabla} := \gamma^\mu \nabla_\mu, \quad \nabla_\mu \Psi := \partial_\mu \Psi + \Gamma_\mu \Psi, \tag{2.2}$$

³The Dirac conjugation is sometimes defined by $\bar{\Psi} = \Psi^\dagger \gamma^0$ [45, sec. 5.3.2], [48, sec. 9.7], but the above object does not transform appropriately under Lorentz transformations (see appendix B).

where the connection reads:

$$\Gamma_\mu := -\frac{i}{4}\omega_\mu\gamma_*, \quad \omega_\mu := \omega_{\mu ab}\epsilon^{ab}, \quad (2.3)$$

with $\omega_{\mu ab}$ is the spin connection, $\epsilon_{01} = 1$ is the antisymmetric Levi-Civita symbol, and $\gamma_* = i\gamma^0\gamma^1$. Curved indices are denoted with Green letters (μ, ν , etc.), and indices of the local flat frame by Latin letters (a, b , etc.). One has to introduce the chirality matrix γ_* in the mass term, since the standard bilinear $\bar{\Psi}\Psi$ vanishes (see appendix B). Finally, let us mention that $i\cancel{\nabla}$ is Hermitian with the usual inner-product, defined below in (2.18), since γ_* is Hermitian and the complex conjugation of the factor i compensates the sign arising from the integration by parts.

Even though the connection term vanishes due to the flip relation (B.16) [49], one needs to keep it when defining the functional integral in order to work with a covariant object. In particular, in the Weyl basis, writing $\Psi = (\bar{\psi}, \psi)$ this action reads

$$S[g, \Psi] = \frac{1}{4\pi} \int d^2z \sqrt{g} (\psi \bar{\partial} \psi - \bar{\psi} \partial \bar{\psi} + 2m \psi \bar{\psi}) \quad (2.4)$$

where $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$ and $\bar{\partial} = \frac{1}{2}(\partial_0 + i\partial_1)$. This is the standard form for the massive Ising model [50, sec. 9.2.2].

2.2 Functional integral

The partition function of the theory is given by the functional integral

$$Z = \int \mathcal{D}g \mathcal{D}\Psi e^{-S[g, \Psi] - S_\mu[g]} := \int \mathcal{D}g Z[g], \quad (2.5)$$

where $S_\mu[g] = \mu \int d^2x \sqrt{g} = \mu A$ is the cosmological constant action. In this paper, we will not be interested in the integration over the metrics but restrict ourselves to the matter part. For this reason, we also do not take into account the Einstein-Hilbert action, which in two dimensions is just a constant proportional to the Euler number of the Riemann surface. $Z[g]$ can be further decomposed as

$$Z[g] = e^{-S_\mu[g]} \int \mathcal{D}\Psi e^{-S[g, \Psi]} := e^{-S_\mu[g]} Z_m[g]. \quad (2.6)$$

The effective action $S_{\text{eff}}[g]$ is defined by

$$Z_m[g] := e^{-S_{\text{eff}}[g]}. \quad (2.7)$$

Computing $S_{\text{eff}}[g]$ directly is very challenging, and it is simpler to obtain the gravitational action

$$S_{\text{grav}}[\hat{g}, g] := S_{\text{eff}}[g] - S_{\text{eff}}[\hat{g}] = -\ln \frac{Z_m[g]}{Z_m[\hat{g}]}, \quad (2.8)$$

where g and \hat{g} are two different metrics. One then has

$$Z[g] = e^{-S_\mu[g]} e^{-S_{\text{grav}}[\hat{g}, g]} Z_m[\hat{g}] \quad (2.9)$$

and the complete action reads

$$S_{\text{cg}}[\hat{g}, g, \Psi] = S_{\text{grav}}[\hat{g}, g] + S_m[\hat{g}, \Psi]. \quad (2.10)$$

Since the action is quadratic, the path integral is Gaussian and reduces to the determinant of the kinetic operator

$$D = i\not{\nabla} + m\gamma^* \tag{2.11}$$

to some power. The operator D is Hermitian since $i\not{\nabla}$ and γ_* are Hermitian. Moreover, it is possible to rewrite the determinant in terms of the square of the operator D

$$D^2 = -\Delta + \frac{R}{4} + m^2, \tag{2.12}$$

where Δ is the spinor Laplacian:

$$\Delta\Psi := g^{\mu\nu} \nabla_\mu \nabla_\nu \Psi. \tag{2.13}$$

Note that D^2 is diagonal only in the Weyl basis (B.22), where $\gamma_* = \text{diag}(1, -1)$. The rest of this section is devoted to the derivation of this result, which is obtained by diagonalizing (formally) the kinetic operator.

2.2.1 Mode expansion

There is no solution to the eigenvalue equation

$$D\Psi = (i\not{\nabla} + m\gamma_*)\Psi \stackrel{?}{=} \lambda\Psi \tag{2.14}$$

with $\lambda \in \mathbb{R}$ (since D is Hermitian) and such that Ψ satisfies the Majorana condition:

$$\Psi^* = C\Psi, \tag{2.15}$$

with C the charge conjugation matrix defined in (B.12). Indeed, by taking the conjugate of the equation and inserting the Majorana condition, one finds

$$D\Psi = (i\not{\nabla} + m\gamma_*)\Psi = -\lambda^*\Psi. \tag{2.16}$$

This problem can be solved by looking for complex eigenvectors to be decomposed into their real and imaginary parts (under the Majorana conjugation). Hence, we are looking for complex eigenfunctions $\Psi_n \in \mathbb{C}$ of D with real eigenvalues λ_n ($n \in \mathbb{Z}$) [51, sec. 13.3]:⁴

$$D\Psi_n = (i\not{\nabla} + m\gamma_*)\Psi_n = \lambda_n \Psi_n, \quad \lambda_n \in \mathbb{R}. \tag{2.17}$$

The inner-product between two spinors ψ_1 and ψ_2 is defined as:

$$\langle \psi_1 | \psi_2 \rangle := \frac{1}{2\pi} \int d^2x \sqrt{g} \psi_1(x)^\dagger \psi_2(x). \tag{2.18}$$

⁴Note that we could work with real eigenmodes by inserting the matrix γ_* on the r.h.s. . However, this complicates all expressions since this matrix would appear in the definition of the inner-product, Green functions, etc.

Note that it vanishes for anti-commuting Majorana spinors (for example, in the Majorana basis: $\psi^\dagger\psi = \psi^t\psi = 0$); however, this is not a problem since the eigenmodes are commuting and complex.⁵The eigenfunctions form a complete set and are taken to be orthonormal:⁶

$$\langle \Psi_m | \Psi_n \rangle = \delta_{mn}. \tag{2.19}$$

By computing the inner-product with an insertion of D , we can easily check that the eigenvalues are real:

$$\begin{aligned} \langle \Psi_n | D \Psi_n \rangle &= \lambda_n \\ &= \langle D \Psi_n | \Psi_n \rangle = \lambda_n^*. \end{aligned} \tag{2.20}$$

One can check that if λ_n is the eigenvalue associated to ψ_n , then $-\lambda_n$ is the eigenvalue of $C^{-1}\Psi_n^*$ for $n \neq 0$:

$$D(C^{-1}\Psi_n^*) = -\lambda_n(C^{-1}\Psi_n^*). \tag{2.21}$$

As a consequence, we define

$$\forall n \in \mathbb{N}^* : \quad \Psi_{-n} := C^{-1}\Psi_n^*, \quad \lambda_{-n} := -\lambda_n. \tag{2.22}$$

The Majorana field is expanded on the modes Ψ_n as:

$$\Psi := \sum_{n \geq 0} (a_n \Psi_n + a_{-n} \Psi_{-n}), \tag{2.23}$$

where the a_n are complex Grassmann variables and satisfy:

$$a_{-n} = a_n^\dagger. \tag{2.24}$$

Note that in this decomposition the coefficients are taken to be Grassmann numbers while the eigenfunctions are commuting functions. As a consequence, the normalization (2.19) would be non-trivial even without inserting γ_* . The coefficient a_n can be recovered by taking the inner-product with Ψ_n :

$$a_n = \langle \Psi_n | \Psi \rangle. \tag{2.25}$$

The Dirac conjugate is:

$$\bar{\Psi} = \sum_{n \geq 0} (a_n \Psi_n^t C + a_{-n} \Psi_{-n}^t C) = \sum_{n \geq 0} (a_n \bar{\Psi}_{-n} + a_{-n} \bar{\Psi}_n), \tag{2.26}$$

⁵It would be necessary to add γ_* in the definition of the inner-product if it also appears in the r.h.s. of the eigenvalue equation. But, as pointed in the previous footnote, this makes all expressions much more complicated. One particular problem is that D is not self-adjoint for this product, instead: $\langle \psi_1 | D \psi_2 \rangle = \langle \tilde{D} \psi_1 | \psi_2 \rangle$, where $\tilde{D} := \gamma_* D \gamma_*$.

⁶In fact, modes with $\lambda_n = m$ (where m is the mass) are degenerate and generically not orthonormal, see section 3.3. However, ignoring this subtlety does not change the computation in general and the fact that zero-modes are not orthonormal will be taken care of when needed.

using that $\bar{\Psi} = \bar{\Psi}^c = \Psi^t C$ since $(C^{-1})^\dagger = (C^*)^\dagger = C^t = C$, and the second equality follows from (2.22). Since $\bar{\Psi} = \Psi^\dagger$, one can also recover the expression (2.24) from the coefficient of $\bar{\Psi}_n$.

Then, we can define real modes (under the Majorana conjugate) for $n \geq 0$:

$$\chi_n := \frac{1}{\sqrt{2}}(\Psi_n + \Psi_{-n}), \quad \phi_n := -\frac{i}{\sqrt{2}}(\Psi_n - \Psi_{-n}), \quad (2.27)$$

such that

$$\chi_n^* = C\chi_n, \quad \phi_n^* = C\phi_n. \quad (2.28)$$

These modes form two orthonormal sets:

$$\langle \chi_m | \chi_n \rangle = \langle \phi_m | \phi_n \rangle = \delta_{mn}, \quad \langle \chi_m | \phi_n \rangle = 0. \quad (2.29)$$

It is then straightforward to check that these modes satisfy the equations:

$$D\chi_n = i\lambda_n \phi_n, \quad D\phi_n = -i\lambda_n \chi_n \quad (2.30)$$

since

$$D\chi_n = \frac{1}{\sqrt{2}}D(\Psi_n + \Psi_{-n}) = \frac{\lambda_n}{\sqrt{2}}\gamma_*(\Psi_n - \Psi_{-n}) = i\lambda_n \phi_n.$$

Squaring this equation gives:

$$D^2\chi_n = \Lambda_n\chi_n, \quad D^2\phi_n = \Lambda_n\phi_n, \quad \Lambda_n := \lambda_n^2. \quad (2.31)$$

This also implies:

$$D^2\Psi_n = \Lambda_n\Psi_n. \quad (2.32)$$

This means that the (χ_n, ϕ_n) are eigenfunctions of the second-order (Laplace-type) kinetic operators, but not of the Dirac operator. Note that it should be related to the decomposition of a Majorana spinor into a Weyl spinor and its complex conjugate [52, sec. 3.4].

The eigenvalues are indexed by $n \in \mathbb{N}$ and sorted by ascending order:

$$0 < m^2 \leq \Lambda_0 \leq \Lambda_1 \leq \dots \quad (2.33)$$

In particular, there is no zero-mode if $m^2 > 0$ (section 3.3).

