



Swell refraction by oceanic currents and multiscale stochastic closures

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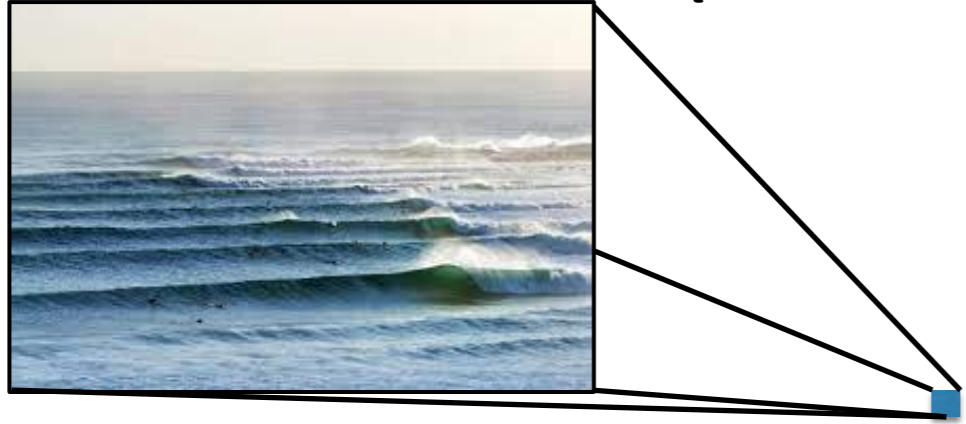
Content

1. Wave refraction due to turbulence
2. Non-Markovian closure
with current multi-scale in space
& in time
3. Markovian closure
with current multi-scale in space
& **white** in time
- Simple cases and analytic solutions -

1.

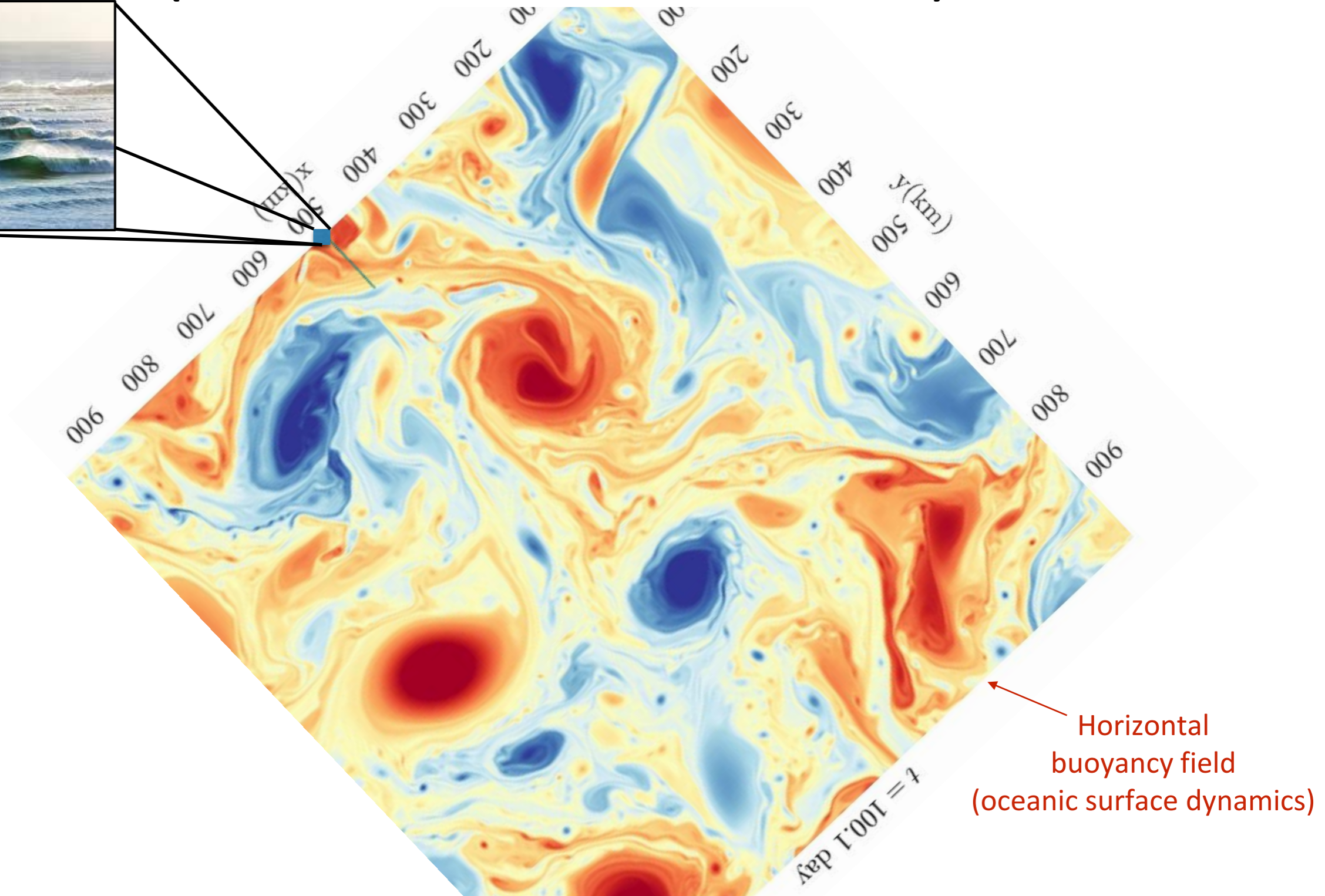
Wave refraction due to turbulence

Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)



Horizontal
buoyancy field
(oceanic surface dynamics)

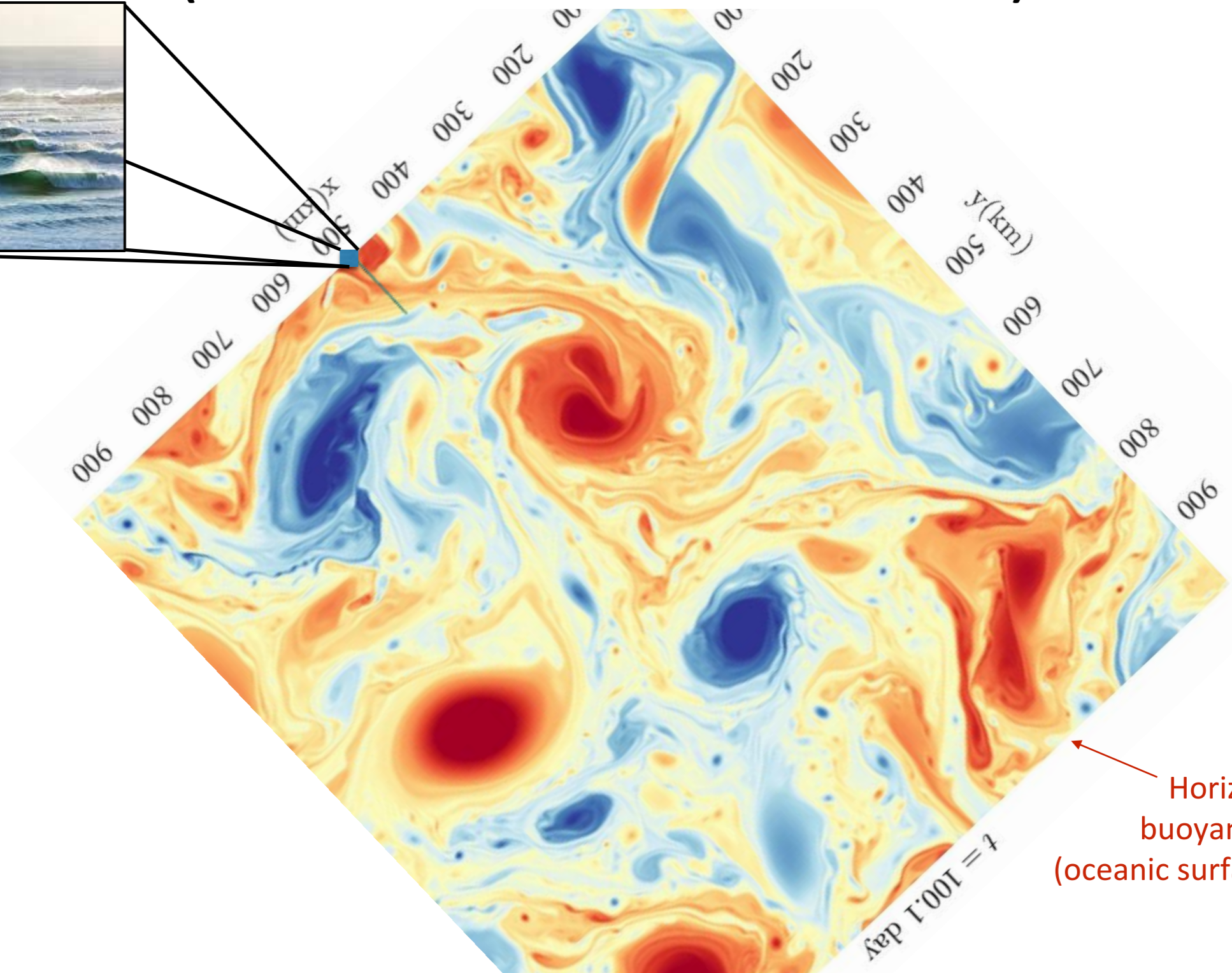
Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)



