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HOEFFDING-TYPE DECOMPOSITION FOR U -STATISTICS ON BIPARTITE NETWORKS

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Abstract. We consider a broad class of random bipartite networks, the distribution of which is invariant under permutation within each type of nodes. We are interested in U -statistics defined on the adjacency matrix of such a network, for which we define a new type of Hoeffding decomposition. This decomposition enables us to characterize non-degenerate U -statistics – which are then asymptotically normal – and provides us with a natural and easy-to-implement estimator of their asymptotic variance. We illustrate the use of this general approach on some typical random graph models and use it to estimate or test some quantities characterizing the topology of the associated network. We also assess the accuracy and the power of the proposed estimates or tests, via a simulation study.

1. Introduction

Networks are popular objects to represent a set of interacting entities. The last decades have witnessed an explosion in the number of networks datasets. The fields of application range from sociology to ecology, from economics to computer science. Understanding the organization of the network is a first step towards a better insight of the system it represents. Several strategies exist to study or describe the topology of a network. Many of them are based on the calculation of one or several numeric quantities (statistics) such as density, clustering coefficients or counts of given motifs to name but a few. These statistics generally rely on several nodes.

The calculation of these numerical quantities on a given network naturally leads to comparing them to a reference value, or to the value obtained on another network. The concept of hypothesis tests naturally meets this expectation. The challenging step of statistical hypothesis testing is identifying the statistic distribution under the null assumption. In particular, one class of statistic considered are the U -statistics which, in the context of network analysis, have complex dependencies.

Networks and dissociated RCE matrices In networks, entities are represented by nodes which are linked by edges when they interact. In bipartite networks, the nodes are divided into two types and the interactions only happen between nodes of the two different types. Some examples of bipartite networks connect users and items in recommender systems [53], papers and scientists in authorship networks [33], or plants and pollinators in ecological interaction networks [16]. The networks are naturally encoded in matrices. In the adjacency matrix Y of a bipartite network (sometimes also called incidence matrix), the two types of nodes are represented by rows and columns, so that Y_{ij} encodes the interaction between entity i of the first type and entity j of the second type. In binary networks, $Y_{ij} = 1$ if i and j interact, else $Y_{ij} = 0$. Some networks are weighted, meaning that Y_{ij} represents the intensity of the interaction.

We consider the asymptotic framework where Y is an infinite adjacency matrix and the adjacency matrix of an observed network of size $m \times n$ is the submatrix extracted from the leading m rows and n columns of Y . Probabilistic models define a joint distribution on the values of the matrix entries. In random graph models, it is common to assume that the nodes of the networks are exchangeable, i.e. that the distribution of the network does not change if its nodes are permuted. For instance, the stochastic blockmodel [47], the random dot product graph model [51] or the latent space model [24] are all node-exchangeable. On the corresponding adjacency matrix of a bipartite network, this assumption implies the row-column exchangeability. Y is said to be row-column exchangeable (RCE) if for any

couple $\Phi = (\sigma_1, \sigma_2)$ of finite permutations of \mathbb{N} ,

$$\Phi Y \stackrel{\mathcal{D}}{=} Y,$$

where $\Phi Y := (Y_{\sigma_1(i)\sigma_2(j)})_{i \geq 1, j \geq 1}$. Many exchangeable random graph models also have a dissociatedness property, i.e. their adjacency matrices are also dissociated [44, 28]. A RCE matrix is said to be dissociated if for all m and n , $(Y_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is independent from $(Y_{ij})_{i > m, j > n}$. In the present work, we only consider RCE dissociated matrices.

U -statistics and Hoeffding decomposition U -statistics are a generalization of the empirical mean to functions of more than one variable. Many estimators fall under the category of U -statistics. Given a sequence of random variables $Y = (Y_i)_{i \geq 1}$ numbered with a unique index a U -statistic $U_n^h(Y)$ of order n and kernel function h is defined as the following average

$$U_n^h(Y) = \binom{n}{k}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} h(Y_{i_1}, Y_{i_2}, \dots, Y_{i_k}),$$

where $h : \mathbb{R}^k \rightarrow \mathbb{R}$ is a symmetric function referred to as the kernel. Denote $\llbracket n \rrbracket := \{1, \dots, n\}$ and for a set A , $\mathcal{P}_k(A)$ the set of all subsets of cardinal k of A . Let $\mathbf{i} = \{i_1, \dots, i_k\} \in \mathcal{P}_k(\llbracket n \rrbracket)$, then by symmetry of h , $h(Y_{i_1}, \dots, Y_{i_k})$ does not depend on the order of the elements of \mathbf{i} . Therefore, we will denote $h(Y_{\mathbf{i}}) := h(Y_{i_1}, \dots, Y_{i_k})$. Finally, the U -statistic $U_n^h(Y)$ can be formulated as follows:

$$U_n^h(Y) = \binom{n}{k}^{-1} \sum_{\mathbf{i} \in \mathcal{P}_k(\llbracket n \rrbracket)} h(Y_{\mathbf{i}}).$$

When Y is an exchangeable sequence, $h(Y_{\mathbf{i}})$ has the same distribution for all $\mathbf{i} \in \mathcal{P}_k(\llbracket n \rrbracket)$, therefore $U_n^h(Y)$ is an unbiased estimate of $h(Y_{\llbracket k \rrbracket})$. The case where Y is an i.i.d. sequence is well-studied: the U -statistics are known to be asymptotically normal [22] and can be used for inference tasks such as estimation and hypothesis testing.

In the i.i.d. case, a useful technique to study the asymptotic behavior of U -statistics is the Hoeffding decomposition, formalized for the first time in [23]. For $1 \leq c \leq k$, define the function $\psi^c h : \mathbb{R}^c \rightarrow \mathbb{R}$ as

$$\psi^c h : (y_1, \dots, y_c) \mapsto \mathbb{E}[h(Y_1, \dots, Y_k) \mid Y_1 = y_1, \dots, Y_c = y_c].$$

Again, by symmetry of h , for some set $\mathbf{i} \in \mathcal{P}_c(\llbracket n \rrbracket)$, we can denote $\psi^c h(Y_{\mathbf{i}}) := \psi^c h(Y_{i_1}, \dots, Y_{i_c})$ since the order of the elements of \mathbf{i} does not matter. Set $p^0 h = \mathbb{E}[h(Y_{\llbracket k \rrbracket})]$ and define recursively

$$p^c h(Y_{\mathbf{i}}) = \psi^c h(Y_{\mathbf{i}}) - \sum_{c'=0}^{c-1} \sum_{\mathbf{i}' \in \mathcal{P}_{c'}(\mathbf{i})} p^{c'} h(Y_{\mathbf{i}'}).$$

for all subsets $\mathbf{i} \in \mathcal{P}_c(\llbracket n \rrbracket)$, for all $1 \leq c \leq k$. Then, for $\mathbf{i} \in \mathcal{P}_k(\llbracket n \rrbracket)$, $h(Y_{\mathbf{i}})$ can be written

$$h(Y_{\mathbf{i}}) = \sum_{0 \leq c \leq k} \sum_{\mathbf{i}' \in \mathcal{P}_c(\mathbf{i})} p^c h(Y_{\mathbf{i}'}).$$

The U -statistic U_n^h can be written as

$$U_n^h(Y) = \sum_{c=0}^k \binom{k}{c} P_n^c h(Y),$$

where for $1 \leq c \leq k$, $P_n^c h(Y) = \binom{n}{c}^{-1} \sum_{\mathbf{i} \in \mathcal{P}_c(\llbracket n \rrbracket)} p^c h(Y_{\mathbf{i}})$.

This decomposition is interesting as all the quantities $p^c h(Y_{\mathbf{i}})$ are orthogonal. By extension the U -statistics $P_n^c h$ are also orthogonal. The leading terms of this decomposition have been used by [22] to prove the asymptotic normality of U -statistics. The decomposition also yields a decomposition of the variance of U -statistics.

By extension, a network U -statistic averages a function h defined over sub-matrices of size $p \times q$. Let Y be an infinite adjacency matrix from which we observe the first m rows and n columns. Let

$h : \mathcal{M}_{p,q}(\mathbb{R}) \rightarrow \mathbb{R}$ be a function defined on $p \times q$ matrices, $1 \leq p \leq m$, $1 \leq q \leq n$, verifying the following symmetry property: for all $(\sigma_1, \sigma_2) \in \mathbb{S}_p \times \mathbb{S}_q$,

$$h(Y_{(i_{\sigma_1(1)}, \dots, i_{\sigma_1(p)}; j_{\sigma_2(1)}, \dots, j_{\sigma_2(q)})}) = h(Y_{(i_1, i_2, \dots, i_p; j_1, j_2, \dots, j_q)}), \quad (1)$$

where $Y_{(i_1, \dots, i_p; j_1, \dots, j_q)}$ is the $p \times q$ sub-matrix consisting of the rows and columns of Y indexed by i_1, \dots, i_p and j_1, \dots, j_q respectively. Therefore, since the order of the elements of $\mathbf{i} = \{i_1, \dots, i_p\}$ and $\mathbf{j} = \{j_1, \dots, j_q\}$ does not matter, we can denote $h(Y_{\mathbf{i}, \mathbf{j}}) := h(Y_{(i_1, i_2, \dots, i_p; j_1, j_2, \dots, j_q)})$. Then the associated U -statistic is

$$U_{m,n}^h(Y) = \binom{m}{p}^{-1} \binom{n}{q}^{-1} \sum_{\substack{\mathbf{i} \in \mathcal{P}_p(\llbracket m \rrbracket) \\ \mathbf{j} \in \mathcal{P}_q(\llbracket n \rrbracket)}} h(Y_{\mathbf{i}, \mathbf{j}}). \quad (2)$$

Note that the assumption on the symmetry of h can be made without loss of generality. Indeed, if $h^0 : \mathcal{M}_{p,q}(\mathbb{R}) \rightarrow \mathbb{R}$ is not symmetric, then $h : \mathcal{M}_{p,q}(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$h(Y_{(i_1, i_2, \dots, i_p; j_1, j_2, \dots, j_q)}) = (p!q!)^{-1} \sum_{(\sigma_1, \sigma_2) \in \mathbb{S}_p \times \mathbb{S}_q} h^0(Y_{(i_{\sigma_1(1)}, \dots, i_{\sigma_1(p)}; j_{\sigma_2(1)}, \dots, j_{\sigma_2(q)})}) \quad (3)$$

verifies Equation (1) and leads to the same U -statistic ($U_{m,n}^{h^0}(Y) = U_{m,n}^h(Y)$).

In the case where Y is a RCE matrix, [29] used a martingale approach to obtain a weak convergence result for $U_{m,n}^h$ when m and n grow to infinity at the same rate. Applying this result requires specific development to get the asymptotic variance. In this paper, we propose an Hoeffding decomposition-based approach. This strategy has the advantage to provide a method to estimate the asymptotic variance of the U -statistics. Indeed, obtaining an estimation of the (asymptotic) variance of U -statistics is a required condition to perform practical inference such as hypothesis testing. However, it remains a complex task and has been tackled with various methods in the literature.

Variance estimation of U -statistics (related work) The standard error of U -statistics is most often computed using resampling techniques such as the jackknife [2] and bootstrap [17, 9] estimators of variance. [41, 42, 11, 40] suggested various estimators of the asymptotic variance. However, all these estimators are biased for both the variance and the asymptotic variances. [11, 40] also discussed unbiased estimators, but they are computationally more demanding than all the previous estimators and they find them to have a positive probability of being negative, which is undesirable.

Contribution We show how U -statistics of size $p \times q$ can be used for exchangeable network inference. In particular, we propose a Hoeffding-type decomposition to identify the asymptotic distribution of these U -statistics and build a consistent estimator of their variance. For that, we assume that the matrix dimensions m_N and n_N grow at the same rate with $m_N/N \rightarrow \rho$ and $n_N/N \rightarrow 1 - \rho$, where $\rho \in]0, 1[$. For simplification, we denote $U_N^h(Y) := U_{m_N, n_N}^h(Y)$. First, we show that the distribution of $\sqrt{N}(U_N^h(Y) - \mathbb{E}[h])$ converges to a Gaussian. Next, we build a consistent estimator for the variance of $U_N^h(Y)$ and we study its properties.

Outline In Section 2, we first extend the Hoeffding decomposition of a sub-graph statistic. The decomposition we propose is based on the Aldous-Hoover-Kallenberg (AHK) representation. Section 3 exploits this Hoeffding decomposition to demonstrate the asymptotic normality of the U -statistics we consider. In Section 4, we use the Hoeffding decomposition to build an estimator of the asymptotic variance of the U -statistics. Section 5 presents some models for RCE matrices and kernels that can be interesting for network analysis and comparisons. Section 6 and 7 are dedicated to simulation studies and an illustration of legislature dataset.

2. Hoeffding decomposition of a submatrix U -statistic

Aldous-Hoover-Kallenberg (AHK) representation Corollary 7.23 of [25] states that for any dissociated RCE matrix Y , there exists $(\xi_i)_{i \geq 1}$, $(\eta_j)_{j \geq 1}$ and $(\zeta_{ij})_{i, j \geq 1}$ arrays of i.i.d. random variables with uniform

distribution over $[0, 1]$ and a real measurable function ϕ such that for all $1 \leq i, j < \infty$, $Y_{ij} \stackrel{a.s.}{=} \phi(\xi_i, \eta_j, \zeta_{ij})$. With such a representation, the kernel function taken on a $p \times q$ submatrix indexed by the rows $\mathbf{i} \in \mathcal{P}_p(\mathbb{N})$ and columns $\mathbf{j} \in \mathcal{P}_q(\mathbb{N})$ can be written $h(Y_{\mathbf{i}, \mathbf{j}}) \stackrel{a.s.}{=} h_\phi((\xi_i)_{i \in \mathbf{i}}; (\eta_j)_{j \in \mathbf{j}}; (\zeta_{ij})_{\substack{i \in \mathbf{i} \\ j \in \mathbf{j}}})$, where h_ϕ is some function depending on h and ϕ .

Note that the AHK decomposition is not unique. In the rest of the paper, we assume that for each dissociated RCE matrix Y , we have picked an AHK representation, i.e. a suitable function ϕ , and suitable i.i.d. random variables $(\xi_i)_{i \geq 1}$, $(\eta_j)_{j \geq 1}$ and $(\zeta_{ij})_{i, j \geq 1}$. In the rest of the paper, we will write, abusively but without ambiguity,

$$Y_{ij} = \phi(\xi_i, \eta_j, \zeta_{ij}).$$

and

$$h(Y_{\mathbf{i}, \mathbf{j}}) = h_\phi((\xi_i)_{i \in \mathbf{i}}; (\eta_j)_{j \in \mathbf{j}}; (\zeta_{ij})_{\substack{i \in \mathbf{i} \\ j \in \mathbf{j}}}).$$

For $\mathbf{i}' \in \mathcal{P}(\mathbb{N})$ and $\mathbf{j}' \in \mathcal{P}(\mathbb{N})$, we define the σ -algebra

$$\mathcal{A}_{\mathbf{i}', \mathbf{j}'} := \sigma((\xi_i)_{i \in \mathbf{i}'}, (\eta_j)_{j \in \mathbf{j}'}, (\zeta_{ij})_{\substack{i \in \mathbf{i}' \\ j \in \mathbf{j}'}}).$$

Therefore, it follows from our notations that

$$\mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{A}_{\mathbf{i}', \mathbf{j}'}] = \mathbb{E}[h_\phi((\xi_i)_{i \in \mathbf{i}}; (\eta_j)_{j \in \mathbf{j}}; (\zeta_{ij})_{\substack{i \in \mathbf{i} \\ j \in \mathbf{j}}}) \mid (\xi_i)_{i \in \mathbf{i}'}; (\eta_j)_{j \in \mathbf{j}'}; (\zeta_{ij})_{\substack{i \in \mathbf{i}' \\ j \in \mathbf{j}'}}].$$

Now, notice that for fixed \mathbf{i}' and \mathbf{j}' , the quantity $\mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{A}_{\mathbf{i}', \mathbf{j}'}]$ only depends on the elements shared by \mathbf{i} and \mathbf{i}' and the elements shared by \mathbf{j} and \mathbf{j}' , and not on the other elements of \mathbf{i} , \mathbf{i}' , \mathbf{j} and \mathbf{j}' . Suppose $r = \text{Card}(\mathbf{i} \cap \mathbf{i}')$ and $c = \text{Card}(\mathbf{j} \cap \mathbf{j}')$. Without loss of generality, we can assume that $\mathbf{i}' \in \mathcal{P}_r(\mathbf{i})$ and $\mathbf{j}' \in \mathcal{P}_c(\mathbf{j})$ so $\mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{A}_{\mathbf{i}', \mathbf{j}'}]$ only depends on the r elements of \mathbf{i}' and the c elements of \mathbf{j}' . Therefore, we can define the quantities $\psi^{r,c}h(Y_{\mathbf{i}', \mathbf{j}'})$ such that

$$\psi^{r,c}h(Y_{\mathbf{i}', \mathbf{j}'}) := \mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{A}_{\mathbf{i}', \mathbf{j}'}],$$

where the choice of \mathbf{i} and \mathbf{j} does not matter as long as $\mathbf{i}' \subset \mathbf{i}$ and $\mathbf{j}' \subset \mathbf{j}$. Note that $\psi^{r,c}h(Y_{\mathbf{i}', \mathbf{j}'})$ is simply a notation and not a function of $Y_{\mathbf{i}', \mathbf{j}'}$. If $\mathbf{i}' = \emptyset$ or $\mathbf{j}' = \emptyset$, we will still use this notation, for example

$$\psi^{r,c}h(Y_{\mathbf{i}', \emptyset}) = \mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{A}_{\mathbf{i}', \emptyset}],$$

despite $Y_{\mathbf{i}', \emptyset}$ being undefined.

Hoeffding-type decomposition of the kernel In the following, for elements of \mathbb{N}^2 , $(x, y) \leq (x', y')$ means that both $x \leq x'$ and $y \leq y'$; $(x, y) < (x', y')$ means that, in addition, $(x, y) \neq (x', y')$.

For all $\mathbf{i} \in \mathcal{P}_r(\mathbb{N})$ and $\mathbf{j} \in \mathcal{P}_c(\mathbb{N})$, we define by recursion the following quantity

$$p^{r,c}h(Y_{\mathbf{i}, \mathbf{j}}) = \psi^{r,c}h(Y_{\mathbf{i}, \mathbf{j}}) - \sum_{(0,0) \leq (r', c') < (r, c)} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}) \\ \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j})}} p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}). \quad (4)$$

Since $\psi^{p,q}h(Y_{\mathbf{i}, \mathbf{j}}) = h(Y_{\mathbf{i}, \mathbf{j}})$ for $\mathbf{i} \in \mathcal{P}_p(\mathbb{N})$ and $\mathbf{j} \in \mathcal{P}_q(\mathbb{N})$, (4) yields the decomposition of the kernel function h

$$h(Y_{\mathbf{i}, \mathbf{j}}) = \sum_{(0,0) \leq (r, c) \leq (p, q)} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_r(\mathbf{i}) \\ \mathbf{j}' \in \mathcal{P}_c(\mathbf{j})}} p^{r,c}h(Y_{\mathbf{i}', \mathbf{j}'}). \quad (5)$$

Remark 1. From this formula, we see that $h(Y_{\mathbf{i}, \mathbf{j}})$ is a linear combination of the $(p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}))_{\substack{0 \leq r' \leq p, 0 \leq c' \leq q \\ \mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}), \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j})}}$, therefore, this is a linear combination of the $(\psi^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}))_{\substack{0 \leq r' \leq r, 0 \leq c' \leq c \\ \mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}), \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j})}}$.

Now, we show that for $\mathbf{i}' \subset \mathbf{i}$ and $\mathbf{j}' \subset \mathbf{j}$, $p^{r,c}h(Y_{\mathbf{i}', \mathbf{j}'})$ is actually the projection of $h(Y_{\mathbf{i}, \mathbf{j}})$ on the subspace generated by L_2 functions of all the AHK variables of $\mathcal{A}_{\mathbf{i}', \mathbf{j}'}$, orthogonally to the sub-spaces generated by L_2 functions of all the variables of $\mathcal{A}_{\mathbf{i}'', \mathbf{j}''}$, for $\mathbf{i}'' \subset \mathbf{i}'$ and $\mathbf{j}'' \subset \mathbf{j}'$. This system of projection is analogous to the Hoeffding decomposition for the kernel functions of usual U -statistics on i.i.d. data. We prove that the following orthogonality properties hold.

Proposition 2.1. Let h_1 and h_2 two kernel functions of respective size $p_1 \times q_1$ and $p_2 \times q_2$.