The Majorana field is expanded on the real modes as

$$\Psi = \sum_{n \geq 0} (b_n \chi_n + c_n \phi_n) \quad (2.34)$$

where (b_n, c_n) are real Grassmann variables such that

$$a_n = \frac{1}{\sqrt{2}}(b_n + ic_n), \quad a_n^\dagger = \frac{1}{\sqrt{2}}(b_n - ic_n), \quad (2.35)$$

and we have the relation

$$a_n^\dagger a_n = \frac{i}{2}(b_n c_n - c_n b_n) = i b_n c_n. \quad (2.36)$$

Let us stress that a_n^\dagger and a_n anti-commute. Note also that the Dirac conjugate is

$$\bar{\Psi} = \sum_{n \geq 0} (b_n \chi_n^t + c_n \phi_n^t) C. \quad (2.37)$$

2.2.2 Evaluation of the functional integral

We want to compute the generating functional with source η :

$$Z[\eta] := \int d\Psi \exp(-S[\Psi] + i\langle \bar{\eta} | \Psi \rangle), \quad (2.38)$$

where the action can be written in terms of the inner-product (2.18) as:

$$S[\Psi] = \langle \Psi | D | \Psi \rangle. \quad (2.39)$$

The source is decomposed as

$$\eta = \sum_{n \geq 0} (u_n \chi_n + v_n \phi_n) = \sum_{n \geq 0} (s_n \Psi_n + s_{-n} \Psi_{-n}), \quad s_{-n} = s_n^\dagger, \quad (2.40)$$

where

$$s_n = \frac{1}{\sqrt{2}}(u_n + iv_n), \quad s_n^\dagger = \frac{1}{\sqrt{2}}(u_n - iv_n). \quad (2.41)$$

We have the relation:

$$s_n^\dagger s_n = iu_n v_n. \quad (2.42)$$

We can evaluate the inner-product which appears in the path integral,

$$\langle \Psi | D | \Psi \rangle = 2i \sum_n \lambda_n b_n c_n = 2 \sum_n \lambda_n a_n^\dagger a_n. \quad (2.43)$$

and also:

$$\langle \eta | \Psi \rangle = \sum_{n \geq 0} (u_n b_n + v_n c_n) = \sum_{n \geq 0} (s_n a_n^\dagger + s_n^\dagger a_n). \quad (2.44)$$

The functional integral reads:

$$Z[\eta] = \int \prod_{n \geq 0} db_n dc_n \exp\left(-i \sum_{n \geq 0} [\lambda_n b_n c_n + u_n b_n + v_n c_n]\right) \quad (2.45a)$$

$$= \int \prod_{n \geq 0} da_n da_n^\dagger \exp\left(\sum_{n \geq 0} [-\lambda_n a_n^\dagger a_n + i s_n a_n^\dagger + i s_n^\dagger a_n]\right). \quad (2.45b)$$

The next step consists in shifting the variables a_n :

$$\bar{a}_n = a_n + \frac{i}{\lambda_n} s_n, \quad \bar{a}_n^\dagger = a_n^\dagger - \frac{i}{\lambda_n} s_n^\dagger \quad (2.46)$$

such that

$$Z[\eta] = \exp\left(\sum_{n \geq 0} \frac{1}{\lambda_n} s_n^\dagger s_n\right) \int \prod_{n \geq 0} d\bar{a}_n d\bar{a}_n^\dagger \exp\left(-\sum_{n \geq 0} \lambda_n \bar{a}_n^\dagger \bar{a}_n\right). \quad (2.47)$$

The integral is a simple Gaussian integral of complex Grassmann variables:

$$Z[\eta] = \exp\left(\sum_{n \geq 0} \frac{1}{\lambda_n} s_n^\dagger s_n\right) \prod_{n \geq 0} \lambda_n. \quad (2.48)$$

Note that only half of the eigenvalues are included because we had to combine the real functions into complex functions, which lifts the double degeneracy that one has with a Dirac fermion. The product of the positive eigenvalues gives the square-root of the determinant:

$$\prod_{n \geq 0} \lambda_n = \sqrt{\prod_{n \geq 0} \lambda_n^2} = \left(\prod_{n \in \mathbb{Z}} \lambda_n^2 \right)^{\frac{1}{4}} = (\det D^2)^{1/4} = \det \left(-\Delta + \frac{R}{4} + m^2 \right)^{1/4}. \quad (2.49)$$

The first equality allows writing squares of eigenvalues, such that one can extend the range to negative n after the second equality since $\lambda_{-n} = -\lambda_n$. Using formal manipulations of determinants, we can rewrite $\sqrt{\det D^2} = \det D$ such that:

$$\prod_{n \geq 0} \lambda_n = \sqrt{\det D} = \sqrt{\det (i\mathcal{V} + m\gamma_*)}. \quad (2.50)$$

Note that the fact that one can take the square-root without ambiguity (up to a sign) is a consequence of the self-adjointness of the operator [53, p. 1470].

The Green function corresponds to

$$\begin{aligned} S(x, y) &:= \langle x | \frac{1}{D} | y \rangle := \langle x | \frac{1}{i\mathcal{V} + m\gamma_*} | y \rangle \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{\lambda_n} \Psi_n(x) \Psi_n(y)^\dagger = i \sum_{n \geq 0} \frac{1}{\lambda_n} (\phi(x) \chi(y)^t - \chi(x) \phi(y)^t). \end{aligned} \quad (2.51)$$

It follows from:

$$\langle x | \frac{1}{D} | y \rangle = \sum_{n \in \mathbb{Z}} \langle x | \frac{1}{D} | \Psi_n \rangle \langle \Psi_n | y \rangle = \sum_{n \in \mathbb{Z}} \frac{1}{\lambda_n} \langle x | \Psi_n \rangle \langle \Psi_n | y \rangle. \quad (2.52)$$

The Green function is antisymmetric and purely imaginary:

$$S_{\alpha\beta}(x, y) = -S_{\beta\alpha}(y, x). \quad (2.53)$$

In full similarity, we obtain the Green function of D^2 :

$$G(x, y) := \langle x | \frac{1}{D^2} | y \rangle = \sum_{n \in \mathbb{Z}} \frac{1}{\Lambda_n} \Psi_n(x) \Psi_n(y)^\dagger. \quad (2.54)$$

Note that S and G are 2-dimensional matrices in terms of Dirac indices. The trace over Dirac indices is denoted by tr_D .

Finally, we have that

$$\langle \eta | S | \eta \rangle = \sum_{n \in \mathbb{Z}} \frac{1}{\lambda_n} s_n^\dagger s_n = 2i \sum_{n \geq 0} \frac{1}{\lambda_n} u_n v_n. \quad (2.55)$$

As a conclusion, we find:

$$Z[\eta] = \exp \left(\frac{1}{2} \langle \eta | S | \eta \rangle \right) \det \left(-\Delta + \frac{R}{4} + m^2 \right)^{1/4}. \quad (2.56)$$

The factor of 1/2 arises because the Green function has a sum $n \in \mathbb{Z}$, but the functional integral gives only $n \geq 0$ in the exponential.

This gives the expression of the effective action

$$S_{\text{eff}} := -\ln Z[0] = -\frac{1}{4} \ln \det D^2 = -\frac{1}{4} \text{tr} \ln D^2 = -\frac{1}{4} \text{tr} \ln \left(-\Delta + \frac{R}{4} + m^2 \right). \quad (2.57)$$

This determinant can be defined using the standard heat kernel and zeta function methods, since D^2 is an operator of Laplace type.

Let us pause to comment on the case where there are zero-modes (vanishing eigenvalues, or massless fermion). In this case, $Z[\eta]$ in (2.48) looks ill-defined because the first term diverges and the product of eigenvalues vanishes. However, the eigenvalues $\lambda_n = 0$ do not appear in the sum in $\langle \Psi | D \Psi \rangle$ such that the product would be only over strictly positive eigenvalues, $n > 0$. Similarly, (2.56) has instead $\det' D$ and \tilde{S} , the determinant and Green functions without zero-modes. Moreover, the integrals over zero-modes would remain to be done: since they are fermionic, it looks like the result would vanish. This is solved by inserting zero-modes in the functional integral and carefully normalizing [53, 54].

3 Spectral analysis

In this section, we review some points of spectral analysis that will be used to compute the gravitational action. In section 3.1, we define the spectral functions (Green's function, heat kernel and zeta functions) and recall some of their properties. Then, section 3.2 describes how the objects are affected by a Weyl transformations. Finally, in section 3.3, we discuss the zero-modes of the Dirac operator.

While it is possible to derive most formulas for both Green functions $S(x, y)$ and $G(x, y)$, we will not need any formula for $S(x, y)$ in our approach, and hence we focus on the Green function $G(x, y)$ of the operator D^2 .

We define the area of the surface with metric g as

$$A[g] = \int d^2x \sqrt{g}. \quad (3.1)$$

3.1 Spectral functions

From now on, we will denote collectively the spinors χ_n and ϕ_n by Ψ_n , since they are associated to the same eigenvalue $\Lambda_n = \lambda_n^2$ of the operator $D^2 = -\Delta + R/4 + m^2$. States associated with the eigenvalue $\Lambda_0 = m^2$ are called zero-modes. Since both $\lambda_0 = \pm m$ lead to the same Λ_0 , we denote by N_0 the number of such pairs, and write $\Psi_{0,i}$ the eigenstates with $\lambda_0 = +m$. We can also understand N_0 as counting the number of complex spinors (under Majorana conjugation), given (2.22), or the number of eigenmodes with positive (or negative) chirality [55]. Indeed, $[(i\cancel{\nabla})^2, \gamma_*] = 0$ such that it is possible to use a basis of zero-modes with definite chirality $\gamma_* \Psi_{0,i,\pm} = \pm \Psi_{0,i,\pm}$.

We will put a tilde on every quantity from which the contributions of the zero-modes have been subtracted, and a subscript “(0)” when $m = 0$. In particular, Λ_n is related to $\Lambda_n^{(0)}$ by

$$\Lambda_n = \Lambda_n^{(0)} + m^2. \quad (3.2)$$

This shows that there is no zero-mode when $m^2 \neq 0$.

Green's function. The Green's function for the operator D^2 is given by (2.54)

$$D^2 G(x, y) = \frac{\delta(x - y)}{\sqrt{g}} \mathbf{I}_2 \quad \Leftrightarrow \quad G(x, y) = \sum_{n \in \mathbb{Z}} \frac{\Psi_n(x) \Psi_n(y)^\dagger}{\Lambda_n}, \quad (3.3)$$

where \mathbf{I}_2 denotes the 2-dimensional identity. G and its hermitian conjugate G^\dagger are related by

$$G(y, x) = G(x, y)^\dagger. \quad (3.4)$$

One can define the Green's function without the zero-modes by

$$\tilde{G}(x, y) = \sum_{n \neq 0} \frac{\Psi_n(x) \Psi_n(y)^\dagger}{\Lambda_n} = G(x, y) - \frac{1}{m^2} P(x, y), \quad (3.5)$$

where

$$P(x, y) = \sum_i \Psi_{0,i}(x) \Psi_{0,i}^\dagger(y) \quad (3.6)$$

is the projector on the zero-mode subspaces, with $\{\psi_i(x)\}$ an orthonormal basis (see section 3.2 for a discussion of this point). It satisfies the following equation

$$D^2 \tilde{G}(x, y) = \frac{\delta(x - y)}{\sqrt{g}} \mathbf{I}_2 - P(x, y), \quad (3.7)$$

and it is orthogonal to the projector

$$\int d^2 z \sqrt{g} P(x, z) \tilde{G}(z, y) = 0. \quad (3.8)$$

The last relation also implies

$$\tilde{G}(x, y) = \int d^2 z \sqrt{g} \tilde{G}(x, z) D_z^2 \tilde{G}(z, y) = \int d^2 z \sqrt{g} D_z \tilde{G}(x, z) D_z \tilde{G}(z, y). \quad (3.9)$$

When x goes to y , the Green's function presents the usual logarithmic singularity

$$G_{\text{sing}} = \tilde{G}_{\text{sing}} = -\frac{1}{4\pi} \ln(\ell(x, y)^2) \quad (3.10)$$

where $\ell(x, y)$ is the geodesic distance between x and y . A regularization of the Green's function at coincident points is then

$$G_R(x) = \lim_{y \rightarrow x} \left(G(x, y) + \frac{1}{4\pi} \ln(\mu^2 \ell^2(x, y)) \right), \quad (3.11)$$

where μ is a mass scale needed for dimensional reasons.

The Green function G is ill-defined when $\Lambda_0 = m = 0$ since it contains a sum over all eigenmodes with coefficients containing Λ_n^{-1} .

