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Macroeconomic Complementarities, Coordination failures and Economic Fluctuations : from multiplicity of equilibria to disequilibrium dynamics¹

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1. A first version of this paper was presented at the workshop Economics and Coordination, Nice 9-10 Sept. 2022. I wish to thank the participants and especially Peter Howitt for his comments. The usual caveats apply.

Introduction

This contribution aims to characterize the main lines of research developed at the end of the 80s and the 90s to model economic fluctuations in the macroeconomic approach involving coordination failures induced by strategic complementarities.

In his 1986 lecture at the Canadian Economics Association, "The Keynesian recovery", Peter Howitt gives a clear and synthetic presentation of this still emerging approach at the time. These models contribute in particular to renewing the Keynesian process of multiplier, "because the non-price interaction involves what Haltiwanger and Waldman (1985) call synergism, or what Cooper and John (1985) call strategic complementarity. That is, an exogenous increase in activity by one set of agents induces other agents to want to increase their activity too, thus reinforcing the initial disturbance². He goes on to mention the two major implications of this approach, "One of the most striking features of these models is their tendency to produce multiple equilibria. If everyone believes that markets will be inactive they will anticipate a high cost of transacting; this will discourage them from undertaking transactions, and the initial beliefs will be self-fulfilling. On the other hand the expectation of a high level of activity can also be self-fulfilling. The low-level equilibria are reminiscent of the chronic states of subnormal activity that Keynes was trying to explain. In simple models they are Pareto-dominated by the high-level equilibria³.

Indeed, the initial results on strategic complementarities came under comparative statics of Pareto-ranked equilibria. Various forms of complementarities were introduced in macroeconomic modelling at that time. These complementarities refer to market participation, non-convexities and technological linkages, synchronization of choices or final demand, imperfect competition playing here a major role. We will thus consider the role of these forms of macroeconomic complementarities as a source of multiplicity of Nash equilibria in the static case.

Very quickly in the mid 80's, deterministic and stochastic dynamic settings were developed. Then, the modeling strategy shifted from the analysis of a multiplicity of equilibria to a multiplicity of steady states or equilibrium paths. Local dynamics in the neighbourhood of multiple steady states may in some configurations become complex, with the possibility of existence of sunspots equilibria and en-

2. Howitt, 1986 p.636. In a footnote he points out that these approach more generally involves what Schelling (1978) in *Micromotives and Macrobehavior* "especially 89-1 10) calls the phenomenon of critical mass." (Howitt, 1986 p.636). As we will see Howitt strongly contributed to the approach both in statics and in dynamics.

3. Ibid.

ogenous cycles with a strong enough degree of complementarity.

Starting with the static setting, we try to identify and characterize analytically these main paths followed to make the approach dynamic.

For obvious reasons, our approach is not intended to be exhaustive. First, we only focus on key or pioneering contributions for each type of modeling. Second, our analysis only concerns short-term dynamics and more particularly business cycles. A more complete approach would also require considering the specific stochastic frameworks implemented for analyzing the impact complementarities and incomplete markets on aggregate growth⁴. In addition, we have deliberately excluded the specific literature on dynamic coordination games and beliefs which mainly developed from the 2000s⁵. Finally, this contribution focuses mainly on the theoretical advances and addresses the empirical aspects only in a tangential way.

The structure of the paper follows directly. The first section is concerned with the static framework. The second one is devoted to the history-dependent selection hypothesis and the business cycle. The third section deals with the introduction of animal-spirit cycles and sun-spot equilibria. The fifth section develops an example of endogenous fluctuations with disequilibrium coordination in the manner of the nonlinear dynamic models of the 90s and the last section concludes.

Strategic complementarities in multiple Nash equilibria models : the static case

The seminal contribution of Cooper and John, "Coordinating coordination failures in Keynesian models"⁶ provides the general framework that has become classic for analyzing the role of strategic complementarities in statics.

The aim of the authors is to propose an analytical framework that can unify different contributions participating in "rational Keynesianism" that appeared in the late 1970s and early 1980s⁷. Then, as they put it, their "more ambitious

4. Notably by Durlauf 1990, 1991

5. On this point see e.g Morris and Shin 2000, Rethinking Multiple Equilibria in Macroeconomic Modeling, NBER Macroeconomics Annual , Vol. 15, pp. 139-161.

6. The finalized contribution appears in a Cowles Foundation Discussion Paper in April 1985, published in 1988 in the *QJE*.

7. On this point see notably Arena and Torre 1992 pp.9-46.

goal is to use this framework to yield further insights into coordination failures and to relate this literature to other models in the Keynesian tradition, including those with fixed prices”⁸. Cooper and John give the following example : Diamond (1982) is based on search theory, while Bryant (1983) emphasises imperfect information and Weitzman (1983) increasing returns. But the similarities between these models and the crucial point of coordination failure have not been well identified. The concept of strategic complementarity is precisely for Cooper and John the unifying element of these approaches.

To begin with, they explain how this concept differs from that of spillovers. In a game theoretic framework, the latter refers to strategic interactions between agents at the level of payoffs, while the former refers to interactions at the level of strategies. Accordingly, in a game with two players spillovers arise if an increase in one player’s strategy affects the payoffs of the other players. Strategic complementarities arise if an increase in one player’s strategy increases the optimal strategy of the other player. They show, on the hand, that ”strategic complementarities are associated with the presence of ”Keynesian features” such as multiple equilibria and a multiplier process⁹. On the other hand, ”spillovers imply that these equilibria generally will be inefficient and can be Pareto-ranked”¹⁰.

Without entering into the details, let recall the general stylized framework exhibiting a multiplicity of symmetric Nash equilibria.

Consider a group or a continuum of identical agents i whose actions have non negligible effects on the payoffs of the others and who behave strategically with respect to one another. Define by $V(e_i, \bar{e}, \theta)$ the payoff of an agent i , where $e_i \in [0, x_{max}]$ is the strategy variable, or ”action”, of agent i , θ is a shift parameter and \bar{e} is the decision taken by all other agents in the economy¹¹. We can also interpret \bar{e} as some aggregate (average) index of other agents strategies. As noticed by Cooper and John, ”this idea that an individual’s payoff may depend upon economy- wide aggregates also often seems to be a feature of Keynesian models”¹².¹³. Let $e_i^*(\bar{e})$ be the optimal response of agent i when the other $j \neq i$ chose \bar{e} .

We put the emphasis on the concept of Strategic complementarity. It means that

8. Cooper and John, 1988, p.442.

9. Cooper and John, 1988, p.442.

10. *ibid.*

11. The function V is continuously differentiable with $\frac{\partial^2 V}{\partial^2 e_i} < 0$ and $\frac{\partial^2 V}{\partial e_i \partial \theta} > 0$.

12. Cooper and John, 1988, p.444

13. The function V is continuously differentiable with $\frac{\partial^2 V}{\partial^2 e_i} < 0$ and $\frac{\partial^2 V}{\partial e_i \partial \theta} > 0$.

an increase in \bar{e} , the action of all agents except i , increases the marginal payoff of agent i 's action, e_i . Hence, e_i is an increasing function of \bar{e} , that is $V_{12}(e_i, \bar{e}) > 0$.¹⁴

It can be shown that Strategic complementarity, i.e. a positive slope for the reaction function, is a necessary but not sufficient condition for multiple symmetric Nash equilibria¹⁵. Indeed, a stronger necessary condition is that "the reaction function must somewhere have slope of at least unity, implying that agent i 's action increases at least one-for-one with other agents' actions"¹⁶. A sufficient condition for multiplicity is a slope strictly greater than one at a symmetric Nash equilibrium¹⁷.

A symmetric Nash equilibrium is a fixed point of this best response function. Such function meeting the conditions mentioned above and displaying three equilibria, e' , e'' , e''' shapes as follows :

14. The game exhibits strategic substitutability if $V_{12}(e_i, \bar{e}) < 0$. The other properties defined by Cooper and John are the following : If $V_{22}(e_i, \bar{e}) \geq > 0$, the game exhibits positive or negative spillovers.

15. Let consider the first order condition for a Nash equilibrium. By total differentiation with $d\theta = 0$ we obtain the slope $\rho = \frac{de}{d\bar{e}} = -\frac{V_{12}}{V_{11}}$. Thus, Strategic complementarity is equivalent to $\rho > 0$ and hence is necessary for multiple equilibria.

16. Cooper and John, 1988 p.447.

17. Which is therefore unstable.

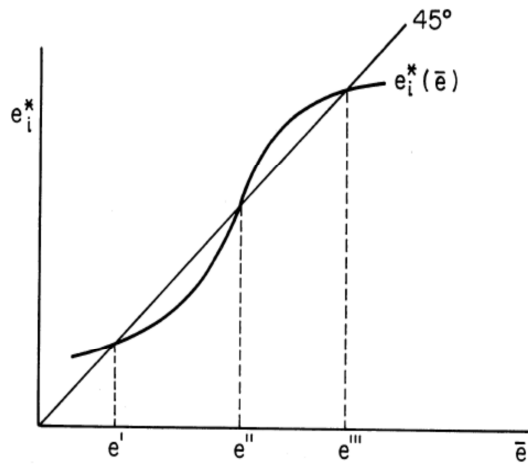


FIGURE I

In this configuration, with multiple equilibria and positive spillovers globally, the three equilibria can be Pareto ranked by the equilibrium action, since equilibria with higher levels of action are preferred by the agents. Consequently, as Cooper and John put it, "the economy can get stuck at an inefficient equilibrium with a low level of "economic activity," even though equilibrium exists. This is a co-ordination failure : if there mechanism for agents to coordinate their activities,

achieve a better (cooperative) equilibrium"¹⁸, and as they conclude, many models with Keynesian features involve at the time these coordination failures.

