# On the questionable use of CIE L* to infer geometric properties of achromatic perception

Michel Berthier, Edoardo Provenzi

# On the questionable use of CIE $L^*$ to infer geometric properties of achromatic perception

Michel Berthier*[1] and Edoardo Provenzi†[2]

[1]Laboratoire MIA, Batiment Pascal, Pôle Sciences et Technologie, Université de La Rochelle, 23, Avenue A. Einstein, BP 33060, 17031 La Rochelle cedex, France
[2]Université de Bordeaux, CNRS, Bordeaux INP, IMB, UMR 5251, F-33400, 351 Cours de la Libération, Talence, France

**Abstract**

In the paper 'The non-Riemannian nature of perceptual color space', the authors analyze the data of visual experiments conduced using triad of CIE $L^*$ achromatic stimuli. The statistical analysis of their results leads them to claim that the $L^*$ axis cannot be endowed with a Riemannian metric coherent with observers' responses. In this communication we provide a mathematical analysis which, in our opinion, confutes the main claim of the quoted paper. Furthermore, we raise questions on the soundness of using an achromatic perceptual coordinate to describe non-chromatic perception. We inform about an alternative in which achromaticity emerges from the very act of observing colors [2, 3].

**Keywords**: Geometry of perceptual color space; Riemannian metrics; Achromatic perception.

## 1 Introduction

In the paper [4], the authors use the psychological Thurstone protocol [15] applied to triads [13] to study the perception of achromatic visual stimuli. The statistical analysis of the large quantity of data that they have gathered reveals a diminishing return effect. From this, they infer that 'the perceptual color space' cannot be Riemannian.

Our primary goal in this contribution is to show that this inference is not mathematically correct. Our counter-argument will be presented in section 3, after having translated in mathematical terms Thurstone's protocol applied to triads in section 2.

To give a concise description of our counter-argument, we recall here that reasoning followed by the authors in [4] is the following:

1. hypothesis: the perceptual color space, let us denote it with $\mathcal{C}$, is supposed to be Riemannian;
2. hypothesis: the triad stimuli are codified by $L^*$ values;
3. hypothesis: the set of $L^*$ values is a geodesic w.r.t. the Riemannian metric of $\mathcal{C}$;
4. mathematical fact: the Riemannian distance induced on a geodesic is additive;
5. proposal by the authors: equip what they call the $L^*$ axis with a Riemannian distance based on the Thurstone map;

*michel.berthier@univ-lr.fr
†edoardo.provenzi@math.u-bordeaux.fr

6. experimental procedure followed by the authors: observers are exposed to triads of $L^*$ stimuli with fixed difference of differences, they have to indicate which one of the test stimuli 'is more different than the reference stimulus';

7. experimental fact: the statistical analysis of the data gathered shows that the observers' responses are not compatible with the additive property of the Riemannian distance proposed in item 5.

8. inference: the $L^*$ axis cannot be endowed with a Riemannian distance compatible with achromatic perception.

Our objection to this reasoning is the following: the experimental procedure followed in [4] involves two different distances on the $L^*$ axis. The first is the one used to define differences of differences in stimuli triads. The latter is the Thurstone Riemannian distance used to test statistical compatibility with observers' responses to the question 'which test stimulus is more different than the reference one'.

As we will show in section 3, for the argument of [4] to be coherent, the distance used to define differences of differences (i.e. the Riemannian one supposed to be defined on the $L^*$ axis) should be the same as that used to test the responses of the observers, which is not the case.

Besides this critical mathematical issue, let us also remark that the authors consider only one particular Riemannian distance on the $L^*$ axis, the one induced by the Thurstone map, whereas many other Riemannian distances can be envisaged.

The second goal of our contribution, developed in section 4, is to further underline the caution expressed by some experienced color scientists about the use of an achromatic coordinate, as $L^*$, to perform psycho-physical tests.

