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Claude-Michel Brauner*†, Peipei Shang‡ and Linwan Zhang§

Abstract

We consider a traveling wave solution of a two-dimensional transmission problem across the $y-$axis modeling autoignition of reactive jets, a limiting version corresponding to large heat loss. A jump of the normal derivative generates a singularity at the origin. We construct an explicit solution to a problem in distributional sense that verifies the transmission system and closely examine the properties of the solution near the singularity. Using the Implicit Function Theorem, we study the level sets, especially the one through the origin. The numerical illustrations are consistent with our theoretical results.

Keywords: Transmission problem, fixed interface, traveling waves, singularity, fundamental solution, Bessel functions, autoignition.
MSC: 35K57, 35C07, 35A21, 35A08, 33C10, 80A25

1 Introduction and main results

1.1 Physical framework

Autoignition, the spontaneous growth of reaction rates in reactive systems, is a classical problem of combustion theory. Mathematical modeling of autoignition (thermal explosion) traces back to 1920's-1930's. During that time, a general approach to study autoignition was developed resulting in what is now called Semnov-Frank-Kamenetskii theory of thermal explosion (see, e.g., [4, 23]). Using the general approach of Semnov and Frank-Kamenetskii multiple models for studying autoignition in different situations were derived. In particular, in a series of works several models for autoignition of reactive jets were proposed and analysed [5, 6, 8, 9].

It is most typical that the autoignition is immediately followed by the flame propagation. This is, however, not the case for laminar co-flow reactive jets as evident from experimental studies of autoignition of hydrothermal flames [16]. In such jets, autoignition results in formation of a localized ignition kernel which then assumes a shape of a

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sausage-like structure propagating downstream. This structure can be called an ignition front as the reaction rate in this front is small in comparison to the well developed flame. A simple model for propagation of cylindrical front of ignition in such co-flow reactive laminar jets was proposed and studied in [7]. There, the authors focused on the prediction of the effect of counterflow propagation of ignition fronts. The model of propagation of the ignition front, upstream of the reactant flow, presented in this paper combined with a model of autoignition presented in [6, 8] provided basic quantitative description of spontaneous formation and propagation of ignition in a co-flow laminar jets. These models can be used for guiding experimental studies for identification of the necessary (formation of the ignition kernel) and sufficient (counterflow propagation) conditions for establishing steady diffusion flames in co-flow jets.

In this paper, we are interested in a two-dimensional reaction diffusion model for the normalized temperature Θ, corresponding to a large heat loss of a three-dimensional model discussed in [7].

At the outset, consider the problem

$$c_p\rho(T_t - vT_z) = \kappa\Delta_{r,z}T - \nu(T - T_0), \tag{1.1}$$

where T is the temperature, c_p the specific heat, ρ the density, v the jet velocity, κ the thermal conductivity; ν is a bulk radiative heat loss parameter, T_0 is an ambient temperature, t the time; z the vertical axis that coincides with the central line of the jet and pointing in the direction opposite to the jet flow, r is the distance from the central line of the jet in the horizontal plane; in (1.1), $\Delta_{r,z} = \dfrac{\partial^2}{\partial z^2} + \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$ is the Laplace operator in cylindrically symmetric coordinates.

At the interface $r = R$, it is assumed continuity of the temperature and discontinuity of the normal component of the temperature gradient[7], that is

$$[T] = 0, \quad \kappa[\nabla T \cdot N] = -QH(T - T_i). \tag{1.2}$$

Throughout the paper, brackets stand for the jump of a quantity when crossing the interface. In (1.2), Q is the reaction intensity, H is the usual Heaviside function, T_i is the ignition temperature and N is the outward unit normal to the interface. It is convenient to introduce the following scaling for the temperature and coordinates

$$\Theta = \frac{T - T_0}{T_b - T_0}, \quad \Theta_i = \frac{T_i - T_0}{T_b - T_0}, \quad \tau = \frac{\kappa t}{c_p\rho R^2}, \quad \xi = \frac{r}{R}, \quad \zeta = \frac{z}{R}, \tag{1.3}$$

as well as for the heat loss parameter and jet velocity

$$h = \frac{R^2\nu}{\kappa}, \quad u = \frac{c_p\rho R v}{\kappa}. \tag{1.4}$$

Then, the rescaled system (1.1) reads:

$$\Theta_\tau - u\Theta_\xi = \Delta_{\xi,\zeta}\Theta - h\Theta, \quad \eta \in \mathbb{R}, \ \xi \in (0,1) \cup (1,+\infty), \tag{1.5}$$

together with the interface conditions at $\xi = 1$:

$$[\Theta] = 0, \quad [\Theta_\xi] = -H(\Theta - \Theta_i). \tag{1.6}$$

We look for a solution $\Theta(\eta,\xi,\tau)$ of system (1.5)–(1.6) in the moving frame coordinate

$$\eta = \zeta - U\tau. \tag{1.7}$$

Setting $c = u + U$, Θ is solution of

$$\begin{cases} \Theta_\tau - c\,\Theta_\xi = \Delta_{\xi,\eta}\Theta - h\Theta, \quad \eta \in \mathbb{R},\ \xi \in (0,1)\cup(1,+\infty), \\ [\Theta] = 0, \quad [\Theta_\xi] = -H(\Theta - \Theta_i). \end{cases} \tag{1.8}$$

Next, performing the following change of variables

$$t = h\,\tau, \quad y = \sqrt{h}\,\eta, \quad x = \sqrt{h}(\xi - 1), \quad \widetilde{\Theta} = \sqrt{h}\,\Theta, \quad \tilde{c} = \frac{c}{\sqrt{h}}, \tag{1.9}$$

it comes

$$\widetilde{\Theta}_t - \tilde{c}\,\widetilde{\Theta}_y = \Delta_{x,y,h}\widetilde{\Theta} - \widetilde{\Theta}, \quad y \in \mathbb{R}, \quad x \in (-\sqrt{h},0)\cup(0,+\infty), \tag{1.10}$$

where

$$\Delta_{x,y,h} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\sqrt{h}}\left(\frac{1}{1+\frac{x}{\sqrt{h}}}\right)\frac{\partial}{\partial x}, \tag{1.11}$$

h being the normalized heat loss parameter, see (1.4). The interface is now at $x = 0$ and the interface conditions read:

$$[\widetilde{\Theta}] = 0, \quad [\widetilde{\Theta}_x] = -H(\widetilde{\Theta} - \widetilde{\Theta}_i), \quad x = 0, \tag{1.12}$$

where $\widetilde{\Theta}_i = \sqrt{h}\,\Theta_i$.

We are interested in the large values of the normalized heat loss parameter h. Let us introduce the small parameter $\varepsilon = 1/\sqrt{h}$. The problem becomes:

$$\begin{cases} \widetilde{\Theta}_t - \tilde{c}\,\widetilde{\Theta}_y = \Delta\widetilde{\Theta} - \widetilde{\Theta} + \varepsilon\left(\dfrac{1}{1+\varepsilon x}\right)\widetilde{\Theta}_x, \quad x \in (-\dfrac{1}{\varepsilon},0)\cup(0,+\infty), \\ [\widetilde{\Theta}] = 0, \quad [\widetilde{\Theta}_x] = -H(\widetilde{\Theta} - \widetilde{\Theta}_i), \quad x = 0,\ y \in \mathbb{R}. \end{cases} \tag{1.13}$$

where $\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}$ is the usual bidimensional Laplace operator. Formally, when $\varepsilon \to 0$, we get at the limit the following two-dimensional problem:

$$\begin{cases} \widetilde{\Theta}_t - \tilde{c}\,\widetilde{\Theta}_y = \Delta\widetilde{\Theta} - \widetilde{\Theta}, \quad x \in \mathbb{R}\backslash\{0\}, \quad y \in \mathbb{R}, \\ [\widetilde{\Theta}] = 0, \quad [\widetilde{\Theta}_x] = -H(\widetilde{\Theta} - \widetilde{\Theta}_i), \quad x = 0,\ y \in \mathbb{R}. \end{cases} \tag{1.14}$$

The evolution problem (1.14) admits clearly two 1D equilibria: the trivial solution $\widetilde{\Theta} = 0$ that corresponds to the absence of any reaction, and a nontrivial steady solution of (1.14), $\widetilde{\Theta}_\infty = 1/2e^{-|x|}$, which verifies

$$\begin{cases} \widetilde{\Theta}''_\infty - \widetilde{\Theta}_\infty = 0, \\ [\widetilde{\Theta}_\infty] = 0, \quad [\widetilde{\Theta}'_\infty] = -1 \ \text{ at } \ x = 0. \end{cases} \tag{1.15}$$

Therefore, omitting the tildes, we look for a traveling front solution (c, Θ) of problem (1.14) satisfying the two-dimensional stationary system

$$\begin{cases} -c\,\Theta_y = \Delta\Theta - \Theta, \quad x \in \mathbb{R}\backslash\{0\}, \quad y \in \mathbb{R}, \\ [\Theta] = 0, \quad [\Theta_x] = -H(\Theta - \Theta_i), \quad x = 0, \ y \in \mathbb{R}, \end{cases} \tag{1.16}$$

and connects the two one-dimensional equilibria $\Theta = 0$ and $\Theta = \Theta_\infty$, namely

$$\Theta \to \Theta_\infty, \ y \to -\infty, \quad \Theta \to 0, \ y \to +\infty. \tag{1.17}$$

Moreover, the following boundary conditions hold

$$\Theta(x, y) \to 0, \ |x| \to +\infty, \ y \in \mathbb{R}. \tag{1.18}$$

This kind of model also occurs in the cases when the productive part of the reaction happens only on a surface in space or on a line on plane but the degradation and the diffusion happen in bulk are important for modeling various biological processes (see [2] and more recently [22]).

1.2 The transmission problem

More specifically, we consider system (1.16)–(1.18) as a steady transmission problem across a fixed interface S, which is the $y-$axis, for a two-dimensional traveling wave solution $\Theta = \Theta(x, y)$ of system (1.14), which propagates at a velocity $c \in \mathbb{R}$ to be determined.