Heat kernel. The heat kernel is defined to be the solution of

$$\left(\frac{d}{dt} + D^2\right) K(t, x, y) = 0, \quad K(t, x, y) \underset{t \rightarrow 0}{\sim} \frac{\delta(x-y)}{\sqrt{g}} \mathbf{I}_2. \quad (3.12)$$

In terms of eigenvectors and eigenvalues, this can be expressed as

$$K(t, x, y) = \sum_{n \in \mathbb{Z}} e^{-\Lambda_n t} \Psi_n(x) \Psi_n(y)^\dagger. \quad (3.13)$$

One also defines the integrated heat kernel by

$$K(t) = \int d^2x \sqrt{g} \operatorname{tr}_D K(t, x, x) = \sum_{n \in \mathbb{Z}} e^{-\Lambda_n t}. \quad (3.14)$$

The corresponding quantities $\tilde{K}(t)$ and $\tilde{K}(t, x, y)$ are obtained by excluding the zero-modes from the sum. In particular, if $m = 0$, one needs to work with these quantities.

For $t > 0$, we see from (3.13) that $K(t, x, y)$ is given by a converging sum and is finite, even as $x \rightarrow y$. For $t \rightarrow 0$ various divergences are recovered, in particular,

$$\int_0^\infty dt K(t, x, y) = G(x, y) \quad (3.15)$$

exhibits the short-distance singularity of the Green's function (3.10).

The behavior of K for small t is related to the asymptotics of the eigenvalues Λ_n and eigenfunctions Ψ_n for large n , which in turn is related to the short-distance properties of the Riemann surface. It is well-known that the small- t asymptotics is given in terms of local expressions of the curvature and its derivatives, and that on a compact manifold without boundaries one has

$$K(t, x, y) \underset{t \rightarrow 0}{\sim} \frac{1}{4\pi t} e^{-\ell^2(x,y)/4t} \sum_{k \geq 0} t^k a_k(x, y). \quad (3.16)$$

The expansion coefficients can be computed recursively using normal coordinates around x (since for small t , the exponential forces ℓ^2 to be small). To compute the gravitational action, we will only need the first ones at coinciding points, which for our operator D^2 turn out to be

$$a_0(x, x) = \mathbf{I}_2 \quad (3.17a)$$

$$a_1(x, x) = \left(-\frac{R}{12} - m^2\right) \mathbf{I}_2. \quad (3.17b)$$

For small t , K is then given by

$$K(t) = \frac{A}{2\pi t} - \frac{1}{24\pi} \int d^2x \sqrt{g} R(x) - \frac{m^2 A}{2\pi} + o(t). \quad (3.18)$$

Another useful relation is

$$\tilde{K}(t) = e^{-m^2 t} \tilde{K}^{(0)}(t) \quad (3.19)$$

using (3.2).

Zeta functions. The zeta function and its integrated version are defined by

$$\zeta(s, x, y) = \sum_n \frac{\Psi_n(x)\Psi_n(y)^\dagger}{\Lambda_n^s}, \quad \zeta(s) = \int d^2x \operatorname{tr}_D \zeta(s, x, x) = \sum_n \frac{1}{\Lambda_n^s}. \quad (3.20)$$

They are related to the heat kernel by a Laplace transform

$$\zeta(s, x, y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt t^{s-1} K(t, x, y), \quad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} dt t^{s-1} K(t). \quad (3.21)$$

Plugging (3.18) into (3.21) enables one to compute $\zeta(0)$:

$$\zeta(0) = -\frac{1}{24\pi} \int d^2x \sqrt{g} R(x) - \frac{m^2 A}{2\pi} = -\frac{1}{24\pi} \int d^2x \sqrt{g} R(x) - \frac{m^2 A}{2\pi} \quad (3.22)$$

In the same way, one shows that the zeta function has a pole for $s = 1$ with residue $\frac{a_0(x,x)}{4\pi} = \frac{I_2}{4\pi}$. Then, using the relation

$$\tilde{\zeta}(0, x, x) = \zeta(0, x, x) - P(x, x), \quad (3.23)$$

we obtain

$$\tilde{\zeta}(0) = \zeta(0) - N_0. \quad (3.24)$$

The zeta function can be used to define a regularized version of the Green's function at coincident points:

$$G_\zeta(x) = \lim_{s \rightarrow 1} \left(\mu^{2s-2} \zeta_{\text{reg}}(s, x, x) \right), \quad \zeta_{\text{reg}}(s, x, x) = \zeta(s, x, x) - \frac{\mu^{2-2s} I_2}{4\pi(s-1)}, \quad (3.25)$$

and the same for $\tilde{\zeta}$ and \tilde{G}_ζ where G_ζ and \tilde{G}_ζ are related by

$$G_\zeta(x) = \tilde{G}_\zeta(x) + \frac{1}{m^2} P(x, x). \quad (3.26)$$

G_R and G_ζ only differ by a constant [56]:

$$G_\zeta(x) = G_R(x) + \alpha. \quad (3.27)$$

Finally, we will use the zeta function to provide a regularized definition of the logarithm of the determinant of the operator D^2 . Let consider

$$S_{\text{eff}}[g] = -\ln Z_m[g] = -\frac{1}{4} \sum_{n \neq 0} \ln \frac{\Lambda_n}{\hat{\mu}^2}, \quad (3.28)$$

where the energy scale $\hat{\mu}$ has been introduced to ensure that the argument of the logarithm is dimensionless. The infinite sum is regularized through an analytic continuation such that:

$$\sum_{n \neq 0} \ln \frac{\lambda_n}{\mu^2} = -\zeta'(0) - \ln \hat{\mu}^2 \zeta(0), \quad (3.29)$$

and then

$$S_{\text{eff}} = \frac{1}{4} \zeta'(0) + \frac{1}{4} \ln \hat{\mu}^2 \zeta(0). \quad (3.30)$$

3.2 Weyl transformations

A Weyl transformation corresponds to a rescaling of the metric

$$g \longrightarrow e^{2\omega} g. \quad (3.31)$$

It can be used to reach the conformal gauge

$$g = e^{2\sigma(x)} \hat{g}, \quad (3.32)$$

where \hat{g} is a fixed metric and σ is the Liouville field. If the latter is small, one finds

$$\delta g = 2\delta\sigma(x)\hat{g}. \quad (3.33)$$

The metric in which the Green functions, heat kernel and zeta functions are defined is indicated by an index, for example $K_g(t)$ and $K_{\hat{g}}(t)$. In the case of a functional, it is given in a bracket. Finally, differential operators receive a hat, for example $\hat{\nabla}_\mu$ is expressed in the metric \hat{g} .

Using the variation of (3.3) under an infinitesimal Weyl transformation and the expression for δD^2 given in (A.26), we see that, at first order in $\delta\sigma$, δG is a solution of

$$D^2 \delta G(x, y) = -2m^2 \delta\sigma(x) G(x, y) + \partial_\nu \delta\sigma(x) \gamma^{\mu\nu} \nabla_\mu G(x, y) + \frac{1}{2} \Delta(\delta\sigma(x)) G(x, y). \quad (3.34)$$

The solution is obtained by convolution with G :

$$\begin{aligned} \delta G(x, y) = & -2m^2 \int d^2 z \sqrt{g} G(x, z) \delta\sigma(z) G(z, y) + \int d^2 z \sqrt{g} G(x, z) \partial_\nu \delta\sigma(z) \gamma^{\mu\nu} \nabla_\mu G(z, y) \\ & + \frac{1}{2} \int d^2 z \sqrt{g} G(x, z) \Delta(\delta\sigma(z)) G(z, y). \end{aligned} \quad (3.35)$$

We can simplify this expression as follows (writing $G(x, y)$ as G_{xy} for concision):

$$\begin{aligned} \delta G_{xy} = & - \int d^2 z \sqrt{g} G_{xz} \left(2m^2 \delta\sigma_z - (\partial_z^\mu \delta\sigma_z) \nabla_{z\mu} + (\not{\partial}_z \delta\sigma_z) \not{\nabla}_z - \frac{1}{2} \Delta_z \delta\sigma_z \right) G_{zy} \\ = & - \int d^2 z \sqrt{g} \delta\sigma_z \left[2m^2 G_{xz} G_{zy} + \cancel{G_{xz} \not{\nabla}_{z\mu} \not{\nabla}_{z\mu} G_{zy}} + G_{xz} \Delta_z G_{zy} \right. \\ & \quad - G_{xz} \not{\nabla}_z \not{\nabla}_z G_{zy} - G_{xz} \not{\nabla}_z^2 G_{zy} - \frac{1}{2} G_{xz} \not{\Delta}_z G_{zy} \\ & \quad \left. - \cancel{G_{xz} \not{\nabla}_{z\mu} \nabla_z^\mu G_{zy}} - \frac{1}{2} G_{xz} \Delta_z G_{zy} \right] \\ = & - \int d^2 z \sqrt{g} \delta\sigma_z \left[m^2 G_{xz} G_{zy} + \frac{1}{2} G_{xz} D_z^2 G_{zy} + \frac{1}{2} G_{xz} \not{D}_z^2 G_{zy} - G_{xz} \not{\nabla}_z \not{\nabla}_z G_{zy} \right] \\ = & - \int d^2 z \sqrt{g} \delta\sigma_z \left[m^2 G_{xz} G_{zy} + \frac{1}{2} G_{xz} \frac{\delta_{zy}}{\sqrt{g}} + \frac{1}{2} \frac{\delta_{xz}}{\sqrt{g}} G_{zy} - G_{xz} \not{\nabla}_z \not{\nabla}_z G_{zy} \right], \end{aligned}$$

where we used $\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu - g_{\mu\nu}$ to get the first equality, integrated by part the derivatives acting on $\delta\sigma_z$ for the second equality, and wrote Δ and $\not{\nabla}^2$ in terms of D^2 for the third equality. We defined $\not{\nabla}_z$ and other similar symbols such that its derivatives act on the

object closest on its left, with gamma matrices staying where the symbol sits. Finally, we can use the Green equation (3.3) to get:

$$\begin{aligned} \delta G(x, y) = & -\frac{1}{2}(\delta\sigma(x) + \delta\sigma(y))G(x, y) - m^2 \int d^2z \sqrt{g} \delta\sigma(z) G(x, z)G(z, y) \\ & + \int d^2z \sqrt{g} \delta\sigma(z) G(x, z) \overleftarrow{\nabla}_z \overleftarrow{\nabla}_z G(z, y). \end{aligned} \quad (3.36)$$

Using the fact that

$$\begin{aligned} \delta \ell^2(x, y) &= \ell^2(x, y)(\delta\sigma(x) + \delta\sigma(y)) + \mathcal{O}(\ell^4(x, y)) \\ \implies \lim_{y \rightarrow x} \delta \ln(\mu^2 \ell^2(x, y)) &= 2\delta\sigma(x), \end{aligned} \quad (3.37)$$

one can compute the variation of the regularized Green functions

$$\delta G_\zeta(x) = \delta G_R(x) = \delta \lim_{y \rightarrow x} \left(G(x, y) + \frac{1}{4\pi} \ln(\mu^2 \ell^2(x, y)) \right). \quad (3.38)$$

The zero-modes transform under a Weyl transformation. Given the transformation law (A.22) of the Dirac operator, it follows directly that if Ψ_0 is a zero-mode of ∇ , then

$$\hat{\Psi}_0 = e^{\sigma/2} \Psi_0 \quad (3.39)$$

is a zero-mode of $\hat{\nabla}$. A first consequence is that a Weyl transformation does not change the number of zero-modes N_0 .

This, together with the fact that the integration is over a different space, implies that the zero-modes in the \hat{g} metric will not be orthonormal even if the zero-modes in the metric g formed an orthonormal basis [53].⁷ As a consequence, the projector (3.6) on the zero-modes must be modified. For this, reason, we introduce the normalization matrix

$$\kappa_{ij}[g] = \int d^2x \sqrt{g} \text{tr}_D \Psi_{0,i}(x)^\dagger \Psi_{0,j}(x). \quad (3.40)$$

Then the projector onto the zero-modes becomes

$$P_g(x, y) = \sum_{i,j} \Psi_{0,i}(x) \kappa^{ij}[g] \Psi_{0,j}^\dagger(y), \quad (3.41)$$

where κ^{ij} denotes the inverse of κ_{ij} . The trace of the projector counts the number of zero-modes

$$\int d^2x \sqrt{g} \text{tr}_D P_g(x, x) = N_0 \quad (3.42)$$

since

$$\int d^2x \sqrt{g} \text{tr}_D P_g(x, x) = \sum_{i,j} \int d^2x \sqrt{g} \text{tr}_D \Psi_{0,i}(x) \kappa^{ij}[g] \Psi_{0,j}^\dagger(x) = \kappa^{ij}[g] \kappa_{ij}[g] = \delta_i^i. \quad (3.43)$$

⁷The scalar field ϕ provides a simple example. Since the scalar Laplacian is Weyl invariant in two dimensions, the zero-mode is also invariant and $\phi_0 = \hat{\phi}_0$. However, if $\phi_0 = 1/\sqrt{A}$ is normalized in the first metric, it is not in the second metric and $\kappa[\hat{g}] = \int d^2x \sqrt{\hat{g}} \phi_0^2 = \hat{A}/A$. It would have been possible to start with the non-normalized zero-mode $\phi_0 = 1$, such that $\kappa[g] = A$ and $\kappa[\hat{g}] = \hat{A}$. This motivates the introduction of the normalization in the projection operator: its absence in [31] was not important because there was a single zero-mode which expression is known, and thus the variation in the normalization could be tracked by hand. The situation is much more complicated for a Majorana field.