Numerical example of wave (swell) traveling inside turbulence (oceanic surface currents)

Wave:

$$ae^{\frac{i}{\epsilon}\phi(\epsilon t, \epsilon x)}$$



Current
velocity:
 v

Wave:
 $a e^{i\phi(\epsilon t, \epsilon x)}$

$k = \nabla\phi$

Dispersion ratio

At the first order in steepness ($=\|\nabla\phi\|a$)

$$\frac{\partial\phi}{\partial t} + v \cdot \underbrace{\nabla\phi}_{=k} = - \underbrace{\sqrt{g\|\nabla\phi\|}}_{=\omega_0(k)}$$

$$- \frac{\partial\phi}{\partial t} = \omega(k) = +v \cdot k + \omega_0(k)$$

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Wave phase transport
with a current velocity v

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Dispersion ratio

At the first order in steepness ($=\|\nabla\phi\|a$)

Wave phase transport
with a current velocity v

$$\frac{\partial\phi}{\partial t} + v \cdot \underbrace{\nabla\phi}_{=k} =$$

Frequency without currents

$$- \underbrace{\sqrt{g\|\nabla\phi\|}}_{=\omega_0(k)}$$

$$- \frac{\partial\phi}{\partial t} = \omega(k) = + v \cdot k + \omega_0(k)$$

Total group velocity u_g :

$$v + C_g^0$$

Wave:

$$ae^{i\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Method of characteristic (ray tracing)

$$\dot{X}_r = \nabla_k \omega = C_g^0 + v \quad | \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\dot{k} = -\nabla_x \omega = -\nabla v^T k \quad | \quad \text{Wave-vector}$$

Conservation of action

$$\dot{A} = -\nabla \cdot u_g A \quad | \quad A = \frac{a^2}{\omega_0} \quad | \quad \text{Amplitude}$$

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Refraction & contraction/dilatation

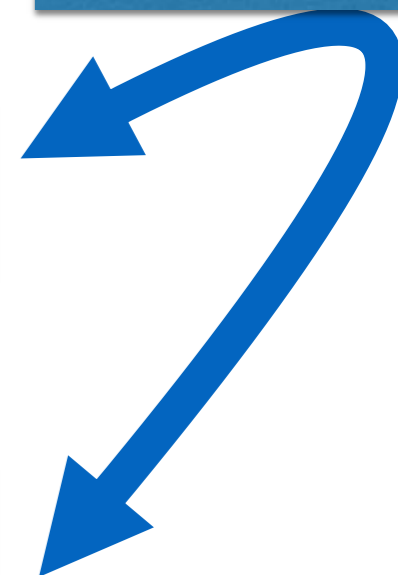
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Fonction of x

Conservation of action

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Non-linear coupling



Total group velocity u_g :

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Wave:

$$ae^{i\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Method of characteristic

(ray tracing)

Fonction of k

Main contribution

Non-linear coupling

$$\dot{X}_r = \nabla_k \omega = C_g^0 + v \quad | \quad \text{Ray}$$

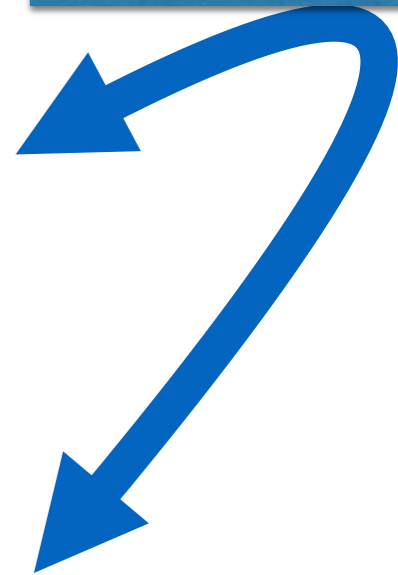
Refraction & contraction/dilatation

$$\dot{k} = -\nabla_x \omega = -\nabla v^T k \quad | \quad \text{Wave-vector}$$

Fonction of x

Conservation of action

$$\dot{A} = -\nabla \cdot u_g A \quad | \quad A = \frac{a^2}{\omega_0} \quad | \quad \text{Amplitude}$$



Method of characteristic (ray tracing)

Total group velocity u_g :
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Wave:
 $a e^{i\phi(\epsilon t, \epsilon x)}$

Group velocity without current
 $C_g^0 = \nabla_k \omega_0$

$\nabla = \nabla_x$

Fonction of k

Main contribution

$$\dot{X}_r = \nabla_k \omega = C_g^0 + v \quad \text{Ray}$$

Change orientation of C_g^0

Refraction & contraction/dilatation

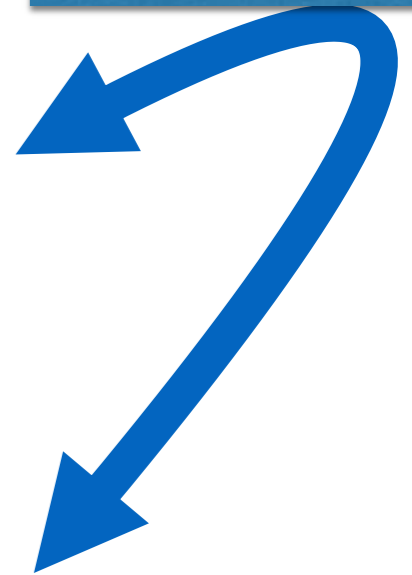
$$\dot{k} = -\nabla_x \omega = -\nabla v^T k \quad \text{Wave-vector}$$

Fonction of x

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$$\dot{A} = -\nabla \cdot u_g A \quad \left| \quad A = \frac{a^2}{\omega_0} \quad \right| \quad \text{Amplitude}$$

Non-linear coupling



Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

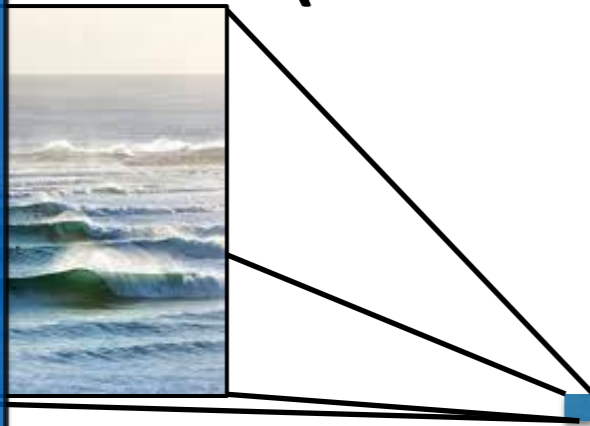
$$v'$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi(\epsilon t, \epsilon x)}$$

Group velocity
without current

$$C_g^0 = \nabla_k \omega_0$$



Numerical example of wave (swell) traveling inside **turbulence** (oceanic surface currents)

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v'$$

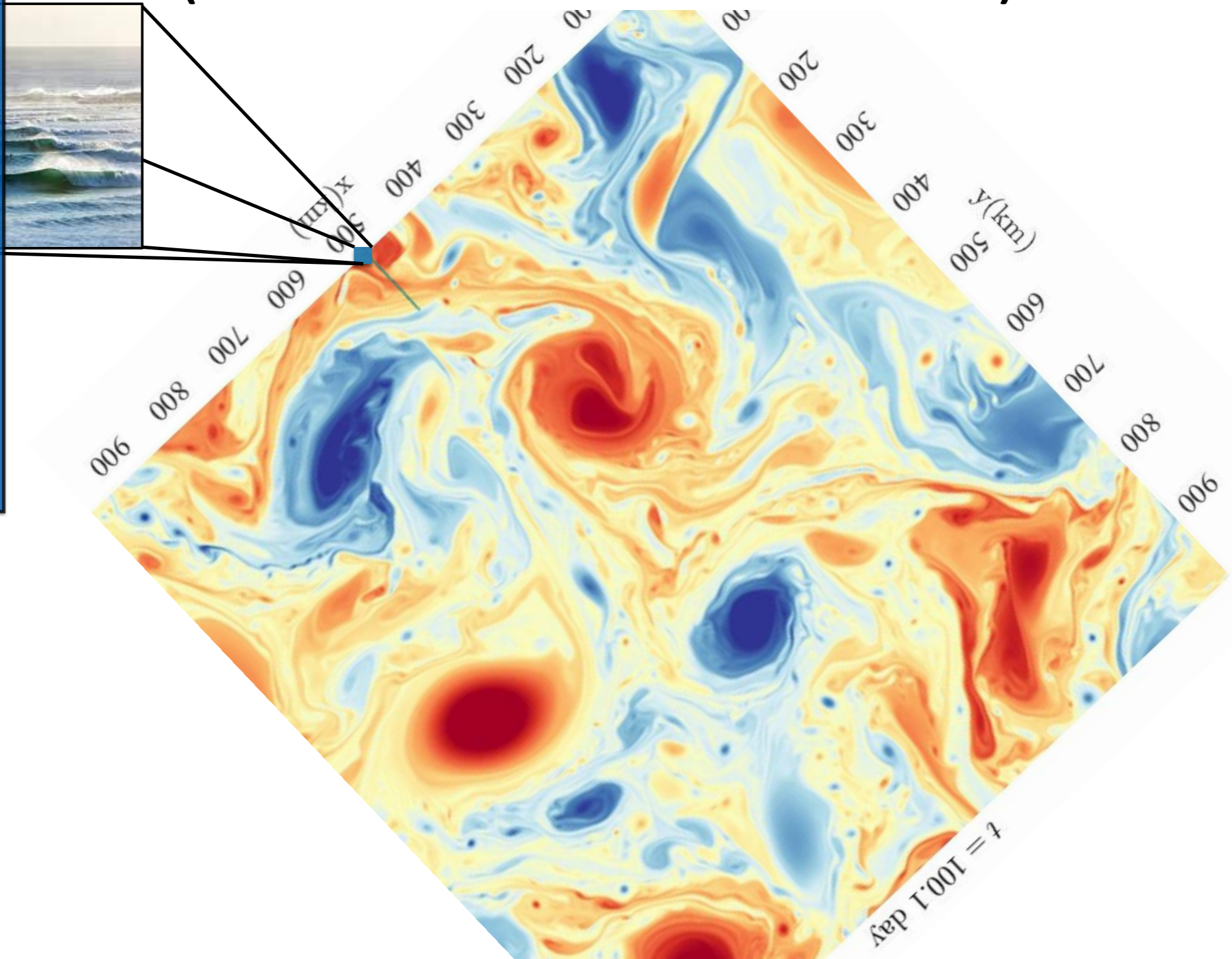
Wave:

$$ae^{i\frac{\omega}{\epsilon}}\phi(\epsilon t, \epsilon x)$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