We adopt the following notation

$$\Omega_- = \{(x, y) \in \mathbb{R}^2, \ x < 0\}, \tag{1.19}$$

$$\Omega_+ = \{(x, y) \in \mathbb{R}^2, \ x > 0\}. \tag{1.20}$$

The transmission problem for Θ^- and Θ^+, respectively restrictions of Θ to $\overline{\Omega}_-$ and $\overline{\Omega}_+$, and the velocity c reads

$$\begin{cases} -\Delta\Theta^- - c\dfrac{\partial\Theta^-}{\partial y} + \Theta^- = 0, \ (x, y) \in \Omega_-, \\ -\Delta\Theta^+ - c\dfrac{\partial\Theta^+}{\partial y} + \Theta^+ = 0, \ (x, y) \in \Omega_+, \end{cases} \tag{1.21}$$

with transmission conditions at the interface $S = \{(0, y), y \in \mathbb{R}\}$,

$$\begin{cases} \Theta^-(0, y) = \Theta^+(0, y), \ y \in \mathbb{R}, \\ \dfrac{\partial\Theta^+}{\partial x}(0, y) - \dfrac{\partial\Theta^-}{\partial x}(0, y) = -H\big(\Theta(0, y) - \Theta_i\big), \ y \in \mathbb{R}, \end{cases} \tag{1.22}$$

and boundary conditions:

$$\begin{cases} \Theta^\pm(x, y) \to 0, \quad x \in \mathbb{R}, \ y \to +\infty, \\ \Theta^\pm(x, y) \to \Theta_\infty(x), \quad x \in \mathbb{R}, \ y \to -\infty, \\ \Theta^-(x, y) \to 0, \quad x \to -\infty, \ y \in \mathbb{R}, \\ \Theta^+(x, y) \to 0, \quad x \to +\infty, \ y \in \mathbb{R}. \end{cases} \tag{1.23}$$

Here, H is the Heaviside function, $\Theta_i > 0$ is the normalized ignition temperature. Furthermore, $\Theta_\infty = 1/2e^{-|x|}$ is the solution of (see (1.15))

$$\Theta''_\infty(x) - \Theta_\infty(x) = 0,\ x \in \mathbb{R},\ x \neq 0, \quad \Theta'_\infty(0^+) - \Theta'_\infty(0^-) = -1. \tag{1.24}$$

Let us make the additional assumption about the normalized ignition temperature, which is physically relevant:

$$0 < \Theta_i < \frac{1}{2}. \tag{1.25}$$

As we will see in this paper (see Lemma 3.2), the mapping $y \mapsto \Theta(0, y)$ is decreasing from $1/2$ to 0. In such a case, thanks to (1.25), there is a unique y_0 such that $\Theta(0, y_0) = \Theta_i$. Owing to the translation invariance along the $y-$axis, without loss of generality, we may assume that $y_0 = 0$, i.e., it holds at the origin

$$\Theta(0, 0) = \Theta_i, \tag{1.26}$$

which will eventually determine the velocity c (see Section 2.3). Then, it comes

$$H(\Theta(0, y) - \Theta_i) = 0 \text{ for } y > 0, \quad H(\Theta(0, y) - \Theta_i) = 1 \text{ for } y < 0. \tag{1.27}$$

Therefore, the transmission conditions (1.22) read:

$$\begin{cases} \Theta(0, 0) = \Theta_i, \quad 0 < \Theta_i < \dfrac{1}{2}, \\ \Theta^-(0, y) = \Theta^+(0, y),\ y \in \mathbb{R}, \\ \dfrac{\partial \Theta^+}{\partial x}(0, y) - \dfrac{\partial \Theta^-}{\partial x}(0, y) = \begin{cases} -1, \text{ if } y < 0, \\ 0, \quad \text{if } y > 0. \end{cases} \end{cases} \tag{1.28}$$

Transmission problems were extensively studied in the 1950s-1960s, particularly under the influence of M. Picone [13], J.-L. Lions [12], G. Stampacchia [18] and M. Schechter [15]. Fixed interface problems in Russian literature go under the name of diffraction problems, see, e.g., [11, 14]. Significant recent work has shown a renewed interest in this area, see [17] and the literature therein. L. A. Caffarelli, M. Soria-Carro and P. R. Stinga [3] have considered a transmission problem with Hölder continuous transmission conditions at the interface.

1.3 Main results

Let us compute the operator $L = -\Delta - c\dfrac{\partial}{\partial y} + I$ in the sense of distributions on $\mathbb{R}^2$. The following application of Green's formula is formal. For $\varphi \in C_0^\infty(\mathbb{R}^2)$, the space of

smooth functions on $\mathbb{R}^2$ with compact support, it comes

$$\begin{aligned}
\langle L[\Theta],\varphi\rangle &= \int_{\mathbb{R}^2} \Theta\, L^*[\varphi]dxdy \\
&= \int_{\Omega_-} \Theta^- L^*[\varphi]dxdy + \int_{\Omega_+} \Theta^+ L^*[\varphi]dxdy \\
&= \int_{\Omega_-} L[\Theta^-]\varphi dxdy + \int_{\Omega_+} L[\Theta^+]\varphi dxdy \\
&\quad + \int_{-\infty}^{+\infty} \left(\Theta^+(0,y) - \Theta^-(0,y)\right) \frac{\partial\varphi}{\partial x}(0,y)dy \\
&\quad - \int_{-\infty}^{+\infty} \left(\frac{\partial\Theta^+}{\partial x}(0,y) - \frac{\partial\Theta^-}{\partial x}(0,y)\right)\varphi(0,y)dy \\
&= -\int_{-\infty}^{+\infty} \left(\frac{\partial\Theta^+}{\partial x}(0,y) - \frac{\partial\Theta^-}{\partial x}(0,y)\right)\varphi(0,y)dy \\
&= \int_{-\infty}^{0} \varphi(0,y)dy,
\end{aligned} \tag{1.29}$$

thanks to (1.21) and the transmission conditions (1.28).

Let us define the tempered distribution T on $\mathbb{R}^2$ as the direct product of the indicator function of the set $\{y<0\}$ with the Dirac Delta distribution at $x=0$:

$$T = \mathbb{1}_{y<0} \otimes \delta_{x=0}. \tag{1.30}$$

It follows from (1.29) that $\langle L[\Theta],\varphi\rangle = \langle T,\varphi\rangle$ for all Schwartz function φ on $\mathbb{R}^2$, i.e.,

$$L[\Theta] = T \tag{1.31}$$

in the space $\mathscr{S}'(\mathbb{R}^2)$ of tempered distribution on $\mathbb{R}^2$.

Hereafter, we call problem (1.31) the problem in the sense of distributions, in contrast to the original transmission problem (1.21), (1.23) and (1.28).

The paper is organized as follows: In Section 2, we show that problem (1.31) has a solution (c,Θ) in the sense of distributions by means of the method of fundamental solutions in constructing a sequence of approximations. Next, we prove some properties of Θ in Section 3: in particular, we show that Θ is continuous on $\mathbb{R}^2$ and verifies the boundary conditions (1.23), see Lemmas 3.3 and 3.4. Other relevant properties proved in Section 3 are listed hereafter, see Theorem 1.1(A). In Section 4, we focus on the main feature of the model, namely the logarithmic singularity at the origin that generates a discontinuity in the transmission condition $\frac{\partial}{\partial x}\Theta^+(0,y) - \frac{\partial}{\partial x}\Theta^-(0,y)$ at the interface which takes values -1 if $y<0$, $-1/2$ if $y=0$ and 0 if $y>0$, see Theorem 1.1(B).

Theorem 1.1. *The transmission problem* (1.21), (1.23) *and* (1.28) *has a unique solution* (c,Θ)

$$c = \cot(2\pi\Theta_i), \quad \Theta(x,y) = \int_y^{+\infty} E(x,u)du, \tag{1.32}$$

where the fundamental solution of operator L

$$E(x,u) = \frac{1}{2\pi} e^{-\frac{c}{2}u} K_0\left(\sqrt{(1+\frac{c^2}{4})(x^2+u^2)}\right) \tag{1.33}$$

is in $L^p(\mathbb{R}^2)$ *for any* $1 \leqslant p < +\infty$ *(see Lemma 2.1) and* $K_0(z)$ *is the modified Bessel function of the second kind.*

A. The solution Θ *enjoys the following properties (see Section 3):*

(i) Regularity: Θ *is continuous on* $\mathbb{R}^2$, $\Theta^{\pm}$ *are smooth on* $\Omega_{\pm}$ *(see Subsection 3.1);*

(ii) $\dfrac{\partial \Theta^+}{\partial x}(x,y) < 0$ *on* Ω_+ *and* $\dfrac{\partial \Theta^-}{\partial x}(x,y) > 0$ *on* Ω_- *(see Lemma 3.5);*

(iii) For $x \in \mathbb{R}$, $\Theta(x,y)$ *is a monotonically decreasing function of* y *and* $\dfrac{\partial \Theta}{\partial y}(x,y) \to 0$ *as* $|y| \to +\infty$ *(see Lemma 3.2).*

B. Moreover, Θ *has a singularity at the origin, which is to be found in the limits at the interface* S *(see Section 4):*

(iv) if $y < 0$, $\lim\limits_{x \to 0^+} \dfrac{\partial \Theta^+}{\partial x}(x,y) = -1/2$, $\lim\limits_{x \to 0^-} \dfrac{\partial \Theta^-}{\partial x}(x,y) = 1/2$ *(see Lemma 4.4);*

(v) if $y = 0$, $\lim\limits_{x \to 0^+} \dfrac{\partial \Theta^+}{\partial x}(x,0) = -1/4$, $\lim\limits_{x \to 0^-} \dfrac{\partial \Theta^-}{\partial x}(x,0) = 1/4$ *(see Lemma 4.5);*

(vi) if $y > 0$, $\lim\limits_{x \to 0^{\pm}} \dfrac{\partial \Theta^{\pm}}{\partial x}(x,y) = 0$ *(see Lemma 3.5);*

(vii) $\lim\limits_{y \to 0^{\pm}} \dfrac{\partial \Theta}{\partial y}(0,y) = -\infty$ *(see Lemma 4.1).*

Uniqueness in Theorem 1.1 is not difficult: let Θ_1 and Θ_2 be two distinct solutions of the transmission problem, $w := \Theta_1 - \Theta_2$ is a continuous function on $\mathbb{R}^2$ such that it holds $Lw^{\pm} = 0$ in $\Omega_{\pm}$ and $w^+(0,y) - w^-(0,y) = \frac{\partial}{\partial x} w^+(0,y) - \frac{\partial}{\partial x} w^-(0,y) = 0$ at the interface, where $w^{\pm}$ are the restrictions of w on $\Omega_{\pm}$ respectively. Following the lines of (1.29), it is easy to see that $Lw = 0$. Then, $w = \delta * w = LE * w = E * Lw = 0$ in the sense of Schwartz distributions which yields $w = 0$ on $\mathbb{R}^2$.

It may be infer therefrom that

Corollary 1.2. *The problem* (1.21)–(1.23) *has a solution* (c, Θ) *which is unique up to translations along the* $y-$*axis.*

Next, in Section 5, we use the Implicit Function Theorem to prove the existence and regularity of the level sets whose graph passes through a point $(0, y_0)$ of the interface.

