

# $W^{2,p}$ -estimates of the Stokes system with traction boundary conditions.

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## Abstract

The article deals with  $W^{2,p}$ -estimates of solutions to the 3D stationary Stokes system under traction boundary conditions. Existence, uniqueness and  $W^{2,p}$ -estimates up to the boundary are established for solutions in interior and exterior domains. The proofs are based on the method of integral equations. With this well known approach, solutions to the boundary value problem are constructed by solving certain integral equations on the boundary of the domain under consideration. This access simultaneously yields an integral representation of the solution of the boundary value problem. The difficulty then is to derive  $W^{2,p}$ -estimates of the integrals appearing in this representation. Ultimately such estimates are reduced to the  $W^{2,p}$ -theory of the Stokes system in bounded domains, under Dirichlet boundary conditions.

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**Key words.** Stokes system, traction boundary conditions, Neumann boundary conditions, regularity, method of integral equations.

## 1 Introduction.

We consider the Stokes system

$$-\Delta U + \nabla P = F, \quad \operatorname{div} U = 0 \tag{1.1} \quad \square_{10}$$

in a bounded domain  $\Omega \subset \mathbb{R}^3$  and in the exterior domain  $\bar{\Omega}^c := \mathbb{R}^3 \setminus \bar{\Omega}$ , where the boundary  $\partial\Omega$  is connected and of class  $C^2$ . System (1.1) is supplemented by traction conditions (also called Neumann boundary conditions)

$$\sum_{k=1}^3 (\partial_j U_k + \partial_k U_j - \delta_{jk} P) n_k^{(\Omega)} = B \quad \text{for } 1 \leq j \leq 3, \tag{1.2} \quad \square_{20}$$

where  $n^{(\Omega)}$  denotes the outward unit normal to  $\Omega$ . The functions  $F$  (volume force) and  $B$  (force orthogonal to the surface) are given, and  $U$  (velocity) and  $P$  (pressure) are the unknowns of problem (1.1), (1.2). Boundary conditions as in (1.2) arise, for example, in incompressible linear elastostatics and in free boundary value problems for incompressible viscous fluids.

If the Stokes system (1.1) in  $\Omega$  or  $\overline{\Omega}^c$  is supplemented by Dirichlet boundary conditions, then  $U$  and  $P$  are respectively  $W^{2,p}$ - and  $W^{1,p}$ -regular up to the boundary, provided the data satisfy suitable assumptions. In the case of the interior domain  $\Omega$ , this is well known since the beginning of the 1960s, due to the pioneering work of Cattabriga [2] and Ladyzhenskaya [18]. These two authors chose quite different accesses to their respective theory. Cattabriga reduced his results to the half-space case, in which a solution to (1.1), (1.2) may be constructed in a rather explicit form. Ladyzhenskaya used the method of integral equations. More precisely, by solving suitable integral equations whose unknowns are functions on  $\partial\Omega$ , she was able to construct a solution, in  $\Omega$  and in  $\overline{\Omega}^c$ , to the Dirichlet problem associated with the Stokes system (1.1). Then she evaluated the integrals appearing in this representation in such a way that  $L^p$ -estimates for Stokes flows in  $\Omega$  under Dirichlet conditions reduce to  $L^p$ -estimates for the Poisson equation in  $\Omega$  with the same type of boundary conditions.  $W^{2,p}$ -regularity for exterior Stokes flows was not treated in either [2] or [18]. To cite a more recent reference, in the monograph [12], Cattabriga's method is used in order to obtain  $W^{2,p}$ -regularity of the velocity and  $W^{1,p}$ -regularity for the pressure up to the boundary for interior and exterior flows under Dirichlet boundary conditions. Reference [7] derives the same type of regularity for the same type of problem for exterior flows, but by applying Ladyzhenskaya's approach.

An  $L^2$ -theory for the Stokes system (1.1) under traction condition (1.2) may be found in literature. In fact, such a theory was derived by Giaquinta, Modica [13]; see [13, Theorem II.1.2].

$L^p$ -theories for the Stokes system under boundary conditions other than Dirichlet ones – but different from (1.2) – are also available in literature. As an example we mention reference [1], where Navier conditions are considered. On the other hand, systems closely related to (1.1) have been studied when associated with boundary condition (1.2). In fact, if the Stokes resolvent equation instead of (1.1) is supplemented by (1.2), Grubb [14]) constructed  $L^p$ -solutions by means of the theory of pseudodifferential operators. Shibata, Shimizu [21] and Shibata [22] obtained similar results by reducing this boundary value problem to the half-space case. The time-dependent Stokes system with (1.2) prescribed as boundary condition has also been solved in an  $L^p$ -framework; see [15] and [19]. However, an  $L^p$ -theory for problem (1.1), (1.2) does not seem to be a corollary of these results on the Stokes resolvent or on the evolutionary Stokes system. And to our knowledge, no such  $L^p$ -theory has been derived in literature.

Our aim here is to fill this gap. To this end we will use the method of integral equations in a similar way as in [18]. This method yields a solution to (1.1), (1.2), and at the same time an integral representation of that solution. By deriving suitable estimates of the integrals appearing in this representation, we then reduce  $L^p$ -estimates of our solution in respectively  $\Omega$  and  $\overline{\Omega}^c$ , to  $L^p$ -estimates of (1.1) in bounded domains, with a Dirichlet condition instead of (1.2) prescribed on the boundary; see the proof of Theorem 6.1 and 6.3. This step is mainly based on the technical results stated as Theorem 5.1 and 5.2, the proof of which constitutes the main difficulty of our work.

We mention that our way of solving (1.1), (1.2) by means of the integral equation method

is not new. A more sophisticated version of this approach was presented by Starita, Tartaglione [23], who dealt even with the case of a non-connected boundary  $\partial\Omega$ , whereas in the work at hand, as in [18],  $\partial\Omega$  is supposed to be connected. However, in [23] classical solutions are constructed without evaluation in any norm, except for a pointwise estimate of  $|\sum_{k=1}^3(\partial_j U_k + \partial_k U_j - \delta_{jk} P)(x \pm \kappa n^{(\Omega)}(x)) n_k^{(\Omega)}(x)|$  with respect to  $x \in \partial\Omega$  and small  $\kappa > 0$ . The  $L^p$ -estimates which are the main feature of the present article pose a major additional obstacle.

We will rely on [18] with respect to a number of auxiliary results on the integral operators appearing in our proofs. However, we will not refer to [18] directly. Instead we will draw on results from [8], where the pertinent sections from [18] are worked out in detail.

Let us state our main results. First consider (1.1) in the interior domain  $\Omega$ . The following theorem summarizes Corollary 6.5 (existence and regularity) and Theorem 7.1 (uniqueness).

(theorem100.1)

**Theorem 1.1** *Define*

$$\begin{aligned}\phi^{(j)}(x) &:= (\delta_{jk})_{1 \leq k \leq 3}, & \phi^{(4)}(x) &:= (x_3, 0, -x_1), & \phi^{(5)}(x) &:= (x_2, -x_1, 0), \\ \phi^{(6)}(x) &:= (0, x_3, -x_2) & \text{for } 1 \leq j \leq 3, & x \in \mathbb{R}^3.\end{aligned}$$

Let  $p \in (1, \infty)$ ,  $f \in L^p(\Omega)$ ,  $b \in W^{1-1/p, p}(\partial\Omega)^3$ . Suppose that  $\int_{\partial\Omega} b \cdot \phi^{(j)} d\sigma_x + \int_{\Omega} f \cdot \phi^{(j)} dx = 0$  for  $1 \leq j \leq 6$ . Then there is a unique pair of functions  $(u, \pi) \in W^{2,p}(\Omega)^3 \times W^{1,p}(\Omega)$  satisfying (1.1) and (1.2), and such that  $\int_{\Omega} u \cdot \phi^{(j)} dx = 0$  for  $1 \leq j \leq 6$ . In addition, the estimate  $\|u\|_{2,p} + \|\pi\|_{1,p} \leq \mathfrak{C}(\|f\|_p + \|b\|_{1-1/p, p})$  holds, with a constant  $\mathfrak{C}$  independent of  $f$  and  $b$ .

The essential features of solutions to (1.1), (1.2) in the exterior domain case may be stated as follows.

(theorem100.2)

**Theorem 1.2** Let  $p_1 \in (1, 3/2)$ ,  $f \in L^{p_1}(\overline{\Omega}^c)^3$ ,  $b \in W^{1-1/p_1, p_1}(\partial\Omega)^3$ . Then there is a unique pair  $(v, \varrho) \in W_{loc}^{2,p_1}(\overline{\Omega}^c)^3 \times W_{loc}^{1,p_1}(\overline{\Omega}^c)$  such that

$$\begin{aligned}\|v\|_{(1/p_1-2/3)^{-1}} + \|\partial_l v\|_{(1/p_1-1/3)^{-1}} + \|\varrho\|_{(1/p_1-1/3)^{-1}} + \|\partial_m \partial_l v\|_{p_1} + \|\partial_l \varrho\|_{p_1} \\ \leq \mathfrak{C}(\|f\|_{p_1} + \|b\|_{1-1/p_1, p_1}) \quad \text{for } 1 \leq j, l, m \leq 3,\end{aligned}$$

and such that the pair  $(v, \varrho)$  satisfies equation (1.1) in  $\overline{\Omega}$  with  $F = f$ , as well as (1.2) with  $B = b$ .

If in addition there is  $p_2 \in (1, 3)$  such that  $f \in L^{p_2}(\overline{\Omega}^c)^3$  and  $b \in W^{1-1/p_2, p_2}(\partial\Omega)^3$ , then

$$\|\partial_l v\|_{(1/p_2-1/3)^{-1}} + \|\varrho\|_{(1/p_2-1/3)^{-1}} + \|\partial_m \partial_l v\|_{p_2} + \|\partial_l \varrho\|_{p_2} \leq \mathfrak{C}(\|f\|_{p_2} + \|b\|_{1-1/p_2, p_2})$$

for  $l, m$  as above. Under the further assumptions that  $f \in L^{p_3}(\overline{\Omega}^c)^3$ ,  $b \in W^{1-1/p_3, p_3}(\partial\Omega)^3$  for some  $p_3 \in (1, \infty)$ , and  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , the estimate

$$\|\partial_m \partial_l v\|_{p_3} + \|\partial_l \varrho\|_{p_3} \leq \mathfrak{C}(\|f\|_{p_3} + \|f|_{B_{2R}^c}\|_{p_2} + \|b\|_{1-1/p_3, p_3}) \quad (1.3) \quad \boxed{\text{T100.2.10}}$$

is valid for  $l, m$  as before. The constants  $\mathfrak{C}$  in these estimates are independent of  $f$  and  $b$ .

More details may be found in Corollary 6.4, and for the case  $f = 0$  in Theorem 6.2. A uniqueness result stronger than the one in the preceding theorem is given in Theorem 7.2.

Concerning the strange-looking term  $\|f|B_{2R}^c\|_{p_2}$  in (1.3), we think an estimate of the form  $\|\partial_m \partial_l v\|_{p_3} + \|\partial_l \varrho\|_{p_3} \leq \mathfrak{C} (\|f\|_{p_3} + \|b\|_{1-1/p_3, p_3})$  cannot hold with the same constant  $\mathfrak{C}$  for all functions  $f \in L^{p_3}(\overline{\Omega}^c)^3$  and  $b \in W^{1-1/p_3, p_3}(\partial\Omega)^3$ . This is indicated by a similar situation in the Dirichlet case. In fact, the inequality  $\|\partial_m \partial_l U\|_{p_3} \leq \mathfrak{C} \|F\|_{p_3}$  does not hold with the same constant  $\mathfrak{C}$  for all  $F \in L^{p_3}(\overline{\Omega}^c)^3$ , even if an associated solution  $(U, P)$  to (1.1) in  $\overline{\Omega}^c$  with  $U|_{\partial\Omega} = 0$  is subject to rather restrictive regularity conditions; see [4, Theorem 1.3].

## 2 Notation. Some auxiliary results.

The symbol  $|\cdot|$  denotes the Euclidean norm of  $\mathbb{R}^n$  for any  $n \in \mathbb{N}$ , and we use it when we write  $|\alpha|$  for the length  $\alpha_1 + \dots + \alpha_n$  of a multi-index  $\alpha \in \mathbb{N}_0^n$ , as well as  $|A|$  for the Borel measure of a measurable subset  $A$  of  $\mathbb{R}^3$ . For  $R \in (0, \infty)$ ,  $x \in \mathbb{R}^3$ , put  $B_R(x) := \{y \in \mathbb{R}^3 : |x - y| < R\}$ . In the case  $x = 0$ , we write  $B_R$  instead of  $B_R(0)$ . An open ball in  $\mathbb{R}^2$  with radius  $R > 0$  and centered in  $\varrho \in \mathbb{R}^2$  is denoted by  $B_R^2(\varrho)$ .

The set  $\Omega \subset \mathbb{R}^3$  and the parameter  $\tau \in (0, \infty)$  introduced in Section 1 will be kept fixed throughout. Recall that  $\Omega$  is open and bounded, with connected Lipschitz boundary, and that  $n^{(\Omega)}$  denotes the outward unit normal to  $\Omega$ . We put  $\Omega_R := B_R \setminus \overline{\Omega}$ .

For  $n \in \mathbb{N}$ ,  $I \subset \mathbb{R}^n$ , let  $\chi_I$  stand for the characteristic function of  $I$  in  $\mathbb{R}^n$ . If  $A \subset \mathbb{R}^3$ , we denote by  $A^c$  the complement  $\mathbb{R}^3 \setminus A$  of  $A$  in  $\mathbb{R}^3$ . Put  $e_l := (\delta_{jl})_{1 \leq j \leq 3}$  for  $1 \leq l \leq 3$  (unit vector in  $\mathbb{R}^3$ ). If  $A$  is some nonempty set and  $\gamma : A \mapsto \mathbb{R}^n$  a function for some  $n \in \mathbb{N}$ , we set  $|\gamma|_\infty := \sup\{|\gamma(x)| : x \in A\}$ .

Let  $\alpha \in (0, 1)$ . For any  $B \subset \mathbb{R}^3$ , we write  $C^\alpha(B)$  for the set of all Hölder continuous functions on  $B$ , that is,  $\psi \in C^\alpha(B)$  iff  $\psi : B \mapsto \mathbb{R}$  with

$$|\psi|_\alpha := |\psi|_\infty + \sup\{|\psi(x) - \psi(y)| |x - y|^{-\alpha} : x, y \in B, x \neq y\} < \infty.$$

If  $B \subset \mathbb{R}^3$  is open, the space  $C^{1,\alpha}(B)$  is to consist of all functions  $\psi \in C^1(B)$  with  $|\psi|_\infty < \infty$  and  $|\partial_l \psi|_\alpha < \infty$  for  $1 \leq l \leq 3$ . We further define  $C^{1,\alpha}(\partial\Omega) := \{\psi|_{\partial\Omega} : \psi \in C^{1,\alpha}(\mathbb{R}^3)\}$ .

Let  $p \in [1, \infty)$ ,  $m \in \mathbb{N}$ . For  $A \subset \mathbb{R}^3$  open, the notation  $\|\cdot\|_p$  stands for the norm of the Lebesgue space  $L^p(A)$ , and  $\|\cdot\|_{m,p}$  for the usual norm of the Sobolev space  $W^{m,p}(A)$  of order  $m$  and exponent  $p$ . If  $A \subset \mathbb{R}^3$  possesses a bounded  $C^2$ -boundary, the Sobolev space  $W^{r,p}(\partial A)$  with  $r \in (0, 2)$  is to be defined as in [11, Section 6.8.6].

If  $\mathcal{A}$  and  $\mathcal{B}$  are vector spaces and  $T : \mathcal{A} \mapsto \mathcal{B}$  is a linear operator, we write  $\ker T$  for the kernel of  $T$  and  $\text{ran } T$  for the range of  $T$ .

Numerical constants are denoted by  $C$ , and constants depending exclusively on parameters  $\gamma_1, \dots, \gamma_n \in [0, \infty)$ , for some  $n \in \mathbb{N}$ , take the form  $C(\gamma_1, \dots, \gamma_n)$ . In most cases it is not worthwhile to list all such parameters, in particular if they are related to  $\partial\Omega$ . Then we use the symbol  $\mathfrak{C}$  for constant whose dependencies – or more importantly, their non-dependency – on certain parameters should be clear from context. Sometimes, in order

to insist that the constant in question depends on  $\gamma_1, \dots, \gamma_n$  among other quantities, we use the notation  $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$ .

In the following theorem, we reproduce the Calderon-Zygmund inequality for even kernels. This well known estimate is restated here because we will need some details on how the upper bound given by this inequality relates to the structure of the kernel.

(theorem0.10)

**Theorem 2.1** *Let  $n \in \mathbb{N}$  and  $K : \mathbb{R}^n \mapsto \mathbb{R}$  a function with  $K(z) = |z|^{-n} K(|z|^{-1} z)$  and  $K(-z) = K(z)$  for  $z \in \mathbb{R}^n \setminus \{0\}$ . Put  $\Lambda(z) := K(z)$  for  $z \in \partial B_1$  and suppose that  $\Lambda \in L^1(\partial B_1)$ .*

*Let  $p \in (1, \infty)$ . Then  $\int_{\mathbb{R}^n \setminus B_\epsilon^n(x)} |K(x-y) f(y)| dy < \infty$  for  $x \in \mathbb{R}^n$ ,  $\epsilon \in (0, \infty)$  and  $f \in L^p(\mathbb{R}^n)$ . Define  $(K_\epsilon * f)(x) := \int_{\mathbb{R}^n \setminus B_\epsilon^n(x)} K(x-y) f(y) dy$  for  $x, \epsilon, f$  as before.*

*Then there is a constant  $C(p, n) > 0$  such that  $\|K_\epsilon * f\|_p \leq C(p, n) \|\Lambda\|_1 \|f\|_p$  for  $\epsilon \in (0, \infty)$  and  $f \in L^p(\mathbb{R}^n)$ .*

**Proof:** [20, p.89, Theorem 2 a) ]. □

We state a lemma which is convenient to handle weakly singular integral operators.

(lemma3.2)

**Lemma 2.1** *Let  $n \in \mathbb{N}$ ,  $A, B \subset \mathbb{R}^n$  nonempty,  $\lambda$  et  $\nu$  measures on  $\sigma$ -algebras over  $A$  and  $B$ , respectively. Further assume that the function  $K : A \times B \mapsto [0, \infty)$  is measurable and the upper bounds  $\mathfrak{A}_1 := \sup\{\int_B K(x, y) d\nu(y) : x \in A\}$  and  $\mathfrak{A}_2 := \sup\{\int_A K(x, y) d\lambda(x) : y \in B\}$  are finite.*

*Then, for  $p \in [1, \infty)$  and  $\phi : B \mapsto \mathbb{R}$  measurable with  $\int_B |\phi|^p d\nu < \infty$ , the integral  $\int_{\partial\Omega} K(x, y) |\phi(y)| do_y$  is finite for  $\lambda$ -a. e.  $x \in A$ , and*

$$\left( \int_A \left( \int_B K(x, y) |\phi(y)| d\nu(y) \right)^p d\lambda(x) \right)^{1/p} \leq \mathfrak{A}_1^{1/p'} \mathfrak{A}_2^{1/p} \left( \int_B |\phi|^p d\nu \right)^{1/p}. \quad (2.1) \quad \boxed{\text{T0.10.1}}$$

*The preceding assumptions hold true if, for example,  $n = 3$ ,  $A = B = \partial\Omega$ ,  $\lambda$  and  $\nu$  are the usual surface measure on  $\partial\Omega$ , and if there are numbers  $C, \kappa \in (0, \infty)$  such that  $K(x, y) \leq C |x - y|^{-2+\kappa}$  for  $x, y \in \Omega$ ,  $x \neq y$ .*

*These assumptions are also valid if  $A, B \subset \mathbb{R}^2$  are open, bounded and nonempty, if  $\lambda$  and  $\nu$  are the usual Borel measure on  $A$  and  $B$  respectively, and if  $K(\varrho, \eta) \leq C |\varrho - \eta|^{-2+\kappa}$  for  $\varrho \in A$ ,  $\eta \in B$ ,  $\varrho \neq \eta$ , with  $C$  and  $\kappa$  given as before.*

**Proof:** Inequality (2.1) is a simple application of Hölder's inequality and Fubini's theorem; see [20, p. 7], for example. The first claim of the lemma follows from (2.1). □

We do not know a precise reference of the following theorem. For the reader's convenience, and since this theorem is a key tool and not so obvious to derive, we give some details of a proof.

(theorem5.10)

**Theorem 2.2** *Let  $J \subset \mathbb{R}^2$  be open, bounded and convex, and  $L : J \times J \mapsto \mathbb{C}$  a measurable function with  $L(\cdot, \eta) \in C^1(J \setminus \{\eta\})$  for  $\eta \in J$ . Suppose there is  $c_0 > 0$  with  $|\partial_\varrho^\alpha L(\varrho, \eta)| \leq c_0 |\varrho - \eta|^{-1-|\alpha|}$  for  $\varrho, \eta \in J$  with  $\varrho \neq \eta$ ,  $\alpha \in \mathbb{N}_0^2$  with  $|\alpha| \leq 1$ .*

Let  $p \in (1, \infty)$  and define  $\mathcal{L}(\psi)(\varrho) := \int_J L(\varrho, \eta) \psi(\eta) d\eta$  for  $\psi \in L^p(J)$ ,  $\varrho \in J$ ; see Lemma 2.1. Then  $\mathcal{L}(\psi) \in W^{1-1/p, p}(J)$  and  $\|\mathcal{L}(\psi)\|_{1-1/p, p} \leq \mathfrak{C} \|\psi\|_p$  for  $\psi$  as before.

**Proof:** Let  $\varrho, \varrho', \eta \in J$  with  $\varrho \neq \eta$ ,  $\varrho' \neq \eta$ . If  $|\varrho - \eta| \leq 2|\varrho - \varrho'|$ , we have  $|\varrho' - \eta| < 3|\varrho - \varrho'|$ , so

$$|L(\varrho, \eta) - L(\varrho', \eta)| \leq c_0 (|\varrho - \eta|^{-1} + |\varrho' - \eta|^{-1}) \leq C(c_0) (|\varrho - \eta|^{-2} + |\varrho' - \eta|^{-2}) |\varrho - \varrho'|.$$

In the case  $|\varrho - \eta| \geq 2|\varrho - \varrho'|$ , we get for  $\vartheta \in [0, 1]$  that  $|\varrho + \vartheta(\varrho' - \varrho) - \eta| \geq |\varrho - \eta|/2$ . Thus, using the equation  $|L(\varrho, \eta) - L(\varrho', \eta)| = |\int_0^1 (\nabla L)(\varrho + \vartheta(\varrho' - \varrho), \eta) d\vartheta \cdot (\varrho - \varrho')|$ , we get in any case that

$$\left| L(\varrho, \eta) - L(\varrho', \eta) \right| \leq C(c_0) (|\varrho - \eta|^{-2} + |\varrho' - \eta|^{-2}) |\varrho - \varrho'| \quad \text{for } \varrho, \varrho', \eta \in J \quad (2.2) \quad \boxed{\text{T5.10.6}}$$

with  $\eta \notin \{\varrho, \varrho'\}$ . Let  $r \in (1, 2)$ ,  $\varrho, \varrho' \in J$ , and put  $J_{\varrho, \varrho'} := \{\eta \in J : |\varrho - \eta| \leq 2|\varrho - \varrho'|\}$ . By splitting the set  $J$  into the parts  $J_{\varrho, \varrho'}$  and  $J \setminus J_{\varrho, \varrho'}$ , it may be shown that the inequality  $\int_J |L(\varrho, \eta) - L(\varrho', \eta)|^r d\eta \leq \mathfrak{C} |\varrho - \varrho'|^{-r+2}$  holds, where the estimate of the integral over  $J \setminus J_{\varrho, \varrho'}$  is based on (2.2). Put  $\epsilon := \min\{1/(2p'), 1/(4p)\}$ . Note that  $1 + \epsilon p' < 2$ . Let  $\psi \in L^p(J)$ . Then we get by Hölder's inequality and by the previous estimate with  $r = 1 + \epsilon p'$  that

$$|\mathcal{L}(\psi)(\varrho) - \mathcal{L}(\psi)(\varrho')|^p \leq \mathfrak{C} \int_J |L(\varrho, \eta) - L(\varrho', \eta)|^{1-\epsilon p} (|\varrho - \varrho'|^{1-\epsilon p'})^{p/p'} |\psi(\eta)|^p d\eta$$

for  $\varrho, \varrho' \in J$ . Set  $\mathfrak{A}(\varrho, \varrho') := |\mathcal{L}(\psi)(\varrho) - \mathcal{L}(\psi)(\varrho')|^p |\varrho - \varrho'|^{-p-1}$ . It follows that

$$\mathfrak{A}(\varrho, \varrho') \leq \mathfrak{C} \int_J |L(\varrho, \eta) - L(\varrho', \eta)|^{1-\epsilon p} |\varrho - \varrho'|^{-2-\epsilon p} |\psi(\eta)|^p d\eta.$$

We integrate both sides of the preceding inequality with respect to  $\varrho \in J$  and  $\varrho' \in J$ , then apply (2.2) and change the order of integration. In this way we arrive at the estimate  $\int_J \int_J \mathfrak{A}(\varrho, \varrho') d\varrho d\varrho' \leq \mathfrak{C} (\mathfrak{B}_1 + \mathfrak{B}_2)$ , where

$$\mathfrak{B}_1 := \int_J \int_J \int_J |\varrho - \eta|^{-2(1-\epsilon p)} |\varrho - \varrho'|^{-1-2\epsilon p} d\varrho d\varrho' |\psi(\eta)|^p d\eta.$$

and where term  $\mathfrak{B}_2$  is to be defined in the same way as  $\mathfrak{B}_1$ , except that the difference  $|\varrho - \eta|$  is replaced by  $|\varrho' - \eta|$ , and the order of integration with respect to  $\varrho$  and  $\varrho'$  is exchanged. Now we use that  $-2(1-\epsilon p)$  and  $-1-2\epsilon p$  belong to  $(-2, 0)$ , and  $-2(1-\epsilon p) - 1 - 2\epsilon p = -3 < -2$ . Thus we get that  $\int_J |\varrho - \eta|^{-2(1-\epsilon p)} |\varrho - \varrho'|^{-1-2\epsilon p} d\varrho \leq \mathfrak{C} |\varrho' - \eta|^{-1}$  for  $\varrho', \eta \in J$  with  $\varrho' \neq \eta$ , as follows by splitting  $J$  into four sets according to four cases, three of them given by the inequalities  $|\varrho - \eta| \leq |\varrho' - \eta|/2$ ,  $|\varrho - \varrho'| \leq |\varrho' - \eta|/2$ ,  $|\varrho - \eta| \geq 2|\varrho' - \eta|$ , and the fourth consisting of the requirement that none of the three preceding conditions holds; compare [10, Lemma 1.4.2]. It follows that  $\mathfrak{B}_1 \leq \mathfrak{C} \|\psi\|_p^p$ . An analogous argument yields that  $\mathfrak{B}_2 \leq \mathfrak{C} \|\psi\|_p^p$ . Therefore the theorem follows from the estimate  $\int_J \int_J \mathfrak{A}(\varrho, \varrho') d\varrho d\varrho' \leq \mathfrak{C} (\mathfrak{B}_1 + \mathfrak{B}_2)$  shown above, and from Lemma 2.1.  $\square$

⟨lemma1.40⟩

**Lemma 2.2** Let  $L : \partial\Omega \times \partial\Omega \mapsto \mathbb{C}$  be measurable. Suppose there is  $C_L \in (0, \infty)$  with  $|L(x, y)| \leq C_L |x - y|^{-1}$  for  $x, y \in \partial\Omega$ ,  $x \neq y$ . Let  $p \in (1, \infty)$ . We may define an operator  $\mathcal{L} : L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$  by setting  $\mathcal{L}(\phi)(x) := \int_{\partial\Omega} L(x, y) \phi(y) do_y$  for  $\phi \in L^p(\partial\Omega)$ ,  $x \in \partial\Omega$ ; see Lemma 2.1.

Then  $\mathcal{L} : L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$  is linear, bounded and compact.

**Proof:** Obviously  $L$  is linear. The boundedness of  $L$  holds according to Lemma 2.1. As for compactness, we remark that for any  $\epsilon \in (0, \infty)$ , the operator  $\mathcal{L}_{p,\epsilon} : L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$ ,  $\mathcal{L}_{p,\epsilon}(\phi)(x) := \int_{\partial\Omega} \chi_{(\epsilon,\infty)}(|x - y|) L(x, y) \phi(y) do_y$  ( $x \in \partial\Omega$ ,  $\phi \in L^p(\partial\Omega)$ ) is compact ([17, p. 275, Theorem 11.6]). On the other hand,

$$\sup \left\{ \int_{\partial\Omega} \chi_{(0,\epsilon)}(|x - y|) |x - y|^{-1} do_y : x \in \partial\Omega \right\} \rightarrow 0 \quad (\epsilon \downarrow 0),$$

so it follows by Lemma 2.1 that  $\mathcal{L}_\epsilon$  converges to  $\mathcal{L}$  with respect to the operator norm of the space of linear bounded operators from  $L^p(\partial\Omega)$  into  $L^p(\partial\Omega)$ . As a consequence,  $\mathcal{L}$  is compact as well.  $\square$

⟨lemma2.10⟩

**Lemma 2.3** Let  $L \in C^1(\mathbb{R}^3 \setminus \{0\})$  and  $C_L \in (0, \infty)$  such that  $|\partial^\alpha L(z)| \leq C_L |z|^{-1-|\alpha|}$  for  $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $\alpha \in \mathbb{N}_0^3$ ,  $|\alpha| \leq 1$ . Let  $\phi \in C^0(\partial\Omega)$  and put  $\mathcal{A}(\phi)(x) := \int_{\partial\Omega} L(x - y) \phi(y) do_y$  for  $x \in \mathbb{R}^3$ . Then  $\mathcal{A}(\phi) \in C^a(\mathbb{R}^3)$  for  $a \in [0, 1)$ .