It is beyond the scope of this paper to review these contributions¹⁹. Let limit ourselves to recalling a few salient themes. A first privileged area is that of production technology and linkages. Accordingly many contributions introduce a form of externality into the production process, non convexities, trading externalities or a technology for matching traders²⁰. Finally, let us emphasize the importance the importance of imperfect competition in generating coordination failures in these approaches²¹.

But, two main limits of this initial framework were quickly raised. The first one is that the existence of multiple equilibria may weaken the predictive power of these models with strategic complementarities. By definition, the static analytical framework is only interested in the conditions of existence of multiple equilibria, leaving aside the question of their selection. Second, the dynamic (stochastic) properties of these models have to be fully explored if this approach wishes to compete with the RBC models then in full development.

The history-dependent selection hypothesis and the business cycle

As mentioned above, the mere conditions for the existence of multiple equilibria do not provide insights into which of the stable equilibria is more or less likely to be observed²². Cooper recalls that it is often argued that the Pareto

18. Op cit. p.448.

19. See notably Colander 1996 and for a synthesis including both theoretical and empirical findings see Cooper and Haltiwanger 1996.

20. For example, markets are replaced by a random matching mechanism in Diamond's (1982) search model while Howitt (1985) introduces costs of trading depending on the level of activity in the economy.

21. This point is emphasized in Cooper and John 1988. Heller (1986) shows that multiple equilibria exist if demand is sufficiently inelastic at low levels of consumption, while other contributions introduce imperfect competition into a multi-sector economy. For a synthesis on specific development in the context of market power, notably related to dynamics see Silvestre 1995.

22. In line with Samuelson's classical principle, it is assumed that only stable equilibria are observable. As mentioned above, the Low and High equilibria are by construction stable in the static frameworks. The issue of observability in a dynamic setting is addressed in Howitt and Mc Afee 1988.

dominant equilibrium may provide a natural focal point and hence is likely to be the outcome of a game with multiple Nash equilibria. But making this assumption would lead to invalidate multiple equilibrium theories of coordination failures.

Thus, Cooper looks for an approach that does not rely on this assumption. His argument is that historical experience may generate a focal point²³ If the economy has been at a particular equilibrium in the past, then as the fundamentals of the economy change due to small disturbances, the new equilibrium is likely to be near the old one. Using techniques of regular economies, Cooper shows that for small disturbances, there will generically exist equilibria close to a given equilibrium point. Thus, one can perform these local comparative statics and apply this selection criterion.

To illustrate this generic selection process, Cooper proposes an example building on Copper and John (1986). In this simple production economy firms have a choice of production techniques which differ by their fixed and marginal costs²⁴

From this standpoint, he considers an economy with three commodities. Two are produced by firms in two sectors, indexed $i=1,2$ and the third one is endowed by a group of agents called "outsiders"²⁵. These outsiders have an aggregate endowment of $2\bar{M}$ which is spent equally on each of the two produced goods and whose variations will be the source of fluctuations in this economy. Firms in sector 1 have to chose both a level of output q and a technology²⁶. Accordingly, two technologies $j = L, H$ are available with the following cost functions :

$$C^j(q) = K_j + \frac{q}{\theta_j}$$

where $K_H > K_L$ and $\theta_H > \theta_L$. Thus H is more productive than L , but requires a higher level of fixed cost K_H .

This choice of technique in sector 1 is at the origin of multiple equilibria in this economy because of the interaction between the choice of technology and

23. The contribution is part of the NBER's research program in Economic Fluctuations. A first version was developed in 1987 in a NBER Working paper and published in the Economic Journal in 1994.

24. Cooper mentions that this was interpreted as a choice of the number of shifts to run a plant. Thus, as in Kiyotaki, the multiplicity derives from technological features as opposed to the structure of demand, like in Heller.

25. The number of firms is finite and given Quantity decisions of firms have non negligible effects on the prices in their sectors of activity.

26. These technics use sector 2 output and the non-produced good.

the prices of the produced goods²⁷. As explained by Cooper, it generates an externality in the gains of utilizing a more productive technology. Indeed, let suppose that all but one firm is using technique H, then prices in the economy will be relatively low and the remaining firm may find it worthwhile to pay the extra fixed cost to operate the more productive technology as well. Alternatively, if all but one firm is using the less productive technology, then prices in the economy will be higher and the returns to producing more for the remaining firm lower. As a consequence, it may not pay the remaining firm to utilize the more productive technology so that an equilibrium with all firms using the less productive technology can emerge as well²⁸.

Firms in sector 2 simply select a level of output for given conjectures about the output decisions of the other firms in their sector and the output technology choices in sector 1. Sector one firms select technologies and output levels given conjectures about the decisions of the other firms in the economy. In equilibrium, these conjectures are correct.

The utility for firms in sector 1 using technique $j = 1, 2$ is given by

$$\Phi(p_2)R(q) - C^j(q)$$

where, $R(q)$ denote the gross earnings obtained by producing q units of output and $\Phi(p_2) = \alpha^\alpha(1 - \alpha)^{(1-\alpha)}(\frac{1}{p_2})^\alpha$ captures the interaction between the technology and the price of good 2. For for sector 2, the utility is given by

$$\Phi(p_1)R(q) - kq$$

where, $k < 1$ is the unitary cost and $\Phi(p_1)$ is defined in the same way as $\Phi(p_2)$.

A symmetric Nash equilibrium in sector 2 is characterized by the couple (q_2^j, p_2^j) for a given choice of technique j in sector 1 and a level E_2 of expenditure in sector 2 from firms in sector 1 and the outsiders²⁹.

Sector one firms select technologies and output levels given conjectures about the decisions of the other firms in the economy. The procedure adopted to characterise an equilibrium is as follows. Suppose first that all firms in sector 1 use technique L and select output that maximizes utility. Second, solve for the equilibrium in the economy under this hypothesis about the choice of technology. Third, check to see whether or not an arbitrary firm in sector 1 would profit by deviating and producing using the H technology. If not, then the equilibrium with

27. Since prices are mark-ups over unit costs, the technique is reflected in prices.

28. Notice that this type of externality is similar as in Shleifer (1986)

29. More specifically, the first order condition for utility maximization leads to $q_2^j = \frac{\Phi(p_1^j)\eta E_2}{kF}$, $p_2^j = \Phi(p_1^j)^{\frac{k}{\eta}}$, with $\eta = 1 - 1/F$ where F is the number of firms in the sector

the L technology is obtained. This process is repeated using the H technology and then checking deviations to the L technology by an arbitrary firm. Then, the symmetric Nash equilibrium in sector 1, (q_1^j, p_1^j) , is obtained given the price in sector 2 and the level of expenditures in sector 1, E_1 ³⁰, which completes the characterization of the equilibrium for this economy given that all firms are using technology $j = H, L$.³¹

Accordingly, two types of symmetric equilibria, L-equilibrium and H-equilibrium may exist³². It is shown that the likelihood of existence of a L or H equilibrium depends on the level of the aggregate endowment of the non-produced good, M . Accordingly, for small \bar{M} , the condition for L-equilibrium is more likely to be met. Conversely, for high levels of \bar{M} , a H-equilibrium is likely to exist.

Multiple equilibria may exist in this economy if α is large enough. This means that the decisions of the firms must be sufficiently sensitive to the prices of consumption goods³³. The intuition given behind this result is that if firms in sector 1 produce with the H technology, this will induce firms in sector 1 to produce more since the gains to more output are high when the price in sector 1 is low. It can be checked that when the price in sector 1 is low, so will be the price in sector 2. This induces the firms in sector 1 to produce more and makes it more likely, that they will adopt the H technology. The same reasoning applies for the L-equilibrium.

Moreover, these equilibria may be Pareto-ranked. In particular, when α is sufficiently large, all agents are better off in the H-equilibrium than in the L-equilibrium.

Finally, assuming that multiple equilibria exist, Cooper briefly addresses the issue of selection dynamics. From this standpoint, it is necessary to view the game as a repeated game in which the firms are only present for a single period and the level of endowment of the outsiders, M , is now an iid random variable³⁴.

30. In formula, $q_1^j = \theta_j \Phi(p_2^j) \eta E_1 F$ and $p_1^j = \frac{\theta_j}{\eta} \Phi(p_2^j)$.

31. As for the equilibrium level of expenditures in each sector E^* , it can be obtained given the preferences and the symmetry of the model with $E^* = \frac{(1+\alpha)\bar{M}}{(1-\alpha)^2}$.

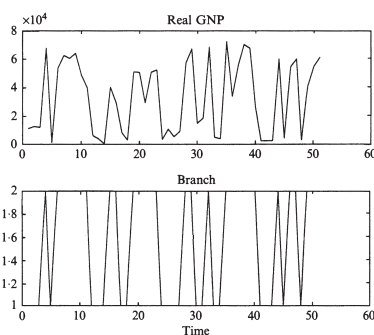
32. Firms within a sector are identical.

