$W^{2,p}$-estimates of the Stokes system with traction boundary conditions.

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Abstract

The article deals with the 3D stationary Stokes system under traction boundary conditions, in interior and exterior domains. In the interior domain case, we obtain solutions with $W^{2,p}$-regular velocity and $W^{1,p}$-regular pressure globally in the domain, under suitable assumptions on the data. In the exterior domain case, following up a conjecture by T. Hishida [16], we construct two solutions. The velocity part of both of them is $W^{2,p}$-regular, and the pressure part $W^{1,p}$-regular, in any vicinity of the boundary, with $p \in (1,\infty)$ determined by the assumptions on the data. In addition the velocity is L^s-integrable near infinity, for some $s > 3$ determined by the data. Moreover the velocity part of one of these solutions satisfies a zero flux condition on the boundary, whereas the pressure part of the other one is L^s-integrable near infinity, for some $s > 3/2$ also determined by the data. Both existence classes are subsets of uniqueness classes.

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1 Introduction.

We consider the Stokes system

$$-\Delta v + \nabla \varrho = f, \ \ \operatorname{div} v = 0 \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ and in the exterior domain $\overline{\Omega}^c := \mathbb{R}^3 \backslash \overline{\Omega}$, where the boundary $\partial\Omega$ is connected and of class C^2. System (1.1) is supplemented by traction conditions (also called Neumann boundary conditions)

$$\sum_{k=1}^{3} (\partial_j v_k + \partial_k v_j - \delta_{jk}\, \varrho)\, n_k^{(\Omega)} = b \ \text{ on } \partial\Omega \quad \text{for } \ 1 \le j \le 3, \tag{1.2}$$

where $n^{(\Omega)}$ denotes the outward unit normal to Ω. The functions f (volume force) and b (force orthogonal to the surface) are given, and u (velocity) and π (pressure) are the unknowns of problem (1.1), (1.2). Boundary conditions as in (1.2) arise, for example, in incompressible elastostatics and in free boundary value problems for incompressible viscous fluids.

If the Stokes system (1.1) in Ω or $\overline{\Omega}^c$ is supplemented by Dirichlet boundary conditions, then u and π are respectively $W^{2,p}$- and $W^{1,p}$-regular up to the boundary, provided the

data satisfy suitable assumptions. In the case of the interior domain Ω, this is well known since the beginning of the 1960s, due to the pioneering work of Cattabriga [3] and Ladyzhenskaya [19]. These two authors chose quite different accesses to their respective theory. Cattabriga reduced his results to the half-space case, in which a solution to (1.1), (1.2) may be constructed in a rather explicit form. Ladyzhenskaya used the method of integral equations, that is, reduction of a boundary value problem to an integral equation on the boundary. In the monograph [13], Cattabriga's method is used to obtain a large range of L^p-estimates for interior and exterior Stokes flows under Dirichlet boundary conditions. Reference [8] derives some of these estimates ($W^{2,p}$-regularity of the velocity, $W^{1,p}$ of the pressure) for exterior Stokes flows, but by applying Ladyzhenskaya's approach.

As concerns the Stokes system (1.1) under traction condition (1.2), an L^2-theory, derived by Giaquinta, Modica [14], is available in literature; see [14, Theorem II.1.2]. Classical solutions were constructed by Starita, Tartaglione [23], via the method of integral equations. In addition these authors estimated the maximum modulus of the traction field in the direction of the normal to the boundary ([23, inequality (5.2)].) Problem (1.1), (1.2) is of a type considered by Agmon, Douglis, Nirenberg [2]. But the parameters associated with this problem in [2] are such that [2, Theorem 10.5], the main result in [2] on L^p-regularity and L^p-estimates, can be applied to solutions of (1.1), (1.2) only if their velocity part is $W^{2,p}$- regular, and their pressure part $W^{1,p}$-regular, in Ω in the interior domain case and in a neighbourhood of $\partial\Omega$ else. This is the same situation as in the Dirichlet case ([24, p. 23-24]). As far as we know, existence of solutions to (1.1), (1.2) with this level of regularity and corresponding estimates have not been established in previous literature if $p \neq 2$.

Our aim here is to fill this gap. Following [19] and [23], we will use the method of integral equations. It yields a solution to (1.1), (1.2), as well as an integral representation of that solution. This representation contains a single layer potential involving a fundamental solution of (1.1) and a "layer function" which solves a certain integral equation on $\partial\Omega$. The key step and main difficulty of our approach consists in estimating the $W^{2-1/p,p}$-norm of the Dirichlet boundary data of this single layer potential against the $W^{1-1/p,p}$-norm of its traction boundary data. In this way we are able to reduce $W^{2,p}$-estimates of solutions to (1.1), (1.2), in Ω and in $\overline{\Omega}^c$, to $W^{2,p}$-estimates of solutions to (1.1) in bounded domains, with Dirichlet conditions instead of (1.2); see the proof of Theorem 6.1 (exterior domain case) and Theorem 6.3 (interior domain case). The Dirichlet-to-Neumann estimate mentioned above is stated in Corollary 5.1, which is a consequence of Theorem 5.1 and 5.2. These two theorem provide the technical basis of Corollary 5.1; their proof requires considerable effort. Section 2 to 4 serve to set up a suitable framework and state auxiliary results. In Section 6, exploiting Corollary 5.1, we present our theory on existence and regularity, and in Section 7 we use some elements from this theory in order to prove uniqueness results.

Following up a conjecture by T. Hishida [16], we show in Section 6 that in general there are two solutions to (1.1), (1.2) in $\overline{\Omega}^c$ with velocity $W^{2,p}$-regular near the boundary and L^p-integrable near infinity, and with pressure $W^{1,p}$-regular near the boundary. The velocity part of one of these two solutions has zero flux through $\partial\Omega$, and the pressure part of the other one is L^p-integrable near infinity. The two solutions coincide only if the sum of b

and the traction boundary data of a solution to (1.1) in the whole space $\mathbb{R}^3$, with the zero extension of f to $\mathbb{R}^3$ as right-hand side, is L^2-orthogonal to the one-dimensional kernel of a certain boundary operator on $\partial\Omega$. Otherwise the difference of the two solutions possesses a non-constant velocity part (Theorem 6.5). These rather unexpected facts seem to have been unobserved up to now. The two existence classes are contained in uniqueness classes (Theorem 7.1).

Solutions to (1.1), (1.2) in interior domains with $W^{2,p}$-regular velocity and $W^{1,p}$-regular pressure are not unique either, but two such solutions only differ by a rigid motion (Theorem 7.2).

The main difficulties with our approach already arise if $f = 0$. Once this case is settled (Theorem 6.2 (exterior domain), 6.3 (interior domain)), the transition to non-vanishing f may be achieved in an obvious way: The Stokes system (1.1) is solved in the whole space $\mathbb{R}^3$, with the zero extension of f to $\mathbb{R}^3$ as right-hand side (Theorem 6.4). Then problem (1.1), (1.2) is solved with $f = 0$ and with the traction boundary data of the whole space solution added to the right-hand side of (1.2) ("boundary correction"). The sum of the two solutions yields the flow field which is looked for (Theorem 6.5 (exterior domain), Corollary 6.5 (interior domain)). Since solving problem (1.1) in $\mathbb{R}^3$ is not linked in any way to traction boundary conditions (1.2), the case $f \neq 0$ is not our main interest here. We deal with it only for completeness and because for the proof of our uniqueness results in Section 7, we need solutions to (1.1), (1.2) with f being an arbitrary C^∞-function in $\overline{\Omega}^c$ with bounded support. It is thus sufficient for our purposes to assume that $f \in L^p(\overline{\Omega}^c)^3$ for some $p \in (1,\, 3/2)$. Then a whole space solution is conveniently given by a volume potential, which is rather easy to handle. Actually the condition $p < 3/2$ may be removed by an approximation argument as in [13, p. 242-243], or by working with Kondratiev spaces in $\mathbb{R}^3$ with weights $(1 + |x|)^\beta$ ($x \in \mathbb{R}^3$), for suitable values $\beta \in \mathbb{R}$; see [21] for example. All this is not an issue when the interior domain is considered, because if $f \in L^p(\Omega)^3$ for whatever $p \in (1, \infty)$, we always have $f \in L^q(\Omega)^3$ for any $q \in [1, p]$.

Pointwise decay estimates of our solutions to (1.1), (1.2) in $\overline{\Omega}^c$ are provided in Corollary 6.4 under the assumption that f has bounded support. These estimates allow to determine L^p-regularity and L^p-estimates of exterior flows in more detail: We may distinguish regularity properties outside large balls from those valid in neighbourhoods of $\partial\Omega$ (Corollary 6.4, Theorem 6.2).

Our results are summarized in the following Theorem 1.1 (exterior flows) and 1.2 (interior flows).

Theorem 1.1 *Let $p \in (1,\, 3/2)$, $f \in L^p(\overline{\Omega}^c)^3$ and $b \in W^{1-1/p,\,p}(\partial\Omega)^3$. Then there is a unique pair $(v, \varrho) \in W^{2,p}_{loc}(\Omega^c)^3 \times W^{1,p}_{loc}(\Omega^c)$ satisfying (1.1), (1.2), the zero flux condition $\int_{\partial\Omega} v \cdot n^{(\Omega)}\, do_x = 0$, as well as the inequality*

$$\begin{aligned} &\|v\|_{(1/p-2/3)^{-1}} + \|\partial_l v\|_{(1/p-1/3)^{-1}} + \|\varrho + c\|_{(1/p-1/3)^{-1}} + \|\partial_m \partial_l v\|_p + \|\partial_l \varrho\|_p \qquad (1.3)\\ &\leq \mathfrak{C}\,(\|f\|_p + \|b\|_{1-1/p,\,p}) \quad (1 \leq l, m \leq 3), \end{aligned}$$

for some $c \in \mathbb{R}$ (which is of course uniquely determined). Supppose <u>in addition</u> that $q \in [3/2,\, 3)$, $f \in L^q(\overline{\Omega}^c)^3$ and $b \in W^{1-1/q,\,q}(\partial\Omega)^3$. Then $(v, \varrho) \in W^{2,q}_{loc}(\Omega^c)^3 \times W^{1,q}_{loc}(\Omega^c)$

and

$$\|\partial_l v\|_{(1/q-1/3)^{-1}} + \|\varrho + c\|_{(1/q-1/3)^{-1}} + \|\partial_m \partial_l v\|_q + \|\partial_l \varrho\|_q \tag{1.4}$$
$$\leq \mathfrak{C}\,(\|f\|_q + \|b\|_{1-1/q,\,q}), \quad \textit{for } l,\ m \textit{ as above.}$$

Further suppose that $r \in [3,\infty)$, $f \in L^r(\overline{\Omega}^c)^3$, $b \in W^{1-1/r,\,r}(\partial\Omega)^3$ *and* $R \in (0,\infty)$ *with* $\overline{\Omega} \subset B_R$. *Then* $(v,\varrho) \in W^{2,r}_{loc}(\Omega^c)^3 \times W^{1,r}_{loc}(\Omega^c)$ *and*

$$\|\partial_m \partial_l v\|_r + \|\partial_l \varrho\|_r \leq \mathfrak{C}\,(\|f\|_r + \|f|B^c_{2R}\|_q + \|b\|_{1-1/r,\,r}) \tag{1.5}$$

for l, m *as before. The constants* $\mathfrak{C}$ *in these estimates do not depend on* f *or* b.

Moreover there is one and only one pair $(\overline{v},\overline{\varrho}) \in W^{2,p}_{loc}(\Omega^c)^3 \times W^{1,p}_{loc}(\Omega^c)$ *such that the preceding statements remain valid with* $\overline{v}$, $\overline{\varrho}$ *in the role of* v *and* ϱ, *respectively, except that* $\int_{\partial\Omega} \overline{v}\cdot n^{(\Omega)}\,do_x \neq 0$ *in general and the constant* c *in (1.3) and (1.4) must be taken as zero.*

If $f = 0$, *the assumption* $p < 3/2$ *may be dropped (hence inequality (1.4) and (1.5) become special cases of (1.3) and therefore are no longer relevant).*

Theorem 1.2 *Define*

$$\phi^{(j)}(x) := (\delta_{jk})_{1\leq k\leq 3}, \quad \phi^{(4)}(x) := (x_3, 0, -x_1), \quad \phi^{(5)}(x) := (x_2, -x_1, 0),$$
$$\phi^{(6)}(x) := (0, x_3, -x_2) \quad \textit{for } 1 \leq j \leq 3,\ x \in \mathbb{R}^3 \quad \textit{("rigid motions").}$$

Let $p \in (1,\infty)$, $f \in L^p(\Omega)^3$, $b \in W^{1-1/p,\,p}(\partial\Omega)^3$ *such that* $\int_{\partial\Omega} b\cdot\phi^{(j)}\,do_x + \int_\Omega f\cdot\phi^{(j)}\,dx = 0$ *for* $1 \leq j \leq 6$. *Then there is a unique pair of functions* $(v,\varrho) \in W^{2,p}(\Omega)^3 \times W^{1,p}(\Omega)$ *satisfying (1.1) and (1.2), as well as the equation* $\int_\Omega v\cdot\phi^{(j)}\,dx = 0$ *for* $1 \leq j \leq 6$. *In addition, the estimate* $\|v\|_{2,p} + \|\varrho\|_{1,p} \leq \mathfrak{C}\,(\|f\|_p + \|b\|_{1-1/p,\,p})$ *holds, with a constant* $\mathfrak{C}$ *independent of* f *and* b.

Concerning the strange term $\|f|B^c_{2R}\|_q$ in (1.5), we think that in the case $r \geq 3$, an estimate of the form $\|\partial_m\partial_l u\|_r + \|\partial_l \pi\|_r \leq \mathfrak{C}\,(\|f\|_r + \|b\|_{1-1/r,\,r})$ cannot hold for all functions $f \in L^r(\overline{\Omega}^c)^3$ with the same constant $\mathfrak{C}$, even if $b = 0$. This is indicated by a similar situation in the Dirichlet case; see [5, Theorem 1.3] and compare with inequality [13, (V.4.46)]. In Theorem 6.5 we present a more detailed version of inequality 1.5, which may serve as starting point for removing the term $\|f|B^c_{2\,R}\|_q$ when (1.1) is solved in $\mathbb{R}^3$ by means of Kondratiev's theory.

We will use the fact, stated in Theorem 4.6, that the traction field of the Stokes double layer potential associated with $\partial\Omega$ is continuous on any line through $\partial\Omega$, provided this line is orthogonal to $\partial\Omega$ and the layer function of the potential is sufficiently smooth. In the context of the Laplace equation, this result is sometimes called "Lyapunov-Tauber theorem". Its version in Theorem 4.6 is frequently applied in analysis and numerics (boundary element method) of incompressible flows, but what is available as proof is not completely satisfying, as explained in the comment following Theorem 4.6. Therefore we find it worthwhile to present a proof; see Appendix.

We will rely on [19] with respect to a number of auxiliary results on the integral operators appearing in our proofs. However, we will not refer to [19] directly. Instead we will draw on results from [9], where the pertinent sections from [19] are worked out in detail.

2 Notation. Some auxiliary results.

The symbol $|\ |$ denotes the Euclidean norm of $\mathbb{R}^n$ for any $n \in \mathbb{N}$, the length $\alpha_1 + ... + \alpha_n$ of a multi-index $\alpha \in \mathbb{N}_0^n$, as well as the Borel measure of a measurable subset A of $\mathbb{R}^3$. For $R \in (0,\infty),\ x \in \mathbb{R}^3$, put $B_R(x) := \{y \in \mathbb{R}^3 : |x-y| < R\}$. In the case $x = 0$, we write B_R instead of $B_R(0)$. An open ball in $\mathbb{R}^2$ with radius $R > 0$ and centered in $\varrho \in \mathbb{R}^2$ is denoted by $B_R^2(\varrho)$.

The set $\Omega \subset \mathbb{R}^3$ introduced in Section 1 will be kept fixed throughout. Recall that Ω is open and bounded, with connected C^2-boundary, and that $n^{(\Omega)}$ denotes the outward unit normal to Ω. We put $\Omega_R := B_R \backslash \overline{\Omega}$.

For $n \in \mathbb{N},\ I \subset \mathbb{R}^n$, let χ_I stand for the characteristic function of I in $\mathbb{R}^n$. If $A \subset \mathbb{R}^3$, we denote by A^c the complement $\mathbb{R}^3 \backslash A$ of A in $\mathbb{R}^3$. Put $e_l := (\delta_{jl})_{1 \le j \le 3}$ for $1 \le l \le 3$ (unit vector in $\mathbb{R}^3$). If A is some nonempty set and $\gamma : A \mapsto \mathbb{R}^n$ a function for some $n \in \mathbb{N}$, we set $|\gamma|_\infty := \sup\{|\gamma(x)| : x \in A\}$.

Let $a \in (0,1)$. For any $B \subset \mathbb{R}^3$, we write $C^a(B)$ for the set of all Hölder continuous functions on B, that is, $\psi \in C^a(B)$ iff $\psi : B \mapsto \mathbb{R}$ with

$$|\psi|_a := |\psi|_\infty + \sup\{|\psi(x) - \psi(y)|\,|x-y|^{-a} : x, y \in B,\ x \neq y\} < \infty.$$

If $B \subset \mathbb{R}^3$ is open, the space $C^{1,a}(B)$ is to consist of all functions $\psi \in C^1(B)$ with $|\psi|_\infty < \infty$ and $|\partial_l \psi|_a < \infty$ for $1 \le l \le 3$. We further define $C^{1,a}(\partial\Omega) := \{\psi|\partial\Omega : \psi \in C^{1,a}(\mathbb{R}^3)\}$.

Let $p \in [1,\infty),\ m \in \mathbb{N}$. For $A \subset \mathbb{R}^3$ open, the notation $\|\ \|_p$ stands for the norm of the Lebesgue space $L^p(A)$, and $\|\ \|_{m,p}$ for the usual norm of the Sobolev space $W^{m,p}(A)$ of order m and exponent p. If $A \subset \mathbb{R}^3$ has a bounded C^2-boundary, the Sobolev space $W^{r,p}(\partial A)$ with $r \in (0,2)$ is to be defined as in [12, Section 6.8.6].

If $\mathcal{A}$ and $\mathcal{B}$ are vector spaces and $T : \mathcal{A} \mapsto \mathcal{B}$ is a linear operator, we write $ker\, T$ for the kernel of T and $ran\, T$ for the range of T.

Numerical constants are denoted by C, and constants depending exclusively on parameters γ_1, ..., $\gamma_n \in [0,\infty)$, for some $n \in \mathbb{N}$, take the form $C(\gamma_1, ..., \gamma_n)$. In most cases it is not possible and of no interest to list all such parameters. Then we use the symbol $\mathfrak{C}$ for constant whose dependencies – or more importantly, their non-dependency – on certain parameters should be clear from context, or are pointed out in the text.

In the following theorem, we reproduce the Calderon-Zygmund inequality for odd kernels. This well known estimate is restated here because we will need some details on how the upper bound given by this inequality relates to the structure of the kernel.

Theorem 2.1 *Let $n \in \mathbb{N}$ and $K : \mathbb{R}^n \mapsto \mathbb{R}$ a function with $K(z) = |z|^{-n} K(|z|^{-1} z)$ and $K(-z) = -K(z)$ for $z \in \mathbb{R}^n \backslash \{0\}$. Put $\Lambda := K|\partial B_1$ and suppose that $\Lambda \in L^1(\partial B_1)$.*

Let $p \in (1,\infty)$. Then $\int_{\{y \in \mathbb{R}^n : |y-x| \ge \epsilon\}} |K(x-y)\, f(y)|\, dy < \infty$ for $x \in \mathbb{R}^n,\ \epsilon \in (0,\infty)$ and $f \in L^p(\mathbb{R}^n)$. Define $(K_\epsilon * f)(x) := \int_{\{y \in \mathbb{R}^n : |y-x| \ge \epsilon\}} K(x-y)\, f(y)\, dy$ for x, ϵ, f as before.

Then $\|K_\epsilon * f\|_p \le C(p,n)\, \|\Lambda\|_1\, \|f\|_p$ for $\epsilon \in (0,\infty)$ and $f \in L^p(\mathbb{R}^n)$.

Proof: [22, p.89, Theorem 2 a)]. □

We state a lemma which is convenient to handle weakly singular integral operators.

Lemma 2.1 *Let $n \in \mathbb{N}$, A, $B \subset \mathbb{R}^n$ nonempty, λ et ν measures on σ-algebras over A and B, respectively. Further assume that the function $K : A \times B \mapsto [0, \infty)$ is measurable and the upper bounds $\mathfrak{A}_1 := \sup\{\int_B K(x,y)\, d\nu(y) \,:\, x \in A\}$ and $\mathfrak{A}_2 := \sup\{\int_A K(x,y)\, d\lambda(x) \,:\, y \in B\}$ are finite.*

Then, for $p \in [1, \infty)$ and $\phi : B \mapsto \mathbb{R}$ measurable with $\int_B |\phi|^p\, d\nu < \infty$, the integral $\int_{\partial\Omega} K(x,y)\, |\phi(y)|\, do_y$ is finite for λ-a. e. $x \in A$, and

$$\Big(\int_A \Big(\int_B K(x,y)\, |\phi(y)|\, d\nu(y)\Big)^p\, d\lambda(x)\Big)^{1/p} \le \mathfrak{A}_1^{1/p'}\, \mathfrak{A}_2^{1/p} \Big(\int_B |\phi|^p\, d\nu\Big)^{1/p}. \tag{2.1}$$

The preceding assumptions hold true if, for example, $n = 3$, $A = B = \partial\Omega$, λ and ν are the usual surface measure on $\partial\Omega$, and if there are numbers c_0, $\kappa \in (0, \infty)$ such that $K(x,y) \le c_0\, |x-y|^{-2+\kappa}$ for $x, y \in \Omega$, $x \ne y$.

These assumptions are also valid if $n \in \{2, 3\}$, A, $B \subset \mathbb{R}^n$ open, bounded and nonempty, λ and ν the usual Borel measure on A and B respectively, and if $K(\varrho, \eta) \le c_0\, |\varrho - \eta|^{-n+\kappa}$ for $\varrho \in A$, $\eta \in B$, $\varrho \ne \eta$, with c_0 and κ given as before.

Proof: Inequality (2.1) is a simple application of Hölder's inequality and Fubini's theorem, as used in [22, p. 7] in the case of convolution kernels. The first claim of the lemma follows from (2.1). □

In Theorem 2.2 and Lemma 2.2 to 2.4 below, we state some properties of weakly singular integral operators on L^p-spaces. In the case of Theorem 2.2, we give a proof for the reader's convenience because we do not know a precise reference.

Theorem 2.2 *Let $J \subset \mathbb{R}^2$ be open, bounded and convex, and $L : J \times J \mapsto \mathbb{R}$ a measurable function with $L(\,\cdot\,, \eta) \in C^1(J \backslash \{\eta\})$ for $\eta \in J$. Suppose there is $c_0 > 0$ with $|\partial_\varrho^\alpha L(\varrho, \eta)| \le c_0\, |\varrho - \eta|^{-1-|\alpha|}$ for ϱ, $\eta \in J$ with $\varrho \ne \eta$, $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le 1$.*

Let $p \in (1, \infty)$ and define $\mathcal{L}(\psi)(\varrho) := \int_J L(\varrho, \eta)\, \psi(\eta)\, d\eta$ for $\psi \in L^p(J)$, $\varrho \in J$; see Lemma 2.1. Then $\mathcal{L}(\psi) \in W^{1-1/p,p}(J)$ and $\|\mathcal{L}(\psi)\|_{1-1/p,\,p} \le \mathfrak{C}\, \|\psi\|_p$ for ψ as before.

Proof: Let ϱ, ϱ', $\eta \in J$ with $\eta \notin \{\varrho, \varrho'\}$. If $|\varrho - \eta| \le 2\, |\varrho - \varrho'|$, we have $|\varrho' - \eta| < 3\, |\varrho - \varrho'|$, so

$$|L(\varrho, \eta) - L(\varrho', \eta)| \le c_0\, (|\varrho - \eta|^{-1} + |\varrho' - \eta|^{-1}) \le 3\, c_0\, (|\varrho - \eta|^{-2} + |\varrho' - \eta|^{-2})\, |\varrho - \varrho'|.$$

In the case $|\varrho - \eta| \ge 2\, |\varrho - \varrho'|$, we get for $\vartheta \in [0, 1]$ that $|\varrho + \vartheta\, (\varrho' - \varrho) - \eta| \ge |\varrho - \eta|/2$. Thus, using the equation $|L(\varrho, \eta) - L(\varrho', \eta)| = |\int_0^1 (\nabla L)\big(\varrho + \vartheta\, (\varrho' - \varrho),\, \eta\big)\, d\vartheta \cdot (\varrho - \varrho')|$, we get in any case that

$$\Big|L(\varrho, \eta) - L(\varrho', \eta)\Big| \le C(c_0)\, (|\varrho - \eta|^{-2} + |\varrho' - \eta|^{-2})\, |\varrho - \varrho'| \quad \text{for } \varrho,\, \varrho',\, \eta \in J \tag{2.2}$$

with $\eta \notin \{\varrho, \varrho'\}$. Let $r \in (1, 2)$, ϱ, $\varrho' \in J$, and put $J_{\varrho,\varrho'} := \{\eta \in J \,:\, |\varrho - \eta| \le 2\, |\varrho - \varrho'|\}$. By splitting the set J into the parts $J_{\varrho,\varrho'}$ and $J \backslash J_{\varrho,\varrho'}$, it may be shown that the inequality $\int_J |L(\varrho, \eta) - L(\varrho', \eta)|^r\, d\eta \le \mathfrak{C}\, |\varrho - \varrho'|^{-r+2}$ holds, where the estimate of the

integral over $J \backslash J_{\varrho,\varrho'}$ is based on (2.2). Note that $|\varrho' - \eta| \geq |\varrho - \varrho'|$ for $\eta \in J \backslash J_{\varrho,\varrho'}$. Put $\epsilon := \min\{1/(2\,p'),\, 1/(4\,p)\}$. Then $1 + \epsilon\, p' < 2$. Let $\psi \in L^p(J)$. By Hölder's inequality, the splitting $1 = 1/p' + \epsilon + 1/p - \epsilon$ and the previous estimate with $r = 1 + \epsilon\, p'$,

$$|\mathcal{L}(\psi)(\varrho) - \mathcal{L}(\psi)(\varrho')|^p \leq \mathfrak{C}\,(|\varrho - \varrho'|^{1-\epsilon\,p'})^{p/p'} \int_J |L(\varrho,\eta) - L(\varrho',\eta)|^{1-\epsilon\,p}\ |\psi(\eta)|^p\, d\eta$$

for $\varrho,\ \varrho' \in J$. Set $\mathfrak{A}(\varrho,\varrho') := |\mathcal{L}(\psi)(\varrho) - \mathcal{L}(\psi)(\varrho')|^p\, |\varrho - \varrho'|^{-p-1}$. It follows that

$$\mathfrak{A}(\varrho,\varrho') \leq \mathfrak{C}\, |\varrho - \varrho'|^{-2-\epsilon\,p} \int_J |L(\varrho,\eta) - L(\varrho',\eta)|^{1-\epsilon\,p}\, |\psi(\eta)|^p\, d\eta.$$

We integrate both sides of the preceding inequality with respect to $\varrho \in J$ and $\varrho' \in J$, then apply (2.2) and change the order of integration. In this way we arrive at the estimate $\int_J \int_J \mathfrak{A}(\varrho,\varrho')\, d\varrho\, d\varrho' \leq \mathfrak{C}\,(\mathfrak{B}_1 + \mathfrak{B}_2)$, where

$$\mathfrak{B}_1 := \int_J \int_J \int_J |\varrho - \eta|^{-2\,(1-\epsilon\,p)}\ |\varrho - \varrho'|^{-1-2\,\epsilon\,p}\, d\varrho\, d\varrho'\, |\psi(\eta)|^p\, d\eta.$$

and where term $\mathfrak{B}_2$ is to be defined in the same way as $\mathfrak{B}_1$, except that the difference $|\varrho - \eta|$ is replaced by $|\varrho' - \eta|$, and the order of integration with respect to ϱ and ϱ' is exchanged. Now we use that $-2\,(1-\epsilon\,p)$ and $-1-2\,\epsilon\,p$ belong to $(-2, 0)$, and $-2\,(1-\epsilon\,p) - 1 - 2\,\epsilon\,p = -3 < -2$. Thus we get that $\int_J |\varrho - \eta|^{-2\,(1-\epsilon\,p)}\ |\varrho - \varrho'|^{-1-2\,\epsilon\,p}\, d\varrho \leq \mathfrak{C}\, |\varrho' - \eta|^{-1}$ for $\varrho',\ \eta \in J$ with $\varrho' \neq \eta$, as follows by splitting J into four sets according to four cases, three of them given by the inequalities $|\varrho - \eta| \leq |\varrho' - \eta|/2,\ |\varrho - \varrho'| \leq |\varrho' - \eta|/2,\ |\varrho - \eta| \geq 2\,|\varrho' - \eta|$, and the fourth consisting of the requirement that none of the three preceding conditions holds; compare [11, Lemma 1.4.2]. It follows that $\mathfrak{B}_1 \leq \mathfrak{C}\, \|\psi\|_p^p$. An analogous argument yields that $\mathfrak{B}_2 \leq \mathfrak{C}\, \|\psi\|_p^p$. Therefore the theorem follows from the estimate $\int_J \int_J \mathfrak{A}(\varrho,\varrho')\, d\varrho\, d\varrho' \leq \mathfrak{C}\,(\mathfrak{B}_1 + \mathfrak{B}_2)$ shown above, and from Lemma 2.1. □

Lemma 2.2 *Let $L : \partial\Omega \times \partial\Omega \mapsto \mathbb{R}$ be measurable. Suppose there is $c_0 \in (0,\infty)$ with $|L(x,y)| \leq c_0\, |x-y|^{-1}$ for $x, y \in \partial\Omega,\ x \neq y$. Let $p \in (1,\infty)$. We may define an operator $\mathcal{L} : L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$ by setting $\mathcal{L}(\phi)(x) := \int_{\partial\Omega} L(x,y)\, \phi(y)\, do_y$ for $\phi \in L^p(\partial\Omega),\ x \in \partial\Omega$; see Lemma 2.1.*

Then $\mathcal{L} : L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$ is linear, bounded and compact.