**Proof:** [8, Lemma 6.1].  $\square$

⟨lemma3.21⟩

**Lemma 2.4** Let  $p \in (1, \infty)$ ,  $a \in (0, 2/(3p))$ ,  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ . Then

$$\left( \int_{B_R} \left( \int_{\partial\Omega} |x - y|^{-2} |\phi(y)| do_y \right)^{(1/p-a/2)^{-1}} do_x \right)^{1/p-a/2} \leq \mathfrak{C}(p, a, R) \|\phi\|_p \quad \text{for } L^p(\partial\Omega).$$

**Proof:** See [6, Lemma 3.2].  $\square$

We will use the fact that a function  $v$  defined in a 3D exterior domain and whose gradient is  $L^q$ -integrable for some  $q \in (1, 3)$  takes a constant boundary value at infinity:

theorem100.81)

**Theorem 2.3** Let  $U \subset \mathbb{R}^3$  be open and bounded, with Lipschitz boundary. Let  $q \in (1, 3)$ . Then, for any  $v \in W_{loc}^{1,1}(\bar{U}^c)$  with  $\nabla v \in L^q(\bar{U}^c)^3$ , there is  $c(v) \in \mathbb{R}$  with  $v + c(v) \in L^{(1/q-1/3)^{-1}}(\bar{U}^c)$ . If such a function  $v$  satisfies the additional condition  $v \in L^s(\bar{U}^c)$  for some  $s \in (1, \infty)$ , then  $c(v) = 0$ .

The estimate  $\|v + c(v)\|_{(1/q-1/3)^{-1}} \leq \mathfrak{C} \|\nabla v\|_q$  holds for this class of functions  $v$ .

**Proof:** See [12, Theorem II.6.1], except as concerns the (rather obvious) criterion for the equation  $c(v) = 0$ , which is treated in [5, Lemma 2.4].  $\square$

The role of the functions  $\phi^{(1)}, \dots, \phi^{(6)}$  from Theorem 1.1 becomes clear by the following theorem.

theorem100.90)

**Theorem 2.4** Let  $U \subset \mathbb{R}^3$  be a domain. Put

$$\mathfrak{L} := \{v \in C^1(U)^3 : \partial_j v_k + \partial_k v_j = 0 \text{ for } 1 \leq j, k \leq 3\}.$$

Then the family  $(\phi^{(1)}|_U, \dots, \phi^{(6)}|_U)$  is a basis of  $\mathfrak{L}$ .

**Proof:** [8, Satz 6.1]. □

We end this section by recalling some properties of solutions to either the Poisson equation or the Stokes system. In particular we state an  $L^p$ -theory for the Stokes system in bounded domains under Dirichlet boundary conditions (Theorem 2.5).

(lemma100.30)

**Lemma 2.5** Let  $U \subset \mathbb{R}^3$  be open and bounded, with  $C^2$ -boundary. Let  $n^{(U)}$  denote the outward unit normal to  $U$ . Suppose that  $u \in C^1(\bar{U})^3 \cap W^{2,1}(U)^3 \cap C^2(U)^3$ ,  $\pi \in C^0(\bar{U}) \cap W^{1,1}(U) \cap C^1(U)$ ,  $\tilde{u} \in C^0(\bar{U})^3 \cap W^{1,1}(U)^3 \cap C^1(U)^3$ . Alternatively, let  $p \in (1, \infty)$ ,  $u \in W^{2,p}(U)^3$ ,  $\pi \in W^{1,p}(U)$ ,  $\tilde{u} \in W^{1,p'}(U)^3$ . In both cases suppose that  $\operatorname{div} \tilde{u} = 0$ . Then

$$\begin{aligned} & \int_U (\Delta u - \nabla \pi + \nabla \operatorname{div} u) \cdot \tilde{u} \, dx + (1/2) \int_U \sum_{j,k=1}^3 (\partial_j u_k + \partial_k u_j) (\partial_j \tilde{u}_k + \partial_k \tilde{u}_j) \, dx \\ &= \int_{\partial U} \sum_{j,k=1}^3 n_k^{(U)} (\partial_k u_j + \partial_j u_k - \delta_{jk} \pi) \tilde{u}_j \, d\sigma_x. \end{aligned}$$

**Proof:** Divergence theorem; see [8, Lemma 3.1]. □

(lemma10.40)

**Lemma 2.6** Let  $U \subset \mathbb{R}^3$  be open and bounded, with  $C^2$ -boundary. Let  $r \in (3/2, \infty)$ ,  $b \in W^{2-1/p,p}(\partial U)$ ,  $f \in L^r(U)$ ,  $u \in C^0(\bar{U})$  with  $u|_U \in C^2(U)$ ,  $\Delta(u|_U) = f$  and  $u|_{\partial U} = b$ .

Then  $u|_U \in W^{2,r}(U)$  and  $(u|_U)|_{\partial U} = b$  in the trace sense.

**Proof:** This theorem is a special case of [6, Lemma 3.4], which, in turn, is based on the  $W^{2,q}$ -theory of the Poisson equation and on the maximum principle. □

theorem100.30)

**Theorem 2.5** Let  $U \subset \mathbb{R}^3$  be a bounded domain with  $C^2$ -boundary. Let  $p \in (1, \infty)$ ,  $F \in L(U)^3$ ,  $B \in W^{2-1/p,p}(U)^3$  with  $\int_{\partial U} B \cdot n^{(U)} \, d\sigma_x = 0$ , where  $n^{(U)}$  denotes the outward unit normal to  $U$ .

Then there are functions  $u \in W^{2,p}(U)^3$ ,  $\pi \in W^{1,p}(U)$  such that  $-\Delta u + \nabla \pi = F$ ,  $\operatorname{div} u = 0$ ,  $u|_{\partial U} = B$ ,  $\int_U \pi \, dx = 0$ . Moreover there is  $C_1 > 0$  such that for  $F, B, u$  and  $\pi$  as before, the estimate  $\|u\|_{2,p} + \|\pi\|_{1,p} \leq C_1 (\|F\|_p + \|B\|_{2-1/p,p})$  holds.

Proof: [12, Theorem IV.6.1].

theorem100.40)

**Theorem 2.6** Let  $U, p, n^{(U)}$  be given as in Theorem 2.5. Assume that  $u \in W^{1,p}(U)^3$ ,  $\pi \in L^p(U)$  such that  $\int_U (\nabla u \cdot \nabla \varphi + \pi \operatorname{div} \varphi) \, dx = 0$  for  $\varphi \in C_0^\infty(U)^3$ . Further suppose that  $\operatorname{div} u = 0$ ,  $u|_{\partial U} = 0$ . Then  $u = 0$ .

**Proof:** [12, Lemma IV.6.2].

### 3 Some results on local coordinates of $\partial\Omega$ .

In this section, we specify the type of local coordinates we will use in what follows

(lemma1.5)

**Lemma 3.1** *There are numbers  $k(\Omega) \in \mathbb{N}$ ,  $\alpha(\Omega) \in (0, \infty)$ , and for any  $t \in \{1, \dots, k(\Omega)\}$  a function  $a_t \in C^2([- \alpha(\Omega), \alpha(\Omega)]^2)$  with  $|\nabla a_t|_\infty < 1/4$ , an orthonormal matrix  $D_t \in \mathbb{R}^{3 \times 3}$  and a vector  $C_t \in \mathbb{R}^3$  such that the following properties hold:*

*Put  $\Delta_\sigma := (-\sigma \alpha(\Omega), \sigma \alpha(\Omega))$  for  $\sigma \in (0, 1]$ ,  $\gamma_t(\eta) := D_t \cdot (\eta, a_t(\eta)) + C_t$  for  $\eta \in \Delta_1$ ,  $t \in \{1, \dots, k(\Omega)\}$ , and  $\Lambda_{t,\sigma} := \{\gamma_t(\eta) : \eta \in \Delta_\sigma\}$ ,*

$$U_{t,\sigma} := \{D_t \cdot (\eta, a_t(\eta) + s) : \eta \in \Delta_\sigma, s \in (-\sigma \alpha(\Omega), \sigma \alpha(\Omega))\}$$

*for  $\sigma, t$  as before. Then*

$$\mathfrak{C}_1 \|f\|_1 \leq \sum_{t=1}^{k(\Omega)} \|f \circ \gamma_t|_{\Delta_\delta}\|_1 \leq \mathfrak{C}_2 \|f\|_1 \text{ for } f \in L^1(\partial\Omega), \delta \in [1/4, 1], \quad (3.1) \quad \boxed{\text{L1.5.100}}$$

*with constants  $\mathfrak{C}_1, \mathfrak{C}_2$  independent of  $\delta$  and  $f$ . The function  $\gamma_t : \Delta_1 \mapsto \Lambda_{t,1}$  is bijective, continuous and with continuous inverse, the set  $U_{t,\sigma}$  is open in  $\mathbb{R}^3$ , and  $\Lambda_{t,\sigma} = U_{t,\sigma} \cap \partial\Omega$  for  $t \in \{1, \dots, k(\Omega)\}$ ,  $\sigma \in (0, 1]$ . Moreover  $\text{dist}(\partial\Omega \setminus \Lambda_{t,\sigma_2}, \Lambda_{t,\sigma_1}) > 0$  for  $t$  as before and  $\sigma_1, \sigma_2 \in (0, 1]$  with  $\sigma_1 < \sigma_2$ .*

*Define  $J_t(\eta) := (1 + \sum_{j=1}^2 \partial_j a_t(\eta)^2)^{1/2}$  for  $\eta \in \Delta_1$ ,  $t$  as before. Then for such  $t$  and for functions  $F : \Lambda_{t,1} \mapsto \mathbb{C}$ , the relation  $F \in L^1(\Lambda_{t,1})$  holds iff  $(F \circ \gamma_t) J_t \in L^1(\Delta_1)$ . In addition*

$$\int_{\Lambda_{t,1}} F \, d\omega_x = \int_{\Delta_1} (F \circ \gamma_t)(\eta) J_t(\eta) \, d\eta \text{ for } F \in L^1(\Lambda_{t,1}). \quad (3.2) \quad \boxed{\text{L1.5.7}}$$

**Proof:** All the statements of the lemma are standard results except perhaps that the local charts  $\gamma_t$  may be chosen in such a way that  $|\nabla a_t|_\infty < 1/4$  for  $1 \leq t \leq k(\Omega)$ . In order to satisfy this condition, the boundary  $\partial\Omega$  has to be split into sufficiently small parts. Details of this procedure are rather technical but straightforward.  $\square$

(lemma1.1)

**Lemma 3.2** *There are constants  $\delta(\Omega), \mathcal{D}, \tilde{\mathcal{D}} \in (0, \infty)$  such that*

$$y + \kappa n^{(\Omega)}(y) \in \bar{\Omega}^c, \quad y - \kappa n^{(\Omega)}(y) \in \Omega \text{ for } \kappa \in (0, \delta(\Omega)], y \in \partial\Omega, \quad (3.3) \quad \boxed{\text{L1.1.8}}$$

$$|x + \kappa n^{(\Omega)}(x) - x' - \kappa' n^{(\Omega)}(x')| \geq \mathcal{D} (|x - x'| + |\kappa - \kappa'|), \quad (3.4) \quad \boxed{\text{L1.1.10}}$$

$$|(x - x') \cdot n^{(\Omega)}(x)| \leq \tilde{\mathcal{D}} |x - x'|^2 \text{ for } x, x' \in \partial\Omega, \kappa, \kappa' \in [-\delta(\Omega), \delta(\Omega)]. \quad (3.5) \quad \boxed{\text{L1.5.10}}$$

*For  $\delta \in (0, \delta(\Omega)]$ , put*

$$\mathfrak{U}_\delta := \{x \in \mathbb{R}^3 : \text{dist}(x, \bar{\Omega}) < \mathcal{D} \delta / 2\}, \quad \mathfrak{U}_{-\delta} := \{x \in \mathbb{R}^3 : \text{dist}(x, \Omega^c) < \mathcal{D} \delta / 2\}.$$

*Note that  $\mathfrak{U}_\delta$  and  $\mathfrak{U}_{-\delta}$  are open and  $\bar{\Omega} \subset \mathfrak{U}_\delta$ ,  $\Omega^c \subset \mathfrak{U}_{-\delta}$  for  $\delta$  as before. The estimates*

$$|x - (y + \delta n^{(\Omega)}(y))| \geq \mathcal{D} \delta / 2, \quad |x' - (y - \delta n^{(\Omega)}(y))| \geq \mathcal{D} \delta / 2 \quad (3.6) \quad \boxed{\text{L1.1.9}}$$

*hold for such  $\delta$ , for  $y \in \partial\Omega$ ,  $x \in \mathfrak{U}_\delta$  and  $x' \in \mathfrak{U}_{-\delta}$ .*

**Proof:** See [8, (2.24), (2.22)] for (3.3), (3.5), respectively, [6, Lemma 2.1] for (3.4), and the proof of [6, Lemma 5.3] for the properties of  $\mathfrak{U}_\delta$  and  $\mathfrak{U}_{-\delta}$ .  $\square$

## 4 Simple and double layer potentials related to the Stokes system or the Poisson equation.

The solutions to (1.1), (1.2) we will consider are given by a sum of simple layer, double layer and volume potentials. The first two types of potentials are introduced and studied in this section. We begin by defining some kernel functions, among them a fundamental solution to (1.1). Put

$$\mathfrak{N}(z) := (4\pi|z|)^{-1}, \quad E_{jk}(z) := (8\pi|z|)^{-1}(\delta_{jk} + z_j z_k |z|^{-2}), \quad (4.1) \quad \boxed{10.29.def}$$

$$S_{jkl} := -\delta_{jk} \partial_l \mathfrak{N} - \partial_k E_{jl} - \partial_j E_{kl} \quad \text{for } z \in \mathbb{R}^3 \setminus \{0\}, \quad 1 \leq j, k, l \leq 3. \quad (4.2) \quad \boxed{10.30.def}$$

The matrix-valued function  $E = (E_{jk})_{1 \leq j, k \leq 3}$  is the velocity part of a fundamental solution to the Stokes system (1.1), with its associated pressure part given by  $-\nabla \mathfrak{N}$ .

In the next lemma, as in similar situations below, the restrictions on  $|\alpha|$  (order of differentiation) may of course be dropped if the constants may depend on  $|\alpha|$ . We will not need this fact, opting instead for such restrictions.

(lemma10.30)

**Lemma 4.1** *The relations  $E_{jk}, S_{jkl}, \mathfrak{N} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  and*

$$-\Delta E_{jk} - \partial_j \partial_k \mathfrak{N} = 0, \quad \sum_{\mu=1}^3 \partial_\mu E_{j\mu} = 0, \quad \Delta \mathfrak{N} = 0 \quad (4.3) \quad \boxed{L1.30.31}$$

hold for  $1 \leq j, k, l \leq 3$ . In addition

$$|\partial^\alpha E_{jk}(z)| + |\partial^\alpha \mathfrak{N}(z)| \leq C |z|^{-1-|\alpha|}, \quad (4.4) \quad \boxed{L1.30.40}$$

for  $j, k$  as before,  $z \in \mathbb{R}^3 \setminus \{0\}$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 3$ . Moreover

$$S_{jkl}(z) = 3/(4\pi) z_j z_k z_l |z|^{-5}, \quad -\Delta S_{jkl} + 2 \partial_j \partial_k \partial_l \mathfrak{N} = 0, \quad \sum_{j=1}^3 \partial_j S_{jkl} = 0 \quad (4.5) \quad \boxed{L1.30.50}$$

for  $j, k, l, z$  as before.

**Proof:** Inequality (4.4) is an obvious consequence of (4.1). The equations in (4.3) follow from (4.1), the first equation in (4.5) is a consequence of the same reference and of (4.2), whereas the other equations in (4.5) may be deduced from (4.3).  $\square$

The pair  $(V(\phi), Q(\phi))$  defined in the following lemma for any  $\phi \in L^1(\partial\Omega)^3$  is the simple layer potential associated with the Stokes system (1.1).

⟨lemma1.50⟩

**Lemma 4.2** *Let  $\phi \in L^1(\partial\Omega)$ . For  $x \in \mathbb{R}^3 \setminus \partial\Omega$ , put*

$$V(\phi)(x) := \int_{\partial\Omega} E(x-y) \cdot \phi(y) \, do_y, \quad Q(\phi)(x) := \int_{\partial\Omega} (-\nabla \mathfrak{N})(x-y) \cdot \phi(y) \, do_y.$$

*The integral  $\int_{\partial\Omega} |E(x-y) \cdot \phi(y)| \, do_y$  is finite for a. e.  $x \in \partial\Omega$ . In particular, for  $x \in \partial\Omega$ , the term  $V(\phi)(x)$  may be defined in the same way as for  $x \in \mathbb{R}^3 \setminus \partial\Omega$ .*

*The restrictions of  $V(\phi)_j$  and  $Q(\phi)$  to  $\mathbb{R}^3 \setminus \partial\Omega$  belong to  $C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$ , for  $1 \leq j \leq 3$ , with  $\partial_x^\alpha V(\phi)(x) = \int_{\partial\Omega} (\partial^\alpha E)(x-y) \cdot \phi(y) \, do_y$  for  $x \in \mathbb{R}^3 \setminus \partial\Omega$ ,  $\alpha \in \mathbb{N}_0^3$ , and with an analogous formula being valid for  $Q(\phi)$ .*

*The pair  $(U, \Pi) = (V(\phi)|_{\mathbb{R}^3 \setminus \partial\Omega}, Q(\phi)|_{\mathbb{R}^3 \setminus \partial\Omega})$  satisfies the Stokes system (1.1) in  $\mathbb{R}^3 \setminus \partial\Omega$  with  $F = 0$ .*

*If  $\phi \in C^0(\partial\Omega)^3$ , then  $V(\phi) \in C^a(\mathbb{R}^3)^3$  for  $a \in [0, 1)$ .*

**Proof:** The term  $V(\phi)(x)$  is well defined also for  $x \in \partial\Omega$  according to Lemma 2.1. The claims related to the differential properties in  $\mathbb{R}^3 \setminus \partial\Omega$  follow from Lebesgue's theorem, Lemma 4.1 and (4.3). The statement on Hölder continuity of  $V(\phi)$  in  $\mathbb{R}^3$  if  $\phi$  is continuous follows from (4.4) and Lemma 2.3. This means in particular that for such  $\phi$ , the continuous extension of  $V(\phi)|_{\mathbb{R}^3 \setminus \partial\Omega}$  to  $\partial\Omega$  is given by the definition of  $V(\phi)|_{\partial\Omega}$  in the lemma. Note that  $Q(\phi)$  is not defined on  $\partial\Omega$  because its kernel is singular with respect to integrals on  $\partial\Omega$ .  $\square$

⟨corollary100.40⟩

**Corollary 4.1** *Let  $p \in (1, \infty)$  and  $R \in (0, \infty)$  with  $\Omega_R \subset B_R$ . Then, if  $r \in [1, 3p/2)$ , the estimate*

$$\|V(\phi)|_{B_R \setminus \partial\Omega}\|_{1,r} + \|Q(\phi)|_{B_R \setminus \partial\Omega}\|_r \leq \mathfrak{C} \|\phi\|_p \quad \text{for } \phi \in L^p(\partial\Omega)^3. \quad (4.6) \quad \boxed{\text{C100.40.10}}$$

*is valid. In particular  $V(\phi)|_\Omega \in W^{1,r}(\Omega)^3$ ,  $V(\phi)|_{\Omega_R} \in W^{1,r}(\Omega_R)^3$  for  $r, \phi$  as above, and  $V(\phi)|_\Omega$  and  $V(\phi)|_{\Omega_R}$  have a trace on  $\partial\Omega$ . Moreover*

$$\|V(\phi)|_{\partial\Omega}\|_p \leq \mathfrak{C} \|\phi\|_p \quad \text{for } \phi \in L^p(\partial\Omega)^3. \quad (4.7) \quad \boxed{\text{C100.40.20}}$$

**Proof:** Let  $r \in (p, 3p/2)$ . Then we obtain from Lemma 2.4 with  $a = 2(1/p - 1/r)$  that

$$\|\partial_m(V(\phi)|_{B_R \setminus \partial\Omega})\|_r + \|Q(\phi)|_{B_R \setminus \partial\Omega}\|_r \leq \mathfrak{C} \|\phi\|_p \quad (1 \leq m \leq 3, \phi \in L^p(\Omega)^3).$$

It follows that even if  $r \in [1, 3p/2)$ , the preceding inequality still holds uniformly in  $m$  and  $\phi$ , but, of course, with a different constant  $\mathfrak{C}$ . Since  $|x-y|^{-1} \leq \mathfrak{C}|x-y|^{-2}$  for  $x, y \in B_R$  with  $x \neq y$ , the same argument implies that if  $r \in [1, 3p/2)$ , we have  $\|V(\phi)|_{B_R \setminus \partial\Omega}\|_r \leq \mathfrak{C} \|\phi\|_p$  for  $m, \phi$  as before. This proves (4.6). As concerns inequality (4.7), we refer to Lemma 2.1.  $\square$

Next we introduce double layer potentials related to the Stokes system. The function  $S_{jkl}$  was introduced in (4.2).

(lemma3.10)

**Lemma 4.3** Let  $\phi \in L^1(\partial\Omega)^3$ . Then for  $x \in \mathbb{R}^3 \setminus \partial\Omega$ ,  $l \in \{1, 2, 3\}$ , put

$$W(\phi)_l(x) := \int_{\partial\Omega} - \sum_{j,k=1}^3 S_{jkl}(x-y) \phi_j(y) n_k^{(\Omega)}(y) do_y,$$

$$\Pi(\phi)(x) := \int_{\partial\Omega} (-2) \sum_{j,k=1}^3 (\partial_j \partial_k \mathfrak{N})(x-y) \phi_j(y) n_k^{(\Omega)}(y) do_y.$$

Then  $W(\phi)_l, \Pi(\phi)$  belong to  $C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$  for  $1 \leq l \leq 3$ , and any derivative commutes with the integration over  $\partial\Omega$  appearing in the definition of these functions. Moreover the pair  $(W(\phi), \Pi(\phi))$  solves (1.1) in  $\mathbb{R}^3 \setminus \partial\Omega$  with  $F = 0$ .

**Proof:** Lebesgue's theorem and the relation  $E_{jk}, \mathfrak{N} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  for  $1 \leq j, k \leq 3$  yield the lemma except its last claim, which follows from (4.5).  $\square$

(lemma-decay)

**Lemma 4.4** Let  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ . Then

$$|\partial^\alpha V(\phi)(x)| \leq \mathfrak{C} \|\phi\|_1 |x|^{-1-|\alpha|}, \quad |\partial^\alpha W(\phi)(x)| \leq \mathfrak{C} \|\phi\|_1 |x|^{-2-|\alpha|} \quad (4.8) \quad \boxed{7.32a}$$

for  $\phi \in L^1(\partial\Omega)^3$ ,  $x \in B_R^c$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 2$ ,

$$|\partial^\alpha Q(\phi)(x)| \leq \mathfrak{C} \|\phi\|_1 |x|^{-2-|\alpha|}, \quad |\partial^\alpha \Pi(\phi)(x)| \leq \mathfrak{C} \|\phi\|_1 |x|^{-3-|\alpha|} \quad (4.9) \quad \boxed{7.32b}$$

for  $\phi, x$  as before, and for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ .

Consequently, if  $r_1 \in (1, \infty)$ ,  $r_2 \in (3/2, \infty)$ ,  $r_3 \in (3, \infty)$ , then for  $1 \leq l, m \leq 3$ ,

$$\begin{aligned} & \|\partial_m \partial_l V(\phi)|_{B_R^c}\|_{r_1} + \|\partial_m Q(\phi)|_{B_R^c}\|_{r_1} + \|\partial_m W(\phi)|_{B_R^c}\|_{r_1} \\ & \quad + \|\Pi(\phi)|_{B_R^c}\|_{r_1} + \|\partial_m \Pi(\phi)|_{B_R^c}\|_{r_1} \leq \mathfrak{C}(R, r_1) \|\phi\|_1, \\ & \|W(\phi)|_{B_R^c}\|_{r_2} + \|\partial_m V(\phi)|(B_R^c)\|_{r_2} + \|Q(\phi)|_{B_R^c}\|_{r_2} \leq \mathfrak{C}(R, r_2) \|\phi\|_1, \\ & \|V(\phi)|_{B_R^c}\|_{r_3} \leq \mathfrak{C}(R, r_3) \|\phi\|_1. \end{aligned}$$

**Proof:** Obviously  $\delta := \text{dist}(\bar{\Omega}, B_R^c) > 0$ . Let  $y \in \partial\Omega \setminus \{0\}$  and put  $z := |y|^{-1} R y$ . Then  $|y| + |z - y| \leq |z| = R$ , and  $|z - y| \geq \delta$ , so  $|y| \leq R - \delta$ . Hence for  $x \in B_R^c$  we get  $|x - y| \geq (\delta/R)|x| + (1 - \delta/R)|x| - |y| \geq (\delta/R)|x| + R - \delta - |y| \geq (\delta/R)|x|$ . Now the lemma follows from (4.4).  $\square$

In the following lemma, we define the double layer potential also on  $\partial\Omega$ , instead of only on  $\mathbb{R}^3 \setminus \partial\Omega$ . The key fact in this respect is the estimate at the beginning of Lemma 4.5. It will turn out that in general, the double layer potential is not continuous in  $\mathbb{R}^3$ , even if the layer function  $\phi$  is smooth; see Theorem 4.4 and 4.5.

(lemma1.60)

**Lemma 4.5** The inequality  $|\sum_{k=1}^3 S_{jkl}(x-y) n_k^{(\Omega)}(x)| \leq \mathfrak{C} |x-y|^{-1}$  holds for  $x, y \in \partial\Omega$  with  $x \neq y$ ,  $1 \leq j, l \leq 3$ . The preceding estimate remains valid if the term  $n_k^{(\Omega)}(x)$  is replaced by  $n_k^{(\Omega)}(y)$ .

If  $\phi \in L^1(\partial\Omega)^3$ ,  $1 \leq j \leq 3$ , we may define

$$\mathfrak{T}^*(\phi)_j(x) := (-2) \int_{\partial\Omega} \sum_{k,l=1}^3 S_{jkl}(x-y) n_k^{(\Omega)}(x) \phi_l(y) do_y$$

for  $x \in \partial\Omega$ . Then, for any  $p \in (1, \infty)$ ,  $\phi \in L^p(\partial\Omega)^3$ , the relation  $\mathfrak{T}^*(\phi) \in L^p(\partial\Omega)^3$  is valid, and the operator  $T_p^* : L^p(\partial\Omega)^3 \mapsto L^p(\partial\Omega)^3$ ,  $T_p^*(\phi) := \mathfrak{T}^*(\phi)$  ( $p \in L^p(\partial\Omega)^3$ ) is linear, bounded and compact.

Define the function  $\mathfrak{T}(\phi)$  by replacing the term  $n_k^{(\Omega)}(x)$  by  $-n_k^{(\Omega)}(y)$  in the definition of  $\mathfrak{T}^*(\phi)$ . An operator  $T_p$  may be associated with  $\mathfrak{T}$  in the same way as  $T_p^*$  is associated with  $\mathfrak{T}^*$ , for  $p \in (1, \infty)$ . Then  $T_p$  is equally linear, bounded and compact.

Let  $I_q : L^q(\partial\Omega)^3 \mapsto L^q(\partial\Omega)^3$  denote the identity mapping of  $L^q(\partial\Omega)^3$ . Then the operators  $\pm I_p + T_p$  and  $\pm I_p + T_p^*$  are Fredholm with index zero.

For  $q \in (1, \infty)$ , the operator  $T_q^*$  is dual to  $T_q$ .

**Proof:** For the estimate of  $|\sum_{k=1}^3 S_{jkl}(x-y) n_k^{(\Omega)}(x)|$  stated in the lemma see (3.5), (4.5). In the case that the term  $n_k^{(\Omega)}(x)$  is replaced by  $-n_k^{(\Omega)}(y)$ , the same references may be used.

Lemma 2.1 yields that  $\mathfrak{T}^*(\phi)$  and  $\mathfrak{T}(\phi)$  are well defined for  $\phi \in L^1(\partial\Omega)^3$ , and  $\mathfrak{T}^*(\phi)$ ,  $\mathfrak{T}(\phi) \in L^p(\partial\Omega)^3$  if  $p \in (1, \infty)$  and  $\phi \in L^p(\partial\Omega)^3$ . Moreover it follows from Lemma 2.2 that  $T_p^*$  and  $T_p$  are linear, bounded and compact. The general theory of Fredholm operators now implies that  $\pm I_p + T_p$  and  $\pm I_p + T_p^*$  are Fredholm with index zero. The last statement of the lemma is a consequence of Fubini's theorem; see [8, Satz 5.1].  $\square$

theorem100.10)

**Theorem 4.1** Let  $p \in (1, \infty)$ . Then with the notations of Lemma 4.5,

$$\ker(-I_p + T_p^*) = \{k n^{(\Omega)} : k \in \mathbb{R}\}, \quad \ker(I_p + T_p) = \{\phi^{(j)}|_{\partial\Omega} : j \in \{1, \dots, 6\}\},$$

with  $\phi^{(j)}$  ( $1 \leq j \leq 6$ ) introduced in Theorem 1.1

**Proof:** [8, Lemma 6.7, 6.5, 6.10].  $\square$

corollary100.10)

**Corollary 4.2** Let  $p \in (1, \infty)$ . Then  $\dim \ker(I_p + T_p^*) = 6$  and  $\dim \ker(-I_p + T_p) = 1$ ,

$$\text{ran}(I_p + T_p^*) = \{v \in L^p(\partial\Omega)^3 : \int_{\partial\Omega} v \cdot \phi^{(j)} do_x = 0 \text{ for } 1 \leq j \leq 6\}, \quad (4.10) \quad \boxed{\text{C100.10.10}}$$

$$\text{ran}(-I_p + T_p^*) = \{v \in L^p(\partial\Omega)^3 : \int_{\partial\Omega} v \cdot \gamma do_x \text{ for } \gamma \in \ker(-I_{p'} + T_{p'}^*)\}, \quad (4.11) \quad \boxed{\text{C100.10.20}}$$

where the functions  $\phi^{(j)}$  for  $1 \leq j \leq 6$  were introduced in Theorem 1.1.