The following relation will be useful:

$$\delta \ln \det \kappa = \int d^2x \sqrt{g} \delta \sigma(x) \operatorname{tr}_D P_g(x, x), \tag{3.44}$$

which follows from

$$\delta \operatorname{tr} \ln \kappa = \operatorname{tr} \kappa^{-1} \delta \kappa = \kappa^{ij} \int d^2x \sqrt{g} \delta \sigma \operatorname{tr}_D \Psi_{0,i}(x) \Psi_{0,j}(x)^\dagger. \tag{3.45}$$

3.3 Spin structure and zero-modes of the Dirac operator

In this section, we discuss the zero-modes of the massless operator $-\Delta + \frac{R}{4}$, which are also the zero-modes of $\not{\nabla}$, in terms of the spin structure.

Sphere. On the sphere, we can take for g the round metric, which has $R > 0$ constant. As $-\Delta$ is a positive operator, we see that $-\Delta + \frac{R}{4}$ cannot have any zero-mode and hence

$$N_0 = 0, \quad P_g(x, y) = 0. \tag{3.46}$$

Torus. On the torus, we can use the flat metric $g = \delta$. Then, a zero-mode satisfies

$$\not{\partial} \Psi_0 = 0 \tag{3.47}$$

and the only solutions are the two constant spinors

$$\Psi_0 = \frac{1}{\sqrt{A}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi'_0 = \frac{1}{\sqrt{A}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.48}$$

where $A := A[\delta]$ is the area of the torus. The solution survives only for the odd spin structure, which has periodic boundary conditions in both directions. If at least one boundary has anti-periodic conditions, then there is no solution.

As a consequence, there is one (complex) zero-mode for the odd spin structure, and no zero-mode for the three even spin structures:

$$N_0 = 0 \quad (\text{even structures}), \quad N_0 = 1 \quad (\text{odd structure}). \tag{3.49}$$

We immediately derive the normalization matrix and projector:

$$P_g(x, y) = \frac{1}{A} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \kappa[g] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.50}$$

such that

$$-\frac{1}{2} \ln \det \frac{\kappa[g]}{\kappa[\hat{g}]} = \ln \frac{A}{\hat{A}}. \tag{3.51}$$

Higher-genus. For a genus $g \geq 2$, there is generically one (resp. no) zero-mode when the spin structure is odd (resp. even). However, there can be up to g zero-modes [55]. Zero-modes can be computed by taking \hat{g} to be the metric such that $\hat{R} = -1$ or to be the canonical metric.

4 Gravitational action

Since it is complicated to evaluate the effective action (3.30) directly, it is simpler to consider the gravitational action (2.8)

$$S_{\text{grav}}[\hat{g}, g] = \frac{1}{4}(\zeta'_g(0) + \ln \mu^2 \zeta_g(0)) - \frac{1}{4}(\zeta'_{\hat{g}}(0) + \ln \mu^2 \zeta_{\hat{g}}(0)). \quad (4.1)$$

To simplify further the computation, one can consider an infinitesimal difference such that

$$\delta S_{\text{grav}}[g] = \frac{1}{4}(\delta \zeta'_g(0) + \ln \mu^2 \delta \zeta_g(0)). \quad (4.2)$$

Finally, it is natural to consider both metrics to be related by a Weyl transformation $g = e^{2\sigma(x)} \hat{g}$, or its infinitesimal version $g = 2\delta\sigma(x) \hat{g}$. The gravitational action will be recovered at the end by integrating over $\delta\sigma$.

4.1 Variation of the zeta function

To compute the gravitational action, we need $\delta\zeta(0)$ and $\delta\zeta'(0)$. The variation of the zeta function is given by

$$\delta\zeta(s) = -s \sum_n \frac{\delta\Lambda_n}{\Lambda_n^{s+1}}. \quad (4.3)$$

As there is no zero-mode contribution to this equation, we can work with the function $\tilde{\zeta}$ instead of ζ :

$$\delta\zeta(s) = \delta\tilde{\zeta}(s). \quad (4.4)$$

Doing this, we do not have to distinguish between the massive and the massless case.

The variation of the non-zero eigenvalues Λ_n can be obtained through usual perturbation theory:

$$\begin{aligned} \delta\Lambda_n &= \langle \Psi_n | \delta D^2 | \Psi_n \rangle \\ &= -2(\Lambda_n - m^2) \langle \Psi_n | \delta\sigma | \Psi_n \rangle - \langle \Psi_n | \partial_\nu(\delta\sigma) \gamma^{\mu\nu} \nabla_\mu | \Psi_n \rangle - \frac{1}{2} \langle \Psi_n | \Delta(\delta\sigma) | \Psi_n \rangle, \end{aligned} \quad (4.5)$$

where we used (A.26). However, the first eigenvalue $\Lambda_0 = m^2$ does not change: $\delta\Lambda_0 = 0$. We then get

$$\begin{aligned} \delta\tilde{\zeta}(s) &= 2s \sum_{n \neq 0} \int d^2x \sqrt{g} \delta\sigma(x) \frac{\Psi_n(x)^\dagger \Psi_n(x)}{\Lambda_n^s} \\ &\quad - 2m^2 s \sum_{n \neq 0} \int d^2x \sqrt{g} \delta\sigma(x) \frac{\Psi_n(x)^\dagger \Psi_n(x)}{\Lambda_n^{s+1}} \\ &\quad + s \sum_{n \neq 0} \int d^2x \sqrt{g} \frac{\partial_\nu(\delta\sigma) \Psi_n(x)^\dagger \gamma^{\mu\nu} \nabla_\mu \Psi_n(x)}{\Lambda_n^{s+1}} \\ &\quad + \frac{s}{2} \sum_{n \neq 0} \int d^2x \sqrt{g} \Delta(\delta\sigma) \frac{\Psi_n(x)^\dagger \Psi_n(x)}{\Lambda_n^{s+1}} \end{aligned} \quad (4.6)$$

$$=: 2s I_1 - 2m^2 s I_2 + s I_3 + \frac{s}{2} I_4. \quad (4.7)$$

One can show that

$$I_1 = \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s, x, x), \quad (4.8a)$$

$$I_2 = \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s+1, x, x), \quad (4.8b)$$

$$I_3 = -\frac{1}{2} \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D (\Delta \tilde{\zeta}(s+1, x, x)), \quad (4.8c)$$

$$I_4 = \int d^2x \sqrt{g} \delta\sigma \text{tr}_D (\Delta \tilde{\zeta}(s+1, x, x)). \quad (4.8d)$$

We define the Laplacian acting on the zeta function below, see (4.14). The first two expressions are immediate consequences of the definition (3.20) of the zeta function. Moreover, I_4 follows directly from an integration by parts, and only I_3 is non-trivial.

The first step is to rewrite the derivative acting on $\delta\sigma$ as a covariant derivative, and then integrate by parts:

$$\begin{aligned} I_3 &= \sum_{n \neq 0} \frac{1}{\Lambda_n^{s+1}} \int d^2x \sqrt{g} \delta\sigma \left(\Psi_n(x)^\dagger \overleftarrow{\nabla}_\nu \gamma^{\mu\nu} \nabla_\mu \Psi_n(x) + \Psi_n(x)^\dagger \gamma^{\mu\nu} \nabla_\nu \nabla_\mu \Psi_n(x) \right) \\ &= \sum_{n \neq 0} \frac{1}{\Lambda_n^{s+1}} \int d^2x \sqrt{g} \delta\sigma \left(\Psi_n(x)^\dagger \overleftarrow{\nabla} \not{\Psi} \Psi_n(x) + \Psi_n(x)^\dagger \overleftarrow{\nabla}^\mu \nabla_\mu \Psi_n(x) \right. \\ &\quad \left. - \frac{R}{4} \Psi_n(x)^\dagger \Psi_n(x) \right). \end{aligned} \quad (4.9)$$

The second equality follows using $\gamma^{\mu\nu} = \gamma^\mu \gamma^\nu - g^{\mu\nu}$ and (A.14). We will now compute the first term, before turning our attention to the second and third terms together.

We notice that, for $n \neq 0$, $\not{\Psi} \Psi_n$ is also an eigenvector of D^2 associated with the eigenvalue Λ_n and normalized such that

$$\int d^2x \sqrt{g} (\not{\Psi} \Psi_n)^\dagger \not{\Psi} \Psi_n = - \int d^2x \sqrt{g} \Psi_n^\dagger \not{\Psi}^2 \Psi_n = \Lambda_n^{(0)} = \Lambda_n - m^2. \quad (4.10)$$

As the heat kernel is uniquely defined by (3.12), (3.21) implies that the zeta function is also uniquely defined and does not depend on the basis of eigenvectors. Then, one has

$$\tilde{\zeta}(s, x, y) = \sum_{n \neq 0} \frac{\not{\Psi} \Psi_n(x) (\not{\Psi} \Psi_n(y))^\dagger}{\Lambda_n^{(0)} \Lambda_n^s} = \sum_{n \neq 0} \frac{\not{\Psi} \Psi_n(x) (\not{\Psi} \Psi_n(y))^\dagger}{\Lambda_n^s (\Lambda_n - m^2)}. \quad (4.11)$$

Since $\Lambda_n = \Lambda_n^{(0)} + m^2$, we have $\Lambda_n \geq m^2$ for all $n \neq 0$, and we can expand the denominator as a series:

$$\tilde{\zeta}(s, x, y) = \sum_{n \neq 0} \frac{\not{\Psi} \Psi_n(x) (\not{\Psi} \Psi_n(y))^\dagger}{\Lambda_n^{s+1}} \sum_{k \geq 0} \left(\frac{m^2}{\Lambda_n} \right)^k = \sum_{\substack{n \neq 0 \\ k \geq 0}} \frac{\not{\Psi} \Psi_n(x) (\not{\Psi} \Psi_n(y))^\dagger}{\Lambda_n^{s+1+k}} m^{2k}. \quad (4.12)$$

The term $k = 0$ corresponds to the first term in (4.9). We also see that shifting s by 1 and multiplying by m^2 gives the same sum, but with a different initial value. This suggests

computing:

$$\begin{aligned}
& \tilde{\zeta}(s, x, y) - m^2 \tilde{\zeta}(s+1, x, y) \\
&= \sum_{\substack{n \neq 0 \\ k \geq 0}} \frac{\not{\nabla} \Psi_n(x) (\not{\nabla} \Psi_n(y))^\dagger}{\Lambda_n^{s+1+k}} m^{2k} - \sum_{\substack{n \neq 0 \\ k \geq 0}} \frac{\not{\nabla} \Psi_n(x) (\not{\nabla} \Psi_n(y))^\dagger}{\Lambda_n^{s+2+k}} m^{2k+2} \\
&= \sum_{n \neq 0} \frac{\not{\nabla} \Psi_n(x) (\not{\nabla} \Psi_n(y))^\dagger}{\Lambda_n^{s+1}}.
\end{aligned} \tag{4.13}$$

Next, we need to evaluate the other terms in (4.9). We expect the second term to be related to $\Delta \tilde{\zeta}$, since we obtain it by distributing one derivative on each spinor inside $\tilde{\zeta}$:

$$\begin{aligned}
\text{tr}_D \Delta \tilde{\zeta}(s, x, x) &= \text{tr}_D \nabla^\mu \nabla_\mu \tilde{\zeta}(s+1, x, x) = \text{tr}_D \nabla^\mu \nabla_\mu \sum_{n \neq 0} \frac{\Psi_n(x) \Psi_n(x)^\dagger}{\Lambda_n^s} \\
&= \text{tr}_D \sum_{n \neq 0} \frac{1}{\Lambda_n^s} \left((\Delta \Psi_n(x)) \Psi_n(x)^\dagger + \Psi_n(x) \Psi_n(x)^\dagger \overleftarrow{\Delta} + 2 \nabla_\mu \Psi_n(x) \Psi_n(x)^\dagger \overleftarrow{\nabla}^\mu \right) \\
&= \text{tr}_D \sum_{n \neq 0} \frac{1}{\Lambda_n^s} \left((\Delta \Psi_n(x)) \Psi_n(x)^\dagger + \Psi_n(x) (\Delta \Psi_n(x))^\dagger + 2 \nabla_\mu \Psi_n(x) (\nabla^\mu \Psi_n(x))^\dagger \right).
\end{aligned} \tag{4.14}$$

In the last line, all covariant derivatives and Laplacians act on spinors. Note that there is no ambiguity on the location of the matrices thanks to the presence of the trace. We can then replace Δ using (2.12), before combining with the third term from (4.9) in R and acting with D^2 on the eigenstates:

$$\begin{aligned}
2 \text{tr}_D \sum_{n \neq 0} \frac{\nabla_\mu \Psi_n(x) (\nabla^\mu \Psi_n(x))^\dagger}{\Lambda_n^s} &= \text{tr}_D \Delta \tilde{\zeta}(s, x, x) + 2 \text{tr}_D \tilde{\zeta}(s-1, x, x) \\
&\quad - 2 \left(\frac{R}{4} + m^2 \right) \text{tr}_D \tilde{\zeta}(s, x, x).
\end{aligned} \tag{4.15}$$

Putting all the pieces together with $s \rightarrow s+1$ gives the expression for I_3 in (4.8).