33. More precisely, it is shown that if α is close to 1, then there will be a range of values of M such that multiple equilibria exist.

34. The assumption that the firms play only once is used to eliminate from discussion the possibility of the cooperative outcome due to repeated play of the game and the assumption that variations in M are uncorrelated is used so that the observed correlations in output are produced internally.

The selection criterion discussed above states that the equilibrium will remain on the branch L (respectively H) of the equilibrium manifold in period $t + 1$ if the economy was there in period t on this branch. Persistence depends on the beliefs of the players and historical experience suggests a focal point to them. Thus, bad times tend to persist since the economy requires a large shock to M in order to move to an H-equilibrium. Conversely, a large negative shock is needed to bring the economy back to an L-equilibrium. Thus, Cooper concludes that this selection criterion can account for the properties of the business cycle without correlated shocks.

A quantitative simulation completes this heuristics in the published version of the contribution³⁵.



An example of real GNP and switching dynamics (Cooper, 1994 p.1119)

For each simulation, the author computes a discrete variable (Branch) that equals 1 in an L-equilibrium and 2 in an H-equilibrium. The figure above displays fluctuations of output for one simulation along one branch until there is a sufficiently large shock that the branch disappears and so forth over time³⁶. In these simulations, the average probability of being on the H-branch in period $t + 1$ given that the economy was on the high branch in period t is 0.55. Similarly, the average

35. Cooper 1994, p.1117-19. He mentions that due to the simplicity of the model, it is not a fully-fledged calibration procedure, but a simple illustrative exercise of the properties of the model. From this standpoint, the value of α is 0.9, $F = 4$ in each sector so that the markup is 0.33, which is at the low range of the estimates reported in the literature.

36. The model was simulated for 500 times, 50 periods per iter

probability of being on the L branch in period $t + 1$ given that the economy was on the low branch in period t is also 0.55. This means that this theoretical selection framework potentially accounts for a certain amount of persistence, but not for asymmetry in persistence of contractions and expansions³⁷.

To conclude, these simulations illustrate how the history-dependent selection hypothesis in the context of multiple equilibria is able to generate serial correlation in data and between output and productivity. Unlike the standard RBC models, these dynamical features do not come from technology shocks, but from variations in the endowments of outsiders assimilated to demand shocks, which propagate in the rest of the economy through intersectoral linkages.

This contribution naturally does not exhaust the dynamic approaches in terms of shocks.

Several contributions underline the facilitating role of complementarities in the amplification and propagation of aggregated shocks, including shocks on preferences which model variations in demand³⁸. In general, these works confirm the intuitive idea that models with complementarities induce a positive correlation between agents for production or employment. On the one hand, specific sectoral shocks, through their influence on aggregate production, lead to an increase in production activity in other sectors. Because of this externality of production, the other sectors of the economy are influenced and a positive correlation is created in the movements of employment and production. On the other hand, the linkages introduced by final or intermediate demand through the production process in a multisectoral framework can also generate positive correlations between sectors. Finally, works in line with Diamond's 1982 discrete choice model, show that with strategic complementarities, agents are encouraged to synchronize their discrete choices³⁹.

37. As for instance in the study by S.Cooper and Durlauf (1992) for the United States 1939-91 with respectively 0.58 and 0.5.

38. Let mention for instance the model with capital accumulation developed by Baxter and King 1991 with instantaneous interactions. Durlauf 1991 analyses the role of learning by doing with local complementarities in which the present level of productivity of agents in a given sector depends on the level of past activities of neighboring sectors and not of all sectors, showing that this environment amplifies the interactions between sectors.

39. This property is analysed notably by Hall 1991 or by Cooper and Haliwanger 1993. Let also mention Shleifer (1986) model of implementation cycles, provides an other example where strategic complementarities induced by aggregate profits between sectors lead to synchronization of innovations and endogenous cycles.

Strategic complementarities and endogenous fluctuations

The existence of endogenous fluctuations with strategic complementarities has also been addressed since the beginning of the 1990s referring to the concept of extrinsic uncertainty and sunspot equilibria. In this perspective, the role of animal spirit was first explored in a framework with multiple stationary equilibria arising from a transaction externality by Howitt and McAfee⁴⁰. The existence of sunspot equilibria proper and endogenous cycles was also analysed in an overlapping-generations setting with imperfectly competitive product markets where strategic complementarities emerge from the interaction of firms in their participation decisions⁴¹. As emphasised by Chatterjee, Cooper and Ravikamur, "apart from suggesting that instability of expectations may be a distinct source of cyclical volatility, sunspot equilibria may also explain why apparently small disturbances in fundamentals lead to large change in macroeconomic aggregates."⁴². Let us consider successively the key elements of these two lines of research.

Transaction externality animal spirits and cycles

The authors deliberately use the term "animal spirits" cycles and not sun-spot equilibria. This is obviously a direct reference to Keynes' account of investment decisions and long term expectations and the trade cycle in the *General theory*⁴³. But it is not the only reason. For them the term sunspot is unfortunate and "misleading in the light of Jevons's view"⁴⁴.

The framework used is a variant in a discrete-time dynamic setting with multiple stationary equilibria arising from a transaction externality⁴⁵.

40. Howitt and McAfee 1990, 1992, building on Howitt and McAfee 1987 and 1988.

41. The reference here is Chatterjee, Cooper and Ravikamur, 1990, 1993. Other overlapping-generations models of coordination failures in a deterministic framework emphasize the effects induced by imperfect competition and final demand in the emergence of endogenous fluctuations. See notably d'Aspremont, Dos Santos Ferreira and Gerard-Varet 1994

42. Chatterjee, Cooper and Ravikamur, 1993, p.795.

43. And more generally a common feature of British trade cycles theories like in Mill, Marshall, Lavington or Pigou.

44. Howitt and McAfee, 1990, p.2. Reacting to the presentation of this paper during the Nice Workshop, Howitt reiterated this view. For him the term sun spot in its modern sense "remains an insult to Jevons who took sun spots seriously", referring, as H.L. Moore latter, to the effective effects of climatic variations induced by these spots on crops.

45. Developed notably in Howitt 1985 and in a continuous time setting in Howitt and McAfee 1988. It follows Diamond's suggestion of deriving multiplicity of equilibria from a transaction externality.

The key element refers to the search-matching mechanism between potential workers and a fixed number of identical firms. Each period, $\delta \in (0, 1)$ newborn workers enter the labor market and begin searching for a firm. A match results in a lifetime employment contract⁴⁶. A firm that wishes hire a fraction θ_t of these searchers must pay a cost $c_t\theta_t$ in the form of output. All firms are identical and compare this cost with the benefit of hiring in period t . This value is $fg(n_t)$, where $f > 0$ is the constant marginal product of labor n_t and $g(n_t)$ is 1 minus the cost per unit of selling output, where g is a continuous increasing function in $[0, 1]$. It is an increasing function of aggregate employment because of an externality, an increase in the number of searching workers allows the firm to make more contacts at no extra cost. Thus, as in Diamond (1982) and Howitt (1985), higher employment n_t means higher aggregate demand, which reduces the marginal cost of contacting a customer⁴⁷. As we will see, this assumption on g plays a crucial role in obtaining the results of the model.

Thus, the dynamics of the level of employment depends on the value of θ_t which can take two possible values 0 or $h > 0$ ⁴⁸.

$$\begin{aligned} n_{t+1} &= n^L(n_t) = (1 - \delta)n_t, \theta_t = 0 \\ n^H(n_t) &= (1 - \delta)[n_t + h(1 - n_t)], \theta_t = h \end{aligned}$$

As shown by Howitt and McAfee in a continuous-time version of this model, there may be many perfect-foresight equilibria starting from the same initial employment level. They focus on two of them, the pessimistic and optimistic paths. Along the pessimistic path, everyone correctly believes that there will be no recruiting, and employment falls gradually to zero. Along the optimistic path, everyone correctly believes that all firms will actively recruit at all times, and employment asymptotically approaches a stationary value $n^H \in (0, 1)$.

Assume that agents have perfect foresight and define by $\lambda^L(n)$, (respectively $\lambda^H(n)$), the value of hiring when employment is equal to n and the economy on the pessimistic (optimistic) path. If for all $n \in [0, n^H]$

$$\lambda^L(n) < \frac{c}{(1 - c)} < \lambda^H(n)$$

46. The worker gives his entire labor endowment to the firm and receives each period a constant fraction w of the current value of the match. Workers are dying with the same probability δ and firms live for ever.

47. The size of the firms is negligible so that they treat n_t as given.

48. The cost parameter c_t is an independent and identically distributed random variable with positive mean c and its realization is not known at the time of the decision.

Then, for all $n_0 \in [0, n^H]$, the pessimistic and optimistic paths are perfect-foresight equilibria.

An animal-spirits cycle is a cycle in which employment switches randomly between these optimistic and pessimistic paths. The selected path at period t depends upon the value of an extrinsic random variable $s_t \in L, H$, called "Animal spirits"⁴⁹. When they are high $s_t = H$, every firm recruits. Conversely, when they are low $s_t = L$, none recruits. They are modeled by two-state Markov process with transition matrix :

$$\begin{pmatrix} 1 - a^L & a^L \\ a^H & 1 - a^H \end{pmatrix}$$

where a^L (respectively a^H) is the probability of change given a High, respectively Low, level of animal spirits⁵⁰.