Proof: Obviously L is linear. The boundedness of L holds according to Lemma 2.1. As for compactness, we remark that for any $\epsilon \in (0,\infty)$, the operator $\mathcal{L}_{p,\epsilon} : L^p(\partial\Omega) \mapsto L^p(\partial\Omega),\ \mathcal{L}_{p,\epsilon}(\phi)(x) := \int_{\partial\Omega} \chi_{(\epsilon,\infty)}(|x-y|)\, L(x,y)\, \phi(y)\, do_y\ \big(x \in \partial\Omega,\ \phi \in L^p(\partial\Omega)\big)$ is compact ([17, p. 275, Theorem 11.6]). On the other hand,

$$\sup\{\int_{\partial\Omega} \chi_{(0,\epsilon)}(|x-y|)\, |x-y|^{-1}\, do_y\ :\ x \in \partial\Omega\} \to 0\ (\epsilon \downarrow 0),$$

so it follows by Lemma 2.1 that $\mathcal{L}_\epsilon$ converges to $\mathcal{L}$ with respect to the operator norm of the space of linear bounded operators from $L^p(\partial\Omega)$ into $L^p(\partial\Omega)$. As a consequence, $\mathcal{L}$ is compact as well. □

Lemma 2.3 *Let $L \in C^1(\mathbb{R}^3\backslash\{0\})$ and $c_0 \in (0,\infty)$ such that $|\partial^\alpha L(z)| \le c_0\,|z|^{-1-|\alpha|}$ for $z \in \mathbb{R}^3\backslash\{0\}$, $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 1$. Let $\phi \in C^0(\partial\Omega)$ and put $\mathcal{A}(\phi)(x) := \int_{\partial\Omega} L(x-y)\,\phi(y)\,do_y$ for $x \in \mathbb{R}^3$. Then $\mathcal{A}(\phi) \in C^a(\mathbb{R}^3)$ for $a \in [0,1)$.*

Proof: [9, Lemma 6.1]. □

Lemma 2.4 *Let $p \in (1,\infty)$, $a \in \big(0,\, 2/(3\,p)\big)$, $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$. Then*

$$\Big(\int_{B_R}\Big(\int_{\partial\Omega} |x-y|^{-2}\,|\phi(y)|\,do_y\Big)^{(1/p-a/2)^{-1}}\,do_x\Big)^{1/p-a/2} \le \mathfrak{C}(p,a,R)\,\|\phi\|_p \quad \text{for } L^p(\partial\Omega).$$

Proof: See [7, Lemma 3.2]. □

We will use the fact that a function v defined in a 3D exterior domain and whose gradient is L^q-integrable for some $q \in (1,3)$ takes a constant boundary value at infinity:

Theorem 2.3 *Let $U \subset \mathbb{R}^3$ be open and bounded, with Lipschitz boundary. Let $q \in (1,3)$. Then for any $v \in W^{1,1}_{loc}(\overline{U}^c)$ with $\nabla v \in L^q(\overline{U}^c)^3$, there is $\tau(v) \in \mathbb{R}$ with $v + \tau(v) \in L^{(1/q-1/3)^{-1}}(\overline{U}^c)$. There is $c_0 > 0$ such that $\|v+\tau(v)\|_{(1/q-1/3)^{-1}} \le c_0\,\|\nabla v\|_q$ for such functions v. In addition, if a function v of this kind belongs to $L^s(\overline{U}^c)$ for some $s \in (1,\infty)$, then the constant $\tau(v)$ vanishes.*

Proof: See [13, Theorem II.6.1], except as concerns the criterion for the equation $\tau(v) = 0$, which is treated in [6, Lemma 2.4]. □

If U is an open, bounded set in $\mathbb{R}^n$ with some regularity of the boundary, the trace of a function $v \in C^\infty(\overline{U})$ is, of course, the restriction of v to ∂U, by the definition of the trace. Several times we will use a slight generalization of this fact, as stated in

Lemma 2.5 *Let $n \in \mathbb{N}$, $U \subset \mathbb{R}^n$ with C^1-boundary, $v \in C^0(\overline{U})$ with $v|U \in W^{1,1}(U)$. Then the trace of $v|U$ on $\partial\Omega$ coincides with $v|\partial U$.*

Proof: The extension operator from [1, 4.26] yields a function $\widetilde{v} \in C_0^0(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$ with $\widetilde{v}|\overline{U} = v$. The lemma then follows via a sequence $(\widetilde{v}_n)$ in $C_0^\infty(\mathbb{R}^n)$ constructed by means of Friedrich's mollifier and converging to $\widetilde{v}$ in $W^{1,1}(\mathbb{R}^n)$ and pointwise uniformly in $x \in \mathbb{R}^n$. □

The role of the functions $\phi^{(1)}$, ..., $\phi^{(6)}$ from Theorem 1.2 becomes clear by the following theorem.

Theorem 2.4 *Let $U \subset \mathbb{R}^3$ be a domain. Put*

$$\mathfrak{L} := \{v \in C^1(U)^3 \,:\, \partial_j v_k + \partial_k v_j = 0 \text{ for } 1 \le j,k \le 3\}.$$

Then the family $(\phi^{(1)}|U,\, ...,\, \phi^{(6)}|U)$ is a basis of $\mathfrak{L}$.

Proof: [9, Satz 6.1]. □

We end this section by recalling some properties of solutions to either the Poisson equation or the Stokes system.

Lemma 2.6 *Let $U \subset \mathbb{R}^3$ be open and bounded, with C^2-boundary. Let $n^{(U)}$ denote the outward unit normal to U. Suppose that $u \in C^1(\overline{U})^3 \cap W^{2,1}(U)^3 \cap C^2(U)^3$, $\pi \in C^0(\overline{U}) \cap W^{1,1}(U) \cap C^1(U)$, $\widetilde{u} \in C^0(\overline{U})^3 \cap W^{1,1}(U)^3 \cap C^1(U)^3$. Alternatively, let $p \in (1,\infty)$, $u \in W^{2,p}(U)^3$, $\pi \in W^{1,p}(U)$, $\widetilde{u} \in W^{1,p'}(U)^3$. Then*

$$\int_U (\Delta u - \nabla \pi + \nabla \, div \, u) \cdot \widetilde{u} \, dx + (1/2) \int_U \sum_{j,k=1}^{3} (\partial_j u_k + \partial_k u_j) \, (\partial_j \widetilde{u}_k + \partial_k \widetilde{u}_j) \, dx$$

$$= \int_{\partial U} \sum_{j,k=1}^{3} n_k^{(U)} (\partial_k u_j + \partial_j u_k - \delta_{jk} \, \pi) \, \widetilde{u}_j \, do_x + \int_U \pi \, div \, \widetilde{u} \, dx.$$

Proof: Apply the Divergence theorem. For functions u, π and $\widetilde{u}$ given as in the first case considered in the lemma, this is possible according to the reasoning in [9, Lemma 3.1]. In the second case, the functions under consideration are such that applying the Divergence theorem in a suitable way is possible due the density of C^∞-functions in Sobolev spaces. □

The ensuing lemma deals with the Poisson equation with data in certain Sobolev and L^p-spaces, respectively. The lemma states that in such a situation, unsurprisingly, a classical solution belongs to a Sobolev space corresponding to the regularity of the data.

Lemma 2.7 *Let $U \subset \mathbb{R}^3$ be open and bounded, with C^2-boundary. Let $r \in (3/2, \infty)$, $\gamma \in W^{2-1/r,\,r}(\partial U)$, $g \in L^r(U)$, $w \in C^0(\overline{U})$ with $w|U \in C^2(U)$, $\Delta(w|U) = g$ and $w|\partial U = \gamma$.*

Then $w|U \in W^{2,r}(U)$ and $w|\partial U = \gamma$ in the trace sense.

Proof: This theorem is a special case of [7, Lemma 3.4], which, in turn, is based on the $W^{2,q}$-theory of the Poisson equation and on the maximum principle. Also see Lemma 2.5. □

We state a result on $W^{2,p}$-resularity of solutions to the Stokes system in bounded domains under Dirichlet boundary conditions:

Theorem 2.5 *Let $U \subset \mathbb{R}^3$ be a bounded domain with C^2-boundary. Let $p \in (1,\infty)$, $f \in L^p(U)^3$, $b \in W^{2-1/p,\,p}(U)^3$ with $\int_{\partial U} b \cdot n^{(U)} \, do_x = 0$, where $n^{(U)}$ denotes the outward unit normal to U.*

Then there are functions $u \in W^{2,p}(U)^3$, $\pi \in W^{1,p}(U)$ such that $-\Delta u + \nabla \pi = f$, $div \, u = 0$, $u|\partial U = b$, $\int_U \pi \, dx = 0$. Moreover there is $c_0 > 0$ such that for f, b, u and π as before, the estimate $\|u\|_{2,p} + \|\pi\|_{1,p} \leq c_0 \, (\|f\|_p + \|b\|_{2-1/p,\,p})$ holds.

Proof: [13, Theorem IV.6.1]. □

In the ensuing theorem, we present a uniqueness result for L^p-weak solutions to the Stokes system in bounded domains, under Dirichlet boundary conditions:

Theorem 2.6 *Let U, p, $n^{(U)}$ be given as in Theorem 2.5. Assume that $u \in W^{1,p}(U)^3$, $\pi \in L^p(U)$ such that $\int_U (\nabla u \cdot \nabla \varphi + \pi \, div \, \varphi) \, dx = 0$ for $\varphi \in C_0^\infty(U)^3$. Further suppose that $div \, u = 0$, $u|\partial U = 0$. Then $u = 0$.*

Proof: [13, Lemma IV.6.2].

Finally we mention a technical result on the difference of two solutions to (1.1) in $\overline{\Omega}^c$, concerning L^q-integrability of the gradient of the pressure near infinity.

Theorem 2.7 *For $j \in \{1, 2\}$, let p_j, $r_j \in (1, \infty)$, $u^{(j)} \in W^{2,p_j}_{loc}(\overline{\Omega}^c)^3$, $\pi^{(j)} \in W^{1,p_j}_{loc}(\overline{\Omega}^c)$ with $\nabla\pi^{(j)}|B_R^c \in L^{r_j}(B_R^c)^3$ for some $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$. Further suppose there are numbers q_1, $q_2 \in (1, \infty)$ such that $u^{(j)}|B_R^c \in L^{q_j}(B_R^c)^3$ for $j \in \{1, 2\}$.*

Put $u := u^{(1)} - u^{(2)}$, $\pi := \pi^{(1)} - \pi^{(2)}$, and suppose that the pair (u, π) solves (1.1) with $f = 0$. Abbreviate $r := \{p_1, p_2, r_1, r_2\}$. Then $\nabla\pi|B_{R+1}^c \in L^s(B_{R+1}^c)^3$ for $s \in (1, r]$.

Proof: We refer to the proof of [5, (3.4)], only adding that due a misprint, there is a reference to [5, (1.5)] instead of [5, (1.3)] in that proof. Note that $\pi|B_R^c$ is a slowly increasing function ([25, p. 150])in view of [13, (II.6.19) and (II.6.24)]. This latter fact is relevant on [5, p. 1524 above]. □

3 Some results on local charts of $\partial\Omega$.

Since the crucial point of our theory is a $W^{2-1/p,\,p}$-estimate on $\partial\Omega$ of a certain boundary potential (Theorem 5.2), precise informations on the local charts we will use are essential for what follows. These informations are specified in this section. We choose a description of the boundary as introduced in [12, p. 304-306].

Lemma 3.1 *There are numbers $k(\Omega) \in \mathbb{N}$, $\alpha(\Omega) \in (0, \infty)$, and for any $t \in \{1, ..., k(\Omega)\}$ a function $a_t \in C^2\big([-\alpha(\Omega), \alpha(\Omega)]^2\big)$ with $|\nabla a_t(\varrho)| < 1/4$ $(\varrho \in \Delta_1)$, an orthonormal matrix $D_t \in \mathbb{R}^{3\times 3}$ and a vector $C_t \in \mathbb{R}^3$ such that the following properties hold:*

Put $\Delta_\sigma := \big(-\sigma\,\alpha(\Omega),\, \sigma\,\alpha(\Omega)\big)^2$ for $\sigma \in (0, 1]$,

$$\gamma_t(\eta) := D_t \cdot \big(\eta,\, a_t(\eta)\big) + C_t \;\; (\eta \in \Delta_1), \quad \Lambda_{t,\sigma} := \{\gamma_t(\eta) \,:\, \eta \in \Delta_\sigma\},$$
$$U_{t,\sigma} := \{D_t \cdot \big(\eta,\, a_t(\eta) + s\big) \,:\, \eta \in \Delta_\sigma,\; s \in \big(-\sigma\,\alpha(\Omega),\, \sigma\,\alpha(\Omega)\big)\}$$

for σ, t as before. Then there is a constant $c_0 \in (0, \infty)$ such that

$$c_0^{-1}\,\|f\|_1 \le \sum_{t=1}^{k(\Omega)} \|f \circ \gamma_t|\Delta_{1/4}\|_1 \le c_0\,\|f\|_1 \;\; \text{for } f \in L^1(\partial\Omega). \tag{3.1}$$

The function $\gamma_t : \Delta_1 \mapsto \Lambda_{t,1}$ is bijective, continuous and with continuous inverse, the set $U_{t,\sigma}$ is open in $\mathbb{R}^3$, and $\Lambda_{t,\sigma} = U_{t,\sigma} \cap \partial\Omega$ for $t \in \{1, ..., k(\Omega)\}$, $\sigma \in (0, 1]$. Moreover $dist(\partial\Omega\backslash\Lambda_{t,\sigma_2},\, \Lambda_{t,\sigma_1}) > 0$ for t as before and σ_1, $\sigma_2 \in (0, 1]$ with $\sigma_1 < \sigma_2$.

Define $J_t(\eta) := \big(1 + \sum_{j=1}^2 \partial_j a_t(\eta)^2\big)^{1/2}$ for $\eta \in \Delta_1$, t as before. Then for such t and for functions $F : \Lambda_{t,1} \mapsto \mathbb{C}$, the relation $F \in L^1(\Lambda_{t,1})$ holds iff $(F \circ \gamma_t)\,J_t \in L^1(\Delta_1)$. In addition

$$\int_{\Lambda_{t,1}} F\,do_x = \int_{\Delta_1} (F \circ \gamma_t)(\eta)\,J_t(\eta)\,d\eta \quad \text{for } \; F \in L^1(\Lambda_{t,1}). \tag{3.2}$$

Proof: All the statements of the lemma are standard results (see [12, Lemma 6.3.9, Definition 6.3.10, Theorem 6.3.12], with additional details in [9, Section 2]), except perhaps the claim that the local charts γ_t may be chosen in such a way that $|\nabla a_t|_\infty < 1/4$ for $1 \le t \le k(\Omega)$. In order to satisfy this condition, the boundary $\partial\Omega$ has to be split into sufficiently small parts. Details of this procedure are rather technical but straightforward. □

Lemma 3.2 *There are constants $\delta(\Omega)$, $\mathcal{D} \in (0,\infty)$ such that*

$$x + \kappa\, n^{(\Omega)}(x) \in \overline{\Omega}^c, \quad x - \kappa\, n^{(\Omega)}(x) \in \Omega, \tag{3.3}$$

$$|x + \kappa\, n^{(\Omega)}(x) - x' - \kappa'\, n^{(\Omega)}(x')| \ge \mathcal{D}\,(|x - x'| + |\kappa - \kappa'|), \tag{3.4}$$

for x, $x' \in \partial\Omega$, κ, $\kappa' \in [-\delta(\Omega), \delta(\Omega)]$. In addition

$$|(x - x') \cdot n^{(\Omega)}(x)| \le \mathfrak{C}\,|x - x'|^2 \qquad \text{for } x,\, x' \in \partial\Omega, \tag{3.5}$$

with $\mathfrak{C}$ independent of x and x'. For $\delta \in \big(0, \delta(\Omega)\big]$, put

$$\mathfrak{U}_\delta := \{x \in \mathbb{R}^3 \,:\, dist(x, \overline{\Omega}) < \mathcal{D}\,\delta/2\}, \quad \mathfrak{U}_{-\delta} := \{x \in \mathbb{R}^3 \,:\, dist(x, \Omega^c) < \mathcal{D}\,\delta/2\}.$$

Note that the sets $\mathfrak{U}_\delta$ and $\mathfrak{U}_{-\delta}$ are open in $\mathbb{R}^3$ and $\overline{\Omega} \subset \mathfrak{U}_\delta$, $\Omega^c \subset \mathfrak{U}_{-\delta}$, for δ as before. The estimates

$$|x - \big(y + \delta\, n^{(\Omega)}(y)\big)| \ge \mathcal{D}\,\delta/2, \quad |x' - \big(y - \delta\, n^{(\Omega)}(y)\big)| \ge \mathcal{D}\,\delta/2 \tag{3.6}$$

hold for δ as before, $y \in \partial\Omega$, $x \in \mathfrak{U}_\delta$ and $x' \in \mathfrak{U}_{-\delta}$.

Proof: See [9, (2.24), (2.22)] for (3.3), (3.5), respectively, [7, Lemma 2.1] for (3.4), and the proof of [7, Lemma 5.3] for the properties of $\mathfrak{U}_\delta$ and $\mathfrak{U}_{-\delta}$. □

4 Simple and double layer potentials related to the Stokes system or the Poisson equation.

The solutions to (1.1), (1.2) we will consider are given by a sum of simple layer, double layer and volume potentials. The first two types of potentials are introduced and studied in this section. We begin by defining some kernel functions, among them a fundamental solution to (1.1). Put

$$\mathfrak{N}(z) := (4\,\pi\,|z|)^{-1}, \quad E_{jk}(z) := (8\,\pi\,|z|)^{-1}\,(\delta_{jk} + z_j\,z_k\,|z|^{-2}), \quad E := (E_{jk})_{1\le j,k\le 3}, \tag{4.1}$$

$$S_{jkl} := -\delta_{jk}\,\partial_l\mathfrak{N} - \partial_k E_{jl} - \partial_j E_{kl} \quad \text{for } z \in \mathbb{R}^3\backslash\{0\},\ 1 \le j,\, k,\, l \le 3. \tag{4.2}$$

The matrix-valued function $E = (E_{jk})_{1\le j,k\le 3}$ is the velocity part of a fundamental solution to the Stokes system (1.1), with its associated pressure part given by $-\nabla\mathfrak{N}$.

The next lemma is an obvious consequence of (4.1) and (4.2). In that lemma, as in similar situations below, the restrictions on $|\alpha|$ (order of differentiation) may of course be dropped if the constants may depend on $|\alpha|$. We will not need this fact, instead limiting the range of $|\alpha|$.

Lemma 4.1 *The relations* E_{jk}, S_{jkl}, $\mathfrak{N} \in C^\infty(\mathbb{R}^3\backslash\{0\})$ *and*

$$-\Delta E_{jk} - \partial_j \partial_k \mathfrak{N} = 0,\ \sum_{\mu=1}^{3} \partial_\mu E_{j\mu} = 0,\ \Delta \mathfrak{N} = 0 \tag{4.3}$$

hold for $1 \le j, k, l \le 3$. *In addition*

$$|\partial^\alpha E_{jk}(z)| + |\partial^\alpha \mathfrak{N}(z)| \le C\,|z|^{-1-|\alpha|}, \tag{4.4}$$

for j, k *as before,* $z \in \mathbb{R}^3\backslash\{0\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 3$. *Moreover, for* j, k, l, z *as before,*

$$S_{jkl}(z) = 3/(4\,\pi)\, z_j\, z_k\, z_l\, |z|^{-5}, \quad -\Delta S_{jkl} + 2\,\partial_j \partial_k \partial_l \mathfrak{N} = 0, \quad \sum_{\nu=1}^{3} \partial_\nu S_{jk\nu} = 0. \tag{4.5}$$

In the following lemma we introduce a simple layer potential associated with the Stokes system (1.1).

Lemma 4.2 *Let* $\phi \in L^1(\partial\Omega)$. *For* $x \in \mathbb{R}^3\backslash\partial\Omega$, *put*

$$V(\phi)(x) := \int_{\partial\Omega} E(x-y)\cdot\phi(y)\,do_y, \quad Q(\phi)(x) := \int_{\partial\Omega} (-\nabla\mathfrak{N})(x-y)\cdot\phi(y)\,do_y.$$

The integral $\int_{\partial\Omega} |E(x-y)\cdot\phi(y)|\,do_y$ *is finite for a. e.* $x \in \partial\Omega$. *In particular, for* $x \in \partial\Omega$, *the term* $V(\phi)(x)$ *may be defined in the same way as for* $x \in \mathbb{R}^3\backslash\partial\Omega$.

The functions $V(\phi)_j|\mathbb{R}^3\backslash\partial\Omega$ *and* $Q(\phi)$ *belong to* $C^\infty(\mathbb{R}^3\backslash\partial\Omega)$, *for* $1 \le j \le 3$, *with* $\partial_x^\alpha V(\phi)(x) = \int_{\partial\Omega} (\partial^\alpha E)(x-y)\cdot\phi(y)\,do_y$ *for* $x \in \mathbb{R}^3\backslash\partial\Omega$, $\alpha \in \mathbb{N}_0^3$, *and with an analogous formula being valid for* $Q(\phi)$.

The pair $\big(V(\phi)|\mathbb{R}^3\backslash\partial\Omega,\, Q(\phi)|\mathbb{R}^3\backslash\partial\Omega\big)$ *satisfies the Stokes system (1.1) with* $f = 0$.

If $\phi \in C^0(\partial\Omega)^3$, *then* $V(\phi) \in C^a(\mathbb{R}^3)^3$ *for* $a \in [0, 1)$.

Note that $Q(\phi)$ is not defined on $\partial\Omega$ because its kernel is singular with respect to integrals on $\partial\Omega$.

Proof of Lemma 4.2: The term $V(\phi)(x)$ is well defined also for $x \in \partial\Omega$ according to Lemma 2.1. The claims related to the differential properties in $\mathbb{R}^3\backslash\partial\Omega$ follow from Lebesgue's theorem and Lemma 4.1. The statement on Hölder continuity of $V(\phi)$ in $\mathbb{R}^3$ if ϕ is continuous is a consequence of (4.4) and Lemma 2.3. □

Corollary 4.1 *Let* $p \in (1, \infty)$ *and* $R \in (0, \infty)$ *with* $\Omega_R \subset B_R$. *Then, if* $r \in [1,\, 3\,p/2)$, *the estimate*

$$\|V(\phi)|B_R\backslash\partial\Omega\|_{1,r} + \|Q(\phi)|B_R\backslash\partial\Omega\|_r \le \mathfrak{C}\,\|\phi\|_p \quad \textit{for}\ \ \phi \in L^p(\partial\Omega)^3. \tag{4.6}$$

is valid. In particular $V(\phi)|\Omega \in W^{1,r}(\Omega)^3$, $V(\phi)|\Omega_R \in W^{1,r}(\Omega_R)^3$ *for* r, ϕ *as above, and* $V(\phi)|\Omega$ *and* $V(\phi)|\Omega_R$ *have a trace on* $\partial\Omega$. *Moreover*

$$\|V(\phi)|\partial\Omega\|_p \le \mathfrak{C}\,\|\phi\|_p \quad \textit{for}\ \ \phi \in L^p(\partial\Omega)^3, \tag{4.7}$$

with $V(\phi)|\partial\Omega$ *as defined in Lemma 4.2. The preceding constants* $\mathfrak{C}$ *are independent of* ϕ.

Proof: Let $\phi \in L^p(\partial\Omega)^3$. The constants $\mathfrak{C}$ in this proof are independent of ϕ. Take $r \in (p,\, 3\,p/2)$. Then we obtain from Lemma 2.4 with $a = 2\,(1/p - 1/r)$ that

$$\|\partial_m(V(\phi)|B_R\backslash\partial\Omega)\|_r + \|Q(\phi)|B_R\backslash\partial\Omega\|_r \leq \mathfrak{C}\,\|\phi\|_p \quad (1 \leq m \leq 3).$$

It follows that even if $r \in [1,\, 3\,p/2)$, the preceding inequality remains valid. Since for $x,\, y \in B_R$ with $x \neq y$, we have $|x-y|^{-1} \leq 2\,R\,|x-y|^{-2}$, the same argument implies that if $r \in [1,\, 3\,p/2)$, the estimate $\|V(\phi)|B_R\backslash\partial\Omega\|_r \leq \mathfrak{C}\,\|\phi\|_p$ holds. This proves (4.6). As concerns inequality (4.7), we refer to Lemma 2.1. □

We turn to the question of how to approximate $V(\phi)|\Omega$ and $V(\phi)|\overline{\Omega}^c$ by functions which are C^∞ in open sets somewhat larger than $\overline{\Omega}$ and Ω^c, respectively.

Lemma 4.3 *Recall the parameter $\delta(\Omega)$ and the sets $\mathfrak{U}_\delta$ and $\mathfrak{U}_{-\delta}$ for $\delta \in \big(0,\, \delta(\Omega)\big]$ introduced in Lemma 3.2. Further recall that $\mathfrak{U}_\delta$ and $\mathfrak{U}_{-\delta}$ are open sets in $\mathbb{R}^3$ with $\overline{\Omega} \subset \mathfrak{U}_\delta$, $\Omega^c \subset \mathfrak{U}_{-\delta}$, for δ as before. Let $\phi \in L^1(\partial\Omega)^3$ and define $V^{(\delta)}(\phi) : \mathfrak{U}_\delta \mapsto \mathbb{C}^3$, $Q^{(\delta)}(\phi) : \mathfrak{U}_\delta \mapsto \mathbb{C}$ by setting*

$$V^{(\delta)}(\phi)(x) := \int_{\partial\Omega} E\big(\,x - [y + \delta\,n^{(\Omega)}(y)]\,\big)\cdot\phi(y)\,do_y,$$
$$Q^{(\delta)}(\phi)(x) := \int_{\partial\Omega} -(\nabla\mathfrak{N})\big(\,x - [y + \delta\,n^{(\Omega)}(y)]\,\big)\cdot\phi(y)\,do_y$$

for $x \in \mathfrak{U}_\delta$, $\delta \in \big(0,\, \delta(\Omega)\big]$. In addition, we introduce the functions $V^{(-\delta)}(\phi) : \mathfrak{U}_{-\delta} \mapsto \mathbb{C}^3$, $Q^{(\delta)} : \mathfrak{U}_{-\delta} \mapsto \mathbb{C}$ by replacing $\mathfrak{U}_\delta$ with $\mathfrak{U}_{-\delta}$ and the term $y + \delta\,n^{(\Omega)}$ with $y - \delta\,n^{(\Omega)}$ in the respective definitions of $V^{(\delta)}(\phi)$ and $Q^{(\delta)}(\phi)$.

Then $V^{(\pm\delta)}(\phi)_j$, $Q^{(\pm\delta)}(\phi)$ belong to $C^\infty(\mathfrak{U}_{\pm\delta})$ for $1 \leq j \leq 3$ and for δ as above. Any derivative of these functions commutes with the integration over $\partial\Omega$ appearing in their definition. The pair $\big(V^{(\pm\delta)}(\phi),\, Q^{(\pm\delta)}(\phi)\big)$ is a solution of (1.1) in $\mathfrak{U}_{\pm\delta}$ with $f = 0$. In addition

$$V^{(\delta)}(\phi)(x) \to V(\phi)(x) \text{ for } x \in \Omega, \quad V^{(-\delta)}(\phi)(x) \to V(\phi)(x) \quad \text{if } \delta \downarrow 0, \text{ for } x \in \overline{\Omega}^c. \quad (4.8)$$

Suppose that $p \in (1,\infty)$, $\phi \in L^p(\partial\Omega)^3$, $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$. Then

$$\int_{\Omega_R} |\partial_x^\alpha\big(\,V^{(-\delta)}(\phi) - V(\phi)\,\big)(x)|^p\,do_x \to 0, \quad (4.9)$$

$$\int_{\Omega} |\partial_x^\alpha\big(\,V^{(\delta)}(\phi) - V(\phi)\,\big)(x)|^p\,do_x \to 0 \quad \text{if } \delta \downarrow 0, \quad \text{for } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1. \quad (4.10)$$

If $\phi \in C^0(\partial\Omega)^3$, then for $x \in \partial\Omega$, $1 \leq j \leq 3$,

$$V^{(\pm\delta)}(\phi)(x) \to V(\phi)(x), \quad (4.11)$$

$$\sum_{k=1}^3 n_k^{(\Omega)}(x)\,\big(\,\partial_j V^{(\pm\delta)}(\phi)_k + \partial_k V^{(\pm\delta)}(\phi)_j - \delta_{jk}\,Q^{(\pm\delta)}(\phi)\,\big)(x) \quad (4.12)$$
$$\to (1/2)\,\big(\,\pm\phi + \mathfrak{T}^*(\phi)\,\big)_j(x) \quad (1 \leq j \leq 3),$$

for $\delta \downarrow 0$, with the convergence in (4.11) and (4.12) being uniform with respect to $x \in \partial\Omega$. The function $V(\phi)|\partial\Omega$ as defined in Lemma 4.2 and the trace of $V(\phi)|\Omega$ and $V(\phi)|\Omega_R$ on $\partial\Omega$ (see Corollary 4.1) coincide.

Proof: We only consider $V^{(-\delta)}(\phi)$ and $Q^{(-\delta)}(\phi)$. If $-\delta$ is replaced by δ, an analogous reasoning is valid.

The differential properties of $V^{(-\delta)}(\phi)$ and $Q^{(-\delta)}(\phi)$ are a consequence of (3.6), the relation E_{jk}, $\mathfrak{N} \in C^\infty(\mathbb{R}^3\backslash\{0\})$ for $1 \le j, k \le 3$, the equations satisfied by E and $\mathfrak{N}$ (see Lemma 4.1) and Lebesgue's theorem.

Let $x \in \overline{\Omega}^c$. For $y \in \partial\Omega,\ \delta \in \big(0,\ \delta(\Omega)\big]$, the relation $|x - [y - \delta\, n^{(\Omega)}(y)]| \ge \mathcal{D}\,\delta/2$ holds by (3.6). Since $E \in C^\infty(\mathbb{R}^3\backslash\{0\})^{3\times 3}$, we may conclude that the claim on $V^{(-\delta)}(\phi)$ in (4.8) follows from Lebesgue's theorem. For the proof of (4.9) and (4.10), we refer to the proof of [7, Lemma 5.4],

Concerning the proof of respectively (4.11) and (4.12), we refer to [9, Lemma 6.3] and [9, (6.20), (4.72)], respectively, as concerns $V^{(-\delta)}(\phi)$. When $V^{(\delta)}(\phi)$ is considered, the relevant references are [9, (6.10)] and [9, (6.19), (4.71)].

We finally note that because of Lemma 2.1, inequality (3.4) and Lebesgue's theorem, the relation $\|V^{(\pm\delta)}(\phi) - V(\phi)|\partial\Omega\|_p \to 0$ $(\delta \downarrow 0)$ holds. The last claim of Lemma 4.3 follows from the preceding relation, (4.9), (4.10), and C^∞-regularity of $V^{(\pm\delta)}(\phi)$ on $\mathfrak{U}_\pm$. □

Lemma 4.4 *The inequality*

$$|\partial_l\big(\, V(\phi)|\mathbb{R}^3\backslash\partial\Omega\,\big)\big(\, x \pm \delta\, n^{(\Omega)}(x)\,\big) - \partial_l V^{(\mp\delta)}(\phi)(x)| \le \mathfrak{C}\,|\phi|_\infty \delta^{1/2}$$

holds for $\phi \in C^0(\partial\Omega)^3,\ x \in \partial\Omega,\ \delta \in \big(0,\ \delta(\Omega)\big]$.

Proof: This lemma follows from (3.4); see the proof of [7, Corollary 5.3] or [9, (6.21), (6.22)]. □

Next we introduce double layer potentials related to the Stokes system.

Lemma 4.5 *Let $\phi \in L^1(\partial\Omega)^3$. Then for $x \in \mathbb{R}^3\backslash\partial\Omega,\ l \in \{1, 2, 3\}$, put*

$$W(\phi)_l(x) := \int_{\partial\Omega} -\sum_{j,k=1}^{3} S_{jkl}(x-y)\,\phi_j(y)\,n_k^{(\Omega)}(y)\,do_y,$$

$$\Pi(\phi)(x) := \int_{\partial\Omega} (-2)\sum_{j,k=1}^{3} (\partial_j\partial_k\mathfrak{N})(x-y)\,\phi_j(y)\,n_k^{(\Omega)}(y)\,do_y.$$

Then $W(\phi)_l$ and $\Pi(\phi)$ belong to $C^\infty(\mathbb{R}^3\backslash\partial\Omega)$ for $1 \le l \le 3$, and any derivative of these functions commutes with the integration over $\partial\Omega$ appearing in their definition. Moreover the pair $\big(W(\phi),\ \Pi(\phi)\big)$ solves (1.1) with $f = 0$. (The functions S_{jkl} and $\mathfrak{N}$ were introduced in (4.2) and (4.1), respectively.)