Theorem 1.3. *The level sets* $\Theta(x,y) = \Theta(0,y_0)$, $y_0 \in \mathbb{R}$, *enjoy the following properties:*

(i) when $y_0 = 0$, *the level set through the singularity* $(0,0)$ *is the graph of a continuously differentiable function* $\varphi_0(x)$ *which is increasing when* $x < 0$ *and decreasing when* $x > 0$ *and satisfies* $\varphi_0'(0) = 0$. *It has* $x = \pm x_i$ *as its asymptotes, where* $x_i = -\ln(2\Theta_i)$;

(ii) when $y_0 < 0$, *the level set through* $(0, y_0)$ *is the graph of a continuous function* $\varphi_{y_0}(x)$, *differentiable except at the origin, increasing when* $x < 0$ *and decreasing when* $x > 0$. *It has the asymptotes* $x = \pm \ln(2\Theta(0,y_0))$. *At* $(0,y_0)$, *the curve has a corner because* $\varphi_{y_0}'(0^+) < 0$ *and* $\varphi_{y_0}'(0^-) > 0$;

(iii) when $y_0 > 0$, the level set through $(0, y_0)$ is the graph of a continuously differentiable function $\varphi_{y_0}(x)$ which is increasing when $x < 0$ and decreasing when $x > 0$ and satisfies $\varphi'_{y_0}(0) = 0$. It has the asymptotes $x = \pm \ln(2\Theta(0, y_0))$.

Because of the singularity of $\Theta(x, y)$ at $(0, 0)$, we can not apply the Implicit Function Theorem at the origin. The idea is to start from a point, e.g., $(x_* > 0, y_* < 0)$ different from the origin, apply the Implicit Function Theorem at this point, and then extend the level set on both sides, with the origin being the endpoint of the left extension. Surprisingly, the level set through the singularity is C^1, however the tangent at the level set $\varphi'_0(0) = 0$ is inherently linked to the logarithmic singularity of $\frac{\partial}{\partial y}\Theta(x, y)$ at the origin, see Section 5.

Our results are highlighted with some numerical results, which confirm in particular that the level set has a horizontal tangent at the origin.

2 The problem in the sense of distributions

In this section, c is a fixed real number. We are going to prove that problem (1.31) has a solution Θ in the space of tempered distributions on $\mathbb{R}^2$.

2.1 Fundamental solution of operator L

To determine a solution of (1.31), we first look for a fundamental solution of the differential operator $L = -\Delta - c\frac{\partial}{\partial y} + I$, namely a distribution E on $\mathbb{R}^2$ solution of

$$L[E] = \delta, \tag{2.1}$$

where δ stands for Dirac Delta distribution on $\mathbb{R}^2$. Because L has constant coefficients, the existence of E follows from Malgrange-Ehrenpreis theorem (see, e.g., [19]); however, it is not difficult to check that E is a regular distribution associated with the function (see, e.g., [1, p. 376])

$$E(x, y) = \frac{1}{2\pi} e^{-\frac{c}{2}y} K_0\left(\sqrt{(1 + \frac{c^2}{4})(x^2 + y^2)}\right), \tag{2.2}$$

where K_0 is a modified Bessel function of the second kind

$$K_0(z) = \int_0^{+\infty} e^{-z\cosh(t)} dt \tag{2.3}$$

defined for $z > 0$. Let us remind that $K_0(z)$ is positive and decreasing,

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \text{ as } z \to +\infty, \tag{2.4}$$

(see, e.g., [1, Section 9.6]) and, as $z \to 0^+$,

$$K_0(z) \sim -\ln z. \tag{2.5}$$

Therefore, we observe that the fundamental solution E is smooth on $\mathbb{R}^2$ except at the origin where it has a logarithmic singularity. In addition, one has

Lemma 2.1. *$E \in L^p(\mathbb{R}^2)$, $1 \leqslant p < +\infty$.*

Proof. Using the change of variables, we have

$$\begin{aligned}\iint\limits_{\mathbb{R}^2} E^p(x,y)dxdy &= \left(\frac{1}{2\pi}\right)^p \iint\limits_{\mathbb{R}^2} e^{-\frac{c}{2}py} K_0^p\left(\sqrt{(1+\frac{c^2}{4})(x^2+y^2)}\right)dxdy \\ &= \left(\frac{1}{2\pi}\right)^p \int_0^{+\infty}\int_0^{2\pi} e^{-\frac{c}{2}pr\sin\theta} K_0^p\left(r\sqrt{1+\frac{c^2}{4}}\right) rd\theta dr \qquad (2.6) \\ &:= J_1 + J_2, \qquad (2.7)\end{aligned}$$

where

$$\begin{aligned}J_1 &= \left(\frac{1}{2\pi}\right)^p \int_0^{\rho}\int_0^{2\pi} e^{-\frac{c}{2}pr\sin\theta} K_0^p\left(r\sqrt{1+\frac{c^2}{4}}\right) rd\theta dr, \\ J_2 &= \left(\frac{1}{2\pi}\right)^p \int_{\rho}^{+\infty}\int_0^{2\pi} e^{-\frac{c}{2}pr\sin\theta} K_0^p\left(r\sqrt{1+\frac{c^2}{4}}\right) rd\theta dr\end{aligned}$$

with some fixed $0<\rho<1$. For J_1, we have

$$J_1 \leqslant \left(\frac{1}{2\pi}\right)^{p-1} \int_0^{\rho} e^{\frac{|c|}{2}p} K_0^p\left(r\sqrt{1+\frac{c^2}{4}}\right) dr. \qquad (2.8)$$

By (2.5) and noticing that the integral

$$\int_0^{\rho} e^{\frac{|c|}{2}p}\left(-\ln(r\sqrt{1+\frac{c^2}{4}})\right)^p dr \qquad (2.9)$$

converges, then we obtain that J_1 converges.

For J_2, we have

$$J_2 \leqslant \left(\frac{1}{2\pi}\right)^{p-1} \int_{\rho}^{+\infty} e^{\frac{|c|}{2}pr} K_0^p\left(r\sqrt{1+\frac{c^2}{4}}\right) rdr. \qquad (2.10)$$

Noticing from (2.4) that

$$K_0(r\sqrt{1+\frac{c^2}{4}}) \sim \sqrt{\frac{\pi}{2r\sqrt{1+\frac{c^2}{4}}}} e^{-r\sqrt{1+\frac{c^2}{4}}}, \quad \text{as} \quad r\to+\infty \qquad (2.11)$$

and as $\sqrt{1+c^2/4} > |c|/2$, one has that

$$\int_{\rho}^{+\infty} e^{pr(\frac{|c|}{2}-\sqrt{1+\frac{c^2}{4}})}\left(\frac{\pi}{2r\sqrt{1+\frac{c^2}{4}}}\right)^{\frac{p}{2}} rdr \qquad (2.12)$$

converges, we then obtain that J_2 converges. The proof is completed. □

Finally, because the fundamental solution E is in $L^p(\mathbb{R}^2)$, it defines of course a tempered distribution. The latter is unique (see [22]), which will result in the uniqueness of the solution to problem (1.31).

2.2 Construction of the solution

We remind that T is given by (1.30) as

$$T = \mathbb{1}_{y<0} \otimes \delta_{x=0}.$$

We now approximate T by a sequence of compactly supported distributions $T_n \in \mathscr{E}'(\mathbb{R}^2)$, defined by

$$T_n = \mathbb{1}_{(-n,0)} \otimes \delta_{x=0}, \tag{2.13}$$

that is, for φ in the Schwartz space $\mathscr{S}(\mathbb{R}^2)$, $\langle T_n, \varphi\rangle = \int_{-n}^{0} \varphi(0,y)dy$. Clearly, $T_n \to T$ in the space $\mathscr{S}'(\mathbb{R}^2)$.

We now define $\Theta_n = T_n * E$ as the unique solution in $\mathscr{S}'(\mathbb{R}^2)$ of

$$L[\Theta_n] = T_n. \tag{2.14}$$

Let φ be a Schwartz function,

$$\langle \Theta_n, \varphi\rangle = \langle T_n * E, \varphi\rangle = \langle T_n, \widetilde{E} * \varphi\rangle, \tag{2.15}$$

where $\widetilde{E}$ is the transpose of E, i.e., $\widetilde{E}(x,y) = E(-x,-y)$. Then,

$$\widetilde{E} * \varphi(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \widetilde{E}(x-x', y-y')\varphi(x',y')dx'dy', \tag{2.16}$$

$$\begin{aligned}\langle T_n, \widetilde{E} * \varphi\rangle &= \int_{-n}^{0}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \widetilde{E}(-x', y-y')\varphi(x',y')dx'dy'dy \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_{-n}^{0} \widetilde{E}(-x', y-y')dy\right)\varphi(x',y')dx'dy'\end{aligned} \tag{2.17}$$

by Fubini theorem. Therefore, we have computed

$$\Theta_n(x',y') = \int_{-n}^{0} \widetilde{E}(-x', y-y')dy, \tag{2.18}$$

that is more convenient to write as:

$$\Theta_n(x,y) = \int_{y}^{y+n} E(x,u)du. \tag{2.19}$$

Lemma 2.2. *As $n \to +\infty$, $\Theta_n \to \Theta$ in the space $\mathscr{S}'(\mathbb{R}^2)$, where*

$$\Theta(x,y) = \int_{y}^{+\infty} E(x,u)du.$$

Proof. Let φ be a Schwartz function on $\mathbb{R}^2$, thanks to Lemma 2.1, we see that

$$\langle \Theta_n, \varphi\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbb{1}_{(y,y+n)}E(x,u)\varphi(x,y)dudxdy \tag{2.20}$$

converges to

$$\langle \Theta, \varphi\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x,u)\varphi(x,y)dudxdy \tag{2.21}$$

by Lebesgue's dominated convergence theorem. □

Finally, it is easy to pass to the limit in (2.14) as $n \to +\infty$ in the space of tempered distributions on $\mathbb{R}^2$, which shows that indeed Θ verifies $L[\Theta] = T$. Summarizing, we have proved the following theorem:

Theorem 2.3. *The function*

$$\Theta(x,y) = \int_y^{+\infty} E(x,u)du = \frac{1}{2\pi}\int_y^{+\infty} e^{-\frac{c}{2}u} K_0\left(\sqrt{(1+\frac{c^2}{4})(x^2+u^2)}\right) du \tag{2.22}$$

is the unique solution in $\mathscr{S}'(\mathbb{R}^2)$ of problem (1.31).

Remark 2.4. *Because $E(x,y) = E(-x,y)$ for all $(x,y) \in \mathbb{R}^2$ and $(x,y) \neq (0,0)$, it is clear that Θ is symmetric with respect to the interface.*

2.3 Computation of the velocity

It is now an exercise to compute the velocity c explicitly (see also [22]).

Lemma 2.5. *It holds for $0 < \Theta_i < 1/2$, i.e., $0 < 2\pi\Theta_i < \pi$,*

$$c = 2\cot(2\pi\Theta_i). \tag{2.23}$$

Consequently, it holds:

(i) when $\Theta_i = \dfrac{1}{4}$, $c = 0$;

(ii) when $\Theta_i \to 0$ (resp. $1/2$), then $c \to +\infty$ (resp. $-\infty$).