Numerical example of wave (swell) traveling inside **turbulence** (oceanic surface currents)



Large scale group velocity:

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Small scale group velocity:

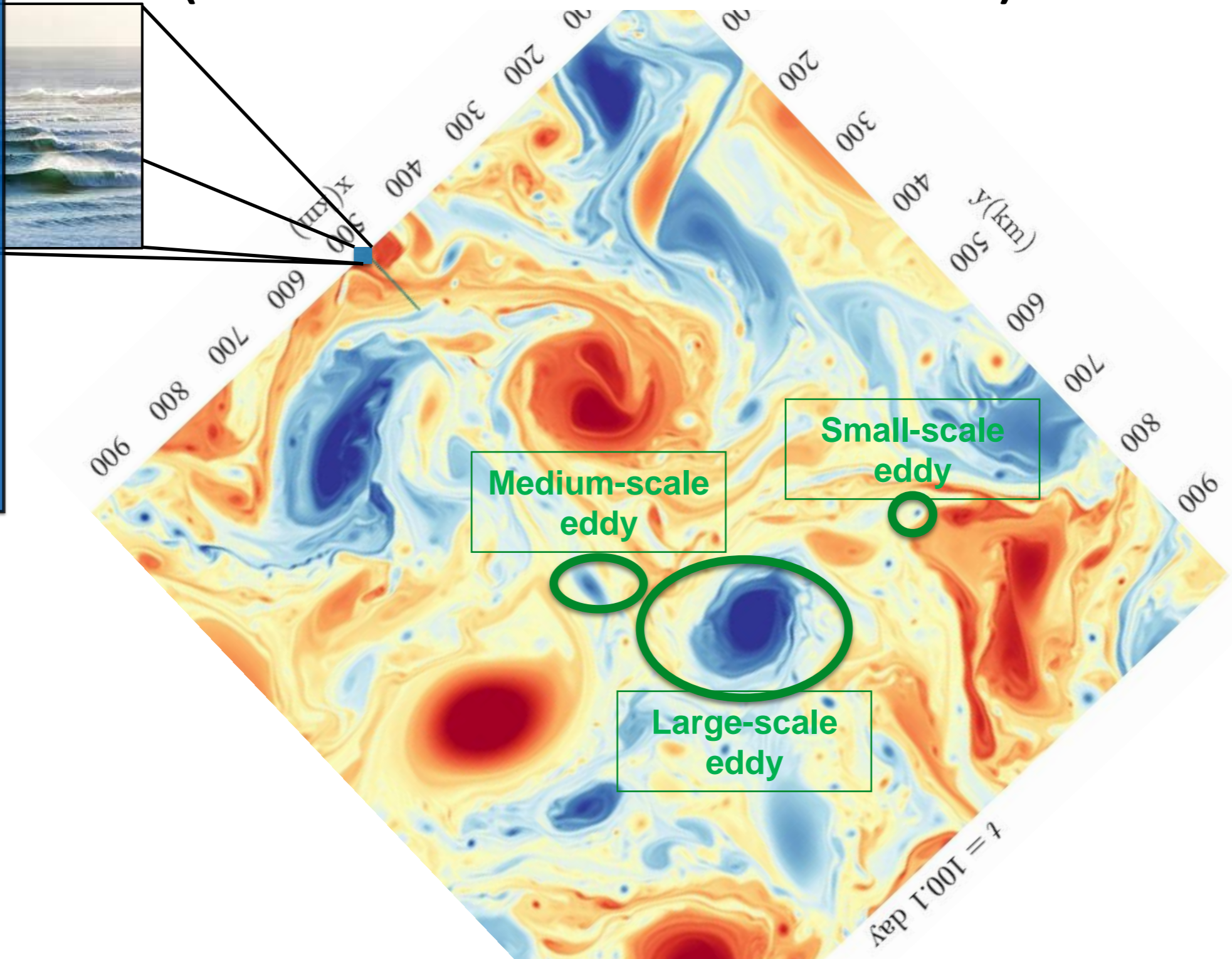
$$v'$$

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Small scale group velocity:

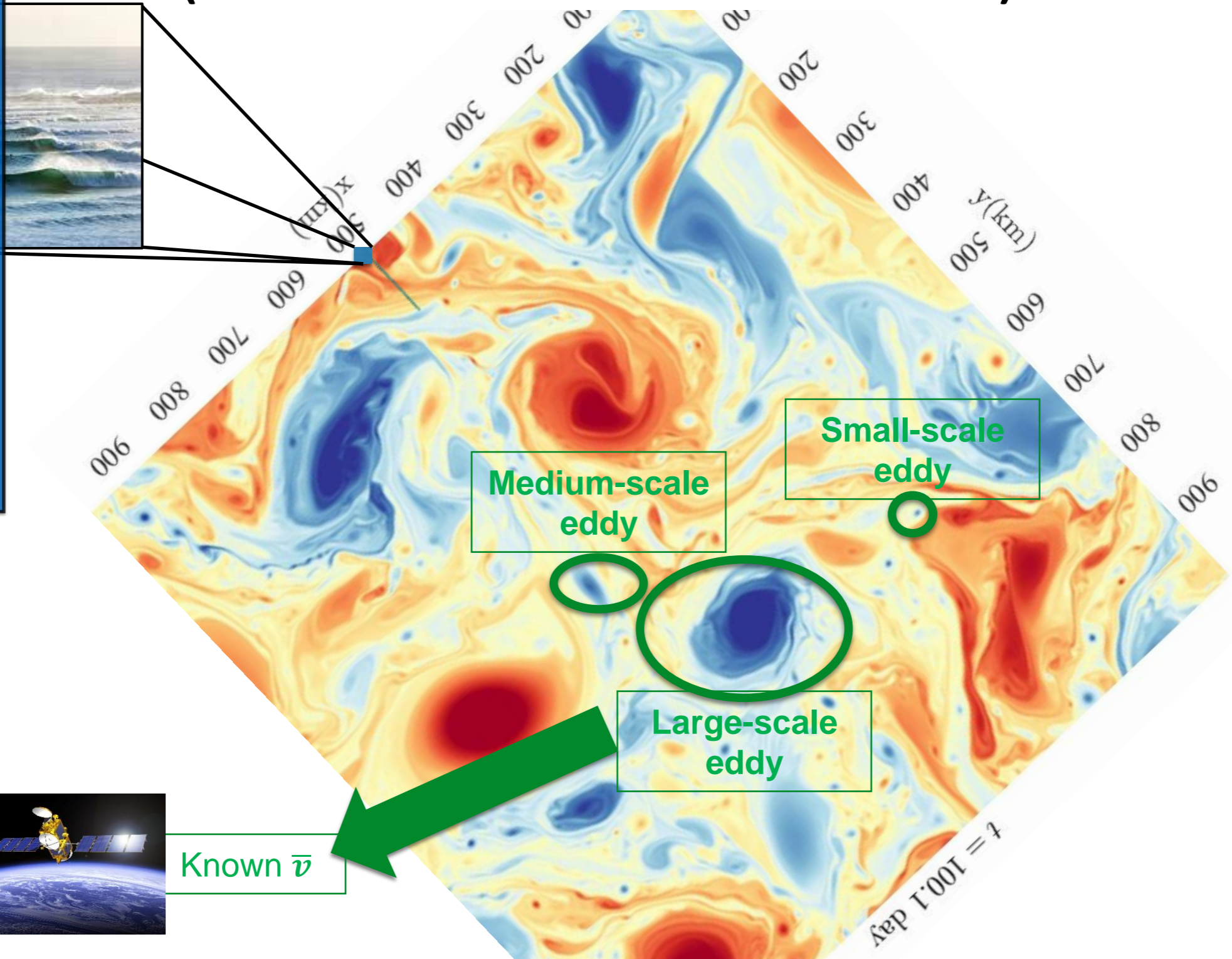
$$v'$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$



Known \bar{v}

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v'$$

Wave:

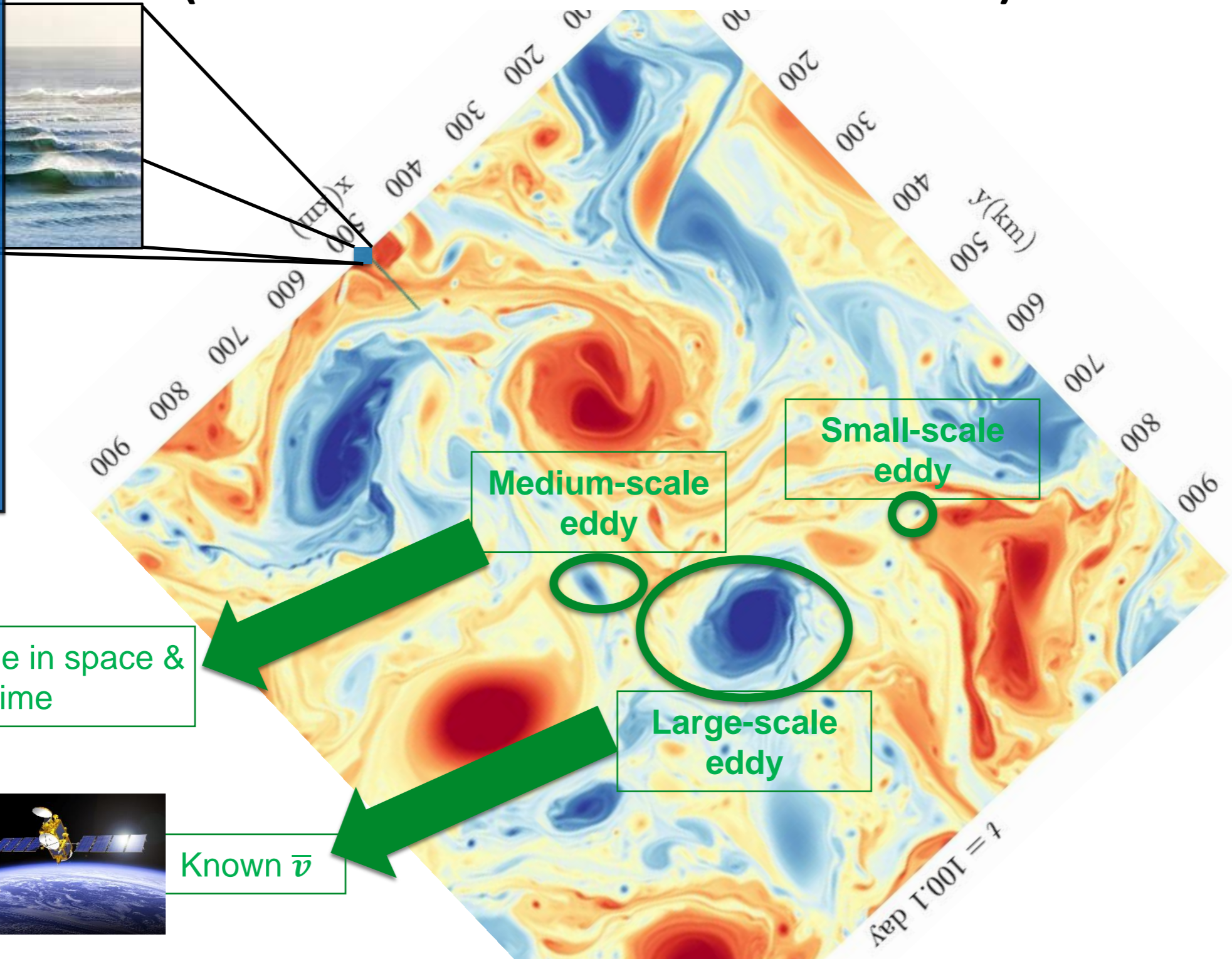
$$ae^{i\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$



Numerical example of wave (swell) traveling inside **turbulence** (oceanic surface currents)



3. Multi-scale in space & in time



Known \bar{v}

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

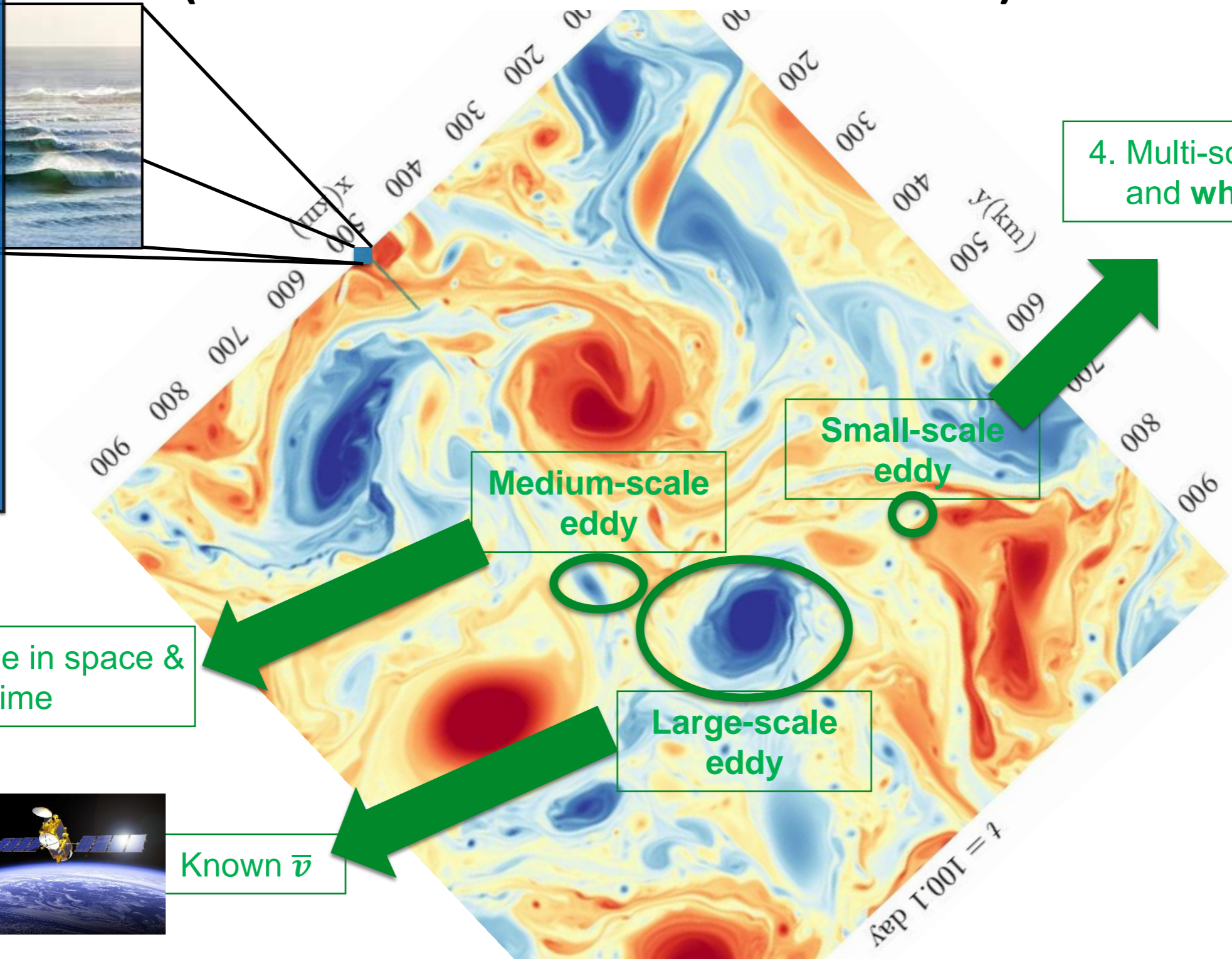
$$v'$$

Wave:

$$ae^{i\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$



4. Multi-scale in space and **white** in time

3. Multi-scale in space & in time



Known \bar{v}

2.

Non-Markovian closure
with current multi-scale in
space & in time

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v'$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' :

Calibration

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

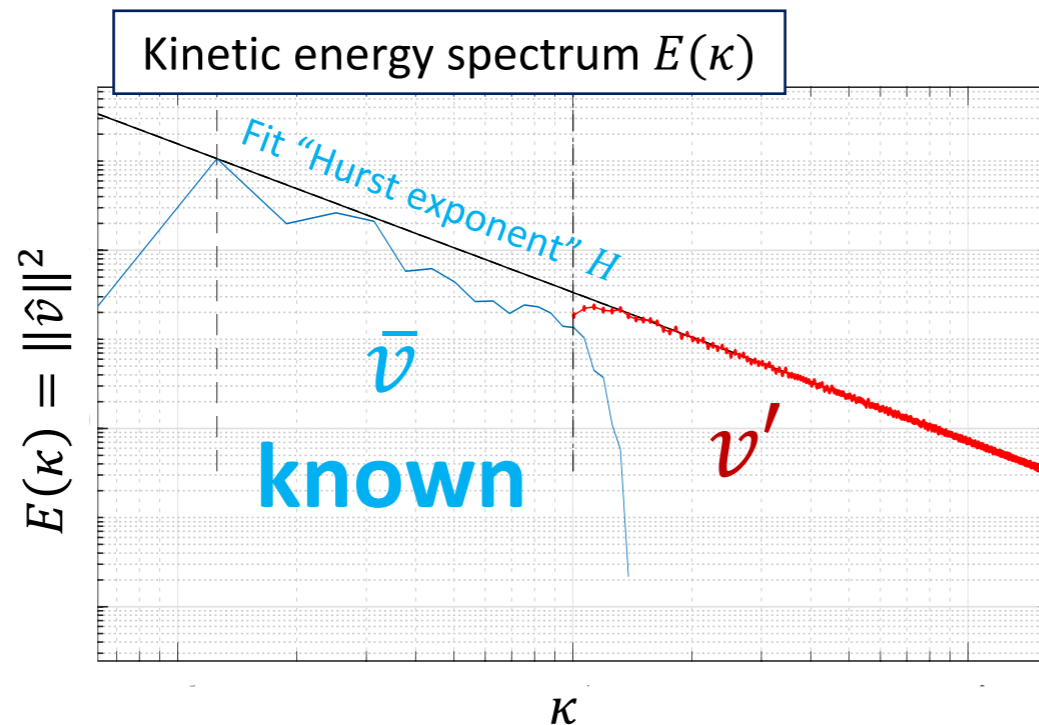
Small scale
group velocity:

$$v'$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' : Calibration



$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

v'