Proof: Lebesgue's theorem and the relation E_{jk}, $\mathfrak{N} \in C^\infty(\mathbb{R}^3\backslash\{0\})$ for $1 \le j, k \le 3$ yield the lemma except its last claim, which follows from (4.5). □

Lemma 4.6 *Let $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$, and put $\delta := dist(\overline{\Omega},\, B_R^c)$. Then*

$$|\partial^\alpha V(\phi)(x)| \le C(\delta,R)\,\|\phi\|_1\,|x|^{-1-|\alpha|},\quad |\partial^\alpha W(\phi)(x)| \le C(\delta,R)\,\|\phi\|_1\,|x|^{-2-|\alpha|} \tag{4.13}$$

$$|\partial^\alpha Q(\phi)(x)| \le C(\delta,R)\,\|\phi\|_1\,|x|^{-2-|\alpha|},\quad |\partial^\alpha \Pi(\phi)(x)| \le C(\delta,R)\,\|\phi\|_1\,|x|^{-3-|\alpha|} \tag{4.14}$$

for $\phi \in L^1(\partial\Omega)^3$, $x \in B_R^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$. Consequently, if $r_1 \in (1,\infty)$, $r_2 \in (3/2,\,\infty)$, $r_3 \in (3,\infty)$, then for $1 \le l,m \le 3$, $\phi \in L^1(\partial\Omega)^3$,

$$\begin{aligned}&\|\partial_m\partial_l V(\phi)|B_R^c\|_{r_1} + \|\partial_m Q(\phi)|B_R^c\|_{r_1} + \|\partial_m W(\phi)|B_R^c\|_{r_1} \\ &\qquad + \|\Pi(\phi)|B_R^c\|_{r_1} + \|\partial_m \Pi(\phi)|B_R^c\|_{r_1} \le C(\delta,R,\,r_1)\,\|\phi\|_1,\end{aligned} \tag{4.15}$$

$$\|W(\phi)|B_R^c\|_{r_2} + \|\partial_m V(\phi)|(B_R)^c\|_{r_2} + \|Q(\phi)|B_R^c\|_{r_2} \le C(\delta,R,\,r_2)\,\|\phi\|_1, \tag{4.16}$$

$$\|V(\phi)|B_R^c\|_{r_3} \le C(\delta,R,\,r_3)\,\|\phi\|_1. \tag{4.17}$$

Proof: Obviously $\delta > 0$. Let $y \in \partial\Omega\backslash\{0\}$ and put $z := |y|^{-1}\,R\,y$. Then $|y| + |z-y| \le |z| = R$, and $|z-y| \ge \delta$, so $|y| \le R - \delta$. Hence for $x \in B_R^c$ we get

$$|x-y| \ge (\delta/R)\,|x| + (1-\delta/R)\,|x| - |y| \ge (\delta/R)\,|x| + R - \delta - |y| \ge (\delta/R)\,|x|.$$

Now the lemma follows from (4.4). □

We introduce two integral operators, denoted by $\mathfrak{T}$ and $\mathfrak{T}^*$ and defined by double layer potentials, which map the space $L^p(\partial\Omega)^3$ into itself, for any $p \in (1,\infty)$. These operators are closely linked with the boundary values of $W(\phi)|\Omega$ and $W(\phi)|\overline{\Omega}^c$, which do not coincide except if $\phi = 0$ (Theorem 4.5).

Lemma 4.7 *The inequality $|\sum_{k=1}^3 S_{jkl}(x-y)\,n_k^{(\Omega)}(x)| \le \mathfrak{C}\,|x-y|^{-1}$ holds for $x,y \in \partial\Omega$ with $x \ne y$, $1 \le j,l \le 3$. The preceding estimate remains valid if the term $n_k^{(\Omega)}(x)$ is replaced by $n_k^{(\Omega)}(y)$.*

If $\phi \in L^1(\partial\Omega)^3$, $1 \le j \le 3$, we may define

$$\mathfrak{T}^*(\phi)_j(x) := (-2)\int_{\partial\Omega} \sum_{k,l=1}^3 S_{jkl}(x-y)\,n_k^{(\Omega)}(x)\,\phi_l(y)\,do_y$$

for $x \in \partial\Omega$. Then, for any $p \in (1,\infty)$, $\phi \in L^p(\partial\Omega)^3$, the relation $\mathfrak{T}^*(\phi) \in L^p(\partial\Omega)^3$ is valid, and the operator $T_p^* : L^p(\partial\Omega)^3 \mapsto L^p(\partial\Omega)^3$, $T_p^*(\phi) := \mathfrak{T}^*(\phi)$ $\big(\phi \in L^p(\partial\Omega)^3\big)$ is linear, bounded and compact.

Define the function $\mathfrak{T}(\phi)$ by replacing the term $n_k^{(\Omega)}(x)$ by $-n_k^{(\Omega)}(y)$ in the definition of $\mathfrak{T}^*(\phi)$. An operator T_p may be associated with $\mathfrak{T}$ in the same way as T_p^* is associated with $\mathfrak{T}^*$, for $p \in (1,\infty)$. Then equally $T_p : L^p(\partial\Omega)^3 \mapsto L^p(\partial\Omega)^3$ is linear, bounded and compact.

Let $I_q : L^q(\partial\Omega)^3 \mapsto L^q(\partial\Omega)^3$ denote the identity mapping of $L^q(\partial\Omega)^3$. Then the operators $\pm I_p + T_p$ and $\pm I_p + T_p^*$ are Fredholm with index zero.

For $q \in (1,\infty)$, the operator T_q^* is dual to $T_{q'}$.

Proof: For the estimate of $|\sum_{k=1}^3 S_{jkl}(x-y)\, n_k^{(\Omega)}(x)|$ stated in the lemma see (3.5) and the first equation in (4.5). In the case that the term $n_k^{(\Omega)}(x)$ is replaced by $-n_k^{(\Omega)}(y)$, the same references may be used.

Lemma 2.1 yields that $\mathfrak{T}^*(\phi)$ and $\mathfrak{T}(\phi)$ are well defined for $\phi \in L^1(\partial\Omega)^3$, and $\mathfrak{T}^*(\phi)$, $\mathfrak{T}(\phi) \in L^p(\partial\Omega)^3$ if $p \in (1,\infty)$ and $\phi \in L^p(\partial\Omega)^3$. Moreover it follows from Lemma 2.2 that T_p^* and T_p are linear, bounded and compact. The general theory of Fredholm operators now implies that $\pm I_p + T_p$ and $\pm I_p + T_p^*$ are Fredholm with index zero. The last statement of the lemma is a consequence of Fubini's theorem. □

Theorem 4.1 *Let $p \in (1,\infty)$. Then with the notation of Lemma 4.7,*

$$ker(-I_p + T_p^*) = \{k\, n^{(\Omega)} \,:\, k \in \mathbb{R}\}, \quad ker(I_p + T_p) = span\big\{\phi^{(j)}|\partial\Omega \,:\, j \in \{1,\, ...,\, 6\}\,\big\}.$$

Proof: [9, Lemma 6.7, 6.5, 6.10]. □

Corollary 4.2 *Let $p \in (1,\infty)$. Then $dim\, ker(I_p + T_p^*) = 6$ and $dim\, ker(-I_p + T_p) = 1$,*

$$ran(I_p + T_p^*) = \{v \in L^p(\partial\Omega)^3 \,:\, \int_{\partial\Omega} v \cdot \phi^{(j)} do_x = 0 \text{ for } 1 \le j \le 6\}, \tag{4.18}$$

$$ran(-I_p + T_p^*) = \{v \in L^p(\partial\Omega)^3 \,:\, \int_{\partial\Omega} v \cdot \gamma\, do_x \ \text{ for } \gamma \in ker(-I_{p'} + T_{p'})\}. \tag{4.19}$$

Proof: By Lemma 4.7, we know that the operator $I_p + T_p^*$ is Fredholm with index zero, so $dim\, ker(I_p + T_p^*) = codim\, ran(I_p + T_p^*)$. On the other hand, the fact that $I_p + T_p^*$ is Fredholm means in particular this operator has closed range. We further recall that $I_{p'} + T_{p'}$ is dual to $I_p + T_p^*$ (Lemma 4.7). It follows with the closed range theorem and Riesz' representation theorem in L^p-spaces that $codim\, ran(I_p + T_p^*) = dim\, ker(I_{p'} + T_{p'})$ and

$$ran(I_p + T_p^*) = \{v \in L^p(\partial\Omega)^3 \,:\, \int_{\partial\Omega} v \cdot \gamma\, do_x = 0 \text{ for } \gamma \in ker(I_{p'} + T_{p'})\}.$$

Equation (4.18) now is a consequence of Theorem 4.1, and the equation $dim\, ker(I_p + T_p^*) = 6$ follows with Theorem 4.1. An analogous reasoning based on Lemma 4.7 and Theorem 4.1 yields that $dim\, ker(-I_p + T_p) = 1$ and that equation (4.19) holds. □

Theorem 4.2 *Let $p \in (1,\infty)$, $a \in [0,1)$, $b \in C^a(\partial\Omega)^3$, $\psi \in L^p(\partial\Omega)^3$ with $\pm\psi + \mathfrak{T}(\psi) = b$ or $\pm\psi + \mathfrak{T}^*(\psi) = b$. Then $\psi \in C^a(\partial\Omega)^3$.*

This means in particular that $ker(\pm I_p + T_p) = ker(\pm I_q + T_q)$ and $ker(\pm I_p + T_p^*) = ker(\pm I_q + T_q^*)$ for $q \in (1,\infty)$.

Proof: See [9, Lemma 5.4]. □

Corollary 4.3 *Let $p \in (1,\infty)$. Then $ker(\pm I_p + T_p^*)$ possesses a topological complement $E_p^{(\pm)}$ in $L^p(\partial\Omega)^3$. The estimate $\|\phi\|_p \le \mathfrak{C}\, \|(\pm\phi + T_p^*)(\phi)\|_p$ holds for $\phi \in E_p^{(\pm)}$.*

If $b \in L^p(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot \phi^{(j)}\, do_x = 0$ for $1 \le j \le 6$, there is a unique function $F_p^{(+)}(b) \in E_p^{(+)}$ with $(1/2)\,\big[\, F_p^{(+)}(b) + \mathfrak{T}^*\big(\, F_p^{(+)}(b)\,\big)\,\big] = b$. (See Theorem 1.2 for the definition of the functions $\phi^{(1)}$, ..., $\phi^{(6)}$.)

Fix a function $\psi^{(0)} \in ker(-I_{p'} + T_{p'})\backslash\{0\}$. (In view of Theorem 4.2, this function $\psi^{(0)}$ belongs to $ker(-I_r + T_r)$ for any $r \in (1,\infty)$.) If $b \in L^p(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot \psi^{(0)}\, do_x = 0$, then there is a unique function $F_p^{(-)}(b) \in E_p^{(-)}$ with $(1/2)\left[-F_p^{(-)}(b) + \mathfrak{T}^*\left(F_p^{(-)}(b) \right) \right] = b$.

Suppose that $q,\, r \in (1,\infty)$ and $b \in L^q(\partial\Omega)^3 \cap L^r(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot \phi^{(j)}\, do_x = 0$ for $1 \le j \le 6$. Then $F_q^{(+)}(b) = F_r^{(+)}(b)$. Similarly $F_q^{(-)}(b) = F_r^{(-)}(b)$ if $b \in L^q(\partial\Omega)^3 \cap L^r(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot \psi^{(0)}\, do_x = 0$. Therefore, in view of (4.18), we will write $F^{(\pm)}(b)$ instead of $F_p^{(\pm)}(b)$ if $b \in ran(\pm I_p + T_p)$.

Proof: Since $dim\, ker(\pm I_p + T_p^*) < \infty$ by Corollary 4.2 and Theorem 4.1, the existence of a complement $E_p^{(\pm)}$ as described in Corollary 4.3 follows by general theory. Obviously $\pm I_p + T_p^* | E_p^{(\pm)}$ is a bijective operator from $E_p^{(\pm)}$ onto $ran(\pm I_p + T_p^*)$. On the other hand, $\pm I_p + T_p^*$ is Fredholm (Lemma 4.7), and thus has closed range. It follows by the open mapping theorem that $\|\phi\|_p \le \mathfrak{C}\, \|(\pm I_p + T_p^*)(\phi)\|_p$ for $\phi \in E_p^{(\pm)}$, hence $\|\phi\|_p \le \mathfrak{C}\, \| \pm \phi + \mathfrak{T}^*(\phi)\|_p$ for such ϕ.

For any $b \in L^p(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot \phi^{(j)}\, do_x = 0$ for $1 \le j \le 6$, we know by (4.18) that $b \in ran(I_p + T_p^*)$. Thus, since $I_p + T_p^* | E_p^{(+)}$ is a bijective operator from $E_p^{(+)}$ onto $ran(I_p + T_p^*)$, there is a unique function $F^{(+)}(b)$ with the properties stated in the corollary.

Since $dim\, ker(-I_{p'} + T_{p'}) = 1$, any function $b \in L^p(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot \psi^{(0)}\, do_x = 0$ verifies the equation $\int_{\partial\Omega} b \cdot \gamma\, do_x = 0$ for any $\gamma \in ker(-I_{p'} + T_{p'})$. Thus an analogous reasoning as in the case of $F^{(+)}(b)$, but based on (4.19) instead of (4.18), yields existence of a unique function $F^{(-)}(b)$ with properties as described in the corollary. The last claim of the corollary follows from Theorem 4.2. □

Lemma 4.8 *Put $M := (\int_{\partial\Omega} \phi^{(j)} \cdot \phi^{(k)}\, do_x)_{1\le j,k\le 3}$ and $\widetilde{M} := (\int_{\Omega} \phi^{(j)} \cdot \phi^{(k)}\, dx)_{1\le j,k\le 3}$, where $\phi^{(1)}$, ..., $\phi^{(6)}$ were introduced in Theorem 1.2. Then the matrices M and $\widetilde{M}$ are invertible.*

Proof: The functions $\phi^{(1)}|\partial\Omega$, ..., $\phi^{(6)}|\partial\Omega$ are linearly independent ([9, Lemma 6.5]). Let $\alpha \in \mathbb{R}^6$ with $M \cdot \alpha = 0$. Then $\sum_{j=1}^6 \alpha_j \sum_{k=1}^6 \int_{\partial\Omega} \phi^{(j)} \cdot \phi^{(k)}\, do_x\, \alpha_k = 0$, that is, $\int_{\partial\Omega} |\sum_{l=1}^6 \alpha_l\, \phi^{(l)}|^2\, do_x = 0$, so $\sum_{l=1}^6 \alpha_l\, \phi^{(l)} = 0$, and finally $\alpha = 0$. This means that M is invertible. Obviously the functions $\phi^{(1)}|\Omega$, ..., $\phi^{(6)}|\Omega$ are linearly independent as well. Thus the same argument as for M yields that $\widetilde{M}$ is invertible, too. □

Lemma 4.9 *Let $p \in (1,\infty)$, $b \in W^{1-1/p,\,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} \phi^{(j)} \cdot b\, do_x = 0$ for $1 \le j \le 6$. Then there exists a sequence (b_n) in $C^0(\partial\Omega)^3$ such that for $n \in \mathbb{N}$, the function b_n belongs to $C^a(\partial\Omega)^3$ for $a \in (0,1)$, hence $b_n \in W^{1-1/r,\,r}(\partial\Omega)^3$ for $r \in (1,\infty)$, $\int_{\partial\Omega} \phi^{(j)} \cdot b_n\, do_x = 0$ for $1 \le j \le 6$, and such that $\|b - b_n\|_{1-1/p,\,p} \to 0$.*

Suppose that $\int_{\partial\Omega} \psi^{(0)} \cdot b\, do_x = 0$, where $\psi^{(0)}$ was introduced in Corollary 4.3. Then there is a sequence (b_n) in $C^0(\partial\Omega)^3$ with the same properties as before, except that the condition $\int_{\partial\Omega} \phi^{(j)} \cdot b_n\, do_x = 0$ $(1 \le j \le 6,\ n \in \mathbb{N})$ is replaced by $\int_{\partial\Omega} \psi^{(0)} \cdot b_n\, do_x = 0$ $(n \in \mathbb{N})$.

Proof: Since $b \in W^{1-1/p,\,p}(\partial\Omega)^3$, there is $B \in W^{1,p}(\Omega)^3$ with $B|\partial\Omega = b$. We may choose a sequence (B_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|B_n - B\|_{1,p} \to 0$, and thus $\|B_n - b\|_{1-1/p,\,p} \to 0$. Obviously $\widetilde{b}_n := B_n|\partial\Omega \in C^a(\partial\Omega)^3$ for $n \in \mathbb{N},\ a \in (0,1)$.

Let $n \in \mathbb{N}$ and set $c_n := M^{-1} \cdot (\int_{\partial\Omega} \widetilde{b}_n \cdot \phi^{(j)}\, do_x)_{1\le j\le 6}$, with the matrix M introduced in Lemma 4.8. Then $\int_{\partial\Omega} \widetilde{b}_n \cdot \phi^{(j)}\, do_x = \sum_{k=1}^{6} \int_{\partial\Omega} \phi^{(j)} \cdot \phi^{(k)}\, do_x\, c_{n,k}$ for $1 \le j \le 6$. Thus, putting $b_n := \widetilde{b}_n - \sum_{k=1}^{6} c_{n,k}\, \phi^{(k)}$, we obtain a function b_n belonging to $C^a(\partial\Omega)^3$ for $a \in (0,1)$ and verifying the relation $\int_{\partial\Omega} \phi^{(j)} \cdot b_n\, do_x = 0$ for $1 \le j \le 6$. Since $\|\widetilde{b}_n - b\|_{1-1/p,\,p} \to 0$ and $\int_{\partial\Omega} \widetilde{b}_n \cdot \phi^{(j)}\, do_x = \int_{\partial\Omega} (\widetilde{b}_n - b) \cdot \phi^{(j)}\, do_x$ for $1 \le j \le 6,\ n \in \mathbb{N}$ by our assumptions on b, hence $|c_n| \le \mathfrak{C}\, \|\widetilde{b}_n - b\|_p$ with $\mathfrak{C}$ independent of n, we get in addition that $\|b_n - b\|_{1-1/p,\,p} \to 0$

The second part of the lemma may be proved in the same way as the first, but the reasoning is somewhat simpler because no matrix is involved. Note that by Theorem 4.2, we have $\psi^{(0)} \in C^a(\partial\Omega)^3$ for $a \in (0,1)$. □

The ensuing theorem constitutes the key point in Ladyzhenskaya's theory [19] on $W^{2,p}$-regularity of solutions to (1.1) under Dirichlet boundary conditions.

Theorem 4.3 *Let $p \in (1,\infty)$. If $\phi \in C^a(\partial\Omega)^3$ for some $a \in (0,1)$ and if $\pm\phi + \mathfrak{T}(\phi) \in C^a(\partial\Omega)^3 \cap W^{2-1/p,\,p}(\partial\Omega)^3$, then ϕ belongs to $W^{2-1/p,\,p}(\partial\Omega)^3$ and the inequality $\|\phi\|_{2-1/p,\,p} \le \mathfrak{C}\,(\|\pm\phi + \mathfrak{T}(\phi)\|_{2-1/p,\,p} + \|\phi\|_p)$ holds.*

Proof: [9, Lemma 7.8]. □

The following "jump relation" combined with Lemma 4.2 and Corollary 4.3 allow to solve (1.1), (1.2) by using the single layer potential $V(\phi)$ and $Q(\phi)$. However, due to the nontrivial kernels of $\pm I_p + T_p^*$ (Theorem 4.1, Corollary 4.2), only data satisfying certain conditions may be admitted. The double layer potentials introduced in Lemma 4.5 will be used in order to eliminate these conditions.

Theorem 4.4 *Recall the parameter $\delta(\Omega)$ from Lemma 3.2. Let $\psi \in C^0(\partial\Omega)^3$ and put $U := V(\psi)|\mathbb{R}^3\backslash\partial\Omega,\ \Pi := Q(\psi)$, where $V(\psi)$ and $Q(\psi)$ were introduced in Lemma 4.2. Then*

$$\sum_{k=1}^{3} n_k^{(\Omega)}(x)\,(\partial_j U_k + \partial_k U_j - \delta_{jk}\,\Pi)\big(\,x \pm \epsilon\, n^{(\Omega)}(x)\,\big) \to (1/2)\,\big(\,\mp\psi + \mathfrak{T}^*(\psi)\,\big)_j(x) \quad (4.20)$$

for $\epsilon \to 0,\ \epsilon \in \big(\,0,\,\delta(\Omega)\,\big]$, uniformly with respect to $x \in \partial\Omega,\ 1 \le j \le 3$.

Proof: The relation in (4.20) holds according to [9, Lemma 4.8]. Note that the definition of $\mathfrak{T}^*(\psi)$ in [9] (see [9, Definition 4.2 and 5.1]) coincides with ours in Lemma 4.7. This follows from (4.5). □

Theorem 4.5 *Let $\phi \in C^0(\partial\Omega)^3$. Then the function $W(\phi)|\overline{\Omega}^c$ admits a continuous extension to Ω^c, denoted by $W_{ex}(\phi)$ and given by $W_{ex}(\phi)|\partial\Omega = (-1/2)\,\big(\,\phi + \mathfrak{T}(\phi)\,\big)$. Similarly the function $W(\phi)|\Omega$ admits a continuous extension to $\overline{\Omega}$, denoted by $W_{in}(\phi)$ and given by $W_{in}(\phi)|\partial\Omega = (-1/2)\,\big(\,-\phi + \mathfrak{T}(\phi)\,\big)$.*

Proof: See [9, Theorem 4.1]. □

Theorem 4.6 *Recall the parameter $\delta(\Omega)$ introduced in Lemma 3.2. Let $a \in (0,1),\ \phi \in$*

$C^{1,a}(\partial\Omega)^3$, $x \in \partial\Omega$, $j \in \{1, 2, 3\}$. *Define the function* $F : [-\delta(\Omega), \delta(\Omega)]\backslash\{0\} \mapsto \mathbb{C}$ *by*

$$F(\delta) := \sum_{k=1}^{3} n_k^{(\Omega)}(x) \big(\partial_j W(\phi)_k + \partial_k W(\phi)_j - \delta_{jk}\, \Pi(\phi) \big) \big(x + \delta\, n^{(\Omega)}(x) \big)$$

for $\delta \in [-\delta(\Omega), \delta(\Omega)]\backslash\{0\}$. *Then the limits* $\lim_{\delta\downarrow 0} F(\delta)$ *and* $\lim_{\delta\downarrow 0} F(\delta)$ *exist and coincide.*

This result is due to Faxén [10, § 11], but the proof in [10] is rather long (10 pages) and in parts somewhat vague. An analogous result for the Laplace double layer potential is shown in a more precise way by Hackbusch [15, Section 8.5.2]. More general elliptic equations (but not systems) are treated by Miranda [20, Theorem II.15.V]. Kupradze [18, §VI.9, Theorem 10] considers two limits analogous to the ones in Theorem 4.6, but with a "stress operator" applied to double layer potentials associated with the Lamé operator. He shows a result somewhat weaker than the one in the theorem above: If one of the two limits exists for all $x \in \partial\Omega$ and is Hölder continuous as a function of such x, then the other limit exists as well and coincides with the first.

Since Faxen's argument [10] leaves a margin for improvement, and because the other preceding references do not address the Stokes case relevant in Theorem 4.6, we present a proof of that theorem in the Appendix. Lemma 3.2 allows us to carry out this proof without using local coordinates.

Theorem 4.7 *Let* $p \in (3, \infty)$, $b \in W^{2-1/p,\,p}(\partial\Omega)^3$, $\phi \in L^p(\partial\Omega)^3$ *such that one of the two equations* $(-1/2)\big(-\phi + \mathfrak{T}(\phi)\big) = b$ *or* $(-1/2)\big(\phi + \mathfrak{T}(\phi)\big) = b$ *holds. Take* $R \in (0, \infty)$ *with* $\overline{\Omega} \subset B_R$.

Then there is $a \in (0, 1)$ *with* $\phi \in C^{1,a}(\partial\Omega)^3$. *Moreover* $\phi \in W^{2-1/p,\,p}(\partial\Omega)^3$, $W(\phi)|U \in W^{2,p}(U)^3$, $\Pi(\phi)|U \in W^{1,p}(U)$ *for* $U \in \{\Omega, \Omega_R\}$, *and* $W_{in}(\phi) \in C^1(\overline{\Omega})^3$, $W_{ex}(\phi) \in C^1(\Omega^c)^3$. *The functions* $\Pi(\phi)|\Omega$ *and* $\Pi(\phi)|\overline{\Omega}^c$ *may be continuously extended to* $\overline{\Omega}$ *and* Ω^c, *respectively. These extensions are denoted by* $\Pi_{in}(\phi)$ *and* $\Pi_{ex}(\phi)$, *respectively. (The functions* $W_{in}(\phi)$ *and* $W_{ex}(\phi)$ *were introduced in Theorem 4.5.)*

Proof: A direct proof of the relation $\phi \in C^{1,a}(\partial\Omega)^3$ for some $a \in (0, 1)$ may be found in [23]; see [23, Lemma 3.1]. In view of the L^p-estimates available here, we may obtain this result in a a shorter way.

In fact, by a trace theorem and an extension theorem, there is $B \in W_0^{2,p}(\mathbb{R}^3)^3$ with $B|\partial\Omega = b$. Since $p > 3$, a Sobolev inequality implies there is $a \in (0, 1)$ such that $B \in C^{1,\alpha}(\mathbb{R}^3)^3$, so $b \in C^{1,\alpha}(\partial\Omega)^3$. Thus $\phi \in C^a(\partial\Omega)^3$ by Theorem 4.2. Moreover, referring to Theorem 4.3, we see that $\phi \in W^{2-1/p,\,p}(\partial\Omega)^3$. Now the same argument as used above for b provides that $\phi \in C^{1,\alpha}(\partial\Omega)^3$. In addition, [9, Lemma 7.15] yields at this point that $\Pi(\phi)|U \in W^{1,p}(U)^3$ for $U \in \{\Omega, \Omega_R\}$. Since $p > 3$, we may again refer to a Sobolev inequality, obtaining that $\Pi(\phi)|U$ may be continuously extended to $\overline{U}$. Since $\Pi(\phi)|\overline{\Omega}^c \in C^\infty(\overline{\Omega}^c)$, this means in particular that $\Pi(\phi)|\overline{\Omega}^c$ may be continuously extended to Ω^c.

Obviously (Lemma 4.5), we have $W(\phi)|U \in C^\infty(U)$ for U as before. Since in particular $\phi \in C^0(\partial\Omega)^3$, we further know by Theorem 4.5 that $W_{in}(\phi) \in C^0(\overline{\Omega})^3$, $W_{ex}(\phi) \in C^0(\Omega^c)^3$, $(-1/2)\big(-\phi + \mathfrak{T}(\phi)\big) = W_{in}(\phi)|\partial\Omega$, $(-1/2)\big(\phi + \mathfrak{T}(\phi)\big) = W_{ex}(\phi)|\partial\Omega$. Note that $(-1/2)\big(\pm\phi + \mathfrak{T}(\phi)\big) \in \{b,\, b - \phi,\, b + \phi\}$. But b, $b - \phi$, $b + \phi \in W^{2-1/p,\,p}(\partial\Omega)^3 \cap C^0(\partial\Omega)^3$

by what was explained above, so $W_{in}(\phi)|\partial\Omega$, $W_{ex}(\phi)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3 \cap C^0(\partial\Omega)^3$. On the other hand, $W(\phi)|\overline{\Omega}^c \in C^\infty(\overline{\Omega}^c)^3$ and $\overline{\Omega} \subset B_R$, so in particular $W(\phi)|\partial B_R \in W^{2-1/p,\,p}(\partial B_R)^3$. Since $W_{ex}(\phi)|\overline{\Omega}^c = W(\phi)|\overline{\Omega}^c$ by the definition of $W_{ex}(\phi)$, we thus get $W_{ex}(\phi)|\partial\Omega_R \in W^{2-1/p,\,p}(\partial\Omega_R)^3 \cap C^0(\partial\Omega_R)^3$. Recall that $\Delta W(\phi) = \nabla\Pi(\phi)$ (Lemma 4.5) and $\Pi(\phi)|U \in W^{1,p}(U)$ for $U \in \{\Omega,\, \Omega_R\}$, as mentioned above. Further recall that $W_{in}(\phi)|\Omega = W(\phi)|\Omega$. Altogether we see that Lemma 2.7 may be applied; it yields that $W(\phi)|U \in W^{2,p}(U)^3$ for $U \in \{\Omega,\, \Omega_R\}$. Due to the assumption $p > 3$ and a Sobolev inequality, it follows that the function $W(\phi)|U$ may be extended to a C^1-function in $\overline{U}$, for U as before, so $W_{in}(\phi) \in C^1(\overline{\Omega})^3$ and $W_{ex}(\phi)|\overline{\Omega_R} \in C^1(\overline{\Omega_R})^3$. But $W_{ex}(\phi)|\overline{\Omega}^c = W(\phi)|\overline{\Omega}^c \in C^\infty(\overline{\Omega}^c)^3$, so $W_{ex}(\phi) \in C^1(\Omega^c)^3$. $\square$

The next lemma indicates how for a given function $b \in C^0(\partial\Omega)^3$, a function $\psi \in C^0(\partial\Omega)$ may be chosen so that the Dirichlet boundary data of $V(b)|\Omega$ and $W(\psi)|\Omega$ coincide. The same question is answered for the boundary values of $V(b)|\overline{\Omega}^c$ and $W(\psi)|\overline{\Omega}^c$.

Lemma 4.10 *Let ϕ, $b \in C^0(\partial\Omega)^3$ with $(1/2)\big(\mp\phi + \mathfrak{T}^*(\phi)\big) = b$. Then $V(\phi) \in C^\kappa(\mathbb{R}^3)^3$ for $\kappa \in [0,1)$ and*

$$(1/2)\big[\mp V(\phi)|\partial\Omega + \mathfrak{T}\big(V(\phi)|\partial\Omega\big)\big] = V(b)|\partial\Omega.$$

Note that the term $(1/2)\big[-V(\phi)|\partial\Omega + \mathfrak{T}\big(V(\phi)|\partial\Omega\big)\big]$ coincides with the Dirichlet boundary data of $-W\big(V(\phi)|\partial\Omega\big)|\Omega$, and the function $(1/2)\big[V(\phi)|\partial\Omega + \mathfrak{T}\big(V(\phi)|\partial\Omega\big)\big]$ with those of $-W\big(V(\phi)|\partial\Omega\big)|\overline{\Omega}^c$ (Theorem 4.5).

Proof: We consider the case $(1/2)\big(-\phi + \mathfrak{T}^*(\phi)\big) = b$. If $(1/2)\big(\phi + \mathfrak{T}^*(\phi)\big) = b$, an analogous reasoning is valid.