1. Let $(0, 0) \leq (r_1, c_1) \leq (p_1, q_1)$ and $(0, 0) \leq (r_2, c_2) \leq (p_2, q_2)$ such that $(r_1, c_1) \neq (r_2, c_2)$. Let $(\mathbf{i}_1, \mathbf{j}_1) \in \mathcal{P}_{r_1}(\llbracket m \rrbracket) \times \mathcal{P}_{c_1}(\llbracket n \rrbracket)$ and $(\mathbf{i}_2, \mathbf{j}_2) \in \mathcal{P}_{r_2}(\llbracket m \rrbracket) \times \mathcal{P}_{c_2}(\llbracket n \rrbracket)$, then

$$\text{Cov}(p^{r_1, c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r_2, c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})) = 0.$$

2. Let (r, c) such that $(0, 0) \leq (r, c) \leq (p_1, q_1)$ and $(0, 0) \leq (r, c) \leq (p_2, q_2)$. Let $(\mathbf{i}_1, \mathbf{j}_1)$ and $(\mathbf{i}_2, \mathbf{j}_2)$ two elements of $\mathcal{P}_r(\llbracket m \rrbracket) \times \mathcal{P}_c(\llbracket n \rrbracket)$. If $(\mathbf{i}_1, \mathbf{j}_1) \neq (\mathbf{i}_2, \mathbf{j}_2)$, then

$$\text{Cov}(p^{r, c} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r, c} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})) = 0.$$

This proposition relies on the fact that the projections are "conditionnally centered". This property is given by the following lemma, the proof of which is provided in Appendix B.

Lemma 2.2. Let h be a kernel function of size $p \times q$. Let $(\mathbf{i}, \mathbf{j}) \in \mathcal{P}_r(\mathbb{N}) \times \mathcal{P}_c(\mathbb{N})$, where $(0, 0) < (r, c) \leq (p, q)$. For all $\underline{\mathbf{i}} \subset \mathbf{i}$ and $\underline{\mathbf{j}} \subset \mathbf{j}$, we have

$$\mathbb{E}[p^{r, c} h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] = 0.$$

Proof of Proposition 2.1. The two properties are proven similarly and derive from the fact that $(\mathbf{i}_1, \mathbf{j}_1) \neq (\mathbf{i}_2, \mathbf{j}_2)$. This is true for both properties.

Consider any (possibly equal) (r_1, c_1) and (r_2, c_2) and associated $(\mathbf{i}_1, \mathbf{j}_1) \neq (\mathbf{i}_2, \mathbf{j}_2)$. Then $\mathbf{i}_1 \neq \mathbf{i}_2$ or $\mathbf{j}_1 \neq \mathbf{j}_2$. Without loss of generality, assume that $\mathbf{i}_1 \neq \mathbf{i}_2$ so there is an element $i_2 \in \mathbf{i}_2$ which is not included in \mathbf{i}_1 . Then

$$\begin{aligned} \mathbb{E}[p^{r_1, c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}) p^{r_2, c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})] &= \mathbb{E}[\mathbb{E}[p^{r_1, c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}) p^{r_2, c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2}) \mid \mathcal{A}_{\{i_2\}, \emptyset}]] \\ &= \mathbb{E}[p^{r_1, c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}) \mathbb{E}[p^{r_2, c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2}) \mid \mathcal{A}_{\{i_2\}, \emptyset}]] \\ &= 0, \end{aligned}$$

since $\mathbb{E}[p^{r_2, c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2}) \mid \mathcal{A}_{\{i_2\}, \emptyset}] = 0$ from Lemma 2.2.

We have proven both properties, since the case $(r_1, c_1) \neq (r_2, c_2)$ corresponds to the first property and the case $(r_1, c_1) = (r_2, c_2)$ corresponds to the second property. \square

Decomposition of U -statistics Using the Hoeffding-type decomposition of kernel functions (5) the U -statistic (2) can be reformulated as:

$$\begin{aligned} U_{m, n}^h(Y) &= \binom{m}{p}^{-1} \binom{n}{q}^{-1} \sum_{\substack{1 \leq i_1 < \dots < i_p \leq m \\ 1 \leq j_1 < \dots < j_q \leq n}} \sum_{(0, 0) \leq (r, c) \leq (p, q)} \sum_{\substack{\mathbf{i} \in \mathcal{P}_r(\{i_1, \dots, i_p\}) \\ \mathbf{j} \in \mathcal{P}_c(\{j_1, \dots, j_q\})}} p^{r, c} h(Y_{\mathbf{i}, \mathbf{j}}) \\ &= \binom{m}{p}^{-1} \binom{n}{q}^{-1} \sum_{(0, 0) \leq (r, c) \leq (p, q)} \binom{m-r}{p-r} \binom{n-c}{q-c} \sum_{\substack{\mathbf{i} \in \mathcal{P}_r(\llbracket m \rrbracket) \\ \mathbf{j} \in \mathcal{P}_c(\llbracket n \rrbracket)}} p^{r, c} h(Y_{\mathbf{i}, \mathbf{j}}) \\ &= \sum_{(0, 0) \leq (r, c) \leq (p, q)} \binom{p}{r} \binom{q}{c} \binom{m}{r}^{-1} \binom{n}{c}^{-1} \sum_{\substack{\mathbf{i} \in \mathcal{P}_r(\llbracket m \rrbracket) \\ \mathbf{j} \in \mathcal{P}_c(\llbracket n \rrbracket)}} p^{r, c} h(Y_{\mathbf{i}, \mathbf{j}}) \\ &= \sum_{(0, 0) \leq (r, c) \leq (p, q)} \binom{p}{r} \binom{q}{c} P_{m, n}^{r, c} h(Y) \end{aligned}$$

where for all $0 \leq r \leq p$ and $0 \leq c \leq q$,

$$P_{m, n}^{r, c} h(Y) = \binom{m}{r}^{-1} \binom{n}{c}^{-1} \sum_{\substack{\mathbf{i} \in \mathcal{P}_r(\llbracket m \rrbracket) \\ \mathbf{j} \in \mathcal{P}_c(\llbracket n \rrbracket)}} p^{r, c} h(Y_{\mathbf{i}, \mathbf{j}})$$

is the U -statistic of kernel function $p^{r, c} h$ taken on the first $m \times n$ rows and columns of the matrix Y . A consequence of the orthogonality of the projections of h is the orthogonality of these U -statistics, as stated in the following corollary which is proven in Appendix B.

Corollary 2.3. Let h_1 and h_2 two kernel functions of respective sizes $p_1 \times q_1$ and $p_2 \times q_2$. Let $(0, 0) \leq (r_1, c_1) < (p_1, q_1)$ and $(0, 0) \leq (r_2, c_2) < (p_2, q_2)$.

1. If $(r_1, c_1) \neq (r_2, c_2)$, then

$$\text{Cov}(P_{m,n}^{r_1, c_1} h_1(Y), P_{m,n}^{r_2, c_2} h_2(Y)) = 0.$$

2. If $(r_1, c_1) = (r_2, c_2) = (r, c)$, then

$$\text{Cov}(P_{m,n}^{r,c} h_1(Y), P_{m,n}^{r,c} h_2(Y)) = \binom{m}{r}^{-1} \binom{n}{c}^{-1} \text{Cov}(p^{r,c} h_1(Y_{\llbracket r \rrbracket, \llbracket c \rrbracket}), p^{r,c} h_2(Y_{\llbracket r \rrbracket, \llbracket c \rrbracket})).$$

The orthogonality between the $P_{m,n}^{r_1, c_1} h_1(Y)$ and $P_{m,n}^{r_2, c_2} h_2(Y)$ allows to decompose the covariance of two U -statistics into a few covariance terms.

Corollary 2.4.

$$\begin{aligned} \text{Cov}(U_{m,n}^{h_1}(Y), U_{m,n}^{h_2}(Y)) &= \sum_{(0,0) < (r,c) \leq (p,q)} \binom{p}{r}^2 \binom{q}{c}^2 \text{Cov}(P_{m,n}^{r,c} h_1(Y), P_{m,n}^{r,c} h_2(Y)) \\ &= \sum_{(0,0) < (r,c) \leq (p,q)} \binom{p}{r}^2 \binom{q}{c}^2 \binom{m}{r}^{-1} \binom{n}{c}^{-1} \text{Cov}(p^{r,c} h_1(Y_{\llbracket r \rrbracket, \llbracket c \rrbracket}), p^{r,c} h_2(Y_{\llbracket r \rrbracket, \llbracket c \rrbracket})). \end{aligned}$$

Corollary 2.4 is a direct consequence of Corollary 2.3. This result will be helpful when considering the asymptotic properties of the U -statistics. Indeed, one can see that each covariance term is associated with a binomial coefficient depending on different orders of m and n .

3. Asymptotic normality of U -statistics

For the following sections, we will use simplified notations, summarizing the couple (m_N, n_N) into N . We recall that $U_N^h(Y) = U_{m_N, n_N}^h(Y)$. We also denote $P_N^{r,c} h(Y) := P_{m_N, n_N}^{r,c} h(Y)$. When this is unambiguous, we will omit to mention Y , so we will simply write U_N^h , $P_N^{r,c} h$, $p_{(i,j)}^{r,c} h$ and $\psi_{(i,j)}^{r,c} h$ instead of $U_N^h(Y)$, $P_N^{r,c} h(Y)$, $p^{r,c} h(Y_{i,j})$ and $\psi^{r,c} h(Y_{i,j})$.

The following theorem is a Central Limit Theorem for U_N^h . Denote $v_h^{r,c} := \mathbb{V}[\psi_{(\llbracket r \rrbracket, \llbracket c \rrbracket)}^{r,c} h]$.

Theorem 3.1. Let Y be a dissociated RCE matrix. Let h be a $p \times q$ kernel function such that $\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})^2] < \infty$. Let $(m_N, n_N)_{N \geq 1}$ be a sequence of dimensions for the U -statistics, such that $\frac{m_N}{N} \xrightarrow{N \rightarrow \infty} \rho$ and $\frac{n_N}{N} \xrightarrow{N \rightarrow \infty} 1 - \rho$, where $\rho \in]0, 1[$. Let $(U_N^h)_{N \geq 1}$ be the sequence of U -statistics associated with h defined by $U_N^h := U_{m_N, n_N}^h$. Set $U_\infty^h = \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})]$ and

$$V^h = \frac{p^2}{\rho} v_h^{1,0} + \frac{q^2}{1-\rho} v_h^{0,1}.$$

If $V^h > 0$, then

$$\sqrt{N}(U_N^h - U_\infty^h) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V^h).$$

This theorem comes from the decomposition of $\sqrt{N}(U_N^h - U_\infty^h)$ into three different terms, the limits of which are given by the following lemmas (proofs in Appendix C).

Lemma 3.2. If $v_h^{1,0} > 0$, then we have

$$\frac{1}{\sqrt{m}} \sum_{i=1}^m p_{(\{i\}, \emptyset)}^{1,0} h \xrightarrow[m \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, v_h^{1,0})$$

and if $v_h^{0,1} > 0$, then we have

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n p_{(\{j\}, \emptyset)}^{0,1} h \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, v_h^{0,1})$$

Lemma 3.3. Let $A_N := \sqrt{N} \sum_{\substack{(0,0) < (r,c) \leq (p,q) \\ (r,c) \neq (1,0) \neq (0,1)}} \binom{p}{r} \binom{q}{c} P_N^{r,c} h$. Then $A_N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$.

Proof of Theorem 3.1. We have

$$\begin{aligned} U_N^h &= \sum_{(0,0) \leq (r,c) \leq (p,q)} \binom{p}{r} \binom{q}{c} P_N^{r,c} h \\ &= P_N^{0,0} h + p P_N^{1,0} h + q P_N^{0,1} h + \sum_{\substack{(0,0) < (r,c) \leq (p,q) \\ (r,c) \neq (1,0) \neq (0,1)}} \binom{p}{r} \binom{q}{c} P_N^{r,c} h. \end{aligned}$$

First, we see that $P_N^{0,0} h = U_\infty^h$. Next, A_N being defined in Lemma 3.3, we have

$$\sqrt{N}(U_N^h - U_\infty^h) = \frac{\sqrt{N}p}{m_N} \sum_{i=1}^{m_N} p_{(\{i\}, \emptyset)}^{1,0} h + \frac{\sqrt{N}q}{n_N} \sum_{j=1}^{n_N} p_{(\emptyset, \{j\})}^{0,1} h + A_N.$$

From Lemma 3.3, $A_N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$. So by Slutsky's theorem, $\sqrt{N}(U_N^h - U_\infty^h)$ has the same limiting distribution as the two main terms of this decomposition. From Lemma 3.2, this is the sum of two centered Gaussians of respective variance $\frac{p^2}{\rho} v_h^{1,0}$ and $\frac{q^2}{1-\rho} v_h^{0,1}$. Furthermore, $\sum_{i=1}^{m_N} p_{(\{i\}, \emptyset)}^{1,0} h$ and $\sum_{j=1}^{n_N} p_{(\emptyset, \{j\})}^{0,1} h$ are independent, so the two Gaussians are independent, which concludes the proof. \square

Remark 2. The expression of V^h could have been predicted with Corollary 2.4. Indeed, this corollary implies that $\mathbb{V}[U_N^h] = \sum_{(0,0) < (r,c) \leq (p,q)} \binom{p}{r}^2 \binom{q}{c}^2 \binom{m_N}{r}^{-1} \binom{n_N}{c}^{-1} \mathbb{V}[p_{(\llbracket r \rrbracket, \llbracket c \rrbracket)}^{r,c} h]$, so $\lim_{N \rightarrow \infty} N \mathbb{V}[U_N^h] = \lim_{N \rightarrow \infty} \left(\frac{p^2 N}{m_N} \mathbb{V}[p_{(\{1\}, \emptyset)}^{1,0} h] + \frac{q^2 N}{n_N} \mathbb{V}[p_{(\emptyset, \{1\})}^{0,1} h] \right) = \frac{p^2}{c} \mathbb{V}[\psi_{(\{1\}, \emptyset)}^{1,0} h] + \frac{q^2}{1-c} \mathbb{V}[\psi_{(\emptyset, \{1\})}^{0,1} h]$.

Now, we show that the limiting distribution of a vector of U -statistics is a multivariate normal distribution under the condition that all the kernel functions are linearly independent. However, if the kernels functions are of different sizes, the notion of linear independence is unclear. We need to define the concept of kernel extension to enunciate the corollary.

Definition 3.4. Let h be a kernel function of size $p \times q$. Let $p' \geq p$ and $q' \geq q$. We define the extension of h to the size $p' \times q'$ by \tilde{h} such that for all $\mathbf{i}' \in \mathcal{P}_{p'}(\mathbb{N})$ and $\mathbf{j}' \in \mathcal{P}_{q'}(\mathbb{N})$,

$$\tilde{h}(Y_{\mathbf{i}', \mathbf{j}'}) = \left[\binom{p'}{p} \binom{q'}{q} \right]^{-1} \sum_{\substack{\mathbf{i} \in \mathcal{P}_p(\mathbf{i}') \\ \mathbf{j} \in \mathcal{P}_q(\mathbf{j}')}} h(Y_{\mathbf{i}, \mathbf{j}}).$$

The extension of a kernel actually shares some properties with its kernel, as shown in Lemma C.1. With this definition, we can define the linear independence of kernel functions needed for the following corollary as the linear independence of their kernel extensions. Denote $c_{h_k, h_\ell}^{r,c} := \text{Cov} \left(\psi_{(\llbracket r \rrbracket, \llbracket c \rrbracket)}^{r,c} h_k, \psi_{(\llbracket r \rrbracket, \llbracket c \rrbracket)}^{r,c} h_\ell \right)$. The proof of this corollary can be found in Appendix C.

Corollary 3.5. Let Y be a dissociated RCE matrix. Let (h_1, h_2, \dots, h_D) be a vector of kernel functions of respective sizes $p_1 \times q_1, p_2 \times q_2, \dots, p_D \times q_D$ such that

1. Theorem 3.1 applies for each kernel function, i.e. $\mathbb{E}[h_k(Y_{(1,2;1,2)})^2] < \infty$ and $U_\infty^{h_k}$ and V^{h_k} are as defined in Theorem 3.1 for each kernel h_k , $1 \leq k \leq D$,
2. for $t \in \mathbb{R}^D$, $\sum_{k=1}^D t_k \tilde{h}_k \equiv 0$ if and only if $t = (0, \dots, 0)$, where for $1 \leq k \leq D$, \tilde{h}_k is the extension of h_k to size $\max_k(p_k) \times \max_k(q_k)$.

Then

$$\sqrt{N} \left(\begin{pmatrix} U_N^{h_1} \\ U_N^{h_2} \\ \dots \\ U_N^{h_D} \end{pmatrix} - \begin{pmatrix} U_\infty^{h_1} \\ U_\infty^{h_2} \\ \dots \\ U_\infty^{h_D} \end{pmatrix} \right) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^{h_1, \dots, h_D}),$$

with

$$\Sigma^{h_1, \dots, h_D} = (C^{h_k, h_\ell})_{1 \leq k, \ell \leq D},$$

where $C^{h_k, h_\ell} = \frac{p^2}{\rho} c_{h_k, h_\ell}^{1,0} + \frac{q^2}{1-\rho} c_{h_k, h_\ell}^{0,1}$ for all $1 \leq k, \ell \leq D$ (and $C^{h_k, h_k} = V^{h_k}$).

Although, through the first condition, the theorem requires the kernel functions of (h_1, h_2, \dots, h_D) to be linearly independent, the corresponding U -statistics are not independent random variables, even asymptotically, because Σ^{h_1, \dots, h_D} is not a diagonal matrix. One consequence of this corollary is that Theorem 3.1 can be extended to differentiable functions of U -statistics.

Corollary 3.6. *Let h_1, \dots, h_D be D kernel functions such that Corollary 3.5 applies and $\Sigma^{h_1, \dots, h_D} = (C^{h_k, h_\ell})_{1 \leq k, \ell \leq D}$. Denote $\theta = (U_\infty^{h_1}, \dots, U_\infty^{h_D})$. Let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be a differentiable function at θ . Denote ∇g the gradient of g and $V^\delta := \nabla g(\theta)^T \Sigma^{h_1, \dots, h_D} \nabla g(\theta)$. If $V^\delta > 0$, then*

$$\sqrt{N}(g(U_N^{h_1}, \dots, U_N^{h_D}) - g(\theta)) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V^\delta).$$

Proof. The first-order Taylor expansion of g at θ is written

$$g(U_N^{h_1}, \dots, U_N^{h_D}) - g(\theta) = \nabla g(\theta)^T ((U_N^{h_1}, \dots, U_N^{h_D}) - \theta) + o_P(\|(U_N^{h_1}, \dots, U_N^{h_D}) - \theta\|).$$

From Corollary 3.5, $\sqrt{N}((U_N^{h_1}, \dots, U_N^{h_D}) - \theta)$ converges to a multivariate normal distribution with asymptotic covariance matrix Σ^{h_1, \dots, h_D} , so the delta method (see Theorem 3.1 of [50]) can be applied to prove this proposition. \square

4. Estimation of the asymptotic variance of a non-degenerated U -statistic

Theorem 3.1 shows the asymptotic normality of RCE submatrix U -statistics. In order to perform statistical inference using these U -statistics, one needs to estimate their variances. We see that

$$\mathbb{V}[U_N^h] = \left[\binom{m}{p} \binom{n}{q} \right]^{-2} \sum_{\substack{\mathbf{i} \in \mathcal{P}_p(\llbracket m \rrbracket) \\ \mathbf{j} \in \mathcal{P}_q(\llbracket n \rrbracket)}} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_p(\llbracket m \rrbracket) \\ \mathbf{j}' \in \mathcal{P}_q(\llbracket n \rrbracket)}} \text{Cov}(h(Y_{\mathbf{i}, \mathbf{j}}), h(Y_{\mathbf{i}', \mathbf{j}'})).$$

By exchangeability, the covariance between the kernels $h(Y_{\mathbf{i}, \mathbf{j}})$ and $h(Y_{\mathbf{i}', \mathbf{j}'})$ only depends on the number of row and column indices they share. Denote $\gamma_h^{r,c} := \text{Cov}(h(Y_{\mathbf{i}, \mathbf{j}}), h(Y_{\mathbf{i}', \mathbf{j}'}))$ where $Y_{\mathbf{i}, \mathbf{j}}$ and $Y_{\mathbf{i}', \mathbf{j}'}$ share r row indices and c column indices: $\text{Card}(\mathbf{i} \cap \mathbf{i}') = r$ and $\text{Card}(\mathbf{j} \cap \mathbf{j}') = c$. We get

$$\mathbb{V}[U_N^h] = \left[\binom{m}{p} \binom{n}{q} \right]^{-1} \sum_{(0,0) \leq (r,c) \leq (p,q)} \binom{p}{r} \binom{q}{c} \binom{m-p}{p-r} \binom{n-q}{q-c} \gamma_h^{r,c}.$$

Each $\gamma_h^{r,c}$, $1 \leq r \leq p, 1 \leq c \leq q$, can be estimated using empirical covariance estimators, between kernel terms that share r rows and c columns and in particular. This leads to an unbiased estimator of $\mathbb{V}[U_N^h]$ similar to the one discussed by [40] for U -statistics of one-dimensional i.i.d. arrays. However, the estimation of these covariances is computationally intensive and the estimators can take negative values, which can lead to a negative variance estimation.

One approach is to estimate the asymptotic variance V^h . The asymptotic variance formula given by Theorem 3.1 is $V^h = \frac{p^2}{\rho} v_h^{1,0} + \frac{q^2}{1-\rho} v_h^{0,1}$. Remark that $v_h^{1,0} = \gamma_h^{1,0}$ and $v_h^{0,1} = \gamma_h^{0,1}$. It is often tedious to

analytically calculate V^h , especially as it depends on h and the distribution of Y , see for example Section 3 of [29].