Proof. From (1.32) or (2.22), it comes

$$\Theta_i = \Theta(0,0) = \frac{1}{2\pi}\int_0^{+\infty} e^{-\frac{c}{2}u} K_0\left(u\sqrt{1+\frac{c^2}{4}}\right) du. \tag{2.24}$$

Using the table integral (see [10, p. 695]), one has

$$\int_0^{+\infty} e^{-\beta x} K_0(\alpha x)dx = \frac{\arccos\frac{\beta}{\alpha}}{\sqrt{\alpha^2-\beta^2}}, \tag{2.25}$$

it easily comes, with $\alpha = \sqrt{1+c^2/4}$ and $\beta = c/2$, $\alpha + \beta > 0$,

$$\Theta_i = \frac{1}{2\pi}\arccos\left(\frac{c}{2\sqrt{1+\frac{c^2}{4}}}\right), \tag{2.26}$$

hence

$$\frac{c}{\sqrt{1+\frac{c^2}{4}}} = 2\cos(2\pi\Theta_i), \tag{2.27}$$

which in turn yields $c = 2\cot(2\pi\Theta_i)$. □

3 Properties of the solution of problem (1.31)

In this section, we will check that the solution Θ of problem (1.31), explicitly given by (2.22), satisfies problem (1.21), (1.23) and (1.28). At the outset, we see that $\Theta^{\pm}$ verify (1.21) in the sense of distributions in Ω_+ and Ω_-, respectively. For convenience, let us denote as above $\alpha = \sqrt{1 + c^2/4}$ in this and the following sections.

3.1 Regularity

In view of (2.22), we see that Θ^- and Θ^+ are smooth respectively on Ω^- and Ω^+. We are now proving the continuity of Θ on the whole plane (property (i) in Theorem 1.1).

Lemma 3.1. *The solution Θ of problem (1.31) is continuous on $\mathbb{R}^2$.*

Proof. Since Θ is obviously continuous on Ω_- and Ω_+, it remains to prove that it is continuous at the interface S. Let us prove the continuity of Θ at the origin, the proof is the same at any fixed point of the $y-$axis.

Recall that

$$\Theta(x,y) = \frac{1}{2\pi}\int_y^{+\infty} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x^2+u^2}\right) du. \tag{3.1}$$

Let $(x_n, y_n) \to (0,0)$ as $n \to +\infty$, let us prove that (see (2.24))

$$\Theta(x_n,y_n) \to \Theta(0,0) = \Theta_i = \frac{1}{2\pi}\int_0^{+\infty} e^{-\frac{c}{2}u} K_0\left(\alpha u\right) du. \tag{3.2}$$

There are two cases.
(i) Case $y_n > 0$:

$$\begin{aligned}\Theta(x_n,y_n) &= \frac{1}{2\pi}\int_{y_n}^{+\infty} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du \\ &= \frac{1}{2\pi}\int_0^{+\infty} \mathbb{1}_{(y_n,+\infty)} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du.\end{aligned} \tag{3.3}$$

We observe that, as $n \to +\infty$, for all $u \in (0,+\infty)$,

$$\mathbb{1}_{(y_n,+\infty)} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) \to e^{-\frac{c}{2}u} K_0(\alpha u), \tag{3.4}$$

and, because K_0 is decreasing,

$$0 < \mathbb{1}_{(y_n,+\infty)} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) \leqslant e^{-\frac{c}{2}u} K_0(\alpha u), \tag{3.5}$$

where $e^{-\frac{c}{2}u} K_0(\alpha u)$ is integrable on $(0,+\infty)$ thanks to (2.4) and (2.5). □

It follows from Lebesgue's dominated convergence theorem that, as $n \to +\infty$,

$$\Theta(x_n,y_n) \to \frac{1}{2\pi}\int_0^{+\infty} e^{-\frac{c}{2}u} K_0(\alpha u) du = \Theta_i. \tag{3.6}$$

(ii) Case $y_n \leqslant 0$:

$$\begin{aligned}\Theta(x_n, y_n) &= \frac{1}{2\pi}\int_{y_n}^{+\infty} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du \\ &= \frac{1}{2\pi}\int_{y_n}^{0} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du + \frac{1}{2\pi}\int_{0}^{+\infty} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du \\ &= I_n + \Theta_i,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}I_n =& \frac{1}{2\pi}\int_{0}^{-y_n} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du \\ =& \frac{1}{2\pi}\int_{0}^{+\infty} \mathbb{1}_{(0,-y_n)} e^{-\frac{c}{2}u} K_0\left(\alpha\sqrt{x_n^2+u^2}\right) du.\end{aligned} \tag{3.8}$$

Following the same procedure as in the case for $y_n > 0$, we prove by Lebesgue's dominated convergence theorem that $I_n \to 0$ as $n \to +\infty$, hence $\Theta(x_n, y_n) \to \Theta_i$.

3.2 Properties with respect to the variable y

In this subsection, we establish the monotonicity of Θ solution of problem (1.31) with respect to y and check that Θ verifies the boundary conditions (1.23) when the variable y tends to $\pm\infty$.

We first prove the following lemma (property (iii) of Theorem 1.1)

Lemma 3.2. *For any fixed $x \in \mathbb{R}$, $\Theta(x, y)$ is a monotonically decreasing function of y and*

$$\lim_{|y|\to+\infty} \frac{\partial\Theta}{\partial y}(x, y) = \lim_{|y|\to+\infty} \frac{\partial^2\Theta}{\partial y^2}(x, y) = 0. \tag{3.9}$$

Proof. For any fixed $(x, y) \in \mathbb{R}^2$ and $(x, y) \neq (0, 0)$, one has

$$\frac{\partial\Theta}{\partial y}(x, y) = \frac{\partial}{\partial y}\int_{y}^{+\infty} E(x, u)du = -E(x, y) < 0, \tag{3.10}$$

moreover, $E(x, y) \to 0$ as $|y| \to +\infty$. Note that the mapping $y \mapsto \Theta(0, y)$ has a singularity at $y = 0$, see Subsection 4.1 below.

Next, $\frac{\partial^2}{\partial y^2}\Theta(x, y) = -\frac{\partial}{\partial y}E(x, y)$; thus, it is easy to differentiate (2.2) with respect to y and see that ([1, Section 9.6])

$$\frac{\partial}{\partial y}E(x, y) = \frac{-1}{2\pi}e^{-\frac{c}{2}y}\left(\frac{c}{2}K_0(\alpha\sqrt{x^2+y^2}) + \frac{\alpha y K_1(\alpha\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\right). \tag{3.11}$$

From (2.4) and the fact that ([1, Section 9.6])

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad \text{as} \quad z \to +\infty, \tag{3.12}$$

and noticing that $\alpha = \sqrt{1 + c^2/4} > |c|/2$, we easily obtain that at fixed x, the derivative $\frac{\partial}{\partial y}E(x, y)$ also converges to 0 as $|y| \to +\infty$. □

Lemma 3.3. *It holds:*

(i) for any fixed $x \in \mathbb{R}$, $\lim_{y\to+\infty} \Theta(x,y) \to 0$*;*

(ii) for any fixed $x \in \mathbb{R}$, $\lim_{y\to-\infty} \Theta(x,y) \to \Theta_\infty$ *(see* (1.24)*).*

Proof. (i) Because Θ is given explicitly by (2.22) and $E \in L^p(\mathbb{R}^2), 1 \leqslant p < +\infty$, it is clear that, for all $x \in \mathbb{R}$,

$$\Theta(x,y) = \int_y^{+\infty} E(x,u)du \to 0, \quad \text{as } y \to +\infty. \tag{3.13}$$

(ii) For the same reason, it holds for all $x \in \mathbb{R}$,

$$\Theta(x,y) \to \ell(x) = \int_{-\infty}^{+\infty} E(x,u)du, \quad \text{as } y \to -\infty. \tag{3.14}$$

Let us prove that $\ell(x) = \Theta_\infty(x)$. Because of the symmetry with respect to the interface, we only need to prove for the case when $x \geqslant 0$.

Let φ be a test function with compact support contained in $(0,+\infty)$. For any positive monotonically decreasing sequence $\{y_n\}$ that tends to $-\infty$ as $n \to +\infty$, it comes

$$\begin{aligned} &-\int_0^{+\infty} \Theta(x,y_n)\frac{d^2\varphi}{dx^2}(x)dx - \int_0^{+\infty} \frac{\partial^2\Theta}{\partial y^2}(x,y_n)\varphi(x)dx \\ &- c\int_0^{+\infty} \frac{\partial\Theta}{\partial y}(x,y_n)\varphi(x)dx + \int_0^{+\infty} \Theta(x,y_n)\varphi(x)dx = 0. \end{aligned} \tag{3.15}$$

As $\Theta(x,y_n) \to \ell(x)$, $\frac{\partial^2}{\partial y^2}\Theta(x,y_n) \to 0$ and $\frac{\partial}{\partial y}\Theta(x,y_n) \to 0$ when $n \to +\infty$ from Lemma 3.2, it follows from Lebesgue's dominated convergence theorem applied on $\mathrm{supp}(\varphi)$ that

$$-\int_0^{+\infty} \ell(x)\frac{d^2\varphi}{dx^2}(x)dx + \int_0^{+\infty} \ell(x)\varphi(x)dx = 0 \tag{3.16}$$

for all $\varphi \in C_0^\infty((0,+\infty))$. Thus, ℓ verifies $d^2\ell/dx^2 = \ell$ in the distribution sense on $(0,+\infty)$. It follows from (3.14) and the properties of E that $\ell(x)$ is bounded when $x \to +\infty$, hence $l(x) = Ce^{-x}$, $C > 0$.

Let us compute $C = \ell(0)$:

$$\begin{aligned} \ell(0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{c}{2}y} K_0(\alpha y)\, dy \\ &= \frac{1}{2\pi}\int_0^{+\infty} e^{-\frac{c}{2}y} K_0(\alpha y)\, dy + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-\frac{c}{2}y} K_0(\alpha y)\, dy \\ &= \frac{1}{2\pi}\int_0^{+\infty} e^{-\frac{c}{2}y} K_0(\alpha y)\, dy + \frac{1}{2\pi}\int_0^{+\infty} e^{\frac{c}{2}y} K_0(\alpha y)\, dy. \end{aligned} \tag{3.17}$$

According to Lemma 2.5, it comes:

$$\begin{aligned} \ell(0) &= \frac{1}{2\pi}\arccos\left(\frac{c}{2\alpha}\right) + \frac{1}{2\pi}\arccos\left(\frac{-c}{2\alpha}\right) \\ &= \frac{1}{2\pi}(2\pi\Theta_i + \pi - 2\pi\Theta_i) = \frac{1}{2}. \end{aligned} \tag{3.18}$$

Therefore, we have proved that $l(x) = 1/2e^{-x}$, i.e., $l(x) = \Theta_\infty(x)$ for all $x \geqslant 0$ and, by symmetry, for all $x \in \mathbb{R}$. □

We refer to Figures 1–2 for numerical illustrations in the case $\Theta_i = 0.25$, $c = 0$.