In formula, an Animal-Spirits Cycle is defined for any initial value $n_0 \in [0, n^H]$ by the random sequence $\{n_{t+1}\}_0^\infty$ satisfying

$$n_{t+1} = n^i(n_t)$$

if $s_t = i$, $i = L, H$

for all $t = 0, 1, \dots$. This sequence remaining in $[0, n^H]$ for ever.

Consequently, on condition that market externalities are strong enough, expectations are governed by animal spirits, but rational and "people may rationally anticipate the waves of optimism and pessimism that keep employment fluctuating forever"⁵¹.

At this point, Howitt and McAfee point out that "a rational-expectations interpretation of mob psychology may seem incongruous. It also begs the question of how anyone would ever arrive at such peculiar expectations"⁵². This leads them to suggest rather to model expectations from an adaptive scheme, "that does not endow firms *ab initio* with beliefs consistent with the model"⁵³. In this perspective, they introduce a learning mechanism with a Bayesian learning rule⁵⁴. The

49. Howitt and McAfee 1992, op.cit. p.498.

50. These probabilities are exogenous and notably independent of the random hiring cost.

51. Howitt and McAfee, op.cit. p.498.

52. Op cit p.499. Referring on this point to Evans (1989) critics on the "the fragility of sunspots and bubbles".

53. Ibid.

54. This analysis of learning is first developed made in a simplified model and then generalized to the initial setting.

intuition is that there is a spurious correlation between the level of animal spirits s_t and the cost of recruitment c_t , which leads firms to condition the intensity of recruitment θ_t on s_t . This spurious correlation eventually disappears, but a correlation between θ_t and s_t remains fossilized in the data. Thus, the firm needs to learn the following four probabilities for $s = L, H$:

$$p^s \equiv Pr(c_t = c^H / s_t = s)$$

$$q^s \equiv Pr(\theta_t = h / s_t = s)$$

In each period t , the expected value of the probabilities, (p^L, q^L) or (p^H, q^H) are updated according to a Bayesian rule using the observation of the state $s_t = L$ or $s_t = H$ and thus (c_t, θ_t) ⁵⁵.

It can be shown that the Animal Spirits Cycle with Rational Expectations is potentially stable under this Bayesian learning process, and there is consequently a positive probability that the cycle will persist forever⁵⁶. As for the condition of existence of these equilibria, the crucial assumption is that strategic complementarities induced by thin-market externality should be large enough.

Therefore, these results differ significantly from the literature on sunspot equilibria and endogenous cycles of the time. The main differences concern the necessary condition of existence of these equilibria and the features of the cyclical fluctuations.

First, as we have seen the business cycle is not related to the expected rate of inflation, but are the consequence of market externality. As Howitt and McAfee argue, in a Keynesian perspective, "what drives the boom is the expectation of rising aggregate demand, not expectations of inflation"⁵⁷. Second, the existence of sunspot equilibria is not based on the assumption of a backward-bending labor-supply or saving function, but precisely related to the presence of large en-

55. According to the Bayesian rule, the independence of priors implies that there is no information for updating the probabilities associated with the state that is not observed. In a rational expectation equilibrium $p^L = p^H = \bar{p}$ with $c = (1 - \bar{p})c^L + \bar{p}c^H$. An Animal Spirits Equilibrium is characterized by $q^L = 0$ and $q^H = 1$, an optimistic equilibrium by $q^L = q^H = 1$ and a pessimistic one by $q^L = q^H = 0$.

56. Howitt and McAfee mention that their learning rule is "almost identical" to the non Bayesian adaptive learning scheme with least-squares used notably by Margaret Bray (1983) to study the emergence of rational expectations. Accordingly, they specify that what differentiates their approach "is not the learning rule, but the demonstration that it can lead to an animal-spirits equilibrium." Howitt and McAfee, op cit p.500.

57. Howitt and McAfee, op cit p.498.

ough strategic complementarity⁵⁸. Thus, real wages can be pro-cyclical, as is the behavior of productivity⁵⁹. Finally, animal-spirits equilibria may exist with small transition probabilities. As a consequence, the model does not display the counterfactual feature of high-frequency oscillations in employment induced by large transition probabilities.

Participation dynamics, sun-spots and cycles in an overlapping-generations framework

The existence of endogenous fluctuations and sun-spots equilibria was also explored by Chatterjee, Cooper and Ravikamur in an overlapping-generations framework. In this setting, strategic complementarities arise from the interaction of firms in their participation decisions to imperfect competitive product markets⁶⁰.

They consider an overlapping-generations structure with fiat money, two periods and two sectors where F agents are born at discrete times $t = 1, 2$ and live for two periods⁶¹. It is assumed that only young agents are capable of production and that agents consume goods produced in other sectors⁶².

A young agent in sector i faces a two-stage decision problem.

The key issue in the first stage is to decide whether or not to participate in production. Each agent f engaged in production suffers an individual cost k_f introduced to limit the number of agents entering into market activity⁶³. If he

58. As we may recall, this kind of assumptions on the labor supply or saving function was generally necessary for the existence of sun spot equilibria and endogenous cycles in OLG frameworks, as in Azariadis (1981), Azariadis and Guesnerie (1982, 1986), Grandmont (1985) and Reichlin (1985). On the contrary, Woodford (1986, 1988) developed a model with infinite-lived agents and financial constraint which did not require these conditions on the elasticity of labor supply. But the expected rate of inflation $\frac{P_{t+1}^e}{P_t}$ still plays an important role in this model. Contrary to Howitt and McAfee, in this literature, cycles were also related to indeterminacy of a stationary state with perfect foresight. On these approaches in relation with Keynes see e.g. Raybaut, 1992).

59. Howitt and McAfee, op .cit p.495. As we know this issue was highly debated in the analysis of economic fluctuations of the 1930s, with this issue re-emerging in the 1990s.

60. Chatterjee, Cooper and Ravikamur, 1990 and 1993. They mention that the dynamic version of the strategic complementarity model developed in Howitt (1990) comes closest to the spirit of their contribution

61. At $t = 1$, there is also an initial generation of M old agents in each sector who die at the end of the period.

62. That is agents specialize in production. The home good is non marketable and all goods are non storable.

63. The agents f are indexed in the increasing order of their opportunity costs i.e. $k_1 \leq$

chooses to be inactive he obtains a utility of k_f . If he chooses to be active and enter as producer in the market he loses k_f and proceeds to the second stage. In this second stage, production and consumption are determined. In making these decisions, he acts as a Cournot competitor as producer, taking the output levels of other active agents and the industry demand function his sector as given ⁶⁴.

We will not go into the details of this optimisation program here and simply recall some key results.

The focus is first on the existence a Symmetric Nash Equilibrium within a sector i for a given number $N_t^i > 1$ of active agents in the sector, a given amount A_t^i spent by the other sectors in i and prices in the other sectors ⁶⁵. It is shown that a within-sector Symmetric Nash Equilibrium with a level of production $y_t^{i*} > 0$ exists and is unique. Let $W_f(N_t^i, A_t^i)$ be the utility that an active agent f receives at this equilibrium ⁶⁶. Then, it is clear that the agent will undertake production if and only if $W_f(N_t^i + 1, A_t^i) > 0$.

Next, the authors consider the whole economy and the conditions of existence of Symmetric Nash Equilibria Steady States. A steady-state is defined by a 4-tuple (y, A, p, N) satisfying the following equilibrium conditions :

- (i) The aggregate level of production y solves the active agent optimization problem
- (ii) The expenditure in any sector is defined by

$$A = \alpha N p y$$

- (iii) The price level clears the market

$$(1 - \alpha) N p y = M$$

- (iv) All active agents find it worthwhile to be active and all inactive agents do not gain from entry

$$\begin{aligned} W_f(N, A, p^{-1}) &\geq 0 & f = 1, 2, \dots, N \\ W_f(N + 1, A, p^{-1}) &< 0 & f = N + 1, N + 2, \dots, F \end{aligned}$$

$k_2 \leq \dots \leq k_F$.

64. Producers act as price takers as consumers of the goods of the other sector.

65. All active agents N_t^i in a sector are identical, except for the exogenous fixed opportunity cost. The authors use the notation "-i" to refer to other sectors than i . Thus, the prices in other sectors are captured by $z_t^i = (p_t^{-i})^{-\alpha} (p_{t+1}^{-i})^{-(1-\alpha)}$.

66. W_f is decreasing with N_t^i . This property characterises "a congestion effect", a larger number of active agents have to share as sellers the same industry demand which implies less utility for all.

Conditions (i) to (iii) are straightforward and (iv) reflects the fact that there is a cutoff value for the cost parameter k such that all agents with $k_f < k$ participate and all others do not, i.e., equilibria are ordered.