**Proof:** By Lemma 4.5, we know that the operator  $I_p + T_p^*$  is Fredholm with index zero, so  $\dim \ker(I_p + T_p^*) = \text{codim } \text{ran}(I_p + T_p^*)$ . On the other hand, the fact that  $I_p + T_p^*$  is Fredholm means in particular this operator has closed range. We further recall that  $I_{p'} + T_{p'}^*$  is dual to  $I_p + T_p^*$  (Lemma 4.5). It follows with the closed range theorem and

Riesz' representation theorem in  $L^p$ -spaces that  $\dim \ker(I_p + T_p^*) = \text{codim } \text{ran}(I_p + T_p^*) = \dim \ker(I_{p'} + T_{p'})$  and

$$\text{ran}(I_p + T_p^*) = \{v \in L^p(\partial\Omega)^3 : \int_{\partial\Omega} v \cdot \gamma \, do_x = 0 \text{ for } \gamma \in \ker(I_{p'} + T_{p'})\}.$$

Equation (4.10) now is a consequence of Theorem 4.1, and the equation  $\dim \ker(I_p + T_p^*) = 6$  follows with Theorem 4.1. An analogous reasoning based on Lemma 4.5 and Theorem 4.1 yields that  $\dim \ker(-I_p + T_p) = 1$  and that equation (4.11) holds.  $\square$

theorem100.20)

**Theorem 4.2** *Let  $p \in (1, \infty)$ ,  $a \in [0, 1)$ ,  $b \in C^a(\partial\Omega)^3$ ,  $\psi \in L^p(\partial\Omega)^3$  with  $\pm\psi + \mathfrak{T}(\psi) = b$  or  $\pm\psi + \mathfrak{T}^*(\psi) = b$ . Then  $\psi \in C^a(\partial\Omega)^3$ .*

*This means in particular that  $\ker(\pm I_p + T_p) = \ker(\pm I_q + T_q)$  and  $\ker(\pm I_p + T_p^*) = \ker(\pm I_q + T_q^*)$  for  $q \in (1, \infty)$ .*

**Proof:** See [8, Lemma 5.4].  $\square$

ollary100.20)

**Corollary 4.3** *Let  $p \in (1, \infty)$ . Then there is a topological complement  $E_p^{(\pm)}$  of  $\ker(\pm I_p + T_p^*)$  in  $L^p(\partial\Omega)^3$ .*

*The estimate  $\|\phi\|_p \leq \mathfrak{C} \|(\pm\phi + T_p^*)(\phi)\|_p$  holds for  $\phi \in E_p^{(\pm)}$ .*

*If  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot \phi^{(j)} \, do_x = 0$  for  $1 \leq j \leq 6$ , there is a unique function  $F^{(+)}(b) \in E_p^{(+)}$  with  $(1/2) [F^{(+)}(b) + \mathfrak{T}^*(F^{(+)}(b))] = b$ . (See Theorem 1.1 for the definition of the functions  $\phi^{(1)}, \dots, \phi^{(6)}$ .)*

*Fix a function  $\phi^{(0)} \in \ker(-I_{p'} + T_{p'}) \setminus \{0\}$ . (In view of Theorem 4.2, this function  $\phi^{(0)}$  belongs to  $\ker(-I_r + T_r)$  for any  $r \in (1, \infty)$ .) If  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot \phi^{(0)} \, do_x = 0$ , then there is a unique function  $F^{(-)}(b) \in E_p^{(-)}$  with  $(1/2) [-F^{(-)}(b) + \mathfrak{T}^*(F^{(-)}(b))] = b$ .*

**Proof:** Since  $\dim \ker(\pm I_p + T_p^*) < \infty$  by Corollary 4.2, the existence of a complement  $E_p^{(\pm)}$  as described in Corollary 4.3 follows by general theory. Obviously  $\pm I_p + T_p^*|_{E_p^{(\pm)}}$  is a bijective operator from  $E_p^{(\pm)}$  onto  $\text{ran}(\pm I_p + T_p^*)$ . On the other hand,  $\pm I_p + T_p^*$  is Fredholm (Lemma 4.5), and thus has closed range. It follows by the open mapping theorem that  $\|\phi\|_p \leq \mathfrak{C} \|(\pm I_p + T_p^*)(\phi)\|_p$  for  $\phi \in E_p^{(\pm)}$ , hence  $\|\phi\|_p \leq \mathfrak{C} \|\pm\phi + \mathfrak{T}^*(\phi)\|_p$  for such  $\phi$ .

For any  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot \phi^{(j)} \, do_x = 0$  for  $1 \leq j \leq 6$ , we know by (4.10) that  $b \in \text{ran}(I_p + T_p^*)$ . Thus, since  $I_p + T_p^*|_{E_p^{(+)}}$  is a bijective operator from  $E_p^{(+)}$  onto  $\text{ran}(I_p + T_p^*)$ , there is a unique function  $F^{(+)}(b)$  with the properties stated in the corollary.

Since  $\dim \ker(-I_{p'} + T_{p'}) = 1$ , any function  $b \in L^p(\partial\Omega)^3$  with  $\int_{\partial\Omega} b \cdot \phi^{(0)} \, do_x = 0$  verifies the equation  $\int_{\partial\Omega} b \cdot \gamma \, do_x = 0$  for any  $\gamma \in \ker(-I_{p'} + T_{p'})$ . Thus an analogous reasoning as in the case of  $F^{(+)}(b)$ , but based on (4.11) instead of (4.10), yields existence of a unique function  $F^{(-)}(b)$  with properties as described in the corollary.  $\square$

lemma100.60)

**Lemma 4.6** *Put  $M := (\int_{\partial\Omega} \phi^{(j)} \cdot \phi^{(k)} \, do_x)_{1 \leq j, k \leq 3}$  and  $\widetilde{M} := (\int_{\Omega} \phi^{(j)} \cdot \phi^{(k)} \, dx)_{1 \leq j, k \leq 3}$ , where  $\phi^{(1)}, \dots, \phi^{(6)}$  were introduced in Theorem 1.1. Then the matrices  $M$  and  $\widetilde{M}$  are invertible.*

**Proof:** The functions  $\phi^{(1)}|_{\partial\Omega}, \dots, \phi^{(6)}|_{\partial\Omega}$  are linearly independent ([8, Lemma 6.5]). Let  $\alpha \in \mathbb{R}^6$  with  $M \cdot \alpha = 0$ . Then  $\sum_{j=1}^6 \alpha_j \sum_{k=1}^6 \int_{\partial\Omega} \phi^{(j)} \cdot \phi^{(k)} d\alpha_k = 0$ , that is,  $\int_{\partial\Omega} |\sum_{l=1}^6 \alpha_l \phi^{(l)}|^2 d\alpha = 0$ , so  $\sum_{l=1}^6 \alpha_l \phi^{(l)} = 0$ , and finally  $\alpha = 0$ . This means that  $M$  is invertible. Obviously the functions  $\phi^{(1)}|_{\Omega}, \dots, \phi^{(6)}|_{\Omega}$  are linearly independent as well. Thus the same argument as for  $M$  yields that  $\tilde{M}$  is invertible, too.  $\square$

(lemma100.20)

**Lemma 4.7** *Let  $p \in (1, \infty)$ ,  $b \in W^{1-1/p, p}(\partial\Omega)^3$  with  $\int_{\partial\Omega} \phi^{(j)} \cdot b d\alpha = 0$  for  $1 \leq j \leq 6$ ; see Theorem 1.1 for the definition of  $\phi^{(1)}, \dots, \phi^{(6)}$ . Then there exists a sequence  $(b_n)$  in  $C^0(\partial\Omega)^3$  such that for  $n \in \mathbb{N}$ , the function  $b_n$  belongs to  $C^a(\partial\Omega)^3$  for  $a \in (0, 1)$ , hence  $b_n \in W^{1-1/r, r}(\partial\Omega)^3$  for  $r \in (1, \infty)$ ,  $\int_{\partial\Omega} \phi^{(j)} \cdot b_n d\alpha = 0$  for  $1 \leq j \leq 6$ , and such that  $\|b - b_n\|_{1-1/p, p} \rightarrow 0$ .*

*Suppose that  $\int_{\partial\Omega} \phi^{(0)} \cdot b d\alpha = 0$ , where  $\phi^{(0)}$  was introduced in Corollary 4.3. Then there is a sequence  $(b_n)$  in  $C^0(\partial\Omega)^3$  with the same properties as before, except that the relation  $\int_{\partial\Omega} \phi^{(j)} \cdot b_n d\alpha = 0$  ( $1 \leq j \leq 6$ ,  $n \in \mathbb{N}$ ) is replaced by  $\int_{\partial\Omega} \phi^{(0)} \cdot b_n d\alpha = 0$  ( $n \in \mathbb{N}$ ).*

**Proof:** Since  $b \in W^{1-1/p, p}(\partial\Omega)^3$ , there is  $B \in W^{1, p}(\Omega)^3$  with  $B|_{\partial\Omega} = b$ . We may choose a sequence  $(B_n)$  in  $C_0^\infty(\mathbb{R}^3)^3$  with  $\|B_n - B\|_{1, p} \rightarrow 0$ , and thus  $\|B_n - b\|_{1-1/p, p} \rightarrow 0$ . Obviously  $\tilde{b}_n := B_n|_{\partial\Omega} \in C^a(\partial\Omega)^3$  for  $n \in \mathbb{N}$ ,  $a \in (0, 1)$ .

Let  $n \in \mathbb{N}$  and set  $c_n := M^{-1} \cdot (\int_{\partial\Omega} \tilde{b}_n \cdot \phi^{(j)} d\alpha)_{1 \leq j \leq 6}$ , with the matrix  $M$  introduced in Lemma 4.6. Then  $\int_{\partial\Omega} \tilde{b}_n \cdot \phi^{(j)} d\alpha = \sum_{k=1}^6 \int_{\partial\Omega} \phi^{(j)} \cdot \phi^{(k)} d\alpha c_{n, k}$  for  $1 \leq j \leq 6$ . Thus, putting  $b_n := \tilde{b}_n - \sum_{k=1}^6 c_{n, k} \phi^{(k)}$ , we obtain a function  $b_n$  belonging to  $C^a(\partial\Omega)^3$  for  $a \in (0, 1)$  and verifying the relation  $\int_{\partial\Omega} \phi^{(j)} \cdot b_n d\alpha = 0$  for  $1 \leq j \leq 6$ . Since  $\|\tilde{b}_n - b\|_p \rightarrow 0$  and  $\int_{\partial\Omega} \tilde{b}_n \cdot \phi^{(j)} d\alpha = \int_{\partial\Omega} (\tilde{b}_n - b) \cdot \phi^{(j)} d\alpha$  for  $1 \leq j \leq 6$ ,  $n \in \mathbb{N}$  by our assumptions on  $b$ , we get in addition that  $\|b_n - b\|_p \rightarrow 0$

The second part of the lemma may be proved in the same way as the first, but the reasoning is somewhat simpler because no matrix is involved. Note that by Theorem 4.2, we have  $\phi^{(0)} \in C^a(\partial\Omega)^3$  for  $a \in (0, 1)$ .  $\square$

(theorem3.14)

**Theorem 4.3** *Let  $p \in (1, \infty)$ . If  $\phi \in C^a(\partial\Omega)^3$  for some  $a \in (0, 1)$  and if  $\pm\phi + \mathfrak{T}(\phi) \in C^a(\partial\Omega)^3 \cap W^{2-1/p, p}(\partial\Omega)^3$ , then  $\phi$  belongs to  $W^{2-1/p, p}(\partial\Omega)^3$  and the inequality  $\|\phi\|_{2-1/p, p} \leq \mathfrak{C}(\|\pm\phi + \mathfrak{T}(\phi)\|_{2-1/p, p} + \|\phi\|_p)$  holds.*

**Proof:** [8, Lemma 7.8].  $\square$

(theorem2.50)

**Theorem 4.4** *Recall the parameter  $\delta(\Omega)$  from Lemma 3.2. Let  $\psi \in C^0(\partial\Omega)^3$  and put  $U := V(\psi)|_{\mathbb{R}^3 \setminus \partial\Omega}$ ,  $\Pi := Q(\psi)$ , where  $V(\psi)$  and  $Q(\psi)$  were introduced in Lemma 4.2. Then*

$$\sum_{k=1}^3 n_k^{(\Omega)}(x) (\partial_j U_k + \partial_k U_j - \delta_{jk} \Pi) (x \pm \epsilon n^{(\Omega)}(x)) \rightarrow (1/2) (\mp\psi + \mathfrak{T}^*(\psi))_j(x) \quad (4.12) \quad \boxed{\text{T2.50.10}}$$

*for  $\epsilon \rightarrow 0$ ,  $\epsilon \in (0, \delta(\Omega)]$ , uniformly with respect to  $x \in \partial\Omega$ ,  $1 \leq j \leq 3$  ("jump relation").*

**Proof:** The relation in (4.12) holds according to [8, Lemma 4.8]. Note that the definition

of  $\mathfrak{T}^*(\psi)$  in [8] (see [8, Definition 4.2 and 5.1]) coincides with ours in Lemma 4.5. This follows from (4.5); compare [8, p. 100, last line].  $\square$

We turn to the question of how to approximate  $V(\phi)|\Omega$  and  $V(\phi)|\overline{\Omega}^c$  by functions which are  $C^\infty$  in open sets somewhat larger than  $\overline{\Omega}$  and  $\Omega^c$ , respectively.

$\langle$ lemma1.2

**Lemma 4.8** *Recall the parameter  $\delta(\Omega)$  and the sets  $\mathfrak{U}_\delta$  and  $\mathfrak{U}_{-\delta}$  for  $\delta \in (0, \delta(\Omega)]$  introduced in Lemma 3.2. Further recall that  $\mathfrak{U}_\delta$  and  $\mathfrak{U}_{-\delta}$  are open sets in  $\mathbb{R}^3$  with  $\overline{\Omega} \subset \mathfrak{U}_\delta$ ,  $\Omega^c \subset \mathfrak{U}_{-\delta}$ , for  $\delta$  as before. Let  $\phi \in L^1(\partial\Omega)^3$  and define  $V^{(\delta)}(\phi) : \mathfrak{U}_\delta \mapsto \mathbb{C}^3$ ,  $Q^{(\delta)}(\phi) : \mathfrak{U}_\delta \mapsto \mathbb{C}$  by setting*

$$V^{(\delta)}(\phi)(x) := \int_{\partial\Omega} E(x - [y + \delta n^{(\Omega)}(y)]) \cdot \phi(y) do_y,$$

$$Q^{(\delta)}(\phi)(x) := \int_{\partial\Omega} -(\nabla \mathfrak{N})(x - [y + \delta n^{(\Omega)}(y)]) \cdot \phi(y) do_y$$

for  $x \in \mathfrak{U}_\delta$ ,  $\delta \in (0, \delta(\Omega)]$ . In addition, we introduce the functions  $V^{(-\delta)}(\phi) : \mathfrak{U}_{-\delta} \mapsto \mathbb{C}^3$ ,  $Q^{(-\delta)}(\phi) : \mathfrak{U}_{-\delta} \mapsto \mathbb{C}$  by replacing  $\mathfrak{U}_\delta$  with  $\mathfrak{U}_{-\delta}$  and the term  $y + \delta n^{(\Omega)}$  with  $y - \delta n^{(\Omega)}$  in the respective definitions of  $V^{(\delta)}(\phi)$  and  $Q^{(\delta)}(\phi)$ .

Then  $V^{(\pm\delta)}(\phi)_j$ ,  $Q^{(\pm\delta)}(\phi)$  belong to  $C^\infty(\mathfrak{U}_{\pm\delta})$  for  $1 \leq j \leq 3$  and for  $\delta$  as above. Any derivative of these functions commutes with the integration over  $\partial\Omega$  appearing in their definition. The pair  $(V^{(\pm\delta)}(\phi), Q^{(\pm\delta)}(\phi))$  is a solution of (1.1) in  $\mathfrak{U}_{\pm\delta}$  with  $F = 0$ . In addition

$$V^{(\delta)}(\phi)(x) \rightarrow V(\phi)(x) \text{ for } x \in \Omega, \quad V^{(-\delta)}(\phi)(x) \rightarrow V(\phi)(x) \text{ if } \delta \downarrow 0, \text{ for } x \in \overline{\Omega}^c. \quad (4.13) \quad \boxed{1.2.43}$$

Suppose that  $p \in (1, \infty)$ ,  $\phi \in L^p(\partial\Omega)^3$ ,  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ . Then

$$\int_{\Omega_R} |\partial_x^\alpha (V^{(-\delta)}(\phi) - V(\phi))(x)|^p do_x \rightarrow 0, \quad (4.14) \quad \boxed{1.2.44}$$

$$\int_{\Omega} |\partial_x^\alpha (V^{(\delta)}(\phi) - V(\phi))(x)|^p do_x \rightarrow 0 \text{ if } \delta \downarrow 0, \text{ for } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \leq 1. \quad (4.15) \quad \boxed{1.2.45}$$

If  $\phi \in C^0(\partial\Omega)^3$ , then for  $x \in \partial\Omega$ ,  $1 \leq j \leq 3$ ,

$$V^{(\pm\delta)}(\phi)(x) \rightarrow V(\phi)(x), \quad (4.16) \quad \boxed{L1.2.20}$$

$$\sum_{k=1}^3 n_k^{(\Omega)}(x) (\partial_j V^{(\pm\delta)}(\phi)_k + \partial_k V^{(\pm\delta)}(\phi)_j - \delta_{jk} Q^{(\pm\delta)}(\phi))(x) \quad (4.17) \quad \boxed{L1.2.21}$$

$$\rightarrow (1/2) (\pm\phi + \mathfrak{T}^*(\phi))_j(x) \quad (1 \leq j \leq 3),$$

for  $\delta \downarrow 0$ , with the convergence in (4.16) and (4.17) being uniform with respect to  $x \in \partial\Omega$ .

The function  $V(\phi)|\partial\Omega$  as defined in Lemma 4.2 and the trace of  $V(\phi)|\Omega$  and  $V(\phi)|\Omega_R$  on  $\partial\Omega$  (see Corollary 4.1) coincide.

**Proof:** We only consider  $V^{(-\delta)}(\phi)$  and  $Q^{(-\delta)}(\phi)$ . If  $-\delta$  is replaced by  $\delta$ , an analogous reasoning is valid.

The differential properties of  $V^{(-\delta)}(\phi)$  and  $Q^{(-\delta)}(\phi)$  are a consequence of (3.6), the relation  $E_{jk}, \mathfrak{N} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$  for  $1 \leq j, k \leq 3$ , the equations satisfied by  $E$  and  $\mathfrak{N}$  (see Lemma 4.1) and Lebesgue's theorem.

Let  $x \in \overline{\Omega}^c$ . For  $y \in \partial\Omega$ ,  $\delta \in (0, \delta(\Omega)]$ , the relation  $|x - [y - \delta n^{(\Omega)}(y)]| \geq \mathcal{D} \delta/2$  holds by (3.6). Since  $E \in C^\infty(\mathbb{R}^3 \setminus \{0\})^{3 \times 3}$ , we may conclude that the claim on  $V^{(-\delta)}(\phi)$  in (4.13) follows from Lebesgue's theorem. For the proof of (4.14) and (4.15), we refer to the proof of [6, Lemma 5.4],

As concerns the proof of respectively (4.16) and (4.17), we refer to [8, Lemma 6.3] and [8, (6.20), (4.72)], respectively, as concerns  $V^{(-\delta)}(\phi)$ . When  $V^{(\delta)}(\phi)$  is considered, the relevant references are [8, (6.10)] and [8, (6.19), (4.71)].

The last claim of the lemma follows from (4.14), (4.15), the regularity properties of  $V^{(-\delta)}(\phi)$  and  $V^{(\delta)}(\phi)$ , and the uniform convergence in (4.16).  $\square$

$\langle$ lemma5.20

**Lemma 4.9** *The inequality*

$$|\partial_l (V(\phi)|_{\mathbb{R}^3 \setminus \partial\Omega})(x \pm \delta n^{(\Omega)}(x)) - \partial_l V^{(\mp\delta)}(\phi)(x)| \leq \mathfrak{C} |\phi|_\infty \delta^{1/2}$$

holds for  $\phi \in C^0(\partial\Omega)^3$ ,  $x \in \partial\Omega$ ,  $\delta \in (0, \delta(\Omega)]$ .

**Proof:** This lemma follows from (3.4); see the proof of [6, Corollary 5.3] or [8, (6.21), (6.22)].  $\square$

$\langle$ theorem3.20

**Theorem 4.5** *Let  $\phi \in C^0(\partial\Omega)^3$ . Then the function  $W(\phi)|_{\overline{\Omega}^c}$  admits a continuous extension to  $\Omega^c$ , denoted by  $W_{ex}(\phi)$  and given by  $W_{ex}(\phi)|_{\partial\Omega} = (-1/2) (\phi + \mathfrak{T}(\phi))$ . Similarly the function  $W(\phi)|_{\Omega}$  admits a continuous extension to  $\overline{\Omega}$ , denoted by  $W_{in}(\phi)$  and given by  $W_{in}(\phi)|_{\partial\Omega} = (-1/2) (-\phi + \mathfrak{T}(\phi))$ .*

**Proof:** See [8, Theorem 4.1].  $\square$

theorem100.70

**Theorem 4.6** *Recall the parameter  $\delta(\Omega)$  introduced in Lemma 3.2. Let  $a \in (0, 1)$ ,  $\phi \in C^{1,a}(\partial\Omega)^3$ ,  $x \in \partial\Omega$ ,  $j \in \{1, 2, 3\}$ . Define the function  $F : [-\delta(\Omega), \delta(\Omega)] \setminus \{0\} \mapsto \mathbb{C}$  by setting  $W := W(\phi)$ ,  $Q := Q(\phi)$ ,*

$$F(\kappa) := \sum_{k=1}^3 n_k^{(\Omega)}(x) (\partial_j W_k + \partial_k W_j - \delta_{jk} \Pi)(x + \kappa n^{(\Omega)}(x)) \text{ for } \kappa \in [-\delta(\Omega), \delta(\Omega)] \setminus \{0\}.$$

*Then the limits  $\lim_{\kappa \downarrow 0} F(\kappa)$  and  $\lim_{\kappa \downarrow 0} F(-\kappa)$  exist and coincide.*

This result is well known; for example see [24, Proposition 3.31]. However, the only justification we know of was given by Faxén [9, § 11], in a rather long (10 pages) and in parts somewhat vague discussion. An analogous result for the Laplace double layer potential is shown in a more precise way in [16, Section 8.5.2], but the Stokes case is notably more difficult to handle than the Laplace one. So, since the preceding theorem is

a key tool in our theory, we sketch a proof in the appendix. Due to Lemma 3.2, we will not need to argue in terms of local coordinates of  $\partial\Omega$ .

theorem100.50)

**Theorem 4.7** *Let  $p \in (3, \infty)$ ,  $b \in W^{2-1/p,p}(\partial\Omega)^3$ ,  $\phi \in L^p(\partial\Omega)^3$  such that one of the two equations  $(-1/2)(-\phi + \mathfrak{T}(\phi)) = b$  or  $(-1/2)(\phi + \mathfrak{T}(\phi)) = b$  holds. Take  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ .*

*Then there is  $a \in (0, 1)$  with  $\phi \in C^{1,a}(\partial\Omega)^3$ . Moreover  $\phi \in W^{2-1/p,p}(\partial\Omega)^3$ ,  $W(\phi)|_U \in W^{2,p}(U)^3$ ,  $\Pi(\phi)|_U \in W^{1,p}(U)$  for  $U \in \{\Omega, \Omega_R\}$ , and  $W_{in}(\phi) \in C^1(\bar{\Omega})^3$ ,  $W_{ex}(\phi) \in C^1(\Omega^c)^3$ . The functions  $\Pi(\phi)|_\Omega$  and  $\Pi(\phi)|_{\bar{\Omega}^c}$  may be continuously extended to  $\bar{\Omega}$  and  $\Omega^c$ , respectively. These extensions are denoted by  $\Pi_{in}(\phi)$  and  $\Pi_{ex}(\phi)$ , respectively. (The functions  $W_{in}(\phi)$  and  $W_{ex}(\phi)$  were introduced in Theorem 4.5.)*

**Proof:** By a trace theorem and an extension theorem, there is  $B \in W_0^{2,p}(\mathbb{R}^3)^3$  with  $B|\partial\Omega = b$ . Since  $p > 3$ , a Sobolev inequality implies there is  $a \in (0, 1)$  such that  $B \in C^{1,\alpha}(\mathbb{R}^3)^3$ , so  $b \in C^{1,\alpha}(\partial\Omega)^3$ . Thus  $\phi \in C^a(\partial\Omega)^3$  by Theorem 4.2. Moreover, referring to Theorem 4.3, we see that  $\phi \in W^{2-1/p,p}(\partial\Omega)^3$ . Now the same argument as used above for  $b$  provides that  $\phi \in C^{1,\alpha}(\partial\Omega)^3$ . In addition, [8, Lemma 7.15] yields at this point that  $\Pi(\phi)|_U \in W^{1,p}(U)^3$  for  $U \in \{\Omega, \Omega_R\}$ . Since  $p > 3$ , we may again refer to a Sobolev inequality, obtaining that  $\Pi(\phi)|_U$  may be continuously extended to  $\bar{U}$ . Since  $\Pi(\phi)|_{\bar{\Omega}^c} \in C^\infty(\bar{\Omega}^c)$ , this means in particular that  $\Pi(\phi)|_{\bar{\Omega}^c}$  may be continuously extended to  $\Omega^c$ .

Obviously (Lemma 4.3), we have  $W(\phi)|_U \in C^\infty(U)$  for  $U$  as before. Since in particular  $\phi \in C^0(\partial\Omega)^3$ , we further know by Theorem 4.5 that  $W_{in}(\phi) \in C^0(\bar{\Omega})^3$ ,  $W_{ex}(\phi) \in C^0(\Omega^c)^3$ ,  $(-1/2)(-\phi + \mathfrak{T}(\phi)) = W_{in}(\phi)|\partial\Omega$ ,  $(-1/2)(\phi + \mathfrak{T}(\phi)) = W_{ex}(\phi)|\partial\Omega$ . Note that  $(-1/2)(\pm\phi + \mathfrak{T}(\phi)) \in \{b, b - \phi, b + \phi\}$ . But  $b, b - \phi, b + \phi \in W^{2-1/p,p}(\partial\Omega)^3 \cap C^0(\partial\Omega)^3$  by what was explained above, so  $W_{in}(\phi)|\partial\Omega, W_{ex}(\phi)|\partial\Omega \in W^{2-1/p,p}(\partial\Omega)^3 \cap C^0(\partial\Omega)^3$ . On the other hand,  $W(\phi)|_{\bar{\Omega}^c} \in C^\infty(\bar{\Omega}^c)^3$  and  $\bar{\Omega} \subset B_R$ , so in particular  $W(\phi)|\partial B_R \in W^{2-1/p,p}(\partial B_R)^3$ . Since  $W_{ex}(\phi)|_{\bar{\Omega}^c} = W(\phi)|_{\bar{\Omega}^c}$  by the definition of  $W_{ex}(\phi)$ , we thus get  $W_{ex}(\phi)|\partial\Omega_R \in W^{2-1/p,p}(\partial\Omega_R)^3 \cap C^0(\partial\Omega_R)^3$ . Recall that  $\Delta W(\phi) = \nabla\Pi(\phi)$  (Lemma 4.3) and  $\Pi(\phi)|_U \in W^{1,p}(U)$  for  $U \in \{\Omega, \Omega_R\}$ , as mentioned above. Further recall that  $W_{in}(\phi)|_\Omega = W(\phi)|_\Omega$ . Altogether we see that Lemma 2.6 may be applied; it yields that  $W(\phi)|_U \in W^{2,p}(U)^3$  for  $U \in \{\Omega, \Omega_R\}$ . Due to the assumption  $p > 3$  and a Sobolev inequality, it follows that the function  $W(\phi)|_U$  may be extended to a  $C^1$ -function in  $\bar{U}$ , for  $U$  as before, so  $W_{in}(\phi) \in C^1(\bar{\Omega})^3$  and  $W_{ex}(\phi)|_{\bar{\Omega}_R} \in C^1(\bar{\Omega}_R)^3$ . But  $W_{ex}(\phi)|_{\bar{\Omega}^c} = W(\phi)|_{\bar{\Omega}^c} \in C^\infty(\bar{\Omega}^c)^3$ , so  $W_{ex}(\phi) \in C^1(\Omega^c)^3$ .  $\square$

The next lemma indicates how for a given function  $b \in C^0(\partial\Omega)^3$ , a function  $\psi \in C^0(\partial\Omega)$  may be chosen so that the Dirichlet boundary data of  $V(b)|_\Omega$  and  $W(\psi)|_\Omega$  coincide. The same question is answered for the boundary values of  $V(b)|_{\bar{\Omega}^c}$  and  $W(\psi)|_{\bar{\Omega}^c}$ .

(lemma5.10)

**Lemma 4.10** *Let  $\phi, b \in C^0(\partial\Omega)^3$  with  $(1/2)(\mp\phi + \mathfrak{T}^*(\phi)) = b$ . Then  $V(\phi) \in C^\kappa(\mathbb{R}^3)^3$  for  $\kappa \in [0, 1)$  and*

$$(1/2) [\mp V(\phi)|\partial\Omega + \mathfrak{T}(V(\phi)|\partial\Omega)] = V(b)|\partial\Omega.$$

*Note that the term  $(1/2) [-V(\phi)|\partial\Omega + \mathfrak{T}(V(\phi)|\partial\Omega)]$  coincides with the Dirichlet boundary data of  $-W(V(\phi)|\partial\Omega)|_\Omega$ , and the function  $(1/2) [V(\phi)|\partial\Omega + \mathfrak{T}(V(\phi)|\partial\Omega)]$  with*

those of  $-W(V(\phi)|\partial\Omega)|\bar{\Omega}^c$  (Lemma 4.3, Theorem 4.5).