# 2 A mathematical framework for the Thurstone protocol applied to triads of $L^*$ values

The psycho-physical protocol chosen in [4] to gather and analyze the perceptual responses to triads of achromatic stimuli is Thurstone's one [15] applied to triads [13]. It is important to underline that, in spite of its importance and wide use, Thurstone's protocol is not the only one available for the kind of experiments conducted in [4], moreover it is not clear if the hypothesis upon which it relies are satisfied in the case of triads of $L^*$ values.

Nevertheless, since the claims of [4] are based on Thurstone's protocol applied to triads of $L^*$ values, we need to build a mathematical formalization of this particular psycho-physical protocol in order to precisely state our counter-argument.

For the sake of a more readable notation, we shall use the letter $x$ to denote $L^*$ values and reserve the letter $L^*$ to denote the set of all such values, called $L^*$ axis (or achromatic axis) in [4]. As a set, $L^*$ can be identified with $[0, 100)$. More information on $L^*$, which is not relevant at this stage for the mathematical analysis, is given in section 4.

A *generic Riemannian metric* on the $L^*$ axis can be expressed as follows:

$$ds = h(x)dx, \tag{1}$$

where $h : L^* \to (0, 100)$.

The *Riemannian distance* induced by this generic Riemannian metric $ds$ is defined in this way:

$$d^h_{L^*}(x_i, x_j) = \int_{x_i}^{x_j} h(x)dx, \tag{2}$$

for all couples of $L^*$ values $x_i$ and $x_j$ supposed to be arranged in such a way that $x_i < x_j$, so that $d^h_{L^*}$ always takes non-negative values. Notice that the superscript underlines that *the Riemannian distance depends on the function* $h$ that appears in eq. (1).

As mentioned before, the experimental protocole of [4] uses the method of triads [13]. In the sequel,

$$t = (x_1, x_{\text{ref}}, x_2), \tag{3}$$

with $x_1 < x_{\text{ref}} < x_2$, denotes a triad of $L^*$ values and $T$ refers to the set of all triads.

$T$ can be endowed with an equivalence relation $R_h$ that depends on the function $h$ and which will be used later on: given two triads $t = (x_1, x_{\text{ref}}, x_2)$ and $t' = (x'_1, x'_{\text{ref}}, x'_2)$,

$$t \sim_{R_h} t' \iff d^h_{L^*}(x_2, x_{\text{ref}}) - d^h_{L^*}(x_1, x_{\text{ref}}) = d^h_{L^*}(x'_2, x'_{\text{ref}}) - d^h_{L^*}(x'_1, x'_{\text{ref}}), \tag{4}$$

i.e., two triads are $R_h-$equivalent if the difference between the Riemannian distances $d^h_{L^*}$ of the extreme values of the triads w.r.t. to the reference is the same for both triads, or equivalently if the differences of differences in both triads are the same.

To have a concrete example at hand, the triads visualized in Fig. 4 of the paper [4], i.e. $t = (20, 50, 92.5)$ and $t' = (45, 50, 67.5)$ are equivalent w.r.t. the equivalence relation $\sim_{R_{\mathbf{1}}}$, where $\mathbf{1} : L^* \to \{1\}$, $x \mapsto \mathbf{1}(x) = 1$ for all $x \in L^*$.

We also denote $T_{x_{\text{ref}}}$ the subset of $T$ that contains all triads with a fixed reference stimulus $x_{\text{ref}}$. Both triads in the previous example belong to $T_{50}$.

The measurement task performed by the observers described in [4] can be modeled through the following function

$$\begin{array}{rccc} \varphi_{x_{\text{ref}}} : & T_{x_{\text{ref}}} & \longrightarrow & \{0,1,2\} \\ & t = (x_1, x_{\text{ref}}, x_2) & \longmapsto & \varphi_{x_{\text{ref}}}(t), \end{array}$$

where

$$\varphi_{x_{\text{ref}}}(t) = \begin{cases} 1 & \text{if the test stimulus } x_1 \text{ is more different from the reference stimulus than } x_2 \\ 2 & \text{if the test stimulus } x_2 \text{ is more different from the reference stimulus than } x_1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, it is important to notice that such a measurement task can be applied to arbitrary triads without taking care of differences of differences and that it does not involve any metric judgment. According to the authors of [4], the judgment "Which one is more different" is more reliable than the judgment "How big is the difference".