Then it is shown that at least one steady-State exists if $W(2) > k_2$. The condition also ensures that a steady-state has at least two active agents. The multiplicity of these equilibria is due to the fact that the utility from participating in market production $W(N)$ is an increasing function of the number of active agent N in the whole economy. That is, multiplicity arises from a market participation externality. The following interpretation is given. If agents anticipate a high level of market participation then the benefit from participating in the market rises, which attracts more agents to the market validating the initial optimism. Thus, this type of the multiplicity is similar to that analysed by Diamond (1982) and Howitt and McAfee (1992) with the difference that it stems from the effects of thick markets on markups and not from matching⁶⁷.

On this basis, Chatterjee, Cooper and Ravikamur introduce a sun-spot driven by a two-state stationary Markov process⁶⁸. Note that the transition probabilities are assumed quite small, so that the economy is relatively unlikely to change states, a feature shared with the contribution of Howitt and McAfee examined above.

For a given transition matrix, a stationary sun-spot equilibrium is then defined by a 8-tuple (y_s, A_s, p_s, N_s) satisfying the stationary equilibrium conditions specified above, but now defined for all state s taken by the sun-spot.

The conditions of existence of these equilibria are linked, as in the overlapping generations models literature of the time, to the preferences of young agents. Accordingly, it is shown that a stationary sun-spot equilibrium can exist when the wealth effect dominates the substitution effect⁶⁹. But this condition alone is no longer sufficient. The degree of strategic complementarity also comes into play because the economy must have at least two steady states.

67. This result may seem surprising as regards the opposite conclusion mentioned previously for an isolated sector. The explanation given by the authors is as follows. As a seller, any increase in the number of active agents in sector i makes each active agent in sector i worse off. However, as the number of active agents in sector $-i$ increases their price, output declines which makes active agents in sector i better off as buyers. It is shown that this second effect dominates. Recall that the statement here concerns a simultaneous increase in the number of active agents in both sectors.

68. Denoting by b the state when there is a sun-spot and r the state in which there is no sun-spot.

69. The utility function of young agents, including here two states s and s' of the sun-spot,

Next, the authors consider the existence of a two-cycle, that is a perfect-foresight equilibrium such that

$$(N_t, y_t, p_t) = \begin{cases} (N_r, y_r, p_r) & t = 1, 3, 5... \\ (N_b, y_b, p_b) & t = 0, 2, 4... \end{cases}$$

Once again, the authors show that these two cycles exist if strategic complementarities are strong enough so that multiple steady states exist in a static setting.

By mobilizing an initial result of Azariadis and Guesnerie, they show that the mere existence of two-cycles implies that sunspot equilibria exist in the neighbourhood of these cycles⁷⁰. However, the fluctuations associated to these sun-spots display a high frequency of switches across low and high states since these sun-spots occur near a cycle of order two, contrary to the sun-spot equilibria obtained in the general case that require low switching probabilities.

Finally, Chatterjee, Cooper and Ravikamur also investigate some empirical implications of these sunspot equilibria. They argue notably, using a qualitative approach, that the model displays several features of aggregate time series such persistent output fluctuations and co-movement across sectors, the pro-cyclical nature of net business formation and the counter-cyclical behaviour of markups.

Thus, the contributions examined in this section suggest that these dynamic approaches with strategic complementarities had a number advantages over the sun-spot equilibria literature of the early 1990s. But as Howitt points out in retrospect, they suffered from a major empirical drawback : the difficulty to grasp in the data what was "a large enough level" of strategic complementarities able to generate these fluctuations⁷¹.

is Cobb- Douglas and given by

$$\frac{[(c_s^{-i})^\alpha (c_{s'}^{-i})^{1-\alpha}]^{1-\sigma} - 1}{1 - \sigma}$$

. Thus, σ controls the relative importance of income and substitution effects. The wealth effect dominates the substitution effect if $\sigma \geq 1$.

70. By continuity arguments on utility and equilibria with respect to transition probabilities, Azariadis and Guesnerie showed that sun spot equilibria exists in the neighborhood of a cycle of order two in OLG models.

71. Comments made by Howitt during the workshop Economics and Coordination, Nice 9-10 Sept. 2022. He adds that this question came up repeatedly during discussions and critics at the time.

Disequilibrium cyclical dynamics with bounded rationality in a nonlinear setting : an example

A different perspective is to consider a cyclical adjustment process in disequilibrium as suggested notably in Franke (2001).

To begin with, he notices that with the exception of very few models with sun-spot equilibria and endogenous cycles examined in the previous section, the interesting dynamics in the strategic complementarity approach relies on large exogenous shocks. Indeed, the transition from one equilibrium to another requires sufficiently large changes in the parameters that temporarily upset the underlying structure of the economy and alter the number of equilibria⁷². Second, Franke points out the implicit assumption that the convergence path from the old to the new equilibrium is monotonic and quasi immediate after a one-time shock⁷³. It therefore seems necessary to develop a strategic complementarity framework presenting a richer dynamics. This type of modeling should in particular be able to exhibit cyclical adjustment paths and overshooting so that the trajectories can enter the basin of attraction of another equilibrium point endogenously or with minor shocks without resorting to 'catastrophes' modifying the equilibrium structure.

But this line of research, which amounts to formalizing disequilibrium coordination proper, also raises questions about rationality and anticipations.

Indeed, the introduction of less rational agents in a strategic complementarity setting had already been analyzed by Haltiwanger and Waldman from the end of the 80s⁷⁴. Building on their previous work on the topics⁷⁵, they investigated

72. From this perspective, he points out the similarity of this argument with the mechanism developed by Kaldor (1940) in his business cycle model, the 'S' shaped best response function echoing Kaldor's investment function. But the key conceptual difference is that in Kaldor (1940), the shifts in the investment (and also the saving) functions are explained endogenously, notably by the dynamics of the stock of capital.

73. Referring on this point to the stability results in the references given by Cooper (1994), notably Furth (1986).

74. Their initial contribution was notably discussed in 1986 at the NBER Summer Institute and published in 1989 in the *QJE*. Let mention that the NBER Economic Fluctuations and Growth Program was composed of small research groups focussing on specialized topics, notably on macroeconomic complementarities. Each group bringing together researchers with common interests, ranging from senior graduate students to well known researchers generally met at the NBER's Summer Institute in July, and sometimes in conjunction with the NBER research meetings.

75. See Haltiwanger and Waldman 1985.

the idea that agents were heterogeneous in terms of their ability to form expectations. Indeed, they consider a macroeconomic model including two types of agents. Sophisticated ones have unlimited abilities to process information and are endowed with rational expectations, while naive agents are limited in their abilities and are assumed to follow an adaptive expectations rule. The analysis, in a dynamic setting, of the deviations from the steady state induced by expectation errors following a shock, confirms their initial conclusion that with strategic complementarity, the less rational agents have a disproportionate impact on equilibrium. Accordingly, they show that a one-time shock results in an adjustment path which is qualitatively similar to what occurs in the pure naive case : a slow movement back to steady state is obtained, rather than the immediate return which would occur if all agents were sophisticated.

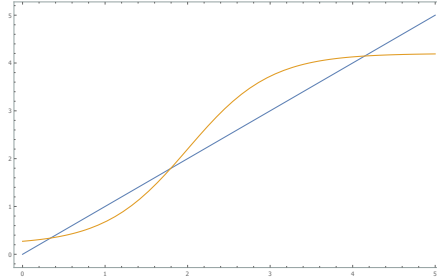
An additional step is to assume that all agents are endowed with bounded rationality, which implies the introduction of at least one ad hoc expectation rule. From this standpoint, Franke (2001) assumes that the agents update their expectations according to a rule that combines two heuristics, 'conservatism' and 'representativeness' and is compatible with psychological and empirical research. The first one, "states that individuals are slow to change beliefs in the face of new evidence. They will therefore liable to under-reactions"⁷⁶. It is captured by a mechanism of purely adaptive expectations. The second one, "is the tendency of people to view events as typical or representative of some specific class and to ignore the laws of probability ... (Thus) people are prone to recognize patterns in a time series, even is a truly random sequence "⁷⁷. As a consequence, this behavior captured by extrapolative forecasts is likely to produce overreaction. On the whole, this combination of adaptive expectations and extrapolative forecasts gives a satisfactory account, at least on first analysis, of the dynamics of expectations in the economy⁷⁸.

Starting from this approach, let consider a simple framework of equilibrium selection with cyclical dynamics. In Franke (2001) the best response function is a piecewise linear function in the level of activity $x(t)$ and the dynamics is explained by a linear delay differential equation. We develop below an example in a nonlinear setting with a continuously differentiable reaction function Φ . The best response function Φ is 'S' shaped and thus similar to that of Cooper and John (1988) :

76. Franke, 2001, p.172.

77. Ibid. Franke refers on this point on the growing advances a the time on behavioral economics and finance, notably with heterogenous interacting agents. For more details see Franke (2001), p.171-74.

78. For more details see Franke (2001), p.171-74.



A Nash equilibrium is as in Cooper and John a fixed-point of this function. In an economy characterized by strategic complementarity, following Cooper and John (1988) and Copper (1987, 94), three of them are assumed given the S shape of the function Φ . Using the standard notation, let denote by x_j^* , with $j = L, M, H$, these equilibria associated with a Low, Medium or High level of activity .