**Proof:** We consider the case  $(1/2)(-\phi + \mathfrak{T}^*(\phi)) = b$ . If  $(1/2)(\phi + \mathfrak{T}^*(\phi)) = b$ , an analogous reasoning is valid.

The relation  $V^{(\lambda)}(\phi)|\partial\Omega \in C^\kappa(\partial\Omega)^3$  for  $\kappa \in [0, 1)$  holds according to Lemma 4.2.

Recall the parameter  $\delta(\Omega) > 0$ , as well as the set  $\mathfrak{U}_\delta$  for  $\delta \in (0, \delta(\Omega)]$  from Lemma 3.2. Put  $V^{(-\delta)} := V^{(-\delta)}(\phi)$ ,  $Q^{(-\delta)} := Q^{(-\delta)}(\phi)$ . These functions were introduced in Lemma 4.8. Note that the set  $\mathfrak{U}_\delta \subset \mathbb{R}^3$  is open and  $\Omega^c \subset \mathfrak{U}_\delta$  (Lemma 3.2),  $V^{(-\delta)} \in C^\infty(\mathfrak{U}_\delta)^3$ ,  $Q^{(-\delta)} \in C^\infty(\mathfrak{U}_\delta)$ , and the pair  $(V^{(-\delta)}, Q^{(-\delta)})$  satisfies (1.1) in  $\mathfrak{U}_\delta$  with  $F = 0$ , for  $\delta \in (0, \delta(\Omega)]$  (Lemma 4.8).

In the following we will use the Stokes fundamental solution  $E$  defined in (4.1).

Consider  $\delta \in (0, \delta(\Omega)]$ . Choose  $R_0 \in (0, \infty)$  such that  $\bar{\Omega} \subset B_{R_0/2}$ . Let  $R \in [R_0, \infty)$ . We write  $n^{(\Omega_R)}$  for the outward unit normal to  $\Omega_R$ , that is,  $n^{(\Omega_R)}(z) = -n^{(\Omega)}(z)$  for  $z \in \partial\Omega$ ,  $n^{\Omega_R}(z) = R^{-1}z$  for  $z \in \partial B_R$ . Then it follows by a standard representation formula for solutions to (1.1) (see [8, (3.6)] for example) that

$$V_l^{(-\delta)}(x) = \int_{\partial\Omega_R} \sum_{j,k=1}^3 (E_{jl}(x-z)(\partial_k V_j^{(-\delta)} + \partial_j \tilde{V}_k^{(-\delta)} - \delta_{jk} Q^{(-\delta)})(z) - S_{jkl}(x-z)V_j^{(-\delta)}(z))n_k^{(\Omega_R)}(z) do_z \quad (4.18) \quad \boxed{\text{L5.10.11}}$$

for  $1 \leq l \leq 3$ ,  $x \in \Omega_R$ . Note that  $V^{(-\delta)}$  is continuous, so the restriction "a. e." on  $x \in \Omega_R$  in [8, (3.6)] may be dropped. Fix some  $x \in \bar{\Omega}^c$  and consider  $R \in [R_0, \infty)$  with  $x \in B_{R/2}$ . Since  $\bar{\Omega} \subset B_{R_0/2} \subset B_{R/2}$ , and because  $y - \delta n^{(\Omega)}(y) \in \Omega$  for  $y \in \partial\Omega$  (Lemma 3.2), we find for  $z \in \partial B_R$ ,  $y \in \partial\Omega$  that  $|z - (y - \delta n^{(\Omega)}(y))| \geq R/2$  and  $|x - z| \geq R/2$ . As a consequence, with (4.4), for  $z \in \partial B_R$ ,  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$ ,  $1 \leq j, l, k \leq 3$ ,

$$|\partial^\alpha V^{(-\delta)}(z)| \leq \mathfrak{C}R^{-1-|\alpha|}, \quad |Q^{(-\delta)}(z)| \leq \mathfrak{C}R^{-2}, \quad |\partial_z^\alpha(E_{jl}(x-z))| \leq \mathfrak{C}R^{-1-|\alpha|}, \\ |(\partial_l \mathfrak{N})(x-z)| + |S_{jkl}(x-z)| \leq \mathfrak{C}R^{-2}.$$

Thus, by letting  $R$  tend to infinity in (4.18), the integral over  $\partial B_R$  implicitly present in that equation tends to zero. Hence the integral over  $\partial\Omega_R$  becomes an integral over  $\partial\Omega$ , with  $n^{(\Omega_R)}$  replaced by  $-n^{(\Omega)}$ . Next we use (4.13), (4.16), (4.17) in order to let  $\delta$  tend to zero. In this way we get that

$$V(\phi)_l(x) = \int_{\partial\Omega} \left( \sum_{j=1}^3 -E_{jl}(x-z)(1/2)(-\phi + \mathfrak{T}^*(\phi))_j(z) + \sum_{j,k=1}^3 S_{jkl}(x-z)V(\phi)_j(z)n_k^{(\Omega)}(z) \right) do_z. \quad (4.19) \quad \boxed{\text{L5.10.20}}$$

We recall that  $(1/2)(-\phi + \mathfrak{T}^*(\phi)) = b$ . Abbreviate  $w := V(\phi)|\partial\Omega$ , and note that  $w \in C^0(\partial\Omega)^3$ ; see the first sentence of this proof. We may then rewrite (4.19) as

$$V(\phi)_l(x) = -V(b)_l(x) - W(w)_l(x), \quad (4.20) \quad \boxed{\text{L5.10.30}}$$

with  $W(w)$  defined in Lemma 4.3. This is true for any  $x \in \overline{\Omega}^c$ . Since  $w \in C^0(\partial\Omega)^3$ , Theorem 4.5 yields that the function  $W(w)$  may be extended continuously to  $\Omega^c$ . This extension is denoted by  $W_{ex}(w)$ ; see Theorem 4.5. By that theorem, we know that  $W_{ex}(w)(x) = (-1/2) (w + \mathfrak{T}(w))(x)$  for  $x \in \partial\Omega$ . Take  $x_0 \in \partial\Omega$ . Thus we may conclude that  $W(w)(x) \rightarrow (-1/2) (w + \mathfrak{T}(w))(x_0)$  for  $x \rightarrow x_0$ ,  $x \in \overline{\Omega}^c$ . On the other hand, since  $b, \phi \in C^0(\partial\Omega)^3$ , we know from Lemma 4.2 that  $V(b), V(\phi) \in C^0(\mathbb{R}^3)^3$ . Thus, by letting  $x$  tend to  $x_0$  in  $\overline{\Omega}^c$ , we may deduce from (4.20) that  $V(\phi)_l(x_0) = -V(b)(x_0) + (1/2) (w + \mathfrak{T}(w))(x_0)$ . But  $V(\phi)(x_0) = w(x_0)$  by the definition of  $w$ , so we finally arrive at the equation  $0 = -V(b)(x_0) + (1/2) (-w + \mathfrak{T}(w))(x_0)$ .  $\square$

## 5 $W^{2-1/p,p}$ -regularity of $V(\phi)$ .

In this section, we address the key element of our theory, that is, the fact that  $V(\phi)|_{\partial\Omega} \in W^{2-1/p,p}(\partial\Omega)^3$  if  $\phi \in W^{1-1/p,p}(\partial\Omega)^3$ . The proof of this relation constitutes the main difficulty we have to put up with, and is split into the proofs of the next two theorems. The result in the first – Theorem 5.1 – amounts to an  $W^{1,p}$ -estimate of  $V(\phi)|_{\partial\Omega}$  against the  $L^p$ -norm of  $\phi$ . (The function  $V(\phi)$  was introduced in Lemma 4.2.)

(theorem7.10) **Theorem 5.1** *Fix numbers  $k(\Omega) \in \mathbb{N}$ ,  $\alpha(\Omega) \in (0, \infty)$ , sets  $\Delta_\kappa, U_{t,\kappa}, \Lambda_{t,\kappa}$  and functions  $a_t, \gamma_t$  for  $\kappa \in (0, 1]$ ,  $1 \leq t \leq k(\Omega)$  as specified in Lemma 3.1.*

*Let  $t \in \{1, \dots, k(\Omega)\}$ . For  $f : \Delta_1 \mapsto \mathbb{C}^3$ , define  $\mathcal{Z}_t(f) : \partial\Omega \mapsto \mathbb{C}^3$  as the zero extension of  $f \circ (\gamma_t)^{-1} : \Lambda_{t,1} \mapsto \mathbb{C}^3$  to  $\partial\Omega$ . Fix a function  $\Psi_t \in C_0^\infty(U_{t,3/4})$  with  $\Psi_t|_{U_{t,1/2}} = 1$ .*

*Let  $p \in (1, \infty)$ . Then, for  $f \in L^p(\Delta_1)^3$ , the function  $\Psi_t \mathcal{Z}_t(f)$  belongs to  $L^p(\partial\Omega)^3$ , the function  $V(\Psi_t \mathcal{Z}_t(f)) \circ \gamma_t$  is in  $W^{1,p}(\Delta_t)^3$ , and*

$$\|V(\Psi_t \mathcal{Z}_t(f)) \circ \gamma_t\|_{1,p} \leq \mathfrak{E} \|f\|_p. \quad (5.1) \quad \boxed{\text{T7.10.1}}$$

**Proof:** Recall there is an orthonormal matrix  $D_t \in \mathbb{R}^{3 \times 3}$ , a vector  $C_t \in \mathbb{R}^3$  and a function  $a_t \in C^2(\overline{\Delta}_1)$  with  $\gamma_t(\eta) = D_t \cdot (\eta, a_t(\eta)) + C_t$  for  $\eta \in \Delta_t$ , and such that  $|\nabla a_t|_\infty < 1/4$ ; see Lemma 3.1. We have  $\Psi_t \in C_0^\infty(U_{t,3/4})$  by the choice of  $\Psi_t$  in the theorem, so  $\Psi_t \circ \gamma_t \in C_0^2(\Delta_{3/4})$  by the definition of  $U_{t,3/4}$  and  $\gamma_t$  in Lemma 3.1. In addition we will use the function  $J_t$  (surface element) introduced in Lemma 3.1, as well as the parameter  $\delta(\Omega)$ , which was fixed in Lemma 3.2. Let  $\sigma_0 \in (0, \infty)$  be so small that  $B_{\sigma_0}^2(\varrho) \subset \Delta_1$  for  $\varrho \in \Delta_{3/4}$ . We introduce some additional notation. For  $\varrho, \eta \in \Delta_1$ ,  $\tilde{\eta} \in \mathbb{R}^2$ ,  $\delta \in [0, \delta(\Omega)]$ , put

$$\begin{aligned} \Gamma(\varrho, \eta, \delta) &:= \Gamma_t(\varrho, \eta, \delta) := \gamma_t(\varrho) - \gamma_t(\eta) - \delta (n^{(\Omega)} \circ \gamma_t)(\eta), \\ \bar{\Gamma}(\varrho, \eta, \delta) &:= \bar{\Gamma}_t(\varrho, \eta, \delta) := \gamma_t(\varrho) - \delta (n^{(\Omega)} \circ \gamma_t)(\varrho) - \gamma_t(\eta), \\ \tilde{\Gamma}(\varrho, \tilde{\eta}) &:= \tilde{\Gamma}_t(\varrho, \tilde{\eta}) := D_t \cdot (\varrho - \tilde{\eta}, \nabla a_t(\varrho) \cdot (\varrho - \tilde{\eta})), \\ \mathfrak{E} &:= (\Psi_t \circ \gamma_t) J_t. \end{aligned}$$

Let  $f \in C^1(\Delta_1)^3 \cap L^p(\Delta_1)^3$ ,  $j \in \{1, 2, 3\}$  and  $\nu \in \{1, 2\}$ . Since  $\Psi_t \circ \gamma_t \in C_0^2(\Delta_{3/4})$ , we have  $\mathfrak{E} \in C_0^1(\Delta_{3/4})^3$ , so  $\mathfrak{E} f \in C_0^1(\Delta_{3/4})^3$ . In particular  $\mathfrak{E} f$  considered as a function with

domain  $\mathbb{R}^2$  belongs to  $C^\kappa(\mathbb{R}^2)^3$  for any  $\kappa \in [0, 1)$ , and to  $L^p(\mathbb{R}^2)^3$ , and we may define

$$\mathcal{F}(f)(\varrho, \eta) := \mathfrak{E}(\eta) f(\eta) - \mathfrak{E}(\varrho) f(\varrho) \text{ for } \varrho, \eta \in \mathbb{R}^2.$$

In addition  $(\gamma_t)^{-1} : \Lambda_{t,1} \mapsto \Delta_1$  is continuous (Lemma 3.1), so  $\Psi_t \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$  and  $\text{supp}(\Psi_t \mathcal{Z}_t(f)) \subset \Lambda_{t,3/4}$ . Note that  $(\Psi_t \mathcal{Z}_t(f)) \circ \gamma_t = (\Psi_t \circ \gamma_t) f$ , so due to equation (3.2),

$$\|\Psi_t \mathcal{Z}_t(f)\|_p = \|(\Psi_t \circ \gamma_t) f\|_p \leq \mathfrak{E} \|f\|_p, \quad (5.2) \quad \boxed{\text{T7.10.6}}$$

with  $\mathfrak{E}$  independent of  $f$ . We consider the function  $V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))$  introduced in Lemma 4.8. According to that reference, this function is  $C^\infty$  in an open set  $\mathfrak{U}_\delta$  containing  $\bar{\Omega}$  as a subset, and

$$\partial_l V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))_j(x) = \int_{\partial\Omega} \sum_{k=1}^3 (\partial_l E_{jk})(x - [y + \delta n^{(\Omega)}(y)]) (\Psi_t \mathcal{Z}_t(f))_k(y) dy \quad (5.3) \quad \boxed{\text{T7.10.3}}$$

for  $\delta \in (0, \delta(\Omega)]$ ,  $x \in \mathfrak{U}_\delta$ ,  $1 \leq l \leq 3$ , with  $(E_{jk})_{1 \leq j, k \leq 3}$  introduced in (4.1). Thus  $V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t$  is a  $C^1$ -function, and we get with (3.2) that

$$\begin{aligned} & \partial_\nu [V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t](\varrho) \\ &= \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} (\partial_l E_{jk})(\Gamma(\varrho, \eta, \delta)) f_k(\eta) \mathfrak{E}(\eta) d\eta = \sum_{\mu=1}^4 F^{(\mu)}(\varrho, \delta), \end{aligned}$$

for  $\varrho \in \Delta_1$ ,  $\delta \in (0, \delta(\Omega)]$ , with

$$\begin{aligned} F^{(1)}(\varrho, \delta) &:= \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} \left( (\partial_l E_{jk})(\Gamma(\varrho, \eta, \delta)) - (\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta)) \right) f_k(\eta) \mathfrak{E}(\eta) d\eta, \\ F^{(2)}(\varrho, \delta) &:= \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} (\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta)) \mathcal{F}(f)(\varrho, \eta)_k d\eta, \\ F^{(3)}(\varrho, \delta) &:= \sum_{k,l=1}^3 f_k(\varrho) J_t(\varrho) \int_{\Delta_1} (\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta)) \\ &\quad \left( \partial_\nu \gamma_t(\varrho)_l (\Psi_t \circ \gamma_t)(\varrho) - \partial_\nu \gamma_t(\eta)_l (\Psi_t \circ \gamma_t)(\eta) \right) d\eta, \\ F^{(4)}(\varrho, \delta) &:= - \sum_{k=1}^3 f_k(\varrho) J_t(\varrho) \int_{\Delta_1} E_{jk}(\bar{\Gamma}(\varrho, \eta, \delta)) \partial_\nu (\Psi_t \circ \gamma_t)(\eta) d\eta. \end{aligned}$$

Note that the definition of  $F^{(4)}(\varrho, \delta)$  involves a partial integration, which is possible due to (3.6). Let  $\zeta \in C_0^\infty(\Delta_1)$ . Since  $\Psi_t \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$ , as mentioned above, we may deduce from the uniform convergence in (4.16) that

$$\int_{\Delta_1} \partial_\nu \zeta(\varrho) [V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t](\varrho) d\varrho \rightarrow \int_{\Delta_1} \partial_\nu \zeta(\varrho) [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t](\varrho) d\varrho$$

for  $\delta \downarrow 0$ . Moreover, also because  $\Psi_t \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$ , Lemma 4.9 yields that the integral  $\int_{\Delta_1} \zeta(\varrho) F^{(1)}(\varrho, \delta) d\varrho$  tends to zero for  $\delta \downarrow 0$ . Since  $\mathfrak{E} f \in C_0^1(\Delta_{3/4})^3$ , as explained above, there is a constant  $c(f) > 0$  with

$$|\mathcal{F}(f)(\varrho, \eta)| \leq c(f) |\varrho - \eta| \quad \text{for } \varrho, \eta \in \Delta_1. \quad (5.4) \quad \boxed{\text{T7.10.20}}$$

In addition, by (4.4) and (3.4),  $|(\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta))| \leq \mathfrak{C} |\varrho - \eta|^{-2}$ , hence

$$|(\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta)) \mathcal{F}(f)(\varrho, \eta)_k| \leq \mathfrak{C} |\varrho - \eta|^{-1} \quad (5.5) \quad \boxed{\text{T7.10.30}}$$

for  $\varrho, \eta \in \Delta_1$  with  $\varrho \neq \eta$ ,  $1 \leq k, l \leq 3$ ,  $\delta \in [0, \delta(\Omega)]$ . We thus see by Lebesgue's theorem that the function  $F^{(2)}(\cdot, \delta) : \Delta_1 \mapsto \mathbb{R}$  is well defined and integrable also for  $\delta = 0$ , and

$$\int_{\Delta_1} \zeta(\varrho) F^{(2)}(\varrho, \delta) d\varrho \rightarrow \int_{\Delta_1} \zeta(\varrho) F^{(2)}(\varrho, 0) d\varrho \quad (\delta \downarrow 0).$$

Since  $\gamma_t$  belongs to  $C^2(\Delta_1)^3$  and has bounded derivatives, and because of (3.4) and the relation  $\Psi_t \circ \gamma_t \in C_0^2(\Delta_{3/4})$ , we have

$$\begin{aligned} |(\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta)) (\partial_\nu \gamma_t(\varrho)_l (\Psi_t \circ \gamma_t)(\varrho) - \partial_\nu \gamma_t(\eta)_l (\Psi_t \circ \gamma_t)(\eta))| &\leq \mathfrak{C} |\varrho - \eta|^{-1}, \\ |E_{jk}(\bar{\Gamma}(\varrho, \eta, \delta)) \partial_\nu (\Psi_t \circ \gamma_t)(\varrho)| &\leq \mathfrak{C} |\varrho - \eta|^{-1} \end{aligned} \quad (5.6) \quad \boxed{\text{T7.10.40}}$$

for  $\varrho, \eta \in \Delta_1$  with  $\varrho \neq \eta$ ,  $1 \leq k, l \leq 3$ ,  $\delta \in [0, \delta(\Omega)]$ . As a consequence, as in the case of  $F^{(2)}(\cdot, \delta)$ , the function  $F^{(\mu)}(\cdot, \delta) : \Delta_1 \mapsto \mathbb{R}$  for  $\mu \in \{3, 4\}$  is well defined and integrable also for  $\delta = 0$ , and

$$\int_{\Delta_1} \zeta(\varrho) F^{(\mu)}(\varrho, \delta) d\varrho \rightarrow \int_{\Delta_1} \zeta(\varrho) F^{(\mu)}(\varrho, 0) d\varrho \quad (\delta \downarrow 0).$$

Altogether we may now conclude that the weak derivative  $\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t]$  exists and

$$\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t](\varrho) = \sum_{\mu=2}^4 F^{(\mu)}(\varrho, 0) \quad \text{for } \varrho \in \Delta_1. \quad (5.7) \quad \boxed{\text{T7.10.70}}$$

We are going to transform  $F^{(2)}(\cdot, 0)$ . Recall the term  $\tilde{\Gamma}(\varrho, \eta)$  introduced at the beginning of this proof. We have

$$|\bar{\Gamma}(\varrho, \eta, 0) - \tilde{\Gamma}(\varrho, \eta)| = |a_t(\varrho) - a_t(\eta) - \nabla a_t(\varrho) \cdot (\varrho - \eta)| \leq \mathfrak{C} |\varrho - \eta|^2,$$

and  $|\bar{\Gamma}(\varrho, \eta, 0) + \vartheta (\bar{\Gamma}(\varrho, \eta, 0) - \tilde{\Gamma}(\varrho, \eta))| \geq |\varrho - \eta|$  for  $\vartheta \in [0, 1]$ ,  $\varrho, \eta \in \Delta_1$ , so with (4.4),

$$|(\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, 0)) - (\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta))| \leq \mathfrak{C} |\varrho - \eta|^{-1} \quad (5.8) \quad \boxed{\text{T7.10.50}}$$

for such  $\varrho, \eta \in \Delta_1$ ,  $\varrho \neq \eta$ , and for  $1 \leq k, l \leq 3$ . Since  $\mathfrak{E}f \in C_0^1(\Delta_{3/4})^3$ , we may thus define

$$G^{(1)}(\varrho) := \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} \left( (\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, 0)) - (\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta)) \right) f_k(\eta) \mathfrak{E}(\eta) d\eta,$$

$$G^{(2)}(\varrho) := - \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l f_k(\varrho) \mathfrak{E}(\varrho) \int_{\Delta_1} \left( (\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, 0)) - (\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta)) \right) d\eta$$

for  $\varrho \in \Delta_1$ . In view of (5.5), we may further define

$$G^{(3)}(\varrho) := \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{B_{\sigma_0}^2(\varrho)} (\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta)) \mathcal{F}(f)(\varrho, \eta)_k d\eta \quad (5.9) \quad \boxed{\text{T7.10.59}}$$

for  $\varrho \in \Delta_1$ . Since

$$\sup\{ |(\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta))| : \varrho, \eta \in \Delta_1, |\varrho - \eta| \geq \sigma_0, 1 \leq k, l \leq 3 \} < \infty, \quad (5.10) \quad \boxed{\text{T7.10.60}}$$

we may set

$$G^{(4)}(\varrho) := \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1 \setminus B_{\sigma_0}^2(\varrho)} (\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta)) f_k(\eta) \mathfrak{E}(\eta) d\eta,$$

$$G^{(5)}(\varrho) := - \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l f_k(\varrho) \mathfrak{E}(\varrho) \int_{\Delta_1 \setminus B_{\sigma_0}^2(\varrho)} (\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta)) d\eta$$

for  $\varrho \in \Delta_1$ . Then

$$F^{(2)}(\varrho, 0) = \sum_{\mu=1}^5 G^{(\mu)}(\varrho) \quad \text{for } \varrho \in \Delta_1. \quad (5.11) \quad \boxed{\text{T7.10.80}}$$

Concerning this equation, note that the domain of integration  $B_{\sigma_0}^2(\varrho)$  in the definition of  $G^{(3)}(\varrho)$  may be replaced by  $\Delta_1 \cap B_{\sigma_0}^2(\varrho)$ , because  $\mathfrak{E}_t f \in C_0^1(\Delta_{3/4})^3$  and due to the choice of  $\sigma_0$  at the beginning of this proof. In view of (5.7) and (5.11), let us estimate the terms  $G^{(\mu)}(\varrho)$  for  $\mu \in \{1, \dots, 5\}$ , as well as  $F^{(3)}(\varrho, 0)$  and  $F^{(4)}(\varrho, 0)$ . The function  $G^{(3)}$  is by far the most difficult to handle since it hides a singular integral. Following [3, (2.1)] (where the term  $(\varrho - \eta)_1^s (\varrho - \eta)_2^{2\nu-s}$  is lacking), we write  $G^{(3)}$  as a series. To this end we recall that  $|\nabla a_t|_\infty \leq 1/4$  by the specifications on  $a_t$  in Lemma 3.1. As a consequence  $|\nabla a_t(\varrho) \cdot (\varrho - \eta)|/|\varrho - \eta| \leq 1/2$  for  $\varrho, \eta \in \Delta_1$  with  $\varrho \neq \eta$ . Hence for  $\tau \in \mathbb{N}$ ,  $\varrho, \eta \in \Delta_1$  with  $\varrho \neq \eta$ ,

$$|\tilde{\Gamma}(\varrho, \eta)|^{-\tau} = |\varrho - \eta|^{-\tau} \left( 1 + (\nabla a_t(\varrho) \cdot (\varrho - \eta))^2 / |\varrho - \eta|^2 \right)^{-\tau/2} \quad (5.12) \quad \boxed{\text{T7.10.90}}$$

$$= \sum_{m=0}^{\infty} \binom{-\tau/2}{m} \sum_{n=0}^{2m} \binom{2m}{n} \partial_1 a_t(\varrho)^n \partial_2 a_t(\varrho)^{2m-n} (\varrho - \eta)_1^n (\varrho - \eta)_2^{2m-n} |\varrho - \eta|^{-\tau-2m}.$$

On the other hand by the definition in (4.1)

$$(\partial_l E_{jk})(z) = (-\delta_{jk} z_l + \delta_{jl} z_k + \delta_{kl} z_j) |z|^{-3} - 3 z_j z_k z_l |z|^{-5} \quad (5.13) \quad \boxed{\text{T7.10.100}}$$

for  $1 \leq k, l \leq 3$ ,  $z \in \mathbb{R}^3 \setminus \{0\}$ . We combine (5.12) and (5.13). To this end we put

$$\mathfrak{A}(\varrho) := D_t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 a_t(\varrho) & \partial_2 a_t(\varrho) \end{pmatrix} \quad \text{for } \varrho \in \Delta_1,$$

with  $D_t$  introduced in Lemma 3.1. Then  $\tilde{\Gamma}(\varrho, \eta) = \mathfrak{A}(\varrho) \cdot (\varrho - \eta)$  ( $\varrho, \eta \in \Delta_t$ ). Put

$$\begin{aligned} \mathfrak{Z}(\varrho)_{klr} &:= -\delta_{jk} \mathfrak{A}(\varrho)_{lr} + \delta_{jl} \mathfrak{A}(\varrho)_{kr} + \delta_{kl} \mathfrak{A}(\varrho)_{jr}, \\ \tilde{\mathfrak{Z}}(\varrho)_{kl\alpha} &:= -3 \mathfrak{A}(\varrho)_{j\alpha_1} \mathfrak{A}(\varrho)_{k\alpha_2} \mathfrak{A}(\varrho)_{l\alpha_3}(\varrho) \end{aligned}$$

for  $k, l \in \{1, 2, 3\}$ ,  $r \in \{1, 2\}$ ,  $\alpha \in \{1, 2\}^3$ ,  $\varrho \in \Delta_t$ . Then we get from (5.13) that

$$\begin{aligned} &(\partial_l E_{jk})(\tilde{\Gamma}(\varrho, \eta)) \quad (5.14) \quad \boxed{\text{T7.10.110}} \\ &= \sum_{r=1}^2 \mathfrak{Z}(\varrho)_{klr} (\varrho - \eta)_r |\tilde{\Gamma}(\varrho, \eta)|^{-3} + \sum_{\alpha \in \{1, 2\}^3} \tilde{\mathfrak{Z}}(\varrho)_{kl\alpha} \prod_{s=1}^3 (\varrho - \eta)_{\alpha_s} |\tilde{\Gamma}(\varrho, \eta)|^{-5} \end{aligned}$$

for  $k, l, \varrho$  as before, and for  $\eta \in \mathbb{R}^2$  with  $\varrho \neq \eta$ . Further put

$$\mathfrak{B}(m, n, r, \kappa) := \kappa_r \kappa_1^n \kappa_2^{2m-n} |\kappa|^{-3-2m}, \quad \tilde{\mathfrak{B}}(m, n, \alpha, \kappa) := \prod_{s=1}^3 \kappa_{\alpha_s} \kappa_1^n \kappa_2^{2m-n} |\kappa|^{-5-2m}$$

for  $r, \alpha$  as above,  $m \in \mathbb{N}$ ,  $n \in \{0, \dots, 2m\}$ ,  $\kappa \in \mathbb{R}^2 \setminus \{0\}$ ,

$$\begin{aligned} \mathfrak{W}(m, n, \varrho) &:= \binom{-3/2}{m} \binom{2m}{n} \partial_1 a_t(\varrho)^n \partial_2 a_t(\varrho)^{2m-n}, \\ \tilde{\mathfrak{W}}(m, n, \varrho) &:= \binom{-5/2}{m} \binom{2m}{n} \partial_1 a_t(\varrho)^n \partial_2 a_t(\varrho)^{2m-n}, \end{aligned}$$

for  $m, n$  as before and  $\varrho \in \Delta_t$ . Then by (5.9), (5.12) and (5.14), it follows that  $G^{(3)}(\varrho) = G^{(3,1)}(\varrho) + G^{(3,2)}(\varrho)$ , with

$$G^{(3,1)}(\varrho) \quad (5.15) \quad \boxed{\text{T7.10.115}}$$

$$= \sum_{k,l=1}^3 \sum_{r=1}^2 \partial_\nu \gamma_t(\varrho)_l \mathfrak{Z}(\varrho)_{klr} \int_{B_{\sigma_0}^2(\varrho)} \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \mathfrak{W}(m, n, \varrho) \mathfrak{B}(m, n, r, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k d\eta,$$

$$G^{(3,2)}(\varrho) = \sum_{k,l=1}^3 \sum_{\alpha \in \{1, 2\}^3} \partial_\nu \gamma_t(\varrho)_l \tilde{\mathfrak{Z}}(\varrho)_{kl\alpha} \int_{B_{\sigma_0}^2(\varrho)} \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \tilde{\mathfrak{W}}(m, n, \varrho) \tilde{\mathfrak{B}}(m, n, \alpha, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k d\eta$$

for  $\varrho \in \Delta_1$ . Since  $|\nabla a_t|_\infty \leq 1/4$ , we obtain with (5.4) that

$$\begin{aligned} & \left| \sum_{n=0}^{2m} \mathfrak{W}(m, n, \varrho) \mathfrak{B}(m, n, r, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k \right| \\ & \leq \left| \binom{-3/2}{m} \right| (1/4)^{2m} \sum_{n=0}^{2m} \binom{2m}{n} |\varrho - \eta|^{-2} |\mathcal{F}(f)(\varrho, \eta)_k| \leq \mathfrak{C} \left| \binom{-3/2}{m} \right| (1/2)^{2m} |\varrho - \eta|^{-1} \end{aligned} \quad (5.16) \quad \boxed{\text{T7.10.120}}$$

for  $m \in \mathbb{N}$ ,  $1 \leq k \leq 3$ ,  $1 \leq r \leq 2$ ,  $\varrho, \eta \in \Delta_1$  with  $\varrho \neq \eta$ . We thus see that the integral in the definition of  $G^{(3,1)}(\varrho)$  may be moved inside the sum with respect to  $m \in \mathbb{N}$ . In this way we arrive at the integral  $\int_{B_{\sigma_0}^2(\varrho)} \mathfrak{B}(m, n, r, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k d\eta$  for  $m, k, r$  as before and for  $0 \leq n \leq 2m$ ,  $\varrho \in \Delta_1$ . Since  $|\mathfrak{B}(m, n, r, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k| \leq \mathfrak{C} |\varrho - \eta|^{-1}$  for  $m, k, r, \varrho, \eta$  as in (5.16) and for  $0 \leq n \leq 2m$ , as follows from (5.4), we obtain

$$\begin{aligned} & \int_{B_{\sigma_0}^2(\varrho) \setminus B_\sigma^2(\varrho)} \mathfrak{B}(m, n, r, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k d\eta \\ & \rightarrow \int_{B_{\sigma_0}^2(\varrho)} \mathfrak{B}(m, n, r, \varrho - \eta) \mathcal{F}(f)(\varrho, \eta)_k d\eta \quad (\sigma \downarrow 0), \quad \text{uniformly in } \varrho \in \Delta_1. \end{aligned} \quad (5.17) \quad \boxed{\text{T7.10.130}}$$