If there exists a Riemannian distance $d^h_{L^*}$ on the $L^*$ axis which is compatible with observers' judgments, then we must have

$$\begin{cases} \varphi_{x_{\text{ref}}}(t) = 1 \iff d^h_{L^*}(x_1, x_{\text{ref}}) > d^h_{L^*}(x_2, x_{\text{ref}}) \\ \varphi_{x_{\text{ref}}}(t) = 2 \iff d^h_{L^*}(x_2, x_{\text{ref}}) > d^h_{L^*}(x_1, x_{\text{ref}}) \\ \varphi_{x_{\text{ref}}}(t) = 0 \iff d^h_{L^*}(x_2, x_{\text{ref}}) = d^h_{L^*}(x_1, x_{\text{ref}}) \end{cases} \tag{5}$$

for all $x_{\text{ref}}$ and all $t$ in $T_{x_{\text{ref}}}$. Investigating this problem in its full generality is untractable. The authors of [4] propose to restrict to five choices of $x_{\text{ref}}$ (30, 40, 50, 60, and 70), and more crucially to fix three possible values for the difference of differences $\Delta$ in triads (2.5, 5, and 10). By making the test stimuli $x_1$ and $x_2$ varying, with fixed $\Delta$, they show that, statistically, observers' judgments are not compatible with the constraints (5) when dealing with the Riemannian metric induced on the $L^*$ axis by the Thurstone map.

We refer the reader to [15] for a full description of what is now known as Thrustone's protocol. This protocol involves stimuli that are differentiated by psychological discriminal processes of unkonwn nature that constitute the discriminal continuum. It is assumed that the correspondence between a stimuli and the corresponding psychological process is subject to noticeable fluctuation, *i.e.* to discriminal dispersion. This dispersion is modelized by a normal density whose mean is the perceptual strength of the given stimuli. In the sequel we denote with $g$ the Thurstone map between the $L^*$ axis and the set $P^*$ of Thurstone perceptual strengths:

$$\begin{array}{rccc} g : & L^* & \longrightarrow & P^* \\ & x & \longmapsto & g(x) = y. \end{array}$$

This map permits to equip the $L^*$ axis with a Riemannian metric that makes it isometric to the space $P^*$ endowed with the metric $dy$. More precisely, the metric on $L^*$ is given by

$$ds = g'(x)dx, \tag{6}$$

which means that, with the notations introduced above, $h(x) = g'(x)$. The fact that $(L^*, ds = g'(x)dx)$ is isometric to $(P^*, dy)$ follows from the equation:

$$\int_{x_1}^{x_2} g'(x)dx = \int_{y_1}^{y_2} dy. \tag{7}$$

This is precisely the *particular* metric that the authors of [4] claim that it is not compatible with the constraints (5).

# 3 The mathematical issue in paper [4]

As mentioned before, only stimuli triads with fixed differences of differences are considered for the measurement tasks. Since the $L^*$ axis is equipped with the Riemannian metric $ds = g'(x)dx$ induced by the Thurstone's map $g$, one should consider, for given $x_{\text{ref}}$ and $\Delta$, the set

$$T_{x_{\text{ref}}}^{\Delta,h} = \{t \in T_{x_{\text{ref}}} : d_{L^*}^h(x_2, x_{\text{ref}}) - d_{L^*}^h(x_1, x_{\text{ref}}) = \Delta\}. \tag{8}$$

with $h = g'$. According to the definition of the equivalence relation $\sim_{R_h}$ given in eq. (4), any two different triads $t$ and $t'$ of $T_{x_{\text{ref}}}^{\Delta,h}$ are equivalent modulo $\sim_{R_h}$.