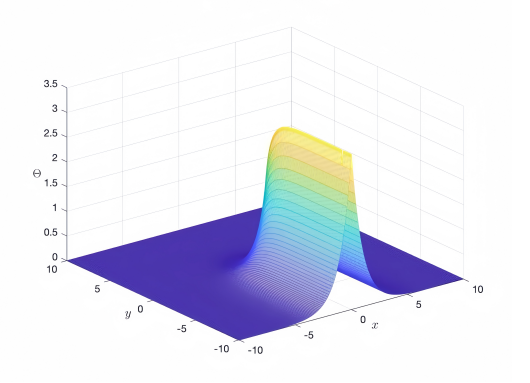

Figure 1: Profile of the solution $\Theta(x, y)$ in the case $\Theta_i = 0.25$, $c = 0$.

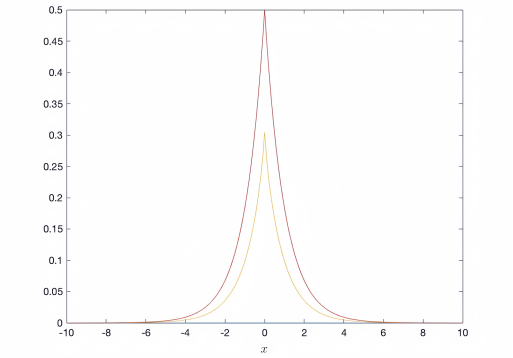

Figure 2: The red curve represents the stationary solution $\Theta_\infty(x)$, while the yellow one is the curve $\Theta(x, -2)$.

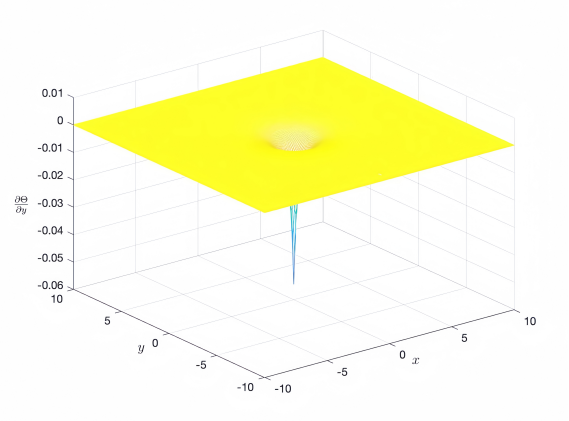

Figure 3: View of the singularity of $\frac{\partial}{\partial y}\Theta(x, y)$ when $\Theta_i = 0.25$, $c = 0$.

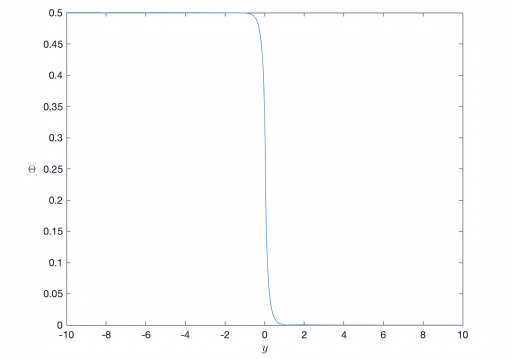

Figure 4: Singular profile of $\frac{\partial}{\partial y}\Theta(0, y)$ near the origin when $\Theta_i = 0.25$, $c = 0$.

3.3 Properties with respect to the variable x

In this subsection, we first establish that the solution of problem (1.31) verifies the boundary conditions (1.23) when the variable x tends to $\pm\infty$.

Lemma 3.4. *For any fixed $y \in \mathbb{R}$, $\Theta(x, y) \to 0$ as $|x| \to +\infty$.*

Proof. The lemma is easily verified using Lebesgue's dominated convergence theorem. □

Next, we prove properties (ii) and (vi) in Theorem 1.1, i.e., the following lemma

Lemma 3.5. *For any fixed $y \in \mathbb{R}$, we have*

$$x > 0, \quad \frac{\partial \Theta^+}{\partial x}(x, y) < 0, \tag{3.19}$$

$$x < 0, \quad \frac{\partial \Theta^-}{\partial x}(x, y) > 0. \tag{3.20}$$

Moreover, for $y > 0$ only, we have

$$\lim_{x \to 0} \frac{\partial \Theta^\pm}{\partial x}(x, y) = 0. \tag{3.21}$$

Proof. When $x \neq 0$, for any $y \in \mathbb{R}$, we have (see [1, Section 9.6])

$$\begin{cases} \dfrac{\partial \Theta}{\partial x}(x, y) = -\dfrac{\alpha x}{2\pi} I(x, y), \\ I(x, y) = \displaystyle\int_y^{+\infty} e^{-\frac{c}{2}u} \frac{K_1\left(\alpha\sqrt{x^2 + u^2}\right)}{\sqrt{x^2 + u^2}} du. \end{cases} \tag{3.22}$$

Since $I(x, y) > 0$, (3.19)–(3.20) follow directly from (3.22). For any $y > 0$, using the monotonicity of K_1, we have

$$I(x, y) \leqslant I(0, y) = \frac{1}{y} \int_y^{+\infty} e^{-\frac{c}{2}u} K_1(\alpha u)\, du. \tag{3.23}$$

Using (3.12) again, $I(0, y)$ converges, thus (3.21) holds and, as $x \to 0$,

$$\frac{\partial \Theta}{\partial x}(x, y) \sim -\frac{\alpha x}{2\pi} I(0, y). \tag{3.24}$$

Remark 3.6. *However, we see that $I(0, y)$ blows up when $y \to 0$. Indeed, one just need to notice from [1, Section 9.6] that*

$$K_1(z) \sim \frac{1}{z}, \quad \text{as} \quad z \to 0. \tag{3.25}$$

□

4 Singularity of the solution

In comparison with (3.21), we have already known that when $y < 0$, the corresponding derivative of the solution Θ has singularity at $x = 0$. In this section, we verify the jump condition in (1.28) and prove properties (iv) and (v) in Theorem 1.1.

4.1 Singularity of $\frac{\partial}{\partial y}\Theta(0, y)$ near the origin

Outside a neighborhood of 0, the mapping $y \mapsto \Theta(0, y) = \int_y^{+\infty} E(0, u) du$ is smooth, however, we observe that $\frac{\partial}{\partial y}\Theta(0, 0) = -\infty$ from Figures 3–4. We are going to give a more precise description of the singularity:

Lemma 4.1. *As $|y| \to 0$, it holds*

$$\frac{\partial \Theta}{\partial y}(0,y) \sim \frac{1}{2\pi} \ln(\alpha|y|). \tag{4.1}$$

Therefore, as $y \to 0^{\pm}$, $\dfrac{\partial \Theta}{\partial y}(0,y) \to -\infty$.

Proof. The result follows immediately from (3.10)

$$\frac{\partial \Theta}{\partial y}(0,y) = -E(0,y) = -\frac{1}{2\pi} e^{-\frac{c}{2}y} K_0(\alpha|y|) \tag{4.2}$$

together with (2.5). □

4.2 Singularity of $\frac{\partial}{\partial x}\Theta(x,y)$ near the origin

We first give the following preliminary lemma

Lemma 4.2. *The two one-sided limits $\lim\limits_{x\to 0^+} \dfrac{\partial \Theta^+}{\partial x}(x,0)$ and $\lim\limits_{x\to 0^-} \dfrac{\partial \Theta^-}{\partial x}(x,0)$ exist.*

Proof. The idea is to firstly prove that $\frac{\partial}{\partial x}\Theta(x,0)$ is bounded in a neighborhood of $x=0$. Then the existence of the two one-sided limits are guaranteed by the monotonicity of $\frac{\partial}{\partial x}\Theta(x,0)$ with respect to x. Indeed, for any $x \neq 0$, let

$$\begin{aligned}
\frac{\partial}{\partial x}\Theta(x,0) =& -\frac{\alpha x}{2\pi} \int_0^{+\infty} e^{-\frac{c}{2}u} \frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}} du \\
=& -\frac{\alpha x}{2\pi} \int_0^{\varepsilon} e^{-\frac{c}{2}u} \frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}} du \\
& -\frac{\alpha x}{2\pi} \int_{\varepsilon}^{+\infty} e^{-\frac{c}{2}u} \frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}} du \\
=:& I_1(x) + I_2(x),
\end{aligned} \tag{4.3}$$

where $\varepsilon > 0$ is any given constant. Looking at the case for $y > 0$, we have verified in Lemma 3.5 that $I_2(x)$ is bounded in a neighborhood of $x = 0$. Let us now deal with the $I_1(x)$ term. Making the change of variable $\alpha\sqrt{x^2+u^2} = v$, we have

$$I_1(x) := -\frac{\alpha x}{2\pi} \int_{\alpha|x|}^{\alpha\sqrt{x^2+\varepsilon^2}} e^{-\frac{c}{2}\sqrt{\frac{v^2}{\alpha^2}-x^2}} \frac{K_1(v)}{\sqrt{v^2-\alpha^2x^2}} dv. \tag{4.4}$$

Making another change of variable $v/\alpha = z$ and one has for any $x \neq 0$ that

$$\begin{aligned}
I_1(x) :=& -\frac{\alpha x}{2\pi} \int_{|x|}^{\sqrt{x^2+\varepsilon^2}} e^{-\frac{c}{2}\sqrt{z^2-x^2}} \frac{K_1(\alpha z)}{\sqrt{z^2-x^2}} dz \\
=& -\frac{\alpha x}{2\pi} \int_{|x|}^{\sqrt{x^2+\varepsilon^2}} e^{-\frac{c}{2}|x|\sqrt{(\frac{z}{x})^2-1}} \frac{K_1(\alpha z)}{|x|\sqrt{(\frac{z}{x})^2-1}} dz \\
\overset{u=\frac{z}{|x|}}{=}& -\frac{\alpha x}{2\pi} \int_1^{\sqrt{1+(\frac{\varepsilon}{x})^2}} e^{-\frac{c}{2}|x|\sqrt{u^2-1}} \frac{K_1(\alpha|x|u)}{\sqrt{u^2-1}} du.
\end{aligned} \tag{4.5}$$

From [20], we know that for any $x > 0$

$$\sqrt{x+\frac{2}{\pi}} < \sqrt{\frac{2}{\pi}} x e^x K_1(x) < \sqrt{x+\frac{3}{4}}.$$

Hence,

$$K_1(\alpha|x|u) < \sqrt{\frac{\pi}{2}} \frac{\sqrt{\alpha|x|u+\frac{3}{4}}}{\alpha|x|u} e^{-\alpha|x|u}, \tag{4.6}$$

which together with (4.5) gives

$$\begin{aligned} |I_1(x)| <& \frac{1}{2\sqrt{2\pi}} \int_1^{\sqrt{1+(\frac{\varepsilon}{x})^2}} e^{-\frac{c}{2}|x|\sqrt{u^2-1}} \cdot e^{-\alpha|x|u} \frac{\sqrt{\alpha|x|u+\frac{3}{4}}}{u\sqrt{u^2-1}} du \\ <& \frac{1}{2\sqrt{2\pi}} (1+e^{-\frac{c}{2}\varepsilon}) \cdot e^{-\alpha|x|} \int_1^{\sqrt{1+(\frac{\varepsilon}{x})^2}} \frac{\sqrt{\alpha\sqrt{x^2+\varepsilon^2}+\frac{3}{4}}}{u\sqrt{u^2-1}} du \\ <& \frac{1}{2\sqrt{2\pi}} (1+e^{-\frac{c}{2}\varepsilon}) \sqrt{\alpha\sqrt{x^2+\varepsilon^2}+\frac{3}{4}} \int_1^{+\infty} \frac{1}{u\sqrt{u^2-1}} du. \end{aligned} \tag{4.7}$$

One notices that the integral $\int_1^{+\infty} \frac{1}{u\sqrt{u^2-1}} du$ converges, thus from (4.7), we know that $I_1(x)$ is bounded in a neighborhood of $x = 0$ ($x \neq 0$).