Wave:

$$ae^{i\frac{z}{\epsilon}\phi}$$

Time-correlated model for v' :

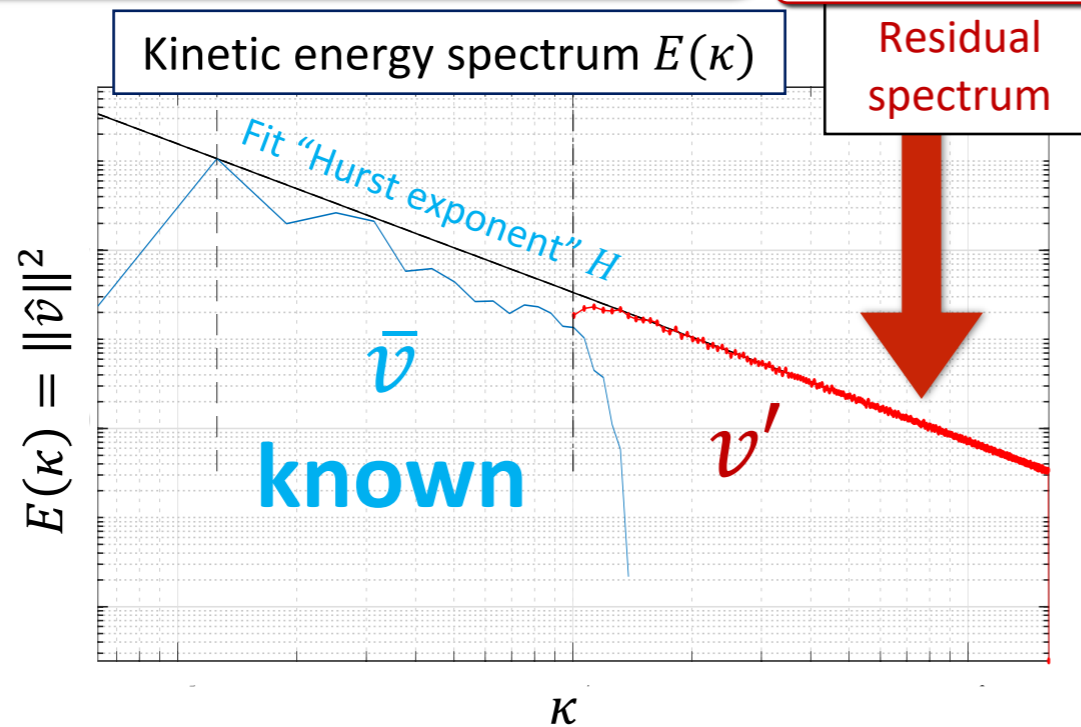
Calibration

Physical scale symmetry

$$\text{On-line fit : } A \|k\|^{-H}$$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$



$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:
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Small scale group velocity:
 v'

Wave:
 $ae^{i\frac{z}{\epsilon}\phi}$

Time-correlated model for v' :

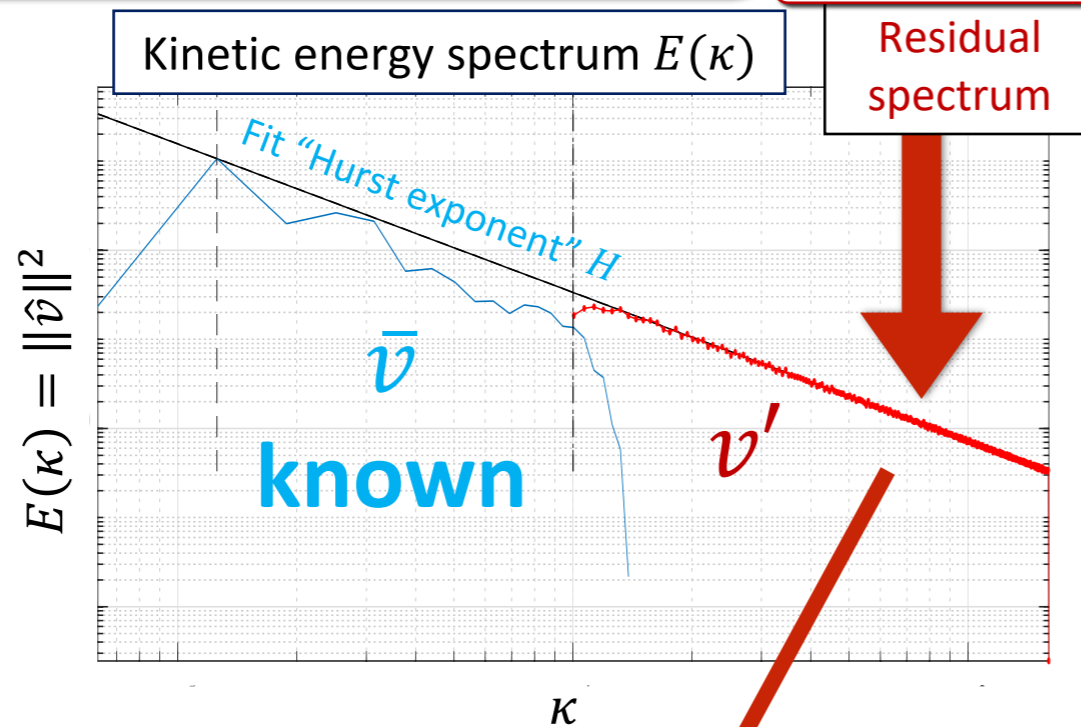
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Time-correlated model for v' :

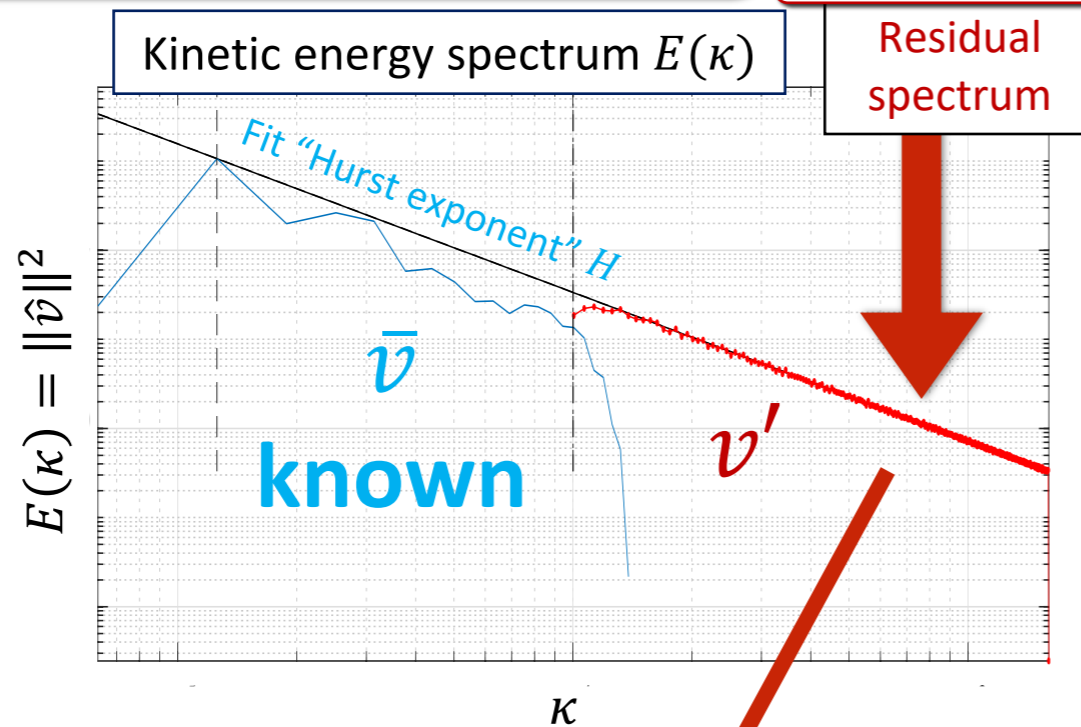
Calibration

Physical scale symmetry

$$\text{On-line fit : } A \|k\|^{-H}$$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$



Filter h

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

v'

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

Time-correlated model for v' :