There is a simple mathematical way to express the system of constraints (5) by means of the equivalence relation: the map $\varphi_{x_{\text{ref}}}$ must 'pass to the quotient', which means that there must exist a function $\tilde{\varphi}_{x_{\text{ref}}}$ that closes the following commutative diagram[1]:

$$\begin{array}{ccc} T_{x_{\text{ref}}}^{\Delta,h} & \xrightarrow{\quad\varphi_{x_{\text{ref}}}\quad} & \{0,1,2\} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\tilde{\varphi}_{x_{\text{ref}}}} & \\ & T_{x_{\text{ref}}}^{\Delta,h}/\sim_{R_h} & \end{array}, \tag{9}$$

where $\pi$ is the canonical projection of $T_{x_{\text{ref}}}^{\Delta,h}$ on the quotient set $T_{x_{\text{ref}}}^{\Delta,h}/\sim_{R_h}$ that maps a triad of $T_{x_{\text{ref}}}^{h,\Delta}$ into the only $\sim_{R_h}$-equivalence class of triads that contains it.

This diagram, which expresses the compatibility of a Riemannian metric on the $L^*$ axis, not only for the one chosen by the authors of [4], with the observers' judgments, clearly shows the importance of defining the difference of differences in a coherent way. Moreover, it also clearly shows that no metric judgment from the observers should be taken into account. This is coherent with the fact that the observers do not provide such judgements as recalled before.

At the beginning of the section "Analysis" of [4], the authors explained that they consider differences of differences in triads computed from $d_i := |L_{\text{ref}}^* - L_{t_i}^*|$, that is that they consider the factorizing problem for the following diagram:

$$\begin{array}{ccc} T_{x_{\text{ref}}}^{\Delta,\mathbf{1}} & \xrightarrow{\quad\varphi_{x_{\text{ref}}}\quad} & \{0,1,2\} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\tilde{\varphi}_{x_{\text{ref}}}} & \\ & T_{x_{\text{ref}}}^{\Delta,\mathbf{1}}/\sim_{R_{\mathbf{1}}} & \end{array}. \tag{10}$$

[1] We recall that a diagram with sets and functions between them is called commutative if all directed paths with the same start and endpoints lead to the same result. Hence, the diagram (9) is commutative if $\tilde{\varphi}_{x_{\text{ref}}}$ satisfies $\varphi_{x_{\text{ref}}} = \tilde{\varphi}_{x_{\text{ref}}} \circ \pi$. If this relationship is not satisfied, then the diagram is not commutative. Such diagrams are constantly used in fields such as Riemannian geometry to better visualize the spaces involved in the problem analyzed and the maps between them. Riemannian geometry is precisely the mathematical domain in which the assertions of the paper of Bujack et al. take place, this is why we consider the use of commutative diagram useful and pertinent.

The statistical analysis proposed in [4] shown that the commutative diagram (10) cannot be closed, however *this fact does not disprove that the commutative diagram* (9)*, which would have been the correct one to study, can be closed for a suitable function* $h$, and in particular for the *chosen* function $h = g'$.

In other words, the authors of [4] prove just that $(L^*, ds = \mathbf{1}(x)dx = dx)$ is not a Riemannian space coherent with the perceptual data that they gathered using the Thurstone protocol described in the previous section. As we have just explained, this fact is very different than saying that the $L^*$ axis cannot be endowed with a Riemannian metric coherent with achromatic perception.

In the next section we will discuss several motivations for which we deem that the choice of $L^*$ is not the correct one to study achromatic visual perception. However, even if the $L^*$ axis were the appropriate choice, the results in [4] would not prove that it is impossible to find a suitable function $h$ such that the Riemannian space $(L^*, ds = h(x)dx)$ is coherent with achromatic perception.

In conclusion, stating that the mathematical analysis of the perceptual data performed in [4] implies that the perceptual color space $\mathcal{C}$ cannot have a Riemannian nature is questionable to say the least.