We present here a kernel-free and model-free estimator of V^h , taking advantage of the Hoeffding decomposition. Indeed, we will first define estimators for the conditional expectations $(\psi_{(\{i\}, \emptyset)}^{1,0} h)_{1 \leq i \leq m}$ and $(\psi_{(\emptyset, \{j\})}^{0,1} h)_{1 \leq j \leq n}$. Then, using the fact that $v_h^{1,0} = \mathbb{V}[\psi_{(\{1\}, \emptyset)}^{1,0} h]$ and $v_h^{0,1} = \mathbb{V}[\psi_{(\emptyset, \{1\})}^{0,1} h]$, we can derive a positive estimator for V^h .

V^h is the asymptotic variance of a single U -statistic. Afterwards, we explain how to generalize this estimator to estimate the asymptotic variance of a function of U -statistics to apply Corollary 3.6.

4.1. Some useful notations and results

First, we introduce further notations and a helpful lemma for this section.

For some $N > 0$, the size of the overall RCE matrix is $m_N \times n_N$. i being a row index means that $1 \leq i \leq m_N$ and j being a column index means that $1 \leq j \leq n_N$.

We denote

$$X_{\mathbf{i}, \mathbf{j}} := h(Y_{\mathbf{i}, \mathbf{j}}).$$

For N such that $m_N \geq p$ and $n_N \geq q$, we further denote

$$\mathcal{S}_N^{p,q} := \{(\mathbf{i}, \mathbf{j}) : \mathbf{i} \in \mathcal{P}_p(\llbracket m_N \rrbracket), \mathbf{j} \in \mathcal{P}_q(\llbracket n_N \rrbracket)\},$$

so that the set of the kernels taken on all the $p \times q$ submatrices can be written as $(X_{\mathbf{i}, \mathbf{j}})_{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_N^{p,q}}$.

Let $\underline{\mathbf{i}}$ be a set of row indices of size \underline{p} such that $0 \leq \underline{p} \leq p$ and $\underline{\mathbf{j}}$ a set of column indices of size \underline{q} such that $0 \leq \underline{q} \leq q$. The subset of $\mathcal{S}_N^{p,q}$ where $\underline{\mathbf{i}}$ is included in the row indices and $\underline{\mathbf{j}}$ is included in the column indices is denoted

$$\mathcal{S}_{N, (\underline{\mathbf{i}}, \underline{\mathbf{j}})}^{p,q} := \{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_N^{p,q} : \underline{\mathbf{i}} \subset \mathbf{i}, \underline{\mathbf{j}} \subset \mathbf{j}\}.$$

For example, the set of the kernels taken on all the $p \times q$ submatrices containing the columns 1 and 2 can be written $(X_{\mathbf{i}, \mathbf{j}})_{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_{N, (\emptyset, \{1, 2\})}^{p,q}}$.

It is obvious that

$$\text{Card}(\mathcal{S}_N^{p,q}) = \binom{m_N}{p} \binom{n_N}{q} \quad \text{and} \quad \text{Card}(\mathcal{S}_{N, (\underline{\mathbf{i}}, \underline{\mathbf{j}})}^{p,q}) = \binom{m_N - \underline{p}}{p - \underline{p}} \binom{n_N - \underline{q}}{q - \underline{q}}.$$

Let $I_K = (\underline{\mathbf{i}}_1, \dots, \underline{\mathbf{i}}_K)$ be a K -uplet of sets of row indices and $J_K = (\underline{\mathbf{j}}_1, \dots, \underline{\mathbf{j}}_K)$ a K -uplet of subsets of column indices. Denote

$$\mathcal{T}_{N, (I_K, J_K)}^{p,q} := \mathcal{S}_{N, (\underline{\mathbf{i}}_1, \underline{\mathbf{j}}_1)}^{p,q} \times \mathcal{S}_{N, (\underline{\mathbf{i}}_2, \underline{\mathbf{j}}_2)}^{p,q} \times \dots \times \mathcal{S}_{N, (\underline{\mathbf{i}}_K, \underline{\mathbf{j}}_K)}^{p,q}.$$

In the rest of the paper, we will often see averages of the type

$$T_N^{p,q}(I_K, J_K) := \frac{1}{\prod_{k=1}^K \text{Card}(\mathcal{T}_{N, (I_K, J_K)}^{p,q})} \sum_{\mathcal{T}_{N, (I_K, J_K)}^{p,q}} \mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} X_{\mathbf{i}_2, \mathbf{j}_2} \dots X_{\mathbf{i}_K, \mathbf{j}_K}]. \quad (6)$$

As a remark, $T_N^{p,q}((\emptyset), (\emptyset)) = U_N^h$.

By exchangeability, the quantities $\mathbb{E}[X_{(\mathbf{i}_1, \mathbf{j}_1)} X_{(\mathbf{i}_2, \mathbf{j}_2)} \dots X_{(\mathbf{i}_K, \mathbf{j}_K)}]$ do not depend on the row indices that do not belong to any pairwise intersection of the $(\mathbf{i}_k)_{1 \leq k \leq K}$. The same holds for column indices that do not belong to any pairwise intersection of the $(\mathbf{j}_k)_{1 \leq k \leq K}$. Therefore, assuming $m_N \geq \text{Card}(\cup_{k=1}^K \mathbf{i}_k)$ and $n_N \geq \text{Card}(\cup_{k=1}^K \mathbf{j}_k)$, we can define

$$\alpha(I_K, J_K) := \mathbb{E}[X_{\bar{\mathbf{i}}_1, \bar{\mathbf{j}}_1} X_{\bar{\mathbf{i}}_2, \bar{\mathbf{j}}_2} \dots X_{\bar{\mathbf{i}}_K, \bar{\mathbf{j}}_K}] \quad (7)$$

where for $1 \leq k \leq K$, the p -uplet $\bar{\mathbf{i}}_k$ only consists of elements of \mathbf{i}_k and elements that are not in any of the other $\mathbf{i}_{k'}$, i.e. the $\bar{\mathbf{i}}_k$ are of the form $\bar{\mathbf{i}}_k = \mathbf{i}_k \cup \tilde{\mathbf{i}}_k$ where $\cap_{k=1}^K \tilde{\mathbf{i}}_k = \emptyset$ and $(\cup_{k=1}^K \tilde{\mathbf{i}}_k) \cap (\cup_{k=1}^K \mathbf{i}_k) = \emptyset$.

The following lemma will be helpful in later proofs as it provides the asymptotic behaviour of $T_N^{p,q}(I_K, J_K)$. It shows that these averages can be reduced to one dominant expectation term given by $\alpha(I_K, J_K)$ and a remainder vanishing as N grows.

Lemma 4.1. *Let $I_K = (\mathbf{i}_1, \dots, \mathbf{i}_K)$ and $J_K = (\mathbf{j}_1, \dots, \mathbf{j}_K)$ be K -uplets of respectively row and column indices. Let $\alpha(I_K, J_K)$ defined by (7). Let :*

- $\underline{p}_k := \text{Card}(\mathbf{i}_k)$ and $\underline{q}_k := \text{Card}(\mathbf{j}_k)$, for $1 \leq k \leq K$,
- $\underline{P} := \sum_{k=1}^K \underline{p}_k$ and $\underline{Q} := \sum_{k=1}^K \underline{q}_k$,
- $\bar{p} := \text{Card}(\cup_{k=1}^K \mathbf{i}_k)$ and $\bar{q} := \text{Card}(\cup_{k=1}^K \mathbf{j}_k)$.

We have

$$T_N^{p,q}(I_K, J_K) = \alpha(I_K, J_K) + O(m_N^{-1} + n_N^{-1}).$$

4.2. Estimation of the conditional expectations

In this paragraph, for all $i \in \llbracket m_N \rrbracket$ and $j \in \llbracket n_N \rrbracket$, we define estimators for $\psi_{(\{i\}, \emptyset)}^{1,0} h = \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) \mid \xi_i]$ and $\psi_{(\emptyset, \{j\}}^{0,1} h = \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) \mid \eta_j]$, where ξ_i and η_j have been defined in Section 2.

Let $\widehat{\mu}_N^{h,(i)}$ be the average of the kernel function applied on the $p \times q$ submatrices containing the row i . Symmetrically, let $\widehat{\nu}_N^{h,(j)}$ be the average of the kernel function applied on the $p \times q$ submatrices containing the column j . This means

$$\widehat{\mu}_N^{h,(i)} := \binom{m_N - 1}{p - 1}^{-1} \binom{n_N}{q}^{-1} \sum_{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_{N, (\{i\}, \emptyset)}^{p,q}} h(Y_{\mathbf{i}, \mathbf{j}}), \quad (8)$$

and

$$\widehat{\nu}_N^{h,(j)} := \binom{m_N}{p}^{-1} \binom{n_N - 1}{q - 1}^{-1} \sum_{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_{N, (\emptyset, \{j\}}^{p,q}} h(Y_{\mathbf{i}, \mathbf{j}}). \quad (9)$$

Now, we establish some properties for these estimators.

Proposition 4.2. *If Y is a RCE matrix, then $\widehat{\mu}_N^{h,(i)}$ and $\widehat{\nu}_N^{h,(j)}$ are both conditionally unbiased $\psi_{(\{i\}, \emptyset)}^{1,0} h$ and $\psi_{(\emptyset, \{j\}}^{0,1} h$, i.e. we have for all $N \in \mathbb{N}$:*

- $\mathbb{E}[\widehat{\mu}_N^{h,(i)} \mid \xi_i] = \psi_{(\{i\}, \emptyset)}^{1,0} h$,
- $\mathbb{E}[\widehat{\nu}_N^{h,(j)} \mid \eta_j] = \psi_{(\emptyset, \{j\}}^{0,1} h$.

Proof. The first result can be found directly by the definition of $\psi_{(\{i\}, \emptyset)}^{1,0} h$, since

$$\begin{aligned} \mathbb{E}[\widehat{\mu}_N^{h,(i)} \mid \xi_i] &= \binom{m_N - 1}{p - 1}^{-1} \binom{n_N}{q}^{-1} \sum_{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_{N, (\{i\}, \emptyset)}^{p,q}} \mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \xi_i] \\ &= \binom{m_N - 1}{p - 1}^{-1} \binom{n_N}{q}^{-1} \sum_{(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_{N, (\{i\}, \emptyset)}^{p,q}} \psi_{(\{i\}, \emptyset)}^{1,0} h \\ &= \psi_{(\{i\}, \emptyset)}^{1,0} h. \end{aligned}$$

The second result can be obtained analogously. □

Proposition 4.3. *If Y is a RCE matrix, then :*

- $\widehat{\mu}_N^{h,(i)} \xrightarrow[N \rightarrow \infty]{a.s., L_1} \psi_{(\{i\}, \emptyset)}^{1,0} h$,

- $\widehat{\mathcal{V}}_N^{h,(j)} \xrightarrow[N \rightarrow \infty]{a.s., L_1} \psi_{(\emptyset, \{j\})}^{0,1} h.$

As a consequence, $\widehat{\mu}_N^{h,(i)}$ and $\widehat{\mathcal{V}}_N^{h,(j)}$ are consistent estimators for $\psi_{(\{i\}, \emptyset)}^{1,0} h$ and $\psi_{(\emptyset, \{j\})}^{0,1} h.$

Proof. Let $N \in \mathbb{N}$ and $\mathcal{F}_N(Y) = \sigma((\widehat{\mu}_K^{h,(i)}(Y))_{K \geq N})$. Let $\Phi_N \in \mathbb{S}_{m_N}^{(i)} \times \mathbb{S}_{n_N}$ where $\mathbb{S}_{m_N}^{(i)}$ is the group of permutations σ_i of $\llbracket m_N \rrbracket$ such that $\sigma_i(i) = i$. If $\Phi_N = (\sigma_i, \tau)$, denote $\Phi_N Y = (Y_{\sigma_i(k)\tau(j)})_{\substack{1 \leq k \leq m_N \\ 1 \leq j \leq n_N}}$.

First, we observe that $\widehat{\mu}_N^{h,(i)}(Y) = \widehat{\mu}_N^{h,(i)}(\Phi_N Y)$, so $\mathcal{F}_N(\Phi_N Y) = \mathcal{F}_N(Y)$. Therefore, by the exchangeability of Y , we have

$$Y \mid \mathcal{F}_N(Y) \stackrel{D}{=} \Phi_N Y \mid \mathcal{F}_N(Y).$$

This assertion is true for all $N \in \mathbb{N}$ and $\Phi_N \in \mathbb{S}_{m_N}^{(i)} \times \mathbb{S}_{n_N}$. Now, note that for all (\mathbf{i}, \mathbf{j}) and $(\mathbf{i}', \mathbf{j}')$ elements of $\mathcal{S}_{N, (\{i\}, \emptyset)}^{p,q}$, we can always find a permutation $\Phi_N \in \mathbb{S}_{m_N}^{(i)} \times \mathbb{S}_{n_N}$ such that $h(\Phi_N Y_{\mathbf{i}, \mathbf{j}}) = h(Y_{\mathbf{i}', \mathbf{j}'})$. Thus, we have $\mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{F}_N(Y)] = \mathbb{E}[h(Y_{\mathbf{i}', \mathbf{j}'}) \mid \mathcal{F}_N(Y)]$. Hence, for any $(\mathbf{i}, \mathbf{j}) \in \mathcal{S}_{N+1, (\{i\}, \emptyset)}^{p,q}$, we deduce that

$$\begin{aligned} \mathbb{E}[\widehat{\mu}_N^{h,(i)}(Y) \mid \mathcal{F}_{N+1}(Y)] &= \mathbb{E}[h(Y_{\mathbf{i}, \mathbf{j}}) \mid \mathcal{F}_{N+1}(Y)] \\ &= \mathbb{E}[\widehat{\mu}_{N+1}^{h,(i)}(Y) \mid \mathcal{F}_{N+1}(Y)] \\ &= \widehat{\mu}_{N+1}^{h,(i)}(Y). \end{aligned}$$

Therefore, $\widehat{\mu}_N^{h,(i)}(Y)$ is a backward martingale with respect to $\mathcal{F}_N(Y)$ (see Appendix A). By Theorem A.3, we have that $\widehat{\mu}_N^{h,(i)}(Y) \xrightarrow[N \rightarrow \infty]{a.s., L_1} \mathbb{E}[\widehat{\mu}_1^{h,(i)}(Y) \mid \mathcal{F}_\infty(Y)]$, where $\mathcal{F}_\infty(Y) = \bigcap_{N=1}^\infty \mathcal{F}_N(Y)$.

Finally, $\mathcal{F}_\infty(Y) = \sigma(\xi_i)$ so Proposition 4.2 implies that $\mathbb{E}[\widehat{\mu}_1^{h,(i)}(Y) \mid \mathcal{F}_\infty(Y)] = \psi_{(\{i\}, \emptyset)}^{1,0} h(Y)$ and thus, $\widehat{\mu}_N^{h,(i)}(Y) \xrightarrow[N \rightarrow \infty]{a.s., L_1} \psi_{(\{i\}, \emptyset)}^{1,0} h(Y)$. \square

4.3. Estimation of V^h

Finally, since we have defined estimators for $\psi_{(\{i\}, \emptyset)}^{1,0} h$ and $\psi_{(\emptyset, \{j\})}^{0,1} h$, natural estimators for $v_h^{1,0} = \mathbb{V}[\psi_{(\{1\}, \emptyset)}^{1,0} h]$ and $v_h^{0,1} = \mathbb{V}[\psi_{(\emptyset, \{j\})}^{0,1} h]$ can be given by :

$$\begin{aligned} \widehat{v}_N^{h;1,0} &= \binom{m_N}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq m_N} \frac{(\widehat{\mu}_N^{h,(i_1)} - \widehat{\mu}_N^{h,(i_2)})^2}{2} \\ \widehat{v}_N^{h;0,1} &= \binom{n_N}{2}^{-1} \sum_{1 \leq j_1 < j_2 \leq n_N} \frac{(\widehat{\mathcal{V}}_N^{h,(j_1)} - \widehat{\mathcal{V}}_N^{h,(j_2)})^2}{2} \end{aligned}$$

Then, an estimator for V^h is

$$\widehat{V}_N^h := \frac{p^2}{\rho} \widehat{v}_N^{h;1,0} + \frac{q^2}{1-\rho} \widehat{v}_N^{h;0,1}.$$

The following theorem shows that \widehat{V}_N^h is a consistent estimator for V^h .

Theorem 4.4. *We have $\widehat{v}_N^{h;1,0} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} v_h^{1,0}$ and $\widehat{v}_N^{h;0,1} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} v_h^{0,1}$. As a consequence, $\widehat{V}_N^h \xrightarrow[N \rightarrow \infty]{\mathbb{P}} V^h$.*

Before proving this theorem, we first need to highlight some properties of $\widehat{v}_N^{h;1,0}$ and $\widehat{v}_N^{h;0,1}$, which are proven in Appendix D.

Proposition 4.5. *We have $\mathbb{E}[\widehat{v}_N^{h;1,0}] = v_h^{1,0} + O(N^{-1})$ and $\mathbb{E}[\widehat{v}_N^{h;0,1}] = v_h^{0,1} + O(N^{-1})$. As a consequence, $\widehat{v}_N^{h;1,0}$ and $\widehat{v}_N^{h;0,1}$ are asymptotically unbiased estimators for $v_h^{1,0}$ and $v_h^{0,1}$.*

Proposition 4.6. *We have $\mathbb{V}[\widehat{v}_N^{h;1,0}] = O(N^{-1})$ and $\mathbb{V}[\widehat{v}_N^{h;0,1}] = O(N^{-1})$.*

Proof of Theorem 4.4. For some $\epsilon > 0$, it follows from Proposition 4.5 that for large enough values of N , $|\mathbb{E}[\widehat{v}_N^{h;1,0}] - v_h^{1,0}| < \epsilon$. The triangular inequality and Chebyshev's inequality states that

$$\mathbb{P}\left(\left|\widehat{v}_N^{h;1,0} - v_h^{1,0}\right| > \epsilon\right) \leq \mathbb{P}\left(\left|\widehat{v}_N^{h;1,0} - \mathbb{E}[\widehat{v}_N^{h;1,0}]\right| \geq \epsilon - \left|\mathbb{E}[\widehat{v}_N^{h;1,0}] - v_h^{1,0}\right|\right) \leq \frac{\mathbb{V}[\widehat{v}_N^{h;1,0}]}{\left(\epsilon - \left|\mathbb{E}[\widehat{v}_N^{h;1,0}] - v_h^{1,0}\right|\right)^2}.$$

Applying Propositions 4.5 and 4.6 to the right-hand side of the inequality ensures that $\mathbb{P}\left(\left|\widehat{v}_N^{h;1,0} - v_h^{1,0}\right| > \epsilon\right) \xrightarrow[N \rightarrow \infty]{} 0$ which concludes the proof. \square

With Theorem 4.4, it is possible to use \widehat{V}_N^h for statistical inference tasks when plugged-in in place of V^h , a asymptotic normality result similar to Theorem 3.1 holds.

Corollary 4.7. *If $V^h > 0$, then*

$$\sqrt{\frac{N}{\widehat{V}_N^h}} (U_N^h - U_\infty^h) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

Proof. First, Theorem 3.1 ensures that $\sqrt{\frac{N}{\widehat{V}_N^h}} (U_N^h - U_\infty^h) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1)$. Second, it can be derived from Theorem 4.4 that $\sqrt{\frac{V^h}{\widehat{V}_N^h}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 1$.

Then, applying Slutsky's theorem, we get $\sqrt{\frac{N}{\widehat{V}_N^h}} (U_N^h - U_\infty^h) = \sqrt{\frac{V^h}{\widehat{V}_N^h}} \times \sqrt{\frac{N}{V^h}} (U_N^h - U_\infty^h) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1)$. \square

4.4. Calculation of the estimator

In practice, the computation of the estimators $(\widehat{\mu}_N^{h,(i)})_{1 \leq i \leq m_N}$ and $(\widehat{v}_N^{h,(j)})_{1 \leq j \leq n_N}$ using their definition is pretty tedious, as they are sums of $O(N^{p+q-1})$ terms. The computation cost of \widehat{V}_N^h is then $O(N^{p+q})$ which is of same order as the computation of U_N^h when naively applying the kernel function to all the $p \times q$ submatrices and averaging.

However, one would actually try to avoid to compute U_N^h in that way if possible, as for simple kernels, it is possible to write U_N^h in the form of operation on matrices, which are more optimized, say $O(N^{a+b})$ with $a \leq p$ and $b \leq q$.

In these cases, it might be time-efficient to compute $\widehat{V}_N^h = \frac{p^2}{\rho} \widehat{v}_N^{h;1,0} + \frac{q^2}{1-\rho} \widehat{v}_N^{h;0,1}$ with the following alternative form.