Note that $I_3 + I_4/2 = 0$, such that the total variation of the zeta function is:

$$\delta \tilde{\zeta}(s) = 2s \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s, x, x) - 2m^2 s \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s+1, x, x) \tag{4.16}$$

while its derivative is

$$\begin{aligned}
\delta \tilde{\zeta}'(s) &= 2s \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}'(s, x, x) + 2 \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s, x, x) \\
&\quad - 2m^2 \left(s \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}'(s+1, x, x) + \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s+1, x, x) \right).
\end{aligned} \tag{4.17}$$

To compute the variation of the gravitational action, one needs the value of $\delta \tilde{\zeta}$ and $\delta \tilde{\zeta}'$ at $s = 0$. They are given by

$$\delta \tilde{\zeta}(0) = -2m^2 \lim_{s \rightarrow 0} s \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s+1, x, x) = -\frac{m^2}{\pi} \int d^2x \sqrt{g} \delta\sigma(x) \tag{4.18}$$

and

$$\begin{aligned} \delta\tilde{\zeta}'(0) &= 2 \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(0, x, x) \\ &\quad - 2m^2 \lim_{s \rightarrow 0} \left[\left(1 + s \frac{d}{ds}\right) \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s+1, x, x) \right]. \end{aligned} \quad (4.19)$$

We recover the same formula as in [31] for the case of the massive scalar field.

In the massive case, to compute the first term in (4.19), we use the fact that

$$\tilde{\zeta}(0, x, x) = \zeta(0, x, x) - P(x, x) = \frac{a_1(x, x)}{4\pi} - P(x, x) \quad (4.20)$$

such that

$$\begin{aligned} \int d^2x \sqrt{g} \delta\sigma \text{tr}_D \tilde{\zeta}(0, x, x) &= -\frac{1}{24\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D P(x, x) \\ &\quad - \frac{m^2}{2\pi} \int d^2x \sqrt{g} \delta\sigma(x). \end{aligned} \quad (4.21)$$

To compute the second term, we recognize ζ_{reg} from (3.25):

$$\lim_{s \rightarrow 0} \left[\left(1 + s \frac{d}{ds}\right) \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}(s+1, x, x) \right] = \int d^2x \sqrt{g} \delta\sigma \tilde{\zeta}_{\text{reg}}(1, x, x). \quad (4.22)$$

Putting the pieces together, we find that

$$\begin{aligned} \delta\zeta'(0) &= -\frac{1}{12\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x) - 2 \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D P(x, x) \\ &\quad - \frac{m^2}{\pi} \int d^2x \sqrt{g} \delta\sigma(x) - 2m^2 \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{\zeta}_{\text{reg}}(1, x, x). \end{aligned} \quad (4.23)$$

The first term is recognized to be the variation of the Liouville action

$$\delta S_L = \frac{1}{4\pi} \int d^2x \sqrt{g} \delta\sigma(x) R(x). \quad (4.24)$$

The second term corresponds to the variation of the zero-mode normalization matrix (3.44). If there is a single constant zero-mode, like for the flat torus (3.50), this term is related to the variation of the area:

$$\int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D P(x, x) = \delta \ln \det \kappa = -2 \frac{\delta A}{A}. \quad (4.25)$$

The third term contains is related to the variation of the Mabuchi action [23]:

$$\delta S_M \propto \int d^2x \sqrt{g} \delta\sigma + \dots \quad (4.26)$$

The infinitesimal variation of the gravitational action is

$$\begin{aligned} \delta S_{\text{grav}} &= -\frac{1}{12} \delta S_L - \frac{1}{2} \delta \ln \det \kappa \\ &\quad - \frac{m^2}{2} \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \left(\tilde{\zeta}_{\text{reg}}(1, x, x) + \frac{\ln \mu^2}{4\pi} + \frac{1}{4\pi} \right). \end{aligned} \quad (4.27)$$

We can express this in terms of the regularized Green's function by noting that

$$\tilde{\zeta}_{\text{reg}}(1, x, x) + \frac{\ln \mu^2}{4\pi} = \tilde{G}_\zeta(x) \quad (4.28)$$

so that

$$\delta S_{\text{grav}} = -\frac{1}{12} \delta S_L - \frac{1}{2} \delta \ln \det \kappa - \frac{m^2}{2} \int d^2x \sqrt{g} \delta \sigma(x) \text{tr}_D \left(\tilde{G}_\zeta(x) + \frac{1}{4\pi} \right) \quad (4.29a)$$

$$= -\frac{1}{12} \delta S_L - \frac{m^2}{2} \int d^2x \sqrt{g} \delta \sigma(x) \text{tr}_D \left(G_\zeta(x) + \frac{1}{4\pi} \right). \quad (4.29b)$$

Now we want to express δS_{grav} as the variation of some functional. For this, we note that

$$\int d^2x \sqrt{g} \delta \sigma \text{tr}_D G_\zeta(x) = \frac{1}{2} \delta \left(\int d^2x \sqrt{g} \text{tr}_D G_\zeta(x) \right) - \frac{1}{2} \int d^2x \sqrt{g} \text{tr}_D \delta G_\zeta(x). \quad (4.30)$$

Using (3.35) and (3.38), computations similar to the previous steps lead to

$$\int d^2x \sqrt{g} \text{tr}_D \delta G_\zeta(x) = -2m^2 \int d^2x \sqrt{g} \delta \sigma(x) \text{tr}_D \zeta(2, x, x) + \frac{1}{2\pi} \int d^2x \sqrt{g} \delta \sigma(x). \quad (4.31)$$

This gives:

$$\delta S_{\text{grav}} = -\frac{1}{12} \delta S_L - \frac{m^2}{4} \delta \left(\int d^2x \sqrt{g} \text{tr}_D G_\zeta(x) \right) - \frac{m^4}{2} \int d^2x \sqrt{g} \delta \sigma(x) \text{tr}_D \zeta(2, x, x). \quad (4.32)$$

To go further, we show that

$$m^4 \int d^2x \sqrt{g} \delta \sigma(x) \text{tr}_D \zeta(2, x, x) = \frac{1}{4} \int_0^\infty dt \frac{e^{m^2 t} - m^2 t - 1}{t} \delta \tilde{K}(t) + \int d^2x \sqrt{g} \delta \sigma(x) \text{tr}_D P(x, x) \quad (4.33)$$

in appendix D. Reporting in (4.32), we get

$$\delta S_{\text{grav}} = -\frac{1}{12} \delta S_L - \frac{1}{2} \delta \ln \det \kappa - \frac{m^2}{4} \delta \left(\int d^2x \sqrt{g} \text{tr}_D G_\zeta(x) \right) - \frac{1}{8} \delta \left(\int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) \tilde{K}(t) \right). \quad (4.34)$$

Because (4.34) is a total variation, it can be integrated:

$$S_{\text{grav}}[\hat{g}, g] = -\frac{1}{12} S_L[\hat{g}, g] - \frac{1}{2} \ln \det \frac{\kappa[g]}{\kappa[\hat{g}]} - \frac{m^2}{4} \int d^2x \text{tr}_D (\sqrt{g} G_{g,\zeta}(x) - \sqrt{\hat{g}} G_{\hat{g},\zeta}(x)) - \frac{1}{8} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) (\tilde{K}_g(t) - \tilde{K}_{\hat{g}}(t)). \quad (4.35)$$

A last simplification can be achieved by noting that the difference between G and \tilde{G} is the same constant, proportional to the number of zero-modes, using (3.5) and (3.42). This, it

cancels and one can replace G by \tilde{G}

$$\begin{aligned}
 S_{\text{grav}}[\hat{g}, g] &= -\frac{1}{12} S_L[\hat{g}, g] - \frac{1}{2} \ln \det \frac{\kappa[g]}{\kappa[\hat{g}]} \\
 &\quad - \frac{m^2}{4} \int d^2x \operatorname{tr}_D (\sqrt{g} \tilde{G}_{g,\zeta}(x) - \sqrt{\hat{g}} \tilde{G}_{\hat{g},\zeta}(x)) \\
 &\quad - \frac{1}{8} \int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) (\tilde{K}_g(t) - \tilde{K}_{\hat{g}}(t)).
 \end{aligned} \tag{4.36}$$

The first two terms, which would be the only ones in the massless case, arise from the conformal anomaly and agree with the standard result, giving the Liouville action for a CFT with central charge $c = 1/2$ [53].

Note that $\ln \det \kappa[g]/\kappa[\hat{g}]$ can be rewritten as a determinant inside the path integral: it takes into account the redefinition of the modes that would become zero-modes in the massless limit. To make contact with the scalar field computation of [31], this expression gives the correct factor $\ln(A/\hat{A})$ using any choice of normalization (see footnote 7). When there is a single constant zero-mode, such as for the flat torus (3.50), we get:

$$-\frac{1}{2} \ln \det \frac{\kappa[g]}{\kappa[\hat{g}]} = \ln \frac{A}{\hat{A}}. \tag{4.37}$$

The expression (4.36) coincides with the result for the scalar field from [31] up to factors $\pm 1/2$. One can guess that the different signs come from the statistics of the fermion, and the factor $1/2$ from the central charge. Studying new models would be useful for checking this last guess. Moreover, it would be very interesting to interpret these facts in view of supersymmetry.

4.2 Small mass expansion

The goal of this section is to study the small mass expansion of (4.36).