# 4 Issues about the use of $L^*$ as a perceptual coordinate

We have shown up to now that the mathematical development proposed in [4] does not permit to justify the main conclusion of this cited work. We discuss now the issue, suggested by the very title of [4], of whether the $L^*$ coordinate is really appropriate to account for achromatic color perception. First, we point out the role of surround in triads and how adaptation to $x_{\text{ref}}$ should lead to the necessity of incorporating Stevens' effect into the Thurstone protocol. Then, we provide some arguments of the existing literature, that, quite inexplicably, some colorimetrists claim to be well-known and some other seem to ignore, which tend to confirm that the $L^*$ coordinate does not enable to describe faithfully perceptual achromatic attributes. Finally, we draw attention to a recent model of color perception based on color measurement expressed via quantum information in which perceptual achromatic attributes, lightness and brightness, are inherently linked to the measuring process.

In order to precisely state our concerns it is not superfluous to recall the birth of $L^*$, which has been manufactured by the CIE in the attempt to provide a more perceptually uniform version of the *luminance* $L$, relative to a reference white. In the CIEXYZ non-perceptual color space, $L$ coincides with the $Y$ coordinate, see e.g. [17] for more details.

When building the CIELAB space, the CIE proposed an interpolated re-scaling of $Y$ in order to better approach human perception. From the experiments of Weber [16] and the eventual mathematical formalization of Fechner [8], it was known that the perceptual response of humans to intensity variations of achromatic light stimuli is a compressive one. Weber-Fechner's law uses a logarithmic function to approximate this compressive behavior. Stevens [14] famously proposed in 1961 to replace the logarithm with a power law with, of course, an exponent between 0 and 1 to achieve compression. Actually, already in 1958, Glasser et al. proposed in [9] a cube-root coordinate, obtained in very restrictive isolated conditions, which was eventually used by the CIE to built the $L^*$ coordinate, along with some ad-hoc parameters to match as well as possible the observational data. As mentioned to us by an anonymous reviewer, it is notable that the CIE publication dealing with colorimetry [1] is silent on the subject of what to do when $Y$ is greater than 100, as it occurs in fluorescent reflectance specimens as well as with self-luminous objects. Once fixed an achromatic illuminant used as reference for white with luminance indicated with $Y_n$, the relationship between the luminance $Y$ of an achromatic stimulus and its $L^*$ coordinate was established through the following empirical definition:

$$L^* = 116\, f\left(\frac{Y}{Y_n}\right) - 16, \tag{11}$$

where, writing $t = Y/Y_n$,

$$f(t) = \begin{cases} \sqrt[3]{t} & \text{if } t > \delta^3 \\ \frac{t}{3\delta^2} + \frac{4}{29} & \text{otherwise} \end{cases}, \tag{12}$$

where $\delta = 6/29$.

Thus, what must be always kept in mind is that $L^*$ *is nothing but a numerical interpolation of how the CIE luminance* $Y$ *is perceived to vary when its magnitude is modified over a uniform background with no surround*... and with several more experimental constraints that must be satisfied for $L^*$ to make sense.

## 4.1 On the influence of Stevens' effect

$L^*$ is the achromatic coordinate of the CIELAB space. Although widely used (and misused) in color science and image processing, it is considered by some experienced researchers not a suitable perceptual quantity because of its inability to cope with appearance effects both in isolation and with a non-trivial surround, see e.g. [11, 7]. This is a crucial issue because, on one side, there exists an appearance effect, Stevens' one, that directly concerns achromatic stimuli and, on the other side and as previously said, the protocol followed in [4] relies on the use of triads of achromatic stimuli, which clearly constitute a non-trivial surround. This remark clearly raises more than a question about the coherence between the use of $L^*$, which was manufactured in a surround-free environment, and the study a visual configuration in which it is precisely the comparison with the surrounding stimulus in a triad that matters.

Let us now come to Stevens' effect, which is described in [7] as the empirically observed fact that contrast increases with luminance, where here the word contrast indicates the perceived difference between stimuli. It is clear that Stevens' effect induces a perceptual bias in the observation of inter-differences in triads, however no mention of this effect and how to balance it can be found in [4].