Proposition 4.8. *An alternative form for $\widehat{v}_N^{h;1,0}$ and $\widehat{v}_N^{h;0,1}$ is given by*

$$\widehat{v}_N^{h;1,0} = \frac{(m_N - p)^2}{p^2(m_N - 1)} \sum_{i=1}^{m_N} \left(U_N^h - U_N^{h,(-i,\emptyset)} \right)^2$$

and

$$\widehat{v}_N^{h;0,1} = \frac{(n_N - q)^2}{q^2(n_N - 1)} \sum_{j=1}^{n_N} \left(U_N^h - U_N^{h,(\emptyset,-j)} \right)^2,$$

where

$$U_N^{h,(-i,\emptyset)} := \left[\binom{m_N - 1}{p} \binom{n_N}{q} \right]^{-1} \sum_{\substack{i \in \mathcal{P}_p(\llbracket m_N \rrbracket \setminus \{i\}) \\ j \in \mathcal{P}_q(\llbracket n_N \rrbracket)}} h(Y_{i,j})$$

and

$$U_N^{h,(\emptyset,-j)} := \left[\binom{m_N}{p} \binom{n_N - 1}{q} \right]^{-1} \sum_{\substack{i \in \mathcal{P}_p(\llbracket m_N \rrbracket) \\ j \in \mathcal{P}_q(\llbracket n_N \rrbracket \setminus \{j\})}} h(Y_{i,j}).$$

Proof. The relations are obtained from the definition of $\widehat{v}_N^{h;1,0}$ and $\widehat{v}_N^{h;0,1}$ and noticing that

$$\binom{m_N-1}{p-1} \binom{n_N}{q} \widehat{\mu}_N^{h,(i)} = \binom{m_N}{p} \binom{n_N}{q} U_N^h - \binom{m_N-1}{p} \binom{n_N}{q} U_N^{h,(-i,\emptyset)}$$

and

$$\binom{m_N}{p} \binom{n_N-1}{q-1} \widehat{\nu}_N^{h,(j)} = \binom{m_N}{p} \binom{n_N}{q} U_N^h - \binom{m_N}{p} \binom{n_N-1}{q} U_N^{h,(\emptyset,-j)}.$$

□

Remark. This is one alternative method to compute \widehat{V}_N^h , but not necessarily the optimal one. With this form, the computational cost of \widehat{V}_N^h is $O(N^{a+b+1})$. This can be outperformed by the naive method in some specific cases, e.g. when $a = p$ and $b = q$. This method is particularly fit when $a + b + 1 < p + q$.

Remark. The alternative form of $\widehat{v}_N^{h;1,0}$ and $\widehat{v}_N^{h;0,1}$ is reminiscent of the jackknife estimator for the variance of U -statistics of one-dimensional arrays [2], but the two are well distinct. In the case where Y is a one-dimensional array, the U -statistic associated to the kernel $h: \mathbb{R}^p \rightarrow \mathbb{R}$ is

$$U_N^h = \binom{N}{p}^{-1} \sum_{\mathbf{i} \in \mathcal{P}_p(\llbracket N \rrbracket)} h(X_{\mathbf{i}}).$$

The jackknife estimator of the asymptotic variance of this U -statistic is

$$\widehat{V}_N^{h,J} = (N-1) \sum_{i=1}^N (U_N^h - U_N^{h,(-i)})^2$$

where

$$U_N^{h,(-i)} = \binom{N}{p}^{-1} \sum_{\mathbf{i} \in \mathcal{P}_p(\llbracket N \rrbracket \setminus \{i\})} h(X_{\mathbf{i}}).$$

In fact, our estimator is closer to Sen's estimator of the asymptotic variance of [41, 42] which depends on the kernel size and is related to the jackknife estimator $\widehat{V}_N^{h,S} = (N-k)^2 / (N-1)^2 \widehat{V}_N^{h,J}$. However, it is unclear how these estimators could be translated in a two-dimensional setup where Y is a matrix instead of a vector, especially how to define the analog of $U_N^{h,(-i)}$.

4.5. Extension to functions of U -statistics

In the case of a function of U -statistics $g(U_N^{h_1}, \dots, U_N^{h_D})$, Corollary 3.6 applies and the asymptotic variance to be estimated is $V^\delta = \nabla g(\theta)^T \Sigma^{h_1, \dots, h_D} \nabla g(\theta) \neq 0$, where $\theta = (U_\infty^{h_1}, \dots, U_\infty^{h_D})$. Similar to the estimator for the asymptotic variance of V^h , we suggest an estimator for the covariance matrix $\Sigma^{h_1, \dots, h_D} = (C^{h_k, h_\ell})_{1 \leq k, \ell \leq D}$.

For each kernel h_k , $1 \leq k \leq D$, let $\widehat{\mu}_N^{h_k,(i)}$ and $\widehat{\nu}_N^{h_k,(j)}$ be the respective estimators of the conditional expectations $\psi_{\{(i), \emptyset\}}^{1,0} h_k$ and $\psi_{\{\emptyset, \{j\}\}}^{0,1} h_k$, as defined in equations (8) and (9).

Now define

$$\widehat{c}_N^{h_k, h_\ell; 1, 0} := \binom{m_N}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq m_N} \frac{(\widehat{\mu}_N^{h_k,(i_1)} - \widehat{\mu}_N^{h_k,(i_2)})(\widehat{\mu}_N^{h_\ell,(i_1)} - \widehat{\mu}_N^{h_\ell,(i_2)})}{2}$$

and

$$\widehat{c}_N^{h_k, h_\ell; 0, 1} := \binom{n_N}{2}^{-1} \sum_{1 \leq j_1 < j_2 \leq n_N} \frac{(\widehat{\nu}_N^{h_k,(j_1)} - \widehat{\nu}_N^{h_k,(j_2)})(\widehat{\nu}_N^{h_\ell,(j_1)} - \widehat{\nu}_N^{h_\ell,(j_2)})}{2}.$$

Then, for two kernels h_k and h_ℓ ,

$$\widehat{C}_N^{h_k, h_\ell} := \frac{p^2}{\rho} \widehat{c}_N^{h_k, h_\ell; 1, 0} + \frac{q^2}{1-\rho} \widehat{c}_N^{h_k, h_\ell; 0, 1},$$

is an estimator of the asymptotic covariance term C^{h_k, h_ℓ} .

With a similar proof as for Theorem 4.4, the following theorem ensures the consistency of this estimator.

Theorem 4.9. For two kernel functions h_k and h_ℓ , we have $\widehat{c}_N^{h_k, h_\ell; 1, 0} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} c_{h_k, h_\ell}^{1, 0}$ and $\widehat{c}_N^{h_k, h_\ell; 0, 1} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} c_{h_k, h_\ell}^{0, 1}$. As a consequence, $\widehat{C}_N^{h_k, h_\ell} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} C^{h_k, h_\ell}$.

Thus, for linearly independent kernel functions (h_1, \dots, h_D) , the entries of the matrix $\widehat{\Sigma}_N^{h_1, \dots, h_D} := \left(\widehat{C}_N^{h_k, h_\ell} \right)_{1 \leq k, \ell \leq D}$ converge to the entries of Σ^{h_1, \dots, h_D} . Set $\widehat{V}_N^\delta := \nabla g(U_N^{h_1}, \dots, U_N^{h_D})^T \widehat{\Sigma}_N^{h_1, \dots, h_D} \nabla g(U_N^{h_1}, \dots, U_N^{h_D})$, then we have the following straightforward result.

Corollary 4.10. We have $\widehat{V}_N^\delta \xrightarrow[N \rightarrow \infty]{\mathbb{P}} V^\delta$.

Finally, the plug-in corollary also stands.

Corollary 4.11. If $V^\delta > 0$, then

$$\sqrt{\frac{N}{\widehat{V}_N^\delta}} (g(U_N^{h_1}, \dots, U_N^{h_D}) - g(\theta)) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

5. RCE models, kernel functions and network comparison

5.1. Examples of RCE models

Bipartite Expected Degree Distribution model The Bipartite Expected Degree Distribution (BEDD) model, suggested by [36], is a binary graph model characterised by two distributions from which the row and column nodes draw a weight. The probability of a connection between two nodes is fully determined by the corresponding row and column weight distributions. The distribution of a graph following a BEDD model can be written using latent variables $(\xi_i)_{i \geq 1}$ and $(\eta_j)_{j \geq 1}$ corresponding to the row and column nodes of the graph :

$$\begin{aligned} \xi_i, \eta_j &\stackrel{iid}{\sim} \mathcal{U}[0, 1] \\ Y_{ij} \mid \xi_i, \eta_j &\sim \mathcal{B}(\lambda f(\xi_i) g(\eta_j)). \end{aligned} \tag{10}$$

where

- f and g are positive, càdlàg, nondecreasing, bounded and normalized ($\int f = \int g = 1$) real functions $[0, 1] \rightarrow \mathbb{R}_+$,
- λ is a positive real number such that $\lambda \leq \|f\|_\infty^{-1} \|g\|_\infty^{-1}$,
- \mathcal{B} is the Bernoulli distribution.

The BEDD model is a RCE version of the Expected Degree Sequence model of [12] but in the BEDD, the row weights $f(\xi_i)$ and column weights $g(\eta_j)$ are exchangeable. Indeed, f and g characterize the weight distributions of the row and column nodes whereas the weights are fixed in the Expected Degree Sequence model. The triplet (λ, f, g) is called the BEDD parameters.

Latent Block model The Latent Block model (LBM) [21] is a binary graph model characterised by a partition of row and column nodes in several groups. It can be considered as a bipartite extension of the Stochastic Block model [34]. The probability of interaction between two nodes is fully determined by the groups to which they belong. All the nodes have the same probability to belong to each group. The distribution of a LBM is most commonly written using independent latent variables for the node attribution in a group $(Z_i)_{i \geq 1}$ and $(W_j)_{j \geq 1}$ corresponding to the row and column nodes of the graph :

$$\begin{aligned} Z_i &\stackrel{iid}{\sim} \mathcal{M}(1; \boldsymbol{\alpha}) \\ W_j &\stackrel{iid}{\sim} \mathcal{M}(1; \boldsymbol{\beta}) \\ Y_{ij} \mid Z_i = k, W_j = \ell &\sim \mathcal{B}(\pi_{k\ell}), \end{aligned} \tag{11}$$

where $\alpha = (\alpha_1, \dots, \alpha_K)$ and $\beta = (\beta_1, \dots, \beta_L)$ are the probability vectors of the rows and the columns and $\pi = (\pi_{k\ell})_{1 \leq k \leq K, 1 \leq \ell \leq L} \in [0, 1]^{KL}$ is a matrix of probabilities.

The LBM is a RCE model, since the group attribution variables of the nodes are exchangeable.

W-graph model Let w be a function of $[0, 1]^2 \rightarrow [0, 1]$. The W-graph model associated to w is defined by

$$\begin{aligned} \xi_i, \eta_j &\stackrel{i.i.d.}{\sim} \mathcal{U}[0, 1] \\ Y_{ij} &| \xi_i, \eta_j \sim \mathcal{B}(w(\xi_i, \eta_j)), \end{aligned} \quad (12)$$

w is sometimes referred to as a graphon. For identification reasons, we assume $\int w(\cdot, \eta) d\eta$ and $\int w(\xi, \cdot) d\xi$ to be càdlàg, nondecreasing and bounded.

Any RCE model can be written as a W-graph model [14] so can the BEDD model and the LBM be expressed with a graphon. For the BEDD model, $w(\xi_i, \eta_j) = \lambda f(\xi_i) g(\eta_j)$ where f and g are those of Equation (10). For the LBM, $w(\xi_i, \eta_j) = \sum_{k,\ell} \pi_{k\ell} \mathbb{1}\{s(\xi_i) = k\} \mathbb{1}\{t(\eta_j) = \ell\}$ where $s(\xi_i) = 1 + \sum_{k=1}^K \mathbb{1}\{\xi_i > \sum_{k'=1}^k \alpha_{k'}\}$, $t(\eta_j) = 1 + \sum_{\ell=1}^L \mathbb{1}\{\eta_j > \sum_{\ell'=1}^{\ell} \beta_{\ell'}\}$ and α , β and π are those of Equation (11).

Extension to weighted graphs All the models presented above are defined for binary graphs. However, one can extend it to weighted graphs by switching the Bernoulli distributions with another. In particular, the Poisson distribution is particularly fitted for count values in \mathbb{N} :

$$\begin{aligned} \xi_i, \eta_j &\stackrel{i.i.d.}{\sim} \mathcal{U}[0, 1] \\ Y_{ij} &| \xi_i, \eta_j \sim \mathcal{P}(w(\xi_i, \eta_j)), \end{aligned} \quad (13)$$

where $w : [0, 1]^2 \rightarrow \mathbb{R}_+$ is the graphon of this model. Similarly, one can define a Poisson-BEDD model or Poisson-LBM.

5.2. Examples of kernel functions

We present three examples of statistics which are U -statistics or functions of U -statistics. Sometimes, a kernel function h has a long expression, especially when it is the symmetric version of some simpler function h^0 , as in (3). In this case, we introduce the kernels of interest with h^0 instead of h , but the U -statistics and asymptotic results always apply to the symmetric version h .

Motif counts Motifs are the name given to small subgraphs. Their occurrences in the complete network can be counted. Motif counts are useful statistics for random graphs as they provide information on the network local structure [43, 26]. Many random network models hinge on motif frequencies. In the Exponential Random Graph Model [19], motif frequencies are sufficient statistics. The dk -random graph model [35] also largely relies on motif frequencies.

Their asymptotic properties are widely studied and a large numbers of studies use motif counts to perform statistical tests [39, 8, 7, 13, 20, 30, 32, 36]. Our framework is particularly well adapted to the use of motif counts for statistical tests as frequencies are in fact U -statistics with kernel functions of the same size as the motifs.

For many applications, motifs are considered as elementary building blocks which can be specifically interpreted. This is the case in molecular biology [43, 37, 1], neurology [52], sociology [6], evolution [38] and ecology [48, 4]. In particular, Figure 3 of [45] lists all the bipartite motifs consisting of from 2 to 6 nodes. For example, their motif 6 represents a 2×2 clique, where every node is connected to all others (Figure 1). Their motif 14 represents a path between two column nodes, passing through another column node and two row nodes (Figure 1). The latter indicates an indirect interaction between the row nodes, hinging on the middle row node. [27] found that motif 6 is over-represented in plant-pollinator interaction networks, while motif 14 is under-represented, compared to Erdős-Rényi graphs of the same density.

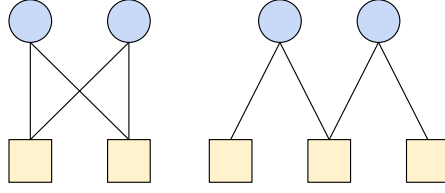


Figure 1: Motif 6 and 14, counted by $U_N^{h_6}$ and $U_N^{h_{14}}$

These motifs can be counted with $U_N^{h_6}$ and $U_N^{h_{14}}$ the U -statistics associated to the kernels h_6 and h_{14} , that we introduce as the symmetric version of the following functions :

$$\begin{aligned} h_6^0(Y_{(i_1, i_2; j_1, j_2)}) &= Y_{i_1 j_1} Y_{i_1 j_2} Y_{i_2 j_1} Y_{i_2 j_2}, \\ h_{14}^0(Y_{(i_1, i_2; j_1, j_2, j_3)}) &= Y_{i_1 j_1} Y_{i_1 j_2} Y_{i_2 j_2} Y_{i_2 j_3} (1 - Y_{i_2 j_1}) (1 - Y_{i_1 j_3}). \end{aligned} \quad (14)$$

Remark. With this example, it is very apparent that introducing simple, non-symmetric kernel functions and then symmetrizing them is way simpler than introducing symmetric kernel functions. h_6^0 is already symmetric, so $h_6 = h_6^0$. However, h_{14}^0 is not symmetric and one can sum over all the permutations of the indices to make it symmetric using formula (3). Because of the automorphisms of motif 14, it involves only 6 permutations instead of $2!3! = 12$.

$$\begin{aligned} h_{14}(Y_{(i_1, i_2; j_1, j_2, j_3)}) &= \frac{1}{6} Y_{i_1 j_1} Y_{i_1 j_2} Y_{i_2 j_2} Y_{i_2 j_3} (1 - Y_{i_2 j_1}) (1 - Y_{i_1 j_3}) + \frac{1}{6} Y_{i_1 j_2} Y_{i_1 j_3} Y_{i_2 j_3} Y_{i_2 j_1} (1 - Y_{i_1 j_1}) (1 - Y_{i_2 j_2}) \\ &+ \frac{1}{6} Y_{i_1 j_3} Y_{i_1 j_1} Y_{i_2 j_1} Y_{i_2 j_2} (1 - Y_{i_1 j_2}) (1 - Y_{i_2 j_3}) + \frac{1}{6} Y_{i_2 j_1} Y_{i_2 j_2} Y_{i_1 j_2} Y_{i_1 j_3} (1 - Y_{i_1 j_1}) (1 - Y_{i_2 j_3}) \\ &+ \frac{1}{6} Y_{i_2 j_2} Y_{i_2 j_3} Y_{i_1 j_3} Y_{i_1 j_1} (1 - Y_{i_2 j_1}) (1 - Y_{i_1 j_2}) + \frac{1}{6} Y_{i_2 j_3} Y_{i_2 j_1} Y_{i_1 j_1} Y_{i_1 j_2} (1 - Y_{i_2 j_2}) (1 - Y_{i_1 j_3}). \end{aligned}$$

Using Corollary 4.7, the following studentized statistics converge to a standard normal distribution

$$\begin{aligned} Z_N^6 &:= \sqrt{\frac{N}{\widehat{V}_N^{h_6}}} (U_N^{h_6} - U_\infty^{h_6}) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1), \\ Z_N^{14} &:= \sqrt{\frac{N}{\widehat{V}_N^{h_{14}}}} (U_N^{h_{14}} - U_\infty^{h_{14}}) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1). \end{aligned}$$

Product graphon The W -graph model encompasses all the dissociated RCE models. One can make use of this model to compare the form of the graphon that has generated a graph to a known form. One interesting form is the product form, which corresponds to an absence of specific interaction between row nodes and column nodes. For example, one can normalize the graphon in the Poisson W -graph model described by (13) such that

$$Y_{ij} \mid \xi_i, \eta_j \sim \mathcal{P}(\lambda \bar{w}(\xi_i, \eta_j)),$$

where $\lambda > 0$ and $\iint \bar{w} = 1$. Call \bar{w} a normalized graphon. Define f and g as the marginals of \bar{w} , i.e.

$$f = \int \bar{w}(\cdot, \eta) d\eta \qquad g = \int \bar{w}(\xi, \cdot) d\xi. \quad (15)$$

A density-free dissimilarity measure between a graphon and its corresponding product form could be the quantity $d(\bar{w})$ defined as

$$d(\bar{w}) := \|\bar{w} - fg\|_2^2 = \iint (\bar{w}(\xi, \eta) - f(\xi)g(\eta))^2 d\xi d\eta. \quad (16)$$

Observe that $d(\bar{w}) = 0$ if and only if \bar{w} is of product form, e.g. $\bar{w}(\xi, \eta) = f(\xi)g(\eta)$ almost everywhere in $[0, 1]^2$. In this case, the W -graph model (5.2) is a BEDD model and f and g have the same role as in (10).

h	$h^0(Y_{(i_1, \dots, i_r; j_1, \dots, j_c)})$	$\mathbb{E}[h(Y_{(i_1, \dots, i_r; j_1, \dots, j_c)})]$
h_A	$h_{A,1}^0(Y_{(i_1, i_2; j_1, j_2)}) - 2h_{A,2}^0(Y_{(i_1, i_2; j_1, j_2)})$	$\lambda^3 \iint \bar{w}(\xi, \eta)(\bar{w}(\xi, \eta) - 2f(\xi)g(\eta))d\xi d\eta$
$h_{A,1}$	$Y_{i_1 j_1}(Y_{i_1 j_1} - 1)Y_{i_2 j_2}$	$\lambda^3 \iint \bar{w}(\xi, \eta)^2 d\xi d\eta$
$h_{A,2}$	$Y_{i_1 j_1} Y_{i_1 j_2} Y_{i_2 j_2}$	$\lambda^3 \iint \bar{w}(\xi, \eta) f(\xi) g(\eta) d\xi d\eta$
h_B	$Y_{i_1 j_1} Y_{i_1 j_2}$	$\lambda^2 \int f(\xi)^2 d\xi$
h_C	$Y_{i_1 j_1} Y_{i_2 j_1}$	$\lambda^2 \int g(\eta)^2 d\eta$
h_D	$Y_{i_1 j_1}$	λ

Table 1: Kernel functions used for the estimation of $d(\bar{w})$. The expectations are given by Lemma E.1 in Appendix.

We suggest an estimator for $d(\bar{w})$. Let h_A, h_B, h_C and h_D be kernel functions of respective size 2×2 , 1×2 , 2×1 and 1×1 defined as in Table 1. These kernel functions are linearly independent, so Corollary 3.5 applies to the associated U -statistics $(U_N^{h_A}, U_N^{h_B}, U_N^{h_C}, U_N^{h_D})$, and they are jointly asymptotically normal with asymptotic covariance matrix $\Sigma^{h_A, h_B, h_C, h_D}$.