But  $\int_{B_{\sigma_0}^2(\varrho) \setminus B_\sigma^2(\varrho)} \mathfrak{B}(m, n, r, \varrho - \eta) d\eta = 0$  for  $\varrho \in \mathbb{R}^2$ ,  $\sigma \in (0, \sigma_0)$ ,  $m, k, r, n$  as before, so we see that the term  $\mathcal{F}(f)(\varrho, \eta)_k$  may be replaced by  $f_k(\eta) \mathfrak{E}(\eta)$  in the integral on the left-hand side of (5.17). Thus with (5.15) and (5.17), we finally arrive at the equation

$$\begin{aligned} G^{(3,1)}(\varrho) &= \sum_{k,l=1}^3 \sum_{r=1}^2 \partial_\nu \gamma_t(\varrho)_l \mathfrak{Z}(\varrho)_{klr} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \mathfrak{W}(m, n, \varrho) \lim_{\sigma \downarrow 0} \int_{B_{\sigma_0}^2(\varrho) \setminus B_\sigma(\varrho)} \mathfrak{B}(m, n, r, \varrho - \eta) f_k(\eta) \mathfrak{E}(\eta) d\eta, \end{aligned} \quad (5.18) \quad \boxed{\text{T7.10.131}}$$

for  $\varrho \in \Delta_1$ , with the limit in this equation being uniform with respect to such  $\varrho$ . An analogous reasoning yields that

$$\begin{aligned} G^{(3,2)}(\varrho) &= \sum_{k,l=1}^3 \sum_{\alpha \in \{1,2\}^3} \partial_\nu \gamma_t(\varrho)_l \tilde{\mathfrak{Z}}(\varrho)_{kl\alpha} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{2m} \tilde{\mathfrak{W}}(m, n, \varrho) \lim_{\sigma \downarrow 0} \int_{B_{\sigma_0}^2(\varrho) \setminus B_\sigma(\varrho)} \tilde{\mathfrak{B}}(m, n, \alpha, \varrho - \eta) f_k(\eta) \mathfrak{E}(\eta) d\eta \end{aligned} \quad (5.19) \quad \boxed{\text{T7.10.131b}}$$

for  $\varrho$  as before, where the limit appearing in this equation is again uniform with respect to  $\varrho \in \Delta_1$ . We note that  $\int_{\partial B_1^2} |\mathfrak{B}(m, n, r, \kappa)| d\sigma_\kappa \leq \int_{\partial B_1^2} |\kappa|^{-2} d\sigma_\kappa = 2\pi$  for  $m \in \mathbb{N}$ ,  $n \in \{0, \dots, 2m\}$ ,  $r \in \{1, 2\}$ , and  $\sum_{m=0}^{\infty} \sum_{n=0}^{2m} |\mathfrak{W}(m, n, \varrho)| \leq \sum_{m=0}^{\infty} \left| \binom{-3/2}{m} \right| (1/2)^{2m}$  for  $\varrho \in \Delta_1$ ; compare (5.16). As explained in the passage preceding (5.2), we have  $\mathfrak{C} f \in L^p(\mathbb{R}^2)^3$ .

At this point Theorem 2.1 allows us to deduce from (5.18) that  $\|G^{(3,1)}\|_p \leq \tilde{\mathfrak{C}} \|\mathfrak{E}f\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$ . Here and in the rest of this proof, constants independent of  $f$  are denoted by  $\tilde{\mathfrak{C}}$ . In an analogous way equation (5.19) leads to the estimate  $\|G^{(3,2)}\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$ , so we finally arrive at the estimate  $\|G^{(3)}\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$ .

All the other relevant functions may be estimated in a rather straightforward way. By (5.6) we see that the absolute value of the integral in the definition of  $F^{(3)}(m, 0)$  and  $F^{(4)}(\cdot, 0)$  is bounded uniformly in  $\varrho \in \Delta_1$ . It follows that  $\|F^{(\mu)}(\cdot, 0)\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$  for  $\mu \in \{3, 4\}$ . An analogous argument, based on (5.8) and (5.10), respectively, instead of (5.6), yields that  $\|G^{(\mu)}\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$  for  $\mu \in \{2, 5\}$ . As for  $G^{(4)}$ , we may use (5.10) to obtain

$$|G^{(4)}(\varrho)| \leq \tilde{\mathfrak{C}} \int_{\Delta_1 \setminus B_{\sigma_0}(\varrho)} |f(\eta)| |\mathfrak{E}(\eta)| d\eta \leq \tilde{\mathfrak{C}} \|f\|_1 \leq \tilde{\mathfrak{C}} \|f\|_p.$$

for  $\varrho \in \Delta_1$ , so that  $\|G^{(4)}\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$ . Concerning  $G^{(1)}$ , inequality (5.8) provides that  $|G^{(1)}(\varrho)| \leq \tilde{\mathfrak{C}} \int_{\Delta_1} |\varrho - \eta|^{-1} |f(\eta)| |\mathfrak{E}(\eta)| d\eta$  for  $\varrho \in \Delta_1$ , so that  $\|G^{(1)}\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$  by Lemma 2.1.

At this point it follows from the representations in (5.7), (5.11) and the previous estimates of the terms  $\|F^{(3)}(\cdot, 0)\|_p$ ,  $\|F^{(4)}(\cdot, 0)\|_p$  and  $\|G^{(\mu)}\|_p$  for  $\mu \in \{1, \dots, 5\}$  that

$$\|\partial_\nu [V(\Psi \mathcal{Z}_t(f))_j \circ \gamma_t]\|_p \leq \tilde{\mathfrak{C}} \|f\|_p \quad \text{for } f \in L^p(\Delta_1)^3 \cap C^1(\Delta_1)^3, j \in \{1, 2, 3\} \quad (5.20) \quad \boxed{\text{T7.10.55}}$$

and  $\nu \in \{1, 2\}$ . Take  $f \in L^p(\Delta_1)^3$ . Obviously inequality (5.2) remains valid for such  $f$ , so Lemma 2.2 implies that  $\|V(\Psi \mathcal{Z}_t(f))\|_p \leq \tilde{\mathfrak{C}} \|\Psi \mathcal{Z}_t(f)\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$ , hence by (3.2),  $\|V(\Psi \mathcal{Z}_t(f)) \circ \gamma_t\|_p \leq \|V(\Psi \mathcal{Z}_t(f))|_{\Lambda_{t,1}}\|_p \leq \|V(\Psi \mathcal{Z}_t(f))\|_p \leq \tilde{\mathfrak{C}} \|f\|_p$ . The theorem follows from this estimate, inequality(5.20) and the density of  $L^p(\Delta_1)^3 \cap C^1(\Delta_1)^3$  in  $L^p(\Delta_1)^3$ .  $\square$

In the next theorem we give a bound of  $V(\phi)|\partial\Omega$  in the norm of  $W^{2-1/p,p}(\partial\Omega)^3$  in terms of the  $W^{1-1/p,p}$ -norm of  $\phi$ . The proof of this inequality is based on the preceding theorem.

(theorem7.20) **Theorem 5.2** *Let  $p \in (1, \infty)$ . Then  $V(\phi)|\partial\Omega \in W^{2-1/p,p}(\partial\Omega)^3$  and  $\|V(\phi)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|\phi\|_{1-1/p,p}$  for  $\phi \in W^{1-1/p,p}(\partial\Omega)^3$ .*

**Proof:** The notation introduced in Theorem 5.1 or at the beginning of the proof of this theorem, up to inequality (5.2), will again be used here, without further notice.

Let  $t \in \{1, \dots, k(\Omega)\}$ ,  $j \in \{1, 2, 3\}$ ,  $f \in L^p(\Delta_1)^3 \cap C^1(\Delta_1)^3$  and  $\nu \in \{1, 2\}$ . Recall that  $\Psi_t \mathcal{Z}_t(f) \in C^0(\partial\Omega)$ . All constants  $\mathfrak{C}$  appearing in this proof are independent of  $f$ .

Let  $\delta \in (0, \delta(\Omega)]$ . Consider the function  $V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))$  introduced in Lemma 4.8. As stated in that lemma, this function is  $C^\infty$  in an open set  $\mathfrak{U}_\delta$  containing  $\bar{\Omega}$  as a subset; see (5.3) as concerns its first order derivatives. Thus by (3.2) and because  $\text{supp}(\Psi_t \mathcal{Z}_t(f)) \subset$

$U_{t,3/4}$  and  $(\Psi_t \mathcal{Z}_t(f)) \circ \gamma_t = (\Psi_t \circ \gamma_t) f$ , we get for  $\varrho \in \Delta_1$  that

$$\begin{aligned} & \partial_\nu [V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t](\varrho) \\ &= \sum_{k,l=1}^3 \partial_\nu \gamma_t(\varrho)_l \int_{\Delta_1} (\partial_l E_{jk})(\Gamma(\varrho, \eta, \delta)) f_k(\eta) \mathfrak{E}(\eta) d\eta = \sum_{\mu=1}^4 H^{(\mu)}(\varrho, \delta), \end{aligned} \quad (5.21) \quad \boxed{\text{T7.20.20b}}$$

with

$$\begin{aligned} H^{(1)}(\varrho, \delta) &:= \sum_{k,l=1}^3 \int_{\Delta_1} (\partial_l E_{jk})(\Gamma(\varrho, \eta, \delta)) (\partial_\nu \gamma_t(\varrho) - \partial_\nu \gamma_t(\eta))_l f_k(\eta) \mathfrak{E}(\eta) d\eta, \\ H^{(2)}(\varrho, \delta) &:= \sum_{k,l=1}^3 \int_{\Delta_1} \left( (\partial_l E_{jk})(\Gamma(\varrho, \eta, \delta)) - (\partial_l E_{jk})(\bar{\Gamma}(\varrho, \eta, \delta)) \right) \\ &\quad \partial_\nu \gamma_t(\eta)_l f_k(\eta) \mathfrak{E}(\eta) d\eta, \\ H^{(3)}(\varrho, \delta) &:= - \sum_{k=1}^3 \int_{\Delta_1} E_{jk}(\bar{\Gamma}(\varrho, \eta, \delta)) f_k(\eta) \partial_\nu \mathfrak{E}(\eta) d\eta, \\ H^{(4)}(\varrho, \delta) &:= - \sum_{k=1}^3 \int_{\Delta_1} E_{jk}(\bar{\Gamma}(\varrho, \eta, \delta)) \partial_\nu f_k(\eta) \mathfrak{E}(\eta) d\eta. \end{aligned}$$

The last two functions arise due to a partial integration with respect to  $\eta$  and because  $\mathfrak{E} \in C_0^1(\Delta_1)$ . Let  $\zeta \in C_0^\infty(\Delta_1)$ . We deduce from the relation  $\Psi_t \mathcal{Z}_t(f) \in C^0(\partial\Omega)^3$  (see further above) and from the uniform convergence in (4.16) that

$$\int_{\Delta_1} \partial_\nu \zeta(\varrho) \left( V^{(\delta)}(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t \right)(\varrho) d\varrho \rightarrow \int_{\Delta_1} \partial_\nu \zeta(\varrho) \left( V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t \right)(\varrho) d\varrho$$

for  $\delta \downarrow 0$ . Lemma 4.9, equation (3.2) and the relation  $\mathfrak{E} \in C_0^1(\Delta_1)$  yield that the integral  $\int_{\Delta_1} \zeta(\varrho) H^{(2)}(\varrho, \delta) d\varrho$  tends to zero for  $\delta \downarrow 0$ . The function  $\gamma_t$  belongs to  $C^2(\bar{\Delta}_1)$  (Lemma 3.1), so with (4.4) and (3.4),

$$|\partial_\varrho^\alpha [(\partial_l E_{jk})(\Gamma(\varrho, \eta, \delta)) (\partial_\nu \gamma_t(\varrho) - \partial_\nu \gamma_t(\eta))_l]| \leq \mathfrak{C} |\varrho - \eta|^{-1-|\alpha|} \quad (5.22) \quad \boxed{\text{T7.20.30}}$$

for  $1 \leq k, l \leq 3$ ,  $\varrho, \eta \in \Delta_1$  with  $\varrho \neq \eta$ ,  $\delta \in [0, \delta(\Omega)]$ ,  $\alpha \in \mathbb{N}_0^2$  with  $|\alpha| \leq 1$ . Since  $\mathfrak{E} \in C_0^1(\Delta_1)$ , we have  $\mathfrak{E} f \in C_0^1(\Delta_1)^3$ , in particular  $|\mathfrak{E} f|_\infty < \infty$ , so we may conclude from (5.22) with  $\alpha = 0$  and from Lebesgue's theorem that the function  $H^{(1)}(\cdot, \delta) : \Delta_1 \mapsto \mathbb{C}$  is well defined and integrable also for  $\delta = 0$ , and  $\int_{\Delta_1} \zeta(\varrho) H^{(1)}(\varrho, \delta) d\varrho \rightarrow \int_{\Delta_1} \zeta(\varrho) H^{(1)}(\varrho, 0) d\varrho$  for  $\delta \downarrow 0$ . We further deduce from (4.4) and (3.4) that

$$|\partial_\varrho^\alpha [E_{jk}(\bar{\Gamma}(\varrho, \eta, \delta))]| \leq \mathfrak{C} |\varrho - \eta|^{-1-|\alpha|} \quad \text{for } k, \varrho, \eta, \delta, \alpha \text{ as in (5.22)}. \quad (5.23) \quad \boxed{\text{T7.20.40}}$$

Taking into account that  $\mathfrak{E} \partial_\nu f$  and  $\partial_\nu \mathfrak{E} f$  belong to  $C_0^0(\Delta_{3/4})^3$ , we see that due to (5.23) with  $\alpha = 0$ , the function  $H^{(\mu)}(\cdot, \delta)$  for  $\mu \in \{3, 4\}$  is well defined and integrable also if

$\delta = 0$ , and  $\int_{\Delta_1} \zeta(\varrho) H^{(\mu)}(\varrho, \delta) d\varrho \rightarrow \int_{\Delta_1} \zeta(\varrho) H^{(\mu)}(\varrho, 0) d\varrho$  ( $\delta \downarrow 0$ ) for such  $\mu$ . At this point we may deduce from (5.21) that the weak derivative  $\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t]$  exists – a fact already known from Theorem 5.1 – and

$$\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t](\varrho) = \sum_{\mu \in \{1, 3, 4\}} H^{(\mu)}(\varrho, 0) \quad \text{for } \varrho \in \Delta_1. \quad (5.24) \quad \boxed{\text{T7.20.50}}$$

Now consider  $f \in L^p(\Delta_1)^3$ . Recalling that  $\partial^\alpha \mathfrak{E} \in C_0^0(\Delta_{3/4})$  for  $\alpha \in \mathbb{N}_0^2$  with  $|\alpha| \leq 1$ , we observe that  $\partial^\alpha \mathfrak{E} f \in L^p(\Delta_1)^3$  and  $\|\partial^\alpha \mathfrak{E} f\|_p \leq \mathfrak{C} \|f\|_p$ . It follows from (5.22), (5.23) with  $\alpha = 0$ ,  $\delta = 0$  and from Lemma 2.1 that if  $\mu \in \{1, 3\}$ , the function  $H^{(\mu)}(\cdot, 0)$  is well defined also with  $f$  as given now, that is,  $f \in L^p(\Delta_1)^3$ , and the estimate  $\|H^{(\mu)}(\cdot, 0)\|_p \leq \mathfrak{C} \|f\|_p$  holds. We recall that according to Theorem 5.1, the weak derivative  $\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t]$  exists also in the case  $f \in L^p(\Delta_1)^3$  considered presently, and inequality (5.1) is valid for this  $f$ . Define

$$\tilde{H}^{(4)} := -H^{(1)}(\cdot, 0) - H^{(3)}(f)(\cdot, 0) + \partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t]. \quad (5.25) \quad \boxed{\text{T7.20.2}}$$

In view of (5.1) and the estimate  $\|H^{(\mu)}(\cdot, 0)\|_p \leq \mathfrak{C} \|f\|_p$  for  $\mu \in \{1, 3\}$  derived above, we see that  $\tilde{H}^{(4)} \in L^p(\Delta_1)^3$  and  $\|\tilde{H}^{(4)}\|_p \leq \mathfrak{C} \|f\|_p$ .

Next take  $f \in W^{1,p}(\Delta_1)^3 \cap C^2(\Delta_1)^3$ . We have  $\tilde{H}^{(4)} = H^{(4)}(\cdot, 0)$  by (5.24), and  $H^{(4)}(\cdot, 0) = -V(\Psi_t \mathcal{Z}_t(\partial_\nu f))_j \circ \gamma_t$  by (3.2). At this point we may refer to Theorem 5.1 to obtain that  $\tilde{H}^{(4)} \in W^{1,p}(\Delta_1)$  and  $\|\partial_r \tilde{H}^{(4)}\|_p \leq \mathfrak{C} \|\partial_\nu f\|_p$  for  $r \in \{1, 2\}$ .

Since we have now shown that  $\|\tilde{H}^{(4)}\|_p \leq \mathfrak{C} \|f\|_p$  for  $f \in L^p(\Delta_1)^3$  and  $\tilde{H}^{(4)} \in W^{1,p}(\Delta_1)^3$ ,  $\|\tilde{H}^{(4)}\|_{1,p} \leq \mathfrak{C} \|f\|_{1,p}$  for  $f \in W^{1,p}(\Delta_1)^3 \cap C^2(\Delta_1)^3$ , we may conclude that  $\tilde{H}^{(4)} \in W^{1,p}(\Delta_1)^3$  and  $\|\tilde{H}^{(4)}\|_{1,p} \leq \mathfrak{C} \|f\|_{1,p}$  for  $f \in W^{1,p}(\Delta_1)^3$ . Therefore interpolation implies that  $\tilde{H}^{(4)} \in W^{1-1/p,p}(\Delta_1)^3$  and  $\|\tilde{H}^{(4)}\|_{1-1/p,p} \leq \mathfrak{C} \|f\|_{1-1/p,p}$  for  $f \in W^{1-1/p,p}(\Delta_1)^3$ .

From (5.22), (5.23) and Theorem 2.2 we obtain that  $H^{(\mu)}(\cdot, 0)$  for  $\mu \in \{1, 3\}$  belongs to  $W^{1-1/p,p}(\Delta_1)^3$  and  $\|H^{(\mu)}(\cdot, 0)\|_{1-1/p,p} \leq \mathfrak{C} \|f\|_p$  if  $f \in L^p(\Delta_1)^3$ . At this point we may refer to equation (5.25) to conclude that  $\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t] \in W^{1-1/p,p}(\Delta_1)$  and  $\|\partial_\nu [V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t]\|_{1-1/p,p} \leq \mathfrak{C} \|f\|_{1-1/p,p}$  for any  $f \in W^{1-1/p,p}(\Delta_1)^3$ ,  $\nu \in \{1, 2\}$ . It follows with Theorem 5.1 that  $V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t$  belongs to  $W^{2-1/p,p}(\Delta_1)$  for  $f \in W^{1-1/p,p}(\Delta_1)^3$ , and  $\|V(\Psi_t \mathcal{Z}_t(f))_j \circ \gamma_t\|_{2-1/p,p} \leq \mathfrak{C} \|f\|_{1-1/p,p}$  for such  $f$ .

Now let  $\phi \in W^{1-1/p,p}(\partial\Omega)^3$ . The constants  $\mathfrak{C}$  appearing in the following are independent of  $\phi$ . Then  $\phi \circ \gamma_t \in W^{1-1/p,p}(\Delta_t)^3$  and  $\mathcal{Z}_t(\phi \circ \gamma_t)|_{\Lambda_{t,1}} = \phi|_{\Lambda_{t,1}}$ . Since  $\text{supp}(\Psi_t) \cap \partial\Omega \subset \Lambda_{t,3/4}$ , we see that  $\Psi_t \mathcal{Z}_t(\phi \circ \gamma_t) = \Psi_t \phi$ , so we now obtain that  $V(\Psi_t \phi)_j \circ \gamma_t \in W^{2-1/p,p}(\Delta_1)$  and

$$\|V(\Psi_t \phi)_j \circ \gamma_t\|_{2-1/p,p} \leq \mathfrak{C} \|\phi \circ \gamma_t\|_{1-1/p,p}. \quad (5.26) \quad \boxed{\text{T7.20.80}}$$

Next we note that  $\text{dist}(\partial\Omega \setminus \Lambda_{t,1/2}, \Lambda_{t,1/4}) > 0$  (Lemma 3.1),  $\text{supp}((1 - \Psi_t)|_{\partial\Omega}) \subset \partial\Omega \setminus \Lambda_{t,1/2}$  and  $\gamma_t(\varrho) \in \Lambda_{t,1/4}$  for  $\varrho \in \Delta_{1/4}$ . It follows with Lebesgue's theorem that

$V((1 - \Psi_t)\phi)_j \circ \gamma_t|_{\Delta_{1/4}} \in C^2(\Delta_{1/4})$  and

$$\begin{aligned} |\partial^\alpha [V((1 - \Psi_t)\phi)_j \circ \gamma_t](\varrho)| &= \left| \int_{\partial\Omega} \sum_{k=1}^3 \partial_\varrho^\alpha [E_{jk}(\gamma_t(\varrho) - y)] (1 - \Psi_t)(y) \phi_k(y) d\varrho_y \right| \\ &\leq \mathfrak{C} \|\phi\|_1 \quad \text{for } \varrho \in \Delta_{1/4}, \alpha \in \mathbb{N}_0^2 \text{ with } |\alpha| \leq 2. \end{aligned}$$

As a consequence  $\|V((1 - \Psi_t)\phi)_j \circ \gamma_t|_{\Delta_{1/4}}\|_{2,p} \leq \mathfrak{C} \|\phi\|_p$ , and therefore

$$\|V((1 - \Psi_t)\phi)_j \circ \gamma_t|_{\Delta_{1/4}}\|_{2-1/p,p} \leq \mathfrak{C} \|\phi\|_p.$$

This estimate combined with (5.26) yields that  $V(\phi)_j \circ \gamma_t|_{\Delta_{1/4}} \in W^{2-1/p,p}(\Delta_{1/4})$  and  $\|V(\phi)_j \circ \gamma_t|_{\Delta_{1/4}}\|_{2-1/p,p} \leq \mathfrak{C} (\|\phi \circ \gamma_t\|_{1-1/p,p} + \|\phi\|_p) \leq \mathfrak{C} \|\phi\|_{1-1/p,p}$ . Since  $j, t, \phi$  were chosen arbitrarily in  $\{1, 2, 3\}$ ,  $\{1, \dots, k\}$  and  $W^{1-1/p,p}(\partial\Omega)^3$ , respectively, the theorem follows with (3.1).  $\square$

The consequence of Theorem 5.2 we are interested in is stated as

Corollary 100.50)

**Corollary 5.1** *Let  $p \in (1, \infty)$ . For  $b \in W^{1-1/p,p}(\partial\Omega)^3$ ,  $\phi \in E_p^{(\mp)}$  with  $(1/2)(\mp\phi + \mathfrak{T}^*(\phi)) = b$ , the relations  $V(\phi)|_{\partial\Omega} \in W^{2-1/p,p}(\partial\Omega)^3$  and  $\|V(\phi)|_{\partial\Omega}\|_{2-1/p,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}$  hold. (The space  $E_p^{(\mp)}$  was introduced in Corollary 4.3.)*

**Proof:** Take  $b$  and  $\phi$  as in the corollary, and consider the case  $(1/2)(\phi + \mathfrak{T}^*(\phi)) = b$ . All the constants  $\mathfrak{C}$  appearing in the following are independent of  $b$ . We have in particular  $b \in \text{ran}(I_p + T_p^*)$ , so by Corollary 4.2  $\int_{\partial\Omega} \phi^{(j)} \cdot b d\varrho_x = 0$  for  $1 \leq j \leq 6$ . Since in addition  $b \in W^{1-1/p,p}(\partial\Omega)^3$ , we may conclude with Lemma 4.7 there is a sequence  $(b_n)$  in  $C^0(\partial\Omega)^3$  with  $b_n \in C^a(\partial\Omega)^3 \cap W^{1-1/p,p}(\partial\Omega)^3$ ,  $\int_{\partial\Omega} \phi^{(j)} \cdot b_n d\varrho_x$  for  $n \in \mathbb{N}$ ,  $a \in (0, 1)$ ,  $1 \leq j \leq 6$ , and such that  $\|b - b_n\|_{1-1/p,p} \rightarrow 0$ .

Let  $n \in \mathbb{N}$ . It follows with Corollary 4.2 that  $b_n \in \text{ran}(I_p + T_p^*)$ , so there is a unique function  $\phi_n \in E_p^{(+)}$  with  $(1/2)(\phi_n + \mathfrak{T}^*(\phi_n)) = b_n$  for  $n \in \mathbb{N}$ ; see Corollary 4.3. Theorem 4.2 yields in particular that  $\phi_n \in C^a(\partial\Omega)^3$  for  $a \in (0, 1)$ . Now we may conclude from Lemma 4.10 that

$$(1/2) [V(\phi_n)|_{\partial\Omega} + \mathfrak{T}(V(\phi_n)|_{\partial\Omega})] = V(b_n)|_{\partial\Omega}. \quad (5.27) \quad \boxed{\text{C100.50.10}}$$

Since  $\|b_n - b\|_p \rightarrow 0$ , we know by (4.7) that  $\|V(b_n) - V(b)|_{\partial\Omega}\|_p \rightarrow 0$ . But

$$\|\phi_n - \phi\|_p \leq \mathfrak{C} \|\phi_n - \phi + \mathfrak{T}^*(\phi_n - \phi)\|_p = \mathfrak{C} \|b_n - b\|_p \quad \text{for } n \in \mathbb{N} \quad (5.28) \quad \boxed{\text{C100.50.5}}$$

according to Corollary 4.3, so  $\|\phi_n - \phi\|_p \rightarrow 0$ , hence  $\|V(\phi_n - \phi)|_{\partial\Omega}\|_p \rightarrow 0$  by (4.7). Now it follows from the boundedness of  $T_p$  (Lemma 4.5) that

$$\|V(\phi_n - \phi)|_{\partial\Omega} + \mathfrak{T}(V(\phi_n - \phi)|_{\partial\Omega})\|_p \rightarrow 0.$$

Altogether we deduce from (5.27) that  $(1/2) [V(\phi)|_{\partial\Omega} + \mathfrak{T}(V(\phi)|_{\partial\Omega})] = V(b)|_{\partial\Omega}$ .

Since  $b_n \in W^{1-1/p,p}(\partial\Omega)^3$ , Theorem 5.2 yields that  $V(b_n)|\partial\Omega \in W^{2-1/p,p}(\partial\Omega)^3$  ( $n \in \mathbb{N}$ ) and

$$\|V(b_n - b_m)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|b_n - b_m\|_{1-1/p,p}, \quad \|V(b_n)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|b_n\|_{1-1/p,p}, \quad (5.29) \quad \boxed{\text{C100.50.6}}$$

for  $m, n \in \mathbb{N}$ . By Lemma 4.2 we further have  $V(b_n)|\partial\Omega \in C^a(\partial\Omega)^3$  for  $0 < a < 1$ ,  $n \in \mathbb{N}$ , so it follows with (5.27) and Theorem 4.3 that the functions  $V(\phi_n - \phi_m)|\partial\Omega$  and  $V(\phi_n)|\partial\Omega$  belong to  $W^{2-1/p,p}(\partial\Omega)^3$  and

$$\begin{aligned} \|V(\phi_n - \phi_m)|\partial\Omega\|_{2-1/p,p} &\leq \mathfrak{C} (\|V(b_n - b_m)|\partial\Omega\|_{2-1/p,p} + \|\phi_n - \phi_m\|_p), \\ \|V(\phi_n)|\partial\Omega\|_{2-1/p,p} &\leq \mathfrak{C} (\|V(b_n)|\partial\Omega\|_{2-1/p,p} + \|\phi_n\|_p) \quad (m, n \in \mathbb{N}). \end{aligned}$$

Due to (5.29) and because of (5.28) and an analogous inequality for  $\|\phi_n\|_p$ , we thus obtain that

$$\|V(\phi_n - \phi_m)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|b_n - b_m\|_{1-1/p,p}, \quad \|V(\phi_n)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|b_n\|_{1-1/p,p} \quad (5.30) \quad \boxed{\text{C100.50.20}}$$

for  $m, n \in \mathbb{N}$ . The first estimate in (5.30) implies there is  $\gamma \in W^{2-1/p,p}(\partial\Omega)^3$  such that  $\|V(\phi_n) - \gamma\|_{2-1/p,p} \rightarrow 0$ . Since  $\|V(\phi_n - \phi)|\partial\Omega\|_p \rightarrow 0$ , as explained following (5.28), we may conclude that  $V(\phi)|\partial\Omega \in W^{2-1/p,p}(\partial\Omega)^3$  and  $\|V(\phi_n - \phi)\|_{2-1/p,p} \rightarrow 0$ . In addition  $\|b_n - b\|_{1-1/p,p} \rightarrow 0$  by the choice of the sequence  $(b_n)$ . At this point the second estimate in (5.30) yields that  $\|V(\phi)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}$ .