The adjustment dynamics results directly from expectations of the level of activity. We have $\dot{x}(t) = x^e(t)$, where $\dot{x}(t) = \frac{dx(t)}{dt}$. Expectations are based on the expectation rule combining adaptive and extrapolative forecasts suggested by Franke (2001) :

$$\dot{x}(t) = x^e(t) = (1 - \omega)x_{Adapt}^e + \omega x_{Extra}^e$$

where $0 < \omega < 1$ is a fixed weighting parameter.

Adaptive expectations, x_{Adapt}^e , are simply captured by a proportion $a > 0$ of the disequilibrium gap, $\Phi(x(t)) - x(t)$. Extrapolative forecasts, x_{Extra}^e , are modeled in proportion $b > 0$ to a perceived trend of the slope of the reaction function Φ over a short rolling period of constant duration $\tau > 0$. That is, $x_{Extra}^e = b \int_{\tau} \frac{\Phi(x(t)) - \Phi(x(t-\tau))}{\tau} d\tau$.

By combining the two and taking a Taylor series of order 2 for small τ , we obtain the following second order differential equation in $x(t)$ ⁷⁹

$$a(1-\omega)[\Phi(x(t))-x(t)] - \frac{1}{4}b\tau\omega[\tau\ddot{x}(t)\Phi'(x(t))+\tau\dot{x}(t)^2\Phi''(x(t))-4\dot{x}(t)\Phi'(x(t))]-\dot{x}(t) = 0$$

Let's make the change of variable $\dot{x}(t) = y(t)$. Then, after a few simple manipulations, this equation can be written in the following form ⁸⁰

79. This approach was largely used in nonlinear business cycles models during the 50's, notably by Japanese contributors, Goodwin and Allais (see Raybaut 2017). For recent formal developments and limitations of this approach see e.g. Insperger (2015) and Doldo and Pender (2020).

80. Recalling that $\Phi'(x(t)) > 0$, $b > 0$, $\tau > 0$ and $\omega > 0$. Notice that contrary to Franke

$$\dot{y}(t) + y(t)f(x(t), y(t)) + g(x(t)) = 0$$

called generalized Lienard or Lienard-Levinson-Smith differential equation where,

$$f(x(t), y(t)) = \frac{4}{\tau} \left(\frac{1}{b\omega\tau^2} - 1 \right) + y(t) \frac{\Phi''(x(t))}{\Phi'(x(t))}$$

$$g(x(t)) = \frac{-4a(1 - \omega)}{b\omega\tau^2} \frac{(\Phi(x(t)) - x(t))}{\Phi'(x(t))}$$

The dynamics of is then equivalently described by the following planar system of nonlinear ordinary differential equations in $(x(t), y(t))$

$$\dot{x}(t) = y(t)$$

$$\dot{y}(t) = -f(x(t), y(t))y(t) - g(x(t))$$

This type of system has been extensively investigated since Levinson and Smith (1942) using qualitative theory of dynamical systems and numerical different techniques. But it is generally accepted that it remains very difficult to find to an exact solution in the general case.⁸¹ In this perspective, it is nevertheless easy to characterize the local dynamic properties of the stationary solutions of this system.

As mentioned above, the reaction function Φ has the usual strategic complementarity property such that three fixed points x_j^* , $j=L,M,H$ exist in \mathbb{R}_+ . Then, it is easy to check that this system admits three stationary solutions, $(x_j^*, 0)$, $j=L,M,H$, where the x_j^* are respectively the Low, Medium and High equilibria values of the level of activity.

The Medium stationary solution $(x_M^*, 0)$ is always a saddle point. The other stationary solutions Low and High, $(x_L^*, 0)$ and $(x_H^*, 0)$, can be either a node or a focus⁸².

2001, this formulation excludes the case $\omega = 0$ where the expectations are only adaptive. The polar case $\omega = 1$ is admissible but would generate extreme dynamics due to the absence of adaptive reactions.

81. Some established results on the existence of periodic solutions exist in the literature when $f(x(t), y(t))$ and $g(x(t))$ are polynomials with specific properties. For recent advances see notably Yang and Tang 2008, Saha and Gangopadhyay 2021, Villari and Zanolin 2021, Adjai and al. 2022.

82. See Appendix **Proof 1**

From this standpoint, it can be shown⁸³ that a stationary point $(x_j^*, 0)$, $x_{j=L,H}$ is a stable or unstable focus and the dynamics cyclical in a neighborhood of this point if and only if

$$a > \tilde{a}_{j=L,H}$$

where

$$\tilde{a} = \frac{b\omega}{(1-\omega)} \frac{\Phi'(x_j^*)}{(1-\Phi')} \left(1 - \frac{1}{\omega b\tau \Phi'(x_j^*)}\right)^2$$

Finally, if $b > \tilde{b}$, where $\tilde{b} = \frac{1}{\omega\tau}$, it exists a critical value for the steepness of the best response function $\tilde{\Phi}' = \frac{1}{b\omega\tau}$ such that the stationary solution $(x_j^*, 0)$ is locally stable (unstable) when $\Phi'(x_j^*) \leq \tilde{\Phi}'$ ⁸⁴.

These local stability properties are summarized in the following tables⁸⁵

- $b \leq \frac{1}{\omega\tau}$

$\Phi'(x_j^*)$	0	1	$\frac{1}{b\omega\tau}$	$+\infty$
x_j^* , $j=M$ x_j^* , $j=L,H$	Locally Stable		Saddle	

- $b > \frac{1}{\omega\tau}$

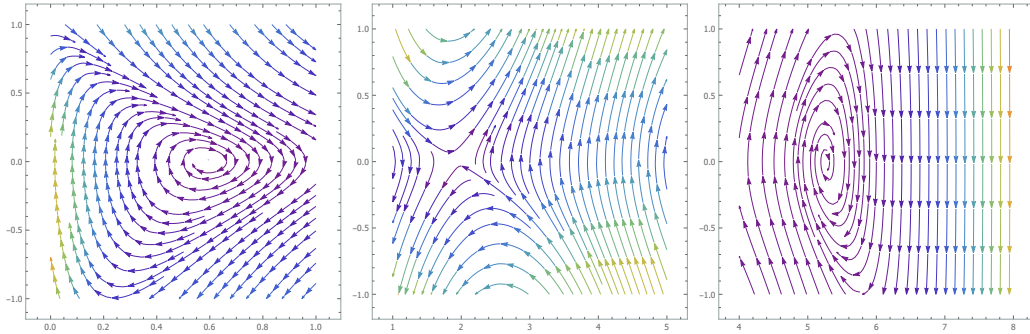
$\Phi'(x_j^*)$	0	$\frac{1}{b\omega\tau}$	1	$+\infty$
x_j^* , $j=M$ x_j^* , $j=L,H$	Locally Stable	Locally Unstable	Saddle	

The figure below shows the phase diagrams with $a > \tilde{a}$ and cyclical dynamics in a neighborhood of two stable Low and High equilibria and the intermediary saddle point , :

83. See Appendix **Proof 2**

84. See Appendix **Proof 3**

85. Recall in addition that for the critical value $\tilde{\Phi}' = \frac{1}{b\omega\tau}$, the solution $(x_j^*, 0)_{j=L,H}$ is a center.



The interpretation of these local properties is quite intuitive. On the one hand, the adjustment dynamics is cyclical if the parameter a is large enough, that is if the speed of adjustment of adaptive expectations based on the disequilibrium gap $\Phi(x(t)) - x(t)$ is sufficiently high. On the other hand, local stability is related to the interplay between the strength of extrapolative forecasts (parameter b) and the degree of strategic complementarities captured by the steepness of the reaction function Φ'^* .

Accordingly, the necessary and sufficient stability property of a low or high equilibrium induced by the structural property of the static frameworks $\Phi'(x_j^*) < 1$, $j=L,H$, loses its general validity. This condition is no longer sufficient, unless for $b < \tilde{b}$, that is when the reaction in terms of extrapolative forecasts is relatively low. Otherwise local stability requires weaker strategic complementarities, i.e. $\Phi'(x_j^*) < \tilde{\Phi}' < 1$.

As noticed by Howitt (1986), the low stationary equilibria are in many respects similar to the persistent states of unemployment depicted by Keynes. Coordination failures may persist when the economy remains stuck on a stable low stationary equilibrium. Thus, unlike the static framework, this scenario requires less strategic complementarities in this dynamic setting. In addition, by a simple argument of comparative static we see that the (local) stability range shrinks as b is increased⁸⁶, which means that the more important the extrapolative expectations are, the less likely this scenario of persistent coordination failures is.

Two additional comments come in order on these stability results.

Howitt and McAfee (1988) have first directly addressed the question of the dynamic stability of these solutions. This issue was discussed in details in an inter-temporal framework with multiple equilibria (low and high) induced by trade externalities. They showed, building on the new price-level-dynamics literature initiated notably by Calvo (1979), that the non saddle stationary equilibria can be

86. Recall that $\tilde{\Phi}' = \frac{1}{b\omega\tau}$

locally stable which implies with perfect-foresight an indeterminacy of equilibrium when the economy begins in the neighborhood of a low level equilibrium.

Let first, recall that the dynamical system discussed in this section is not the result of an inter-temporal optimization process and there is no transversality conditions nor perfect foresight. Thus, the reasoning on saddle point economic stability, local stability and indeterminacy does not apply. Second, locally unstable solutions cannot be dismissed out of hand, particularly on the basis of the argument of their non-observability. Due to the nonlinear structure of the dynamical system, self contained oscillations, limit cycles or homoclinic orbits can emerge around these unstable solutions as exemplified in the numerical examples below.