Calibration

Physical scale symmetry

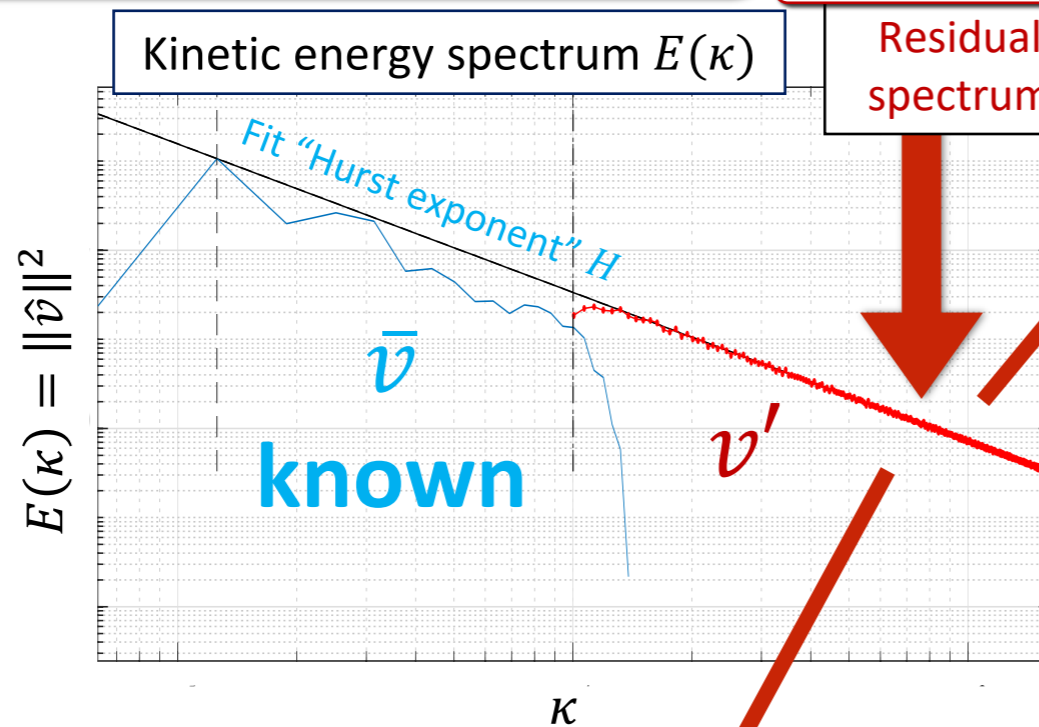
On-line fit : $A \|k\|^{-H}$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$

Current correlation time

$$\tau(\kappa) = \tau_{cur}(\kappa) = \frac{1/\kappa}{v_k} = \frac{1/\kappa}{\sqrt{\kappa E(\kappa)}}$$



Kinetic energy spectrum $E(\kappa)$

Residual spectrum

Fit "Hurst exponent" H

\bar{v}

known

v'

κ

Filter h

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:
 $\bar{v} + C_g^0$

Small scale group velocity:
 v'

Wave:
 $ae^{i\epsilon\phi}$

Time-correlated model for v' :

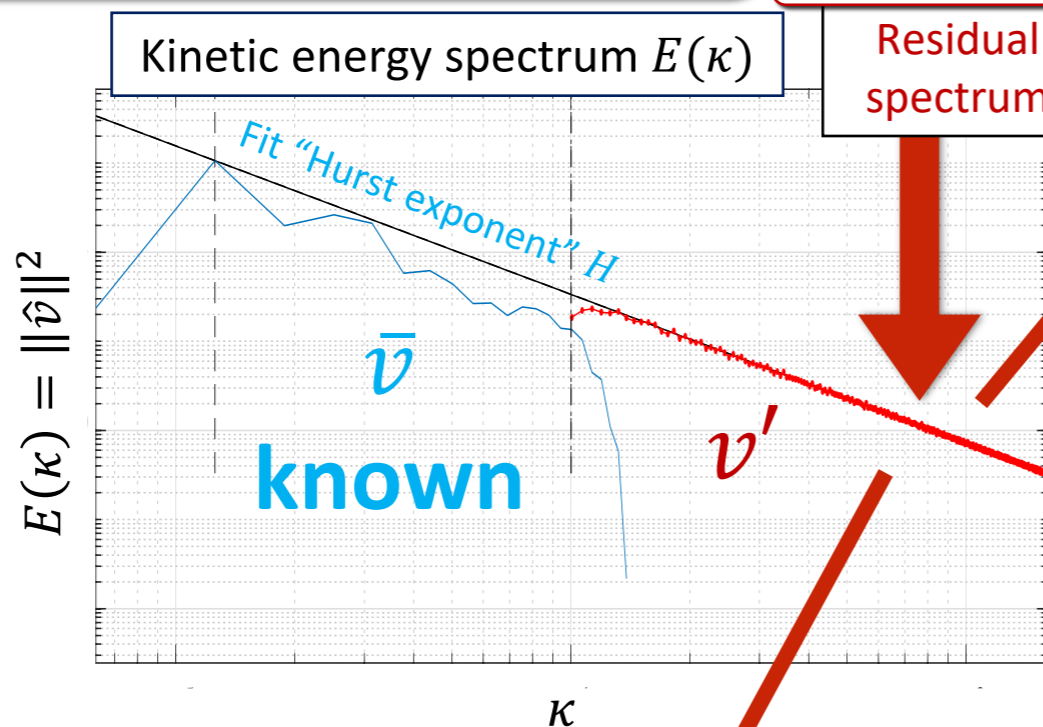
Calibration

Physical scale symmetry

On-line fit : $A \|k\|^{-H}$

(Similar to fractional Brownian motion in space)

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Current correlation time

$$\tau(k) = \tau_{cur}(k) = \frac{1/k}{v_k} = \frac{1/k}{\sqrt{k E(k)}}$$

Multiscale noise B_t^c

From $B_t^c(x) = \iint \widehat{B}_t^c(k) e^{ik \cdot x} dx$,
 Ornstein–Uhlenbeck process in time,
 with correlation time $\tau(k)$ at the spatial scale k :

$$d\widehat{B}_t^c(k) = -\frac{1}{\tau(k)} \widehat{B}_t^c(k) dt + \sqrt{\frac{2}{\tau(k)}} (\widehat{dB}_t)(k)$$

and dB_t/dt "spatiotemporal white noise"
 (cylindrical Wiener Process)

Filter h

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Time-correlated model for v' :

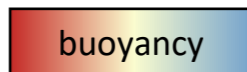
Method of characteristic

$$v = \bar{v} + v'$$

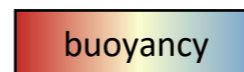
$$v = \bar{v}$$

$$v = \bar{v} + v'_{\text{cor}}$$

wave group amplitude



wave group amplitude



wave group amplitude



Deterministic
reference:

wave groups in
512 x 512 flow

Deterministic
benchmark:

wave groups in
smoothed flow \bar{v} (**16 x 16**)

Our random
model

wave groups in
smoothed flow \bar{v} (**16 x 16**)

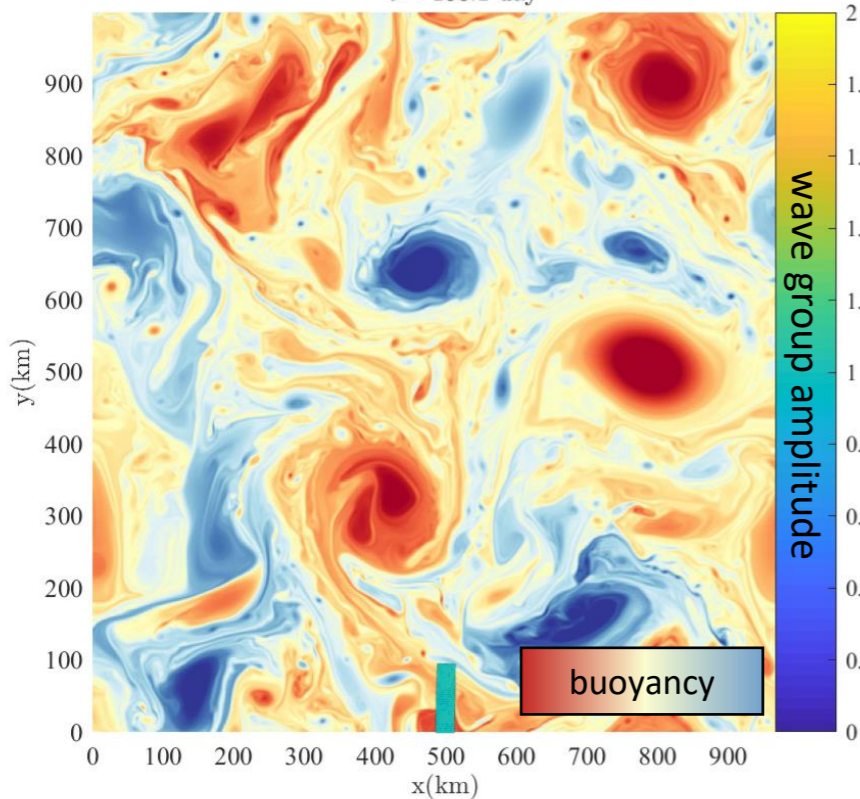
+ time-correlated v'

Time-correlated model for v' :