The relation $V^{(\lambda)}(\phi)|\partial\Omega \in C^\kappa(\partial\Omega)^3$ for $\kappa \in [0,1)$ holds according to Lemma 4.2.

Recall the parameter $\delta(\Omega) > 0$, as well as the set $\mathfrak{U}_{-\delta}$ for $\delta \in \big(0,\, \delta(\Omega)\big]$ from Lemma 3.2. Put $V^{(-\delta)} := V^{(-\delta)}(\phi)$, $Q^{(-\delta)} := Q^{(-\delta)}(\phi)$. These functions were introduced in Lemma 4.3. Note that the set $\mathfrak{U}_{-\delta} \subset \mathbb{R}^3$ is open and $\Omega^c \subset \mathfrak{U}_{-\delta}$ (Lemma 3.2), $V^{(-\delta)} \in C^\infty(\mathfrak{U}_{-\delta})^3$, $Q^{(-\delta)} \in C^\infty(\mathfrak{U}_{-\delta})$, and and the pair $(V^{(-\delta)},\, Q^{(-\delta)})$ satisfies (1.1) in $\mathfrak{U}_{-\delta}$ with $f = 0$, for $\delta \in \big(0,\, \delta(\Omega)\big]$ (Lemma 4.3).

Take $l \in \{1,\, 2,\, 3\}$, $x \in \overline{\Omega}^c$ and $\delta \in \big(0,\, \delta(\Omega)\big]$. Let $R \in (0,\infty)$ such that $\{x\} \cup \overline{\Omega} \subset B_{R/2}$. We write $n^{(\Omega_R)}$ for the outward unit normal to Ω_R, that is, $n^{(\Omega_R)}(z) = -n^{(\Omega)}(z)$ for $z \in \partial\Omega$, $n^{\Omega_R}(z) = R^{-1}\, z$ for $z \in \partial B_R$. Then it follows by a standard representation formula for solutions to (1.1) (see [9, (3.6)] for example) that

$$V_l^{(-\delta)}(x) = \int_{\partial\Omega_R} \sum_{j,k=1}^{3} \big(\, E_{jl}(x-z)\,(\partial_k V_j^{(-\delta)} + \partial_j \widetilde{V}_k^{(-\delta)} - \delta_{jk}\, Q^{(-\delta)})(z) \qquad (4.21)$$

$$-S_{jkl}(x-z)\, V_j^{(-\delta)}(z)\big)\, n_k^{(\Omega_R)}(z)\, do_z$$

for $1 \le l \le 3$, $x \in \Omega_R$. Note that $V^{(-\delta)}$ is continuous, so the restriction "a. e." on $x \in \Omega_R$ in [9, (3.6)] may be dropped.

Since $\{x\} \cup \overline{\Omega} \subset B_{R/2}$, and because $y - \delta\, n^{(\Omega)}(y) \in \Omega$ for $y \in \partial\Omega$ (Lemma 3.2), we find for $z \in \partial B_R$, $y \in \partial\Omega$ that $|z - \big(y - \delta\, n^{(\Omega)}(y)\big)| \ge R/2$ and $|x - z| \ge R/2$. As a consequence,

with (4.4), for $z \in \partial B_R,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1,\ 1 \le j,l,k \le 3$,

$$|\partial^\alpha V^{(-\delta)}(z)| \le \mathfrak{C}\, R^{-1-|\alpha|},\ |Q^{(-\delta)}(z)| \le \mathfrak{C}\, R^{-2},\ |\partial_z^\alpha \big(E_{jl}(x-z)\,\big)| \le \mathfrak{C}\, R^{-1-|\alpha|},$$
$$|(\partial_l \mathfrak{N})(x-z)| + |S_{jkl}(x-z)| \le \mathfrak{C}\, R^{-2}.$$

Thus, by letting R tend to infinity in (4.21), the integral over ∂B_R implicitly present in that equation tends to zero. Hence the integral over $\partial\Omega_R$ becomes an integral over $\partial\Omega$, with $n^{(\Omega_R)}$ replaced by $-n^{(\Omega)}$. Next we use (4.8), (4.11), (4.12) in order to let δ tend to zero. In this way we get that

$$V(\phi)_l(x) = \int_{\partial\Omega} \Big(\sum_{j=1}^3 -E_{jl}(x-z)\,(1/2)\,\big(-\phi + \mathfrak{T}^*(\phi)\,\big)_j(z) \tag{4.22}$$
$$+ \sum_{j,k=1}^3 S_{jkl}(x-z)\, V(\phi)_j(z)\, n_k^{(\Omega)}(z) \Big)\, do_z.$$

We recall that $(1/2)\,\big(-\phi + \mathfrak{T}^*(\phi)\,\big) = b$. Abbreviate $w := V(\phi)|\partial\Omega$, and note that $w \in C^0(\partial\Omega)^3$; see at the beginning of this proof. We may then rewrite (4.22) as

$$V(\phi)_l(x) = -V(b)_l(x) - W(w)_l(x), \tag{4.23}$$

with $W(w)$ defined in Lemma 4.5. This is true for any $x \in \overline{\Omega}^c$. Since $w \in C^0(\partial\Omega)^3$, Theorem 4.5 yields that the function $W(w)$ may be extended continuously to Ω^c. This extension was denoted by $W_{ex}(w)$; see Theorem 4.5. By that theorem, we know that $W_{ex}(w)(z) = (-1/2)\,\big(w + \mathfrak{T}(w)\,\big)(z)$ for $z \in \partial\Omega$. Take $x_0 \in \partial\Omega$. Thus we may conclude that $W(w)(x) \to (-1/2)\,\big(w + \mathfrak{T}(w)\,\big)(x_0)$ for $x \to x_0$, $x \in \overline{\Omega}^c$. On the other hand, since $b,\ \phi \in C^0(\partial\Omega)^3$, we know from Lemma 4.2 that $V(b),\ V(\phi) \in C^0(\mathbb{R}^3)^3$. Thus, by letting x tend to x_0 in $\overline{\Omega}^c$, we get from (4.23) that $V(\phi)_l(x_0) = -V(b)_l(x_0) + (1/2)\,\big(w + \mathfrak{T}(w)\,\big)_l(x_0)$. But $V(\phi)(x_0) = w(x_0)$ by the definition of w, so we finally arrive at the equation $0 = -V(b)_l(x_0) + (1/2)\,\big(-w + \mathfrak{T}(w)\,\big)_l(x_0)$. □

5 $W^{2-1/p,\,p}$-regularity of $V(\phi)|\partial\Omega$.

In this section, we address the key element of our theory, that is, the fact that $V(\phi)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3$ if $\phi \in W^{1-1/p,\,p}(\partial\Omega)^3$. The proof of this relation constitutes the main difficulty we have to put up with, and is split into the proofs of the next two theorems. The result in the first – Theorem 5.1 – amounts to an $W^{1,p}$-estimate of $V(\phi)|\partial\Omega$ against the L^p-norm of ϕ. (The function $V(\phi)$ was introduced in Lemma 4.2.)

Theorem 5.1 *Fix numbers $k(\Omega) \in \mathbb{N},\ \alpha(\Omega) \in (0,\infty)$, sets $\Delta_\kappa,\ U_{t,\kappa},\ \Lambda_{t,\kappa}$ and functions $a_t,\ \gamma_t$ for $\kappa \in (0,1],\ 1 \le t \le k(\Omega)$ as specified in Lemma 3.1.*

Let $t \in \{1,\ ...,\ k(\Omega)\}$. For $f : \Delta_1 \mapsto \mathbb{C}^3$, define $\mathcal{Z}_t(f) : \partial\Omega \mapsto \mathbb{C}^3$ as the zero extension of $f \circ (\gamma_t)^{-1} : \Lambda_{t,1} \mapsto \mathbb{C}^3$ to $\partial\Omega$. Fix a function $\Psi_t \in C_0^\infty(U_{t,\,3/4})$ with $\Psi_t | U_{t,\,1/2} = 1$.

Let $p \in (1,\infty)$. Then, for $f \in L^p(\Delta_1)^3$, the function $\Psi_t\, \mathcal{Z}_t(f)$ belongs to $L^p(\partial\Omega)^3$, the function $V\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big) \circ \gamma_t$ is in $W^{1,p}(\Delta_t)^3$, and

$$\|V\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big) \circ \gamma_t\|_{1,p} \le \mathfrak{C}\, \|f\|_p. \tag{5.1}$$

Proof: Recall that $a_t \in C^2(\overline{\Delta_1})$, $|\nabla a_t|_\infty < 1/4$, and there is an orthonormal matrix $D_t \in \mathbb{R}^{3\times 3}$ and a vector $C_t \in \mathbb{R}^3$ with $\gamma_t(\eta) = D_t \cdot \big(\,\eta,\, a_t(\eta)\,\big) + C_t$ for $\eta \in \Delta_t$; see Lemma 3.1. We have $\Psi_t \in C_0^\infty(U_{t,\,3/4})$ by the choice of Ψ_t in the theorem, so $\Psi_t \circ \gamma_t \in C_0^2(\Delta_{3/4})$ by the definition of $U_{t,\,3/4}$ and γ_t in Lemma 3.1. In addition we will use the function J_t (surface element) introduced in Lemma 3.1, as well as the parameter $\delta(\Omega)$, which was fixed in Lemma 3.2. Let $\sigma_0 \in (0,\infty)$ be so small that $B^2_{\sigma_0}(\varrho) \subset \Delta_1$ for $\varrho \in \Delta_{3/4}$. We introduce some additional notation. For $\varrho,\, \eta \in \Delta_1,\ \widetilde{\eta} \in \mathbb{R}^2,\ \delta \in \big[\,0,\, \delta(\Omega)\,\big]$, put

$$\begin{aligned}
&\Gamma(\varrho,\eta,\delta) := \Gamma_t(\varrho,\eta,\delta) := \gamma_t(\varrho) - \gamma_t(\eta) - \delta\,(n^{(\Omega)} \circ \gamma_t)(\eta),\\
&\overline{\Gamma}(\varrho,\eta,\delta) := \overline{\Gamma}_t(\varrho,\eta,\delta) := \gamma_t(\varrho) - \delta\,(n^{(\Omega)} \circ \gamma_t)(\varrho) - \gamma_t(\eta),\\
&\widetilde{\Gamma}(\varrho,\widetilde{\eta}) := \widetilde{\Gamma}_t(\varrho,\widetilde{\eta}) := D_t \cdot \big(\,\varrho - \widetilde{\eta},\, \nabla a_t(\varrho)\cdot(\varrho-\widetilde{\eta})\,\big),\\
&\mathfrak{E} := (\Psi_t \circ \gamma_t)\, J_t.
\end{aligned}$$

Let $f \in C^1(\Delta_1)^3 \cap L^p(\Delta_1)^3,\ j \in \{1,\,2,\,3\}$ and $\nu \in \{1,2\}$. Since $\Psi_t \circ \gamma_t \in C_0^2(\Delta_{3/4})$, we have $\mathfrak{E} \in C_0^1(\Delta_{3/4})^3$, so $\mathfrak{E}\, f \in C_0^1(\Delta_{3/4})^3$. In particular $\mathfrak{E}\, f$ considered as a function with domain $\mathbb{R}^2$ belongs to $C^\kappa(\mathbb{R}^2)^3$ for any $\kappa \in [0,1)$, and to $L^p(\mathbb{R}^2)^3$, and we may define

$$\mathcal{F}(f)(\varrho,\eta) := \mathfrak{E}(\eta)\, f(\eta) - \mathfrak{E}(\varrho)\, f(\varrho) \ \text{ for } \ \varrho,\, \eta \in \mathbb{R}^2.$$

In addition $(\gamma_t)^{-1} : \Lambda_{t,1} \mapsto \Delta_1$ is continuous (Lemma 3.1), so $\Psi_t\, \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$ and $supp\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big) \subset \Lambda_{t,\,3/4}$. Note that $\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big) \circ \gamma_t = (\Psi_t \circ \gamma_t)\, f$, so due to equation (3.2),

$$\|\Psi_t\, \mathcal{Z}_t(f)\|_p = \|\,|(\Psi_t \circ \gamma_t)\, f|^p\, J_t\|_1 \le \mathfrak{C}\, \|f\|_p, \tag{5.2}$$

with $\mathfrak{C}$ independent of f. We consider the function $V^{(\delta)}\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)$ introduced in Lemma 4.3. According to that reference, this function is C^∞ in an open set $\mathfrak{U}_\delta$ containing $\overline{\Omega}$ as a subset, and

$$\partial_l V^{(\delta)}\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)_j(x) = \int_{\partial\Omega} \sum_{k=1}^3 (\partial_l E_{jk})(x - [y + \delta\, n^{(\Omega)}(y)])\, \big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)_k(y)\, do_y \tag{5.3}$$

for $\delta \in \big(\,0,\, \delta(\Omega)\,\big]$, $x \in \mathfrak{U}_\delta$, $1 \le l \le 3$, with $(E_{jk})_{1\le j,k\le 3}$ introduced in (4.1). Thus $V^{(\delta)}\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t$ is a C^1-function, and we get with (3.2) and (5.3) that

$$\begin{aligned}
&\partial_\nu\big[\,V^{(\delta)}\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big](\varrho)\\
&= \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} (\partial_l E_{jk})\big(\,\Gamma(\varrho,\eta,\delta)\,\big)\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta = \sum_{\mu=1}^4 F^{(\mu)}(\varrho,\delta),
\end{aligned}$$

for $\varrho \in \Delta_1,\ \delta \in \big(0,\ \delta(\Omega)\,\big]$, with

$$F^{(1)}(\varrho,\delta) := \sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} \Big((\partial_l E_{jk})\big(\,\Gamma(\varrho,\eta,\delta)\,\big) - (\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\Big)\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta,$$

$$F^{(2)}(\varrho,\delta) := \sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} (\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big) \mathcal{F}(f)(\varrho,\eta)_k\, d\eta,$$

$$F^{(3)}(\varrho,\delta) := \sum_{k,l=1}^{3} f_k(\varrho)\, J_t(\varrho) \int_{\Delta_1} (\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big) \Big(\partial_\nu \gamma_t(\varrho)_l\, (\Psi_t \circ \gamma_t)(\varrho) - \partial_\nu \gamma_t(\eta)_l\, (\Psi_t \circ \gamma_t)(\eta)\,\Big)\, d\eta,$$

$$F^{(4)}(\varrho,\delta) := \sum_{k=1}^{3} f_k(\varrho)\, J_t(\varrho) \int_{\Delta_1} E_{jk}\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\, \partial_\nu (\Psi_t \circ \gamma_t)(\eta)\,\Big)\, d\eta.$$

The form chosen for the definition of $F^{(4)}(\varrho,\delta)$ arises after a partial integration with respect to $\eta \in \Delta_1$, which is possible due to (3.6). Let $\zeta \in C_0^\infty(\Delta_1)$. Since $\Psi_t\, \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$, as mentioned above, we may deduce from the uniform convergence in (4.11) that

$$\int_{\Delta_1} \partial_\nu \zeta(\varrho)\, \big[\, V^{(\delta)}\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big](\varrho)\, d\varrho \to \int_{\Delta_1} \partial_\nu \zeta(\varrho)\, \big[\, V\big(\,\Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big](\varrho)\, d\varrho$$

for $\delta \downarrow 0$. Also because $\Psi_t\, \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$, we may refer to Lemma 4.4 and equation (3.2) to obtain that the integral $\int_{\Delta_1} \zeta(\varrho)\, F^{(1)}(\varrho,\delta)\, d\varrho$ tends to zero for $\delta \downarrow 0$. As explained above, $\mathfrak{E}\, f \in C_0^1(\Delta_{3/4})^3$, so

$$|\mathcal{F}(f)(\varrho,\eta)| \le \mathfrak{C}\, |\varrho - \eta| \quad \text{for } \ \varrho,\eta \in \Delta_1. \tag{5.4}$$

Here and until further notice, constants $\mathfrak{C}$ are independent of $\varrho,\ \eta \in \Delta_1$ and (not relevant in the case of (5.4)) $\delta \in [0,\ \delta(\Omega)]$. By (4.4) and (3.4), we have $|(\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)| \le \mathfrak{C}\, |\varrho - \eta|^{-2}$, hence

$$|(\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\, \mathcal{F}(f)(\varrho,\eta)_k| \le \mathfrak{C}\, |\varrho - \eta|^{-1} \tag{5.5}$$

for $\varrho,\ \eta \in \Delta_1$ with $\varrho \ne \eta,\ 1 \le k,l \le 3,\ \delta \in \big[\,0,\ \delta(\Omega)\,\big]$. We thus see by Lebesgue's theorem that also for $\delta = 0$, a function $F^{(2)}(\,\cdot\,,\delta) : \Delta_1 \mapsto \mathbb{R}$ may be defined in the same way as in the case $\delta \ne 0$. It further follows that $F^{(2)}(\,\cdot\,,\delta)$ is integrable for $\delta \in [0,\ \delta(\Omega)]$ and

$$\int_{\Delta_1} \zeta(\varrho)\, F^{(2)}(\varrho,\delta)\, d\varrho \to \int_{\Delta_1} \zeta(\varrho)\, F^{(2)}(\varrho,0)\, d\varrho\ (\delta \downarrow 0).$$

Since γ_t belongs to $C^2(\Delta_1)^3$ and has bounded derivatives, and because of (3.4) and the relation $\Psi_t \circ \gamma_t \in C_0^2(\Delta_{3/4})$, we have

$$|(\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\, \big(\,\partial_\nu \gamma_t(\varrho)_l\, (\Psi_t \circ \gamma_t)(\varrho) - \partial_\nu \gamma_t(\eta)_l\, (\Psi_t \circ \gamma_t)(\eta)\,\big)| \le \mathfrak{C}\, |\varrho - \eta|^{-1}, \tag{5.6}$$

$$|E_{jk}\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\, \partial_\nu (\Psi_t \circ \gamma_t)(\varrho)| \le \mathfrak{C}\, |\varrho - \eta|^{-1}$$

for $\varrho,\, \eta \in \Delta_1$ with $\varrho \neq \eta,\ 1 \le k,l \le 3,\ \delta \in \big[\, 0,\, \delta(\Omega) \,\big]$. As a consequence, as in the case of $F^{(2)}(\,\cdot\,,\delta)$, the function $F^{(\mu)}(\,\cdot\,,\delta) : \Delta_1 \mapsto \mathbb{R}$ for $\mu \in \{3,\, 4\}$ may be defined as above also for $\delta = 0$. In addition the function $F^{(\mu)}(\,\cdot\,,\delta)$ is integrable for any $\delta \in [0, \delta(\Omega)]$, and

$$\int_{\Delta_1} \zeta(\varrho)\, F^{(\mu)}(\varrho,\delta)\, d\varrho \to \int_{\Delta_1} \zeta(\varrho)\, F^{(\mu)}(\varrho,0)\, d\varrho \ (\delta \downarrow 0).$$

Altogether we may conclude that the weak derivative $\partial_\nu \big[\, V\big(\, \Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t \,\big]$ exists and

$$\partial_\nu \big[\, V\big(\, \Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t \,\big](\varrho) = \sum_{\mu=2}^{4} F^{(\mu)}(\varrho,0) \quad \text{for } \ \varrho \in \Delta_1. \tag{5.7}$$

We are going to transform $F^{(2)}(\,\cdot\,,0)$. Recall the term $\widetilde{\Gamma}(\varrho,\eta)$ introduced at the beginning of this proof. We have

$$|\overline{\Gamma}(\varrho,\eta,0) - \widetilde{\Gamma}(\varrho,\eta)| = |a_t(\varrho) - a_t(\eta) - \nabla a_t(\varrho)\cdot(\varrho-\eta)| \le \mathfrak{C}|\varrho-\eta|^2,$$

and $|\widetilde{\Gamma}(\varrho,\eta) + \vartheta\,\big(\,\overline{\Gamma}(\varrho,\eta,0) - \widetilde{\Gamma}(\varrho,\eta)\,\big)| \ge |\varrho-\eta|$ for $\vartheta \in [0,1],\ \varrho,\, \eta \in \Delta_1$, so with (4.4),

$$|(\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,0)\,\big) - (\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big)| \le \mathfrak{C}\,|\varrho-\eta|^{-1} \tag{5.8}$$

for $\varrho,\, \eta \in \Delta_1,\ \varrho \neq \eta$, and for $1 \le k,l \le 3$. Since $\mathfrak{E}\, f \in C^1_0(\Delta_{3/4})^3$, we may thus define

$$G^{(1)}(\varrho) := \sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} \Big((\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,0)\,\big) - (\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big) \Big)\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta,$$

$$G^{(2)}(\varrho) := -\sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l\, f_k(\varrho)\, \mathfrak{E}(\varrho) \int_{\Delta_1} \Big((\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,0)\,\big) - (\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big) \Big)\, d\eta$$

for $\varrho \in \Delta_1$. Inequality (5.5) holds with $\overline{\Gamma}(\varrho,\eta,\delta)$ replaced by $\widetilde{\Gamma}(\varrho,\eta)$, so we may further define

$$G^{(3)}(\varrho) := \sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l \int_{B^2_{\sigma_0}(\varrho)} (\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big)\, \mathcal{F}(f)(\varrho,\eta)_k\, d\eta \tag{5.9}$$

for $\varrho \in \Delta_1$. Since

$$\sup\{|(\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big)| \,:\, \varrho,\, \eta \in \Delta_1,\ |\varrho-\eta| \ge \sigma_0,\ 1 \le k,l \le 3\} < \infty, \tag{5.10}$$

we may set

$$G^{(4)}(\varrho) := \sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1 \setminus B^2_{\sigma_0}(\varrho)} (\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big)\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta,$$

$$G^{(5)}(\varrho) := -\sum_{k,l=1}^{3} \partial_\nu \gamma_t(\varrho)_l\, f_k(\varrho)\, \mathfrak{E}(\varrho) \int_{\Delta_1 \setminus B^2_{\sigma_0}(\varrho)} (\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big)\, d\eta$$

for $\varrho \in \Delta_1$. Then

$$F^{(2)}(\varrho,0) = \sum_{\mu=1}^{5} G^{(\mu)}(\varrho) \quad \text{for } \varrho \in \Delta_1. \tag{5.11}$$

Concerning this equation, note that the domain of integration $B^2_{\sigma_0}(\varrho)$ in the definition of $G^{(3)}(\varrho)$ may be replaced by $\Delta_1 \cap B^2_{\sigma_0}(\varrho)$, because $\mathfrak{E}_t f \in C^1_0(\Delta_{3/4})^3$ and due to the choice of σ_0 at the beginning of this proof. In view of (5.7) and (5.11), let us estimate the terms $G^{(\mu)}(\varrho)$ for $\mu \in \{1, ..., 5\}$, as well as $F^{(3)}(\varrho,0)$ and $F^{(4)}(\varrho,0)$. The function $G^{(3)}$ is by far the most difficult to handle since it hides a singular integral. Following [4, (2.1)] (where the term $(\varrho-\eta)_1^s\,(\varrho-\eta)_2^{2\nu-s}$ is lacking), we write $G^{(3)}$ as a series. To this end we recall that $|\nabla a_t(\varrho)| \le 1/4$ ($\varrho \in \Delta_1$) by the specifications on a_t in Lemma 3.1. As a consequence $|\nabla a_t(\varrho)\cdot(\varrho-\eta)|/|\varrho-\eta| \le 1/4 < 1/2$ for $\varrho,\, \eta \in \Delta_1$ with $\varrho \ne \eta$. (The upper bound $1/2$ is sufficient here.) Hence for $\tau \in \mathbb{N},\ \varrho,\, \eta \in \Delta_1$ with $\varrho \ne \eta$,

$$|\widetilde{\Gamma}(\varrho,\eta)|^{-\tau} = |\varrho-\eta|^{-\tau}\left(1 + \big(\,\nabla a_t(\varrho)\cdot(\varrho-\eta)\,\big)^2/|\varrho-\eta|^2\right)^{-\tau/2} \tag{5.12}$$

$$= \sum_{m=0}^{\infty}\binom{-\tau/2}{m}\sum_{n=0}^{2m}\binom{2m}{n}\,\partial_1 a_t(\varrho)^n\,\partial_2 a_t(\varrho)^{2m-n}\,(\varrho-\eta)_1^n\,(\varrho-\eta)_2^{2m-n}\,|\varrho-\eta|^{-\tau-2m}.$$

On the other hand by the definition in (4.1)

$$(\partial_l E_{jk})(z) = (-\delta_{jk}\,z_l + \delta_{jl}\,z_k + \delta_{kl}\,z_j)\,|z|^{-3} - 3\,z_j\,z_k\,z_l\,|z|^{-5} \tag{5.13}$$

for $1 \le k,l \le 3,\ z \in \mathbb{R}^3\backslash\{0\}$. We combine (5.12) and (5.13). To this end we put

$$\mathfrak{A}(\varrho) := D_t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 a_t(\varrho) & \partial_2 a_t(\varrho) \end{pmatrix} \quad \text{for } \varrho \in \Delta_1,$$

with D_t introduced in Lemma 3.1. Then $\widetilde{\Gamma}(\varrho,\eta) = \mathfrak{A}(\varrho)\cdot(\varrho-\eta)$ ($\varrho,\, \eta \in \Delta_t$). Put

$$\mathfrak{Z}(\varrho)_{klr} := -\delta_{jk}\,\mathfrak{A}(\varrho)_{lr} + \delta_{jl}\,\mathfrak{A}(\varrho)_{kr} + \delta_{kl}\,\mathfrak{A}(\varrho)_{jr},$$
$$\widetilde{\mathfrak{Z}}(\varrho)_{kl\alpha} := -3\,\mathfrak{A}(\varrho)_{j\,\alpha_1}\,\mathfrak{A}(\varrho)_{k\,\alpha_2}\,\mathfrak{A}(\varrho)_{l\,\alpha_3}(\varrho)$$

for $k,\, l \in \{1,\, 2,\, 3\},\ r \in \{1,\, 2\},\ \alpha \in \{1,\, 2\}^3,\ \varrho \in \Delta_t$. Then we get from (5.13) that

$$(\partial_l E_{jk})\big(\,\widetilde{\Gamma}(\varrho,\eta)\,\big) \tag{5.14}$$
$$= \sum_{r=1}^{2}\mathfrak{Z}(\varrho)_{klr}\,(\varrho-\eta)_r\,|\widetilde{\Gamma}(\varrho,\eta)|^{-3} + \sum_{\alpha\in\{1,2\}^3}\widetilde{\mathfrak{Z}}(\varrho)_{kl\alpha}\prod_{s=1}^{3}(\varrho-\eta)_{\alpha_s}\,|\widetilde{\Gamma}(\varrho,\eta)|^{-5}$$

for $k,\, l,\, \varrho$ as before, and for $\eta \in \mathbb{R}^2$ with $\varrho \ne \eta$. Further put

$$\mathfrak{B}(m,n,r,\kappa) := \kappa_r\,\kappa_1^n\,\kappa_2^{2m-n}\,|\kappa|^{-3-2m}, \quad \widetilde{\mathfrak{B}}(m,n,\alpha,\kappa) := \prod_{s=1}^{3}\kappa_{\alpha_s}\,\kappa_1^n\,\kappa_2^{2m-n}\,|\kappa|^{-5-2m}$$

for r, α as above, $m \in \mathbb{N},\ n \in \{0,\, ...,\, 2\,m\},\ \kappa \in \mathbb{R}^2 \backslash \{0\}$,

$$\mathfrak{W}(m,n,\varrho) := \binom{-3/2}{m} \binom{2\,m}{n} \partial_1 a_t(\varrho)^n\, \partial_2 a_t(\varrho)^{2\,m-n},$$
$$\widetilde{\mathfrak{W}}(m,n,\varrho) := \binom{-5/2}{m} \binom{2\,m}{n} \partial_1 a_t(\varrho)^n\, \partial_2 a_t(\varrho)^{2\,m-n},$$

for m, n as before and $\varrho \in \Delta_t$. Then by (5.9), (5.12) and (5.14), it follows that $G^{(3)}(\varrho) = G^{(3,1)}(\varrho) + G^{(3,2)}(\varrho)$, with

$$G^{(3,1)}(\varrho) \tag{5.15}$$
$$= \sum_{k,l=1}^{3} \sum_{r=1}^{2} \partial_\nu \gamma_t(\varrho)_l\, \mathfrak{Z}(\varrho)_{klr} \int_{B^2_{\sigma_0}(\varrho)} \sum_{m=0}^{\infty} \sum_{n=0}^{2\,m} \mathfrak{W}(m,n,\varrho)\, \mathfrak{B}(m,n,r,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k\, d\eta,$$

$$G^{(3,2)}(\varrho)$$
$$= \sum_{k,l=1}^{3} \sum_{\alpha \in \{1,\,2\}^3} \partial_\nu \gamma_t(\varrho)_l \widetilde{\mathfrak{Z}}(\varrho)_{kl\alpha} \int_{B^2_{\sigma_0}(\varrho)} \sum_{m=0}^{\infty} \sum_{n=0}^{2\,m} \widetilde{\mathfrak{W}}(m,n,\varrho)\, \widetilde{\mathfrak{B}}(m,n,\alpha,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k\, d\eta$$

for $\varrho \in \Delta_1$. Since $|\nabla a_t(\varrho)| \le 1/4$, we obtain with (5.4) that

$$\Big| \sum_{n=0}^{2\,m} \mathfrak{W}(m,n,\varrho)\, \mathfrak{B}(m,n,r,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k \Big| \tag{5.16}$$
$$\le \Big| \binom{-3/2}{m} \Big|\, (1/4)^{2\,m} \sum_{n=0}^{2\,m} \binom{2\,m}{n} |\varrho-\eta|^{-2}\, |\mathcal{F}(f)(\varrho,\eta)_k| \le \mathfrak{C} \Big| \binom{-3/2}{m} \Big|\, (1/2)^{2\,m}\, |\varrho-\eta|^{-1}$$

for $m \in \mathbb{N},\ 1 \le k \le 3,\ 1 \le r \le 2,\ \varrho,\, \eta \in \Delta_1$ with $\varrho \ne \eta$, where $\mathfrak{C}$ is independent not only of ϱ and η, but also of m. We thus see that the integral in the definition of $G^{(3,1)}(\varrho)$ may be moved inside the sum with respect to $m \in \mathbb{N}$. In this way we arrive at the integral $\int_{B^2_{\sigma_0}(\varrho)} \mathfrak{B}(m,n,r,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k\, d\eta$ for m, k, r as before and for $0 \le n \le 2\,m,\ \varrho \in \Delta_1$. Since $|\mathfrak{B}(m,n,r,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k| \le \mathfrak{C}\, |\varrho-\eta|^{-1}$ for m, k, r, ϱ, η as in (5.16) and for $0 \le n \le 2\,m$, as again follows from (5.4), we obtain

$$\int_{B^2_{\sigma_0(\varrho)} \backslash B^2_\sigma(\varrho)} \mathfrak{B}(m,n,r,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k\, d\eta \tag{5.17}$$
$$\to \int_{B^2_{\sigma_0}(\varrho)} \mathfrak{B}(m,n,r,\ \varrho-\eta)\, \mathcal{F}(f)(\varrho,\eta)_k\, d\eta \quad (\sigma \downarrow 0), \quad \text{uniformly in } \ \varrho \in \Delta_1.$$

But $\int_{B^2_{\sigma_0}(\varrho) \backslash B^2_\sigma(\varrho)} \mathfrak{B}(m,n,r,\ \varrho-\eta)\, d\eta = 0$ for $\varrho \in \mathbb{R}^2,\ \sigma \in (0,\sigma_0),\ m$, k, r, n as before, so we see that the term $\mathcal{F}(f)(\varrho,\eta)_k$ may be replaced by $f_k(\eta)\, \mathfrak{E}(\eta)$ in the integral on the

left-hand side of (5.17). Thus with (5.15) and (5.17), we finally arrive at the equation

$$G^{(3,1)}(\varrho) = \sum_{k,l=1}^{3} \sum_{r=1}^{2} \partial_\nu \gamma_t(\varrho)_l \, \mathfrak{Z}(\varrho)_{klr} \tag{5.18}$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{2\,m} \mathfrak{W}(m,n,\varrho) \lim_{\sigma \downarrow 0} \int_{B^2_{\sigma_0}(\varrho) \setminus B_\sigma(\varrho)} \mathfrak{B}(m,n,r,\, \varrho - \eta)\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta,$$

for $\varrho \in \Delta_1$, with the limit of the integral over $B^2_{\sigma_0} \setminus B_\sigma$ for $\sigma \downarrow 0$ being uniform with respect to such ϱ. An analogous reasoning yields that

$$G^{(3,2)}(\varrho) = \sum_{k,l=1}^{3} \sum_{\alpha \in \{1,\,2\}^3} \partial_\nu \gamma_t(\varrho)_l \, \widetilde{\mathfrak{Z}}(\varrho)_{kl\alpha} \tag{5.19}$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{2\,m} \widetilde{\mathfrak{W}}(m,n,\varrho) \lim_{\sigma \downarrow 0} \int_{B^2_{\sigma_0}(\varrho) \setminus B_\sigma(\varrho)} \widetilde{\mathfrak{B}}(m,n,\alpha,\, \varrho - \eta)\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta$$

for ϱ as before, where the limit of the integral in this equation is again uniform with respect to $\varrho \in \Delta_1$. We note that $\int_{\partial B^2_1} |\mathfrak{B}(m,n,r,\kappa)|\, do_\kappa \le \int_{\partial B^2_1} |\kappa|^{-2}\, do_\kappa = 2\,\pi$ for $m \in \mathbb{N},\ n \in \{0,\, ...,\, 2\,m\},\ r \in \{1,\, 2\}$, and $\sum_{m=0}^{\infty} \sum_{n=0}^{2\,m} |\mathfrak{W}(m,n,\varrho)| \le \sum_{m=0}^{\infty} \Big| \binom{-3/2}{m} \Big|\, (1/2)^{2\,m}$ for $\varrho \in \Delta_1$; compare (5.16). As explained in the passage preceding (5.2), we have $\mathfrak{E}\, f \in L^p(\mathbb{R}^2)^3$. At this point, due to the uniform convergence of the integral in (5.18), we may refer to Theorem 2.1 to deduce from (5.18) that $\|G^{(3,1)}\|_p \le \mathfrak{C}\, \|\mathfrak{E}\, f\|_p \le \mathfrak{C}\, \|f\|_p$. Here and in the rest of this proof, $\mathfrak{C}$ stands for constants independent of f, and also of $\varrho \in \Delta_1$ if such a variable ϱ is involved. In an analogous way as equation (5.18) leads to the previous estimate of $G^{(3,1)}$, it may be deduced from equation (5.19) that $\|G^{(3,2)}\|_p \le \mathfrak{C}\, \|f\|_p$, so we finally obtain that $\|G^{(3)}\|_p \le \mathfrak{C}\, \|f\|_p$.