Stevens' effect could be incorporated in the mathematical framework described in the previous sections by allowing the function $g$ to vary with each triad $t$, becoming a parameterized function $g_t$, to take into account the contrast bias generated by the luminance of the stimuli appearing in each triad. Notice that in [4] it is supposed that $g_t \equiv g$ for all triads $t \in T$.

Taking into account Stevens' effect, it may happen that, for certain $t$,

$$g_t(x_{\text{ref}}) - g_t(x_i) \neq g(x_{\text{ref}}) - g(x_i), \qquad i = 1, 2, \tag{13}$$

which would lead to the need of replacing the commutative diagram (9) with the following one:

$$\begin{array}{ccc} T^{\Delta,g'_t}_{x_{\text{ref}}} & \xrightarrow{\quad\varphi_{x_{\text{ref}}}\quad} & \{0,1,2\} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\tilde{\varphi}_{x_{\text{ref}}}} & \\ & T^{\Delta,g'_t}_{x_{\text{ref}}} \Big/ \sim_{g'_t} & \end{array} \quad . \tag{14}$$

We can also analyze this issue form the point of view of triads in the following way: let $t \in T^{\Delta,h}_{x_{\text{ref}}}$, then we can consider the *normalized triad* defined as

$$\bar{t} = \left( \frac{x_1}{x_{\text{ref}}}, 1, \frac{x_2}{x_{\text{ref}}} \right), \tag{15}$$

which belongs to the set $T_1^{\Delta/x_{\text{ref}},h} = T_1^{\Delta,h/x_{\text{ref}}}$. This would mean that the Riemannian metric on the $L^*$ axis to be searched is

$$ds = \frac{h(x)}{x_{\text{ref}}} dx. \tag{16}$$

## 4.2 Does an achromatic coordinate really exist?

From the official CIE website `https://cie.co.at/e-ilv` we can read these two definitions.

- *Brightness*: attribute of a visual sensation according to which an area appears to be more or less intense or according to which the area in which the visual stimulus is present appears to emit more or less light. Variations in brightness range from bright to dim.
- *Lightness*: attribute of a visual sensation according to which the area in which the visual stimulus is presented appears to emit more or less light in proportion to that emitted by a similarly illuminated area perceived as a white stimulus. In a sense, lightness may be referred to as relative brightness. Variations in lightness range from light to dark.

The CIE self-defined $L^*$ to be the lightness and that became it. Nowadays only few 'free thinkers' allow themselves to disagree with this definition.

One of them is M. Fairchild, who, in section 10.4 of his book [7], entitled 'Why not use just CIELAB?', pointed out that $L^*$ is completely incapable of predicting or explaining appearance phenomena and it incorporates no background or surround dependency. Fairchild's conclusion is that $L^*$ '*does not provide a correlate for the absolute appearance attributes of brightness*'.

Fairchild's objection about $L^*$ is far from having only a linguistic nature: color attributes are intertwined and a true codification of them must take their interrelations into account to provide reliable quantitative data during color perception experiments.

In a publication that dates back to 1953, A. Dresler already wrote [6]: '*Recent papers* [...] *indicate that some of the authors are not wholly familiar with the practical difficulties that inevitably occur when the brightnesses of two adjacent surfaces of different colour have to be matched by eye, and the result then used to express certain luminous properties of the two surfaces (or of the two light sources from which they receive their light) in terms of luminance or any other of the photometric quantities*', and also '*It should, however, be stressed that this definition of luminance and the corresponding linear photometric system do not postulate that the human eye should under all circumstances be capable of measuring luminances by assessing brightnesses. Just the reverse is true, for it is expressly stated in the relevant CIE recommendation of 1939 that the results of photometric measurements must always comply with the definition of luminance quoted above*'.