First, one notes that

$$\int_0^\infty \frac{dt}{t} (e^{m^2 t} - m^2 t - 1) K(t) = O(m^4). \tag{4.38}$$

Then, because of the factor m^2 in front of \tilde{G}_ζ , it is sufficient to consider the limit $m \rightarrow 0$ of \tilde{G} , which is well-defined since we removed the zero-modes. As a consequence, the action (4.36) reduces to

$$\begin{aligned}
 S_{\text{grav}}[\hat{g}, g] &= -\frac{1}{12} S_L[\hat{g}, g] - \frac{1}{2} \ln \det \frac{\kappa[g]}{\kappa[\hat{g}]} \\
 &\quad - \frac{m^2}{4} \int d^2x \operatorname{tr}_D (\sqrt{g} G_{g,\zeta}^{(0)}(x) - \sqrt{\hat{g}} G_{\hat{g},\zeta}^{(0)}(x)) + O(m^4).
 \end{aligned} \tag{4.39}$$

The symbol $O(m^4)$ is omitted from now on. It can be useful to work directly with the variation

$$\delta S_{\text{grav}} = -\frac{1}{12} \delta S_L - \frac{1}{2} \delta \ln \det \kappa - \frac{m^2}{4} \delta \left(\int d^2x \sqrt{g} \operatorname{tr}_D \tilde{G}_\zeta^{(0)}(x) \right) \tag{4.40a}$$

$$= -\frac{1}{12} \delta S_L - \frac{1}{2} \delta \ln \det \kappa - \frac{m^2}{2} \int d^2x \sqrt{g} \delta \sigma(x) \operatorname{tr}_D \left(\tilde{G}_\zeta^{(0)}(x) + \frac{1}{4\pi} \right) \tag{4.40b}$$

$$= -\frac{1}{12} \delta S_L - \frac{m^2}{2} \int d^2x \sqrt{g} \delta \sigma(x) \operatorname{tr}_D \left(G_\zeta^{(0)}(x) + \frac{1}{4\pi} \right). \tag{4.40c}$$

If we directly follow the scalar case, then one can introduce the averaged integrated Green function

$$\Psi_G[g] = \frac{1}{A} \int d^2x \sqrt{g} \operatorname{tr}_D G_{g,\zeta}^{(0)}(x). \quad (4.41)$$

In this case, the above expression becomes

$$S_{\text{grav}}[\hat{g}, g] = -\frac{1}{12} S_L[\hat{g}, g] - \frac{1}{2} \ln \det \frac{\kappa[g]}{\kappa[\hat{g}]} - \frac{m^2 A}{4} (\Psi_G[g] - \Psi_G[\hat{g}]) + \frac{m^2}{4} (A - \hat{A}) \Psi_G[\hat{g}]. \quad (4.42)$$

The last term is independent of g and contributes to the cosmological constant. The second term becomes the usual $\ln A/\hat{A}$ when there is a single constant zero-mode, see (3.50).

Note that the variation of the last term in (4.40) can also be integrated directly to $\sigma e^{2\sigma}$, which shows that the $O(m^2)$ term should contain the Mabuchi action [23, 41]:

$$S_M[\hat{g}, \sigma] = \frac{4}{A} \int d^2x \sqrt{\hat{g}} \sigma e^{2\sigma} + \dots \quad (4.43)$$

Note that the coefficient is $-m^2 A/16\pi$, to be compared with a factor $+m^2 A/16\pi$ for a scalar field [23, 41]. As noted above, this may be related to supersymmetry.

We refrain from providing explicit formulas for (4.42) in terms of the \hat{g} -dependent Green functions, since they are not particularly insightful. However, we highlight the procedure to compute the finite variation of $\Psi_G[g]$. From (4.41), we see that we need the finite variation of the regularized Green function $G_{g,\zeta}(x)$, which can be obtained from the variation of the Green function $G(x, y)$. The latter can be found from the finite transformation of the Green function $\tilde{S}^{(0)}(x, y)$ using [57, eq. (3.32)]:

$$\begin{aligned} \tilde{S}_{g,xy}^{(0)} &= e^{-\frac{\phi_x}{2}} \tilde{S}_{\hat{g},xy}^{(0)} e^{-\frac{\phi_y}{2}} + \int d^2w \sqrt{g} \int d^2z \sqrt{g} P_{g,xw} e^{-\frac{\phi_w}{2}} \tilde{S}_{\hat{g},wz}^{(0)} e^{-\frac{\phi_z}{2}} P_{g,zy} \\ &\quad - \int d^2z \sqrt{g} P_{g,xz} e^{-\frac{\phi_z}{2}} \tilde{S}_{\hat{g},zy}^{(0)} e^{-\frac{\phi_y}{2}} - \int d^2z \sqrt{g} e^{-\frac{\phi_x}{2}} \tilde{S}_{\hat{g},xz}^{(0)} e^{-\frac{\phi_z}{2}} P_{g,zy} \end{aligned} \quad (4.44)$$

(to simplify the notations, we write the coordinate dependence as a subscript). This is because the Green function must be orthogonal to the zero-modes, so we need $S \sim (1 - P_{\hat{g}})\Sigma(1 - P_{\hat{g}})$, where $\Sigma = e^{-(\sigma(x)+\sigma(y))/2} S_{\hat{g}}$ since the Dirac operator transforms covariantly under Weyl transformations. This generalizes the formula found in [58–62] for the massless scalar. The projector (3.41) can be written in terms of the \hat{g} zero-modes as:

$$P_g(x, y) = e^{-\frac{\sigma(x)}{2} - \frac{\sigma(y)}{2}} \sum_{i,j} \hat{\Psi}_{0,i}(x) \kappa^{ij}[g] \hat{\Psi}_{0,j}^\dagger(y), \quad (4.45)$$

but there is no simple formula in terms of $P_{\hat{g}}$ because of the presence of $\kappa[g]$.

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A Formulas and conventions

Greek indices refer to the curved frame and Latin indices (a, b , etc.) to the local flat frame. Explicit indices are denoted by letters in the first case, $\mu = (t, x)$, and numbers in the second case, $a = 0, 1$. We globally follow the conventions of [49, 52].

The metric of the two-dimensional compact curved manifold with Euclidean metric $g_{\mu\nu}$, where $\mu = 0, 1$. Zweibeine e_a^μ are introduced to convert curved indices to local indices:

$$\delta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu} = \text{diag}(1, 1). \quad (\text{A.1})$$

The antisymmetric Levi-Civita symbol ϵ_{ab} is normalized as

$$\epsilon_{01} = \epsilon^{01} = 1. \quad (\text{A.2})$$

Complex coordinates are defined by

$$z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1 \quad (\text{A.3})$$

such that

$$ds^2 = dzd\bar{z}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = \frac{1}{2}. \quad (\text{A.4})$$

The derivatives with respect to the complex coordinates are:

$$\partial := \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \quad \bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1). \quad (\text{A.5})$$

The integration measure is

$$d^2z = 2d^2x, \quad d^2z := dzd\bar{z}, \quad d^2x := dx^0 dx^1. \quad (\text{A.6})$$

A.1 Covariant derivatives

The covariant derivative is denoted by

$$\nabla_\mu := \partial_\mu + \Gamma_\mu, \quad (\text{A.7})$$

where Γ_μ is the connection which contains a combination of the Levi-Civita $\Gamma_{\mu\nu}^\rho$ and spin $\omega_{\mu a}{}^b$ connections depending on the object it acts on. The Laplacian is defined by

$$\Delta := g^{\mu\nu} \nabla_\mu \nabla_\nu. \quad (\text{A.8})$$

The covariant derivative of a fermion reads

$$\nabla_\mu \Psi = \left(\partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \right) \Psi = \left(\partial_\mu - \frac{i}{4} \omega_\mu \gamma_* \right) \Psi, \quad (\text{A.9})$$

where γ_* is defined in (B.5) and:

$$\omega_\mu := \omega_{\mu ab} \epsilon^{ab}. \quad (\text{A.10})$$

The covariant derivative of gamma matrices vanishes:

$$\nabla_\mu \gamma^\nu = 0. \tag{A.11}$$

The square of the Dirac operator $\not{\nabla}$ is:

$$\not{\nabla}^2 = \Delta - \frac{R}{4}, \tag{A.12}$$

where the spinorial Laplacian is

$$\Delta \Psi = g^{\mu\nu} \left[\left(\partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \right) \nabla_\nu \Psi - \Gamma_{\mu\nu}^\rho \nabla_\rho \Psi \right]. \tag{A.13}$$

We have used:

$$\gamma^{\mu\nu} [\nabla_\mu, \nabla_\nu] = -\frac{R}{2}. \tag{A.14}$$

The Dirac operator together with a mass term is denoted by

$$D := i\not{\nabla} + m\gamma_*. \tag{A.15}$$

The square of this operator reads

$$D^2 = -\Delta + \frac{R}{4} + m^2, \tag{A.16}$$

which follows from $\gamma_*^2 = 1$ and $[\gamma^\mu, \gamma_*] = 0$. Due to the additional γ_* in the mass term, and contrary to the usual case, it is not necessary to conjugate the D operator by γ_* in order to obtain the r.h.s. .

A.2 Conformal variations

The conformal variation of the metric reads

$$g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu}, \tag{A.17}$$

which implies:

$$e_\mu^a = e^\sigma \hat{e}_\mu^a. \tag{A.18}$$

Then, the affine et spin connections transform as:

$$\Gamma_{\nu\rho}^\mu = \hat{\Gamma}_{\nu\rho}^\mu + \delta_\nu^\mu \partial_\rho \sigma + \delta_\rho^\mu \partial_\nu \sigma - \hat{g}_{\nu\rho} \partial^\mu \sigma, \tag{A.19}$$

$$\omega_{\mu ab} = \hat{\omega}_{\mu ab} + \hat{e}_{\mu a} \hat{e}_b^\nu \partial_\nu \sigma - \hat{e}_{\mu b} \hat{e}_a^\nu \partial_\nu \sigma. \tag{A.20}$$

This gives the expression for the covariant derivative for a spinor:

$$\nabla_\mu \Psi = \hat{\nabla}_\mu \Psi + \frac{1}{2} \hat{e}_{\mu a} \hat{e}_b^\nu \partial_\nu \sigma \gamma^{ab} \Psi, \tag{A.21}$$

such that

$$\not{\nabla} = e^{-\sigma} \left(\hat{\not{\nabla}} + \frac{1}{2} \partial_\mu \sigma \hat{\gamma}^\mu \right). \tag{A.22}$$

The transformation of the spinor Laplacian Δ is:

$$\Delta = e^{-2\sigma} \left(\hat{\Delta} + \partial_\nu \sigma \hat{\gamma}^{\mu\nu} \hat{\nabla}_\mu - \frac{1}{4} \partial_\mu \sigma \partial^\mu \sigma \right). \quad (\text{A.23})$$

Finally, from the transformation of the Ricci scalar

$$R = e^{-2\sigma} (\hat{R} - 2 \hat{\Delta} \sigma), \quad (\text{A.24})$$

we find:

$$\nabla^2 = e^{-2\sigma} \left(-\hat{\Delta} + \frac{1}{4} \hat{R} - \partial_\nu \sigma \hat{\gamma}^{\mu\nu} \hat{\nabla}_\mu + \frac{1}{4} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} \hat{\Delta} \sigma \right). \quad (\text{A.25})$$

From the last formula, we can deduce the infinitesimal variation of the operator D^2 :

$$\delta D^2 = -2 \delta \sigma \left(-\Delta + \frac{R}{4} \right) - \partial_\nu (\delta \sigma) \gamma^{\mu\nu} \nabla_\mu - \frac{1}{2} \Delta (\delta \sigma). \quad (\text{A.26})$$

B Two-dimensional spinors

This appendix summarizes the main properties of spinors and gamma matrices. For general references, the reader is referred to [49, app. 7.5, 8.5], [52, chap. 2, 3], [63], [51, sec. 13.2]. In this appendix, we will work with local indices.

B.1 Clifford algebra and gamma matrices

The two-dimensional identity is denoted by

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{B.1})$$

or with 1 when no confusion is possible.

The SO(2) Clifford algebra is generated by the two gamma matrices γ^a satisfying the anticommutation relation

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}. \quad (\text{B.2})$$

Both matrices are taken to be unitary, and as a consequence Hermitian

$$(\gamma^a)^\dagger = \gamma^a. \quad (\text{B.3})$$

The last element of the algebra corresponds to the antisymmetric product

$$\gamma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b] = -i \epsilon^{ab} \gamma_*, \quad (\text{B.4})$$

which is proportional to the chirality matrix

$$\gamma_* = i \gamma^0 \gamma^1 = \frac{i}{2} \epsilon_{ab} \gamma^{ab}. \quad (\text{B.5})$$

It corresponds to the generator of the SO(2) group

$$M^{ab} = \frac{i}{2} \gamma^{ab} = \epsilon^{ab} \gamma_*. \quad (\text{B.6})$$

The chirality matrix is Hermitian, unitary and anticommutes with other gamma matrices

$$\gamma_*^2 = 1, \quad (\gamma_*)^\dagger = \gamma_*, \quad \{\gamma_*, \gamma^a\} = 0. \quad (\text{B.7})$$

Some useful identities are

$$\gamma^a \gamma_b \gamma_a = 0 \quad (\text{B.8a})$$

$$\gamma^a \gamma^b = \eta^{ab} - i \varepsilon^{ab} \gamma_* \quad (\text{B.8b})$$

$$\gamma^{ab} \gamma_{bc} = \delta_c^a \quad (\text{B.8c})$$

$$\gamma_* \gamma^a = \varepsilon^{ab} \gamma_b \quad (\text{B.8d})$$

$$[\gamma^{ab}, \gamma^c] = 2\varepsilon^{ab} \varepsilon^{cd} \gamma_d. \quad (\text{B.8e})$$

B.2 Dirac and Majorana spinors

A Dirac spinor Ψ is a 2-dimensional complex vector with anticommuting components that forms a reducible representation of the Clifford algebra. Such a spinor transforms under a Lorentz transformation as

$$\delta\Psi = -\frac{1}{4} \lambda_{ab} M^{ab} \Psi = -\frac{\lambda}{4} \gamma_* \Psi, \quad \lambda := \lambda_{ab} \varepsilon^{ab}. \quad (\text{B.9})$$

The Dirac conjugation corresponds to Hermitian conjugation

$$\bar{\Psi} = \Psi^\dagger. \quad (\text{B.10})$$

This object transforms as

$$\delta\bar{\Psi} = \frac{\lambda}{4} \bar{\Psi} \gamma_* \quad (\text{B.11})$$

such that $\bar{\Psi}\Psi$ is a scalar.