This numerical example gives some insights on the global dynamical properties of this approach.

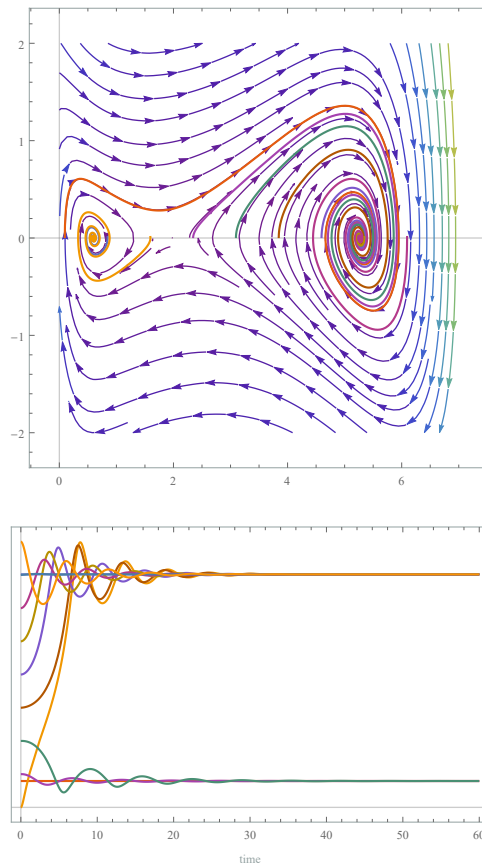
The 'S' shaped reaction function is specified by $\Phi(x) = \theta + 5(1 + \exp[s - x])^{-\frac{1}{\sigma}}$ with $\theta = 0.5$, $s = 0.2$ and $\sigma = 0.13$. We further assume that the weight parameter ω is equal to 0.5 and plays as such a neutral role. In addition let $a = 6$ and $\tau = 1$.

Consequently, the stationary values are $x_L^* = 0.59$, $x_M^* = 1.93$ and $x_H^* = 5.26$ and the steepness of the reaction function at the low and high stationary points is $\Phi'(x_L^*) = 0.293$ and $\Phi'(x_H^*) = 0.230$. The stability threshold is $\tilde{\Phi}' = \frac{1}{b\omega\tau} = \frac{2}{b}$, where $b > \tilde{b}$ is a free parameter and $\tilde{b} = \frac{1}{\omega\tau} = 2$.

Let consider the following three configurations :

- Case 1 $\Phi'(x_H^*) < \Phi'(x_L^*) < \tilde{\Phi}'$

With $b = 3$, we have $\tilde{\Phi}' = 0.6667$ and $\Phi'(x_H^*) = 0.230 < \Phi'(x_L^*) = 0.293 < 0.6667$. Thus, both the low and high equilibrium are locally stable. The first figure shows the phase diagram and the other displays the time paths of $x(t)$ for different initial conditions.

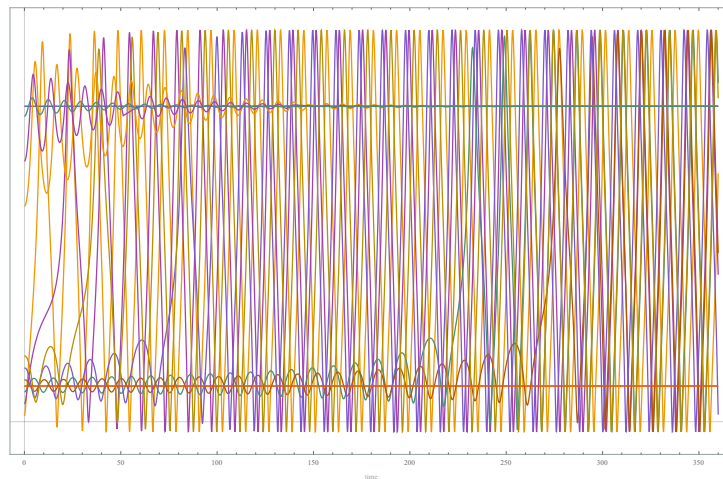
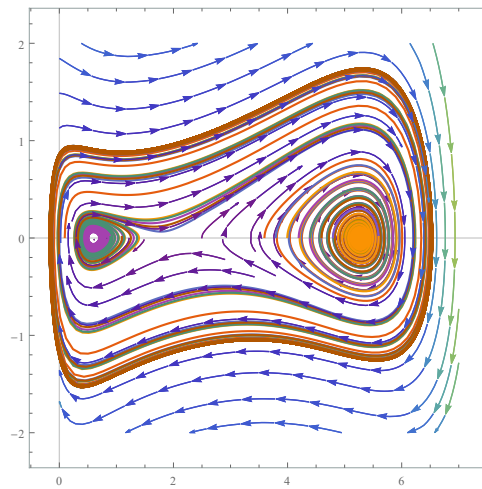


In this case, all initial conditions starting in the basins of attraction of the low and high solutions separated by the saddle point converge respectively to these solutions. But, a transition from the low to high regime is possible outside as exemplified on the figures with $x(0) \approx 0$ ⁸⁷.

- Case 2 $\Phi'(x_H^*) < \tilde{\Phi}' < \Phi'(x_L^*)$

With $\gamma = 7$, we have $\tilde{\Phi}' = 0.286$ and $\Phi'(x_H^*) = 0.230 < 0.286 < \Phi'(x_L^*) = 0.293$. Thus, the low equilibrium is locally unstable and the high equilibrium locally stable. The first figure shows the phase diagram and the other displays the time paths of $x(t)$ for different initial conditions.

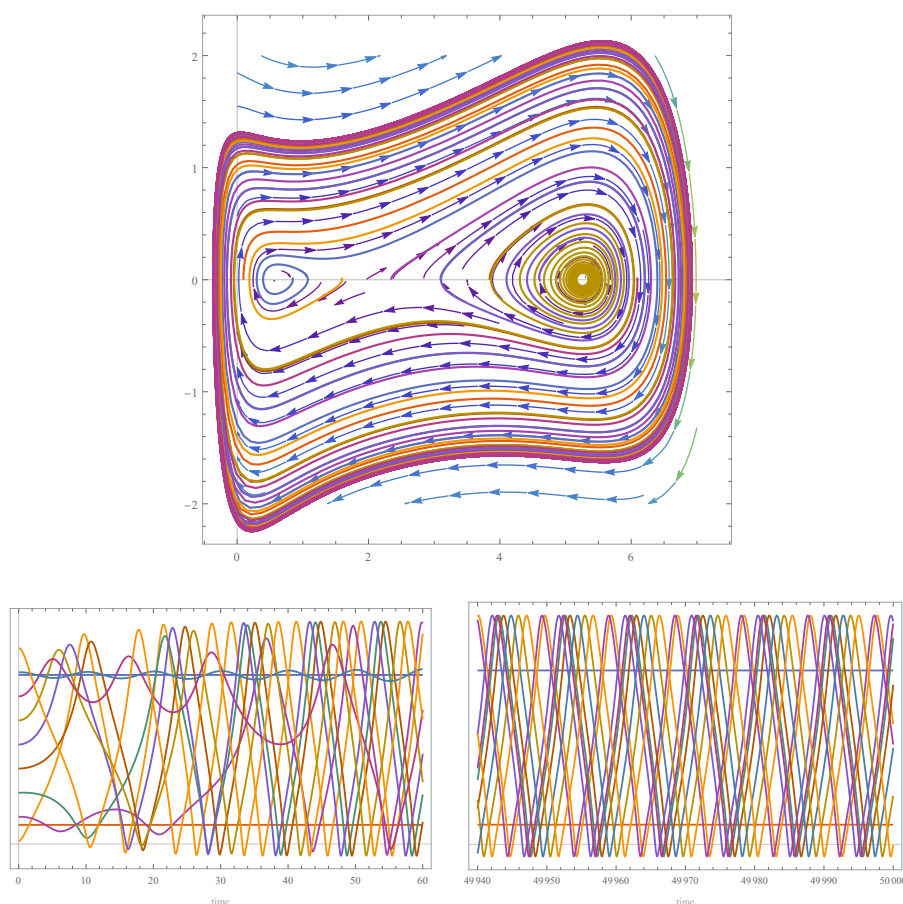
⁸⁷. The same is true for initial conditions much higher than the high solution which then converge to larger



In this case, an economy starting from initial conditions close to the low equilibrium is not stuck in this position, but settles on closed trajectories alternating between the low and high levels of activity. On the contrary, the economy starting from initial conditions close to the high equilibrium converges to this position.

- Case 3 $\tilde{\Phi}' < \Phi'(x_H^*) < \Phi'(x_L^*)$

With $\beta = 10$, we have $\tilde{\Phi}' = 0.2$ and $0.2 < \Phi'(x_H^*) = 0.230 < \Phi'(x_L^*) = 0.293$. The two equilibria are both locally unstable. The figures below display the phase diagram and the trajectories of $x(t)$ for t close to $t = 0$ and at long times.



As exemplified in the phase diagram the two equilibria are locally unstable, but in this generalised Lienard-Levison-Smith dynamical system, the trajectories eventually remain in a closed set⁸⁸. The economy has now the possibility to switch from the neighbourhood of an equilibrium to another one converging to periodic orbits⁸⁹.