All the other relevant functions may be estimated in a rather straightforward way. By (5.6) we see that the absolute value of the integral in the definition of $F^{(3)}(\,m\,,0)$ and $F^{(4)}(\,\cdot\,,0)$ is bounded uniformly in $\varrho \in \Delta_1$. It follows that $\|F^{(\mu)}(\,\cdot\,,0)\|_p \le \mathfrak{C}\, \|f\|_p$ for $\mu \in \{3,\, 4\}$. An analogous argument, based on (5.8) and (5.10), respectively, instead of (5.6), yields that $\|G^{(\mu)}\|_p \le \mathfrak{C}\, \|f\|_p$ for $\mu \in \{2,\, 5\}$. As for $G^{(4)}$, we may use (5.10) to obtain

$$|G^{(4)}(\varrho)| \le \mathfrak{C} \int_{\Delta_1 \setminus B_{\sigma_0}(\varrho)} |f(\eta)|\, |\mathfrak{E}(\eta)|\, d\eta \le \mathfrak{C}\, \|f\|_1 \le \mathfrak{C}\, \|f\|_p.$$

for $\varrho \in \Delta_1$, so that $\|G^{(4)}\|_p \le \mathfrak{C}\, \|f\|_p$. Concerning $G^{(1)}$, inequality (5.8) provides that $|G^{(1)}(\varrho)| \le \mathfrak{C} \int_{\Delta_1} |\varrho - \eta|^{-1}\, |f(\eta)|\, |\mathfrak{E}(\eta)|\, d\eta$ for $\varrho \in \Delta_1$, so that $\|G^{(1)}\|_p \le \mathfrak{C}\, \|f\|_p$ by Lemma 2.1.

At this point it follows from the representations in (5.7), (5.11) and the previous estimates of the terms $\|F^{(3)}(\,\cdot\,,0)\|_p$, $\|F^{(4)}(\,\cdot\,,0)\|_p$ and $\|G^{(\mu)}\|_p$ for $\mu \in \{1,\, ...,\, 5\}$ that

$$\|\partial_\nu \big[\, V\big(\, \Psi\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t \,\big]\|_p \le \mathfrak{C}\, \|f\|_p \quad \text{for} \ \ f \in L^p(\Delta_1)^3 \cap C^1(\Delta_1)^3,\ j \in \{1,\, 2,\, 3\} \tag{5.20}$$

and $\nu \in \{1, 2\}$. Take $f \in L^p(\Delta_1)^3$. Obviously inequality (5.2) remains valid for such f, so Lemma 2.2 implies that $\|V\big(\Psi\, \mathcal{Z}_t(f)\big)\|_p \leq \mathfrak{C}\, \|\Psi\, \mathcal{Z}_t(f)\|_p \leq \mathfrak{C}\, \|f\|_p$, hence by (3.2), $\|V\big(\Psi\, \mathcal{Z}_t(f)\big) \circ \gamma_t\|_p \leq \|V\big(\Psi\, \mathcal{Z}_t(f)\big)|\Lambda_{t,1}\|_p \leq \|V\big(\Psi\, \mathcal{Z}_t(f)\big)\|_p \leq \mathfrak{C}\, \|f\|_p$. The theorem follows from this estimate, inequality(5.20) and the density of $L^p(\Delta_1)^3 \cap C^1(\Delta_1)^3$ in $L^p(\Delta_1)^3$. $\square$

The next theorem states that the $W^{2-1/p,\,p}$-norm of $V(\phi)|\partial\Omega$ is bounded by the $W^{1-1/p,\,p}$-norm of ϕ. This is the key result of our theory. Its proof is based on the preceding theorem.

Theorem 5.2 *Let $p \in (1, \infty)$. Then $V(\phi)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3$ and $\|V(\phi)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\, \|\phi\|_{1-1/p,\,p}$ for $\phi \in W^{1-1/p,\,p}(\partial\Omega)^3$.*

Proof: The notation introduced either in Theorem 5.1 itself or at the beginning of the proof of that theorem, up to inequality (5.2), will be used here again, without further notice.

Let $t \in \{1, ..., k(\Omega)\},\ j \in \{1, 2, 3\},\ f \in L^p(\Delta_1)^3 \cap C^1(\Delta_1)^3$ and $\nu \in \{1, 2\}$. Recall that $\Psi_t\, \mathcal{Z}_t(f) \in C^0(\partial\Omega)$.

Let $\delta \in \big(0,\, \delta(\Omega)\big]$. Consider the function $V^{(\delta)}\big(\Psi_t\, \mathcal{Z}_t(f)\big)$ introduced in Lemma 4.3. As stated in that lemma, this function is C^∞ in an open set $\mathfrak{U}_\delta$ containing $\overline{\Omega}$ as a subset; see (5.3) as concerns its first order derivatives. As in the proof of Theorem 5.1, we split $\partial_\nu\big[\, V^{(\delta)}\big(\Psi_t\, \mathcal{Z}_t(f)\big)_j \circ \gamma_t\,\big](\varrho)$ into a sum of several terms, but in a way different from that in the previous proof. In fact, by (3.2) and because $supp\big(\Psi_t\, \mathcal{Z}_t(f)\big) \subset U_{t,\,3/4}$ and $\big(\Psi_t\, \mathcal{Z}_t(f)\big) \circ \gamma_t = (\Psi_t \circ \gamma_t)\, f$, we get for $\varrho \in \Delta_1$ that

$$\begin{aligned} &\partial_\nu\big[\, V^{(\delta)}\big(\Psi_t\, \mathcal{Z}_t(f)\big)_j \circ \gamma_t\,\big](\varrho) \qquad (5.21)\\ &= \sum_{k,l=1}^{3} \partial_\nu\gamma_t(\varrho)_l \int_{\Delta_1} (\partial_l E_{jk})\big(\,\Gamma(\varrho,\eta,\delta)\,\big)\,\big]\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta = \sum_{\mu=1}^{4} H^{(\mu)}(\varrho,\delta), \end{aligned}$$

with

$$\begin{aligned} H^{(1)}(\varrho,\delta) &:= \sum_{k,l=1}^{3} \int_{\Delta_1} (\partial_l E_{jk})\big(\,\Gamma(\varrho,\eta,\delta)\,\big)\, \big(\,\partial_\nu\gamma_t(\varrho) - \partial_\nu\gamma_t(\eta)\,\big)_l\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta,\\ H^{(2)}(\varrho,\delta) &:= \sum_{k,l=1}^{3} \int_{\Delta_1} \Big((\partial_l E_{jk})\big(\,\Gamma(\varrho,\eta,\delta)\,\big) - (\partial_l E_{jk})\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\Big)\\ &\qquad\qquad \partial_\nu\gamma_t(\eta)_l\, f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta,\\ H^{(3)}(\varrho,\delta) &:= \sum_{k=1}^{3} \int_{\Delta_1} E_{jk}\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\, f_k(\eta)\, \partial_\nu\mathfrak{E}(\eta)\, d\eta,\\ H^{(4)}(\varrho,\delta) &:= \sum_{k=1}^{3} \int_{\Delta_1} E_{jk}\big(\,\overline{\Gamma}(\varrho,\eta,\delta)\,\big)\, \partial_\nu f_k(\eta)\, \mathfrak{E}(\eta)\, d\eta. \end{aligned}$$

The last two functions arise due to a partial integration with respect to $\eta \in \Delta_1$, which is possible due to (3.6) and because $\mathfrak{E} \in C_0^1(\Delta_1)$. Let $\zeta \in C_0^\infty(\Delta_1)$. We deduce from the relation $\Psi_t\, \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$ (see further above) and from the uniform convergence in (4.11) that

$$\int_{\Delta_1} \partial_\nu \zeta(\varrho) \Big(V^{(\delta)}\big(\Psi_t\, \mathcal{Z}_t(f) \big)_j \circ \gamma_t \Big)(\varrho)\, d\varrho \to \int_{\Delta_1} \partial_\nu \zeta(\varrho) \Big(V\big(\Psi_t\, \mathcal{Z}_t(f) \big)_j \circ \gamma_t \Big)(\varrho)\, d\varrho$$

for $\delta \downarrow 0$. Lemma 4.4, equation (3.2) and the relation $\mathfrak{E} \in C_0^1(\Delta_1)$ yield that the integral $\int_{\Delta_1} \zeta(\varrho)\, H^{(2)}(\varrho,\delta)\, d\varrho$ tends to zero for $\delta \downarrow 0$. The function γ_t belongs to $C^2(\overline{\Delta_1})$ (Lemma 3.1), so with (4.4) and (3.4),

$$|\partial_\varrho^\alpha \big[\, (\partial_l E_{jk})\big(\, \Gamma(\varrho,\eta,\delta)\,\big)\, \big(\, \partial_\nu \gamma_t(\varrho) - \partial_\nu \gamma_t(\eta)\,\big)_l \,\big]| \le \mathfrak{C}\, |\varrho-\eta|^{-1-|\alpha|} \tag{5.22}$$

for $1 \le k,l \le 3$, $\varrho, \eta \in \Delta_1$ with $\varrho \ne \eta$, $\delta \in \big[\, 0,\, \delta(\Omega)\,\big]$, $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le 1$. Here and in inequality (5.23) below, $\mathfrak{C}$ is independent of ϱ and η. Since $\mathfrak{E} \in C_0^1(\Delta_1)$, we have $\mathfrak{E}\, f \in C_0^1(\Delta_1)^3$, in particular $|\mathfrak{E}\, f|_\infty < \infty$, so we may conclude from (5.22) with $\alpha = 0$ and from Lebesgue's theorem that the function $H^{(1)}(\,\cdot\,,\delta) : \Delta_1 \mapsto \mathbb{C}$ is well defined and integrable also for $\delta = 0$, and $\int_{\Delta_1} \zeta(\varrho)\, H^{(1)}(\varrho,\delta)\, d\varrho \to \int_{\Delta_1} \zeta(\varrho)\, H^{(1)}(\varrho,0)\, d\varrho$ for $\delta \downarrow 0$. We further deduce from (4.4) and (3.4) that

$$|\partial_\varrho^\alpha \big[\, E_{jk}\big(\, \overline{\Gamma}(\varrho,\eta,\delta)\,\big)\,\big]| \le \mathfrak{C}\, |\varrho-\eta|^{-1-|\alpha|} \quad \text{for } k,\ \varrho,\ \eta,\ \delta,\ \alpha \text{ as in (5.22).} \tag{5.23}$$

Taking into account that $\mathfrak{E}\, \partial_\nu f$ and $\partial_\nu \mathfrak{E}\, f$ belong to $C_0^0(\Delta_{3/4})^3$, we see that due to (5.23) with $\alpha = 0$, the function $H^{(\mu)}(\,\cdot\,,\delta)$ for $\mu \in \{3,\, 4\}$ is well defined and integrable also if $\delta = 0$, and $\int_{\Delta_1} \zeta(\varrho)\, H^{(\mu)}(\varrho,\delta)\, d\varrho \to \int_{\Delta_1} \zeta(\varrho)\, H^{(\mu)}(\varrho,0)\, d\varrho$ ($\delta \downarrow 0$) for such μ. At this point we may deduce from (5.21) that the weak derivative $\partial_\nu \big[\, V\big(\, \Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big]$ exists – a fact already known from Theorem 5.1 – and

$$\partial_\nu \big[\, V\big(\, \Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big](\varrho) = \sum_{\mu \in \{1,\,3,\,4\}} H^{(\mu)}(\varrho,0) \quad \text{for } \ \varrho \in \Delta_1. \tag{5.24}$$

Now consider $f \in L^p(\Delta_1)^3$. In the following, the constants $\mathfrak{C}$ are independent of f. Recalling that $\partial^\alpha \mathfrak{E} \in C_0^0(\Delta_{3/4})$ for $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \le 1$, we observe that $\partial^\alpha \mathfrak{E}\, f \in L^p(\Delta_1)^3$ and $\|\partial^\alpha \mathfrak{E}\, f\|_p \le \mathfrak{C}\, \|f\|_p$. It follows from (5.22), (5.23) with $\alpha = 0$, $\delta = 0$ and from Lemma 2.1 that if $\mu \in \{1,\, 3\}$, the function $H^{(\mu)}(\,\cdot\,,0)$ is well defined also with f as given now, that is, $f \in L^p(\Delta_1)^3$, and the estimate $\|H^{(\mu)}(\,\cdot\,,0)\|_p \le \mathfrak{C}\, \|f\|_p$ holds. We recall that according to Theorem 5.1, the weak derivative $\partial_\nu \big[\, V\big(\, \Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big]$ exists also in the case $f \in L^p(\Delta_1)^3$ considered presently, and inequality (5.1) is valid for this f. Define

$$\widetilde{H}^{(4)} := -H^{(1)}(\,\cdot\,,0) - H^{(3)}(f)(\,\cdot\,,0) + \partial_\nu \big[\, V\big(\, \Psi_t\, \mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big]. \tag{5.25}$$

In view of (5.1) and the estimate $\|H^{(\mu)}(\,\cdot\,,0)\|_p \le \mathfrak{C}\, \|f\|_p$ for $\mu \in \{1,\, 3\}$ derived above, we see that $\widetilde{H}^{(4)} \in L^p(\Delta_1)^3$ and $\|\widetilde{H}^{(4)}\|_p \le \mathfrak{C}\, \|f\|_p$.

Next take $f \in W^{1,p}(\Delta_1)^3 \cap C^2(\Delta_1)^3$. We have $\widetilde{H}^{(4)} = H^{(4)}(\,\cdot\,,0)$ by (5.24), and $H^{(4)}(\,\cdot\,,0) = -V\big(\, \Psi_t\, \mathcal{Z}_t(\partial_\nu f)\,\big)_j \circ \gamma_t$ by (3.2). At this point we may refer to Theorem 5.1 to obtain that $\widetilde{H}^{(4)} \in W^{1,p}(\Delta_1)$ and $\|\partial_r \widetilde{H}^{(4)}\|_p \le \mathfrak{C}\, \|\partial_\nu f\|_p$ for $r \in \{1,\, 2\}$.

Since we have now shown that $\|\widetilde{H}^{(4)}\|_p \le \mathfrak{C}\,\|f\|_p$ for $f \in L^p(\Delta_1)^3$ and $\widetilde{H}^{(4)} \in W^{1,p}(\Delta_1)^3$, $\|\widetilde{H}^{(4)}\|_{1,p} \le \mathfrak{C}\,\|f\|_{1,p}$ for $f \in W^{1,p}(\Delta_1)^3 \cap C^2(\Delta_1)^3$, we may conclude that the two preceding relations remain valid under the condition $f \in W^{1,p}(\Delta_1)^3$ instead of $f \in W^{1,p}(\Delta_1)^3 \cap C^2(\Delta_1)^3$. Therefore interpolation implies that $\widetilde{H}^{(4)} \in W^{1-1/p,\,p}(\Delta_1)^3$ and $\|\widetilde{H}^{(4)}\|_{1-1/p,\,p} \le \mathfrak{C}\,\|f\|_{1-1/p,\,p}$ for $f \in W^{1-1/p,\,p}(\Delta_1)^3$.

From (5.22), (5.23) and Theorem 2.2 we obtain that $H^{(\mu)}(\,\cdot\,,0)$ for $\mu \in \{1,\,3\}$ belongs to $W^{1-1/p,\,p}(\Delta_1)^3$ and $\|H^{(\mu)}(\,\cdot\,,0)\|_{1-1/p,\,p} \le \mathfrak{C}\,\|f\|_p$ if $f \in L^p(\Delta_1)^3$. At this point we may refer to equation (5.25) to conclude that $\partial_\nu\big[\,V\big(\,\Psi_t\,\mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big] \in W^{1-1/p,\,p}(\Delta_1)$ and $\|\partial_\nu\big[\,V\big(\,\Psi_t\,\mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\,\big]\|_{1-1/p,\,p} \le \mathfrak{C}\,\|f\|_{1-1/p,\,p}$ for any $f \in W^{1-1/p,\,p}(\Delta_1)^3$, $\nu \in \{1,\,2\}$. It follows with Theorem 5.1 that for $f \in W^{1-1/p,\,p}(\Delta_1)^3$,

$$V\big(\,\Psi_t\,\mathcal{Z}_t(f)\,\big)_j \circ \gamma_t \in W^{2-1/p,\,p}(\Delta_1), \quad \|V\big(\,\Psi_t\,\mathcal{Z}_t(f)\,\big)_j \circ \gamma_t\|_{2-1/p,\,p} \le \mathfrak{C}\,\|f\|_{1-1/p,\,p}. \tag{5.26}$$

Let $\phi \in W^{1-1/p,\,p}(\partial\Omega)^3$. In the rest of this proof, constants $\mathfrak{C}$ are independent of ϕ. We have $\phi \circ \gamma_t \in W^{1-1/p,\,p}(\Delta_t)^3$ and $\mathcal{Z}_t(\phi \circ \gamma_t)|\Lambda_{t,1} = \phi|\Lambda_{t,1}$. Since $supp(\Psi_t) \cap \partial\Omega \subset \Lambda_{t,\,3/4}$, we see that $\Psi_t\,\mathcal{Z}_t(\phi \circ \gamma_t) = \Psi_t\,\phi$. Thus the relations in (5.26) hold with $\phi \circ \gamma_t$ in the place of f. Moreover we observe that $dist(\partial\Omega\backslash\Lambda_{t,\,1/2},\,\Lambda_{t,\,1/4}) > 0$ (Lemma 3.1), $supp\big(\,(1-\Psi_t)|\partial\Omega\,\big) \subset \partial\Omega\backslash\Lambda_{t,\,1/2}$ and $\gamma_t(\varrho) \in \Lambda_{t,\,1/4}$ for $\varrho \in \Delta_{1/4}$. If follows with Lebesgue's theorem that $V\big(\,(1-\Psi_t)\,\phi\big)_j \circ \gamma_t|\Delta_{1/4} \in C^2(\Delta_{1/4})$ and

$$|\partial^\alpha\big[\,V\big(\,(1-\Psi_t)\,\phi\,\big)_j \circ \gamma_t\,\big](\varrho)| = \Big|\int_{\partial\Omega} \sum_{k=1}^{3} \partial_\varrho^\alpha\big[\,E_{jk}\big(\,\gamma_t(\varrho) - y\,\big)\,\big]\,(1-\Psi_t)(y)\,\phi_k(y)\,do_y\Big|$$

$$\le \mathfrak{C}\,\|\phi\|_1 \quad \text{for} \;\; \varrho \in \Delta_{1/4},\; \alpha \in \mathbb{N}_0^2 \text{ with } |\alpha| \le 2,$$

with $\mathfrak{C}$ being independent of ϕ and ϱ. Therefore $\|V\big(\,(1-\Psi_t)\,\phi\,\big)_j \circ \gamma_t|\Delta_{1/4}\|_{2,p} \le \mathfrak{C}\,\|\phi\|_p$, in particular $\|V\big(\,(1-\Psi_t)\,\phi\,\big)_j \circ \gamma_t|\Delta_{1/4}\|_{2-1/p,\,p} \le \mathfrak{C}\,\|\phi\|_p$. The preceding estimate and (5.26) with $f = \phi \circ \gamma_t$ yield that $V(\phi)_j \circ \gamma_t|\Delta_{1/4} \in W^{2-1/p,\,p}(\Delta_{1/4})$ and $\|V(\phi)_j \circ \gamma_t|\Delta_{1/4}\|_{2-1/p,\,p} \le \mathfrak{C}\,(\|\phi \circ \gamma_t\|_{1-1/p,\,p} + \|\phi\|_p) \le \mathfrak{C}\,\|\phi\|_{1-1/p,\,p}$. Since j, t, ϕ were chosen arbitrarily in the sets $\{1,\,2,\,3\}$, $\{1,\,...,\,k\}$ and $W^{1-1/p,\,p}(\partial\Omega)^3$, respectively, the theorem follows with (3.1). $\square$

The consequence of Theorem 5.2 we are interested in is stated as

Corollary 5.1 *Let $p \in (1,\infty)$. For $b \in W^{1-1/p,\,p}(\partial\Omega)^3$, $\phi \in E_p^{(\mp)}$ with $(1/2)\,\big(\,\mp\phi + \mathfrak{T}^*(\phi)\,\big) = b$, the relations $V(\phi)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3$ and $\|V(\phi)|\partial\Omega\|_{2-1/p,\,p} \le \mathfrak{C}\,\|b\|_{1-1/p,\,p}$ hold. (The space $E_p^{(\mp)}$ was introduced in Corollary 4.3.)*

Note that in the situation of the preceding corollary, we have $\phi = F^{\pm}(b)$; see Corollary 4.2 and 4.3. Thus it follows by Corollary 6.1 and Theorem 6.3 below that b is the traction boundary data of the pair $\big(\,V(\phi)|\overline{\Omega}^c,\,Q(\phi)|\overline{\Omega}^c\,\big)$ (exterior domain case; $\phi = F^{(-)}(b)$) and $\big(\,V(\phi)|\Omega,\,Q(\phi)|\Omega\,\big)$ (interior domain case; $\phi = F^{(+)}(b)$), respectively.

Proof of Corollary 5.1: Take b and ϕ as in the corollary, and consider the case $(1/2)\,\big(\,\phi + \mathfrak{T}^*(\phi)\,\big) = b$. All the constants $\mathfrak{C}$ appearing in the following are independent of b. We have in particular $b \in ran(I_p + T_p^*)$, so by Corollary 4.2 $\int_{\partial\Omega} \phi^{(j)} \cdot b\,do_x = 0$ for $1 \le j \le 6$. Since

in addition $b \in W^{1-1/p,\,p}(\partial\Omega)^3$, we may conclude with Lemma 4.9 there is a sequence (b_n) in $C^0(\partial\Omega)^3$ with $b_n \in C^a(\partial\Omega)^3 \cap W^{1-1/p,\,p}(\partial\Omega)^3$, $\int_{\partial\Omega} \phi^{(j)} \cdot b_n \, do_x$ for $n \in \mathbb{N}$, $a \in (0,1)$, $1 \leq j \leq 6$, and such that $\|b - b_n\|_{1-1/p,\,p} \to 0$.

Let $n \in \mathbb{N}$. It follows with Corollary 4.2 that $b_n \in ran(I_p + T_p^*)$, so there is a unique function $\phi_n \in E_p^{(+)}$ with $(1/2)\big(\phi_n + \mathfrak{T}^*(\phi_n)\big) = b_n$ for $n \in \mathbb{N}$; see Corollary 4.3. Theorem 4.2 yields in particular that $\phi_n \in C^a(\partial\Omega)^3$ for $a \in (0,1)$. Now we may conclude from Lemma 4.10 that

$$(1/2)\big[\, V(\phi_n)|\partial\Omega + \mathfrak{T}\big(\, V(\phi_n)|\partial\Omega\,\big)\,\big] = V(b_n)|\partial\Omega. \tag{5.27}$$

Since $\|b_n - b\|_p \to 0$, we know by (4.7) that$\|V(b_n) - V(b)|\partial\Omega\|_p \to 0$. But

$$\|\phi_n - \phi\|_p \leq \mathfrak{C}\,\|\phi_n - \phi + \mathfrak{T}^*(\phi_n - \phi)\|_p = \mathfrak{C}\,\|b_n - b\|_p \quad \text{for } \ n \in \mathbb{N} \tag{5.28}$$

according to Corollary 4.3, so $\|\phi_n - \phi\|_p \to 0$, hence $\|V(\phi_n - \phi)|\partial\Omega\|_p \to 0$ by (4.7). Now it follows from the boundedness of T_p (Lemma 4.7) that

$$\|V(\phi_n - \phi)|\partial\Omega + \mathfrak{T}\big(\, V(\phi_n - \phi)|\partial\Omega\,\big)\|_p \to 0.$$

Altogether we deduce from (5.27) that $(1/2)\big[\, V(\phi)|\partial\Omega + \mathfrak{T}\big(\, V(\phi)|\partial\Omega\,\big)\,\big] = V(b)|\partial\Omega$.

Since $b_n \in W^{1-1/p,\,p}(\partial\Omega)^3$, Theorem 5.2 yields that $V(b_n)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3$ $(n \in \mathbb{N})$ and

$$\|V(b_n - b_m)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\|b_n - b_m\|_{1-1/p,\,p}, \ \ \|V(b_n)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\|b_n\|_{1-1/p,\,p}, \tag{5.29}$$

for $m, n \in \mathbb{N}$. By Lemma 4.2 we further have $V(b_n)|\partial\Omega \in C^a(\partial\Omega)^3$ for $0 < a < 1$, $n \in \mathbb{N}$, so it follows with (5.27) and Theorem 4.3 that the functions $V(\phi_n - \phi_m)|\partial\Omega$ and $V(\phi_n)|\partial\Omega$ belong to $W^{2-1/p,\,p}(\partial\Omega)^3$ and

$$\begin{aligned}
&\|V(\phi_n - \phi_m)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\big(\,\|V(b_n - b_m)|\partial\Omega\|_{2-1/p,\,p} + \|\phi_n - \phi_m\|_p\,\big),\\
&\|V(\phi_n)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\big(\,\|V(b_n)|\partial\Omega\|_{2-1/p,\,p} + \|\phi_n\|_p\,\big) \quad (m, n \in \mathbb{N}).
\end{aligned}$$

Due to (5.29) and because of (5.28) and an analogous inequality for $\|\phi_n\|_p$, we thus obtain that

$$\|V(\phi_n - \phi_m)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\|b_n - b_m\|_{1-1/p,\,p}, \ \ \|V(\phi_n)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\|b_n\|_{1-1/p,\,p} \tag{5.30}$$

for $m, n \in \mathbb{N}$. The first estimate in (5.30) implies there is $\gamma \in W^{2-1/p,\,p}(\partial\Omega)^3$ such that $\|V(\phi_n) - \gamma\|_{2-1/p,\,p} \to 0$. Since $\|V(\phi_n - \phi)|\partial\Omega\|_p \to 0$, as explained following (5.28), we may conclude that $V(\phi)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3$ and $\|V(\phi_n - \phi)\|_{2-1/p,\,p} \to 0$. In addition $\|b_n - b\|_{1-1/p,\,p} \to 0$ by the choice of the sequence (b_n). At this point the second estimate in (5.30) yields that $\|V(\phi)|\partial\Omega\|_{2-1/p,\,p} \leq \mathfrak{C}\,\|b\|_{1-1/p,\,p}$.

Analogous arguments are valid if $(1/2)\big(-\phi + \mathfrak{T}^*(\phi)\,\big) = b$ if we note that the function $\psi^{(0)}$ introduced in Corollary 4.3 belongs to $C^a(\partial\Omega)^3$ for $a \in (0,1)$; see Theorem 4.2. $\square$

6 Existence and $W^{2,p}$-regularity of solutions to (1.1), (1.2).

To begin with, here is an overview of some key notation used in this section and the following one. The functions $\phi^{(1)}$, ..., $\phi^{(6)}$ were introduced in Theorem 1.2, $V(\phi)$ and $Q(\phi)$ in Lemma 4.2, $W(\phi)$ and $\Pi(\phi)$ in Lemma 4.5, the operators $F^{(+)}$ and $F^{(-)}$ and the function $\psi^{(0)}$ in Corollary 4.3. The constants γ_0 and $\gamma(b)$ will be defined in Theorem 6.2 below, and the functions $\mathfrak{R}(f)$ and $\mathfrak{S}(f)$ in Theorem 6.4.

In the ensuing theorem we consider (1.1) with $f = 0$ (homogeneous Stokes system) in $\overline{\Omega}^c$ and (1.2) with Neumann data satisfying a side condition. This theorem is proved by reducing it to Corollary 5.1 and to the L^p-theory of the Stokes system in bounded domains under Dirichlet boundary conditions.