Next, we prove that $\frac{\partial}{\partial x}\Theta(x,0)$ is monotonically increasing on each side of $x = 0$. Taking the same series of change of variables as from (4.3) to (4.5), one has

$$\frac{\partial \Theta}{\partial x}(x,0) = -\frac{\alpha x}{2\pi} \int_1^{+\infty} e^{-\frac{c}{2}|x|\sqrt{u^2-1}} \frac{K_1(\alpha|x|u)}{\sqrt{u^2-1}} du. \tag{4.8}$$

When $x \neq 0$, we have (see [1, Section 9.6])

$$\frac{\partial^2 \Theta}{\partial x^2}(x,0) = -\frac{\alpha}{2\pi} \int_1^{+\infty} e^{-\frac{c}{2}|x|\sqrt{u^2-1}} I \, du, \tag{4.9}$$

with the integrand

$$I := -\frac{c}{2}|x| K_1(\alpha|x|u) + \frac{K_1(\alpha|x|u)}{\sqrt{u^2-1}} - \alpha|x|u \frac{K_2(\alpha|x|u)}{\sqrt{u^2-1}}. \tag{4.10}$$

Noticing that (see [21])

$$\frac{\nu+\sqrt{x^2+\nu^2}}{x} < \frac{K_{\nu+1}(x)}{K_\nu(x)} < \frac{\nu+\frac{1}{2}+\sqrt{x^2+(\nu+\frac{1}{2})^2}}{x} \tag{4.11}$$

holds for all $x > 0$ and $\nu \geqslant 0$. We get that for any $u > 0$

$$K_2(\alpha|x|u) > \frac{1+\sqrt{(\alpha|x|u)^2+1}}{\alpha|x|u} K_1(\alpha|x|u). \tag{4.12}$$

Replacing (4.12) to (4.10), one immediately obtains that

$$I = \left(-\frac{c}{2}|x| - \frac{\sqrt{(\alpha|x|u)^2+1}}{\sqrt{u^2-1}} \right) K_1(\alpha|x|u) < 0 \tag{4.13}$$

for any fixed c provided that $|x|$ is sufficiently small, which, together with (4.9) gives that $\frac{\partial}{\partial x}\Theta(x,0)$ is increasing with respect to x on each side of $x = 0$. □

Based on Lemma 4.2, we prove furthermore that for any $y < 0$

Lemma 4.3. *The two one-sided limits* $\lim_{x\to 0^+} \frac{\partial}{\partial x}\Theta^+(x,y)$ *and* $\lim_{x\to 0^-} \frac{\partial}{\partial x}\Theta^-(x,y)$ *exist.*

Proof. From Lemma 3.5, we recall that for any fixed $y > 0$

$$\lim_{x\to 0} \frac{\partial \Theta}{\partial x}(x,y) = 0 \tag{4.14}$$

holds for any constant c. When $y = 0$, from Lemma 4.2, assume that

$$\lim_{x\to 0^+} \frac{\partial \Theta^+}{\partial x}(x,0) = A^+, \quad \lim_{x\to 0^-} \frac{\partial \Theta^-}{\partial x}(x,0) = A^-. \tag{4.15}$$

We separate the integral as

$$\begin{aligned}
\frac{\partial \Theta}{\partial x}(x,0) =& -\frac{\alpha x}{2\pi}\int_0^{\bar{y}} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du \\
&-\frac{\alpha x}{2\pi}\int_{\bar{y}}^{+\infty} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du \\
=& -\frac{\alpha x}{2\pi}\int_0^{\bar{y}} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du + \frac{\partial \Theta}{\partial x}(x,\bar{y})
\end{aligned} \tag{4.16}$$

for any $\bar{y} > 0$. Noticing from (4.14) that $\lim_{x\to 0} \frac{\partial}{\partial x}\Theta(x,\bar{y}) = 0$. Taking the limit $x \to 0$ in the above equation and noticing (4.15), one has

$$\lim_{x\to 0^+} -\frac{\alpha x}{2\pi}\int_0^{\bar{y}} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du = A^+, \tag{4.17}$$

$$\lim_{x\to 0^-} -\frac{\alpha x}{2\pi}\int_0^{\bar{y}} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du = A^- \tag{4.18}$$

for any constant c and any $\bar{y} > 0$. On the other hand, we consider the case where $y < 0$, we decompose

$$\begin{aligned}
\frac{\partial}{\partial x}\Theta(x,y) =& -\frac{\alpha x}{2\pi}\int_y^{+\infty} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du \\
=& -\frac{\alpha x}{2\pi}\int_y^{0} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du - \frac{\alpha x}{2\pi}\int_0^{+\infty} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du \\
=& -\frac{\alpha x}{2\pi}\int_y^{0} e^{-\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du + \frac{\partial}{\partial x}\Theta(x,0) \\
=& -\frac{\alpha x}{2\pi}\int_0^{-y} e^{\frac{c}{2}u}\frac{K_1\left(\alpha\sqrt{x^2+u^2}\right)}{\sqrt{x^2+u^2}}du + \frac{\partial}{\partial x}\Theta(x,0).
\end{aligned}$$

By (4.17)–(4.18) and noticing (4.14), we obtain the existence of $\lim_{x\to 0^+} \frac{\partial}{\partial x}\Theta^+(x,y)$ and $\lim_{x\to 0^-} \frac{\partial}{\partial x}\Theta^-(x,y)$. Moreover, one has

$$\lim_{x\to 0^+} \frac{\partial \Theta^+}{\partial x}(x,y) = 2A^+, \tag{4.19}$$

$$\lim_{x\to 0^-} \frac{\partial \Theta^-}{\partial x}(x,y) = 2A^-. \tag{4.20}$$

From the continuity of $\Theta(x,y)$ at $(0,y)$, we obtain that for any $y<0$

$$\lim_{x\to 0^+} \frac{\partial \Theta^+}{\partial x}(x,y) = \lim_{x\to 0^+} \frac{\Theta(x,y)-\Theta(0,y)}{x}, \tag{4.21}$$

$$\lim_{x\to 0^-} \frac{\partial \Theta^-}{\partial x}(x,y) = \lim_{x\to 0^-} \frac{\Theta(x,y)-\Theta(0,y)}{x}. \tag{4.22}$$

□

Noticing that (4.21)–(4.22) show the existence of $\frac{\partial}{\partial x}\Theta^+(0,y)$ and $\frac{\partial}{\partial x}\Theta^-(0,y)$, we can now verify the jump condition in (1.28) and moreover, by the symmetry of the solution Θ, we have

Lemma 4.4.

$$\lim_{x\to 0^+} \frac{\partial \Theta^+}{\partial x}(x,y) = -\frac{1}{2}, \quad \lim_{x\to 0^-} \frac{\partial \Theta^-}{\partial x}(x,y) = \frac{1}{2}. \tag{4.23}$$

Proof. We denote by $\mathbb{R}^2_-$ the half-space $\{(x,y)\in\mathbb{R}^2,\, y<0\}$. We also denote by Q_- and Q_+ respectively the third and fourth quadrants of the plan, namely $Q_- = \{(x,y)\in \mathbb{R}^2_-,\, x<0\}$ and $Q_+ = \{(x,y)\in\mathbb{R}^2_-,\, x>0\}$.

Let $\varphi\in C_0^\infty(\mathbb{R}^2_-)$, i.e., φ is an element of $C_0^\infty(\mathbb{R}^2_-)$ whose support is a compact subset of $\mathbb{R}^2_-$. We proceed as in (1.29):

$$\begin{aligned}
\langle L[\Theta],\varphi\rangle &= \int_{\mathbb{R}^2_-} \Theta\, L^*[\varphi]dxdy \\
&= \int_{Q_-} \Theta^- L^*[\varphi]dxdy + \int_{Q_+} \Theta^+ L^*[\varphi]dxdy \\
&= \int_{Q_-} L[\Theta^-]\varphi dxdy + \int_{Q_+} L[\Theta^+]\varphi dxdy \\
&\quad + \int_{-\infty}^0 \left(\Theta^+(0,y)-\Theta^-(0,y)\right)\frac{\partial\varphi}{\partial x}(0,y)dy \\
&\quad - \int_{-\infty}^0 \left(\frac{\partial\Theta^+}{\partial x}(0,y)-\frac{\partial\Theta^-}{\partial x}(0,y)\right)\varphi(0,y)dy \\
&= -\int_{-\infty}^0 \left(\frac{\partial\Theta^+}{\partial x}(0,y)-\frac{\partial\Theta^-}{\partial x}(0,y)\right)\varphi(0,y)dy \\
&= <T,\varphi> = \int_{-\infty}^0 \varphi(0,y)dy,
\end{aligned} \tag{4.24}$$

where T is defined as (1.30). Note that the integrals along the $y-$axis hold only on the intersection (if any) of the support of φ with the axis; the latter is contained in some

interval $[a,b]$, $-\infty < a < b < 0$, and therefore bounded away from the origin. According to (4.19)–(4.22), it holds for all $\varphi \in C_0^\infty(\mathbb{R}^2_-)$

$$-\int_{-\infty}^{0} 2\left(A^+ - A^-\right)\varphi(0,y)dy = \int_{-\infty}^{0} \varphi(0,y)dy, \tag{4.25}$$

that obviously yields: $A^+ - A^- = -\dfrac{1}{2}$. Moreover, it follows from Lemma 3.5 that $A^+ < 0$ and $A^- > 0$, and, by symmetry (see Remark 2.4), $A^- = -A^+ = \dfrac{1}{4}$. Noticing (4.19) and (4.20), we obtain (4.23). This completes the proof. □

By (4.15), we verified meanwhile property (v) in Theorem 1.1, i.e.,

Lemma 4.5. $\lim\limits_{x\to 0^+} \dfrac{\partial \Theta^+}{\partial x}(x,0) = -\dfrac{1}{4}$ *and* $\lim\limits_{x\to 0^-} \dfrac{\partial \Theta^-}{\partial x}(x,0) = \dfrac{1}{4}$.