Analogous arguments are valid if  $(1/2)(-\phi + \mathfrak{T}^*(\phi)) = b$  if we note that the function  $\phi_p^{(0)}$  introduced in Corollary 4.3 belongs to  $C^a(\partial\Omega)^3$  for  $a \in (0, 1)$  by Theorem 4.2.  $\square$

## 6 Existence and $W^{2,p}$ -regularity of solutions to (1.1), (1.2).

In the ensuing theorem we consider (1.1) with  $F = 0$  (homogeneous Stokes system) and (1.2) with Neumann data satisfying a side condition. This theorem combined with Corollary 6.1 below state that in such a situation, problem (1.1), (1.2) admits a  $W^{2,p}$ -regular solution in the exterior domain  $\bar{\Omega}^c$ . This result is proved by reducing it to Corollary 5.1 and to the  $L^p$ -theory of the Stokes system in bounded domains under Dirichlet boundary conditions.

(theorem7.30)

**Theorem 6.1** *Let  $p \in (1, \infty)$  and  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ ,  $b \in W^{1-1/p,p}(\partial\Omega)^3$  with  $\int_{\partial\Omega} \phi^{(0)} \cdot b \, do_x = 0$ . Abbreviate  $\phi := F^-(b)$ ,  $V := V(\phi)|\bar{\Omega}^c$ ,  $Q := Q(\phi)|\bar{\Omega}^c$ .*

*(The functions  $\phi^{(0)}$  and  $F^-(b)$  were introduced in Corollary 4.3, and the functions  $V(\phi)$  and  $Q(\phi)$  in Lemma 4.2.)*

*Then  $V \in C^\infty(\bar{\Omega}^c)^3$ ,  $Q \in C^\infty(\bar{\Omega}^c)$ , the pair  $(V, Q)$  solves (1.1) in  $\bar{\Omega}^c$  with  $F = 0$ . Let*

$r_1 \in [1, 3p/2)$ ,  $r_2 \in (3, \infty)$ ,  $r_3 \in (3/2, \infty)$ ,  $r_4 \in (1, \infty)$ . Then

$$\|V|\Omega_R\|_{1,r_1} + \|Q|\Omega_R\|_{r_1} \leq \mathfrak{C} \|b\|_p, \quad (6.1) \quad \boxed{\text{T7.30.20}}$$

$$\|V|\Omega_R\|_{2,p} + \|Q|\Omega_R\|_{1,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}, \quad (6.2) \quad \boxed{\text{T7.30.30}}$$

$$\|V|B_R^c\|_{r_2} \leq \mathfrak{C} \|b\|_p, \quad \|\partial_m V|B_R^c\|_{r_3} + \|Q|B_R^c\|_{r_3} \leq \mathfrak{C} \|b\|_p \text{ for } 1 \leq m \leq 3, \quad (6.3) \quad \boxed{\text{T7.30.40}}$$

$$\|\partial_l \partial_m V|B_R^c\|_{r_4} + \|\partial_m Q|B_R^c\|_{r_4} \leq \mathfrak{C} \|b\|_p \text{ for } 1 \leq l, m \leq 3. \quad (6.4) \quad \boxed{\text{T7.30.60}}$$

In particular, if  $r \in (3, 3p/2)$ , then  $\|\partial_m V\|_r + \|Q\|_r \leq \mathfrak{C} \|b\|_p$  for  $1 \leq m \leq 3$ . All the constants  $\mathfrak{C}$  appearing in the preceding estimates are independent of  $b$ .

**Proof:** The relations  $V \in C^\infty(\overline{\Omega}^c)^3$ ,  $Q \in C^\infty(\overline{\Omega}^c)$  and equation (1.1) with  $(V, Q)$  in the place of  $(U, \Pi)$  and with  $F = 0$  are valid according to Lemma 4.2. Moreover Corollary 4.1 yields that  $\|V(\phi)|\Omega_R\|_{1,r_1} + \|Q(\phi)|\Omega_R\|_{r_1} \leq \mathfrak{C} \|\phi\|_p$ . Here and in the following, the constants denoted by  $\mathfrak{C}$  do not depend on  $\phi$  or  $b$ . By Lemma 4.4, we have

$$\begin{aligned} \|V(\phi)|B_R^c\|_{r_2} &\leq \mathfrak{C} \|\phi\|_p, & \|\partial_m V(\phi)|B_R^c\|_{r_3} + \|Q(\phi)|B_R^c\|_{r_3} &\leq \mathfrak{C} \|\phi\|_p, \\ \|\partial_l \partial_m V(\phi)|B_R^c\|_{r_4} + \|\partial_m Q(\phi)|B_R^c\|_{r_4} &\leq \mathfrak{C} \|\phi\|_p \text{ for } 1 \leq l, m \leq 3. \end{aligned}$$

On the other hand, due to Corollary 4.3 and by the definition of  $\phi$ , the relations

$$(1/2) (-\phi + \mathfrak{T}^*(\phi)) = b, \quad \|\phi\|_p \leq \mathfrak{C} \|b\|_p \quad (6.5) \quad \boxed{\text{T7.30.100}}$$

hold. The preceding inequalities imply (6.1) and (6.3) – (6.4).

Let us show (6.2). From (6.5) and Corollary 5.1, we find that  $V(\phi)|\partial\Omega \in W^{2-1/p,p}(\partial\Omega)^3$  and

$$\|V(\phi)|\partial\Omega\|_{2-1/p,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}. \quad (6.6) \quad \boxed{\text{T7.30.110}}$$

Recall that  $V(\phi)(x)$  for  $x \in \partial\Omega$  was given by a direct definition in Lemma 4.2. On the other hand, inequality (6.1) means in particular that  $V(\phi)|\Omega_R \in W^{1,p}(\Omega_R)^3$ , and by the last statement in Lemma 4.8, we know that the trace of  $V(\phi)|\Omega_R$  on  $\partial\Omega$  coincides with  $V(\phi)|\partial\Omega$  as defined in Lemma 4.2. Moreover estimates (6.3) – (6.4) yield in particular that  $V(\phi)|B_{2R} \setminus \overline{B_R} \in W^{2,p}(B_{2R} \setminus \overline{B_R})^3$  and  $\|V(\phi)|B_{2R} \setminus \overline{B_R}\|_{2,p} \leq \mathfrak{C} \|\phi\|_p$ , so the  $C^\infty$ -regularity of  $V$  in  $\overline{\Omega}^c$  mentioned above and a standard trace theorem yield that  $V(\phi)|\partial B_R \in W^{2-1/p,p}(\partial B_R)^3$  and  $\|V(\phi)|\partial B_R\|_{2-1/p,p} \leq \mathfrak{C} \|\phi\|_p$ . Therefore we may conclude with (6.6) that  $V(\phi)|\partial\Omega_R \in W^{2-1/p,p}(\partial\Omega_R)^3$  and

$$\|V(\phi)|\partial\Omega_R\|_{2-1/p,p} \leq \mathfrak{C} (\|b\|_{1-1/p,p} + \|\phi\|_p) \leq \mathfrak{C} \|b\|_{1-1/p,p}, \quad (6.7) \quad \boxed{\text{T7.30.120}}$$

where the last inequality follows from (6.5). Since  $\text{div}(V(\phi)|\mathbb{R}^3 \setminus \partial\Omega) = 0$ , and again by the relation  $V(\phi)|\Omega_R \in W^{1,p}(\Omega_R)^3$ , we get that  $\int_{\partial\Omega_R} V(\phi)(x) \cdot n^{(\Omega,R)}(x) d\sigma_x = 0$ , with  $n^{(\Omega,R)}$  denoting the outward unit normal to  $\Omega_R$ . At this point we may apply Theorem 2.5, which yields functions  $u \in W^{2,p}(\Omega_R)^3$ ,  $\pi \in W^{1,p}(\Omega_R)$  with  $-\Delta u + \nabla \pi = 0$ ,  $u|_{\partial\Omega_R} = V(\phi)|_{\partial\Omega_R}$ ,  $\int_{\Omega_R} \pi dx = 0$  and  $\|u\|_{2,p} + \|\pi\|_{1,p} \leq \mathfrak{C} \|V(\phi)|_{\partial\Omega_R}\|_{2-1/p,p}$ . The latter inequality

and (6.7) imply that  $\|u\|_{2,p} + \|\pi\|_{1,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}$ . But  $u = V(\phi)|_{\Omega_R}$  and  $\pi = Q(\phi)|_{\Omega_R} + c$ , with  $c := -|\Omega_R|^{-1} \int_{\Omega_R} Q(\phi)(x) dx$ . This follows from Theorem 2.6 and the properties of  $V$  and  $Q$  stated at the beginning of this proof, and because  $V(\phi)|_{\Omega_R} \in W^{1,p}(\Omega_R)^3$  and  $Q(\phi)|_{\Omega_R} \in L^p(\Omega_R)$  according to (6.1). Thus inequality (6.2) is proved.  $\square$

ollary100.60)

**Corollary 6.1** *Consider the same situation as in Theorem 6.1. In particular recall the notation  $V := V(\phi)|_{\overline{\Omega}^c}$ ,  $Q := Q(\phi)|_{\overline{\Omega}^c}$ . Moreover write  $(\partial_j V)^{tr}$  and  $Q^{tr}$  for the trace of  $V$  and  $Q$  on  $\partial\Omega$  ( $1 \leq j \leq 3$ ). Then*

$$\sum_{k=1}^3 n_k^{(\Omega)} \left( (\partial_j V)_k^{tr} + (\partial_k V)_j^{tr} - \delta_{jk} Q^{tr} \right) = b_j \quad \text{for } 1 \leq j \leq 3. \quad (6.8) \quad \boxed{\text{C100.60.80}}$$

**Proof:** Since  $(1/2) (-\phi + \mathfrak{T}^*(\phi)) = b$  by the definition of  $\phi$  and the choice of  $F^-(b)$ , we have  $\phi \in \text{ran}(-I_p + T_p^*)$ , so by Corollary 4.2 and Lemma 4.7, we may choose a sequence  $(b_n)$  in  $C^0(\partial\Omega)^3$  such that  $b_n \in C^a(\partial\Omega)^3 \cap W^{1-1/r,r}(\partial\Omega)^3$  and  $\int_{\partial\Omega} \phi^{(0)} \cdot b_n do_x = 0$  ( $n \in \mathbb{N}$ ,  $a \in (0, 1)$ ,  $r \in (1, \infty)$ ), and  $\|b_n - b\|_{1-1/p,p} \rightarrow 0$ .

Let  $n \in \mathbb{N}$ . Since  $\int_{\partial\Omega} \phi^{(0)} \cdot b_n do_x = 0$ , the function  $\phi_n := F^-(b_n) \in E_p^-$  from Corollary 4.3 is well defined. By definition it satisfies the equation  $(1/2) (-\phi_n + \mathfrak{T}^*(\phi_n)) = b_n$ . Let  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_{R/2}$ . By our choice of  $b_n$ , we have  $b_n \in W^{1-1/r,r}(\partial\Omega)^3$  for  $r \in (1, \infty)$ , so Theorem 6.1 implies that  $V(\phi_n)|_{\Omega_R} \in W^{2,r}(\Omega_R)^3$ ,  $Q(\phi_n)|_{\Omega_R} \in W^{1,r}(\Omega_R)$  for such  $r$ . It follows by a Sobolev inequality that  $V(\phi_n)|_{\Omega_R}$  may be continuously extended to a function from  $C^1(\overline{\Omega}_R)^3$ , and  $Q(\phi_n)|_{\Omega_R}$  admits a continuous extension to  $\overline{\Omega}_R$ . Since both  $V(\phi_n)$  and  $Q(\phi_n)$  are  $C^\infty$  in  $\overline{\Omega}^c$  (Lemma 4.2), we may conclude there are functions  $V_n \in C^1(\Omega^c)^3$  and  $Q_n \in C^0(\Omega^c)$  such that  $V(\phi_n)|_{\overline{\Omega}^c} = V_n|_{\overline{\Omega}^c}$  and  $Q(\phi_n)|_{\overline{\Omega}^c} = Q_n|_{\overline{\Omega}^c}$ . On the other hand, since  $b_n \in C^a(\partial\Omega)^3$  for  $a \in (0, 1)$ , Theorem 4.2 yields in particular that  $\phi_n \in C^0(\partial\Omega)^3$ . Recalling that  $\partial_j V_n \in C^0(\Omega^c)^3$ ,  $Q_n \in C^0(\Omega^c)$ ,  $V_n|_{\overline{\Omega}^c} = V(\phi_n)|_{\overline{\Omega}^c}$  and  $Q_n|_{\overline{\Omega}^c} = Q(\phi_n)|_{\overline{\Omega}^c}$ , we may thus deduce from Theorem 4.4 and the equation  $(1/2) (-\phi_n + \mathfrak{T}^*(\phi_n)) = b$  that

$$\sum_{k=1}^3 n_k^{(\Omega)}(x) (\partial_j V_{n,k} + \partial_k V_{n,j} - \delta_{jk} Q_n)(x) = b_{n,j}(x) \quad \text{for } x \in \partial\Omega, 1 \leq j \leq 3. \quad (6.9) \quad \boxed{\text{C100.60.10}}$$

Here  $n$  was arbitrary from  $\mathbb{N}$ . But by our choice of  $(b_n)$  and  $(\phi_n)$ , Theorem 6.1 yields that

$$\|V(\phi_n - \phi)|_{\Omega_R}\|_{2,p} + \|Q(\phi_n - \phi)|_{\Omega_R}\|_{1,p} \leq \mathfrak{C} \|b_n - b\|_{1-1/p,p} \quad \text{for } n \in \mathbb{N}.$$

Thus, using the notation for the trace of  $V(\phi)|_{\overline{\Omega}^c}$  and  $Q(\phi)|_{\overline{\Omega}^c}$ , respectively, introduced in the corollary, and taking into account that  $\partial_j V_n \in C^0(\Omega^c)^3$ ,  $Q_n \in C^0(\Omega^c)$ , we get that

$$\|(\partial_j V)^{tr} - \partial_j V_n\|_p + \|Q^{tr} - Q_n\|_p \leq \mathfrak{C} \|b_n - b\|_{1-1/p,p} \quad (n \in \mathbb{N}, 1 \leq j \leq 3).$$

Since  $\|b_n - b\|_{1-1/p,p} \rightarrow 0$  by the choice of the sequence  $(b_n)$ , equation (6.8) now follows from (6.9).  $\square$

It is well known (see [23]) that the side condition imposed on the boundary data  $b$  in the preceding corollary may be eliminated by using the double layer potentials from Lemma 4.3. In order to check how this steps works out in our  $L^p$ -theory, we give some details of a proof.

theorem100.60)

**Theorem 6.2** Consider the function  $\phi^{(0)}$  from Corollary 4.3. There is  $a \in (0, 1)$  with  $\phi^{(0)} \in C^{1,a}(\partial\Omega)^3$ . Moreover  $\int_{\partial\Omega} \phi^{(0)} \cdot n^{(\Omega)} d\sigma_x \neq 0$ , and there is  $c_0 = c_0(\phi^{(0)}) \in \mathbb{R}^3 \setminus \{0\}$  such that

$$\sum_{j,k=1}^3 n_k^{(\Omega)}(x) (\partial_j W(\phi^{(0)})_k + \partial_k W(\phi^{(0)})_j - \delta_{jk} \Pi(\phi^{(0)})) (x - \kappa n^{(\Omega)}(x)) \rightarrow -c_0 n_j^{(\Omega)}(x) (\kappa \downarrow 0)$$

for  $1 \leq j \leq 3$ ,  $x \in \partial\Omega$ . (The functions  $W(\phi^{(0)})$  and  $\Pi(\phi^{(0)})$  were introduced in Lemma 4.3.) Let  $p \in (1, \infty)$ ,  $b \in W^{1-1/p,p}(\partial\Omega)^3$ . Put

$$\gamma(b) := - \int_{\partial\Omega} \phi^{(0)} \cdot b d\sigma_x \left( \int_{\partial\Omega} \phi^{(0)} \cdot n^{(\Omega)} d\sigma_x \right)^{-1}, \quad \tilde{b} := b + \gamma(b) n^{(\Omega)}.$$

Then  $\int_{\partial\Omega} \phi^{(0)} \cdot \tilde{b} d\sigma_x = 0$ . Put  $\phi := F^{-1}(\tilde{b})$ , with  $F^{-1}(\tilde{b})$  from Corollary 4.3,

$$u := V(\phi) + c_0^{-1} \gamma(b) W(\phi^{(0)})|_{\overline{\Omega}^c}, \quad \pi := Q(\phi) + c_0^{-1} \gamma(b) \Pi(\phi^{(0)})|_{\overline{\Omega}^c}.$$

(See Lemma 4.2 for the definition of  $V(\phi)$  and  $Q(\phi)$ .)

Then  $u \in C^\infty(\overline{\Omega}^c)^3$ ,  $\pi \in C^\infty(\overline{\Omega}^c)$ , and the pair  $(u, \pi)$  solves (1.1) in  $\overline{\Omega}^c$  with  $F = 0$ .

Let  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ ,  $r_1 \in [1, 3p/3)$ ,  $r_2 \in (3, \infty)$ ,  $r_3 \in (3/2, \infty)$ ,  $r_4 \in (1, \infty)$ . Then, with constants  $\mathfrak{C}$  independent of  $b$ ,

$$\|u|_{\Omega_R}\|_{r_1} \leq \mathfrak{C} \|b\|_p, \quad \|u|_{\Omega_R}\|_{2,p} + \|\pi|_{\Omega_R}\|_{1,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}, \quad (6.10) \quad \boxed{\text{T100.60.10a}}$$

$$\|u|_{B_R^c}\|_{r_2} \leq \mathfrak{C} \|b\|_p, \quad \|\partial_n u|_{B_R^c}\|_{r_3} + \|\pi|_{B_R^c}\|_{r_3} \leq \mathfrak{C} \|b\|_p \text{ for } 1 \leq n \leq 3, \quad (6.11) \quad \boxed{\text{T100.60.30a}}$$

$$\|\partial_m \partial_n u|_{B_R^c}\|_{r_4} + \|\partial_n \pi|_{B_R^c}\|_{r_4} \leq \mathfrak{C} \|b\|_p \text{ for } 1 \leq m, n \leq 3, \quad (6.12) \quad \boxed{\text{T100.60.50a}}$$

in particular  $\|\partial_m \partial_n u\|_p + \|\partial_n \pi\|_p \leq \mathfrak{C} \|b\|_p$ . The pair  $(u, \pi)$  solves (1.2) in the trace sense.

**Proof:** By the choice of  $\phi^{(0)}$  in Corollary 4.3, we have  $-\phi^{(0)} + \mathfrak{T}(\phi^{(0)}) = 0$ . Thus we may apply Theorem 4.7 with  $b = 0$  and  $\phi = \phi^{(0)}$ , and with  $p = r$  for any  $r \in (3, \infty)$ . Abbreviating  $W := W(\phi^{(0)})$ ,  $W_{in} := W_{in}(\phi^{(0)})$ ,  $W_{ex} := W_{ex}(\phi^{(0)})$ ,  $\Pi := \Pi(\phi^{(0)})$ ,  $\Pi_{in} := \Pi_{in}(\phi^{(0)})$  and  $\Pi_{ex} := \Pi_{ex}(\phi^{(0)})$ , this theorem combined with Theorem 4.6 and Lemma 4.3 allow us to conclude that  $\phi^{(0)} \in C^{1,a}(\partial\Omega)^3$  for some  $a \in (0, 1)$ ,  $\phi^{(0)} \in W^{2-1/p,p}(\partial\Omega)^3$ ,

$$W_{in} \in C^1(\overline{\Omega})^3, \quad W_{ex} \in C^1(\Omega^c)^3, \quad \Pi_{in} \in C^0(\overline{\Omega}), \quad \Pi_{ex} \in C^0(\Omega^c), \quad (6.13) \quad \boxed{\text{T100.60.10}}$$

$$W_{in}|_{\partial\Omega} = (-1/2) (-\phi^{(0)} + \mathfrak{T}(\phi^{(0)})) = 0, \quad W_{ex}|_{\partial\Omega} = (-1/2) (\phi^{(0)} + \mathfrak{T}(\phi^{(0)})) = -\phi^{(0)},$$

$$W_{in}|_{\Omega} = W|_{\Omega} \in W^{2,p}(\Omega)^3, \quad W_{ex}|_{\overline{\Omega}^c} = W|_{\overline{\Omega}^c}, \quad W \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)^3, \quad \Pi \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega),$$

$$\Delta W + \nabla \Pi = 0, \quad \operatorname{div} W = 0, \quad \Pi_{in}|_{\Omega} = \Pi|_{\Omega} \in W^{1,p}(\Omega), \quad \Pi_{ex}|_{\overline{\Omega}^c} = \Pi|_{\overline{\Omega}^c},$$

$$W|_{\Omega_R} \in W^{2,p}(\Omega_R)^3, \quad \Pi|_{\Omega_R} \in W^{1,p}(\Omega_R).$$

(The functions  $W_{in}(\phi^{(0)})$  and  $W_{ex}(\phi^{(0)})$  were introduced in Theorem 4.5, and  $\Pi_{in}(\phi^{(0)})$  and  $\Pi_{ex}(\phi^{(0)})$  in Theorem 4.7. The parameter  $R$  was fixed in the theorem above.) Theorem

4.5 and the relation  $\phi^{(0)} \in C^{1,a}(\partial\Omega)^3$  provide the equation

$$\begin{aligned} & \sum_{k=1}^3 n_k^{(\Omega)} (\partial_j W_{in,k} + \partial_k W_{in,j} - \delta_{jk} \Pi_{in}) \\ &= \sum_{k=1}^3 n_k^{(\Omega)} (\partial_j W_{ex,k} + \partial_k W_{ex,j} - \delta_{jk} \Pi_{ex}) \quad \text{for } 1 \leq j \leq 3. \end{aligned} \quad (6.14) \quad \boxed{\text{T100.60.20}}$$

Among the relations in (6.13), we next use that  $W_{in} \in C^1(\bar{\Omega})^3$ ,  $W_{in}|_{\partial\Omega} = 0$ ,  $W_{in}|_{\Omega} = W|_{\Omega} \in W^{2,p}(\Omega)^3$ ,  $\Pi_{in} \in C^0(\bar{\Omega})$ ,  $\Pi_{in}|_{\Omega} = \Pi|_{\Omega} \in L^p(\Omega)$ ,  $W \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)^3$ ,  $\Pi \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$ ,  $\Delta W + \nabla \Pi = 0$ ,  $\text{div } W = 0$ . From this and Theorem 2.6, it follows that  $W_{in} = 0$ , hence  $\partial_j W_{in} = 0$  ( $1 \leq j \leq 3$ ) and  $\nabla \Pi|_{\Omega} = 0$ . But  $\Omega$  is a domain and  $\Pi$  is in particular continuous, so there is  $c_0 = c_0(\phi^{(0)}) \in \mathbb{R}$  with  $\Pi|_{\Omega} = c_0$ , hence  $\Pi_{in} = c_0$ . Thus the left-hand side of (6.14) equals  $-c_0 n_j^{(\Omega)}$ , and therefore also its right-hand side, so that

$$\sum_{k=1}^3 n_k^{(\Omega)} (\partial_j W_{ex,k} + \partial_k W_{ex,j} - \delta_{jk} \Pi_{ex}) = -c_0 n_j^{(\Omega)} \quad (1 \leq j \leq 3). \quad (6.15) \quad \boxed{\text{T100.60.25}}$$

Now suppose for a contradiction that

$$c_0 = 0 \quad \text{or} \quad \int_{\partial\Omega} \phi^{(0)} \cdot n^{(\Omega)} d\sigma_x = 0. \quad (6.16) \quad \boxed{\text{T100.60.30}}$$

Then let  $S \in [R, \infty)$ . We use Lemma 2.5 with  $U = \Omega_S$ ,  $u = \tilde{u} = W_{ex}|_{\Omega_S}$ ,  $\pi = \Pi_{ex}$ . This choice is possible according to (6.13). Also according to (6.13), we have  $W_{ex}|_{\partial\Omega} = -\phi^{(0)}$ . Thus Lemma 2.5), equation (6.15) and assumption (6.16) imply that

$$\int_{\Omega_S} \sum_{j,k=1}^3 |\partial_j W_k + \partial_k W_j|^2 dx = \int_{\partial B_S} \sum_{j,k=1}^3 (x_k/S) (\partial_j W_k + \partial_k W_j - \delta_{jk} \Pi)(x) W_j(x) d\sigma_x.$$

But the surface integral on the right-hand side of preceding equation tends to zero for  $S \rightarrow \infty$  due to (4.8), (4.9). It follows that  $\int_{\bar{\Omega}^c} \sum_{j,k=1}^3 |\partial_j W_k + \partial_k W_j|^2 dx = 0$ . This means that  $\partial_j W_k + \partial_k W_j|_{\bar{\Omega}^c} = 0$  for  $1 \leq j, k \leq 3$ . Turning to Theorem 2.4, we now conclude there are numbers  $\alpha_1, \dots, \alpha_6 \in \mathbb{R}$  such that  $W|_{\bar{\Omega}^c} = \sum_{j=1}^6 \alpha_j \phi^{(j)}|_{\bar{\Omega}^c}$ . If there were an index  $j \in \{1, \dots, 6\}$  with  $\alpha_j \neq 0$ , we might choose a sequence  $(x_n)$  in  $\bar{\Omega}^c$  such that  $|x_n| \rightarrow \infty$  and  $\sum_{j=1}^6 \alpha_j \phi^{(j)}(x_n) \not\rightarrow 0$ . For example, if  $\alpha_6 \neq 0$ , a suitable choice would be  $x_n = (0, R+n, 0)$  for  $n \in \mathbb{N}$ . But on the other hand, for any sequence  $(x_n)$  in  $\bar{\Omega}^c$  with  $|x_n| \rightarrow \infty$ , inequality (4.8) implies  $|W(x_n)| \rightarrow 0$ . Thus we may conclude that  $\alpha_j = 0$  for  $1 \leq j \leq 6$ , hence  $W|_{\bar{\Omega}^c} = 0$ , and so  $W_{ex} = 0$ . But  $W_{ex}|_{\partial\Omega} = -\phi^{(0)}$ , so  $\phi^{(0)} = 0$ , in contradiction to the choice of  $\phi^{(0)}$  in Corollary 4.3. Thus none of the equations in (6.16) can be true. As a consequence  $c_0 \neq 0$  and  $\int_{\partial\Omega} \phi^{(0)} \cdot n^{(\Omega)} d\sigma_x \neq 0$ .

Since  $\Omega$  is  $C^2$ -bounded, we have  $n^{(\Omega)} \in C^a(\partial\Omega)^3$  for any  $a \in (0, 1)$ . This means in particular that  $n^{(\Omega)} \in W^{1-1/p,p}(\partial\Omega)^3$ , so  $\tilde{b} \in W^{1-1/p,p}(\partial\Omega)^3$ . (The function  $\tilde{b}$  was introduced in the

theorem.) Obviously  $\int_{\partial\Omega} \phi^{(0)} \cdot \tilde{b} \, do_x = 0$ . Recall that  $\phi = F^-(\tilde{b})$ , with  $F^-(\tilde{b})$  from Corollary 4.3; see the definitions in the theorem. Therefore we see that all statements of of Theorem 6.1 and Corollary 6.1 are true with  $\tilde{b}$  in the role of  $b$ . But  $|\gamma(b)| \leq \mathfrak{C} \|b\|_p$ , so we find that

$$\|\tilde{b}\|_p \leq \mathfrak{C} \|b\|_p (1 + \|n^{(\Omega)}\|_p) \leq \mathfrak{C} \|b\|_p, \quad (6.17) \quad \boxed{\text{T100.60.50}}$$

$$\|\tilde{b}\|_{1-1/p,p} \leq \mathfrak{C} (\|b\|_{1-1/p,p} + |\gamma(b)| \|n^{(\Omega)}\|_{1-1/p,p}) \leq \mathfrak{C} (\|b\|_{1-1/p,p} + \|b\|_p) \leq \mathfrak{C} \|b\|_{1-1/p,p}.$$

Thus the estimates in Theorem 6.1 are valid as stated, that is, with the function  $b$  on their right-hand side. Recall that in (6.13) we noted in particular that

$$\begin{aligned} W_{ex} \in C^1(\Omega^c)^3, \quad W_{ex}|_{\overline{\Omega}^c} = W|\overline{\Omega}^c \in C^\infty(\overline{\Omega}^c)^3, \quad \Pi_{ex} \in C^0(\Omega^c), \quad \Pi_{ex}|_{\overline{\Omega}^c} = \Pi|\overline{\Omega}^c \\ \in C^\infty(\overline{\Omega}^c), \quad -\Delta W + \nabla \Pi = 0, \quad \operatorname{div} W = 0, \quad W|_{\Omega_R} \in W^{2,p}(\Omega_R)^3, \quad \Pi|_{\Omega_R} \in W^{1,p}(\Omega_R). \end{aligned} \quad (6.18) \quad \boxed{\text{T100.60.60}}$$

As a first consequence, the trace of  $\partial_j(W|\overline{\Omega}^c)$  and  $\Pi|\overline{\Omega}^c$  on  $\partial\Omega$  exists and equals  $W_{ex}|_{\partial\Omega}$  and  $\Pi_{ex}|_{\partial\Omega}$ , respectively. Observing that  $\tilde{b} - \gamma(b) n^{(\Omega)} = b$ , recalling that Corollary 6.1 is valid here with  $\tilde{b}$  in the place of  $b$ , and taking into account (6.15), we may thus conclude that the pair  $(u, \pi)$  satisfies (1.2) in the trace sense. From (6.18) and Theorem 6.1, it further follows that  $u \in C^\infty(\overline{\Omega}^c)^3$ ,  $\pi \in C^\infty(\overline{\Omega}^c)$ , and the pair  $(u, \pi)$  solves (1.1) in  $\overline{\Omega}^c$  with  $F = 0$ . Again using that  $|\gamma(b)| \leq \mathfrak{C} \|b\|_p$ , once more referring to (6.18), Theorem 6.1 and (6.17), and noting that Lemma 2.4 and (4.4) imply that  $W|_{\Omega_R} \in L^1(\Omega_R)^3$ , we may conclude that inequality (6.10) – (6.12) hold.  $\square$

Next we turn to the interior domain case.