Theorem 6.1 *Let $p \in (1,\infty)$ and $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$, $b \in W^{1-1/p,\,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} \psi^{(0)} \cdot b \, do_x = 0$. Abbreviate $\phi := F^{(-)}(b)$, $v := V(\phi)|\overline{\Omega}^c$, $\varrho := Q(\phi)|\overline{\Omega}^c$.*

Then $v \in C^\infty(\overline{\Omega}^c)^3$, $\varrho \in C^\infty(\overline{\Omega}^c)$, and the pair (v,ϱ) solves (1.1) with $f = 0$. Let $r_1 \in [1,\, 3\,p/2)$. Then, with constants $\mathfrak{C}$ independent of b.

$$\|\phi\|_p \le \mathfrak{C}\,\|b\|_p, \quad \|v|\Omega_R\|_{1,r_1} + \|\varrho|\Omega_R\|_{r_1} \le \mathfrak{C}\,\|b\|_p, \tag{6.1}$$

$$\|v|\Omega_R\|_{2,p} + \|\varrho|\Omega_R\|_{1,p} \le \mathfrak{C}\,\|b\|_{1-1/p,\,p}. \tag{6.2}$$

Proof: We refer to Lemma 4.2 for the relations $v \in C^\infty(\overline{\Omega}^c)^3$, $\varrho \in C^\infty(\overline{\Omega}^c)$ and the fact that the pair (v,ϱ) solves (1.1) with $f = 0$. Moreover Corollary 4.1 yields that $\|V(\phi)|\Omega_R\|_{1,r_1} + \|Q(\phi)|\Omega_R\|_{r_1} \le \mathfrak{C}\,\|\phi\|_p$. Here and in the following, the constants denoted by $\mathfrak{C}$ are independent of b, and therefore of ϕ as well. Due to Corollary 4.3 and by the definition of ϕ in Theorem 6.1, the relations $\phi \in E_p^{(-)}$, $(1/2)\,\big(-\phi + \mathfrak{T}^*(\phi)\,\big) = b$, $\|\phi\|_p \le \mathfrak{C}\,\|b\|_p$ hold. The preceding inequalities imply (6.1). Let us show (6.2). Since $\phi \in E_p^{(-)}$ and $(1/2)\,\big(-\phi + \mathfrak{T}^*(\phi)\,\big) = b$, Corollary 5.1 yields that $V(\phi)|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3$ and

$$\|V(\phi)|\partial\Omega\|_{2-1/p,\,p} \le \mathfrak{C}\,\|b\|_{1-1/p,\,p}. \tag{6.3}$$

Recall that $V(\phi)(x)$ for $x \in \partial\Omega$ was given by a direct definition in Lemma 4.2. On the other hand, inequality (6.1) means in particular that $V(\phi)|\Omega_R \in W^{1,p}(\Omega_R)^3$, and by the last statement in Lemma 4.3, we know that the trace of $V(\phi)|\Omega_R$ on $\partial\Omega$ coincides with $V(\phi)|\partial\Omega$ as defined in Lemma 4.2. Moreover estimate (4.15) – (4.17) yield in particular that $V(\phi)|B_{2R}\backslash\overline{B_R} \in W^{2,p}(B_{2R}\backslash\overline{B_R})^3$ and $\|V(\phi)|B_{2R}\backslash\overline{B_R}\|_{2,p} \le \mathfrak{C}\,\|\phi\|_p$, so the C^∞-regularity of V in $\overline{\Omega}^c$ mentioned above and a standard trace theorem yield that $V(\phi)|\partial B_R \in W^{2-1/p,\,p}(\partial B_R)^3$ and $\|V(\phi)|\partial B_R\|_{2-1/p,\,p} \le \mathfrak{C}\,\|\phi\|_p$. Therefore we may conclude with (6.3) and the estimate $\|\phi\|_p \le \mathfrak{C}\,\|b\|_p$ already shown that the directly defined function $V(\phi)|\partial\Omega_R$ is the trace of $V(\phi)|\Omega_R$ on $\partial\Omega_R$, belongs to $W^{2-1/p,\,p}(\partial\Omega_R)^3$, and

$$\|V(\phi)|\partial\Omega_R\|_{2-1/p,\,p} \le \mathfrak{C}\,(\|b\|_{1-1/p,\,p} + \|\phi\|_p) \le \mathfrak{C}\,\|b\|_{1-1/p,\,p}. \tag{6.4}$$

Since $div\big(\,V(\phi)|\mathbb{R}^3\backslash\partial\Omega\,\big) = 0$, and again by the relation $V(\phi)|\Omega_R \in W^{1,p}(\Omega_R)^3$, we get that $\int_{\partial\Omega_R} V(\phi)(x) \cdot n^{(\Omega_R)}(x)\, do_x = 0$, with $n^{(\Omega_R)}$ denoting the outward unit normal to Ω_R. At this point we may apply Theorem 2.5, which yields functions $u \in$

$W^{2,p}(\Omega_R)^3$, $\pi \in W^{1,p}(\Omega_R)$ with $-\Delta u + \nabla \pi = 0$, $u|\partial\Omega_R = V(\phi)|\partial\Omega_R$, $\int_{\Omega_R} \pi \, dx = 0$ and $\|u\|_{2,p} + \|\pi\|_{1,p} \leq \mathfrak{C} \, \|V(\phi)|\partial\Omega_R\|_{2-1/p,\,p}$. The latter inequality and (6.4) imply that $\|u\|_{2,p} + \|\pi\|_{1,p} \leq \mathfrak{C} \, \|b\|_{1-1/p,\,p}$. But $u = V(\phi)|\Omega_R$ and $\pi = Q(\phi)|\Omega_R + c$, with $c := -|\Omega_R|^{-1} \int_{\Omega_R} Q(\phi)(x) \, dx$. This follows from Theorem 2.6 and the properties of v and ϱ stated at the beginning of this proof, and because $V(\phi)|\Omega_R \in W^{1,p}(\Omega_R)^3$ and $Q(\phi)|\Omega_R \in L^p(\Omega_R)$ according to (6.1). Thus inequality (6.2) is proved. □

Corollary 6.1 *Consider assumptions and notation as in Theorem 6.1. Then the pair (v, ϱ) satisfies (1.2) in the trace sense.*

Proof: We have $b \in L^p(\partial\Omega)^3$, and $\int_{\partial\Omega} \psi^{(0)} \cdot b \, do_x = 0$ by the assumptions on b. Thus Lemma 4.9 yields a sequence (b_n) in $C^0(\partial\Omega)^3$ such that $b_n \in C^a(\partial\Omega)^3 \cap W^{1-1/r,\,r}(\partial\Omega)^3$ and $\int_{\partial\Omega} \psi^{(0)} \cdot b_n \, do_x = 0$ for $n \in \mathbb{N}$, $a \in (0,1)$, $r \in (1,\infty)$, and such that $\|b_n - b\|_{1-1/p,\,p} \to 0$.

Let $n \in \mathbb{N}$. Since $\int_{\partial\Omega} \psi^{(0)} \cdot b_n \, do_x = 0$, the function $\phi_n := F^{(-)}(b_n) \in E_p^-$ from Corollary 4.3 is well defined. By definition it satisfies the equation $(1/2)\,\big(-\phi_n + \mathfrak{T}^*(\phi_n)\big) = b_n$. Let $R \in (0,\infty)$ with $\overline{\Omega} \subset B_{R/2}$. By our choice of b_n, we have $b_n \in W^{1-1/r,\,r}(\partial\Omega)^3$ for $r \in (1,\infty)$, so Theorem 6.1 implies that $V(\phi_n)|\Omega_R \in W^{2,r}(\Omega_R)^3$, $Q(\phi_n)|\Omega_R \in W^{1,r}(\Omega_R)$ for such r. It follows by a Sobolev inequality that $V(\phi_n)|\Omega_R$ may be continuously extended to a function from $C^1(\overline{\Omega_R})^3$, and $Q(\phi_n)|\Omega_R$ admits a continuous extension to $\overline{\Omega_R}$. Since both $V(\phi_n)$ and $Q(\phi_n)$ are C^∞ in $\overline{\Omega}^c$ (Lemma 4.2), we may conclude there are functions $V_n \in C^1(\Omega^c)^3$ and $Q_n \in C^0(\Omega^c)$ such that $V(\phi_n)|\overline{\Omega}^c = V_n|\overline{\Omega}^c$ and $Q(\phi_n)|\overline{\Omega}^c = Q_n|\overline{\Omega}^c$. On the other hand, since $b_n \in C^a(\partial\Omega)^3$ for $a \in (0,1)$, Theorem 4.2 yields in particular that $\phi_n \in C^0(\partial\Omega)^3$. Recalling that $\partial_j V_n \in C^0(\Omega^c)^3$, $Q_n \in C^0(\Omega^c)$, $V_n|\overline{\Omega}^c = V(\phi_n)|\overline{\Omega}^c$ and $Q_n|\overline{\Omega}^c = Q(\phi_n)|\overline{\Omega}^c$, we deduce from (3.3), Theorem 4.4 and the equation $(1/2)\,\big(-\phi_n + \mathfrak{T}^*(\phi_n)\big) = b_n$ that

$$\sum_{k=1}^{3} n_k^{(\Omega)}(x) \, (\partial_j V_{n,k} + \partial_k V_{n,j} - \delta_{jk} \, Q_n)(x) = b_{n,j}(x) \quad \text{for } x \in \partial\Omega, \ 1 \leq j \leq 3, \ n \in \mathbb{N}. \tag{6.5}$$

But by our choice of b, ϕ, (b_n) and (ϕ_n), Theorem 6.1 yields that

$$\|V(\phi_n - \phi)|\Omega_R\|_{2,p} + \|Q(\phi_n - \phi)|\Omega_R\|_{1,p} \leq \mathfrak{C} \, \|b_n - b\|_{1-1/p,\,p} \ \text{ for } n \in \mathbb{N}.$$

Thus, recalling Lemma 2.5 and the definition of v and ϱ, and taking into account that $\partial_j V_n \in C^0(\Omega^c)^3$, $V_n|\Omega_R = V(\phi_n)|\Omega_R \in W^{2,r}(\Omega_R)^3$, $Q_n \in C^0(\Omega^c)$, $Q_n|\Omega_R = Q(\phi_n)|\Omega_R \in W^{1,r}(\Omega_R)$ for $r \in (1,\infty)$, as explained above, we conclude that

$$\|\partial_j v - \partial_j V_n|\partial\Omega\|_p + \|\varrho - Q_n|\partial\Omega\|_p \leq \mathfrak{C} \, \|b_n - b\|_{1-1/p,\,p} \ \ (n \in \mathbb{N}, \ 1 \leq j \leq 3),$$

where the boundary values are taken in the trace sense. Since $\|b_n - b\|_{1-1/p,\,p} \to 0$ by the choice of the sequence (b_n), it follows with (6.5) that the pair (v, ϱ) fulfills (1.2) in the trace sense. □

It is well known (see [23]) that the side condition imposed on the boundary data b in the preceding corollary may be eliminated by using the double layer potentials from Lemma 4.5. In addition there is a second way to work around this condition. In the following theorem we derive L^p-estimates in both cases, obtaining a velocity satisfying a zero flux condition on $\partial\Omega$, or a pressure which is L^p-integrable near infinity,

Theorem 6.2 *There is $a \in (0,1)$ with $\psi^{(0)} \in C^{1,a}(\partial\Omega)^3$. Moreover $\int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)}\, do_x \neq 0$, and there is $\gamma_0 = \gamma_0(\psi^{(0)}) \in \mathbb{R}^3 \backslash \{0\}$ such that*

$$\sum_{j,k=1}^{3} n_k^{(\Omega)}(x) \big(\partial_j W(\psi^{(0)})_k + \partial_k W(\psi^{(0)})_j - \delta_{jk}\, \Pi(\psi^{(0)}) \big) \big(x - \kappa\, n^{(\Omega)}(x) \big) \to -\gamma_0\, n_j^{(\Omega)}(x) \ (\kappa \downarrow 0)$$

for $1 \leq j \leq 3$, $x \in \partial\Omega$. Let $p \in (1,\infty)$, $b \in W^{1-1/p,\,p}(\partial\Omega)^3$. Put

$$\gamma(b) := -\int_{\partial\Omega} \psi^{(0)} \cdot b\, do_x \big(\int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)}\, do_x \big)^{-1}, \quad \widetilde{b} := b + \gamma(b)\, n^{(\Omega)}.$$

Then $\int_{\partial\Omega} \psi^{(0)} \cdot \widetilde{b}\, do_x = 0$.

Define $\phi := F^{(-)}(\widetilde{b})$, $u := V(\phi)|\overline{\Omega}^c$, $\pi := Q(\phi) + \gamma(b)|\overline{\Omega}^c$. Then $u \in C^\infty(\overline{\Omega}^c)^3 \cap W^{2,p}_{loc}(\Omega^c)^3$, $\pi \in C^\infty(\overline{\Omega}^c) \cap W^{1,p}_{loc}(\Omega^c)$, and the pair (u,π) solves the Stokes system (1.1) with $f = 0$, and verifies the boundary condition (1.2) in the trace sense. In addition the zero flux condition $\int_{\partial\Omega} u \cdot n^{(\Omega)}\, do_x = 0$ holds.

Let $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$, $r_1 \in [1,\, 3\,p/2)$, $r_2 \in (3,\infty)$, $r_3 \in (3/2,\, \infty)$, $r_4 \in (1,\infty)$. Then with constants $\mathfrak{C}$ independent of b,

$$\|u|\Omega_R\|_{1,r_1} + \|\pi|\Omega_R\|_{r_1} \leq \mathfrak{C}\, \|b\|_p, \quad \|u|\Omega_R\|_{2,p} + \|\pi|\Omega_R\|_{1,p} \leq \mathfrak{C}\, \|b\|_{1-1/p,\,p}, \tag{6.6}$$

$$|\partial^\alpha u(x)| \leq \mathfrak{C}\, \|b\|_p\, |x|^{-1-|\alpha|}, \quad |\partial^\alpha \big(\pi - \gamma(b) \big)(x)| \leq \mathfrak{C}\, \|b\|_p\, |x|^{-2-|\alpha|} \tag{6.7}$$

for $x \in B_R^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$,

$$\|u|B_R^c\|_{r_2} \leq \mathfrak{C}\, \|b\|_p, \quad \|\partial_n u|B_R^c\|_{r_3} + \|\pi - \gamma(b)|B_R^c\|_{r_3} \leq \mathfrak{C}\, \|b\|_p, \tag{6.8}$$

$$\|\partial_m \partial_n u|B_R^c\|_{r_4} + \|\partial_n \pi|B_R^c\|_{r_4} \leq \mathfrak{C}\, \|b\|_p \quad \textit{for } 1 \leq m,\, n \leq 3, \tag{6.9}$$

in particular $\|\partial_m \partial_n u\|_p + \|\partial_n \pi\|_p \leq \mathfrak{C}\, \|b\|_p$. Further define

$$\overline{u} := V(\phi) + \gamma_0^{-1}\, \gamma(b)\, W(\psi^{(0)})|\overline{\Omega}^c,\ \overline{\pi} := Q(\phi) + \gamma_0^{-1}\, \gamma(b)\, \Pi(\psi^{(0)})|\overline{\Omega}^c.$$

If $\gamma(b) = 0$, that is, if $\int_{\partial\Omega} \psi^{(0)} \cdot b\, do_x = 0$, then $u = \overline{u}$ and $\pi = \overline{\pi}$.

Suppose that $\gamma(b) \neq 0$. Then the previous results remain valid with u, π replaced by $\overline{u}$ and $\overline{\pi}$, respectively, with the following exceptions: The zero flux condition does not hold for $\overline{u}$. The first estimate in (6.6) is replaced by $\|\overline{u}|\Omega_R\|_{r_1} \leq \mathfrak{C}\, \|b\|_p$, the second estimate in (6.7) by $|\partial^\alpha \overline{\pi}(x)| \leq \mathfrak{C}\, \|b\|_p\, |x|^{-2-|\alpha|}$ for x and α as in (6.7), and the second estimate in (6.8) by $\|\partial_n \overline{u}|B_R^c\|_{r_3} + \|\overline{\pi}|B_R^c\|_{r_3} \leq \mathfrak{C}\, \|b\|_p$, again with constants $\mathfrak{C}$ independent of b.

There is $R_0 \in [R,\infty)$ with $|(\pi - \overline{\pi})(x)| \geq |\gamma(b)|/2$ for $x \in B_{R_0}^c$. Moreover $u - \overline{u}|\partial\Omega \in ker(I_p - T_p)\backslash\{0\}$, and the function $u - \overline{u}$ is not constant.

Proof: First we prove the claims about γ_0 and $\int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)}\, do_x$. By the choice of $\psi^{(0)}$ in Corollary 4.3, we have $-\psi^{(0)} + \mathfrak{T}(\psi^{(0)}) = 0$. Thus we may apply Theorem 4.7 with $b = 0$ and $\psi = \psi^{(0)}$, and with $p = r$ for any $r \in (3,\infty)$. Abbreviating $W := W(\psi^{(0)})$, $W_{in} := W_{in}(\psi^{(0)})$, $W_{ex} := W_{ex}(\psi^{(0)})$, $\Pi := \Pi(\psi^{(0)})$, $\Pi_{in} := \Pi_{in}(\psi^{(0)})$ and $\Pi_{ex} := \Pi_{ex}(\psi^{(0)})$,

this theorem combined with Theorem 4.5 and Lemma 4.5 allows us to conclude that $\psi^{(0)} \in C^{1,a}(\partial\Omega)^3$ for some $a \in (0,1)$, $\psi^{(0)} \in W^{2-1/p,\,p}(\partial\Omega)^3$,

$$
\begin{aligned}
&W_{in} \in C^1(\overline{\Omega})^3,\ W_{ex} \in C^1(\Omega^c)^3,\ \Pi_{in} \in C^0(\overline{\Omega}),\ \Pi_{ex} \in C^0(\Omega^c), \qquad (6.10)\\
&W_{in}|\partial\Omega = (-1/2)\left(-\psi^{(0)} + \mathfrak{T}(\psi^{(0)})\right) = 0,\ W_{ex}|\partial\Omega = (-1/2)\left(\psi^{(0)} + \mathfrak{T}(\psi^{(0)})\right) = -\psi^{(0)},\\
&W_{in}|\Omega = W|\Omega \in W^{2,p}(\Omega)^3,\ W_{ex}|\overline{\Omega}^c = W|\overline{\Omega}^c,\ W \in C^\infty(\mathbb{R}^3\backslash\partial\Omega)^3,\ \Pi \in C^\infty(\mathbb{R}^3\backslash\partial\Omega),\\
&\Delta W + \nabla\Pi = 0,\ \operatorname{div} W = 0,\ \Pi_{in}|\Omega = \Pi|\Omega \in W^{1,p}(\Omega),\ \Pi_{ex}|\overline{\Omega}^c = \Pi|\overline{\Omega}^c,\\
&W|\Omega_R \in W^{2,p}(\Omega_R)^3,\ \Pi|\Omega_R \in W^{1,p}(\Omega_R).
\end{aligned}
$$

(The functions $W_{in}(\psi^{(0)})$ and $W_{ex}(\psi^{(0)})$ were introduced in Theorem 4.5, and $\Pi_{in}(\psi^{(0)})$ and $\Pi_{ex}(\psi^{(0)})$ in Theorem 4.7. The parameter R was fixed in Theorem 6.2.) Theorem 4.6 and the relation $\psi^{(0)} \in C^{1,a}(\partial\Omega)^3$ provide the equation

$$
\sum_{k=1}^{3} n_k^{(\Omega)} \left(\partial_j W_{in,\,k} + \partial_k W_{in,\,j} - \delta_{jk}\,\Pi_{in}\right) = \sum_{k=1}^{3} n_k^{(\Omega)} \left(\partial_j W_{ex,\,k} + \partial_k W_{ex,\,j} - \delta_{jk}\,\Pi_{ex}\right) \quad (6.11)
$$

for $1 \le j \le 3$. Among the relations in (6.10), we next use that $W_{in} \in C^1(\overline{\Omega})^3$, $W_{in}|\partial\Omega = 0$, $W_{in}|\Omega = W|\Omega \in W^{2,p}(\Omega)^3$, $\Pi_{in} \in C^0(\overline{\Omega})$, $\Pi_{in}|\Omega = \Pi|\Omega \in L^p(\Omega)$, $W \in C^\infty(\mathbb{R}^3\backslash\partial\Omega)^3$ and $\Pi \in C^\infty(\mathbb{R}^3\backslash\partial\Omega)$, with $\Delta W + \nabla\Pi = 0$, $\operatorname{div} W = 0$. From this and Theorem 2.6, it follows that $W_{in} = 0$, hence $\partial_j W_{in} = 0$ $(1 \le j \le 3)$ and $\nabla\Pi|\Omega = 0$. But Ω is a domain and Π is in particular continuous, so there is $\gamma_0 = \gamma_0(\psi^{(0)}) \in \mathbb{R}$ with $\Pi|\Omega = \gamma_0$, hence $\Pi_{in} = \gamma_0$. Thus the left-hand side of (6.11) equals $-\gamma_0\, n_j^{(\Omega)}$, and therefore its right-hand side as well, so that

$$
\sum_{k=1}^{3} n_k^{(\Omega)} \left(\partial_j W_{ex,\,k} + \partial_k W_{ex,\,j} - \delta_{jk}\,\Pi_{ex}\right) = -\gamma_0\, n_j^{(\Omega)} \quad (1 \le j \le 3). \qquad (6.12)
$$

For the relation $\int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)}\, do_x \neq 0$ we may refer to [23, Remark 3.3]. However, since this reference would not shorten the argument here, we prove this relation here, too. In fact, suppose for a contradiction that

$$
\gamma_0 = 0 \quad \text{or} \quad \int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)}\, do_x = 0. \qquad (6.13)
$$

Then let $S \in [R,\infty)$. We use Lemma 2.6 with $U = \Omega_S$, $u = \widetilde{u} = W_{ex}|\overline{\Omega_S}$, $\pi = \Pi_{ex}|\overline{\Omega_S}$. This choice in Lemma 2.6 is possible according to (6.10). Also by (6.10), we have $W_{ex}|\partial\Omega = -\psi^{(0)}$. Thus Lemma 2.6), equation (6.12) and assumption (6.13) imply that

$$
\int_{\Omega_S} \sum_{j,k=1}^{3} |\partial_j W_k + \partial_k W_j|^2\, dx = \int_{\partial B_S} \sum_{j,k=1}^{3} (x_k/S)\left(\partial_j W_k + \partial_k W_j - \delta_{jk}\,\Pi\right)(x)\, W_j(x)\, do_x.
$$

But the surface integral on the right-hand side of preceding equation tends to zero for $S \to \infty$ due to (4.13), (4.14). It follows that $\int_{\overline{\Omega}^c} \sum_{j,k=1}^{3} |\partial_j W_k + \partial_k W_j|^2\, dx = 0$. This means that $\partial_j W_k + \partial_k W_j|\overline{\Omega}^c = 0$ for $1 \le j,k \le 3$. Turning to Theorem 2.4, we now

conclude there are numbers α_1, ..., $\alpha_6 \in \mathbb{R}$ such that $W|\overline{\Omega}^c = \sum_{j=1}^6 \alpha_j \, \phi^{(j)}|\overline{\Omega}^c$. If there were an index $j \in \{1, ..., 6\}$ with $\alpha_j \neq 0$, we might choose a sequence (x_n) in $\overline{\Omega}^c$ such that $|x_n| \to \infty$ and $\sum_{j=1}^6 \alpha_j \, \phi^{(j)}(x_n) \nrightarrow 0$. For example, if $\alpha_6 \neq 0$, a suitable choice would be $x_n = (0, \, R+n, \, 0)$ for $n \in \mathbb{N}$. But on the other hand, for any sequence (x_n) in $\overline{\Omega}^c$ with $|x_n| \to \infty$, inequality (4.13) implies $|W(x_n)| \to 0$. Thus we may conclude that $\alpha_j = 0$ for $1 \leq j \leq 6$, hence $W|\overline{\Omega}^c = 0$, and so $W_{ex} = 0$. But $W_{ex}|\partial\Omega = -\psi^{(0)}$ by (6.10), so $\psi^{(0)} = 0$, in contradiction to the choice of $\psi^{(0)}$ in Corollary 4.3. Thus none of the equations in (6.13) can be true. As a consequence $\gamma_0 \neq 0$ and $\int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)} \, do_x \neq 0$.

Since Ω is C^2-bounded, we have $n^{(\Omega)} \in C^a(\partial\Omega)^3$ for any $a \in (0,1)$. This means in particular that $n^{(\Omega)} \in W^{1-1/p,\,p}(\partial\Omega)^3$, so $\widetilde{b} \in W^{1-1/p,\,p}(\partial\Omega)^3$. (The function $\widetilde{b}$ was introduced in Theorem 6.2.) Obviously $\int_{\partial\Omega} \psi^{(0)} \cdot \widetilde{b} \, do_x = 0$. Recall that $\phi = F^{(-)}(\widetilde{b})$; see the definitions in Theorem 6.2. It immediately follows from Theorem 6.1 and Corollary 6.1 that the claims about u and π in Theorem 6.2 up to but excluding the zero flux condition hold true. As concerns that latter condition, we note that by Corollary 4.1 and the last statement of Lemma 4.3, the traces of $V(\phi)|\overline{\Omega}^c$ and $V(\phi)|\Omega$ coincide. So in view of Lemma 4.2, we obtain that $\int_{\partial\Omega} u \cdot n^{(\Omega)} \, do_x = \int_\Omega \operatorname{div} V(\phi) \, dx = 0$. We further find that

$$|\gamma(b)| \leq \mathfrak{C} \, \|b\|_p, \quad \|\widetilde{b}\|_p \leq \|b\|_p + \gamma(b) \, \|n^{(\Omega)}\|_p \leq \mathfrak{C} \, \|b\|_p, \tag{6.14}$$
$$\|\widetilde{b}\|_{1-1/p,\,p} \leq \|b\|_{1-1/p,\,p} + \mathfrak{C} \, |\gamma(b)| \, \|n^{(\Omega)}\|_{1-1/p,\,p} \leq \|b\|_{1-1/p,\,p} + \mathfrak{C} \, \|b\|_p \leq \mathfrak{C} \, \|b\|_{1-1/p,\,p}.$$

Thus we may deduce (6.6) by referring to the second estimate in (6.1) and to (6.2), each time with b replaced by $\widetilde{b}$, and by using (6.14). Moreover, in view of the first estimate in (6.1) with b replaced by $\widetilde{b}$, and due to (6.14), we obtain that $\|\phi\|_1 \leq \mathfrak{C} \, \|\phi\|_p \leq \mathfrak{C} \, \|\widetilde{b}\|_p \leq \mathfrak{C} \, \|b\|_p$. Inequality (6.7) follows from the preceding estimate, (4.13) and (4.14). Similarly the preceding estimate and (4.15)– (4.17) imply (6.8) and (6.9).

For the rest of this proof we suppose that $\gamma(b) \neq 0$. Turning to $\overline{u}$ and $\overline{\pi}$, we recall that according to (6.10) we have in particular that

$$W_{ex} \in C^1(\Omega^c)^3, \; W_{ex}|\overline{\Omega}^c = W|\overline{\Omega}^c \in C^\infty(\overline{\Omega}^c)^3, \; \Pi_{ex} \in C^0(\Omega^c), \; \Pi_{ex}|\overline{\Omega}^c = \Pi|\overline{\Omega}^c \tag{6.15}$$
$$\in C^\infty(\overline{\Omega}^c), \; -\Delta W + \nabla\Pi = 0, \; \operatorname{div} W = 0, \; W|\Omega_R \in W^{2,p}(\Omega_R)^3, \; \Pi|\Omega_R \in W^{1,p}(\Omega_R).$$

As a first consequence, we see that $\partial_j(W|\overline{\Omega}^c)$ for $1 \leq j \leq 3$ and $\Pi|\overline{\Omega}^c$ have a trace on $\partial\Omega$, and this trace equals $\partial_j W_{ex}|\partial\Omega$ and $\Pi_{ex}|\partial\Omega$, respectively (Lemma 2.5). Recalling that Theorem 6.1 and Corollary 6.1 are valid with $\widetilde{b}$ in the place of b, and taking into account (6.12), we may therefore conclude that $\overline{u} \in C^\infty(\overline{\Omega}^c)^3 \cap W^{2,p}_{loc}(\Omega^c)^3$, $\overline{\pi} \in C^\infty(\overline{\Omega}^c) \cap W^{1,p}_{loc}(\Omega^c)$, and the pair $(\overline{u}, \overline{\pi})$ solves the Stokes system (1.1) with $f = 0$ and fulfills (1.2) in the trace sense. We further note that $\overline{u} = u + \gamma_0^{-1} \, \gamma(b) \, W$, $\overline{\pi} = \pi - \gamma(b) + \gamma_0^{-1} \, \gamma(b) \, \Pi$. Therefore the estimates which according to Theorem 6.2 are satisfied by $\overline{u}$ and $\overline{\pi}$ follow from (6.6) – (6.9), (6.15), the inequality $|\gamma(b)| \leq \mathfrak{C} \, \|b\|_p$, and the relation $W \in L^r(\Omega_R)^3$ for $r \in [1, \, 3\,p/2)$, which is a consequence of Lemma 2.4.

Existence of $R_0 \in [R, \infty)$ such that $|(\pi - \overline{\pi})(x)| \geq |\gamma(b)|/2$ for $x \in B^c_{R_0}$ follows from (6.7) and from the estimate of $|\partial^\alpha \overline{\pi}(x)|$ stated in Theorem 6.2. We recall that $W_{ex}|\partial\Omega = -\psi^{(0)} \in ker(I_p - T_p) \backslash \{0\}$ according to (6.10) and Corollary 4.3. Thus we may deduce

from the relations $W_{ex} \in C^0(\Omega^c)^3,\ W_{ex}|\overline{\Omega}^c = W|\overline{\Omega}^c$ (see (6.15)) and $\overline{u} = u + \gamma_0^{-1}\,\gamma(b)\,W$ that $u - \overline{u}|\partial\Omega \in ker(I_p - T_p)\backslash\{0\}$. Since it was shown that $\int_{\partial\Omega} \psi^{(0)} \cdot n^{(\Omega)}\, do_x \neq 0$ and $\int_{\partial\Omega} u \cdot n^{(\Omega)}\, do_x = 0$, it further follows that $\int_{\partial\Omega} \overline{u} \cdot n^{(\Omega)}\, do_x \neq 0$ and $\int_{\partial\Omega} (u - \overline{u}) \cdot n^{(\Omega)}\, do_x \neq 0$. In particular $u - \overline{u}|\partial\Omega$ and hence $u - \overline{u}$ are not constant. □

Next we turn to the interior domain case.

Theorem 6.3 *Let $p \in (1,\infty)$ and $b \in W^{1-1/p,\,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} \phi^{(j)} \cdot b\, do_x = 0$ for $1 \leq j \leq 6$. Abbreviate $\phi := F^{(+)}(b),\ v := V(\phi)|\Omega,\ \varrho := Q(\phi)|\Omega$.*

Then $v \in C^\infty(\Omega)^3,\ \varrho \in C^\infty(\Omega)$, the pair (v,ϱ) solves (1.1) with $f = 0$. Moreover equation (1.2) holds in the trace sense, and

$$\|v\|_{1,r} + \|\varrho\|_r \leq \mathfrak{C}\,\|b\|_p \quad \text{for } r \in [1,\, 3p/2), \qquad \|v\|_{2,p} + \|\varrho\|_{1,p} \leq \mathfrak{C}\,\|b\|_{1-1/p,\,p}.$$

The constants $\mathfrak{C}$ appearing in the preceding estimates are independent of b.