Introducing the charge conjugation matrix C such that

$$(\gamma^\mu)^* = C\gamma^\mu C^{-1}, \quad (\gamma^\mu)^t = C\gamma^\mu C^{-1}, \quad (\text{B.12})$$

the charge conjugated spinor and its Dirac conjugate (giving the Majorana conjugate) are defined by

$$\Psi^c = C^{-1}\Psi^*, \quad \bar{\Psi}^c = \Psi^t C. \quad (\text{B.13})$$

Both spinors transform respectively as Ψ and $\bar{\Psi}$. Note that C is unitary and symmetric.

From a Dirac spinor, one can obtain two different irreducible representations: a Weyl (or chiral) spinor or a Majorana (or real) spinor. The latter is given by the reality condition

$$\Psi^c = \Psi \implies \Psi^* = C\Psi. \quad (\text{B.14})$$

This implies in particular that the Dirac and Majorana conjugations coincide. A Weyl spinor is obtained from a Dirac spinor by projecting it on its positive or negative chirality

$$\Psi = P_\pm \Psi, \quad P_\pm = \frac{1}{2}(1 \pm \gamma_*). \quad (\text{B.15})$$

One should note that it is not possible to have Majorana-Weyl fermion in Euclidean signature (contrary to what happens in Lorentz signature).

Given two Majorana spinors Ψ_1 and Ψ_2 , the Majorana flip relations read [51]

$$\bar{\Psi}_1 \Psi_2 = -\bar{\Psi}_2 \Psi_1, \quad \bar{\Psi}_1 \gamma_\mu \Psi_2 = -\bar{\Psi}_2 \gamma_\mu \Psi_1, \quad \bar{\Psi}_1 \gamma_* \Psi_2 = \bar{\Psi}_2 \gamma_* \Psi_1. \quad (\text{B.16})$$

B.3 Gamma matrix representations

B.3.1 Majorana basis

In the Majorana representation, the Dirac matrices read

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_* = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\text{B.17})$$

which implies that the charge conjugation is the identity

$$C = 1. \quad (\text{B.18})$$

In this basis, a Majorana spinor has real components

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1^* = \psi_1, \quad \psi_2^* = \psi_2, \quad (\text{B.19})$$

The scalar bilinears are

$$\bar{\Psi}\Psi = (\psi_1)^2 + (\psi_2)^2 = 0, \quad \bar{\Psi}\gamma_*\Psi = -2i\psi_1\psi_2, \quad (\text{B.20})$$

while the kinetic operator reads

$$i\gamma^\mu\partial_\mu = i\begin{pmatrix} \partial_1 & \partial_0 \\ \partial_0 & -\partial_1 \end{pmatrix}. \quad (\text{B.21})$$

B.3.2 Weyl basis

In the Majorana representation, the Dirac matrices read

$$\Gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_* = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{B.22})$$

which implies that the charge conjugation is:

$$C = \Gamma^1. \quad (\text{B.23})$$

In this basis, a Majorana spinor has complex components conjugated to each other

$$\Psi = \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix}, \quad \psi^* = \bar{\psi}, \quad \bar{\psi}^* = \psi. \quad (\text{B.24})$$

The relation with the Weyl components is

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2). \quad (\text{B.25})$$

The scalar bilinears are

$$\bar{\Psi}\Psi = \psi\bar{\psi} + \bar{\psi}\psi = 0, \quad \bar{\Psi}\gamma_*\Psi = 2\psi\bar{\psi}, \quad (\text{B.26})$$

while the kinetic operator reads

$$i\gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \bar{\partial} \\ -\partial & 0 \end{pmatrix}. \quad (\text{B.27})$$

C DDK ansatz

In their seminal papers [6, 7], David, Distler and Kawai (DDK) addressed two questions of importance: the rewriting of the functional integral in terms of a free-field measure for the Liouville mode⁸ and the gravitational dressing of matter operator. The latter has been used to provide an ansatz for the action associated to a CFT deformed by primary operators coupled to gravity or on a curved background (see [8, 39, 40, 64–75] for a selection of references). In practice, the ansatz is found by performing a conformal deformation of the gravitational action of the CFT with the additional constraint that these deformations have a conformal weight equal to one.

The first point received strong supports by explicit computations [33–35] and indirect approaches showing the consistency of the theory with a free-field measure [17, 20, 36–38]. However, the validity of the second is more questionable. We will discuss this and set it in the context of this paper in appendix C.2 after giving a general review of the DDK ansatz in appendix C.1.

C.1 Review of the DDK ansatz

The matter theory is described by the action S_{cft} of a CFT deformed by primary operators

$$S_m[g, \psi] = S_{\text{cft}}[g, \psi] + S_p[g, \psi], \tag{C.1a}$$

$$S_p[g, \psi] = \sum_i \lambda^i \int d^2x \sqrt{g} \mathcal{O}_i(\psi), \tag{C.1b}$$

where ψ denotes collectively the fields, λ^i the coupling constants, and $\mathcal{O}_i(\psi)$ is a set of primary operators⁹ with conformal weight h_i and built from the fields ψ . To simplify the discussion, we consider only spinless fields such that the conformal dimension is $2h_i$. Hence, a Weyl transformation acts as

$$g = e^{2\omega} \hat{g}, \quad \mathcal{O}_i = e^{-2h_i\omega} \hat{\mathcal{O}}_i. \tag{C.2}$$

The action S_{cft} is conformally invariant on flat space $g = \delta$, and we assume it invariant under Weyl transformations (see [80] for a discussion of this topic). On the other hand, the action S_p is not invariant if $h_i \neq 1$ for at least one operator. The trace of the energy-momentum tensor for the perturbation is

$$T^{(p)} = 4\pi \sum_i \lambda^i \mathcal{O}_i \tag{C.3}$$

(in the absence of explicit metric dependence in the operators \mathcal{O}_i) and displays the breaking of the conformal invariance by the deformations.

⁸It would be particularly interesting to extend this analysis to the case of non-conformal matter, in particular for the case of the Liouville-Mabuchi gravity. We are grateful to E. D’Hoker for discussions on this topic.

⁹While the operator \mathcal{O}_i can depend explicitly on the metric there is also an implicit metric dependence which is due to the regularization needed to remove self-contraction. This is discussed for example in [35, 76, 77], [78, sec. 3.6], [79] for the case of the scalar field.

The DDK ansatz proposes the total action in the conformal gauge to be given by

$$S_{\text{DDK}}[g, \sigma, \psi] = Q^2 S_L[\hat{g}, \sigma] + S_{\text{cft}}[\hat{g}, \psi] + S_{\text{DDK}}^{(p)}[\hat{g}, \sigma, \psi], \quad (\text{C.4a})$$

$$S_{\text{DDK}}^{(p)}[\hat{g}, \sigma, \psi] = \sum_i \lambda^i \int d^2x \sqrt{\hat{g}} e^{2a_i Q \sigma} \hat{\mathcal{O}}_i(\psi). \quad (\text{C.4b})$$

We recall that Q is related to the central charge by $6Q^2 = 26 - c_m$ (26 arises from the ghost contribution). The a_i are chosen such that each term has a conformal weight 1 and is thus invariant under the Weyl transformation

$$\hat{g} = e^{2\omega} \hat{g}', \quad \sigma = \sigma' - \omega, \quad \hat{\mathcal{O}}_i = e^{-2h_i \omega} \hat{\mathcal{O}}_i, \quad (\text{C.5})$$

which leads to the condition

$$a_i(Q - a_i) + h_i = 1 \quad (\text{C.6})$$

(the a_i^2 term comes from the regularization of the exponential). The solution is

$$a_i = \frac{Q}{2} - \sqrt{\frac{Q^2}{4} + h_i - 1} \quad (\text{C.7})$$

where the sign is found by matching to the semi-classical solution $Q \rightarrow \infty$

$$a_i \sim \frac{1}{Q}(1 - h_i). \quad (\text{C.8})$$

The multiplication of the matter primary by a Liouville primary is called gravitational dressing.

A derivation of (C.4) has been suggested in [72, sec. 3]. Starting from the matter partition function

$$Z_m[g] = \int \mathcal{D}\psi e^{-S_{\text{cft}}[g, \psi] - S_p[g, \psi]} = Z_{\text{cft}}[g] \left\langle \exp \left(-\lambda \int d^2\sigma \sqrt{g} \mathcal{O} \right) \right\rangle_{g, \text{cft}}, \quad (\text{C.9})$$

where $Z_{\text{cft}}[g]$ arises from the normalization of the correlation function, the exponential can be expanded perturbatively in the coupling constant

$$\frac{Z_m[g]}{Z_{\text{cft}}[g]} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{g, \text{cft}} \prod_{i=1}^n \sqrt{g(x_i)} d^2x_i. \quad (\text{C.10})$$

Since the correlation functions are computed in the CFT, one can use the relations (C.2) in order to express the quantities in the conformal gauge

$$\frac{Z_m[g]}{Z_{\text{cft}}[g]} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int \langle \hat{\mathcal{O}}(x_1) \cdots \hat{\mathcal{O}}(x_n) \rangle_{\hat{g}, \text{cft}} \prod_{i=1}^n e^{2(1-h_i)\sigma(x_i)} \sqrt{\hat{g}(x_i)} d^2x_i. \quad (\text{C.11})$$

The expansion can be resummed

$$\frac{Z_m[g]}{Z_{\text{cft}}[g]} = \frac{1}{Z_{\text{cft}}[\hat{g}]} \int d_{\hat{g}}\psi e^{-S_{\text{cft}}[\hat{g}, \psi] - S_{\text{DDK}}^{(p)}[\hat{g}, \sigma, \psi]}. \quad (\text{C.12})$$

Finally, (C.4) is recovered by using the relation

$$\ln \frac{Z_{\text{cft}}[g]}{Z_{\text{cft}}[\hat{g}]} = Q^2 S_L[\hat{g}, \sigma]. \tag{C.13}$$

The semi-classical result can be recovered by a direct computation with the action

$$S_p[g, \psi] = \sum_i \lambda^i \int d^2x \sqrt{g} \mathcal{O}_i = \sum_i \lambda^i \int d^2x \sqrt{\hat{g}} e^{2(1-h_i)\sigma} \hat{\mathcal{O}}_i = S_{\text{DDK}}^{(p)}[\hat{g}, \sigma, \psi]. \tag{C.14}$$

Note that this does not follow from a direct integration of the quantum expectation value of the trace (C.3)

$$\langle T^{(p)} \rangle_g = 4\pi \sum_i \lambda^i \langle \mathcal{O}_i \rangle_g \sim 4\pi \sum_i \lambda^i \mathcal{O}_{i,\text{cl}} \tag{C.15}$$

because one finds an additional factor of $(1 - h_i)^{-1}$.

C.2 Discussion

As mentioned in the introduction of this section, arguments showing that it is not clear whether the DDK ansatz of the gravitational action for a deformed CFT is valid can already be found in the existing literature or derived from elementary facts:

1. The action $S_{\text{DDK}}^{(p)}[g, \sigma, \psi]$ is not a Wess-Zumino action since it depends on both the matter fields and the Liouville mode. This means that the ansatz does not fit directly inside the conformal gauge formalism despite the appearance.