We define the estimator of $d(\bar{w})$ as $\hat{d}_N = t(U_N^{h_A}, U_N^{h_B}, U_N^{h_C}, U_N^{h_D}) := U_N^{h_A} / (U_N^{h_D})^3 + U_N^{h_B} U_N^{h_C} / (U_N^{h_D})^4$. We have $t(U_N^{h_A}, U_N^{h_B}, U_N^{h_C}, U_N^{h_D}) = \iint \bar{w}(\xi, \eta)(\bar{w}(\xi, \eta) - 2f(\xi)g(\eta))d\xi d\eta + \int f(\xi)^2 d\xi \times \int g(\eta)^2 d\eta = d(\bar{w})$. Then, Corollary 4.11 ensures that the studentized statistic converge to a standard normal distribution

$$Z_N^d := \sqrt{\frac{N}{\widehat{V}_N^d}} (\hat{d}_N - d(\bar{w})) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

where $\widehat{V}_N^d = \nabla t(U_N^{h_A}, U_N^{h_B}, U_N^{h_C}, U_N^{h_D})^T \widehat{\Sigma}_N^{h_A, h_B, h_C, h_D} \nabla t(U_N^{h_A}, U_N^{h_B}, U_N^{h_C}, U_N^{h_D})$ and $\widehat{\Sigma}_N^{h_A, h_B, h_C, h_D}$ is defined as in Corollary 4.11.

Heterogeneity of the rows of a network The degree distributions of a network hold significant information about its topology. For a binary network, denote $D_i := \sum_j Y_{ij}$ the degree of the i -th row, then $\mathbb{E}[D_i | \xi_i] = f(\xi_i)\mathbb{E}[D_i]$, where f is given by (15). Therefore, the marginals f and g accounts for the expected relative degree distributions of the binary network, which is known to characterize networks. In the BEDD model, these distributions (and the density of the network) fully characterize the model.

Although the sum of the edge weights and the number of edges stemming from a nodes are equivalent for binary networks, for weighted networks, they are two different quantities. The sum of the edge weights stemming from a node is sometimes called its strength [5], which do not depend entirely on the number of edges. In a network with weighted edges, $f(\xi_i)$ corresponds to expected relative strength of the i -th row node, instead of its expected relative degree.

One way to characterize the degree/strength distributions of a network is to calculate their variance. In particular, the variance of the degree distribution of a network quantifies its heterogeneity (i.e. the unbalance between strongly interacting nodes and the others) and can be used as an index to characterize networks [46]. In our framework with random graph models, rather than directly study the empirical distribution of row degrees/strengths, one would like to retrieve information on f and g , the distributions of the expected degrees/strengths specified by the model. For the BEDD model, f and g are directly given by the model. The variance of the row expected relative degree/strength distribution is $F_2 - 1$, where $F_2 := \int f^2(\xi) d\xi$.

F_2 can be estimated using the estimator $\widehat{F}_{2,N} := \kappa(U_N^{h_1}, U_N^{h_2}) = U_N^{h_1} / U_N^{h_2}$, using the U -statistics associated to the submatrix kernels functions h_1 and h_2 defined as

$$h_1(Y_{(i; j_1, j_2)}) = Y_{i j_1} Y_{i j_2}$$

and

$$h_2(Y_{(i_1, i_2; j_1, j_2)}) = \frac{1}{2}(Y_{i_1 j_1} Y_{i_2 j_2} + Y_{i_1 j_2} Y_{i_2 j_1}).$$

By Lemma E.2 in Appendix, we have $U_\infty^{h_1} = \lambda^2 F_2$ and $U_\infty^{h_2} = \lambda^2$. Given that $\nabla \kappa(U_N^{h_1}, U_N^{h_2}) = (1/U_N^{h_2}, -U_N^{h_1}/(U_N^{h_2})^2)$, the application of Corollary 4.11 yields

$$Z_N^{F_2} := \sqrt{\frac{N}{\widehat{V}_N^{F_2}}} \left(\widehat{F}_{2,N} - F_2 \right) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1), \quad (17)$$

where

$$\widehat{V}_N^{F_2} = \frac{1}{(U_N^{h_2})^2} \widehat{V}_N^{h_1} - \frac{2\widehat{F}_{2,N}}{(U_N^{h_2})^2} \widehat{C}_N^{h_1, h_2} + \frac{(\widehat{F}_{2,N})^2}{(U_N^{h_2})^2} \widehat{V}_N^{h_2}.$$

In a similar way, one can define the analogous estimator $\widehat{G}_{2,N}$ for $G_2 := \int g^2(\eta) d\eta$ and its associated variance estimator to quantify the heterogeneity of the columns of networks.

5.3. Network comparison

Network comparison has a long history in network analysis literature [18, 49]. Few comparison methods use random network models [3, 30, 29]. However, there are several advantages to using model-based approaches. Indeed, random models define a probability distribution of networks which can be used to derive statistical guarantees. Also, models can be used to control the sources of heterogeneity in networks. The comparison can be made with respect to a quantity of interest, which makes it easier to interpret. Now, we show how to define a test statistic for model-based network comparison with our framework. Naturally, U -statistics define network statistics characterizing a network. They can be used to analyze a single network, but their use is easily extended to compare different networks.

Let Y^A and Y^B be two independent networks of respective sizes $m_N^A \times n_N^A$ and $m_N^B \times n_N^B$. Define a network quantity of interest θ . These two networks are generated by two models, leading to different values θ^A and θ^B of this quantity for the two models. Let $\widehat{\theta}_N$ be an estimator for this quantity of interest. If $\widehat{\theta}_N$ is a U -statistic or a function of U -statistics, then as previously, we have both

$$\sqrt{N}(\widehat{\theta}_N(Y^A) - \theta^A) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V^A)$$

and

$$\sqrt{N}(\widehat{\theta}_N(Y^B) - \theta^B) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V^B),$$

where V^A and V^B are the asymptotic variances.

To compare θ^A and θ^B , we can confront the test hypotheses $\mathcal{H}_0 : \theta^A = \theta^B$ and $\mathcal{H}_1 : \theta^A \neq \theta^B$. Because Y^A and Y^B are independent, the previous convergence results give

$$\sqrt{\frac{N}{V^A + V^B}} \left(\widehat{\delta}_N(Y^A, Y^B) - (\theta^A - \theta^B) \right) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1),$$

where $\widehat{\delta}_N(Y^A, Y^B) := \widehat{\theta}_N(Y^A) - \widehat{\theta}_N(Y^B)$. Therefore, using the consistent estimators \widehat{V}_N^A and \widehat{V}_N^B of the asymptotic variances, the test statistic

$$Z_N(Y^A, Y^B) := \sqrt{\frac{N}{\widehat{V}_N^A + \widehat{V}_N^B}} \widehat{\delta}_N(Y^A, Y^B)$$

admits the convergence result

$$Z_N(Y^A, Y^B) - \sqrt{\frac{N}{\widehat{V}_N^A + \widehat{V}_N^B}} (\theta^A - \theta^B) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1).$$

Under \mathcal{H}_0 , we have $\theta^A - \theta^B = 0$, so $Z_N(Y^A, Y^B)$ converges in distribution to a standard Gaussian variable.

6. Simulations

In this section, we illustrate our theoretical results, in particular Corollaries 4.7 and 4.11, using the previous examples of graph models and kernel functions. For each example of kernel function, we check the asymptotic normality of the studentized statistic for networks simulated under different configurations (different models, values of N and ρ). We give the resulting Q-Q plots. We also examine the coverage probabilities of the confidence intervals built with the estimates or the standard deviations of the statistics, depending on whether they are fitted for estimation or statistical testing.

6.1. Motif counts

Model I This graph model is a LBM with 2 row groups and 2 columns groups of equal proportion. Using the notations of (11), this means $\alpha = \beta = (0.5, 0.5)$. The probability matrix π has size 2×2 . We set $\pi_{k\ell} = 0.5$ for all $1 \leq k, \ell \leq 2$ except $\pi_{11} = 0.95$.

Under Model I, for each value $N \in \{32, 45, 64, 90, 128, 181, 256, 362, 512, 724, 1024, 1448, 2048\}$ and $\rho \in \{1/8, 1/2\}$, we have simulated $K = 500$ networks of size $m_N \times n_N$ where $m_N = \lfloor \rho N \rfloor$ and $n_N = N - m_N$. The relative frequencies of motifs 6 and 10 of [45] are respectively given by $U_N^{h_6}$ and $U_N^{h_{14}}$ where h_6 and h_{14} were defined by (14).

The Q-Q plots of the studentized statistic associated to h_6 and h_{14} are given in Figures 2 and 3. For $\rho = 1/2$, we observe that the empirical distribution of both statistics converge and become close to a normal distribution as $N \gtrsim 128$. Figure 4 gives the respective coverage probabilities for $U_\infty^{h_6}$ and $U_\infty^{h_{14}}$ of the intervals $\left[U_N^{h_6} - \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{h_6}}}, U_N^{h_6} + \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{h_6}}} \right]$ and $\left[U_N^{h_{14}} - \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{h_{14}}}}, U_N^{h_{14}} + \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{h_{14}}}}} \right]$ respectively, where Φ is the quantile function of the standard normal distribution. The coverage probabilities converge to α but with different speeds depending on the motif. We also observe that a larger number of nodes is needed to reach the target coverage probabilities with $\rho = 1/8$ (rectangular matrix) than $\rho = 1/2$ (square matrix).

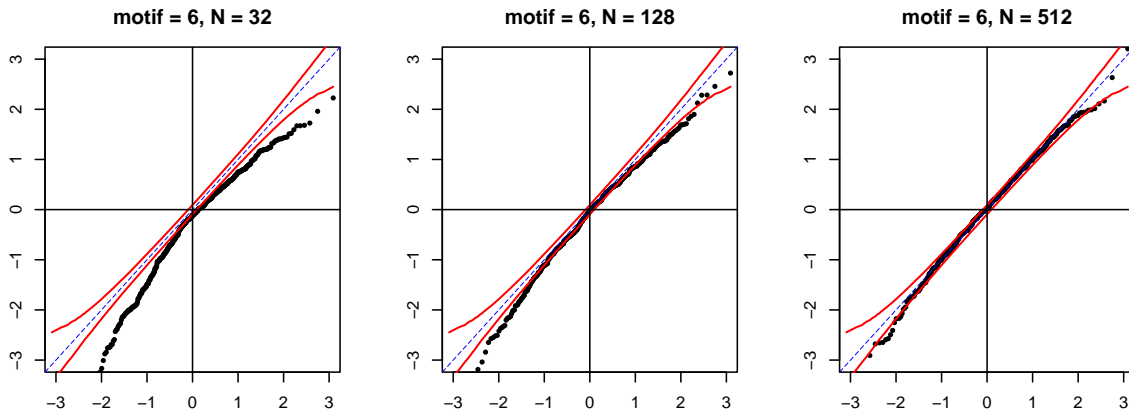


Figure 2: Q-Q plots for Z_N^6 the studentized statistic associated with $U_N^{h_6}$, with $\rho = 0.5$.

6.2. Graphon product distance

Model II(ϵ) We consider a Poisson-LBM. As this is a weighted graph model, π is a matrix of weights rather than probabilities, i.e. $(\pi_{k\ell})_{1 \leq k \leq K, 1 \leq \ell \leq L}$ are real non-negative numbers. We consider 2 row groups and 2 column groups of equal proportion. Let π^0 be a weight matrix. We set $\pi_{11}^0 = 4$, $\pi_{12}^0 = \pi_{21}^0 = 2$ and $\pi_{22}^0 = 1$. The LBM with parameters α , β and π is also a BEDD model. Indeed, its graphon has a product form and can be written $w_0(\xi, \eta) = \lambda f_0(\xi)g_0(\eta)$ where $\lambda = 9/4$, $f_0 = g_0$ both take values $4/3$ on $[0, 0.5]$, $2/3$ on $]0.5, 1]$.

Now, let τ be the 2×2 matrix where $\tau_{11} = \tau_{22} = 2$ and $\tau_{12} = \tau_{21} = 0$. For $\epsilon \geq 0$, we define Model II(ϵ)

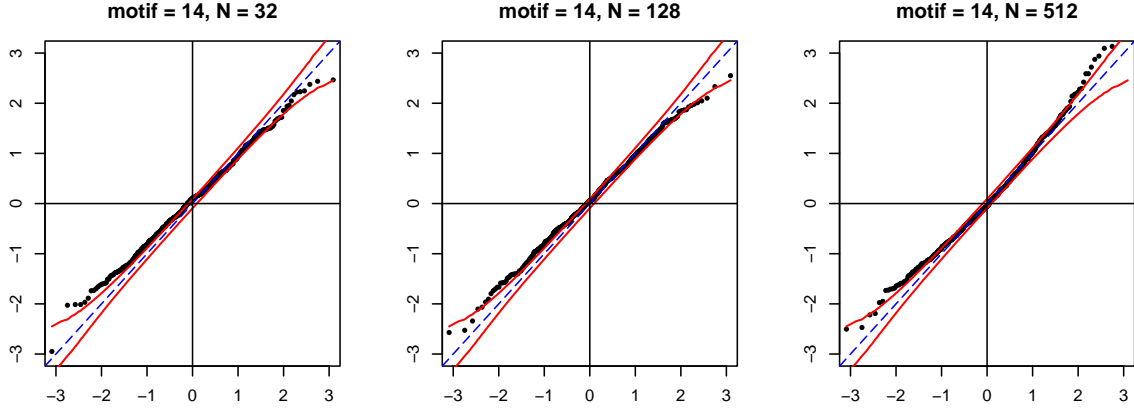


Figure 3: Q-Q plots for Z_N^{14} the studentized statistic associated with $U_N^{h_{14}}$, with $\rho = 0.5$.

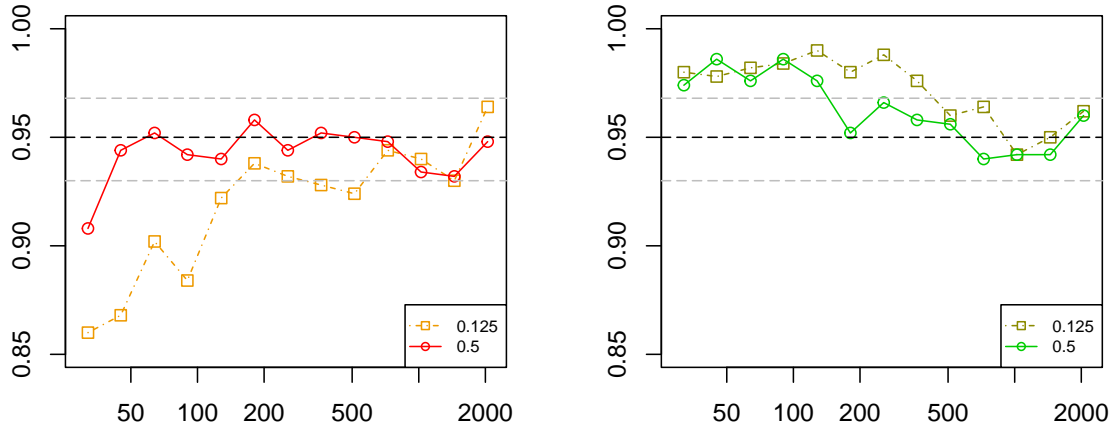


Figure 4: Empirical coverage probabilities for the asymptotic confidence intervals at level $\alpha = 0.95$ of $U_N^{h_6}$ (left) and $U_N^{h_{14}}$ (right) for different values of N (x-axis), $\rho \in \{1/8, 1/2\}$. Grey dashed lines represent the confidence interval at level 0.95 of the frequency $Z = X/K$, if X follows the binomial distribution with parameters K and $\alpha = 0.95$.

as a LBM with group probabilities α and β and with weight matrix $\pi^\epsilon = \frac{\lambda}{\lambda + \epsilon}(\pi^0 + \epsilon\tau)$. Thus, Model II(0) is a LBM with product form (BEDD model) and for $\epsilon > 0$, Model II(ϵ) with graphon $w_\epsilon = \lambda\bar{w}_\epsilon$ is a perturbed version and rescaled to preserve the same density λ . As ϵ grows, the graphon of Model II(ϵ) strays further from a BEDD model. Indeed, one can show that $d(\bar{w}_\epsilon) = 64\epsilon^2(5 + 2\epsilon)^2/(9 + 4\epsilon)^4$, which is an increasing function for $\epsilon \geq 0$.

Under Model II(ϵ), for each value $N \in \{32, 45, 64, 90, 128, 181, 256, 362, 512, 724, 1024, 1448, 2048\}$, $\rho \in \{1/8, 1/2\}$, $\epsilon \in \{0.5, 1, 1.5, 2, 2.5, 3\}$, we have simulated $K = 500$ networks of size $m_N \times n_N$ where $m_N = \lfloor \rho N \rfloor$ and $n_N = N - m_N$. For each network, we have computed \hat{d}_N and \hat{V}_N^d as estimates of $d(\bar{w}_\epsilon)$ and V^d respectively. The Q-Q plots of the studentized statistic Z_N^d are given in Figures 5. It is apparent that Z_N^d is not centered, especially for $N \lesssim 512$. This is due to the fact that \hat{d}_N is obtained via the delta method, so it is a biased estimator of $d(\bar{w}_\epsilon)$ for finite values of N . However, the bias converges to 0 when N grows, so we find that the statistic achieves normality for $N \gtrsim 2048$. Figure 6 gives for different values of ϵ , the coverage probability for $d(\bar{w}_\epsilon)$ of the interval $\left[\hat{d}_N - \Phi\left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{N}{\hat{V}_N^d}}, \hat{d}_N + \Phi\left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{N}{\hat{V}_N^d}} \right]$ where Φ is the quantile function of the standard normal distribution. We observe that convergence is fastest when $\rho = 1/2$ but also when ϵ is larger, which seems to indicate that estimation of $d(\bar{w})$ is more precise for square matrices and when w is further away from a product model.

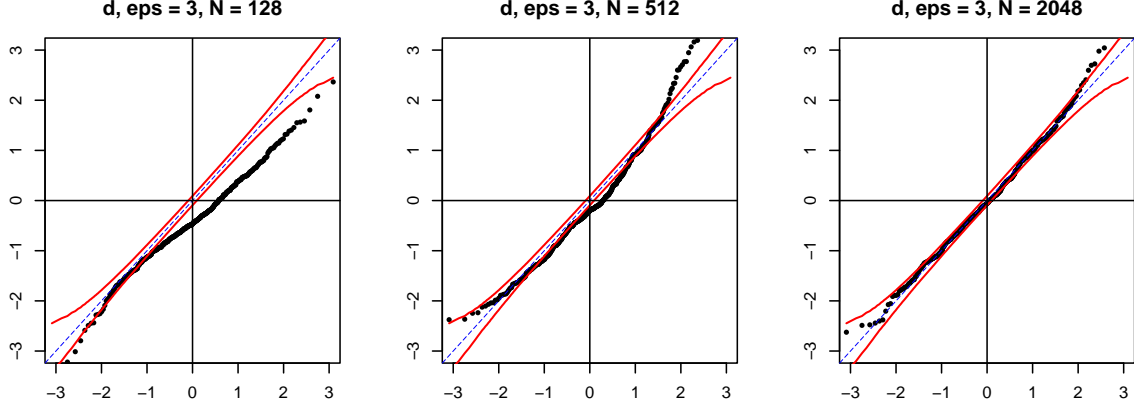


Figure 5: Q-Q plot for Z_N^d the studentized statistic associated with \widehat{d}_N , $\epsilon = 3$, $\rho = 0.5$

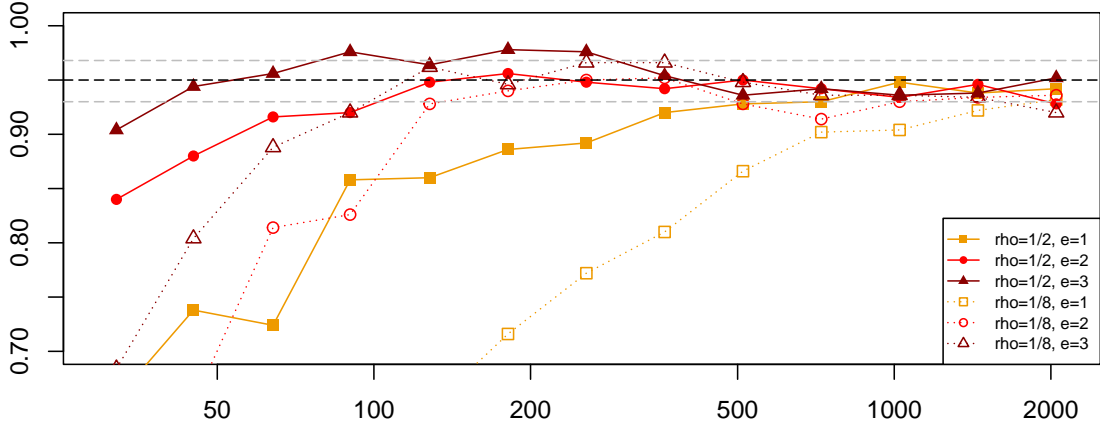


Figure 6: Empirical coverage probabilities for the asymptotic confidence intervals at level $\alpha = 0.95$ of \widehat{d}_N for different values of N (x-axis), $\rho \in \{1/8, 1/2\}$, $\epsilon \in \{1, 2, 3\}$. Grey dashed lines represent the confidence interval at level 0.95 of the frequency $Z = X/K$, if X follows the binomial distribution with parameters K and $\alpha = 0.95$.