Proof: Theorem 6.3 is proved by an analogous reasoning as used in the proof of Theorem 6.1 and Corollary 6.1, with the role of Ω_R now played by Ω. The argument is somewhat more simple. For example there is no analogue to (6.4) which would come up. All the references used in the proof of Theorem 6.1 and Corollary 6.1 are such that they also cover the situation in Theorem 6.3. □

Corollary 6.2 below shows that the condition on b in Theorem 6.3 is necessary. The ensuing lemma is needed for the proof of this corollary.

Lemma 6.1 *Let $p \in (1,\infty),\ u \in W^{2,p}(\Omega)^3,\ \pi \in W^{1,p}(\Omega)$. Then*

$$\int_{\partial\Omega} \sum_{j=1}^3 \phi_j^{(l)} \sum_{k=1}^3 n_k^{(\Omega)}\,(\partial_j u_k + \partial_k u_j - \delta_{jk}\,\pi)\, do_x = \int_\Omega \sum_{j=1}^3 \phi_j^{(l)}\,(\partial_j\, div\, u + \Delta u_j - \partial_j \pi)\, dx$$

for $1 \leq l \leq 6$.

Proof: The lemma follows from the Divergence theorem and the fact that

$$\sum_{j,k=1}^3 \partial_k \phi_j^{(l)}\,(\partial_j u_k + \partial_k u_j - \delta_{jk}\,\pi) = 0 \quad \text{for } 1 \leq l \leq 6. \qquad \square$$

Corollary 6.2 *Let $p \in (1,\infty),\ f \in L^p(\Omega)^3,\ b \in W^{1-1/p,\,p}(\partial\Omega)^3,\ u \in W^{2,p}(\Omega)^3,\ \pi \in W^{1,p}(\Omega)$ such that the pair (u,π) satisfies (1.1) in Ω as well as (1.2). Then the equation $\int_{\partial\Omega} \phi^{(l)} \cdot b\, do_x + \int_\Omega \phi^{(l)} \cdot f\, dx = 0$ holds for $1 \leq l \leq 6$.*

Proof: Combine Lemma 6.1 with (1.2) and (1.1). □

In the rest of this section, we consider solutions to (1.1), (1.2) in the case $f \neq 0$.

Theorem 6.4 *Let $A \subset \mathbb{R}^3$ be measurable and $p \in (1,\, 3/2)$. If $f \in L^p(A)^3$, the integral $\int_A \sum_{k=1}^3 |(\partial^\alpha E_{jk})(x-y)\, f_k(y)|\, dy$ is finite for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1,\ 1 \leq j \leq 3$ and a. e. $x \in \mathbb{R}^3$, so we may define*

$$\mathfrak{R}(f)(x) := \Big(\int_A \sum_{k=1}^3 E_{jk}(x-y)\, f_k(y)\, dy\Big)_{1 \leq j \leq 3} \quad (x \in \mathbb{R}^3).$$

For such f, the relations $\mathfrak{R}(f) \in W^{2,p}_{loc}(\mathbb{R}^3)^3$, $div\mathfrak{R}(f) = 0$ and $\|\mathfrak{R}(f)\|_{(1/p-2/3)^{-1}} \le C(p)\,\|f\|_p$ hold, and in addition $\partial_l\mathfrak{R}(f)_j(x) = \int_A \sum_{k=1}^3 (\partial_l E_{jk})(x-y)\,f_k(y)\,dy$ for $x \in \mathbb{R}^3$, $1 \le j,l \le 3$.

Let $q \in (1,3)$. If $f \in L^q(A)^3$, the integral $\int_A \sum_{k=1}^3 |\partial_k\mathfrak{N}(x-y)\,f_k(y)|\,dy$ is finite for a. e. $x \in \mathbb{R}^3$, so we may define

$$\mathfrak{S}(f)(x) := \int_A \sum_{k=1}^3 (-\partial_k\mathfrak{N})(x-y)\,f_k(y)\,dy \quad \text{for } x \in \mathbb{R}^3,$$

and we have $\mathfrak{S}(f) \in W^{1,q}_{loc}(\mathbb{R}^3)^3$, $\|\mathfrak{S}(f)\|_{(1/q-1/3)^{-1}} \le C(q)\,\|f\|_q$.

If $A = \mathbb{R}^3$ and $f \in L^p(A)^3$, we have $-\Delta\mathfrak{R}(f) + \nabla\mathfrak{S}(f) = f$.

In the case $f \in L^p(A)^3 \cap L^q(A)^3$ the estimate $\|\partial_l\mathfrak{R}(f)\|_{(1/q-1/3)^{-1}} \le C(q)\,\|f\|_q$ is valid.

Let $r \in (1,\infty)$. If $f \in L^p(A)^3 \cap L^r(A)^3$, then $\|\partial_l\partial_m\mathfrak{R}(f)\|_r \le C(r)\,\|f\|_r$ $(1 \le l,m \le 3)$, and in the case $f \in L^q(A)^3 \cap L^r(A)^3$, the estimate $\|\partial_l\mathfrak{S}(f)\|_r \le C(r)\,\|f\|_r$ holds for $1 \le l \le 3$.

Proof: The theorem follows from the Hardy-Littlewood-Sobolev inequality and from the Calderon-Zygmund inequality; see [9, Satz 1.4] and the proof of [13, Theorem IV.2.1]. □

Corollary 6.3 *Let $r \in (1,\infty)$, $R, S \in (0,\infty)$, $f \in L^r(B_R)^3$. Then $\|\mathfrak{R}(f)|B_S\|_{2,r} + \|\mathfrak{S}(f)|B_S\|_{1,r} \le C(r,R,S)\,\|f\|_r$.*

Proof: Obviously $f \in L^p(B_R)^3$ for any $p \in (1,\min\{r,3/2\})$, so $\mathfrak{R}(f)$ and $\mathfrak{S}(f)$ are well defined. Lemma 2.1 and (4.4) yield that $\|\partial^\alpha\mathfrak{R}(f)|B_S\|_r + \|\mathfrak{S}(f)|B_S\|_r \le C(r,R,S)\,\|f\|_r$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. If $|\alpha| = 2$ and $l \in \{1,2,3\}$, we know by Theorem 6.4 that $\|\partial^\alpha\mathfrak{R}(f)\|_r + \|\partial_l\mathfrak{S}(f)\|_r \le C(r)\,\|f\|_r$. Altogether we obtain the estimate stated in the corollary. □

In the ensuing theorem, which is a more detailed version of Theorem 1.1, we solve (1.1), (1.2) in exterior domains also in the case $f \ne 0$.

Theorem 6.5 *Let $p \in (1,3/2)$, $f \in L^p(\overline{\Omega}^c)^3$, and $b \in W^{1-1/p,\,p}(\partial\Omega)^3$. Then*

$$\overline{b}_j := -\sum_{k=1}^3 n_k^{(\Omega)} \left(\,\partial_j\mathfrak{R}(f)_k + \partial_k\mathfrak{R}(f)_j - \delta_{jk}\,\mathfrak{S}(f)\,\right) \quad \text{for } j \in \{1,2,3\}.$$

is well defined in the trace sense for $1 \le j \le 3$. Further define $\gamma(b+\overline{b})$ and ϕ as in Theorem 6.2, but with b replaced by $b+\overline{b}$. Finally put

$$\begin{aligned} u &:= V(\phi)|\overline{\Omega}^c, \quad \pi := Q(\phi) + \gamma(b+\overline{b})|\overline{\Omega}^c, \\ \overline{u} &:= V(\phi) + \gamma_0^{-1}\,\gamma(b+\overline{b})\,W(\psi^{(0)})|\overline{\Omega}^c, \quad \overline{\pi} := Q(\phi) + \gamma_0^{-1}\,\gamma(b+\overline{b})\,\Pi(\psi^{(0)})|\overline{\Omega}^c, \\ v &:= u + \mathfrak{R}(f), \quad \varrho := \pi + \mathfrak{S}(f), \quad \overline{v} := \overline{u} + \mathfrak{R}(f), \quad \overline{\varrho} := \overline{\pi} + \mathfrak{S}(f). \end{aligned} \tag{6.16}$$

Note that the functions u, $\overline{u}$, π and $\overline{\pi}$ coincide with the corresponding functions in Theorem 6.2, except that b is replaced here by $b+\overline{b}$.

The function ϕ is independent of choice of p, (last statement of Corollary 4.3), so the same is true for the functions defined in (6.16).

The pairs (v, ϱ) and $(\overline{v}, \overline{\varrho})$ satisfy all the claims stated about them in Theorem 1.1, with $c = -\gamma(b + \overline{b})$. Let $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$. Then inequality (1.5) may be replaced by the more detailed estimate

$$\begin{aligned} &\|\partial_m \partial_l v\|_r + \|\partial_l \varrho\|_r \qquad (6.17)\\ &\le \mathfrak{C}\,\big(\,\|f\|_r + \|\nabla \mathfrak{R}(f|B^c_{2\,R})\,|\,\Omega_R\|_r + \|\mathfrak{S}(f|B^c_{2\,R})\,|\,\Omega_R\|_r + \|b\|_{1-1/r,\,r}\,\big)\\ &\le \mathfrak{C}\,(\|f\|_r + \|f|B^c_{2R}\|_q + \|b\|_{1-1/r,\,r}) \quad (1 \le l, m \le 3,\ R \in (0,\infty)\ \textit{with}\ \overline{\Omega} \subset B_R), \end{aligned}$$

with the constant $\mathfrak{C}$ independent of f and b, and hence also of v and ϱ. This estimate remains valid with the same type of constant when $(u,\ \varrho)$ is replaced by $(\overline{v}, \overline{\varrho})$.

If $\gamma(b + \overline{b}) = 0$, then $(\overline{v}, \overline{\varrho}) = (v, \varrho)$. Suppose that $\gamma(b + \overline{b}) \neq 0$. Then $\int_{\partial\Omega} n^{(\Omega)} \cdot \overline{v}\, do_x \neq 0$, and there is $R_0 \in [R, \infty)$ with $|(\varrho - \overline{\varrho})(x)| \ge |\gamma(b + \overline{b})|/2$ for $x \in B^c_{R_0}$. In addition $v - \overline{v}|\partial\Omega \in ker(I_p - T_p)\backslash\{0\}$ and $v - \overline{v}$ is not constant.

Proof: Let $p_1 \in (1, 3),\ p_2 \in (1, \infty)$, and suppose that $f \in L^s(\overline{\Omega}^c)^3,\ b \in W^{1-1/s,\,s}(\partial\Omega)^3$ for $s \in \{p,\, p_1,\, p_2\}$.

Since $\mathfrak{R}(f) \in W^{2,p}_{loc}(\mathbb{R}^3)^3$ by Theorem 6.4, the traces of $\mathfrak{R}(f)|\Omega$ and $\mathfrak{R}(f)|\overline{\Omega}^c$ on $\partial\Omega$ exist and coincide, so $\int_{\partial\Omega} \mathfrak{R}(f) \cdot n^{(\Omega)}\, do_x = \int_\Omega \operatorname{div} \mathfrak{R}(f)\, dx = 0$, with the last equation due to Theorem 6.4. The same reference additionally yields that $\mathfrak{S}(f) \in W^{1,p}_{loc}(\mathbb{R}^3)$, so we may conclude that $\mathfrak{R}(f)|\overline{\Omega}^c \in W^{2,p}_{loc}(\Omega^c)^3,\ \mathfrak{S}(f)|\overline{\Omega}^c \in W^{1,p}_{loc}(\Omega^c)$, hence the trace of $\partial_l \mathfrak{R}(f)_j|\Omega_R$ and $\mathfrak{S}(f)|\Omega_R$ exists and belongs to $W^{1-1/p,\,p}(\partial\Omega)$, for $1 \le j, l \le 3$. Therefore $\overline{b}$ is well defined and in $W^{1-1/p,\,p}(\partial\Omega)^3$. Thus we may deduce from Theorem 6.2 with $b + \overline{b}$ in the place of b and from Theorem 6.4 and (6.16) that the pair (v, ϱ) belongs to $W^{2,p}_{loc}(\Omega^c)^3 \times W^{1,p}_{loc}(\Omega^c)$, solves (1.1), and satisfies (1.2) in the trace sense. In addition it follows from Theorem 6.2 and by the results on $\mathfrak{R}(f)$ mentioned above that $\int_{\partial\Omega} v \cdot n^{(\Omega)}\, do_x = 0$.

All the constants $\mathfrak{C}$ appearing in the following are independent of f and b. Let $s \in \{p,\, p_2,\, p_3\}$. Theorem 6.4, applied with q, r replaced by p and s, respectively, yields that

$$\|\partial_l \partial_m \mathfrak{R}(f)\|_s + \|\partial_l \mathfrak{S}(f)\|_s \le C(s)\,\|f\|_s \quad \text{for } 1 \le l, m \le 3. \qquad (6.18)$$

Let $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$. Consider the case $s = p_2$. Let $l \in \{1,\, 2,\, 3\}$. The estimate

$$\|\partial_l \mathfrak{R}(f|\Omega_{2\,R})\,|\,\Omega_R\|_{p_2} + \|\mathfrak{S}(f|\Omega_{2\,R})\,|\,\Omega_R\|_{p_2} \le C(p_2, R)\,\|f|\Omega_{2\,R}\|_{p_2} \le C(p_2, R)\,\|f\|_{p_2} \quad (6.19)$$

holds by Corollary 6.3. Moreover, for $x \in B_R$ and $y \in B^c_{2\,R}$, we have $|x - y| \ge |y|/2$, so

$$\begin{aligned} &\|\partial_l \mathfrak{R}(f|B^c_{2\,R})\,|\,\Omega_R\|_{p_2} + \|\mathfrak{S}(f|B^c_{2\,R})\,|\,\Omega_R\|_{p_2} \le C\,|\Omega_R|^{1/p_2} \int_{B^c_{2\,R}} |y|^{-2}\,|f(y)|\,dy \qquad (6.20)\\ &\qquad \le C(p_1, p_2, R)\,\|f|B^c_{2\,R}\|_{p_1}, \end{aligned}$$

where the last inequality holds due to Hölder's inequality and because $p_1 < 3$. A trace theorem and inequality (6.18) – (6.20) imply that $\overline{b} \in W^{1-1/p_2,\,p_2}(\partial\Omega)^3$ and

$$\begin{aligned} &\|\overline{b}\|_{1-1/p_2,\,p_2} \le \mathfrak{C}\,\big(\,\|\nabla \mathfrak{R}(f)|\Omega_R\|_{1,p_2} + \|\mathfrak{S}(f)|\Omega_R\|_{1,p_2}\,\big) \qquad (6.21)\\ &\le \mathfrak{C}\,\big(\,\|f\|_{p_2} + \|\nabla \mathfrak{R}(f|B^c_{2\,R})\,|\,\Omega_R\|_{p_2} + \|\mathfrak{S}(f|B^c_{2\,R})\,|\,\Omega_R\|_{p_2}\,\big) \le \mathfrak{C}\,(\|f\|_{p_2} + \|f|B^c_{2\,R}\|_{p_1}). \end{aligned}$$

On the other hand, by the definitions in (6.16), and from (6.9) and the second estimate in (6.6) with $b+\overline{b}$ in the role of b and p_2 in that of p, we get that $\|\partial_m\partial_l u\|_{p_2}+\|\partial_l\pi\|_{p_2}\leq \mathfrak{C}\,\|b+\overline{b}\|_{1-1/p_2,\,p_2}$ for $1\leq l,m\leq 3$. Inequality (6.17) with p_2, p_1 in the role of respectively r and q follows from this estimate, (6.21), (6.18) and again the definitions in (6.16). At this point [13, Lemma II.6.1] yields that $v\in W^{2,p_2}_{loc}(\Omega^c)^3,\ \varrho\in W^{1,p_2}_{loc}(\Omega^c)$.

Consider the case $s\in\{p,\,p_1\}$. Let $l\in\{1,\,2,\,3\}$. We know from Theorem 6.4 that

$$\|\partial_l\mathfrak{R}(f)\|_{(1/s-1/3)^{-1}}+\|\mathfrak{S}(f)\|_{(1/s-1/3)^{-1}}\leq C(s)\,\|f\|_s. \tag{6.22}$$

As a consequence $\|\partial_l\mathfrak{R}(f)|\Omega_R\|_s+\|\mathfrak{S}(f)|\Omega_R\|_s\leq C(s,R)\,\|f\|_s$, so with (6.18) we see that $\|\partial_l\mathfrak{R}(f)|\Omega_R\|_{1,s}+\|\mathfrak{S}(f)|\Omega_R\|_{1,s}\leq C(s,R)\,\|f\|_s$. Again using a standard trace theorem, we thus obtain that $\overline{b}\in W^{1-1/s,s}(\partial\Omega)^3$ and $\|\overline{b}\|_{1-1/s,\,s}\leq\mathfrak{C}\,\|f\|_s$, hence

$$\|b+\overline{b}\|_{1-1/s,\,s}\leq\mathfrak{C}\,(\|b\|_{1-1/s,\,s}+\|f\|_s). \tag{6.23}$$

At this point we note that $|\gamma(b+\overline{b})|\leq\mathfrak{C}\,\|b+\overline{b}\|_s$. Therefore from (6.6) with b replaced by $b+\overline{b}$ and p by s, and from (6.23) we find that

$$\|u|\Omega_R\|_{2,s}+\|\pi-\gamma(b+\overline{b})|\Omega_R\|_{1,s}\leq\mathfrak{C}\,\|b+\overline{b}\|_{1-1/s,\,s}\leq\mathfrak{C}\,(\|f\|_s+\|b\|_{1-1/s,\,s}). \tag{6.24}$$

Suppose that $s=p_1$ Then from (6.24) by a Sobolev inequality,

$$\|\partial_l u|\Omega_R\|_{(1/p_1-1/3)^{-1}}+\|\pi-\gamma(b+\overline{b})|\Omega_R\|_{(1/p_1-1/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_{p_1}+\|b\|_{1-1/p_1,\,p_1}). \tag{6.25}$$

On the other hand, because $(1/p_1-1/3)^{-1}>3/2$, the second inequality in (6.8) with $b+\overline{b},\ p_1$ in the place of respectively b and p, together with (6.23), provide that

$$\|\partial_l u|B^c_R\|_{(1/p_1-1/3)^{-1}}+\|\pi-\gamma(b+\overline{b})|B^c_R\|_{(1/p_1-1/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_{p_1}+\|b\|_{1-1/p_1,\,p_1}).$$

The preceding estimate and (6.25) imply that

$$\|\partial_l u\|_{(1/p_1-1/3)^{-1}}+\|\pi-\gamma(b+\overline{b})\|_{(1/p_1-1/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_{p_1}+\|b\|_{1-1/p_1,\,p_1}),$$

hence with (6.22) and the definitions in (6.16),

$$\|\partial_l v\|_{(1/p_1-1/3)^{-1}}+\|\varrho-\gamma(b+\overline{b})\|_{(1/p_1-1/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_{p_1}+\|b\|_{1-1/p_1,\,p_1}). \tag{6.26}$$

Now take $s=p$. We have $\|u|\Omega_R\|_{(1/p-2/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_p+\|b\|_{1-1/p,\,p})$ due to (6.24) and a Sobolev inequality. Moreover, since $(1/p_1-2/3)^{-1}>3$, the first inequality in (6.8) with $b+\overline{b}$ in the place of b together with (6.23) yields that $\|u|B^c_R\|_{(1/p-2/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_p+\|b\|_{1-1/p,\,p})$. It follows with Theorem 6.4 and (6.16) that $\|v\|_{(1/p-2/3)^{-1}}\leq\mathfrak{C}\,(\|f\|_p+\|b\|_{1-1/p,\,p})$.

We recall that (6.17) has been shown above to hold with r and q replaced by respectively p_2 and p_1. Take $p_2=p_1=p$. Then we may conclude from this variant of (6.17), the preceding estimate and (6.26) that inequality (1.3) holds with $c=-\gamma(b+\overline{b})$.

Consider the case $p_2=p_1=q$. Then (6.26), our variant of (6.17) and the relations $v\in W^{2,p_2}_{loc}(\Omega^c)^3,\ \varrho\in W^{1,p_2}_{loc}(\Omega^c)$ also shown above yield inequality (1.4) and the regularity statement preceding it. Finally, setting $r=p_3,\ q=p_2$, we obtain inequality (6.17) and

the relations $v \in W^{2,r}_{loc}(\Omega^c)^3$, $\varrho \in W^{1,r}_{loc}(\Omega^c)$ from the variant of this inequality already proved, and from the regularity property of v an ϱ just mentioned.

The statements in Theorem 1.2 with respect to $\overline{v}$ and $\overline{\varrho}$ follow by analogous arguments, again based on Theorem 6.2 and 6.4.

If $\gamma(b+\overline{b}) = 0$, we deduce from Theorem (6.2 with $b+\overline{b}$ in the place of b, and from (6.16) that $(v, \varrho) = (\overline{v}, \overline{\varrho})$. Suppose that $\gamma(b+\overline{b}) \neq 0$. Since $\int_{\partial\Omega} n^{(\Omega)} \cdot \mathfrak{R}(f)\, do_x = 0$, as shown at the beginning of this proof, it follows from Theorem 6.2 with b replaced by $b+\overline{b}$, and from (6.16) that $\int_{\partial\Omega} n^{(\Omega)} \cdot \overline{v}\, do_x \neq 0$. It further follows from these references that there exists R_0 with properties as mentioned in Theorem 6.5, and that $v - \overline{v}|\partial\Omega \in ker(I_p - T_p)\backslash\{0\}$ and $v - \overline{v}$ is not constant. □

Corollary 6.4 *Let $p \in (1,\infty)$, $R \in (0,\infty)$, $f \in L^p(B_R)^3$, $b \in W^{1-1/p,p}(\partial\Omega)^3$ and $S \in (R,\infty)$ with $\overline{\Omega} \subset B_S$. In particular $f \in L^r(B_R)^3$ for any $r \in (1,p]$. Then, with the notation of Theorem 6.5, and with a constant $\mathfrak{C}$ independent of f, b and x,*

$$|x|^{1+|\alpha|}\,\big[\,|\partial^\alpha v(x)| + |\partial^\alpha \overline{v}(x)|\,\big] \;+\; |x|^{2+|\alpha|}\,\big[\,|\partial^\alpha\big(\,\varrho - \gamma(b+\overline{b})\,\big)(x)| + |\partial^\alpha\overline{\varrho}(x)|\,\big] \qquad (6.27)$$
$$\leq \mathfrak{C}\,(\|f\|_1 + \|b\|_p) \quad \textit{for } x \in B_S^c,\ \alpha \in \mathbb{N}_0^3,\ |\alpha| \leq 2.$$

As a consequence, inequalities (6.8), (6.9) and their analogues for $(\overline{u}, \overline{\pi})$ as described in Theorem 6.2 remain valid when (u,π), $(\overline{u},\overline{\pi})$ are replaced by (v,ϱ) and $(\overline{v},\overline{\varrho})$, respectively, and the factor $\|b\|_p$ by $\|b\|_p + \|f\|_1$. In addition, if $\gamma(b+\overline{b}) \neq 0$, there is $R_0 \in [S,\infty)$ with $|\varrho(x)| \geq |\gamma(b+\overline{b})|/2$ for $x \in B^c_{R_0}$.

Proof: It may be deduced from Lebesgue's theorem and Lemma 4.1 that $\mathfrak{R}(f)_j|B_S^c$ and $\mathfrak{S}(f)|B_S^c$ belong to $C^2(B_S^c)$ $(1 \leq j \leq 3)$, and the derivatives appearing in (6.27) commute with the integrals defining $\mathfrak{R}(f)$ and $\mathfrak{S}(f)$ (Theorem 6.4). Thus it may be shown in the same way as Lemma 4.6 that

$$|x|^{1+|\alpha|}\,|\partial^\alpha \mathfrak{R}(f)(x)| \;+\; |x|^{2+|\alpha|}\,|\partial^\alpha \mathfrak{S}(f)(x)| \leq \mathfrak{C}\,\|f\|_1 \quad \text{for } x,\ \alpha \text{ as in (6.27)},$$

with $\mathfrak{C}$ independent of f and x. Inequality (6.27) now follows with (6.7). The other statements of Corollary 6.4 are obvious consequences of (6.27). □

The ensuing corollary provides additional details about the existence and regularity results in Theorem 1.2, which deals with solutions in interior domains.

Corollary 6.5 *Let $p \in (1,\infty)$, $f \in L^p(\Omega)^3$, $b \in W^{1-1/p,p}(\partial\Omega)^3$ with*

$$\int_{\partial\Omega} \phi^{(l)} \cdot b\, do_x + \int_\Omega \phi^{(l)} \cdot f\, dx = 0 \quad \textit{for } 1 \leq l \leq 6.$$

Put $\overline{b}_j := -\sum_{k=1}^3 n_k^{(\Omega)}\,\big(\,\partial_j \mathfrak{R}(f)_k + \partial_k \mathfrak{R}(f)_j - \delta_{jk}\,\mathfrak{S}(f)\,\big)$ for $1 \leq j \leq 3$. Then for l as above, the equation $\int_{\partial\Omega} \phi^{(l)} \cdot (b+\overline{b})\, do_x = 0$ holds.

Define $\phi := F^{(+)}(b+\overline{b})$, $u := V(\phi) + \mathfrak{R}(f)|\Omega$, $\varrho := Q(\phi) + \mathfrak{S}(f)|\Omega$. With the matrix $\widetilde{M}$ from Lemma 4.8, set $\alpha := \widetilde{M}^{-1} \cdot (\int_\Omega u \cdot \phi^{(l)}\, dx)_{1\leq l\leq 6}$, $v := u - \sum_{l=1}^6 \alpha_l\, \phi^{(l)}$.

Then $v \in W^{2,p}(\Omega)^3$, $\varrho \in W^{1,p}(\Omega)$, the pair (v,ϱ) satisfies (1.1) and (1.2), and v verifies the relation $\int_\Omega v \cdot \phi^{(l)}\, dx = 0$ for $1 \leq l \leq 6$. Moreover $\|v\|_{2,p} + \|\varrho\|_{1,p} \leq \mathfrak{C}\,(\|f\|_p + \|b\|_{1-1/p,\,p})$, with a constant $\mathfrak{C}$ independent of f and b.

Proof: By Corollary 6.3, we have $\mathfrak{R}(f)|\Omega \in W^{2,p}(\Omega)^3$, $\mathfrak{S}(f)|\Omega \in W^{1,p}(\Omega)$, so $\overline{b} \in W^{1-1/p,\,p}(\partial\Omega)^3$. Let $l \in \{1,\,...,\,6\}$. Then we find by Theorem 6.4 and Lemma 6.1 that

$$\int_{\partial\Omega} \phi^{(l)} \cdot (b + \overline{b})\, do_x = \int_{\partial\Omega} \phi^{(l)} \cdot b\, do_x + \int_{\Omega} \phi^{(l)} \cdot f\, dx,$$

so $\int_{\partial\Omega} \phi^{(l)} \cdot (b+\overline{b})\, do_x = 0$ by our assumptions on f and b. Therefore we may apply Theorem 6.3 with b replaced by $b + \overline{b}$, to obtain that $V(\phi)|\Omega \in W^{2,p}(\Omega)^3$, $Q(\phi)|\Omega \in W^{1,p}(\Omega)$, the pair $\big(V(\phi)|\Omega,\, Q(\phi)|\Omega\,\big)$ solves (1.1) with $f = 0$, and equation (1.2) holds with $b+\overline{b}$ in the place of b. In addition this theorem implies that $\|V(\phi)|\Omega\|_{2,p} + \|Q(\phi)|\Omega\|_{1,p} \le \mathfrak{C}\,\|b+\overline{b}\|_{1,p}$. (The constants $\mathfrak{C}$ in this proof are independent of f and b.) A trace estimate yields that

$$\|\overline{b}\|_{1-1/p,\,p} \le \mathfrak{C}\,\Big(\sum_{j=1}^{3} \|\partial_j(\mathfrak{R}(f)|\Omega)\|_{1,p} + \|\mathfrak{S}(f)|\Omega\|_{1,p}\Big) \le \mathfrak{C}\,\big(\,\|\mathfrak{R}(f)|\Omega\|_{2,p} + \mathfrak{S}(f)|\Omega\|_{1,p}\,\big).$$

Since $\|\mathfrak{R}(f)|\Omega\|_{2,p} + \mathfrak{S}(f)|\Omega\|_{1,p} \le \mathfrak{C}\,\|f\|_p$ again by Corollary 6.3, we thus arrive at the inequality $\|u\|_{2,p} + \|\varrho\|_{1,p} \le \mathfrak{C}\,(\|f\|_p + \|b\|_{1-1/p,\,p})$. Due to Theorem 6.4 and by what was stated above on the pair $\big(V(\phi)|\Omega,\, Q(\phi)|\Omega\,\big)$, the pair (u, ϱ) solves (1.1), (1.2).

For any $l \in \{1,\,...,\,6\}$, the function $\phi^{(l)}|\overline{\Omega}$ belongs to $C^\infty(\overline{\Omega})^3$, and the pair $(\phi^{(l)}, 0)$ is a solution of (1.1), (1.2) with $f = 0$ and $b = 0$. Recall the vector $\alpha \in \mathbb{R}^6$ introduced in Corollary 6.5. Since $|\alpha_l| \le \mathfrak{C}\,\|u\|_p$ for $1 \le l \le 6$, and by the properties of u, ϱ and $\widetilde{M}$, the pair (v, ϱ) fulfills the claims stated in that corollary. □

7 Some uniqueness results.

The claims on uniqueness in this section imply in particular what is stated on uniqueness in Theorem 1.1 and 1.2. First we consider the exterior domain case.

Theorem 7.1 *Let $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$. For $j \in \{1,\,2\}$, let p_j, r_j, $s_j \in (1, \infty)$, $u^{(j)} \in W^{2,s_j}_{loc}(\Omega^c)^3$, $\pi^{(j)} \in W^{1,s_j}_{loc}(\Omega^c)$ with $u^{(j)}|B_R^c \in L^{p_j}(B_R^c)^3$ and $\nabla\pi^{(j)}|B_R^c \in L^{r_j}(B_R^c)^3$.*

Further assume that either $\int_{\partial\Omega} u^{(j)} \cdot n^{(\Omega)}\, do_x = 0$ for $j \in \{1,\,2\}$, or that there are numbers q_1, $q_2 \in (1, \infty)$ such that $\pi^{(j)}|B_R^c \in L^{q_j}(B_R^c)$ for $j \in \{1,\,2\}$.

Put $u := u^{(1)} - u^{(2)}$, $\pi := \pi^{(1)} - \pi^{(2)}$. Suppose the pair (u, π) satisfies (1.1) with $f = 0$ as well as (1.2) with $b = 0$. Then $u = 0$ and $\pi = 0$.

Proof: Put $r := \min\{r_1,\, r_2,\, s_1,\, s_2\}$. By Theorem 2.7 we know that $\nabla\pi|B_{R+1}^c \in L^s(B_{R+1}^c)^3$ for $s \in (1, r]$. Theorem 2.3 then implies there is $\tau(\pi) \in \mathbb{R}$ such that $\pi + \tau(\pi)|B_{R+1}^c \in L^{(1/s-1/3)^{-1}}(B_{R+1}^c)$ for $s \in (1, r]$ if $r < 3$, and for any $s \in (1, 3)$ else. Note that $\tau(\pi)$ is independent of s, as follows from the criterion for the case $\tau(v) = 0$ in Theorem 2.3. As a consequence there is $\widetilde{p} \in (3/2,\, \infty]$ such that $\pi + \tau(\pi)|B_{R+1}^c \in L^p(B_{R+1}^c)$ for any $p \in (3/2,\, \widetilde{p})$. In fact, we may choose $\widetilde{p} := (1/r - 1/3)^{-1}$ in the case $r < 3$, and $\widetilde{p} = \infty$ else.