(theorem7.31)

**Theorem 6.3** *Let  $p \in (1, \infty)$  and  $b \in W^{1-1/p,p}(\partial\Omega)^3$  with  $\int_{\partial\Omega} \phi^{(j)} \cdot b \, do_x = 0$  for  $1 \leq j \leq 6$ . Abbreviate  $\phi := F^+(b)$ ,  $V := V(\phi)|_\Omega$ ,  $Q := Q(\phi)|_\Omega$ .*

*(The functions  $\phi^{(1)}, \dots, \phi^{(6)}$  were introduced in Theorem 1.1, the function  $F^+(b)$  in Corollary 4.3, and the functions  $V(\phi)$  and  $Q(\phi)$  in Lemma 4.2.)*

*Then  $V \in C^\infty(\Omega)^3$ ,  $Q \in C^\infty(\Omega)$ , the pair  $(V, Q)$  solves (1.1) in  $\Omega$  with  $F = 0$ . Moreover equation (6.8) holds with  $(\partial_j V)^{\text{tr}}$  and  $Q^{\text{tr}}$  again denoting the trace of  $\partial_j V$  and  $Q$ , respectively, on  $\partial\Omega$  (but with a different meaning of  $V$  and  $Q$  compared to Corollary 6.1), and*

$$\|V\|_{1,r} + \|Q\|_r \leq \mathfrak{C} \|b\|_p \quad \text{for } r \in [1, 3p/2), \quad \|V\|_{2,p} + \|Q\|_{1,p} \leq \mathfrak{C} \|b\|_{1-1/p,p}.$$

*The constants  $\mathfrak{C}$  appearing in the preceding estimates are independent of  $b$ .*

**Proof:** Theorem 6.3 is proved by an analogous reasoning as Theorem 6.1 and Corollary 6.1, with only that part of the proof of Theorem 6.1 being relevant which relates to  $\Omega_R$ .  $\square$

Corollary 6.2 below shows that the condition on  $b$  in Theorem 6.3 is necessary. The ensuing lemma is needed for the proof of this corollary.

(lemma100.50)

**Lemma 6.1** *Let  $p \in (1, \infty)$ ,  $u \in W^{2,p}(\Omega)^3$ ,  $\pi \in W^{1,p}(\Omega)$ . Then*

$$\int_{\partial\Omega} \sum_{j=1}^3 \phi_j^{(l)} \sum_{k=1}^3 n_k^{(\Omega)} (\partial_j u_k + \partial_k u_j - \delta_{jk} \pi) \, do_x = \int_{\Omega} \sum_{j=1}^3 \phi_j^{(l)} (\partial_j \operatorname{div} u + \Delta u_j - \partial_j \pi) \, dx$$

for  $1 \leq l \leq 6$ .

**Proof:** The lemma follows from the Divergence theorem and the fact that

$$\sum_{k=1}^3 \partial_k \phi_j^{(l)} (\partial_j u_k + \partial_k u_j - \delta_{jk} \pi) = 0 \quad \text{for } 1 \leq j \leq 6. \quad \square$$

ollary100.55)

**Corollary 6.2** Let  $p \in (1, \infty)$ ,  $f \in L^p(\Omega)^3$ ,  $b \in W^{1-1/p, p}(\partial\Omega)^3$ ,  $u \in W^{2, p}(\Omega)^3$ ,  $\pi \in W^{1, p}(\Omega)$  such that the pair  $(u, \pi)$  satisfies (1.1) in  $\Omega$  as well as (1.2). Then the equation  $\int_{\partial\Omega} \phi^{(l)} \cdot b \, d\sigma_x + \int_{\Omega} \phi^{(l)} \cdot f \, dx = 0$  holds for  $1 \leq l \leq 6$ .

**Proof:** Combine Lemma 6.1 with (1.2) and (1.1). □

In the rest of this section, we consider solutions to (1.1), (1.2) in the case  $F \neq 0$ .

theorem100.71)

**Theorem 6.4** Let  $A \subset \mathbb{R}^3$  be measurable and  $p \in (1, 3/2)$ . If  $f \in L^p(A)^3$ , the integral  $\int_A \sum_{k=1}^3 |(\partial^\alpha E_{jk})(x-y) f_k(y)| \, dy$  is finite for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$  and for a. e.  $x \in \mathbb{R}^3$ , so we may define

$$\mathfrak{R}(f)(x) := \left( \int_A \sum_{k=1}^3 E_{jk}(x-y) f_k(y) \, dy \right)_{1 \leq j \leq 3} \quad (x \in \mathbb{R}^3).$$

For such  $f$ , the relations  $\mathfrak{R}(f) \in W_{loc}^{2, p}(\mathbb{R}^3)^3$ ,  $\operatorname{div} \mathfrak{R}(f) = 0$  and  $\|\mathfrak{R}(f)\|_{(1/p-2/3)^{-1}} \leq C(p) \|f\|_p$  hold, and in addition  $\partial_l \mathfrak{R}(f)_j(x) = \int_A \sum_{k=1}^3 (\partial_l E_{jk})(x-y) f_k(y) \, dy$  for  $x \in \mathbb{R}^3$ ,  $1 \leq j, l \leq 3$ .

Let  $q \in (1, 3)$ . If  $f \in L^q(A)^3$ , the integral  $\int_A \sum_{k=1}^3 |\partial_k \mathfrak{R}(x-y) f_k(y)| \, dy$  is finite for a. e.  $x \in \mathbb{R}^3$ , so we may define

$$\mathfrak{S}(f)(x) := \int_A \sum_{k=1}^3 (-\partial_j \mathfrak{R})(x-y) f_k(y) \, dy \quad \text{for } x \in \mathbb{R}^3,$$

and we have  $\mathfrak{S}(f) \in W_{loc}^{1, q}(\mathbb{R}^3)^3$ ,  $\|\mathfrak{S}(f)\|_{(1/q-1/3)^{-1}} \leq C(q) \|f\|_q$ .

If  $A = \mathbb{R}^3$  and  $f \in L^p(A)^3$ , we have  $-\Delta \mathfrak{R}(f) + \nabla \mathfrak{S}(f) = f$ .

In the case  $f \in L^p(A)^3 \cap L^q(A)^3$  the estimate  $\|\partial_l \mathfrak{R}(f)\|_{(1/q-1/3)^{-1}} \leq C(q) \|f\|_q$  is valid.

Let  $r \in (1, \infty)$ . If  $f \in L^p(A)^3 \cap L^r(A)^3$ , then  $\|\partial_l \mathfrak{R}(f)\|_r \leq C(r) \|f\|_r$  ( $1 \leq l, m \leq 3$ ), and in the case  $f \in L^q(A)^3 \cap L^r(A)^3$ , the estimate  $\|\partial_l \mathfrak{S}(f)\|_r \leq C(r) \|f\|_r$  holds for  $1 \leq l \leq 3$ .

**Proof:** The theorem follows from the Hardy-Littlewood-Sobolev inequality and from the Calderon-Zygmund inequality; see [8, Satz 1.4] and compare the proof of [12, Theorem IV.2.1]. □

ollary100.24)

**Corollary 6.3** Let  $r \in (1, \infty)$ ,  $R, S \in (0, \infty)$ ,  $f \in L^r(B_R)^3$ . Then  $\|\mathfrak{R}(f)|_{B_S}\|_{2, r} + \|\mathfrak{S}(f)|_{B_S}\|_{1, r} \leq C(r, R, S) \|f\|_r$ .

**Proof:** Obviously  $f \in L^p(B_R)^3$  for any  $p \in (1, \min\{r, 3/2\})$ , so  $\mathfrak{R}(f)$  and  $\mathfrak{S}(f)$  are well defined. Hölder's inequality and (4.4) yield for  $\alpha \in \mathbb{N}_0^3$  with  $|\alpha| \leq 1$  that

$$\|\partial^\alpha \mathfrak{R}(f)|_{B_S}\|_r \leq \left( \int_{B_S} \left( \int_{B_R} |x-y|^{-1-|\alpha|} dy \right)^{r-1} \int_{B_R} |x-y|^{-1-|\alpha|} |f(y)|^r dy dx \right)^{1/r}.$$

The integral  $\int_{B_R} |x-y|^{-1-|\alpha|} dy$  is bounded uniformly in  $x \in B_S$ , and  $\int_{B_S} |x-y|^{-1-|\alpha|} dx$  uniformly in  $y \in B_R$ . In view of the last estimate in Theorem 6.4, it follows that  $\|\mathfrak{R}(f)|_{B_S}\|_{2,r} \leq C(r, R, S) \|f\|_r$ . An analogous reasoning is valid for  $\mathfrak{S}(f)$ .  $\square$

Now we are in a position to give a more detailed version of the existence and regularity results in Theorem 1.2.

Corollary 100.25)

**Corollary 6.4** *Let  $p_1 \in (1, 3/2)$ ,  $p_2 \in (1, 3)$ ,  $p_3 \in (1, \infty)$ , and let  $f$  belong to  $L^s(\overline{\Omega}^c)^3$  and  $b$  to  $W^{1-1/s, s}(\partial\Omega)^3$  for  $s \in \{p_1, p_2, p_3\}$ . Put*

$$\bar{b}_j := - \sum_{k=1}^3 n_k^{(\Omega)} (\partial_j \mathfrak{R}(f)_k + \partial_k \mathfrak{R}(f)_j - \delta_{jk} \mathfrak{S}(f)) \quad \text{for } j \in \{1, 2, 3\}.$$

Then  $\bar{b} \in W^{1-1/s, s}(\partial\Omega)^3$  for  $s \in \{p_1, p_2, p_3\}$ . Further define

$$\gamma(b + \bar{b}) := - \int_{\partial\Omega} \phi^{(0)} \cdot (b + \bar{b}) do_x \left( \int_{\partial\Omega} \phi^{(0)} \cdot n^{(\Omega)} do_x \right)^{-1}, \quad \tilde{b} := b + \bar{b} + \gamma(b + \bar{b}) n^{(\Omega)},$$

with  $\phi^{(0)}$  from Corollary 4.3; see Theorem 6.2 for the fact that  $\int_{\partial\Omega} \phi^{(0)} \cdot n^{(\Omega)} do_x \neq 0$ .

Then  $\int_{\partial\Omega} \phi^{(0)} \cdot \tilde{b} do_x = 0$ , so  $\phi := F^{-1}(\tilde{b}) \in L^{\max\{p_1, p_2, p_3\}}(\partial\Omega)^3$  (Corollary 4.3) is well defined. Finally put

$$\begin{aligned} u &:= V(\phi) + c_0^{-1} \gamma(b + \bar{b}) W(\phi^{(0)})|_{\overline{\Omega}^c}, \quad \pi := Q(\phi) + c_0^{-1} \gamma(b + \bar{b}) \Pi(\phi^{(0)})|_{\overline{\Omega}^c}, \\ v &:= u + \mathfrak{R}(f), \quad \varrho := \pi + \mathfrak{S}(f), \end{aligned}$$

where  $c_0$  was introduced in Theorem 6.2. The functions  $V(\phi)$  and  $Q(\phi)$  are from Lemma 4.2,  $W(\phi^{(0)})$  and  $\Pi(\phi^{(0)})$  from Lemma 4.3, and  $\mathfrak{R}(f)$  and  $\mathfrak{S}(f)$  from Theorem 6.4. Note that the functions  $u$  and  $\pi$  coincide with the corresponding functions in Theorem 6.2, except that  $b$  is replaced here by  $b + \bar{b}$ .

Then  $(v, \varrho) \in W_{loc}^{2, p_1}(\overline{\Omega}^c)^3 \times W_{loc}^{1, p_1}(\overline{\Omega}^c)$  and

$$v \in L^{(1/p_1 - 2/3)^{-1}}(\overline{\Omega}^c)^3, \quad \partial_l v_j, \varrho \in L^{(1/s - 1/3)^{-1}}(\overline{\Omega}^c), \quad \partial_m \partial_l v_j, \partial_l \varrho \in L^r(\overline{\Omega}^c)$$

for  $s \in \{p_1, p_2\}$ ,  $r \in \{p_1, p_2, p_3\}$ ,  $1 \leq j, l, m \leq 3$ , in particular  $v|_{\Omega_R} \in W^{2, p_1}(\Omega_R)^3$  and  $\varrho|_{\Omega_R} \in W^{1, p_1}(\Omega_R)$  for  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , so  $\partial_j v$  ( $1 \leq j \leq 3$ ) and  $\varrho$  have a trace on  $\partial\Omega$ . The pair  $(v, \varrho)$  satisfies equation (1.1) in  $\overline{\Omega}$  with  $F = f$ , and (1.2) with  $B = b$ .

Moreover  $\|v\|_{(1/p_1 - 2/3)^{-1}} \leq \mathfrak{C}(\|f\|_{p_1} + \|b\|_{1-1/p_1, p_1})$ . If  $s \in \{p_1, p_2\}$ , then

$$\|\partial_l v\|_{(1/s - 1/3)^{-1}} + \|\varrho\|_{(1/s - 1/3)^{-1}} + \|\partial_m \partial_l v\|_s + \|\partial_l \varrho\|_s \leq \mathfrak{C}(\|f\|_s + \|b\|_{1-1/s, s})$$

for  $1 \leq l, m \leq 3$ . In addition, if  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ , then

$$\|\partial_m \partial_l v\|_{p_3} + \|\partial_l \varrho\|_{p_3} \leq \mathfrak{C}(R) (\|f\|_{p_3} + \|f|_{B_{2R}^c}\|_{p_2} + \|b\|_{1-1/p_3, p_3})$$

for  $l, m$  as before. The constants  $\mathfrak{C}$  in these estimates are independent of  $f$  and  $b$ .

A comment of the term  $\|f|_{B_{2R}^c}\|_{p_2}$  in the preceding estimate may be found in the passage following Theorem 1.2

**Proof of Corollary 6.4:** All the constants  $\mathfrak{C}$  appearing in the following are independent of  $f$  and  $b$ . Let  $s \in \{p_1, p_2, p_3\}$ . Theorem 6.4, applied with  $p = q = p_1$ ,  $r = s$ , yields that

$$\|\partial_l \partial_m \mathfrak{R}(f)\|_s + \|\partial_l \mathfrak{S}(f)\|_s \leq C(s) \|f\|_s \quad \text{for } 1 \leq l, m \leq 3. \quad (6.19) \quad \boxed{\text{C100.25.5}}$$

Let  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ . By a trace inequality and the preceding estimate we get

$$\begin{aligned} \|\bar{b}\|_{1-1/s, s} &\leq \mathfrak{C} \left( \sum_{l=1}^3 \|\partial_l \mathfrak{R}(f)|_{\Omega_R}\|_{1,s} + \|\mathfrak{S}(f)|_{\Omega_R}\|_{1,s} \right) \\ &\leq \mathfrak{C} \left( \sum_{l=1}^3 \|\partial_l \mathfrak{R}(f)|_{\Omega_R}\|_s + \|\mathfrak{S}(f)|_{\Omega_R}\|_s + \|f\|_s \right). \end{aligned} \quad (6.20) \quad \boxed{\text{C100.25.20}}$$

Consider the case  $s = p_3$ . The estimate

$$\|\partial_l \mathfrak{R}(f|_{\Omega_{2R}})|_{\Omega_R}\|_{p_3} + \|\mathfrak{S}(f|_{\Omega_{2R}})|_{\Omega_R}\|_{p_3} \leq \mathfrak{C} \|f|_{\Omega_{2R}}\|_{p_3} \leq \mathfrak{C} \|f\|_{p_3}$$

holds by Corollary 6.3. Moreover, for  $x \in B_R$  and  $y \in B_{2R}^c$ , we have  $|x - y| \geq |y|/2$ , so

$$\|\partial_l \mathfrak{R}(f|_{B_{2R}^c})|_{\Omega_R}\|_{p_3} + \|\mathfrak{S}(f|_{B_{2R}^c})|_{\Omega_R}\|_{p_3} \leq \mathfrak{C} |\Omega_R|^{1/p_3} \int_{B_{2R}^c} |y|^{-2} |f(y)| dy. \quad (6.21) \quad \boxed{\text{C100.25.25}}$$

Since  $p_2 < 3$ , we have  $p'_2 > 3/2$ , so  $\int_{B_{2R}^c} |y|^{-2p'_2} dy < \infty$ . Therefore, due to Hölder's inequality, the left-hand side of (6.21) may be bounded by  $\mathfrak{C} \|f|_{B_{2R}^c}\|_{p_2}$ . Altogether, if  $G \in \{\partial_l \mathfrak{R}(f) : 1 \leq l \leq 3\}$  or  $G = \mathfrak{S}(f)$ , we get  $\|G|_{\Omega_R}\|_{p_3} \leq \mathfrak{C} (\|f\|_{p_3} + \|f|_{B_{2R}^c}\|_{p_2})$ , so it follows from (6.20) that  $\bar{b} \in W^{1-1/p_3, p_3}(\partial\Omega)^3$  and  $\|\bar{b}\|_{1-1/p_3, p_3} \leq \mathfrak{C} (\|f\|_{p_3} + \|f|_{B_{2R}^c}\|_{p_2})$ . This inequality combined with (6.12) the second estimate in (6.10) yield that

$$\|\partial_m \partial_l u\|_{p_3} + \|\partial_l \pi\|_{p_3} \leq \mathfrak{C} \|b + \bar{b}\|_{1-1/p_3, p_3} \leq \mathfrak{C} (\|f\|_{p_3} + \|f|_{B_{2R}}\|_{p_2} + \|b\|_{1-1/p_3, p_3}).$$

(Recall that  $u$  and  $\pi$  are taken from Theorem 6.2 with  $b$  replaced by  $b + \bar{b}$ .) The estimate at the end of Corollary 6.4 thus follows with (6.19).

Now suppose that  $s \in \{p_1, p_2\}$ . We know from Theorem 6.4 that

$$\|\partial_l \mathfrak{R}(f)\|_{(1/s-1/3)^{-1}} + \|\mathfrak{S}(f)\|_{(1/s-1/3)^{-1}} \leq C(s) \|f\|_s \quad (1 \leq l \leq 3). \quad (6.22) \quad \boxed{\text{C100.25.26}}$$

Thus for  $G$  as above, we have  $\|G|_{\Omega_R}\|_s \leq C(R, s) \|G|_{\Omega_R}\|_{(1/s-1/3)^{-1}} \leq C(R, s) \|f\|_s$ , so we may conclude from (6.20) that  $\bar{b} \in W^{1-1/s, s}(\partial\Omega)^3$  and

$$\|\bar{b}\|_{1-1/s, s} \leq \mathfrak{C} \|f\|_s. \quad (6.23) \quad \boxed{\text{C100.25.27}}$$

The second inequality in (6.10) and the preceding estimate imply that

$$\|u|_{\Omega_R}\|_{2,s} + \|\pi|_{\Omega_R}\|_{1,s} \leq \mathfrak{C} \|b + \bar{b}\|_{1-1/s,s} \leq \mathfrak{C} (\|f\|_s + \|b\|_{1-1/s,s}). \quad (6.24) \quad \boxed{\text{C100.25.30}}$$

Inequality (6.24) combined with (6.19) and (6.22) show in particular that  $\partial_l v_j|_{\Omega_R}$  and  $\varrho|_{\Omega_R}$  – and hence also  $\partial_l v_j$  and  $\varrho$  – have a trace on  $\partial\Omega$ . Due to the definition of  $\bar{b}$  and by Theorem 6.2 with  $b$  replaced by  $b + \bar{b}$ , the pair  $(v, \varrho)$  satisfies (1.2) with  $B = b$ . Moreover we may conclude from Theorem 6.2 and 6.4 that the pair  $(v, \varrho)$  belongs to  $W_{loc}^{2,p_1}(\bar{\Omega}^c)^3 \times W_{loc}^{1,p_1}(\bar{\Omega}^c)$  and satisfies (1.1) with  $F = f$ . Due to (6.24), (6.12) with  $b$  replaced by  $b + \bar{b}$ , (6.23) and (6.19), we obtain that  $\|\partial_m \partial_l v\|_s + \|\partial_l \varrho\|_s \leq \mathfrak{C} (\|f\|_s + \|b\|_{1-1/s,s})$  for  $1 \leq l, m \leq 3$ . Since  $s < 3$ , a Sobolev inequality and inequality (6.24) yield that

$$\|\partial_l u|_{\Omega_R}\|_{(1/s-1/3)^{-1}} + \|\pi|_{\Omega_R}\|_{(1/s-1/3)^{-1}} \leq \mathfrak{C} (\|f\|_s + \|b\|_{1-1/s,s}) \quad (1 \leq l \leq 3). \quad (6.25) \quad \boxed{\text{C100.25.30a}}$$

Again because  $s < 3$ , hence  $(1/s - 1/3)^{-1} > 3/2$ , and in view of the second inequality in (6.11) with  $b + \bar{b}$  instead of  $b$ , we get  $\|\partial_l u|_{B_R^c}\|_{(1/s-1/3)^{-1}} + \|\pi|_{B_R^c}\|_{(1/s-1/3)^{-1}} \leq \mathfrak{C} \|b + \bar{b}\|_s$  for  $l$  as before. The preceding estimate, (6.25), (6.23) and (6.22) yield that

$$\|\partial_l v\|_{(1/s-1/3)^{-1}} + \|\varrho\|_{(1/s-1/3)^{-1}} \leq \mathfrak{C} (\|f\|_s + \|b\|_{1-1/s,s}) \quad (1 \leq l \leq 3).$$

Taking into account that  $p_1 < 3/2$ , hence  $(1/p_1 - 2/3)^{-1} > 3$ , and by the first inequality in (6.11), as well as inequality (6.24) with  $s = p_1$ , a Sobolev inequality, (6.23) and Theorem 6.4, we obtain in an analogous way that  $\|v\|_{(1/p_1-2/3)^{-1}} \leq \mathfrak{C} (\|f\|_{p_1} + \|b\|_{1-1/p_1,p_1})$ , so the proof of Corollary 6.4 is complete.  $\square$

The ensuing corollary elaborates the existence and regularity results in Theorem 1.1.

Corollary 100.26)

**Corollary 6.5** *Let  $p \in (1, \infty)$ ,  $f \in L^p(\Omega)^3$ ,  $b \in W^{1-1/p,p}(\partial\Omega)^3$  with*

$$\int_{\partial\Omega} \phi^{(l)} \cdot b \, d\sigma_x + \int_{\Omega} \phi^{(l)} \cdot f \, dx = 0 \quad \text{for } 1 \leq l \leq 6.$$

*Put  $\bar{b}_j := -\sum_{k=1}^3 n_k^{(\Omega)} (\partial_j \mathfrak{R}(f)_k + \partial_k \mathfrak{R}(f)_j - \delta_{jk} \mathfrak{S}(f))$  for  $1 \leq j \leq 3$ . Then for  $l$  as above the equation  $\int_{\partial\Omega} \phi^{(l)} \cdot (b + \bar{b}) \, d\sigma_x = 0$  holds, so the function  $\phi := F^+(b + \bar{b}) \in E_p^{(+)}(\partial\Omega)$  (Corollary 4.3) is well defined. Put  $v := V(\phi) + \mathfrak{R}(f)|_{\Omega}$ ,  $\varrho := Q(\phi) + \mathfrak{S}(f)|_{\Omega}$  (see Lemma 4.2 and Theorem 6.4),  $\alpha := \widetilde{M}^{-1}(\int_{\Omega} v \cdot \phi^{(j)} \, dx)_{1 \leq j \leq 6}$ ,  $u := v - \sum_{j=1}^6 \alpha_j \phi^{(j)}$ ,  $\pi := \varrho$ , with  $\widetilde{M}$  from Lemma 4.6.*

*Then  $u \in W^{2,p}(\Omega)^3$ ,  $\pi \in W^{1,p}(\Omega)$ , the pair  $(u, \pi)$  satisfies (1.1) in  $\Omega$  with  $F = f$ , as well as (1.2) with  $B = b$ . In addition  $\int_{\Omega} u \cdot \phi^{(j)} \, dx = 0$  for  $1 \leq j \leq 6$ . Moreover  $\|u\|_{2,p} + \|\pi\|_{1,p} \leq \mathfrak{C} (\|f\|_p + \|b\|_{1-1/p,p})$ , with a constant  $\mathfrak{C}$  independent of  $f$  and  $b$ .*

**Proof:** By Corollary 6.3, we have  $\mathfrak{R}(f)|_{\Omega} \in W^{2,p}(\Omega)^3$ ,  $\mathfrak{S}(f)|_{\Omega} \in W^{1,p}(\Omega)$ , so  $\bar{b} \in W^{1-1/p,p}(\partial\Omega)^3$ . Let  $l \in \{1, \dots, 6\}$ . Then we find by Theorem 6.4 and Lemma 6.1 that

$$\int_{\partial\Omega} \phi^{(l)} \cdot (b + \bar{b}) \, d\sigma_x = \int_{\partial\Omega} \phi^{(l)} \cdot b \, d\sigma_x + \int_{\Omega} \phi^{(l)} \cdot f \, dx,$$

so by our assumptions on  $f$  and  $b$ ,  $\int_{\partial\Omega} \phi^{(l)} \cdot (b + \bar{b}) \, do_x = 0$ . Therefore the function  $\phi := F^+(b + \bar{b})$  (Corollary 4.3) is in fact well defined, and we may apply Theorem 6.3 with  $b$  replaced by  $b + \bar{b}$ , to obtain that  $V(\phi)|\Omega \in W^{2,p}(\Omega)^3$ ,  $Q(\phi)|\Omega \in W^{1,p}(\Omega)$ , the pair  $(V(\phi)|\Omega, Q(\phi)|\Omega)$  solves (1.1) in  $\Omega$  with  $F = 0$  and (1.2) with  $B = b + \bar{b}$ . In addition this theorem implies that  $\|V(\phi)|\Omega\|_{2,p} + \|Q(\phi)|\Omega\|_{1,p} \leq \mathfrak{C} \|b + \bar{b}\|_{1,p}$ . On the other hand, with a trace estimate,

$$\|\bar{b}\|_{1-1/p,p} \leq \mathfrak{C} \left( \sum_{j=1}^3 \|\partial_j \mathfrak{R}(f)|\Omega\|_{1,p} + \|\mathfrak{S}(f)|\Omega\|_{1,p} \right) \leq \mathfrak{C} (\|\mathfrak{R}(f)|\Omega\|_{2,p} + \|\mathfrak{S}(f)|\Omega\|_{1,p}).$$

Since  $\|\mathfrak{R}(f)|\Omega\|_{2,p} + \|\mathfrak{S}(f)|\Omega\|_{1,p} \leq \mathfrak{C} \|f\|_p$  again by Corollary 6.3, we thus arrive at the inequality  $\|v\|_{2,p} + \|\varrho\|_{1,p} \leq \mathfrak{C} (\|f\|_p + \|b\|_{1-1/p,p})$ . Due to Theorem 6.4 and by what was stated above on the pair  $(V(\phi)|\Omega, Q(\phi)|\Omega)$ , the pair  $(v, \varrho)$  solves (1.1), (1.2) with  $F, B$  replaced by  $f$  and  $b$ , respectively.

For any  $j \in \{1, \dots, 6\}$ , the function  $\phi^{(j)}|\bar{\Omega}$  belongs to  $C^\infty(\bar{\Omega})^3$ , and the pair  $(\phi^{(j)}, 0)$  is a solution of (1.1), (1.2) with  $F = 0$  and  $B = 0$ . Recall the vector  $\alpha \in \mathbb{R}^6$  introduced in Corollary 6.5. Since  $|\alpha_j| \leq \mathfrak{C} \|v\|_p$  for  $1 \leq j \leq 6$ , and by the properties of  $v$  and  $\varrho$  and the definition of  $\widetilde{M}$ , the pair  $(u, \pi)$  fulfills the claims stated in that corollary.  $\square$

## 7 Some uniqueness results.

The claims on uniqueness in this section imply what is stated on uniqueness in Theorem 1.1 and 1.2. First we consider the interior domain case.

theorem100.91)

**Theorem 7.1** *Let  $p \in (1, \infty)$ ,  $v \in W^{2,p}(\Omega)^3$ ,  $\varrho \in W^{1,p}(\Omega)$  such that the pair  $(v, \varrho)$  satisfies (1.1), (1.2) with  $F = 0$  and  $B = 0$ , and such that  $\int_{\Omega} v \cdot \phi^{(j)} \, dx = 0$  for  $1 \leq j \leq 6$ . Then  $v$  and  $\varrho$  vanish.*

**Proof:** In the case  $p \geq 2$ , we could use Lemma 2.5 with  $p = 2$  and Theorem 2.4. But Theorem 7.1 is not restricted to that case. For a proof of this fact, let  $g \in C_0^\infty(\Omega)^3$ , and put  $\alpha := \widetilde{M}^{-1} (\int_{\Omega} g \cdot \phi^{(j)} \, dx)_{1 \leq j \leq 6}$ , with the matrix  $\widetilde{M}$  defined in Lemma 4.6. Define  $f := g - \sum_{j=1}^6 \alpha_j \cdot \phi^{(j)}$ . Then  $f \in C^0(\bar{\Omega})^3 \subset L^{p'}(\Omega)^3$  and  $\int_{\Omega} f \cdot \phi^{(k)} \, dx = 0$  for  $1 \leq k \leq 6$ . Therefore, by Corollary 6.5, there are functions  $w \in W^{2,p}(\Omega)^3$ ,  $\sigma \in W^{1,p}(\Omega)$  such that the pair  $(w, \sigma)$  satisfies (1.1), (1.2) with  $F = f$ ,  $B = 0$ .