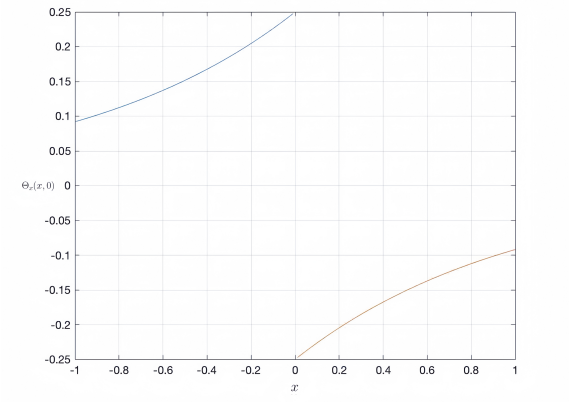

Figure 5: Behavior of $\frac{\partial}{\partial x}\Theta(x,0)$ in a neighborhood of $x = 0$.

We can also refer to Figure 5 that $\frac{\partial}{\partial x}\Theta(x,0)$ is increasing with respect to x on both sides of $x = 0$, and tends to $-1/4$ and $1/4$ on each side.

5 Properties of the level sets

Based on various properties of the solution Θ, in this section, we pursue the properties of the level sets and prove Theorem 1.3. To For simplicity, we write Θ for Θ^+ or Θ^-. We will use the Implicit Function Theorem (IFT) to prove the existence, uniqueness and regularity of the level sets. Firstly in Subsection 5.1, we study the existence and properties of the level set $\Theta(x,y) = \Theta_i$, whose graph passes through the singularity $(0,0)$, and is the most intricate case. Then in Subsections 5.2–5.3, we study the other level sets through $(0,y_0)$.

Still denoting by $Q_+ = \{(x,y) \in \mathbb{R}^2_-,\ x > 0\}$ the fourth quadrant, we recall from Lemma 3.2 and Lemma 3.5 that for any $(x,y) \in Q_+$

$$\frac{\partial \Theta}{\partial x}(x,y) < 0, \quad \frac{\partial \Theta}{\partial y}(x,y) < 0. \tag{5.1}$$

5.1 The level set through the singularity

We will use the IFT to prove the existence, uniqueness and regularity of the level set $\Theta(x, y) = \Theta_i$. It should be emphasized here that we can not apply the IFT at the origin because of the singularity of $\Theta(x, y)$ at $(0, 0)$, see Section 4. The idea is to start from a point different from the origin, applying the IFT at this point, and then extend the singular level set.

Let $1/2e^{-x} = \Theta_i,\ 0 < \Theta_i < 1/2$, the corresponding x_i can be obtained by solving the equation: $x_i = -\ln(2\Theta_i)$. Then, we have the following lemma

Lemma 5.1. *Let $x_i = -\ln(2\Theta_i)$, then for any fixed $x_* \in (0, x_i)$, there exists a unique y_* such that $\Theta(x_*, y_*) = \Theta_i$.*

Proof. Let $x_* \in (0, x_i)$ be fixed, because $x \mapsto \Theta(x, 0)$ is decreasing, so we have $\Theta(x_*, 0) < \Theta(0, 0) = \Theta_i$. From the fact that for any fixed $x > 0$, $\Theta(x, y) \to \Theta_\infty(x)$ $(y \to -\infty)$, we get immediately that $\Theta(x_*, -\infty) = \Theta_\infty(x_*)$. Noticing that $\Theta_\infty(x_*) > \Theta_\infty(x_i) = \Theta_i$, we conclude that there exists $y_* < 0$ such that $\Theta(x_*, y_*) = \Theta_i$ by the continuity of $\Theta(x_*, y)$ with respect to y. Moreover, as for any fixed $x_* \in (0, x_i)$, $y \mapsto \Theta(x_*, y)$ is decreasing, we have the uniqueness of y_*. □

Set $F(x, y) = \Theta(x, y) - \Theta_i$, from Lemma 5.1, we can choose any fixed point (x_*, y_*) such that $F(x_*, y_*) = 0$. We have

Lemma 5.2. *For any fixed $(x_*, y_*) \in Q_+$ such that $F(x_*, y_*) = 0$, there exists a neighborhood U of (x_*, y_*) in Q_+, an open interval V containing x_* and a real valued function $\varphi : V \to \mathbb{R}$, continuously differentiable on V such that for any $x \in V$, $(x, \varphi(x)) \in U$ and*

$$F(x, \varphi(x)) \equiv 0, \quad \varphi(x_*) = y_*. \tag{5.2}$$

Moreover, $\varphi(x)$ is monotonically decreasing with respect to x.

Proof. Noticing that $F(x_*, y_*) = \Theta(x_*, y_*) - \Theta_i = 0$. By definition of $F(x, y)$, we have $\frac{\partial}{\partial x}F(x, y) = \frac{\partial}{\partial x}\Theta(x, y)$, $\frac{\partial}{\partial y}F(x, y) = \frac{\partial}{\partial y}\Theta(x, y)$. And from Theorem 1.1, $\frac{\partial}{\partial x}F(x, y)$ and $\frac{\partial}{\partial y}F(x, y)$ are continuous with respect to their variables and we have $\frac{\partial}{\partial y}F(x_*, y_*) < 0$. Applying the IFT, we can obtain that $F(x, y) = 0$ uniquely determined a continuous differentiable function $y = \varphi(x)$ with $y_* = \varphi(x_*)$ in a neighborhood V of (x_*, y_*). And from (5.1),

$$\varphi'(x) = -\frac{\partial F}{\partial x}(x, \varphi(x)) \left(\frac{\partial F}{\partial y}(x, \varphi(x))\right)^{-1} = -\frac{\partial \Theta}{\partial x}(x, \varphi(x)) \left(\frac{\partial \Theta}{\partial y}(x, \varphi(x))\right)^{-1} < 0. \tag{5.3}$$

□

With the above lemma in hand, we can prove the global existence of the level set by extension.

Lemma 5.3. *The level set $\Theta(x, y) = \Theta_i$ is the graph of a continuously differentiable function $\varphi_0(x)$ which is increasing when $x < 0$ and decreasing when $x > 0$. It has $x = \pm x_i$ as its asymptotes, where $x_i = -\ln(2\Theta_i)$.*

Proof. Since $\Theta(x,y)$ is continuous at any point $(x,y) \in Q_+$ and for all fixed $x > 0$, $\frac{\partial}{\partial y}\Theta(x,y) < 0$ and is continuous with respect to its variables in Q_+, we may reiterate the IFT and extend the level set. Let us assume that there is an end-point $(x_L, y_L) \in Q_+$ on the left. By continuity, $F(x_L, y_L) = 0$. The maximum left extension $\widetilde{\varphi}_L(x)$ is a decreasing C^1 function

$$\widetilde{\varphi}_L : (x_L, x_*] \to [y_*, y_L).$$

We prove in the following that $(x_L, y_L) = (0,0)$. Indeed, if $x_L > 0,\ y_L < 0$, as $F(x_L, y_L) = 0$, then we can apply the IFT again, this contradicts with the maximality of $\widetilde{\varphi}_L$. If $x_L = 0,\ y_L < 0$, then $F(0, y_L) = 0$ gives that $\Theta(0, y_L) = \Theta_i$, which contradicts with $\frac{\partial}{\partial y}\Theta(0,y) < 0$ and $\Theta(0,0) = \Theta_i$. Finally, if $x_L > 0,\ y_L = 0$, then $F(x_L, 0) = 0$ gives that $\Theta(x_L, 0) = \Theta_i$, which contradicts with $\frac{\partial}{\partial x}\Theta(x,0) < 0$ and $\Theta(0,0) = \Theta_i$. Above all, we have the conclusion that $x_L = y_L = 0$, i.e., the singular level set $\Theta(x,y) = \Theta_i$ has its end-point at the singularity $(0,0)$.

Next, we prove that for the right extension $\widetilde{\varphi}_R(x)$,

$$\lim_{x \to x_i^-} \widetilde{\varphi}_R(x) = -\infty, \tag{5.4}$$

which means that $x = x_i$ is an asymptote of the singular level set, see Figure 6 for the numerical illustration. To that end, we first prove that $\widetilde{\varphi}_R(x)$ is well defined on (x_*, x_i).

Assume by contradiction that the end-point for the right extension is $(x_R, y_R) \in Q_+$. If $x_* < x_R < x_i$, then $F(x_R, y_R) = 0$ and we can apply the IFT again, this contradicts with the maximality of $\widetilde{\varphi}_R$. Now if $x_R \geqslant x_i$, then $F(x_R, y_R) = 0$ gives that

$$\Theta(x_R, y_R) = \Theta_i. \tag{5.5}$$

Noticing that $\frac{\partial}{\partial x}\Theta(x, y_R) < 0$, then

$$\Theta(x_R, y_R) \leqslant \Theta(x_i, y_R). \tag{5.6}$$

Using again $\frac{\partial}{\partial y}\Theta(x_R, y) < 0$, one has $\Theta(x_i, y_R) < \Theta(x_i, -\infty) = \Theta_i$, which contradicts with (5.5) and (5.6).

Thanks to Lemma 5.2, $\widetilde{\varphi}_R$ is a decreasing function on (x_*, x_i). Assume now by contradiction that (5.4) does not hold, then we can construct an increasing sequence $\{x_n\}$ satisfying $x_n \neq x_i$ and $\lim_{n \to +\infty} x_n = x_i$, such that

$$\widetilde{\varphi}_R(x_n) \geqslant -M \tag{5.7}$$

for some constant $M > 0$. As

$$\Theta(x_n, \widetilde{\varphi}_R(x_n)) = \Theta_i, \tag{5.8}$$

by the continuity of $\Theta(x,y)$ and $\widetilde{\varphi}_R$ with respect to their variables, let $n \to +\infty$ in (5.8), one obtains

$$\Theta(x_i, \widetilde{\varphi}_R(x_i^-)) = \Theta_i. \tag{5.9}$$

But noticing from (5.7) that

$$\widetilde{\varphi}_R(x_i^-) \geqslant -M, \tag{5.10}$$

this, together with (5.9) contradicts with $\frac{\partial}{\partial y}\Theta(x_i, y) < 0$ and $\Theta(x_i, -\infty) = \Theta_i$, thus (5.4) holds. According to the symmetry of Θ, the implicit function is extended to the whole $x-$axis and hereafter denoted by $\varphi_0(x)$. Moreover, from Lemma 5.2, $\varphi_0(x)$ is increasing when $x < 0$ and decreasing when $x > 0$. □

Based on Lemma 4.4, we now prove $\varphi_0'(0) = 0$, see also Figures 7–9 for numerical illustrations.