Method of characteristic

$$v = \bar{v} + v'$$

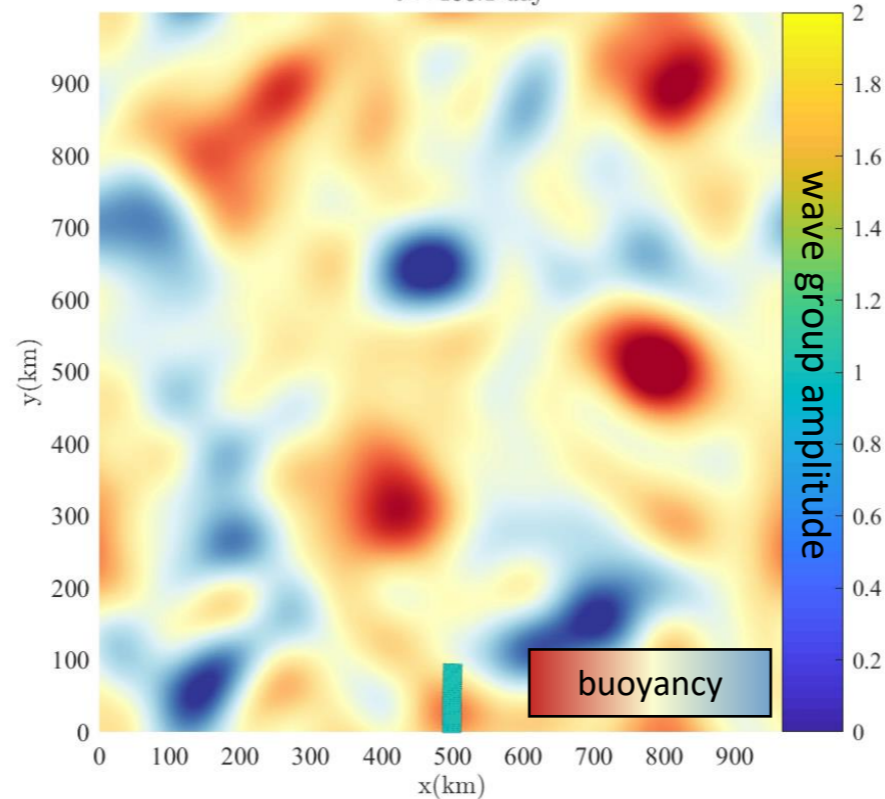
$t = 100.1$ day



Deterministic
reference:
wave groups in
512 x 512 flow

$$v = \bar{v}$$

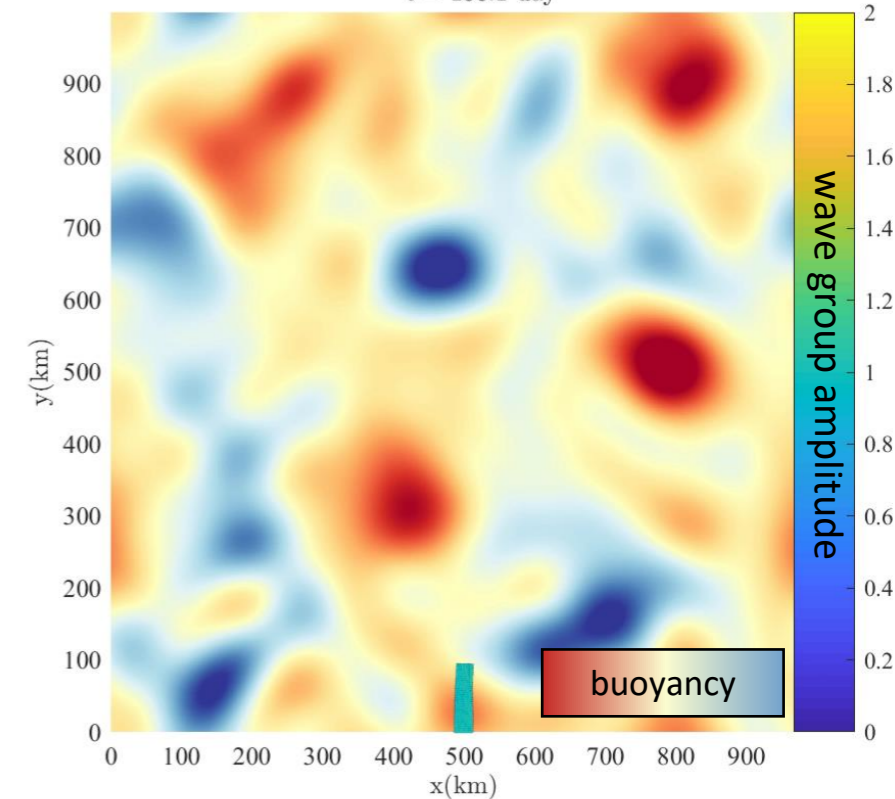
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Deterministic
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wave groups in
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$$v = \bar{v} + v'_{\text{cor}}$$

$t = 100.1$ day



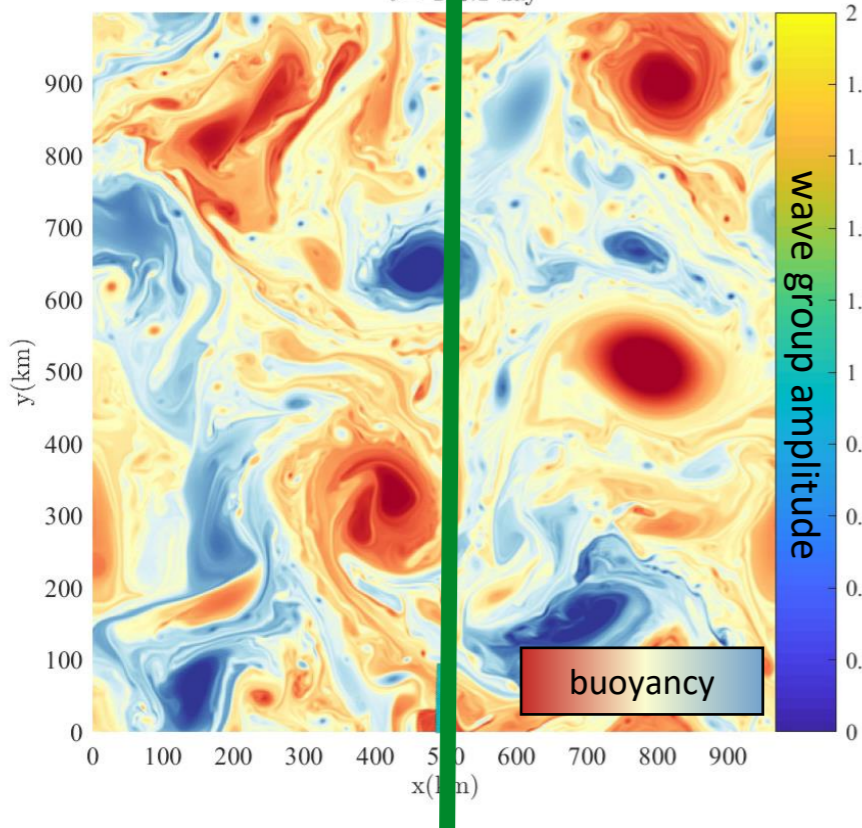
Our random
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wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v' :