In this situation we twice use the formula in Lemma 2.5, once with  $(u, \pi) = (v, \varrho)$ ,  $\tilde{u} = w$ , and then with  $(u, \pi) = (w, \sigma)$ ,  $\tilde{u} = v$ . It follows that  $\int_{\Omega} f \cdot v \, dx = \int_{\Omega} (\Delta v - \nabla \varrho) \cdot w \, dx = 0$ . But by our assumptions,  $v$  is orthogonal to the functions  $\phi^{(j)}$ . Therefore  $\int_{\Omega} g \cdot v \, dx = 0$ . Since  $g$  was an arbitrary function from  $C_0^\infty(\Omega)^3$ , we may conclude that  $v = 0$ , hence there is  $c \in \mathbb{R}$  with  $\varrho = c$  a. e. Equation (1.2) and the assumption  $\varrho \in W^{1,p}(\Omega)$  now yield that  $\varrho = 0$ .  $\square$

Now we turn to solutions in the exterior domain  $\bar{\Omega}^c$ . The uniqueness result we show in this case is more general than the one in Theorem 1.2.

**Theorem 7.2** Take  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ . For  $j \in \{1, 2\}$ , let  $p_j, q_j, r_j, s_j \in (1, \infty)$ ,  $u^{(j)} \in W_{loc}^{2,1}(\bar{\Omega}^c)^3$ ,  $\pi^{(j)} \in W_{loc}^{1,1}(\bar{\Omega}^c)$  such that  $u^{(j)}|_{B_R^c} \in L^{p_j}(B_R^c)^3$ ,  $\pi^{(j)}|_{B_R^c} \in L^{q_j}(B_R^c)$ ,  $\nabla \pi^{(j)}|_{B_R^c} \in L^{r_j}(B_R^c)^3$ , and  $u^{(j)}|_{\Omega_T} \in W^{2,s_j}(\Omega_T)^3$ ,  $\pi^{(j)}|_{\Omega_T} \in W^{1,s_j}(\Omega_T)$  for  $T \in (0, \infty)$  with  $\bar{\Omega} \subset B_T$ . Put  $u := u^{(1)} - u^{(2)}$ ,  $\pi := \pi^{(1)} - \pi^{(2)}$ . Suppose in addition that the pair  $(u, \pi)$  satisfies (1.1) with  $F = 0$  as well as (1.2) with  $B = 0$ . Then  $u = 0$  and  $\pi = 0$ .

Note that the condition  $\pi^{(j)}|_{B_R^c} \in L^{q_j}(B_R^c)$  for  $j \in \{1, 2\}$  is necessary. In fact, let  $(u^{(1)}, \pi^{(1)})$  be the solution of (1.1) with  $F = 0$  and (1.2) with  $B = n^{(\Omega)}$  provided by Theorem 6.2, and put  $u^{(2)} := 0$ ,  $\pi^{(2)} := 1$ . Then the pairs  $(u^{(1)}, \pi^{(1)})$ ,  $(u^{(2)}, \pi^{(2)})$  verify all the assumptions of Theorem 7.2 except the relation  $\pi^{(2)}|_{B_R^c} \in L^{q_2}(B_R^c)$  for some  $q_2 \in (1, \infty)$  and  $R \in (0, \infty)$  with  $\bar{\Omega} \subset B_R$ . These two pairs cannot coincide because  $\pi^{(1)}|_{B_R^c} \in L^r(B_R^c)$  for any  $r \in (3/2, \infty)$ .

**Proof of Theorem 7.2:** Put  $r := \min\{r_1, r_2\}$ . By the proof of [4, Theorem 1.1], where Dirichlet boundary conditions are considered, we know that  $\nabla \pi|_{B_{R+1}^c} \in L^s(B_{R+1}^c)^3$  for  $s \in (1, r]$ . Boundary conditions play no role in the argument leading to that result. Theorem 2.3 then implies there is  $c(\pi) \in \mathbb{R}$  such that  $\pi + c(\pi)|_{B_{R+1}^c} \in L^{(1/s-1/3)^{-1}}(B_{R+1}^c)$  for  $s \in (1, r]$  if  $r < 3$ , and for any  $s \in (1, 3)$  else. Note that  $c(\pi)$  is independent of  $p$ , as follows from the criterion for the case  $c(v) = 0$  in Theorem 2.3.

Let us show that  $c(\pi) = 0$ . To this end put  $A_n := B_{2n} \setminus B_n$  for  $n \in \mathbb{N}$ . Take  $s \in (1, r)$  with  $s < \min\{q_1, q_2, 3\}$ . Then for  $n \in \mathbb{N}$  with  $n \geq R + 1$ , because  $|A_n| = 28n^3/3$ ,

$$\begin{aligned} (28|c(\pi)|\pi n^3/3)^{1/s} &= \|c(\pi)\chi_{A_n}\|_s \leq \|\pi + c(\pi)|_{A_n}\|_s + \sum_{j=1}^2 \|\pi^{(j)}|_{A_n}\|_s \quad (7.1) \quad \boxed{\text{T100.80.10}} \\ &\leq |A_n|^{1/3} \|\pi + c(\pi)|_{A_n}\|_{(1/s-1/3)^{-1}} + \sum_{j=1}^2 |A_n|^{1/s-1/p_j} \|\pi^{(j)}|_{A_n}\|_{p_j}. \end{aligned}$$

Recall that  $s \in (1, r)$  by the choice of  $s$ , and  $\pi + c(\pi)|_{B_{R+1}^c} \in L^{(1/s-1/3)^{-1}}(B_{R+1}^c)$ , as explained above. In addition  $\pi^{(j)}|_{B_R^c} \in L^{p_j}(B_R^c)$  by our assumptions. Thus, after dividing both sides of (7.1) by  $n^{3/s}$ , we get a left-hand side which is independent of  $n$ , and a right-hand side tending to zero for  $n$  tending to infinity. Therefore the assumption  $c(\pi) \neq 0$  would lead to a contradiction, hence  $c(\pi) = 0$ . As a consequence, in view of the integrability properties of  $\pi + c(\pi)$  proved above, there is  $\tilde{p} \in (3/2, \infty]$  such that  $\pi|_{B_{R+1}^c} \in L^{\tilde{p}}(B_{R+1}^c)$  for any  $p \in (3/2, \tilde{p})$ . In fact, we may choose  $\tilde{p} := (1/r - 1/3)^{-1}$  in the case  $r < 3$ , and  $\tilde{p} = \infty$  else.

Fix a function  $\varphi \in C_0^\infty(B_2)$  with  $\varphi|_{B_1} = 1$  and  $0 \leq \varphi \leq 1$ . Put  $\varphi_n(x) := \varphi(n^{-1}x)$  for  $x \in \mathbb{R}^3$ ,  $n \in \mathbb{N}$ . Then  $\varphi_n \in C_0^\infty(B_{2n})$ ,  $\varphi_n|_{B_n} = 1$ ,  $0 \leq \varphi_n \leq 1$ ,  $\text{supp}(\nabla \varphi_n) \subset \overline{B_{2n}} \setminus B_n$  for  $n \in \mathbb{N}$ ,  $\|\nabla \varphi_n\|_p \rightarrow 0$  ( $n \rightarrow \infty$ ) for  $p \in (3, \infty)$ , and  $\|\partial_l \partial_m \varphi_n\|_p \rightarrow 0$  ( $n \rightarrow \infty$ ) for  $p \in (3/2, \infty)$ ,  $1 \leq l, m \leq 3$ .

Let  $\Phi \in C_0^\infty(\bar{\Omega}^c)^3$ , and choose the pair  $(v, \varrho) \in W_{loc}^{2,1}(\bar{\Omega}^c)^3 \times W_{loc}^{1,1}(\bar{\Omega}^c)$  as in Corollary 6.4 with  $f = \Phi$  and  $b = 0$ . This means in particular that  $v_j \in L^{p_1}(\bar{\Omega}^c)$ ,  $\partial_k v_j, \pi \in$

$L^{p_2}(\overline{\Omega}^c)$ ,  $\partial_m \partial_k v_j, \partial_l \pi \in L^{p_3}(\overline{\Omega}^c)$  for any  $p_1 \in (3, \infty)$ ,  $p_2 \in (3/2, \infty)$ ,  $p_3 \in (1, \infty)$ ,  $1 \leq l, m \leq 3$ . This further means that  $(v, \varrho)$  solves (1.1) in  $\overline{\Omega}^c$  with  $F = \Phi$ , as well as (1.2) with  $B = 0$ . For  $n \in \mathbb{N}$ , put  $v_n := \varphi_n v$ ,  $\varrho_n := \varphi_n \varrho$ . We claim that

$$\|u \cdot (-\Delta v_n + \nabla \varrho_n - \Phi)\|_1 \rightarrow 0, \|u \cdot \nabla \operatorname{div} v_n\|_1 \rightarrow 0, \|\pi \operatorname{div} v_n\|_1 \rightarrow 0 \quad (n \rightarrow \infty). \quad (7.2) \quad \boxed{\text{T100.80.20}}$$

In fact, concerning the first of these relations, recall that  $\operatorname{supp}(\nabla \varphi_n) \subset \overline{B_{2n}} \setminus B_n \subset B_R^c$  for  $n \in \mathbb{N}$ ,  $n \geq R$ , and  $\overline{\Omega} \subset B_R$ , so by Hölder's inequality

$$\|u_k^{(j)} \partial_l v_k \partial_l \varphi_n\|_1 \leq \|u_k^{(j)}|_{B_R^c}\|_{p_j} \|\partial_l v_k\|_{3p_j/2} \|\partial_l \varphi_n\|_{3p_j'} \quad \text{for } n \text{ as before, } j \in \{1, 2\},$$

$1 \leq k, l \leq 3$ . Further recall that  $u^{(j)}|_{B_R^c} \in L^{p_j}(B_R^c)^3$ ,  $\partial_l v \in L^p(\overline{\Omega}^c)^3$  for  $p \in (3/2, \infty)$ , and  $\|\nabla \varphi_n\|_{3p_j'} \rightarrow 0$  when  $n$  tends to  $\infty$ . It follows that  $\|u_k^{(j)} \partial_l v_k \partial_l \varphi_n\|_1 \rightarrow 0$  ( $n \rightarrow \infty$ ) for  $j, k$  and  $l$  as before. Similarly  $\|u^{(j)} \cdot v \Delta \varphi_n\|_1 \rightarrow 0$  and  $\|u^{(j)} \varrho \nabla \varphi_n\|_1 \rightarrow 0$  for  $n \rightarrow \infty$ . Altogether, since  $-\Delta v + \nabla \varrho = \Phi$ , we get that  $\|u \cdot (\Delta v_n + \nabla \varrho_n - \Phi)\|_1 \rightarrow 0$  ( $n \rightarrow \infty$ ). Moreover  $\operatorname{div} v = 0$ , so a variant of the preceding argument yields that  $\|u \cdot \nabla \operatorname{div} v_n\|_1 \rightarrow 0$  ( $n \rightarrow \infty$ ). Since  $\pi|_{B_{R+1}^c} \in L^p(B_{R+1}^c)$  for  $p \in (3/2, \tilde{p})$ , as shown above, and because  $v|_{B_{R+1}^c} \in L^s(B_{R+1}^c)^3$  for  $s \in (3, \infty)$ , we may choose  $p \in (3/2, \tilde{p})$  and  $s \in (3, \infty)$  so close to respectively  $3/2$  and  $3$  that  $1 - 1/p - 1/s < 1/3$ , hence  $(1 - 1/p - 1/s)^{-1} > 3$ . As a consequence  $\|\nabla \varphi_n\|_{(1-1/p-1/s)^{-1}} \rightarrow 0$  ( $n \rightarrow \infty$ ), hence we get in view of the equation  $\operatorname{div} = 0$  that  $\|\pi \operatorname{div} v_n\|_1 \rightarrow 0$  ( $n \rightarrow \infty$ ). This completes the proof of (7.2). Recalling the first and second relation in (7.2), as well as the fact that the pair  $(v, \varrho)$  satisfies (1.2) with  $B = 0$ , and noting that  $\operatorname{div} u = 0$  by (1.1),  $\varphi_n|_{B_n} = 1$ ,  $\operatorname{supp}(\varphi_n) \subset B_{2n}$  and  $\overline{\Omega} \subset B_n$  for  $n \in \mathbb{N}$  with  $n \geq R$ , we obtain by Lemma 2.5 that

$$\begin{aligned} \int_{\overline{\Omega}^c} u \cdot \Phi \, dx &= \lim_{n \rightarrow \infty} \int_{\overline{\Omega}^c} u \cdot (-\Delta v_n + \nabla \varrho_n - \nabla \operatorname{div} v_n) \, dx \\ &= \lim_{n \rightarrow \infty} \int_{\overline{\Omega}^c} \sum_{j,k=1}^3 (\partial_k u_j \partial_k v_{n,j} + \partial_k u_j \partial_j v_{n,k}) \, d\omega_x. \end{aligned} \quad (7.3) \quad \boxed{\text{T100.80.30}}$$

Next we use the third relation in (7.2) and then the assumption that the pair  $(u, \pi)$  is a solution of (1.1) with  $F = 0$  and (1.2) with  $B = 0$ . It follows from (7.3) that

$$\int_{\overline{\Omega}^c} u \cdot \Phi \, dx = \lim_{n \rightarrow \infty} \int_{\partial \Omega} \sum_{j,k=1}^3 (\partial_k u_j + \partial_j u_k - \delta_{jk} \pi) n_k^{(\Omega)} v_{n,j} \, d\omega_x = 0.$$

At this point we may conclude that  $u = 0$ , hence  $\pi = 0$  by (1.1) and (1.2).  $\square$

## 8 Appendix: Indications on the proof of Theorem 4.6.

We only consider the limit  $\lim_{\kappa \downarrow 0} F(-\kappa)$ . The limit  $\lim_{\kappa \downarrow 0} F(\kappa)$  is easier to determine because the integrals on  $\Omega_R$  appearing below can be replaced by integrals on  $\Omega$ , so the parameter  $R$  and the difficulties related to it do not arise.

Let  $\kappa \in (0, \delta(\Omega)]$  and put  $z := x - \kappa n^{(\Omega)}(x)$ . Note that  $z \in \Omega$  (see (3.3)) and  $|z - b| \geq \mathcal{D}\kappa/2$  for  $b \in \mathfrak{U}_{-\kappa}$  (see (3.6)), where the open set  $\mathfrak{U}_{-\kappa}$  and the constant  $\mathcal{D}$  were introduced in Lemma 3.2.

It follows that for  $n \in \mathbb{N}$ , the function  $|z - y|^{-n}$  ( $y \in \mathfrak{U}_{-\kappa}$ ) is  $C^\infty$  in  $\mathfrak{U}_{-\kappa}$ . Since  $\overline{\Omega_R} \subset \Omega^c \subset \mathfrak{U}_{-\kappa}$  for  $R \in (0, \infty)$  (see Lemma 3.2), this  $C^\infty$ -regularity in  $\mathfrak{U}_{-\kappa}$  will allow us in the following to apply the Divergence theorem in  $\Omega_R$ .

Let  $j \in \{1, 2, 3\}$  and put  $K_{klm} := -\partial_j S_{klm} - \partial_k S_{jlm} + 2\delta_{jk} \partial_l \partial_m \mathfrak{N}$  for  $k, l, m \in \{1, 2, 3\}$ , where  $S_{\nu lm}$  for  $\nu \in \{1, 2, 3\}$  and  $\mathfrak{N}$  were introduced at the beginning of Section 4; see (4.1) and (4.2). Then by the definitions in Lemma 4.3,

$$F(-\kappa)_j = \sum_{k,l,m=1}^3 n_k^{(\Omega)}(x) \int_{\partial\Omega} K_{klm}(z-y) n_l^{(\Omega)}(y) \phi_m(y) do_y. \quad (8.4) \quad \boxed{\text{Ap10}}$$

Let  $R \in (0, \infty)$  with  $\overline{\Omega} \subset B_R$ , and let  $n^{(\Omega_R)}$  denote the outward unit normal to  $\Omega_R$ , that is,  $n^{(\Omega_R)}|_{\partial\Omega} = -n^{(\Omega)}|_{\partial\Omega}$ ,  $n^{(\Omega_R)}(y) = R^{-1}y$  for  $y \in \partial B_R$ . By the definition of  $C^{1,\alpha}(\partial\Omega)^3$  (see at the beginning of Section 2), there is  $\tilde{\phi} \in C^{1,\alpha}(\mathbb{R}^3)^3$  with  $\tilde{\phi}|_{\partial\Omega} = \phi$ . We will write  $\phi$  instead of  $\tilde{\phi}$  in the following. Then we get from (8.4) that  $F(-\kappa)_j = \mathfrak{A}_1 + \mathfrak{B}_1(R) + \mathring{\mathfrak{A}} + \mathfrak{A}$ , where  $\mathfrak{A}_1$  is given by the right-hand side of (8.4), but with the term  $\phi_m(y)$  replaced by  $\phi_m(y) - \phi_m(x) - \sum_{\nu=1}^3 \partial_\nu \phi_m(x) (y-x)_\nu$ , for  $1 \leq m \leq 3$ . Moreover

$$\begin{aligned} \mathfrak{B}_1(R) &:= \sum_{k,l,m=1}^3 \phi_m(x) n_k^{(\Omega)}(x) \int_{\partial B_R} K_{klm}(z-y) y_l/R do_y, \\ \mathfrak{A} &:= \sum_{k,l,m,\nu=1}^3 \partial_\nu \phi_m(x) n_k^{(\Omega)}(x) \int_{\partial\Omega} K_{klm}(z-y) n_l^{(\Omega)}(y) (y-x)_\nu do_y. \end{aligned}$$

The term  $\mathring{\mathfrak{A}}$  differs from  $\mathfrak{B}_1(R)$  insofar as the integration extends over  $\partial\Omega_R$  instead of  $\partial B_R$ , and  $-n_l^{(\Omega_R)}(y)$  takes the place of  $y_l/R$ , for  $1 \leq l \leq 3$ . Due the Divergence theorem, (4.3) and (4.5), we get that  $\mathring{\mathfrak{A}} = 0$ . Next we observe that  $\mathfrak{A} = \mathfrak{A}_2 + \mathfrak{B}_2(R) + \mathfrak{B}_3(R) + \tilde{\mathfrak{A}}$ , where  $\mathfrak{A}_2$  is defined in the same way as  $\mathfrak{A}$ , but with  $n_k^{(\Omega)}(x) n_l^{(\Omega)}(y)$  for  $1 \leq k, l \leq 3$  replaced by  $n_k^{(\Omega)}(x) n_l^{(\Omega)}(y) - n_k^{(\Omega)}(y) n_l^{(\Omega)}(x)$ . Moreover

$$\begin{aligned} \tilde{\mathfrak{A}} &:= - \sum_{k,l,m,\nu=1}^3 \partial_\nu \phi_m(x) n_l^{(\Omega)}(x) \int_{\partial\Omega_R} K_{klm}(z-y) (y-x)_\nu n_k^{(\Omega_R)}(y) do_y, \quad (8.5) \quad \boxed{\text{Ap40}} \\ \mathfrak{B}_3(R) &:= \sum_{k,l,m,\nu=1}^3 \partial_\nu \phi_m(x) n_l^{(\Omega)}(x) \int_{\partial B_R} K_{klm}(-y) y_\nu y_k/R do_y, \end{aligned}$$

and with  $\mathfrak{B}_2(R)$  chosen as  $\mathfrak{B}_3(R)$ , but with  $K_{klm}(z-y) (y-x)_\nu - K_{klm}(-y) y_\nu$  in the role of  $K_{klm}(-y) y_\nu$ , where  $1 \leq k, l, m, \nu \leq 3$ . Now the Divergence theorem is applied to  $\tilde{\mathfrak{A}}$ , transforming the integral over  $\partial\Omega_R$  in (8.5) into an integral over  $\Omega_R$ . Note that due to (4.5), (4.3), the sum  $\sum_{k=1}^3 \partial y_k (K_{klm}(z-y) (y-x)_\nu)$  reduces to  $K_{\nu lm}(z-y)$ ; see

the definition of  $K_{\nu lm}$  at the beginning of this proof ( $\nu, l, m$  as before). In view of this definition, we may again apply the Divergence theorem, this time in order to retransform the integral over  $\Omega_R$  into an integral over  $\partial\Omega_R$ , which we split according to the equation  $\tilde{\mathfrak{A}} = \mathfrak{A} + \mathfrak{B}$ , with

$$\begin{aligned} \overline{\mathfrak{A}} := & \sum_{l,m,\nu=1}^3 \partial_\nu \phi_m(x) n_l^{(\Omega)}(x) \int_{\partial\Omega} (S_{\nu lm}(z-y) n_j^{(\Omega)}(y) + S_{jlm}(z-y) n_\nu^{(\Omega)}(y) \\ & - 2 \delta_{\nu j} (\partial_l \mathfrak{N})(z-y) n_m^{(\Omega)}(y)) do_y, \end{aligned}$$

$$\begin{aligned} \overline{\mathfrak{B}} := & \sum_{l,m,\nu=1}^3 \partial_\nu \phi_m(x) n_l^{(\Omega)}(x) \int_{\partial B_R} (-S_{\nu lm}(z-y) y_j/R - S_{jlm}(z-y) y_\nu/R \\ & + 2 \delta_{\nu j} (\partial_l \mathfrak{N})(z-y) y_m/R) do_y. \end{aligned}$$

As a consequence  $\tilde{\mathfrak{A}} = \mathfrak{A}_3 + \mathfrak{B}_4(R) + \mathfrak{B}_5(R) + \hat{\mathfrak{A}}$ , where  $\mathfrak{A}_3$  coincides with  $\overline{\mathfrak{A}}$  except that  $n_l^{(\Omega)}(x) n_b^{(\Omega)}(y) - n_l^{(\Omega)}(y) n_b^{(\Omega)}(x)$  takes the role of  $n_l^{(\Omega)}(x) n_b^{(\Omega)}(y)$ , for  $b, l \in \{1, 2, 3\}$ . Moreover  $\mathfrak{B}_4(R)$  and  $\mathfrak{B}_5(R)$  correspond to  $\overline{\mathfrak{B}}(R)$ , but we put  $-S_{blm}(z-y) + S_{blm}(-y)$  and  $(\partial_l \mathfrak{N})(z-y) - (\partial_l \mathfrak{N})(-y)$  in the place of  $-S_{blm}(z-y)$  and  $(\partial_l \mathfrak{N})(z-y)$ , respectively, in the case of  $\mathfrak{B}_4(R)$ , whereas  $\mathfrak{B}_5(R)$  arises from  $\overline{\mathfrak{B}}(R)$  by substituting  $-S_{blm}(-y)$  and  $(\partial_l \mathfrak{N})(-y)$  for  $-S_{blm}(z-y)$  and  $(\partial_l \mathfrak{N})(z-y)$ , respectively ( $1 \leq b, l, m \leq 3$ ). The term  $\hat{\mathfrak{A}}$  is given by

$$\begin{aligned} \hat{\mathfrak{A}} := & \sum_{l,m,\nu=1}^3 \partial_\nu \phi_m(x) \left( n_j^{(\Omega)}(x) \int_{\partial\Omega} (S_{\nu lm}(z-y) n_l^{(\Omega)}(y) do_y \right. \\ & \left. + n_\nu^{(\Omega)}(x) \int_{\partial\Omega} S_{jlm}(z-y) n_l^{(\Omega)}(y) do_y - 2 \delta_{\nu j} n_m^{(\Omega)}(x) \int_{\partial\Omega} (\partial_l \mathfrak{N})(z-y) n_l^{(\Omega)}(y) do_y \right). \end{aligned}$$

We finally observe that  $\hat{\mathfrak{A}} = \mathfrak{B}_6(R) + \mathfrak{B}_7(R) + \underline{\mathfrak{A}}$ , where  $\underline{\mathfrak{A}}$  differs from  $\hat{\mathfrak{A}}$  insofar as the domain of integration now is  $\partial\Omega_R$  instead of  $\partial\Omega$ , and the term  $-n_l^{(\Omega_R)}(y)$  is substituted for  $n_l^{(\Omega)}(y)$  ( $1 \leq l \leq 3$ ). As for  $\mathfrak{B}_6(R)$  and  $\mathfrak{B}_7(R)$ , they are also defined in a way analogous to the definition of  $\hat{A}$ . In their case integration extends over  $\partial B_R$ , and  $y_l/R$  stands in for  $n_l^{(\Omega)}(y)$ . In addition the terms  $S_{blm}(z-y)$  and  $(\partial_l \mathfrak{N})(z-y)$  are replaced by  $S_{blm}(z-y) - S_{blm}(-y)$  and  $(\partial_l \mathfrak{N})(z-y) - (\partial_l \mathfrak{N})(-y)$  as concerns  $\mathfrak{B}_6(R)$ , and by  $S_{blm}(-y)$  and  $(\partial_l \mathfrak{N})(-y)$  as for  $\mathfrak{B}_7(R)$  ( $b, l, m \in \{1, 2, 3\}$ ). By (4.5) and (4.3) we see that  $\underline{\mathfrak{A}} = 0$ .

The splitting of  $F(-\kappa)_j$ ,  $\mathfrak{A}$ ,  $\tilde{A}$  and  $\hat{\mathfrak{A}}$  considered above, and the equations  $\underline{\mathfrak{A}} = \mathring{\mathfrak{A}} = 0$  may be subsumed into a single equation, that is,  $F(-\kappa)_j = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \sum_{\nu=1}^7 \mathfrak{B}_\nu(R)$ . Since  $\overline{\Omega} \subset B_{R/2}$  and  $z \in \Omega$  (see (3.3)), we have  $|\vartheta z - y| \geq |y|/2$  for  $\vartheta \in [0, 1]$ ,  $y \in \partial B_R$ , so  $|\mathfrak{B}_b(R)| \leq \mathfrak{C} \int_{\partial B_R} |y|^{-3} do_y \leq \mathfrak{C} R^{-1}$  for  $b \in \{1, 2, 4, 6\}$ , with  $\mathfrak{C}$  independent of  $R$ .

Moreover we indicate that  $\sum_{\nu=1}^3 \alpha_\nu \int_{\partial B_1} y_j y_l y_m y_\nu do_y = (\alpha_j \delta_{lm} + \alpha_\nu \delta_{jm} + \alpha_m \delta_{jl}) 4\pi/15$  for  $\alpha \in \mathbb{R}^3$ ,  $l, m \in \{1, 2, 3\}$ . The factor  $4\pi/15$  arises due to the equation  $\int_{\partial B_1} y_l^2 y_m^2 do_y =$

$(1 + 2\delta_{lm})4\pi/15$  for  $l, m$  as before. It follows that  $\sum_{b \in \{3, 5, 7\}} \mathfrak{B}_b(R) = 0$ . Up to this point the parameter  $R$  was fixed. Letting  $R$  tend to infinity, we may conclude from the preceding results on  $\mathfrak{B}_b(R)$  that  $\sum_{b=1}^8 \mathfrak{B}_b(R) \rightarrow 0$  ( $R \rightarrow \infty$ ), hence  $F(-\kappa)_j = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3$ .

By the definition of  $\mathfrak{A}_1, \mathfrak{A}_2$  and  $\mathfrak{A}_3$ , it is obvious that for  $b \in \{1, 2, 3\}$ , there is a function  $\mathfrak{Z}_b = \mathfrak{Z}_b^{(j,x,\kappa)} : \partial\Omega \mapsto \mathbb{R}$  such that  $\mathfrak{A}_b = \int_{\partial\Omega} \mathfrak{Z}_b(y) do_y$ . Since by (3.4), we further have  $|z - y| \geq \mathcal{D}|x - y|$  for  $y \in \partial\Omega$ , with the constant  $\mathcal{D}$  introduced in Lemma 3.2, and because  $\phi \in C^{1,\alpha}(\mathbb{R}^3)$  and  $n^{(\Omega)}$  is in particular Lipschitz continuous on  $\partial\Omega$ , we further get that  $|\mathfrak{Z}_b(y)| \leq \mathfrak{C}|x - y|^{-2+\alpha}$  for  $y \in \partial\Omega$ ,  $1 \leq b \leq 3$ , where  $\mathfrak{C} > 0$  does not depend on  $y$ . But  $\int_{\partial\Omega} |x - y|^{-2+\alpha} do_y < \infty$ , so it follows by Lebesgue's theorem and the equation  $F(-\kappa)_j = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3$  that  $F(-\kappa)_j$  converges for  $\kappa \downarrow 0$ , with the limit being the integral arising if in the definition of  $\mathfrak{A}_b$  ( $1 \leq b \leq 3$ ) the parameter  $\kappa$  is set equal to zero. A similar but markedly simpler reasoning (see the remark at the beginning of this proof) yields that the limit of  $F(\kappa)_j$  for  $\kappa \downarrow 0$  exists as well, and its value coincides with  $\lim_{\kappa \downarrow 0} F(-\kappa)_j$ .  $\square$

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