6.3. Heterogeneity in the row weights of a network

Model III In this example, we consider a weighted BEDD model with power-law strength distributions, i.e. the marginals f and g have the form $f(\xi) = (\alpha_f + 1)\xi^{\alpha_f}$ and $g(\eta) = (\alpha_g + 1)\eta^{\alpha_g}$, where α_f and α_g are real non-negative numbers. α_f is directly related to $F_2 = \int f(\xi)^2 d\xi = (\alpha_f + 1)^2 / (2\alpha_f + 1)$.

Under Model III, for each value $N \in \{32, 45, 64, 90, 128, 181, 256, 362, 512, 724, 1024, 1448, 2048\}$ and $\rho \in \{1/8, 1/2, 7/8\}$, we have simulated $K = 500$ networks of size $m_N \times n_N$ where $m_N = \lfloor \rho N \rfloor$ and $n_N = N - m_N$. We estimate F_2 and G_2 using $\widehat{F}_{2,N}$ and its symmetric counterpart $\widehat{G}_{2,N}$. We also compute $\widehat{V}_N^{F_2}$ and $\widehat{V}_N^{G_2}$ to obtain the studentized statistics $Z_N^{F_2}$ and $Z_N^{G_2}$. Q-Q plots of $Z_N^{F_2}$ are given in Figures 7. Figure 8 gives the respective coverage probabilities for F_2 and G_2 of the intervals $\left[\widehat{F}_{2,N} - \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{F_2}}}, \widehat{F}_{2,N} + \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{F_2}}} \right]$ and $\left[\widehat{G}_{2,N} - \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{G_2}}}, \widehat{G}_{2,N} + \Phi\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{N}{\widehat{V}_N^{G_2}}} \right]$ respectively, where Φ is the quantile function of the standard normal distribution. Despite the bias due to the delta method, apparent for $N = 64$ in Figure 7, we find that the coverage probabilities fall in the confidence intervals even for small values of N . We would expect that for F_2 , the worst case correspond to $\rho = 1/8$ (there are less rows) and that for G_2 , the worst case correspond to $\rho = 7/8$ (there are less columns), but Figure 8 does not show a clear difference between the three values of ρ .

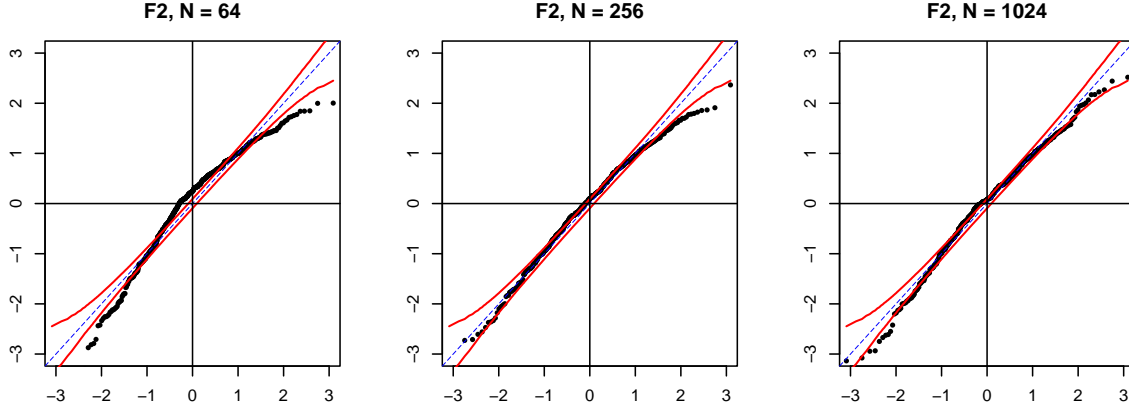


Figure 7: Q-Q plot for $Z_N^{F_2}$ the studentized statistic associated with $\widehat{F}_{2,N}$, $\rho = 0.5$

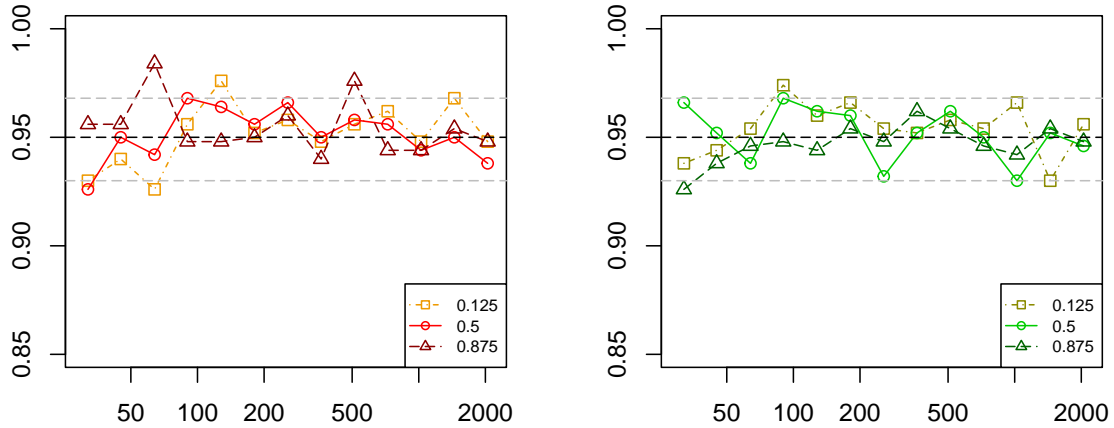


Figure 8: Empirical coverage probabilities for the asymptotic confidence intervals at level $\alpha = 0.95$ of $Z_N^{F_2}$ (left) and $Z_N^{G_2}$ (right) for different values of N (x-axis), $\rho \in \{1/8, 1/2, 7/8\}$. Grey dashed lines represent the confidence interval at level 0.95 of the frequency $Z = X/K$, if X follows the binomial distribution with parameters K and $\alpha = 0.95$.

7. Illustrations

To illustrate the use and interpretation of some of the U -statistics introduced in this paper, we considered the set of law-makers networks compiled by [31]. The database contains networks arising from different fields (ecology, social sciences, life sciences). We focused on the subset of so-called 'legislature' networks both because of their sizes and because network comparison is of interest for this dataset.

Data description. Four law-maker assemblies were considered: the European Parliament ('EP'), the General Assembly of the United Nations ('UN'), the US House of Representatives ('USH') and the US Senate ('USS'). One network has been recorded each year for each parliament; we considered the 26 years from 1979 to 2004, for which the data are available for all the four assemblies. The network recorded for a given assembly in a given year consists of the votes (yes or no) of the different members (rows of the adjacency matrix) for the different proposed laws (columns of the adjacency matrix).

Figure 9 gives the dimensions and densities of the 26 networks collected in each assembly: the main difference is that the European Parliament is both larger (both in terms of members and laws) and sparser than the three others.

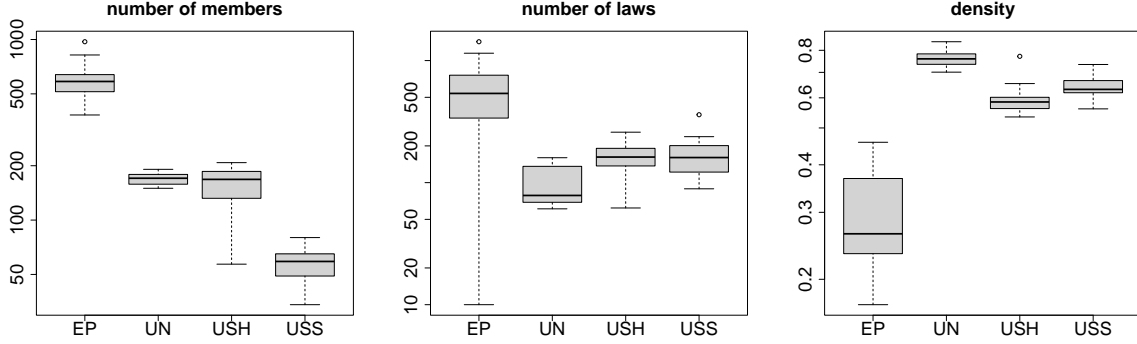


Figure 9: Distribution of the number of members (left), number of laws (center) and density (right) of the four lawmakers networks across the 26 years (in log-scale).

Degree imbalance. We then focused on the degree of imbalance among the rows (resp. columns), which, under the weighted BEDD model defined in Equation (10), can be measured by the U -statistic F_2 (resp. G_2). Figure 10 gives the evolution of each of the two indicators along the years for each parliament. We observe that, for each of them, both F_2 and G_2 remain above 1, all along the period: as expected no uniformity exist, neither among the members ($F_2 > 1$), nor among the laws ($G_2 > 1$). Regarding the US networks (USH and USS), the imbalanced if more marked among the laws than between the members. As expected also, the confidence intervals are narrower for the largest networks (EP). No systematic pattern is observed, except the shift in the imbalance among resolutions voted at the General Assembly of the United Nations (UN) that is observed in 1992 (and which happens to coincide with the UN membership of former soviet republics).

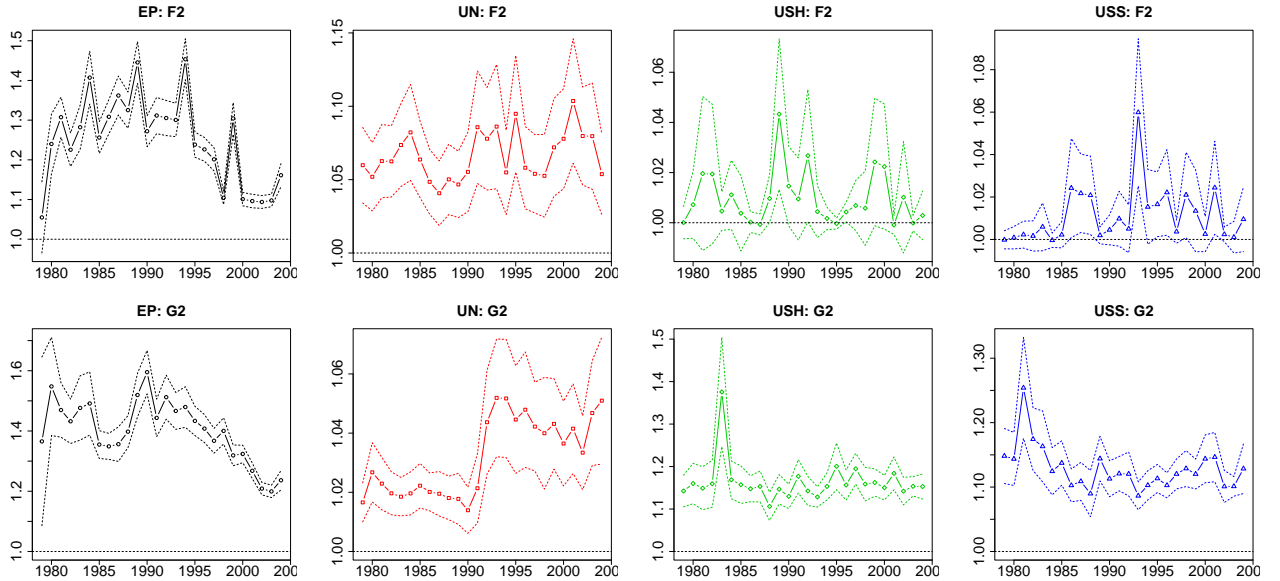


Figure 10: Evolution of degree imbalance in each assembly along the years. Top: F_2 U -statistics, bottom: G_2 . From left to right: European Parliament, General Assembly of the United Nations, US House of Representatives and US Senate. Solid line: U -statistic as an estimate of F_2 (resp. G_2). Dotted line: 95%-confidence interval for F_2 (resp. G_2).

Network comparison. For each available year, we then compared the networks of the four assemblies in terms of degree imbalance (F_2 and G_2) and frequency of topological motifs 6 (as given in Figure 1). We chose this motif as it constitutes a clique, characterizing a group behavior, in which close members are

in favor of the same laws. For each of these parameters, we use the comparison test procedure described in Section 5.3.

Figure 11 display the results. We observe no significant difference between the two US assemblies, which are also the smallest ones: the absence of significant differences can therefore result from a weak power of the tests when considering small networks. We also observe a higher heterogeneity among the members of the European Parliament with respect to all other assemblies, as well as a higher heterogeneity among the member of the United Nations assembly, with respect to the two US chambers. A different picture is obtained for the heterogeneity among the laws, which is significantly higher in the UN assembly and significantly lower in the EP assembly.

The frequency of motif 6 is interesting, as it may reveal a specific socio-political behavior. To this respect, the group structure turns out to be much stronger in the UN than in the EP, the members of which represent both different political orientations and different nations.

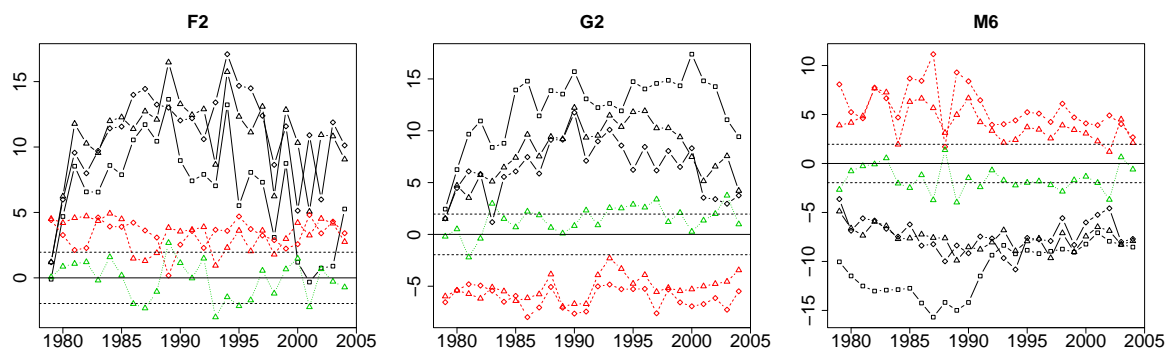


Figure 11: Network comparison for the U -statistics, F_2 , G_2 and for the count the motif 6 (M6). EP-UN= —□—, EP-USS= —◇—, EP-USH= —△—, UN-USS= - -◇- -, UN-USH= - -△- -, USS-USH= ...△... Horizontal lines = standard normal quantiles with level .025 and .975.

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A. Backward martingales

Here, we present the backward martingales and their convergence theorem, which is used to prove the convergence of some estimators. The proof of Theorem A.3 can be found in [15], Section 7, Theorem 4.2. We recall beforehand the definition of a decreasing filtration.

Definition A.1. A decreasing filtration is a decreasing sequence of σ -fields $\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$, i.e. such that for all $n \geq 1$, $\mathcal{F}_{n+1} \subset \mathcal{F}_n$.

Definition A.2. Let $\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$ be a decreasing filtration and $M = (M_n)_{n \geq 1}$ a sequence of integrable random variables adapted to \mathcal{F} . $(M_n, \mathcal{F}_n)_{n \geq 1}$ is a backward martingale if and only if for all $n \geq 1$, $\mathbb{E}[M_n | \mathcal{F}_{n+1}] = M_{n+1}$.

Theorem A.3. Let $(M_n, \mathcal{F}_n)_{n \geq 1}$ be a backward martingale. Then, $(M_n)_{n \geq 1}$ is uniformly integrable, and, denoting $M_\infty = \mathbb{E}[M_1 | \mathcal{F}_\infty]$ where $\mathcal{F}_\infty = \bigcap_{n=1}^\infty \mathcal{F}_n$, we have

$$M_n \xrightarrow[n \rightarrow \infty]{a.s., L_1} M_\infty.$$

B. Proofs of the results presented in Section 2

Proof of Lemma 2.2. We prove this lemma by induction on (r, c) the sizes of \mathbf{i} and \mathbf{j} . For $(r, c) = (1, 0)$ and $(r, c) = (0, 1)$, we have

$$\mathbb{E}[p^{1,0} h(Y_{\mathbf{i}, \emptyset}) | \mathcal{A}_{\emptyset, \emptyset}] = \mathbb{E}[\psi^{1,0} h(Y_{\mathbf{i}, \emptyset})] - \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})] = 0$$

and

$$\mathbb{E}[p^{0,1} h(Y_{\emptyset, \mathbf{j}}) | \mathcal{A}_{\emptyset, \emptyset}] = \mathbb{E}[\psi^{0,1} h(Y_{\emptyset, \mathbf{i}})] - \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})] = 0.$$

Suppose that the lemma is true for all $(0, 0) < (r', c') < (r, c)$. Let $(\mathbf{i}, \mathbf{j}) \in \mathcal{P}_r(\mathbb{N}) \times \mathcal{P}_c(\mathbb{N})$, $\underline{\mathbf{i}} \subset \mathbf{i}$ and $\underline{\mathbf{j}} \subset \mathbf{j}$. Denote $\underline{r} = \text{Card}(\underline{\mathbf{i}})$ and $\underline{c} = \text{Card}(\underline{\mathbf{j}})$. We can write

$$\begin{aligned} \mathbb{E}[p^{r,c} h(Y_{\mathbf{i}, \mathbf{j}}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] &= \mathbb{E}[\psi^{r,c} h(Y_{\mathbf{i}, \mathbf{j}}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] - \mathbb{E}[p^{\underline{r}, \underline{c}} h(Y_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] \\ &\quad - \sum_{(0,0) < (r', c') < (r, c)} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}), \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j}) \\ (\mathbf{i}', \mathbf{j}') \neq (\underline{\mathbf{i}}, \underline{\mathbf{j}})}} \mathbb{E}[p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] \\ &= \sum_{(0,0) \leq (r', c') < (\underline{r}, \underline{c})} \sum_{\mathbf{i}' \in \mathcal{P}_{r'}(\underline{\mathbf{i}}), \mathbf{j}' \in \mathcal{P}_{c'}(\underline{\mathbf{j}})} \mathbb{E}[p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] \\ &\quad - \sum_{(0,0) < (r', c') < (r, c)} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}), \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j}) \\ (\mathbf{i}', \mathbf{j}') \neq (\underline{\mathbf{i}}, \underline{\mathbf{j}})}} \mathbb{E}[p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] \\ &= - \sum_{(0,0) < (r', c') < (r, c)} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}), \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j}) \\ \mathbf{i}' \neq \underline{\mathbf{i}}, \mathbf{j}' \neq \underline{\mathbf{j}}}} \mathbb{E}[p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}) | \mathcal{A}_{\mathbf{i}' \cap \underline{\mathbf{i}}, \mathbf{j}' \cap \underline{\mathbf{j}}}]. \end{aligned}$$

where we have used the fact that the $p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'})$ are measurable by their respective $\mathcal{A}_{\mathbf{i}', \mathbf{j}'}$ so that $\mathbb{E}[p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] = \mathbb{E}[p^{r', c'} h(Y_{\mathbf{i}', \mathbf{j}'}) | \mathcal{A}_{\mathbf{i}' \cap \underline{\mathbf{i}}, \mathbf{j}' \cap \underline{\mathbf{j}}}]$.

Since the last sum excludes the case $\mathbf{i}' = \underline{\mathbf{i}}$ and $\mathbf{j}' = \underline{\mathbf{j}}$, then the induction hypothesis ensures that all the terms are equal to 0, so $\mathbb{E}[p^{r,c} h(Y_{\mathbf{i}, \mathbf{j}}) | \mathcal{A}_{\underline{\mathbf{i}}, \underline{\mathbf{j}}}] = 0$, which concludes the proof by induction. \square

Proof of Corollary 2.3. First, we see that for some (r_1, c_1) and (r_2, c_2) ,

$$\text{Cov}(P_{m,n}^{r_1,c_1} h_1(Y), P_{m,n}^{r_2,c_2} h_2(Y)) = \left[\binom{m}{r_1} \binom{m}{r_2} \binom{n}{c_1} \binom{n}{c_2} \right]^{-1} \sum_{\substack{\mathbf{i}_1 \in \mathcal{P}_{r_1}(\llbracket m \rrbracket) \\ \mathbf{j}_1 \in \mathcal{P}_{c_1}(\llbracket n \rrbracket)}} \sum_{\substack{\mathbf{i}_2 \in \mathcal{P}_{r_2}(\llbracket m \rrbracket) \\ \mathbf{j}_2 \in \mathcal{P}_{c_2}(\llbracket n \rrbracket)}} \text{Cov}(p^{r_1,c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r_2,c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})).$$

If $(r_1, c_1) \neq (r_2, c_2)$, then from Proposition 2.1, all the terms $\text{Cov}(p^{r_1,c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r_2,c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})) = 0$, so $\text{Cov}(P_{m,n}^{r_1,c_1} h_1(Y), P_{m,n}^{r_2,c_2} h_2(Y)) = 0$ and that is the first part of the corollary.