To be more explicit, compare the DDK ansatz (C.4) with the conformal gauge action (2.10) where the gravitational action would be of the form

$$S_{\text{grav}}[\hat{g}, \sigma] = Q^2 S_L[\hat{g}, \sigma] + S_{\text{grav}}^{(p)}[\hat{g}, \sigma]. \tag{C.16}$$

The discrepancy between S_{DDK} and S_{cg} can be summarised by the fact that the non-conformal contribution to the gravitational $S_{\text{grav}}^{(p)}$ and matter S_p actions have been replaced by a single term $S_{\text{DDK}}^{(p)}$.

Moreover, the definition of Q^2 in (C.4) contains the central charge of the ghosts and of every matter sector, which means that the different CFT sectors are not decoupled anymore. The conformal gauge action (2.10) does not show any sign of such coupling.

2. From the previous point, it follows that the DDK action cannot be recovered by integrating the trace anomaly or by computing the ratio of the partition functions in two different metrics.
3. The ansatz was proposed by requiring the total action to be conformally invariant term by term (whereas the validity of the conformal gauge approach asks only for the invariance of the total action). In the DDK approach, this translates into the fact that the gravitationally dressed deformations should be exactly marginal operators. However, this is generically not the case beyond the leading order in the coupling [39, 40, 81–85]. This can be understood from the fact that the presence of the operators inside the

action modifies the renormalization conditions and thus one cannot expect a tree-level condition to hold beyond the semi-classical level. This indicates that the DDK ansatz, beyond the first order approximation, should be supplemented with an infinite number of terms [83]. This may lead to a well-defined gravitational action, but computing the higher-order corrections is a formidable task and not more easy than a direct computation of the gravitational action in the conformal gauge (in particular for weights far from 1).

Let us illustrate this at the second order. Given the OPE coefficients C_{ij}^k

$$V_i(x)V_j(y) \sim C_{ij}^k |x - y|^{2(h_k - h_i - h_j)} V_k(y) \tag{C.17}$$

the beta functions for operators with $h_i \sim 1$ and for vanishing cosmological constant are

$$\beta^i = (\Delta_i^j - 2\delta_i^j)\lambda^j + \pi C_{jk}^i \lambda^j \lambda^k + O(\lambda^3) \tag{C.18}$$

where Δ_i^j is the normalization of the two-point function (the $O(\lambda^3)$ term has been computed in [83, 84]). One can add a term

$$S_{\text{DDK}}^{(p,2)} = \frac{\pi}{Q(1 + 4a_k)} C_{ij}^k \lambda^i \lambda^j \int d^2 \sqrt{\hat{g}} \sigma e^{2a_k Q \sigma} \hat{\mathcal{O}}_k(\psi) \tag{C.19}$$

to ensure that the beta functions vanish at quadratic order (in practice this implies that Δ_i^j becomes coupling-dependent). Note that, without this additional term, the vanishing of the beta functions at $O(\lambda)$ is obtained thanks to the gravitational dressing.

4. The derivation from [72] presented in the previous section may yield an incorrect result because of the formal manipulations in the functional integral. There are three possible sources of errors: 1) the correlation functions have logarithmic singularities [40, p. 4] which are not regularized, 2) the functional integral and infinite sum are exchanged (twice), 3) the infinite series (with divergent terms) is directly resummed. A rigorous computation would require to regularize the correlation functions, for example by introducing a connection on the CFT space [86–89].

Since the massive Majorana theory is a CFT with $c = 1/2$ deformed by a primary operator

$$S_{\text{cft}} = \frac{1}{4\pi} \int d^2 x \sqrt{g} \bar{\Psi} i \not{\partial} \Psi, \quad S_p[g, \psi] = \frac{m}{4\pi} \int d^2 x \sqrt{g} \bar{\Psi} \gamma_* \Psi, \tag{C.20}$$

the current paper provides an explicit example to test the DDK ansatz. The conformal weights of the holomorphic and anti-holomorphic components are respectively $(h, \bar{h}) = (1/2, 0)$ and $(h, \bar{h}) = (0, 1/2)$, such that the weight of the energy $\bar{\Psi} \gamma_* \Psi$ is $(h, \bar{h}) = (1/2, 1/2)$. Then, the DDK ansatz gives

$$\begin{aligned} S_{\text{DDK}}[g, \sigma, \psi] &= -Q^2 S_L[\hat{g}, \sigma] + S_{\text{cft}}[\hat{g}, \psi] + S_{\text{DDK}}^{(p)}[\hat{g}, \sigma, \psi], \\ S_{\text{DDK}}^{(p)}[\hat{g}, \sigma, \psi] &= \frac{m}{4\pi} \int d^2 x \sqrt{g} e^{2a Q \sigma} \bar{\Psi} \gamma_* \Psi, \end{aligned} \tag{C.21}$$

We do not provide the Q^2 and a because their values are model dependent (for example, if one adds other matter sectors). This action looks different from (4.36), which we obtained by the

rigorous heat kernel method. In particular, the DDK action (C.4b) is linear in the coupling constant while the gravitational action for the massive fermion is not. The action (4.36) is by construction consistent with the conformal gauge, see (2.10) and none of the points listed in the list applies to it. According to (4.39), the first-order correction will be proportional to m^2 instead of m for the DDK ansatz.

This can be summarized by saying that a *deformation of the matter CFT* does not seem to be equivalent to a *deformation of the Liouville plus matter CFT*, at least beyond the semi-classical approximation.¹⁰ A possible explanation is that symmetries are not sufficient to determine the form of the action when locality is lost, which is certainly the case in a theory of gravity with massive matter (for example, the Mabuchi action is non-local when expressed in terms of the Liouville field).

It would be interesting to establish our result using CFT methods, for example by performing the computations from [72] using the tools from [86–89].

Finally, note that these problems might just be apparent and result from a “bad” parametrization of the action and functional integrals. This would amount to prove that

$$\int \mathcal{D}\sigma \mathcal{D}\psi e^{-S_{\text{cft}}[\hat{g},\psi] - S_{\text{DDK}}^{(p)}[\hat{g},\sigma,\psi]} = \int \mathcal{D}\sigma \mathcal{D}\psi e^{-S_{\text{cft}}[\hat{g},\psi] - S_p[\hat{g},\psi] - S_{\text{grav}}^{(p)}[\hat{g},\psi]}. \quad (\text{C.22})$$

It is also possible that such a relation would hold only after incorporating all the corrections to the DDK action needed to make the beta functions vanish. While establishing directly this identity might be difficult, one may also prove it either by showing that the correlation functions computed from both sides satisfy the same Ward identities, or by computing numerically the correlation functions and showing that they agree.

Different interpretations can be given to the DDK ansatz, which is reflected by its various uses in the literature. The previous discussion applies *only* when it is used to postulate an action for a deformed CFT coupled to gravity. These comments do not apply when the DDK action is used by itself as a model (for example of a statistical system); instances of such approaches (even if there is a confusion with the previous case) include [6, 65, 71, 72, 74, 90]. It should be clear that this approach gives up the link with gravity and study the action and the associated functional integral just for themselves: as a consequence, any constraint originating from gravity can be relaxed (in particular the Wess-Zumino form and the marginal scaling conditions) and our comments do not apply. The fact that it defines a well-defined statistical system can also be motivated from the mapping to matrix models [73, 74], [39, sec. 4]. Similarly, the gravitational dressing of operators in correlation functions such as

$$\left\langle \prod_i e^{2a_i Q \sigma(x_i)} \mathcal{O}_i(x_i) \right\rangle \quad (\text{C.23})$$

do not pose any problems since the scaling dimensions are not modified in this case (for a discussion see [7, 23, 34, 91, 92]).

¹⁰See also [75, pp. 19–20] for related discussions.

D Proof of various identities

The goal of this section is to express $\int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \zeta(2, x, x)$ in terms of the heat kernel, i.e. to prove (4.33). The variation of $K(t)$ is given by

$$\begin{aligned} \delta K(t) &= \delta \tilde{K}(t) = -t \sum_{n \neq 0} e^{-\Lambda_n t} \delta \Lambda_n \\ &= -2t \left(\frac{d}{dt} + m^2 \right) \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{K}(t, x, x) \\ &\quad - t \sum_{n \neq 0} e^{-\Lambda_n t} \int d^2x \sqrt{g} \Psi_n^\dagger(x) \partial_\nu(\delta\sigma) \gamma^{\mu\nu} \nabla_\mu \Psi_n(x) \\ &\quad - \frac{t}{2} \int d^2x \sqrt{g} \Delta(\delta\sigma(x)) \text{tr}_D \tilde{K}(t, x, x). \end{aligned} \quad (\text{D.1})$$

To compute the second term, we introduce the generalized heat kernel

$$K(s, t, x, y) = \sum_n e^{-\Lambda_n t} \frac{\Psi_n(x) \Psi_n(y)}{\Lambda_n^s}. \quad (\text{D.2})$$

In particular

$$K(0, t, x, y) = K(t, x, y). \quad (\text{D.3})$$

We can then redo the computations we have done for the variation of the zeta function. For instance, the computation of I_3 in (4.8) can be generalized as

$$\sum_{n \neq 0} e^{-\Lambda_n t} \int d^2x \sqrt{g} \frac{\partial_\nu(\delta\sigma) \Psi_n^\dagger(x) \gamma^{\mu\nu} \nabla_\mu \Psi_n(x)}{\Lambda_n^{s+1}} = -\frac{1}{2} \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D(\Delta \tilde{K}(s+1, t, x, x)). \quad (\text{D.4})$$

Taking $s = -1$, we find that

$$\sum_{n \neq 0} e^{-\Lambda_n t} \int d^2x \sqrt{g} \partial_\nu(\delta\sigma) \Psi_n^\dagger(x) \gamma^{\mu\nu} \nabla_\mu \Psi_n(x) = -\frac{1}{2} \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D(\Delta \tilde{K}(t, x, x)). \quad (\text{D.5})$$

We then have

$$\delta K(t) = -2t \left(\frac{d}{dt} + m^2 \right) \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{K}(t, x, x). \quad (\text{D.6})$$

Note that the differential equations satisfied by K and $K^{(0)}$ imply that

$$K(t, x, y) = e^{-m^2 t} K^{(0)}(t, x, y). \quad (\text{D.7})$$

The eigenfunctions expansions show that the same is true between \tilde{K} and $\tilde{K}^{(0)}$, which implies

$$\delta K(t) = -2te^{-m^2 t} \frac{d}{dt} \int d^2x \sqrt{g} \delta\sigma(x) \text{tr}_D \tilde{K}^{(0)}(t, x, x). \quad (\text{D.8})$$

We then have

$$\begin{aligned}
\int_0^\infty dt \frac{e^{m^2 t} - m^2 t - 1}{t} \delta \tilde{K}(t) &= -2 \left[\int d^2 x \sqrt{g} \delta \sigma \operatorname{tr}_D \tilde{K}^{(0)}(t, x, x) \right]_0^{+\infty} \\
&\quad + 2 \int_0^{+\infty} dt (m^2 t + 1) \left(m^2 + \frac{d}{dt} \right) \int d^2 x \sqrt{g} \delta \sigma \operatorname{tr}_D \tilde{K}(t, x, x) \\
&= -2 \int d^2 x \sqrt{g} \delta \sigma \operatorname{tr}_D \left[\tilde{K}^{(0)}(t, x, x) - \tilde{K}(t, x, x) \right]_0^{+\infty} \\
&\quad + 2m^2 \int dt \int d^2 x \sqrt{g} \delta \sigma \operatorname{tr}_D \tilde{K}(t, x, x) \\
&\quad + 2m^2 \int dt t \frac{d}{dt} \int d^2 x \sqrt{g} \delta \sigma \operatorname{tr}_D \tilde{K}(t, x, x) \\
&\quad + 2m^4 \int dt t \int d^2 x \sqrt{g} \delta \sigma \operatorname{tr}_D \tilde{K}(t, x, x) \\
&= 4m^4 \int d^2 x \sqrt{g} \delta \sigma(x) \operatorname{tr}_D \tilde{\zeta}(2, x, x) \\
&= 4m^4 \int d^2 x \sqrt{g} \delta \sigma(x) \operatorname{tr}_D \zeta(2, x, x) \\
&\quad - 4 \int d^2 x \sqrt{g} \delta \sigma(x) \sum_i \Psi_{0,i}^\dagger(x) \Psi_{0,i}(x). \tag{D.9}
\end{aligned}$$

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