These qualitative behaviors follow naturally from the nonlinear structure of the model. Cooper and Haltiwanger (1996) already suggested the idea that in a dynamic setting, economies with complementarities could easily generate nonlinear behaviors through changes of regimes, the different regimes representing the multiple equilibria of the static frameworks. The results shown above, admittedly in a different context, directly illustrate this remark. But, the main limitation of this

⁸⁸. On the formal method for localizing limit sets of such dynamical systems, see notably Styrts and Krishchenko 2017

⁸⁹. As we know, these closed periodic solutions are not unique. Each initial condition is associated with an unique path but the analysis says nothing about how these conditions are selected.

type of qualitative approaches remained empirical, with the difficulty of calibrating the different various parameters and threshold related to expectations and, here again, the sufficient levels of strategic complementarities.

The next step would obviously be to introduce adaptive learning⁹⁰ and to make the parameter ω endogenous. However, with this perspective we would quickly leave the initial framework of coordination failure models with strategic complementarities considered in this paper for an approach with interacting heterogeneous agents and agent-based computational economics as suggested in Howitt (2008).

Conclusion

This contribution has tried to give an overview of some main advances on dynamic models with strategic complementarities that emerged between the end of the 80s and the 90s.

Several theoretical, empirical or experimental contributions mobilizing various forms of complementarities have continued to develop until today. But these works have strayed from the original project on coordination failures. Indeed, the latter aimed at developing a specific Keynesian analysis in which wage and price rigidities played no role, particularly in the downturns. Undeniably, from the second half of the 1990s, this view lost its specificity and influence or was diluted within the new developments of macroeconomic analysis⁹¹.

Several reasons may explain this tendency. The refinement of dynamics approaches has tended progressively to blur the differences with other advances, particularly with regard to the literature on indeterminacy, sun-spot equilibria and learning in overlapping generations or growth frameworks for which there was no really impenetrable barrier between contributors.

By the same token, let also mention the role played by the New Keynesians focusing on diverse forms of nominal rigidities and imperfections. Indeed, as emphasized by Ball and Romer,

"models with nominal rigidities and models with coordination failures are often presented as competing paradigms. [We] show that this

90. For a synthesis of the issue of multiplicity and learning in macroeconomics in the early 2000s see e.g. Evans, Honkapoja and Romer 1998 and Evans and Honkapoja, 2001.

91. Notice that the same observation can be drawn for the literature on nonlinear dynamics and endogenous cycles.

*view is incorrect. We take a step toward unifying the foundations of Keynesian economics by showing that the two sets of ideas are highly complementary. Nominal rigidity arises from a failure to coordinate price changes. This failure has the essential features of coordination failures in previous models. Flexibility in one firm's price increases the incentives for other firms to make their prices flexible. This strategic complementarity leads to multiple equilibria in the degree of nominal rigidity."*⁹²

All these approaches contribute to the analysis of both nominal rigidities and complementarities in the same framework. The range of the Keynesian phenomena explained is broadened, but to the detriment of the initial analytical framework.

In addition, the dynamic models were rather sophisticated, the approach fragmented and there did not exist a dynamic canonical model, which contrasts with the stochastic inter-temporal framework implemented at the time by the RBC program, then adopted by the New-Synthesis. A central feature in the study of macroeconomic fluctuations is then the emphasis put on the quantitative dimension of the business cycles. This quantitative shift can also be observed in the NBER group specially dedicated to the understanding of the macroeconomic implications of complementarities⁹³. Indeed, Cooper mentions that the group continued to explore additional domains in which macroeconomic complementarities were present, including financial instability, debt rollovers, exchange rate and debt crises and informational complementarities including diverse forms of social learning. But for the most part, the group has intensified quantitative researches linking these diverse topics to aggregate, microeconomic and then experimental data.

From this perspective, Cooper and Haltiwanger made a rather pessimistic conclusion regarding the empirical significance of dynamic models with strategic complementarities. The main characteristics of the aggregated time series agreed with the behavior of the complementarity models. But, the difficulty was precisely that many simpler and more tractable dynamic models without complementarities were equally consistent with these data⁹⁴. Notice that these approaches nevertheless continue to differ on the criterion of their policy implications since

92. Ball and Romer, 1991, p.

93. This group directed by Cooper was part of the larger NBER program on economic fluctuations and growth. Major contributions on strategic complementarities were developed in this context. Accordingly, as mentioned in the NBER documentation, the program "featured many spirited scientific discussions but never debates between schools of thought".

94. Cooper and Haltiwanger, 1995 p.193.

there are real gains to coordination in the different models with complementarities⁹⁵.

At the same time, the research agenda on coordination mechanisms with disequilibrium and adaptive dynamics has also evolved a great deal, shifting from conventional models with strategic complementarities to the analysis of social learning, heterogenous interacting agents and agent-based computational economics⁹⁶.

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95. As Cooper put it, "the multiplicity of equilibria allows for a positive coordinating role for the government. In fact, this need not be an active role in the sense of spending and taxation. Instead, the government can provide confidence and thus influence the choice of an equilibrium in the event of multiplicity. At the same time, the presence of the government may itself provide an additional source of instability.", Cooper, 1998, p.227.

96. As suggested for instance by Howitt in his "plea for a more diverse ecology of approaches to macroeconomic theory, one that finds room for agent-based computational economics as well as for more conventional equilibrium theories." (Howitt 2012, p.12

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Annex : Local stability analysis

The jacobian matrix of the system

$$\dot{x}(t) = y(t)$$

$$\dot{y}(t) = -f(x(t), y(t))y(t) - g(x(t))$$

evaluated at a stationary point x_j^* for $j=L,M,H$ is

$$\begin{pmatrix} 0 & 1 \\ -g'(x_j^*) & -f(x_j^*, 0) \end{pmatrix}$$

The determinant of this jacobian matrix writes

$$Det^* = g'(x_j^*) = \frac{4a(1-\omega)(1-\Phi'(x_j^*))}{b\tau^2\omega\Phi'(x_j^*)}$$

The trace evaluated at a stationary point is

$$Tr^* = \frac{4}{\tau} \left(1 - \frac{1}{b\tau\omega\Phi'(x_j^*)}\right)$$

Consequently, the following proofs are directly obtained.

Proof 1

i) With a S shaped Φ and the strategic complementarity assumption we know that $0 < \Phi'(x_j^*) < 1$, for $x_{j=L,H}$ and $\Phi'(x_j^*) > 1$ for $j=M$.

ii) Let consider the determinant of the jacobian matrix of the system evaluated at a stationary point $(x_j^*, 0)$, $x_{j=L,H,M}$. It is assumed that $a > 0, b > 0, \tau > 0$, $0 < \omega < 1$ and $\Phi' > 0$. Then, using (i) we obtain directly that $(x_M^*, 0)$ is always a saddle point and $(x_j^*, 0)$, $j=L,M$ is a node or a focus. ■

Proof 2

A stationary point $(x_j^*, 0)$, $x_{j=L,H}$ is a focus if and only if the jacobian evaluated at this point admits two complex roots. That is if and only if

$$\frac{16}{\tau^2} \left(1 - \frac{1}{b\tau\omega\Phi'(x_j^*)}\right)^2 - \frac{16a(1-\omega)(1-\Phi'(x_j^*))}{b\tau^2\omega\Phi'(x_j^*)} < 0$$

With $b > 0$, $0 < \omega < 1$, $\tau > 0$ and $0 < \Phi'(x_j^*) < 1$, let define

$$\tilde{a} = \frac{b\omega}{(1-\omega)} \frac{\Phi'(x_j^*)}{(1-\Phi')} \left(1 - \frac{1}{\omega b \tau \Phi'(x_j^*)}\right)^2$$

Then, the jacobian evaluated at $(x_j^*, 0)$, $x_{j=L,H}$ admits two complex roots if and only if

$$a > \tilde{a}$$

$(x_j^*, 0)$, $x_{j=L,H}$ is a focus and the dynamics is cyclical. ■

Proof 3

i) Local stability (instability) in a neighborhood of a stationary point $(x_j^*, 0)$, $x_{j=L,H}$ is determined by the sign of the trace of the jacobian matrix evaluated at this stationary point. That is by

$$Sign[Tr^*] = Sign\left[\frac{4}{\tau} \left(1 - \frac{1}{b\tau\omega\Phi'(x_j^*)}\right)\right]$$

where $b > 0$, $0 < \omega < 1$, $\tau > 0$ and $\Phi'(x_j^*) > 0$. Thus, it exists a critical value $\tilde{\Phi}' = \frac{1}{b\omega\tau} > 0$ such that $\Phi'(x_j^*) \leq \tilde{\Phi}' \iff Tr^* \leq 0$ and the stationary solution $(x_j^*, 0)$ is locally stable (unstable).

ii) Let us now recall that a stationary point $(x_j^*, 0)$, $x_{j=L,H}$ verifies $\Phi'(x_j^*) < 1$. Consequently, this point can meet the local instability criterion only if $\frac{1}{b\omega\tau} < 1$, that is if $b > \tilde{b}$, where $\tilde{b} = \frac{1}{\omega\tau}$. ■