If $(r_1, c_1) = (r_2, c_2) = (r, c)$, then from Proposition 2.1, the terms $\text{Cov}(p^{r,c} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r,c} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})) = 0$ if $(\mathbf{i}_1, \mathbf{j}_1) \neq (\mathbf{i}_2, \mathbf{j}_2)$. Using this fact and the exchangeability of Y , we have

$$\begin{aligned} \text{Cov}(P_{m,n}^{r,c} h_1(Y), P_{m,n}^{r,c} h_2(Y)) &= \left[\binom{m}{r} \binom{n}{c} \right]^{-2} \sum_{\substack{\mathbf{i}_1 \in \mathcal{P}_r(\llbracket m \rrbracket) \\ \mathbf{j}_1 \in \mathcal{P}_c(\llbracket n \rrbracket)}} \sum_{\substack{\mathbf{i}_2 \in \mathcal{P}_r(\llbracket m \rrbracket) \\ \mathbf{j}_2 \in \mathcal{P}_c(\llbracket n \rrbracket)}} \text{Cov}(p^{r_1,c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r_2,c_2} h_2(Y_{\mathbf{i}_2, \mathbf{j}_2})) \\ &= \left[\binom{m}{r} \binom{n}{c} \right]^{-2} \sum_{\substack{\mathbf{i}_1 \in \mathcal{P}_r(\llbracket m \rrbracket) \\ \mathbf{j}_1 \in \mathcal{P}_c(\llbracket n \rrbracket)}} \text{Cov}(p^{r_1,c_1} h_1(Y_{\mathbf{i}_1, \mathbf{j}_1}), p^{r_2,c_2} h_2(Y_{\mathbf{i}_1, \mathbf{j}_1})) \\ &= \left[\binom{m}{r} \binom{n}{c} \right]^{-1} \text{Cov}(p^{r,c} h_1(Y_{\llbracket r \rrbracket, \llbracket c \rrbracket}), p^{r,c} h_2(Y_{\llbracket r \rrbracket, \llbracket c \rrbracket})), \end{aligned}$$

which proves the second part of the corollary. \square

C. Proofs of the results presented in Section 3

Proof of Lemma 3.2. We only prove the first convergence result, since the second can be deduced by analogy.

By definition,

$$p_{(\{i\}, \emptyset)}^{1,0} h = \psi_{(\{i\}, \emptyset)}^{1,0} h - \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})].$$

First, we see that the $p_{(\{i\}, \emptyset)}^{1,0} h(Y)$ only depends on ξ_i , so all the $p_{(\{i\}, \emptyset)}^{1,0} h(Y)$ are independent. Because h is symmetric and Y is RCE, they are also identically distributed.

By the tower rule, we also have $\mathbb{E}[\psi_{(\{i\}, \emptyset)}^{1,0} h] = \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})]$ so $\mathbb{E}[p_{(\{i\}, \emptyset)}^{1,0} h] = 0$.

Finally, we have, from the Aldous-Hoover representation theorem, then Cauchy-Schwarz's inequality, and the fact that $\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})^2] < \infty$,

$$\begin{aligned} \mathbb{V}[p_{(\{i\}, \emptyset)}^{1,0} h] &= v_h^{1,0} \\ &= \mathbb{V}[\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) \mid \xi_i]] \\ &= \text{Cov}(h(Y_{(1, \dots, p; 1, \dots, q)}), h(Y_{(1, p+1, \dots, 2p-1; 1, \dots, q)})) \\ &< \mathbb{V}[h(Y_{(1, \dots, p; 1, \dots, q)})] \\ &< \infty. \end{aligned}$$

The classical CLT gives the desired result. \square

Proof of Lemma 3.3. For all $\mathbf{i} \in \mathcal{P}_r(\llbracket m \rrbracket)$ and $\mathbf{j} \in \mathcal{P}_c(\llbracket n \rrbracket)$ and $(r, c) \in \mathbb{N}^2$, we have $\mathbb{E}[\psi_{(\mathbf{i}, \mathbf{j})}^{r,c} h] = \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})]$.

By recursion, we have $\mathbb{E}[A_N] = \mathbb{E}[P_N^{r,c} h] = 0$ for all $(r, c) > (0, 0)$ since

$$\begin{aligned} \mathbb{E}[P_N^{r,c} h] &= \mathbb{E}[p_{(\mathbf{i}, \mathbf{j})}^{r,c} h] \\ &= \mathbb{E}[\psi_{(\mathbf{i}, \mathbf{j})}^{r,c} h] - \sum_{(0,0) \leq (r', c') < (r, c)} \sum_{\substack{\mathbf{i}' \in \mathcal{P}_{r'}(\mathbf{i}) \\ \mathbf{j}' \in \mathcal{P}_{c'}(\mathbf{j})}} \mathbb{E}[p_{(\mathbf{i}', \mathbf{j}')}^{r', c'} h]. \end{aligned}$$

Now, Corollary 2.4 imply that $\text{Cov}(P_N^{r,c}h, P_N^{r',c'}h) = 0$ unless $(r, c) = (r', c')$. So

$$\begin{aligned}
\mathbb{V}[A_N] &= N \sum_{\substack{(0,0) < (r,c) \leq (p,q) \\ (r,c) \neq (1,0) \neq (0,1)}} \sum_{\substack{(0,0) < (r',c') \leq (p,q) \\ (r',c') \neq (1,0) \neq (0,1)}} \binom{p}{r} \binom{p}{r'} \binom{q}{c} \binom{q}{c'} \text{Cov}(P_N^{r,c}h, P_N^{r',c'}h) \\
&= N \sum_{\substack{(0,0) < (r,c) \leq (p,q) \\ (r,c) \neq (1,0) \neq (0,1)}} \binom{p}{r}^2 \binom{q}{c}^2 \mathbb{V}[P_N^{r,c}h] \\
&= N \sum_{\substack{(0,0) < (r,c) \leq (p,q) \\ (r,c) \neq (1,0) \neq (0,1)}} \binom{p}{r}^2 \binom{q}{c}^2 \binom{m_N}{r}^{-1} \binom{n_N}{c}^{-1} \mathbb{V}[p_{\llbracket r \rrbracket, \llbracket c \rrbracket}^{r,c} h].
\end{aligned} \tag{18}$$

From equation 5, we see that $h(Y_{(1,\dots,p,1,\dots,q)})$ is a linear combination of all the $(p_{\mathbf{i},\mathbf{j}}^{r,c} h)_{\substack{0 \leq r \leq p, 0 \leq c \leq q \\ \mathbf{i} \in \mathcal{P}_r(\llbracket p \rrbracket), \mathbf{j} \in \mathcal{P}_c(\llbracket q \rrbracket)}}$. So $\mathbb{E}[h(Y_{(1,\dots,p,1,\dots,q)})^2] < \infty$ ensures that $\mathbb{V}[p_{\llbracket r \rrbracket, \llbracket c \rrbracket}^{r,c} h] < \infty$. Therefore $\mathbb{V}[A_N] = O(Nm_N^{-r}n_N^{-c}) = O(N^{1-r-c})$.

Thus, we deduce from Markov's inequality that $A_N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$. \square

- Lemma C.1.**
1. $U_N^{\tilde{h}} = U_N^h$,
 2. $\mathbb{E}[\tilde{h}(Y_{(1,\dots,p';1,\dots,q')})] = \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})]$,
 3. $\mathbb{E}[\tilde{h}(Y_{(1,\dots,p';1,\dots,q')})^2] \leq \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})^2]$.

Proof of Lemma C.1. The two first properties are straightforward. The third property stems from

$$\begin{aligned}
\mathbb{E}[\tilde{h}(Y_{(1,\dots,p';1,\dots,q')})^2] &= \left[\binom{p'}{p} \binom{q'}{q} \right]^{-2} \sum_{\substack{\mathbf{i}_1 \subset \{i_1, \dots, i_{p'}\} \\ \mathbf{j}_1 \subset \{j_1, \dots, j_{q'}\}}} \sum_{\substack{\mathbf{i}_2 \subset \{i_1, \dots, i_{p'}\} \\ \mathbf{j}_2 \subset \{j_1, \dots, j_{q'}\}}} \mathbb{E}[h(Y_{\mathbf{i}_1, \mathbf{j}_1})h(Y_{\mathbf{i}_2, \mathbf{j}_2})] \\
&= \left[\binom{p'}{p} \binom{q'}{q} \right]^{-2} \sum_{\substack{\mathbf{i}_1 \subset \{i_1, \dots, i_{p'}\} \\ \mathbf{j}_1 \subset \{j_1, \dots, j_{q'}\}}} \sum_{\substack{\mathbf{i}_2 \subset \{i_1, \dots, i_{p'}\} \\ \mathbf{j}_2 \subset \{j_1, \dots, j_{q'}\}}} \left(\mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})^2] - \frac{1}{2} \mathbb{E}[(h(Y_{\mathbf{i}_1, \mathbf{j}_1}) - h(Y_{\mathbf{i}_2, \mathbf{j}_2}))^2] \right) \\
&\leq \left[\binom{p'}{p} \binom{q'}{q} \right]^{-2} \sum_{\substack{\mathbf{i}_1 \subset \{i_1, \dots, i_{p'}\} \\ \mathbf{j}_1 \subset \{j_1, \dots, j_{q'}\}}} \sum_{\substack{\mathbf{i}_2 \subset \{i_1, \dots, i_{p'}\} \\ \mathbf{j}_2 \subset \{j_1, \dots, j_{q'}\}}} \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})^2] \\
&\leq \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})^2].
\end{aligned}$$

\square

Proof of Corollary 3.5. First, we show that $\lim_{N \rightarrow +\infty} N \text{Cov}(U_N^{h_k}, U_N^{h_\ell})$

Let $(Z^{h_k})_{1 \leq k \leq D}$ be a vector of random variables following a centered multivariate Gaussian distribution with covariance matrix Σ^{h_1, \dots, h_D} defined in the theorem. Then $Z^{h_k} \sim \mathcal{N}(0, V^{h_k})$ for all $1 \leq k \leq D$ and $\text{Cov}(Z^{h_k}, Z^{h_\ell}) = C^{h_k, h_\ell}$ for all $1 \leq k \leq D$ and $1 \leq \ell \leq D$.

Denote $p' := \max_k(p_k)$ and $q' := \max_k(q_k)$. For some $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$, we set $\tilde{h}_t := t_1 \tilde{h}_1 + t_2 \tilde{h}_2 + \dots + t_n \tilde{h}_D$. \tilde{h}_t is a kernel function of size $p' \times q'$.

First, assume that $t \neq (0, \dots, 0)$. Then by hypothesis, $\tilde{h}_t \neq 0$, therefore Proposition C.1 implies that $\sum_{k=1}^D t_k U_N^{h_k} = \sum_{k=1}^D t_k U_N^{\tilde{h}_k} = U_N^{\tilde{h}_t}$, the U -statistic with quadruplet kernel \tilde{h}_t (of size $p' \times q'$). Using Cauchy-Schwarz inequality and the fact that from Proposition C.1, $\mathbb{E}[\tilde{h}_k(Y_{(1,\dots,p';1,\dots,q')})^2] \leq \mathbb{E}[h_k(Y_{(1,\dots,p;1,\dots,q)})^2] <$

∞ for all $1 \leq k \leq D$, we have furthermore

$$\begin{aligned} \mathbb{E}[\tilde{h}_t(Y_{(1,\dots,p';1,\dots,q')})^2] &= \sum_{k=1}^n t_k^2 \mathbb{E}[\tilde{h}_k(Y_{(1,\dots,p';1,\dots,q')})^2] + 2 \sum_{1 \leq k \neq \ell \leq D} t_k t_\ell \mathbb{E}[\tilde{h}_k(Y_{(1,\dots,p';1,\dots,q')}) \tilde{h}_\ell(Y_{(1,\dots,p';1,\dots,q')})], \\ &\leq \sum_{k=1}^n t_k^2 \mathbb{E}[\tilde{h}_k(Y_{(1,\dots,p';1,\dots,q')})^2] + 2 \sum_{1 \leq k \neq \ell \leq D} t_k t_\ell \sqrt{\mathbb{E}[\tilde{h}_k(Y_{(1,\dots,p';1,\dots,q')})^2] \mathbb{E}[\tilde{h}_\ell(Y_{(1,\dots,p';1,\dots,q')})^2]}, \\ &< \infty. \end{aligned}$$

Therefore, Theorem 3.1 also applies for $U_N^{\tilde{h}_t}$ and $\sqrt{N}(U_N^{\tilde{h}_t} - U_\infty^{\tilde{h}_t}) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, V^{\tilde{h}_t})$, where :

- $U_\infty^{\tilde{h}_t} = \sum_{k=1}^D t_k U_\infty^{h_k}$,
- $V^{\tilde{h}_t} = t^T \Sigma^{h_1, \dots, h_D} t$.

The second point comes from the fact that $V^{\tilde{h}_t} = \lim_{N \rightarrow +\infty} N \sum_{k=1}^D \sum_{\ell=1}^D t_k t_\ell \text{Cov}(U_N^{h_k}, U_N^{h_\ell})$ and by Corollary 2.4, $\lim_{N \rightarrow +\infty} N \text{Cov}(U_N^{h_k}, U_N^{h_\ell}) = \frac{p^2}{c} c_{h_k, h_\ell}^{1,0} + \frac{q^2}{1-c} c_{h_k, h_\ell}^{0,1} = \sum_{k\ell}^{h_1, \dots, h_D}$. Therefore, we can conclude that $\sqrt{N} \sum_{k=1}^D t_k (U_N^{h_k} - U_\infty^{h_k}) = \sqrt{N} (U_N^{\tilde{h}_t} - U_\infty^{\tilde{h}_t}) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \sum_{k=1}^D t_k Z^{h_k}$.

Now assume that $t = (0, \dots, 0)$. Then $\tilde{h}_t \equiv 0$ so $U_N^{\tilde{h}_t} = 0 = \sum_{k=1}^D t_k Z^{h_k}$. Therefore, $\sqrt{N} \sum_{k=1}^D t_k (U_N^{h_k} - U_\infty^{h_k}) = \sqrt{N} (U_N^{\tilde{h}_t} - U_\infty^{\tilde{h}_t}) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \sum_{k=1}^D t_k Z^{h_k}$ is still true.

We have proven that $\sqrt{N} \sum_{k=1}^D t_k (U_N^{h_k} - U_\infty^{h_k}) \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \sum_{k=1}^D t_k Z^{h_k}$ for all $t \in \mathbb{R}^n$, so we can finally apply the Cramér-Wold theorem (Theorem 29.4 of [10]) which states that $\sqrt{N} (U_N^{h_k} - U_\infty^{h_k})_{1 \leq k \leq D}$ converges jointly in distribution to $(Z^{h_k})_{1 \leq k \leq D}$, which is a centered multivariate Gaussian with covariance matrix Σ^{h_1, \dots, h_D} . \square

D. Proofs of the results presented in Section 4

Lemma D.1. *Let $k \in \mathbb{N}$. Then as $n \rightarrow \infty$,*

$$\binom{n}{k} = \frac{n^k}{k!} + O(n^{k-1})$$

Proof of Lemma 4.1. The sum $T_N^{p,q}(I_K, J_K)$ defined by equation (6) is a sum over T_N expectation terms where

$$T_N := \text{Card}(\mathcal{T}_{N, (I_K, J_K)}^{p,q}) = \prod_{k=1}^K \text{Card}(\mathcal{S}_{N, (\mathbf{i}_k, \mathbf{j}_k)}^{p,q}) = \prod_{k=1}^K \binom{m_N - \underline{p}_k}{p - \underline{p}_k} \binom{n_N - \underline{q}_k}{q - \underline{q}_k}. \quad (19)$$

These expectation terms $\mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}]$ only depend on the number of times elements appear in pairwise intersections between the $(\mathbf{i}_k, \mathbf{j}_k)$, $1 \leq k \leq K$. In particular, $\mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}] = \alpha(I_K, J_K)$ when all the \mathbf{i}_k contains the elements of \mathbf{i}_k and all the other elements do not appear in any other $\mathbf{i}_{k'}$ (see equation 7). Denote A_N the number of terms of $T_N^{p,q}(I_K, J_K)$ where $\mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}] = \alpha(I_K, J_K)$, then

$$T_N^{p,q}(I_K, J_K) = \frac{A_N}{T_N} \alpha(I_K, J_K) + \frac{1}{T_N} \sum_{\substack{\mathcal{T}_{N, (I_K, J_K)}^{p,q} \\ \mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}] \neq \alpha(I_K, J_K)}} \mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} X_{\mathbf{i}_2, \mathbf{j}_2} \dots X_{\mathbf{i}_K, \mathbf{j}_K}]. \quad (20)$$

Using Jensen's inequality, Hölder's inequality and the exchangeability of the submatrices, we have for all the expectation terms,

$$0 \leq |\mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}]| \leq \mathbb{E}[|X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}|] \leq \prod_{k=1}^K \mathbb{E}[|X_{\mathbf{i}_k, \mathbf{j}_k}|^K]^{\frac{1}{K}} = \mathbb{E}[|X_{(1,\dots,p;1,\dots,q)}|^K].$$

In particular, this holds for all $(T_N - A_N)$ terms of the remaining sum in Equation (20) when $\mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}] \neq \alpha(I_K, J_K)$, so

$$0 \leq |T_N^{p,q}(I_K, J_K) - \frac{A_N}{T_N} \alpha(I_K, J_K)| \leq \left(1 - \frac{A_N}{T_N}\right) \mathbb{E}[|X_{(1, \dots, p; 1, \dots, q)}|^K]. \quad (21)$$

We need to calculate T_N and A_N to use this inequality and conclude the proof. We organize the rest of the proof in 3 parts.

1. We find an expression for T_N . Since for all $1 \leq k \leq K$, from Lemma D.1,

$$\binom{m_N - \underline{p}_k}{p - \underline{p}_k} \binom{n_N - \underline{q}_k}{q - \underline{q}_k} = \frac{m_N^{p - \underline{p}_k} n_N^{q - \underline{q}_k}}{(p - \underline{p}_k)! (q - \underline{q}_k)!} (1 + O(m_N^{-1} + n_N^{-1})),$$

we have from Equation (19)

$$T_N = \left(\prod_{k=1}^K (p - \underline{p}_k)! (q - \underline{q}_k)! \right)^{-1} m_N^{Kp - P} n_N^{Kq - Q} (1 + O(m_N^{-1} + n_N^{-1})). \quad (22)$$

2. We find an expression for A_N . The summation over $(\mathbf{i}_k, \mathbf{j}_k) \in \mathcal{S}_{N, (\mathbf{i}_k, \mathbf{j}_k)}^{p, q}$ is in fact a sum over the $p - \underline{p}_k$ elements of \mathbf{i}_k and $q - \underline{q}_k$ elements of \mathbf{j}_k that are not restricted by \mathbf{i}_k and \mathbf{j}_k .

- We can pick the first \mathbf{i}_1 choosing the $p - \underline{p}_1$ unrestricted indices among the $m_N - \bar{p}$ values of $\{1, \dots, m_N\}$ excluding $\cup_{k=1}^K \mathbf{i}_k$. The same follows for the pick of \mathbf{j}_1 , so there are $\binom{m_N - \bar{p}}{p - \underline{p}_1} \binom{n_N - \bar{q}}{q - \underline{q}_1}$ possible picks for $(\mathbf{i}_1, \mathbf{j}_1)$.
- The pick of \mathbf{i}_2 consists in choosing the $p - \underline{p}_2$ unrestricted indices among the $m_N - \bar{p} - (p - \underline{p}_1)$ values of $\{1, \dots, m_N\}$ excluding $\cup_{k=1}^K \mathbf{i}_k$ and the elements already taken by \mathbf{i}_1 . We deduce that there are $\binom{m_N - \bar{p} - (p - \underline{p}_1)}{p - \underline{p}_2} \binom{n_N - \bar{q} - (q - \underline{q}_1)}{q - \underline{q}_2}$ possible picks for $(\mathbf{i}_2, \mathbf{j}_2)$.
- Iteratively, for all $1 \leq k \leq K$, we find that there are $\binom{m_N - \bar{p} - \sum_{k'=1}^{k-1} (p - \underline{p}_{k'})}{p - \underline{p}_k} \binom{n_N - \bar{q} - \sum_{k'=1}^{k-1} (q - \underline{q}_{k'})}{q - \underline{q}_k}$ possible picks for $(\mathbf{i}_k, \mathbf{j}_k)$.

We deduce that the number of possible picks for all the $(\mathbf{i}_k, \mathbf{j}_k)$, $1 \leq k \leq K$ so that $\mathbb{E}[X_{\mathbf{i}_1, \mathbf{j}_1} \dots X_{\mathbf{i}_K, \mathbf{j}_K}] = \alpha(I_K, J_K)$ is

$$A_N = \prod_{k=1}^K \binom{m_N - \bar{p} - \sum_{k'=1}^{k-1} (p - \underline{p}_{k'})}{p - \underline{p}_k} \binom{n_N - \bar{q} - \sum_{k'=1}^{k-1} (q - \underline{q}_{k'})}{q - \underline{q}_k}.$$

But we see that from Lemma D.1, for $1 \leq k \leq K$,

$$\binom{m_N - \bar{p} - \sum_{k'=1}^{k-1} (p - \underline{p}_{k'})}{p - \underline{p}_k} \binom{n_N - \bar{q} - \sum_{k'=1}^{k-1} (q - \underline{q}_{k'})}{q - \underline{q}_k} = \frac{m_N^{p - \underline{p}_k} n_N^{q - \underline{q}_k}}{(p - \underline{p}_k)! (q - \underline{q}_k)!} (1 + O(m_N^{-1} + n_N^{-1})).$$

So we have

$$A_N = \left(\prod_{k=1}^K (p - \underline{p}_k)! (q - \underline{q}_k)! \right)^{-1} m_N^{Kp - P} n_N^{Kq - Q} (1 + O(m_N^{-1} + n_N^{-1})).$$

3. Now with the expressions of T_N and A_N , we can deduce that

$$T_N - A_N = O\left(m_N^{Kp - P} n_N^{Kq - Q} (m_N^{-1} + n_N^{-1})\right),$$

so

$$1 - \frac{A_N}{T_N} = \frac{O\left(m_N^{Kp - P} n_N^{Kq - Q} (m_N^{-1} + n_N^{-1})\right)}{\left(\prod_{k=1}^K (p - \underline{p}_k)! (q - \underline{q}_k)!\right)^{-1} m_N^{Kp - P} n_N^{Kq - Q} (1 + O(m_N^{-1} + n_N^{-1}))} = O(m_N^{-1} + n_N^{-1}).$$

Therefore, we can finally conclude using Equation (21),

$$\begin{aligned} T_N^{p,q}(I_K, J_K) &= (1 + O(m_N^{-1} + n_N^{-1})) \alpha(I_K, J_K) + O(m_N^{-1} + n_N^{-1}) \\ &= \alpha(I_K, J_K) + O(m_N^{-1} + n_N^{-1}). \end{aligned}$$

□

Proof of Proposition 4.5. In this proof, for the estimator defined by equation (8), we write $\widehat{\mu}_N^{(i)}$ instead of $\widehat{\mu}_N^{h,(i)}$ to simplify the notation without ambiguity.