Lemma 5.4. *Let φ_0 be the extended implicit function constructed in Lemma 5.3. Then, it holds $\varphi_0'(0) = 0$, i.e., the level set through the singularity has a horizontal tangent at the origin.*

Proof. By symmetry of the solution Θ, we only need to prove $\lim_{x\to 0^+} \varphi_0'(x) = 0$. Since $\varphi_0(x)$ is monotone, $\lim_{x\to 0^+} \varphi_0'(x)$ must exist, assume by contradiction that $\lim_{x\to 0^+} \varphi_0'(x) \neq 0$. Then, there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exists $0 < x_n < 1/n$ and $\lim_{n\to+\infty} x_n = 0$ such that

$$|\varphi_0'(x_n)| \geqslant \varepsilon_0. \tag{5.11}$$

From (4.23) in Lemma 4.4, one has $\lim_{x\to 0^+} \frac{\partial}{\partial x}\Theta(x, y) = -1/2$ for any $y < 0$, hence, there exists $n_0 > 0$ such that for any $n > n_0$, one has

$$|\frac{\partial\Theta}{\partial x}(x_n, \varphi_0(x_n))| < \frac{2}{3} \tag{5.12}$$

by noticing that $\varphi_0(x_n) < 0$. On the other hand, from (3.10) and (2.5), we obtain that

$$\frac{\partial\Theta}{\partial y}(x, y) \sim \frac{1}{2\pi} \ln(\alpha\sqrt{x^2 + y^2})$$

as $(x, y) \to (0, 0)$. Noticing $\varphi_0(x) \in C^1((0, +\infty))$ and $\lim_{x\to 0^+} \varphi_0(x) = 0$, there exists $n_1 > n_0$ such that for any $n > n_1$

$$|\frac{\partial\Theta}{\partial y}(x_n, \varphi_0(x_n))| > \frac{2}{3\varepsilon_0}, \tag{5.13}$$

this, together with (5.12) gives that for any $n > n_1$

$$|\varphi_0'(x_n)| = \left|\frac{\partial\Theta}{\partial x}(x_n, \varphi_0(x_n)) \left(\frac{\partial\Theta}{\partial y}(x_n, \varphi_0(x_n))\right)^{-1}\right| < \varepsilon_0, \tag{5.14}$$

which leads to contradiction with (5.11). □

5.2 Level sets through $(0, y_0)$ with $y_0 < 0$

To begin with, we assume that $x > 0$. Let $y_0 < 0$ be fixed, for any $(x, y) \in Q_+$, we set $F_{y_0}(x, y) = \Theta(x, y) - \Theta(0, y_0)$, hence $F_{y_0}(0, y_0) = 0$ and

$$\frac{\partial F_{y_0}}{\partial x}(0^+, y_0) = -\frac{1}{2} \tag{5.15}$$

thanks to Lemma 4.4. For $\delta > 0$ and $\eta > 0$ fixed, $\eta > 0$ small enough, set $D_{y_0} = [0, \eta] \times [y_0 - \delta, y_0 + \delta]$. According to (4.23) and (3.10), $F_{y_0}(x, y)$ is at least of class C^1 in D_{y_0}. We can extended it to $\widehat{D}_{y_0} = [-\eta, \eta] \times [y_0 - \delta, y_0 + \delta]$ as a function $\widehat{F}_{y_0} \in C^1(\widehat{D}_{y_0})$, such that $\frac{\partial}{\partial x}\widehat{F}_{y_0}(0, y_0) = -1/2$.

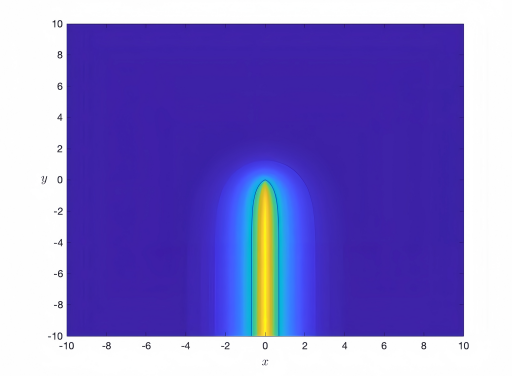

Figure 6: Picture of different level sets $\Theta(x, y) = \Theta(0, y_0)$, for $0.05 \leqslant |y_0| \leqslant 0.45$ in steps of 0.5. Here, $\Theta_i = 0.25$, $c = 0$.

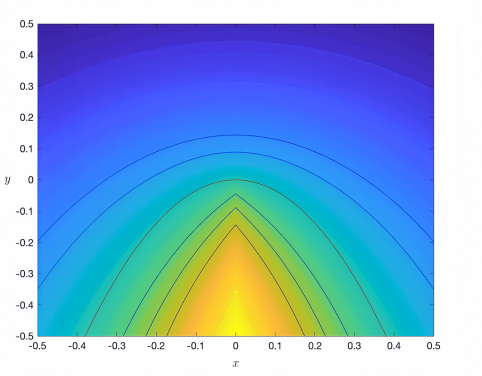

Figure 7: Enlargement of the level sets in Fig. 6 at $x = 0$. The red curve corresponds to $\Theta(x, y) = \Theta_i$ with horizontal tangent at $x = 0$.

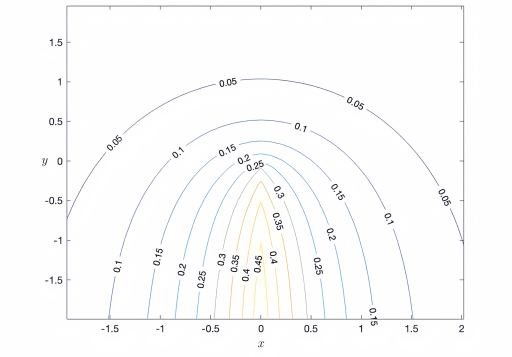

Figure 8: Representation of the implicit functions φ_{y_0} for $0.05 \leqslant |y_0| \leqslant 0.45$ in steps of 0.5. Here, $\Theta_i = 0.25$, $c = 0$.

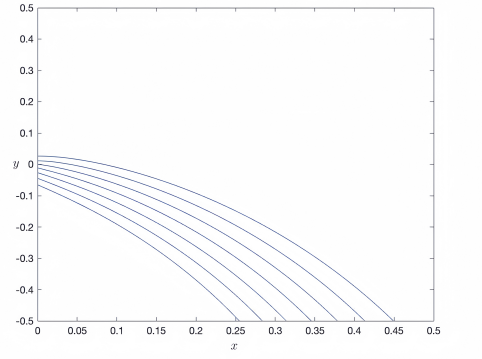

Figure 9: Enlargement of the curves in Fig. 8 for $x \geqslant 0$.

Applying the IFT, there exist $0 < \delta_1 < \delta$, $0 < \eta_1 < \eta$ and a unique function $\varphi_{y_0} \in C^1((-\eta_1, \eta_1) \times (y_0 - \delta_1, y_0 + \delta_1))$ such that, if $(x, y) \in (-\eta_1, \eta_1) \times (y_0 - \delta_1, y_0 + \delta_1)$ and $\widehat{F}_{y_0}(x, y) = 0$, then $y = \varphi_{y_0}(x)$. To simplify the notation, we still denote by φ_{y_0} the restriction of φ_{y_0} to $[0, \eta_1) \times (y_0 - \delta_1, y_0 + \delta_1)$ and $\varphi_{y_0} \in C^1([0, \eta_1) \times (y_0 - \delta_1, y_0 + \delta_1))$.

Note that φ_{y_0} is decreasing, because from (3.10) it holds in $(0, \eta_1) \times (y_0 - \delta_1, y_0 + \delta_1)$ that

$$\begin{aligned}\varphi'_{y_0}(x) &= -\frac{\partial F_{y_0}}{\partial x}(x, \varphi_{y_0}(x)) \left(\frac{\partial F_{y_0}}{\partial y}(x, \varphi_{y_0}(x))\right)^{-1} = -\frac{\partial \Theta}{\partial x}(x, \varphi_{y_0}(x)) \left(\frac{\partial \Theta}{\partial y}(x, \varphi_{y_0}(x))\right)^{-1} \\ &= \frac{\frac{\partial}{\partial x}\Theta(x, \varphi_{y_0}(x))}{E(x, \varphi_{y_0}(x))} < 0. \end{aligned} \tag{5.16}$$

We now proceed as in the above proof of property (i) in Theorem 1.3, i.e., reiterating the IFT and extending the implicit function in the right direction. For simplicity, we still denote the extended implicit function by $\varphi_{y_0}(x)$. By symmetry, a similar result holds when $(x, y) \in Q_-$. We finally obtain

Lemma 5.5. *For $y_0 < 0$, the level set $\Theta(x, y) = \Theta(0, y_0)$ is the graph of a continuous function $\varphi_{y_0}(x)$, differentiable except at $x = 0$, increasing when $x < 0$ and decreasing when $x > 0$. The level set has the asymptotes $x = \pm \ln(2\Theta(0, y_0))$. And at $x = 0$, it holds from* (5.15) *and* (5.16) *that*

$$\varphi'_{y_0}(0^-) = \frac{1}{2E(0, y_0)} > 0, \quad \varphi'_{y_0}(0^+) = -\frac{1}{2E(0, y_0)} < 0, \tag{5.17}$$

which determines the symmetric corner at the interface.

5.3 Level sets through $(0, y_0)$ with $y_0 > 0$

The study of this case is simpler than the previous one, because it turns out that the level set is differentiable at $(0, y_0)$ if $y_0 > 0$. Therefore, we omit the proof of the following

Lemma 5.6. *For $y_0 > 0$, the level set $\Theta(x, y) = \Theta(0, y_0)$ is the graph of a continuous differentiable function $\varphi_{y_0}(x)$, increasing when $x < 0$ and decreasing when $x > 0$. The level set has the asymptotes $x = \pm \ln(2\Theta(0, y_0))$. And at $x = 0$, it holds that $\varphi'_{y_0}(0) = 0$, i.e., the tangent to the level set is horizontal.*

We conclude this section with the following observation:

Remark 5.7. *Let us remind that, at fixed x, the mapping $y \mapsto \Theta(x, y)$ is decreasing, see Lemma 3.2. Therefore, it holds for any $x \in \mathbb{R}$ and $\underline{y}_0 < 0 < \overline{y}_0$ that*

$$\varphi_{\overline{y}_0}(x) < \varphi_0(x) < \varphi_{\underline{y}_0}(x), \tag{5.18}$$

i.e., the level sets are ordered. And it is not difficult to prove $\varphi'_0(0) = 0$ in an alternative way using the inequality (5.18).

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