Fix a function $\varphi \in C_0^\infty(B_2)$ with $\varphi|B_1 = 1$ and $0 \le \varphi \le 1$. Put $\varphi_m(x) := \varphi(m^{-1}\,x)$ for $x \in \mathbb{R}^3$, $m \in \mathbb{N}$. Then $\varphi_m \in C_0^\infty(B_{2\,m})$, $\varphi|B_m = 1$, $0 \le \varphi_m \le 1$, $supp(\nabla\varphi_m) \subset \overline{B_{2\,m}}\backslash B_m$

for $m \in \mathbb{N},\ \|\nabla\varphi_m\|_p \to 0\,(m \to \infty)$ for $p \in (3,\infty)$, and $\|\partial_l\partial_k\varphi_m\|_p \to 0\,(m \to \infty)$ for $p \in (3/2,\,\infty),\ 1 \le l,k \le 3$.

Let $\Phi \in C_0^\infty(\overline{\Omega}^c)^3$. By Theorem 6.5, we may choose a pair $(v,\varrho) \in W^{2,1}_{loc}(\Omega^c)^3 \times W^{1,1}_{loc}(\Omega)$ and a constant $c(\varrho) \in \mathbb{R}$ such that this pair solves (1.1) with $f = \Phi$ and (1.2) with $b = 0$, and such that $v_j \in L^{p_1}(\overline{\Omega}^c),\ \partial_k v_j,\ \varrho + c(\varrho) \in L^{p_2}(\overline{\Omega}^c),\ \partial_m\partial_k v_j,\ \partial_l\varrho \in L^{p_3}(\overline{\Omega}^c)$ for any $p_1 \in (3,\infty),\ p_2 \in (3/2,\,\infty),\ p_3 \in (1,\infty),\ 1 \le j,k,l \le 3$. According to that corollary, we may additionally require that $\int_{\partial\Omega} v\cdot n^{(\Omega)}\,do_x$ or $c(\varrho) = 0$.

For $n \in \mathbb{N}$, put $v_m := \varphi_m\, v,\ \varrho_m := \varphi_m\big(\,\varrho + c(\varrho)\,\big)$. We claim that

$$\|u\cdot(-\Delta v_m + \nabla\varrho_m - \Phi)\|_1 \to 0,\ \|u\cdot\nabla\mathrm{div}\, v_m\|_1 \to 0,\ \|\big(\,\pi + c(\pi)\,\big)\,\mathrm{div}\, v_m\|_1 \to 0 \quad (7.1)$$

for $m \to \infty$. In fact, concerning the first of these relations, recall that $supp(\nabla\varphi_m) \subset \overline{B_{2\,m}}\backslash B_m \subset B_R^c$ for $m \in \mathbb{N},\ m \ge R$, and that $\overline{\Omega} \subset B_R$. Therefore by Hölder's inequality

$$\|u_k^{(j)}\,\partial_l v_k\,\partial_l\varphi_m\|_1 \le \|u_k^{(j)}|B_R^c\|_{p_j}\,\|\partial_l v_k\|_{3\,p_j'/2}\,\|\partial_l\varphi_m\|_{3\,p_j'} \quad \text{for } m \text{ as before},\ j \in \{1,\,2\},$$

$1 \le k,l \le 3$. Further recall that $\partial_l v \in L^p(\overline{\Omega}^c)^3$ for $p \in (3/2,\,\infty),\ u^{(j)}|B_R^c \in L^{p_j}(B_R^c)^3$ and $\|\nabla\varphi_m\|_{3\,p_j'} \to 0\ (m \to \infty)$ for $j \in \{1,\,2\}$. It follows that $\|u_k^{(j)}\,\partial_l v_k\,\partial_l\varphi_m\|_1 \to 0\ (m \to \infty)$ for j, k and l as before. Similarly $\|u^{(j)}\cdot v\,\Delta\varphi_m\|_1 \to 0$ and $\|u^{(j)}\,\big(\,\varrho + c(\varrho)\,\big)\,\nabla\varphi_m\|_1 \to 0$ for $m \to \infty$. Altogether, since $-\Delta v + \nabla\varrho = \Phi$, we get that $\|u\cdot(\Delta v_m + \nabla\varrho_m - \Phi)\|_1 \to 0\ (m \to \infty)$. Moreover $\mathrm{div}\, v = 0$, so a variant of the preceding argument yields that $\|u\cdot\nabla\mathrm{div}\, v_m\|_1 \to 0\ (m \to \infty)$. Since $\pi + \tau(\pi)|B_{R+1}^c \in L^p(B_{R+1}^c)$ for $p \in (3/2,\,\widetilde{p})$, as shown above, and because $v|B_{R+1}^c \in L^s(B_{R+1}^c)^3$ for $s \in (3,\infty)$, we may choose $p \in (3/2,\,\widetilde{p})$ and $s \in (3,\infty)$ so close to respectively $3/2$ and 3 that $1 - 1/p - 1/s < 1/3$, hence $(1 - 1/p - 1/s)^{-1} > 3$. As a consequence $\|\nabla\varphi_m\|_{(1-1/p-1/s)^{-1}} \to 0\ (m \to \infty)$, so we get in view of the equation $\mathrm{div}\, v = 0$ that $\|\big(\,\pi + \tau(\pi)\,\big)\,\mathrm{div}\, v_m\|_1 \to 0\ (m \to \infty)$. This completes the proof of (7.1). We apply the first and second relation in (7.1), obtaining that $\int_{\overline{\Omega}^c} u\cdot\Phi\,dx = \lim_{m\to\infty}\int_{\overline{\Omega}^c} u\cdot(-\Delta v_m + \nabla\varrho_m - \nabla\mathrm{div}\, v_m)\,dx$. At this point we recall that the pair (v,ϱ) satisfies (1.2) with $b = 0$, and that $\mathrm{div}\, u = 0,\ \varphi_m|B_m = 1,\ supp(\varphi_m) \subset B_{2\,m}$ and $\overline{\Omega} \subset B_m$ for $m \in \mathbb{N}$ with $m \ge R$. Thus we have $\partial_j v_{m,k}|\partial\Omega = \partial_j v_k|\partial\Omega,\ \varrho_m|\partial\Omega = \varrho + c(\varrho)|\partial\Omega$, and we may apply Lemma 2.6 with $\Omega_{2\,m}$, $(v_m,\varrho_m)|\Omega_{2\,m}$, $u|\Omega_{2\,m}$ in the role of U, (u,π) and $\widetilde{u}$, respectively. It follows from the preceding equation for $\int_{\overline{\Omega}^c} u\cdot\Phi\,dx$ that

$$\begin{aligned}&\int_{\overline{\Omega}^c} u\cdot\Phi\,dx \qquad (7.2)\\ &= (1/2)\lim_{m\to\infty}\int_{\overline{\Omega}^c}\sum_{j,k=1}^3(\partial_k u_j + \partial_j u_k)\,(\partial_k v_{m,j} + \partial_j v_{m,k})\,dx - c(\varrho)\int_{\partial\Omega} n^{(\Omega)}\cdot u\,do_x.\end{aligned}$$

Next we again use Lemma 2.6, this time with $\big(\,u,\ \pi + \tau(\pi)\,\big)|\Omega_{2\,m},\ v_m|\Omega_{2\,m}$ corresponding to (u,π), $\widetilde{u}$, respectively. In addition we apply the third relation in (7.1) and the assumption that the pair (u,π) is a solution of (1.1) with $f = 0$ and of (1.2) with $b = 0$. Once more we take into account that $\varphi_m|B_m = 1,\ supp(\varphi_m) \subset B_{2\,m}$ and $\overline{\Omega} \subset B_m$ for $m \in \mathbb{N}$ with $m \ge R$. In this way we may deduce from (7.2) that

$$\int_{\overline{\Omega}^c} u\cdot\Phi\,dx = -c(\varrho)\int_{\partial\Omega} n^{(\Omega)}\cdot u\,do_x + \tau(\pi)\int_{\partial\Omega} n^{(\Omega)}\cdot v\,do_x. \qquad (7.3)$$

Now suppose that $\int_{\partial\Omega} n^{(\Omega)} \cdot u \, do_x = 0$. Then we choose the function v in such a way that it additionally satisfies the condition $\int_{\partial\Omega} n^{(\Omega)} \cdot v \, do_x = 0$. As mentioned above, such a choice is possible according to Theorem 6.5. In this way we get from (7.3) that $\int_{\overline{\Omega}^c} u \cdot \Phi \, dx = 0$.

In view of the second uniqueness criterion in Theorem 7.1, let us suppose there are numbers q_1, $q_2 \in (1, \infty)$ such that $\pi^{(j)}|B_R^c \in L^{q_j}(B_R^c)$ for $j \in \{1, 2\}$. Then $\tau(\pi) = 0$. In fact, due to the choice of $\tau(\pi)$ at the beginning of this proof, there is $s \in (1, \infty)$ with $\pi + \tau(\pi)|B_{R+1}^c \in L^s(B_{R+1}^c)^3$, so $\int_{B_1(m)} |\pi + \tau(\pi)|^s \, dx \to 0$ $(m \to \infty)$ by Lebegue's theorem, hence $\int_{B_1(m)} \pi + \tau(\pi) \, dx \to 0$. Since $\pi^{(j)}|B_R^c \in L^{q_j}(B_R^c)$ for $j \in \{1, 2\}$, the same reasoning yields that $\int_{B_1(m)} \pi^{(j)} \, dx \to 0$ $(m \to \infty)$ for $j \in \{1, 2\}$. But $\pi - \pi^{(1)} + \pi^{(2)} = \tau(\pi)$, so we conclude that $\tau(\pi) = 0$. On the other hand, according to Theorem 6.5, we may require that $c(\varrho) = 0$, as already stated at the beginning of this proof. Returning to (7.3), with this choice of ϱ we obtain once more that $\int_{\overline{\Omega}^c} \Phi \cdot u \, dx = 0$.

Therefore this equation is valid in any of the two cases considered in Theorem 7.1. Since Φ was taken arbitrarily from $C_0^\infty(\overline{\Omega}^c)^3$, it follows in both cases that $u = 0$, so $\pi = 0$ due to (1.1), (1.2) and the assumption that $\partial\Omega$, and hence also Ω and $\overline{\Omega}^c$, are connected. □

Finally let us consider uniqueness of solutions in Ω.

Theorem 7.2 *Let $p \in (1, \infty)$, $v \in W^{2,p}(\Omega)^3$, $\varrho \in W^{1,p}(\Omega)$ such that the pair (v, ϱ) satisfies (1.1), (1.2) with $f = 0$ and $b = 0$, and such that $\int_\Omega v \cdot \phi^{(j)} \, dx = 0$ for $1 \le j \le 6$. Then v and ϱ vanish.*

Proof: Let $g \in C_0^\infty(\Omega)^3$, and put $\alpha := \widetilde{M}^{-1} (\int_\Omega g \cdot \phi^{(j)} \, dx)_{1 \le j \le 6}$, with the matrix $\widetilde{M}$ defined in Lemma 4.8. Define $f := g - \sum_{j=1}^6 \alpha_j \cdot \phi^{(j)}$. Then $f \in C^0(\overline{\Omega})^3 \subset L^{p'}(\Omega)^3$ and $\int_\Omega f \cdot \phi^{(k)} \, dx = 0$ for $1 \le k \le 6$. Therefore, by Corollary 6.5, there are functions $w \in W^{2,p}(\Omega)^3$, $\sigma \in W^{1,p}(\Omega)$ such that the pair (w, σ) satisfies (1.1) with the function f defined above, and (1.2) with $b = 0$.

In this situation we twice use the formula in Lemma 2.6, first with $(u, \pi) = (w, \sigma)$, $\widetilde{u} = v$, and then with $(u, \pi) = (v, \varrho)$, $\widetilde{u} = w$. It follows that $\int_\Omega f \cdot v \, dx = 0$. But by our assumptions, v is orthogonal to the functions $\phi^{(j)}$. Therefore $\int_\Omega g \cdot v \, dx = 0$ in view of the definition of f. Since g was an arbitrary function from $C_0^\infty(\Omega)^3$, we may conclude that $v = 0$, hence there is $c \in \mathbb{R}$ with $\varrho = c$ a. e. Equation (1.2) now yields that $\varrho = 0$. □

8 Appendix: Proof of Theorem 4.6.

We only consider the limit $\lim_{\delta \downarrow 0} F(-\delta)$ for the function F from Theorem 4.6. The limit $\lim_{\delta \downarrow 0} F(\delta)$ is easier to determine because then the integrals on Ω_R appearing below are replaced by ones on Ω, so the parameter R and the difficulties related to it do not arise.

Let $\delta \in \big(0, \, \delta(\Omega)\big]$ and put $z := x - \delta \, n^{(\Omega)}(x)$. Note that $z \in \Omega$ (see (3.3)) and $|z - b| \ge \mathcal{D} \, \delta/2$ for $b \in \mathfrak{U}_{-\delta}$ (see (3.6)). (The open set $\mathfrak{U}_{-\delta}$ and the constants $\delta(\Omega)$ and $\mathcal{D}$ were introduced in Lemma 3.2.) It follows that for $m \in \mathbb{N}$, the function $|z - y|^{-m}$ $(y \in \mathfrak{U}_{-\delta})$ is C^∞ in $\mathfrak{U}_{-\delta}$. Since $\overline{\Omega_R} \subset \Omega^c \subset \mathfrak{U}_{-\delta}$ for $R \in (0, \infty)$ (see Lemma 3.2), this C^∞-regularity in

$\mathfrak{U}_{-\delta}$ will allow us in the following to apply the Divergence theorem in Ω_R.

Let $j \in \{1, 2, 3\}$ and put $K_{klm} := -\partial_j S_{klm} - \partial_k S_{jlm} + 2\,\delta_{jk}\,\partial_l\partial_m \mathfrak{N}$ for $k,\, l,\, m \in \{1, 2, 3\}$, where $S_{\nu lm}$ for $\nu \in \{1, 2, 3\}$ and $\mathfrak{N}$ were introduced in (4.2) and (4.1), respectively. Then by the definitions in Theorem 4.6 and Lemma 4.5,

$$F(-\delta) = \sum_{k,l,m=1}^{3} n_k^{(\Omega)}(x) \int_{\partial\Omega} K_{klm}(z-y)\, n_l^{(\Omega)}(y)\, \phi_m(y)\, do_y. \tag{8.4}$$

Let $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$, and let $n^{(\Omega_R)}$ denote the outward unit normal to Ω_R, that is, $n^{(\Omega_R)}|\partial\Omega = -n^{(\Omega)}|\partial\Omega$, $n^{(\Omega_R)}(y) = R^{-1}\, y$ for $y \in \partial B_R$. By the definition of $C^{1,\alpha}(\partial\Omega)^3$ (see at the beginning of Section 2), there is $\widetilde{\phi} \in C^{1,\alpha}(\mathbb{R}^3)^3$ with $\widetilde{\phi}|\partial\Omega = \phi$. We will write ϕ instead of $\widetilde{\phi}$ in the following. Then we get from (8.4) that $F(-\kappa)_j = \mathfrak{A}_1 + \mathfrak{B}_1(R) + \mathring{\mathfrak{A}} + \mathfrak{A}$, where $\mathfrak{A}_1$ is given by the right-hand side of (8.4), but with the term $\phi_m(y)$ replaced by $\phi_m(y) - \phi_m(x) - \sum_{\nu=1}^{3} \partial_\nu\phi_m(x)\,(y-x)_\nu$, for $1 \le m \le 3$. Moreover

$$\mathfrak{B}_1(R) := \sum_{k,l,m=1}^{3} \phi_m(x)\, n_k^{(\Omega)}(x) \int_{\partial B_R} K_{klm}(z-y)\, y_l/R\, do_y,$$

$$\mathfrak{A} := \sum_{k,l,m,\nu=1}^{3} \partial_\nu\phi_m(x)\, n_k^{(\Omega)}(x) \int_{\partial\Omega} K_{klm}(z-y)\, n_l^{(\Omega)}(y)\, (y-x)_\nu\, do_y.$$

The term $\mathring{\mathfrak{A}}$ differs from $\mathfrak{B}_1(R)$ insofar as the integration extends over $\partial\Omega_R$ instead of ∂B_R, and $-n_l^{(\Omega_R)}(y)$ takes the place of y_l/R, for $1 \le l \le 3$. Due the Divergence theorem, (4.3) and (4.5), we get that $\mathring{\mathfrak{A}} = 0$. Next we observe that $\mathfrak{A} = \mathfrak{A}_2 + \mathfrak{B}_2(R) + \mathfrak{B}_3(R) + \widetilde{\mathfrak{A}}$, where $\mathfrak{A}_2$ is defined in the same way as $\mathfrak{A}$, but with $n_k^{(\Omega)}(x)\, n_l^{(\Omega)}(y)$ for $1 \le k, l \le 3$ replaced by $n_k^{(\Omega)}(x)\, n_l^{(\Omega)}(y) - n_k^{(\Omega)}(y)\, n_l^{(\Omega)}(x)$. Moreover

$$\widetilde{\mathfrak{A}} := -\sum_{k,l,m,\nu=1}^{3} \partial_\nu\phi_m(x)\, n_l^{(\Omega)}(x) \int_{\partial\Omega_R} K_{klm}(z-y)\, (y-x)_\nu\, n_k^{(\Omega_R)}(y)\, do_y, \tag{8.5}$$

$$\mathfrak{B}_3(R) := \sum_{k,l,m,\nu=1}^{3} \partial_\nu\phi_m(x)\, n_l^{(\Omega)}(x) \int_{\partial B_R} K_{klm}(-y)\, y_\nu\, y_k/R\, do_y,$$

and with $\mathfrak{B}_2(R)$ chosen as $\mathfrak{B}_3(R)$, but with $K_{klm}(z-y)\,(y-x)_\nu - K_{klm}(-y)\,y_\nu$ in the role of $K_{klm}(-y)\,y_\nu$, where $1 \le k, l, m, \nu \le 3$. Now the Divergence theorem is applied to $\widetilde{\mathfrak{A}}$, transforming the integral over $\partial\Omega_R$ in (8.5) into an integral over Ω_R. Note that due to (4.5), (4.3), the sum $\sum_{k=1}^{3} \partial y_k \big(K_{klm}(z-y)\,(y-x)_\nu \big)$ reduces to $K_{\nu lm}(z-y)$; see the definition of $K_{\nu lm}$ at the beginning of this proof (ν, l, m as before). In view of this definition, we may again apply the Divergence theorem, this time in order to retransform the integral over Ω_R into an integral over $\partial\Omega_R$, which we split according to the equation $\widetilde{\mathfrak{A}} = \overline{\mathfrak{A}} + \overline{\mathfrak{B}}$, with

$$\overline{\mathfrak{A}} := \sum_{l,m,\nu=1}^{3} \partial_\nu\phi_m(x)\, n_l^{(\Omega)}(x) \int_{\partial\Omega} \big(S_{\nu lm}(z-y)\, n_j^{(\Omega)}(y) + S_{jlm}(z-y)\, n_\nu^{(\Omega)}(y) \\ -2\,\delta_{\nu j}\,(\partial_l\mathfrak{N})(z-y)\, n_m^{(\Omega)}(y) \big)\, do_y,$$

$$\overline{\mathfrak{B}} := \sum_{l,m,\nu=1}^{3} \partial_\nu \phi_m(x)\, n_l^{(\Omega)}(x) \int_{\partial B_R} \big(-S_{\nu lm}(z-y)\, y_j/R - S_{jlm}(z-y)\, y_\nu/R \\ +2\,\delta_{\nu j}\, (\partial_l \mathfrak{N})(z-y)\, y_m/R \big)\, do_y.$$

As a consequence $\widetilde{\mathfrak{A}} = \mathfrak{A}_3 + \mathfrak{B}_4(R) + \mathfrak{B}_5(R) + \widehat{\mathfrak{A}}$, where $\mathfrak{A}_3$ coincides with $\overline{\mathfrak{A}}$ except that $n_l^{(\Omega)}(x)\, n_b^{(\Omega)}(y) - n_l^{(\Omega)}(y)\, n_b^{(\Omega)}(x)$ takes the role of $n_l^{(\Omega)}(x)\, n_b^{(\Omega)}(y)$, for $b, l \in \{1, 2, 3\}$. Moreover $\mathfrak{B}_4(R)$ and $\mathfrak{B}_5(R)$ correspond to $\overline{\mathfrak{B}}(R)$, but we put $-S_{blm}(z-y) + S_{blm}(-y)$ and $(\partial_l \mathfrak{N})(z-y) - (\partial_l \mathfrak{N})(-y)$ in the place of $-S_{blm}(z-y)$ and $(\partial_l \mathfrak{N})(z-y)$, respectively, in the case of $\mathfrak{B}_4(R)$, whereas $\mathfrak{B}_5(R)$ arises from $\overline{\mathfrak{B}}(R)$ by substituting $-S_{blm}(-y)$ and $(\partial_l \mathfrak{N})(-y)$ for $-S_{blm}(z-y)$ and $(\partial_l \mathfrak{N})(z-y)$, respectively ($1 \le b, l, m \le 3$). The term $\widehat{\mathfrak{A}}$ is given by

$$\widehat{\mathfrak{A}} := \sum_{m,\nu=1}^{3} \partial_\nu \phi_m(x) \Big(n_j^{(\Omega)}(x) \int_{\partial\Omega} \sum_{l=1}^{3} S_{\nu lm}(z-y)\, n_l^{(\Omega)}(y)\, do_y \\ + n_\nu^{(\Omega)}(x) \int_{\partial\Omega} \sum_{l=1}^{3} S_{jlm}(z-y)\, n_l^{(\Omega)}(y)\, do_y - 2\,\delta_{\nu j}\, n_m^{(\Omega)}(x) \int_{\partial\Omega} \sum_{l=1}^{3} (\partial_l \mathfrak{N})(z-y)\, n_l^{(\Omega)}(y)\, do_y \Big).$$

We finally observe that $\widehat{\mathfrak{A}} = \mathfrak{B}_6(R) + \mathfrak{B}_7(R) + \underline{\mathfrak{A}}$, where $\underline{\mathfrak{A}}$ differs from $\widehat{\mathfrak{A}}$ insofar as the domain of integration now is $\partial\Omega_R$ instead of $\partial\Omega$, and the term $-n_l^{(\Omega_R)}(y)$ is substituted for $n_l^{(\Omega)}(y)$ ($1 \le l \le 3$). Moreover

$$\mathfrak{B}_7(R) := \sum_{m,\nu=1}^{3} \partial_\nu \phi_m(x) \Big(n_j^{(\Omega)}(x) \int_{\partial B_R} \sum_{l=1}^{3} S_{\nu lm}(-y)\, y_l/R\, do_y \\ + n_\nu^{(\Omega)}(x) \int_{\partial B_R} \sum_{l=1}^{3} S_{jlm}(-y)\, y_l/R\, do_y - 2\,\delta_{\nu j}\, n_m^{(\Omega)}(x) \int_{\partial B_R} \sum_{l=1}^{3} (\partial_l \mathfrak{N})(-y)\, y_l/R\, do_y \Big).$$

As for the term $\mathfrak{B}_6(R)$, it is defined in a way analogous to the definition of $\mathfrak{B}_7(R)$, but the terms $S_{blm}(z-y)$ and $(\partial_l \mathfrak{N})(z-y)$ are replaced by $S_{blm}(z-y) - S_{blm}(-y)$ and $(\partial_l \mathfrak{N})(z-y) - (\partial_l \mathfrak{N})(-y)$, respectively ($b \in \{1, 2, 3\}$). By (4.5) and (4.3) we have $\underline{\mathfrak{A}} = 0$.

The splitting of $F(-\delta)$, $\mathfrak{A}$, $\widetilde{A}$ and $\widehat{\mathfrak{A}}$ considered above, and the equations $\underline{\mathfrak{A}} = \mathring{\mathfrak{A}} = 0$ may be subsumed into a single equation, that is, $F(-\delta) = \sum_{\nu=1}^{3} \mathfrak{A}_\nu + \sum_{\nu=1}^{7} \mathfrak{B}_\nu(R)$. Recall the abbreviation $z = x - \delta\, n^{(\Omega)}(x)$ introduced at the beginning of this proof. Since $\overline{\Omega} \subset B_{R/2}$ and $z \in \Omega$ (see (3.3)), we have $|\vartheta\, z - y| \ge |y|/2$ for $\vartheta \in [0, 1]$, $y \in \partial B_R$, so $|\mathfrak{B}_b(R)| \le \mathfrak{C} \int_{\partial B_R} |y|^{-3}\, do_y \le \mathfrak{C}\, R^{-1}$ for $b \in \{1, 2, 4, 6\}$, with $\mathfrak{C}$ independent of R. Moreover we indicate that $\sum_{\nu=1}^{3} \alpha_\nu \int_{\partial B_1} y_j\, y_l\, y_m\, y_\nu\, do_y = (\alpha_j\, \delta_{lm} + \alpha_\nu\, \delta_{jm} + \alpha_m\, \delta_{jl})\, 4\,\pi/15$ for $\alpha \in \mathbb{R}^3$, $l, m \in \{1, 2, 3\}$. The factor $4\,\pi/15$ arises due to the equation $\int_{\partial B_1} y_l^2\, y_m^2\, do_y = (1 + 2\,\delta_{lm})\, 4\,\pi/15$, for l, m as before. It follows that $\sum_{b \in \{3, 5, 7\}} \mathfrak{B}_b(R) = 0$. Up to this point the parameter R was fixed. Letting R tend to infinity, we may conclude from the preceding remarks on $\mathfrak{B}_b(R)$ that $\sum_{b=1}^{8} \mathfrak{B}_b(R) \to 0$ ($R \to \infty$), hence $F(-\delta) = \sum_{\nu=1}^{3} \mathfrak{A}_\nu$.

By the definition of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$, it is obvious that for $b \in \{1, 2, 3\}$, there is a function $\mathfrak{Z}_b = \mathfrak{Z}_b^{(j,x,\delta)} : \partial\Omega \mapsto \mathbb{R}$ such that $\mathfrak{A}_b = \int_{\partial\Omega} \mathfrak{Z}_b(y)\, do_y$. As explained at the beginning of

this proof, we have $|z-y| \geq \mathcal{D}\,|x-y|$ for $y \in \partial\Omega$. In addition ϕ belongs to $C^{1,\alpha}(\mathbb{R}^3)$, and $n^{(\Omega)}$ is in particular Lipschitz continuous on $\partial\Omega$. For these reasons we obtain that $|\mathfrak{Z}_b(y)| \leq \mathfrak{C}|x-y|^{-2+\alpha}$ for $y \in \partial\Omega$, $1 \leq b \leq 3$, where $\mathfrak{C} > 0$ does not depend on y. But $\int_{\partial\Omega} |x-y|^{-2+\alpha}\, do_y < \infty$, so it follows by Lebesgue's theorem and the equation $F(-\delta) = \sum_{\nu=1}^{3} \mathfrak{A}_\nu$ that $F(-\delta)$ converges for $\delta \downarrow 0$, with the limit being the integral arising if z is replaced by x in the definition of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$.

A similar but much simpler reasoning (see the remark at the beginning of this proof) yields that the limit of $F(\delta)$ for $\delta \downarrow 0$ exists, too, and its value coincides with $lim_{\delta\downarrow 0} F(-\delta)$. $\square$

References

[1] Adams, R. A.: Sobolev spaces. Academic Press, New York e.a., 1975.

[2] Agmon, S.; Douglis, A.; Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. Comm. Pure Appl. Math. **17** (1964), 35-92.

[3] Cattabriga, L.: Su un problema al contorno relativo al sistema di equazioni di Stokes. Rend. Sem. Mat. Padova **31** (1961), 308-340.

[4] Deuring, P.: An integral operator related to the Stokes system in exterior domains. Math. Methods Appl. Sci. **13** (1990), 323–333. Addendum, Math. Methods Appl. Sci. **14** (1991), 445.

[5] Deuring, P.: The Stokes system in exterior domains: existence, uniqueness, and regularity of solutions in L^p-spaces. Comm. Part. Diff. Equ. **16** (1991), 1513-1528.

[6] Deuring, P.: Pointwise spatial decay of time-dependent Oseen flows: the case of data with noncompact support. Discr. Cont. Dyn. Syst. Ser. A **33** (2013), 2757-2776.

[7] Deuring, P.: L^p-estimates of the Stokes resolvent with nonhomogeneous boundary conditions in 3D exterior domains. Math. Meth. Appl. Sci. **44**, 13252-13272 (2021).

[8] Deuring, P., von Wahl, W.: Das lineare Stokes-System in $\mathbb{R}^3$. II. Das Außenraumproblem. Bayreuther Math. Schr. **28**, 1-252 (1989).

[9] Deuring, P., von Wahl, W., Weidemaier, P.: Das lineare Stokes-System in $\mathbb{R}^3$. I. Vorlesung über das Innenraumproblem. Bayreuther Math. Schr. **27**, 1-252 (1988).

[10] Faxén, H.: Fredholmsche Integralgleichungen zu der Hydrodynamik zäher Flüssigkeiten 1. Ark. Mat. Astr. Fys. **21A, 14** (1929), 1-40.

[11] Friedman, A.: Partial differential equations of parabolic type. Prentice-Hall, Englewood Cliffs, N.J., 1964.

[12] Fučik, S., John, O., Kufner, A.: Function spaces. Noordhoff, Leyden 1977.

[13] Galdi, G. P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems, 2nd edition. Springer, New York e.a., 2011.

[14] Giaquinta, M., Modica, G.: Non linear systems of the type of the stationary Navier-Stokes system. J. Reine Angew. Math. **330** (1982), 173-214.

[15] Hackbusch, W.: Integral equations. Theory and numerical treatment. International series on numerical mathematics, vol. 120. Birkhäuser, Basel e.a., 1995.

[16] Hishida, T.: Private communication. University of Nagoya.

[17] Jörgens, K.: Linear integral operators. Pitman, Boston e. a., 1982.

[18] Kupradze, V. D.: Potential methods in the theory of elasticity. Daniel Davey & Co, New York, 1965.

[19] Ladyzhenskaya, O. A.: The mathematical theory of viscous incompressible flow. Gordon and Breach, New York e.a., 1969.

[20] Miranda, C.: Partial differential equations of elliptic type. Springer, 1970.

[21] Nazarov, S. A.; Specovius-Neugebauer, M.: Approximation of exterior boundary value problems for the Stokes system. Asymptotic An. **14** (1997), 223-255.

[22] Neri, U.: Singular integrals. Lecture Notes in Mathematics 200. Springer, Berlin e.a., 1971.

[23] Starita, G.; Tartaglione, A.: On the traction problem for the Stokes system. Math. Mod. Meth. Appl. Sci. **12** (2002), 813-834.

[24] Teman, R.: Navier-Stokes equations. Theory and numerical analysis. AMS Chelsea Publishing, Providence R.I., 2001.

[25] Yoshida, K.: Functional analysis (6th ed.). Springer, Berlin e.a., 1980.