Notice that

$$\begin{aligned}\mathbb{E}[\widehat{v}_N^{h;1,0}] &= \frac{1}{2}\mathbb{E}[(\widehat{\mu}_N^{(1)} - \widehat{\mu}_N^{(2)})^2] \\ &= \mathbb{E}[(\widehat{\mu}_N^{(1)})^2] - \mathbb{E}[\widehat{\mu}_N^{(1)}\widehat{\mu}_N^{(2)}].\end{aligned}\tag{23}$$

- First, we calculate $\mathbb{E}[(\widehat{\mu}_N^{(1)})^2]$:

$$\begin{aligned}\mathbb{E}[(\widehat{\mu}_N^{(1)}(Y))^2] &= \binom{m_N - 1}{p - 1}^{-2} \binom{n_N}{q}^{-2} \sum_{(\mathbf{i}_1, \mathbf{j}_1) \in \mathcal{S}_{N,(\{1\},\emptyset)}^{p,q}} \sum_{(\mathbf{i}_2, \mathbf{j}_2) \in \mathcal{S}_{N,(\{1\},\emptyset)}^{p,q}} \mathbb{E}[h(Y_{\mathbf{i}_1, \mathbf{j}_1})h(Y_{\mathbf{i}_2, \mathbf{j}_2})] \\ &= T_N^{p,q}(I_2, J_2)\end{aligned}$$

where $I_2 = (\{1\}, \{1\})$ and $J_2 = (\emptyset, \emptyset)$. Applying Lemma 4.1, with $\underline{P} = \text{Card}(\{1\}) + \text{Card}(\{1\}) = 2$, $\underline{Q} = \text{Card}(\emptyset) + \text{Card}(\emptyset) = 0$ and

$$\begin{aligned}\alpha(I_2, J_2) &= \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})h(Y_{(1,p+1,\dots,2p-1;q+1,\dots,2q)})] \\ &= \mathbb{E}[\mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})h(Y_{(1,p+1,\dots,2p-1;q+1,\dots,2q)}) \mid \xi_1]] \\ &= \mathbb{E}[\mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)}) \mid \xi_1]^2] \\ &= \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h^2].\end{aligned}$$

we find

$$\mathbb{E}[(\widehat{\mu}_N^{(1)})^2] = \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h^2] + O(N^{-1}).\tag{24}$$

- Next, we calculate $\mathbb{E}[\widehat{\mu}_N^{(1)}\widehat{\mu}_N^{(2)}]$:

$$\begin{aligned}\mathbb{E}[\widehat{\mu}_N^{(1)}\widehat{\mu}_N^{(2)}] &= \binom{m_N - 1}{p - 1}^{-2} \binom{n_N}{q}^{-2} \sum_{(\mathbf{i}_1, \mathbf{j}_1) \in \mathcal{S}_{N,(\{1\},\emptyset)}^{p,q}} \sum_{(\mathbf{i}_2, \mathbf{j}_2) \in \mathcal{S}_{N,(\{2\},\emptyset)}^{p,q}} \mathbb{E}[h(Y_{\mathbf{i}_1, \mathbf{j}_1})h(Y_{\mathbf{i}_2, \mathbf{j}_2})] \\ &= T_N^{p,q}(I'_2, J'_2)\end{aligned}$$

where $I'_2 = (\{1\}, \{2\})$ and $J'_2 = (\emptyset, \emptyset)$. Applying Lemma 4.1 with

$$\begin{aligned}\alpha(I'_2, J'_2) &= \mathbb{E}[h(Y_{(1,3,\dots,p+1;1,\dots,q)})h(Y_{(2,p+2,\dots,2p;q+1,\dots,2q)})] \\ &= \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})]^2 \\ &= \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h]^2,\end{aligned}$$

we find

$$\mathbb{E}[\widehat{\mu}_N^{(1)}\widehat{\mu}_N^{(2)}] = \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h]^2 + O(N^{-1}).\tag{25}$$

Finally, we can combine Equations (23), (24) and (25) to obtain

$$\mathbb{E}[\widehat{v}_N^{h;1,0}] = \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h^2] - \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h]^2 + O(N^{-1}) = v_h^{1,0} + O(N^{-1}),$$

which proves the theorem. □

Proof of Proposition 4.6. In this proof, we write $\widehat{\mu}_N^{(i)}$ instead of $\widehat{\mu}_N^{h,(i)}$ to simplify the notation without ambiguity.

Notice that

$$\begin{aligned}
\mathbb{E}[(\widehat{v}_N^{h;1,0})^2] &= \binom{m_N}{2}^{-2} \sum_{1 \leq i_1 < i_2 \leq m_N} \sum_{1 \leq i'_1 < i'_2 \leq m_N} \mathbb{E} \left[\frac{(\widehat{\mu}_N^{(i_1)} - \widehat{\mu}_N^{(i_2)})^2 (\widehat{\mu}_N^{(i'_1)} - \widehat{\mu}_N^{(i'_2)})^2}{4} \right] \\
&= \binom{m_N}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq m_N} \frac{1}{4} \mathbb{E}[(\widehat{\mu}_N^{(1)} - \widehat{\mu}_N^{(2)})^2 (\widehat{\mu}_N^{(3)} - \widehat{\mu}_N^{(4)})^2] + O(N^{-1}) \\
&= \mathbb{E}[(\widehat{\mu}_N^{(1)})^2 (\widehat{\mu}_N^{(2)})^2] - 2\mathbb{E}[(\widehat{\mu}_N^{(1)})^2 \widehat{\mu}_N^{(2)} \widehat{\mu}_N^{(3)}] + \mathbb{E}[\widehat{\mu}_N^{(1)} \widehat{\mu}_N^{(2)} \widehat{\mu}_N^{(3)} \widehat{\mu}_N^{(4)}] \\
&\quad + O(N^{-1}).
\end{aligned} \tag{26}$$

Now, we calculate each of the three expectation terms in this equation.

- First, we calculate $\mathbb{E}[(\widehat{\mu}_N^{(1)})^2 (\widehat{\mu}_N^{(2)})^2]$:

$$\begin{aligned}
\mathbb{E}[(\widehat{\mu}_N^{(1)})^2 (\widehat{\mu}_N^{(2)})^2] &= \binom{m_N - 1}{p - 1}^{-4} \binom{n_N}{q}^{-4} \sum_{(\mathbf{i}_1, \mathbf{j}_1) \in \mathcal{S}_{N, \{1\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_2, \mathbf{j}_2) \in \mathcal{S}_{N, \{1\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_3, \mathbf{j}_3) \in \mathcal{S}_{N, \{2\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_4, \mathbf{j}_4) \in \mathcal{S}_{N, \{2\}, \emptyset}^{p, q}} \\
&\quad \mathbb{E}[h(Y_{\mathbf{i}_1, \mathbf{j}_1}) h(Y_{\mathbf{i}_2, \mathbf{j}_2}) h(Y_{\mathbf{i}_3, \mathbf{j}_3}) h(Y_{\mathbf{i}_4, \mathbf{j}_4})] \\
&= T_N^{p, q}(I_4, J_4)
\end{aligned}$$

where $I_4 = (\{1\}, \{1\}, \{2\}, \{2\})$ and $J_4 = (\emptyset, \emptyset, \emptyset, \emptyset)$. Applying Lemma 4.1 with

$$\begin{aligned}
\alpha(I_4, J_4) &= \mathbb{E}[h(Y_{(1,3, \dots, p+1; 1, \dots, q)}) h(Y_{(1, p+2, \dots, 2p; q+1, \dots, 2q)}) h(Y_{(2, 2p+1, \dots, 3p-1; 2q+1, \dots, 3q)}) h(Y_{(2, 3p, \dots, 4p-2; 3q+1, \dots, 4q)})] \\
&= \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) h(Y_{(1, p+1, \dots, 2p-1; q+1, \dots, 2q)})]^2 \\
&= \mathbb{E}[\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) h(Y_{(1, p+1, \dots, 2p-1; q+1, \dots, 2q)}) \mid \xi_1]]^2 \\
&= \mathbb{E}[\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) \mid \xi_1]^2]^2 \\
&= \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h^2]^2.
\end{aligned}$$

we find

$$\mathbb{E}[(\widehat{\mu}_N^{(1)})^2 (\widehat{\mu}_N^{(2)})^2] = \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h^2]^2 + O(N^{-1}). \tag{27}$$

- Next, we calculate $\mathbb{E}[(\widehat{\mu}_N^{(1)})^2 \widehat{\mu}_N^{(2)} \widehat{\mu}_N^{(3)}]$:

$$\begin{aligned}
\mathbb{E}[(\widehat{\mu}_N^{(1)})^2 \widehat{\mu}_N^{(2)} \widehat{\mu}_N^{(3)}] &= \binom{m_N - 1}{p - 1}^{-4} \binom{n_N}{q}^{-4} \sum_{(\mathbf{i}_1, \mathbf{j}_1) \in \mathcal{S}_{N, \{1\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_2, \mathbf{j}_2) \in \mathcal{S}_{N, \{1\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_3, \mathbf{j}_3) \in \mathcal{S}_{N, \{2\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_4, \mathbf{j}_4) \in \mathcal{S}_{N, \{3\}, \emptyset}^{p, q}} \\
&\quad \mathbb{E}[h(Y_{\mathbf{i}_1, \mathbf{j}_1}) h(Y_{\mathbf{i}_2, \mathbf{j}_2}) h(Y_{\mathbf{i}_3, \mathbf{j}_3}) h(Y_{\mathbf{i}_4, \mathbf{j}_4})] \\
&= T_N^{p, q}(I'_4, J'_4)
\end{aligned}$$

where $I'_4 = (\{1\}, \{1\}, \{2\}, \{3\})$ and $J'_4 = (\emptyset, \emptyset, \emptyset, \emptyset)$. Applying Lemma 4.1 with

$$\begin{aligned}
\alpha(I'_4, J'_4) &= \mathbb{E}[h(Y_{(1,4, \dots, p+2; 1, \dots, q)}) h(Y_{(1, p+3, \dots, 2p+1; q+1, \dots, 2q)}) h(Y_{(2, 2p+2, \dots, 3p; 2q+1, \dots, 3q)}) h(Y_{(3, 3p+1, \dots, 4p-1; 3q+1, \dots, 4q)})] \\
&= \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) h(Y_{(1, p+1, \dots, 2p-1; q+1, \dots, 2q)})]^2 \mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)})]^2 \\
&= \mathbb{E}[\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) h(Y_{(1, p+1, \dots, 2p-1; q+1, \dots, 2q)}) \mid \xi_1]] \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h(Y)]^2 \\
&= \mathbb{E}[\mathbb{E}[h(Y_{(1, \dots, p; 1, \dots, q)}) \mid \xi_1]^2] \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h]^2 \\
&= \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h(Y)^2] \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h]^2.
\end{aligned}$$

we find

$$\mathbb{E}[(\widehat{\mu}_N^{(1)}(Y))^2 \widehat{\mu}_N^{(2)}(Y) \widehat{\mu}_N^{(3)}(Y)] = \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h(Y)^2] \mathbb{E}[\psi_{(\{1\}, \emptyset)}^{1,0} h(Y)]^2 + O(N^{-1}). \tag{28}$$

- Now, we calculate $\mathbb{E}[\widehat{\mu}_N^{(1)}(Y) \widehat{\mu}_N^{(2)}(Y) \widehat{\mu}_N^{(3)}(Y) \widehat{\mu}_N^{(4)}(Y)]$:

$$\begin{aligned}
\mathbb{E}[\widehat{\mu}_N^{(1)}(Y) \widehat{\mu}_N^{(2)}(Y) \widehat{\mu}_N^{(3)}(Y) \widehat{\mu}_N^{(4)}(Y)] &= \binom{m_N - 1}{p - 1}^{-4} \binom{n_N}{q}^{-4} \sum_{(\mathbf{i}_1, \mathbf{j}_1) \in \mathcal{S}_{N, \{1\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_2, \mathbf{j}_2) \in \mathcal{S}_{N, \{2\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_3, \mathbf{j}_3) \in \mathcal{S}_{N, \{3\}, \emptyset}^{p, q}} \sum_{(\mathbf{i}_4, \mathbf{j}_4) \in \mathcal{S}_{N, \{4\}, \emptyset}^{p, q}} \\
&\quad \mathbb{E}[h(Y_{\mathbf{i}_1, \mathbf{j}_1}) h(Y_{\mathbf{i}_2, \mathbf{j}_2}) h(Y_{\mathbf{i}_3, \mathbf{j}_3}) h(Y_{\mathbf{i}_4, \mathbf{j}_4})] \\
&= T_N^{p, q}(I''_4, J''_4)
\end{aligned}$$

where $I'_4 = (\{1\}, \{2\}, \{3\}, \{4\})$ and $J'_4 = (\emptyset, \emptyset, \emptyset, \emptyset)$. Applying Lemma 4.1 with

$$\begin{aligned}\alpha(I''_4, J''_4) &= \mathbb{E}[h(Y_{(1,5,\dots,p+3;1,\dots,q)})h(Y_{(2,p+4,\dots,2p+2;q+1,\dots,2q)})h(Y_{(3,2p+3,\dots,3p+1;2q+1,\dots,3q)})h(Y_{(4,3p+2,\dots,4p;3q+1,\dots,4q)})] \\ &= \mathbb{E}[h(Y_{(1,\dots,p;1,\dots,q)})]^4 \\ &= \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)]^4.\end{aligned}$$

we find

$$\mathbb{E}[\widehat{\mu}_N^{(1)}(Y)\widehat{\mu}_N^{(2)}(Y)\widehat{\mu}_N^{(3)}(Y)\widehat{\mu}_N^{(4)}(Y)] = \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)]^4 + O(N^{-1}). \quad (29)$$

Finally, injecting the calculated expressions (27), (28) and (29) in (26), we obtain

$$\begin{aligned}\mathbb{E}[(\widehat{v}_N^{h;1,0})^2] &= \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)^2]^2 - 2\mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)^2]\mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)]^2 + \mathbb{E}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)]^4 + O(N^{-1}) \\ &= \mathbb{V}[\psi_{(\{1\},\emptyset)}^{1,0} h(Y)^2]^2 + O(N^{-1}) \\ &= (v_h^{1,0})^2 + O(N^{-1}) \\ &= \mathbb{E}[\widehat{v}_N^{h;1,0}]^2 + O(N^{-1})\end{aligned}$$

where we have applied Proposition 4.5 in the last step. This proves that $\mathbb{V}[\widehat{v}_N^{h;1,0}] = O(N^{-1})$, concluding the proof. \square

E. Proofs of the results presented in Section 5

Lemma E.1. *Let Y be a matrix sampled from a Poisson W -graph model. Let \bar{w} , f and g be defined as in Section 5.2. For the kernel functions defined in Table 1, we have*

- $\mathbb{E}[h_{A,1}(Y_{(i_1,i_2;j_1,j_2)})] = \lambda^3 \iint \bar{w}(\xi, \eta)^2 d\xi d\eta,$
- $\mathbb{E}[h_{A,2}(Y_{(i_1,i_2;j_1,j_2)})] = \lambda^3 \iint \bar{w}(\xi, \eta) f(\xi) g(\eta) d\xi d\eta,$
- $\mathbb{E}[h_B(Y_{(i_1;j_1,j_2)})] = \lambda^2 \int f(\xi)^2 d\xi,$
- $\mathbb{E}[h_C(Y_{(i_1,i_2;j_1)})] = \lambda^2 \int g(\eta)^2 d\eta,$
- $\mathbb{E}[h_D(Y_{(i_1;j_1)})] = \lambda.$

Proof. The result for h_D is straightforward. For the other kernel functions :

$$\begin{aligned}\mathbb{E}[h_{A,1}(Y_{(i_1,i_2;j_1,j_2)})] &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} (Y_{i_1 j_1} - 1) Y_{i_2 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\ &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} (Y_{i_1 j_1} - 1) \mid \boldsymbol{\xi}, \boldsymbol{\eta}] \mathbb{E}[Y_{i_2 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\ &= \mathbb{E}[\lambda^2 \bar{w}(\xi_{i_1}, \eta_{j_1})^2 \times \lambda] \\ &= \lambda^3 \iint \bar{w}(\xi, \eta)^2 d\xi d\eta.\end{aligned}$$

$$\begin{aligned}\mathbb{E}[h_{A,2}(Y_{(i_1,i_2;j_1,j_2)})] &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} Y_{i_1 j_2} Y_{i_2 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\ &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} \mid \boldsymbol{\xi}, \boldsymbol{\eta}] \mathbb{E}[Y_{i_1 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}] \mathbb{E}[Y_{i_2 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\ &= \mathbb{E}[\lambda \bar{w}(\xi_{i_1}, \eta_{j_1}) \times \lambda \bar{w}(\xi_{i_1}, \eta_{j_2}) \times \lambda \bar{w}(\xi_{i_2}, \eta_{j_2})] \\ &= \lambda^3 \iint \left[\bar{w}(\xi_{i_1}, \eta_{j_2}) \left(\int \bar{w}(\xi_{i_1}, \eta_{j_1}) d\eta_{j_1} \right) \left(\int \bar{w}(\xi_{i_2}, \eta_{j_2}) d\xi_{i_2} \right) \right] d\xi_{i_1} d\eta_{j_2} \\ &= \lambda^3 \iint \bar{w}(\xi, \eta) f(\xi) g(\eta) d\xi d\eta.\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[h_B(Y_{(i_1; j_1, j_2)})] &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} Y_{i_1 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\
&= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} \mid \boldsymbol{\xi}, \boldsymbol{\eta}] \mathbb{E}[Y_{i_1 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\
&= \mathbb{E}[\lambda \bar{w}(\xi_{i_1}, \eta_{j_1}) \times \lambda \bar{w}(\xi_{i_1}, \eta_{j_2})] \\
&= \lambda^2 \int \left[\left(\int \bar{w}(\xi_{i_1}, \eta_{j_1}) d\eta_{j_1} \right) \left(\int \bar{w}(\xi_{i_1}, \eta_{j_2}) d\eta_{j_2} \right) \right] d\xi_{i_1} \\
&= \lambda^2 \int f(\xi)^2 d\xi.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[h_C(Y_{(i_1, i_2; j_1)})] &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} Y_{i_2 j_1} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\
&= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} \mid \boldsymbol{\xi}, \boldsymbol{\eta}] \mathbb{E}[Y_{i_2 j_1} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\
&= \mathbb{E}[\lambda \bar{w}(\xi_{i_1}, \eta_{j_1}) \times \lambda \bar{w}(\xi_{i_2}, \eta_{j_1})] \\
&= \lambda^2 \int \left[\left(\int \bar{w}(\xi_{i_1}, \eta_{j_1}) d\xi_{i_1} \right) \left(\int \bar{w}(\xi_{i_2}, \eta_{j_1}) d\xi_{i_2} \right) \right] d\eta_{j_1} \\
&= \lambda^2 \int g(\eta)^2 d\eta.
\end{aligned}$$

□

Lemma E.2. *Let Y be a matrix sampled from a W -graph model. Let h_1 and h_2 be the kernel functions defined as in Section 5.2. We have*

- $\mathbb{E}[h_1(Y_{(i_1; j_1, j_2)})] = \lambda^2 \int f(\xi)^2 d\xi = \lambda^2 F_2,$
- $\mathbb{E}[h_2(Y_{(i_1, i_2; j_1, j_2)})] = \lambda^2.$

Proof. The proof for h_1 is identical to that for h_B in the proof of Lemma E.1. For h_2 ,

$$\begin{aligned}
\mathbb{E}[h_2(Y_{(i_1, i_2; j_1, j_2)})] &= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} Y_{i_2 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\
&= \mathbb{E}[\mathbb{E}[Y_{i_1 j_1} \mid \boldsymbol{\xi}, \boldsymbol{\eta}] \mathbb{E}[Y_{i_2 j_2} \mid \boldsymbol{\xi}, \boldsymbol{\eta}]] \\
&= \mathbb{E}[\lambda \times \lambda] \\
&= \lambda^2.
\end{aligned}$$

□