The conclusion drawn by Dresler in his paper is the following: '*We cannot expect a linear system involving a single function to define accurately the relative luminous values of spectral and non-spectral lights under the varying conditions of practical application. But I want to make clear that I have approached the whole problem as a photometrist, a photometrist moreover who has been slightly irritated by people who persist in the belief that the eye should behave in conformity with the mystic "hypothesis of the CIE", and who express surprise and even disbelief when they find that it does not overlook the fact that our "linear" photometric system is a matter of expediency and not a matter of visual judgment*'.

J. Koenderink describes luminance as '*a purely formal entity*' in [12], adding '*it doesn't have any meaning in terms of the perceptual attributes*' and '*it doesn't belong to colorimetry proper*'.

D. Hubel, awarded with the Nobel prize for his studies on the visual system, when discussing Hering's theory in his book [10], underlines the intrinsic role played by chromatic opponency in color perception against the indefiniteness of achromatic stimuli.

## 4.3 Perceptual achromatic attributes emerging from measurements

The belief of the renowned color scientists quoted above regarding the lack of intrinsic meaning of an 'achromatic coordinate' is also part of a recent color perception model, see e.g. [3, 2] and the references therein for more details. Here, we just want to underline that, in this model, color perception is interpreted from the viewpoint of quantum information and observers are exposed to visual scenes prepared in certain *chromatic states* and *the achromatic information* (brightness and lightness) *can be acquired by the observer only after a perceptual measurement, it does not exist as a prior entity, or a coordinate in a space.* Quite remarkably, this way of interpreting the acquisition

of achromatic information permits to give a mathematical explanation to the phenomenon of lightness constancy, see [2]. Giving a more precise description is out of the scope of this work since it requires to introduce many notions and tools from quantum information, such as Lüders operations, quantum relative entropy, generalized states, and so on, the interested reader can refer to [2] for all the details.

# 5 Conclusion

In this communication we have provided a mathematical formalization of the Thurstone protocol applied to triads used by the authors of the paper [4] and we have exploited this rigorous framework to point out a mistake in the mathematical reasoning that leads the authors of [4] to the claim that the space of perceptual colors is not Riemannian.

We have also underlines several reasons why we think that experiments based on the use of the CIE achromatic coordinate $L^*$ to gather perceptual information may be flawed. Independently of the agreement reached between the Bureau of Weights and Measures and the CIE, we consider that the CIE theory must not be treated in a dogmatic way. As researchers, we have the right and duty to question its validity, as we would do with any other theory, in spite of its pervasive presence in all fields related to color.

We would like to end this conclusion with two sentences written by the astronomer and earth scientist S. C. Chandler in his celebrated 1892 paper [5] in which he proved the existence of a remarkable discrepancy in the polar motion, nowadays called Chandler wobble, from what predicted by illustrious predecessors, among them Newton and Euler: '*Naturally, then, I am not much dismayed by the argument of conflict with dynamic laws; since all that such a phrase means must refer merely to the existent state of the theory at any given time*' and '*Are we so sure yet of a complete knowledge of all the forces at work as to exclude the chance of a vera causa?*'.

The words of Chandler incarnate beautifully the awareness, largely diffused among scientists and intellectuals of the final part of the nineteenth century and the beginning of the twentieth century, that *the validity of a sentence must always be referred to the current state of the art of a theory* and that, in order to find the *vera causa* underlying a phenomenon and widen the scientific knowledge, *sometimes a radical rethinking of the available theories is necessary.* This kind of questioning is exactly what drives the research that is being developed in the quantum color perception model [3, 2]. This state of mind has pervaded theoretical physics and persists nowadays, however it does not seem the case in color science, dominated as it is by the CIE model. Unfortunately, this attitude may slow down or even prevent advancements in our current understanding of the mechanisms underlying color perception.


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# Authors' biographies

Michel Berthier is professor of mathematics at Université de La Rochelle. His research interests are related to geometric and algebraic methods and their applications to image processing and vision.

Edoardo Provenzi is professor of mathematics at Université de Bordeaux. He is interested in variational and geometric models of mathematical physics to study vision and its applications to image processing.

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