Method of characteristic

$$v = \bar{v} + v'$$

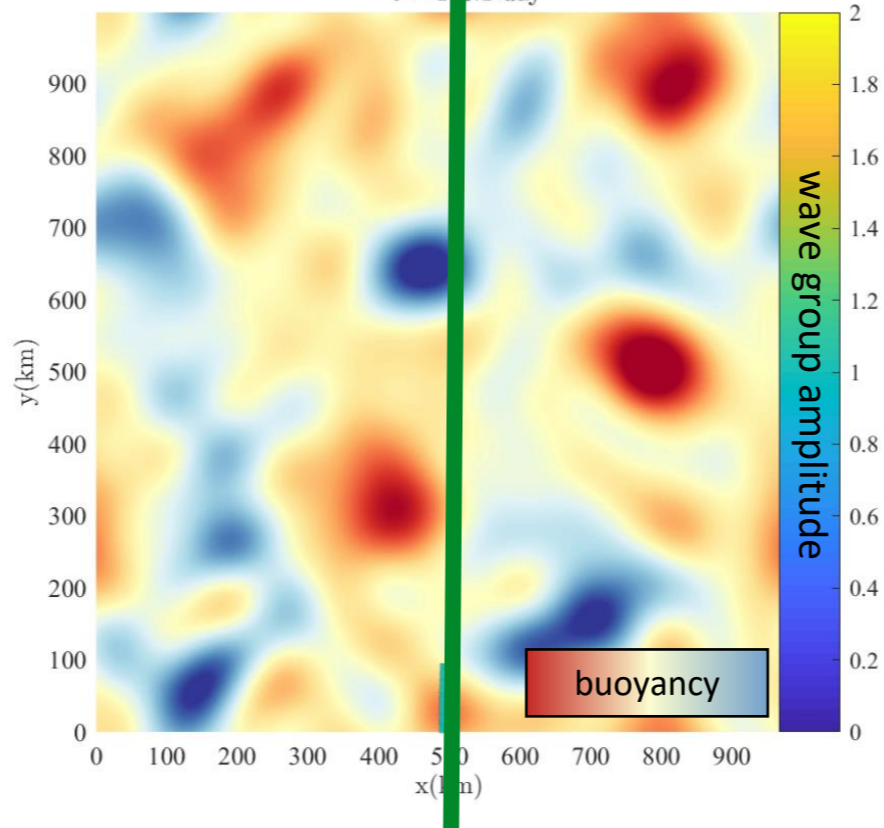
$t = 4.5$ day



Deterministic
reference:
wave groups in
512 x 512 flow

$$v = \bar{v}$$

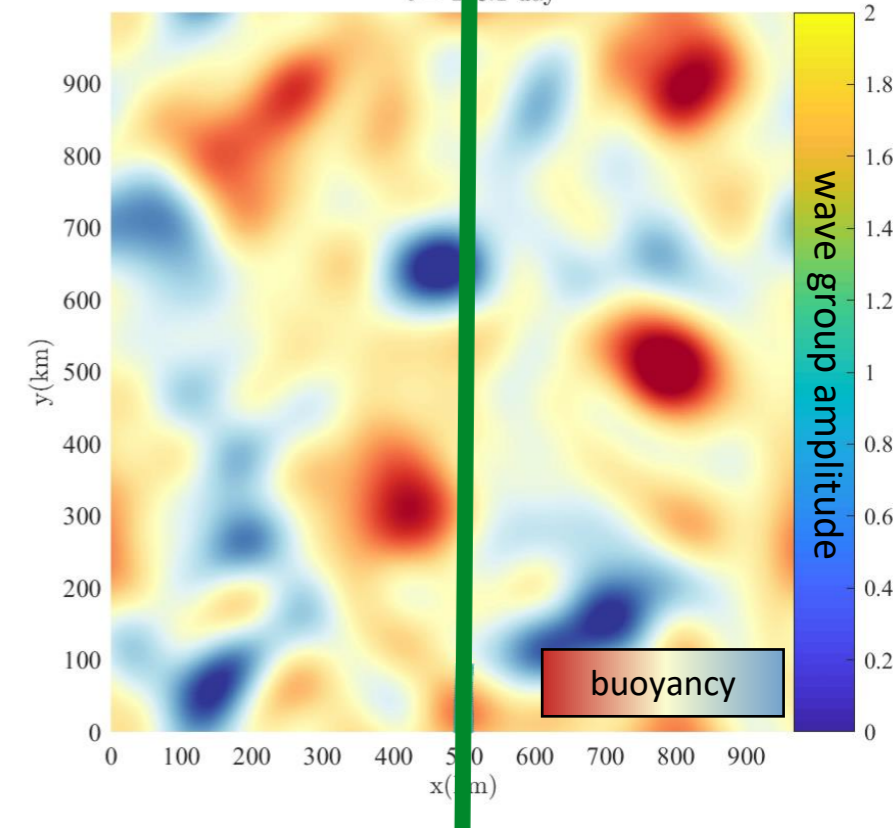
$t = 4.5$ day



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

$$v = \bar{v} + v'_{\text{cor}}$$

$t = 4.5$ day



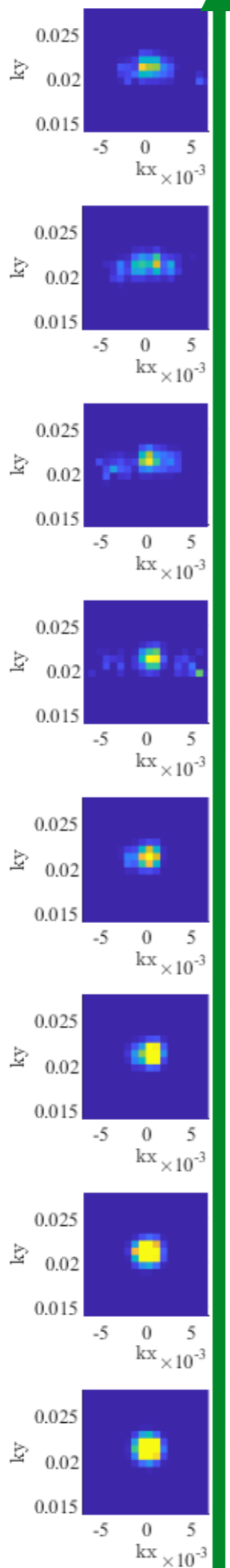
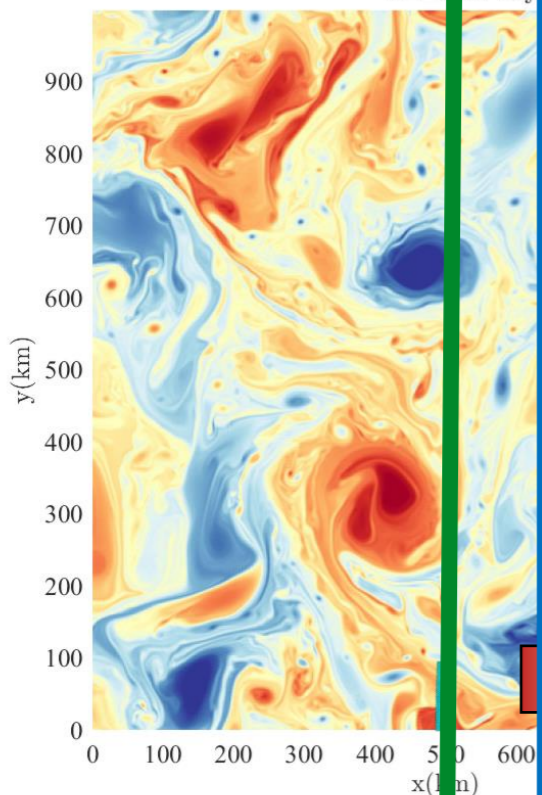
Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v' :

method of characteristics

$$v = \bar{v} + v'$$

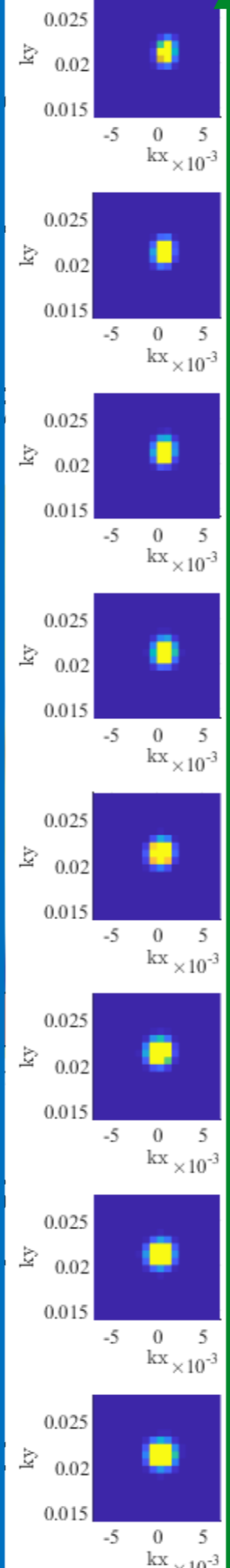
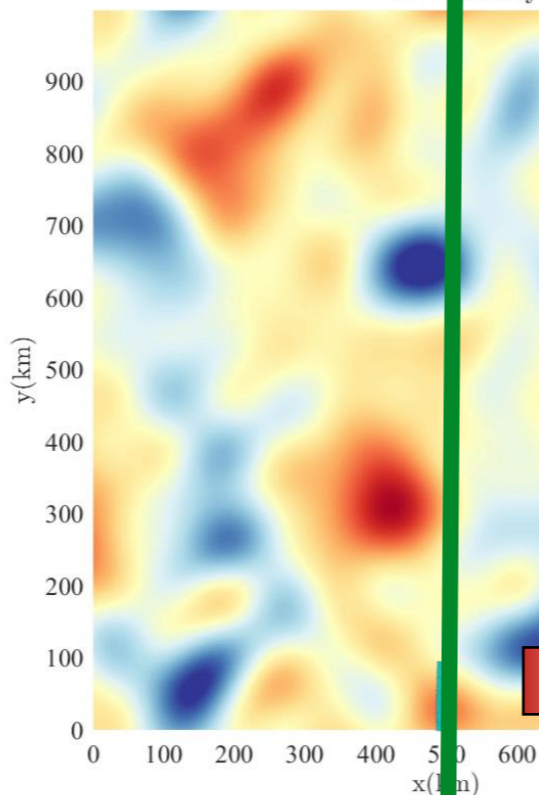
$t = 4.1$ day



Deterministic
reference
wave group
512 x 512 ft

$$v = \bar{v}$$

$t = 4.1$ day

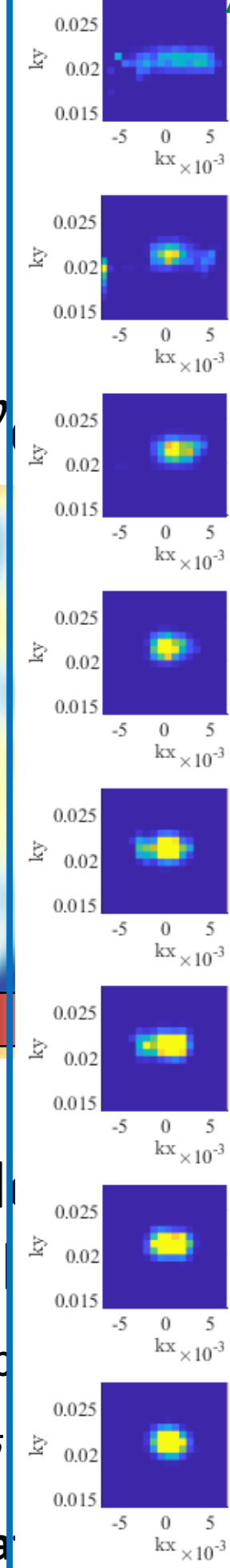
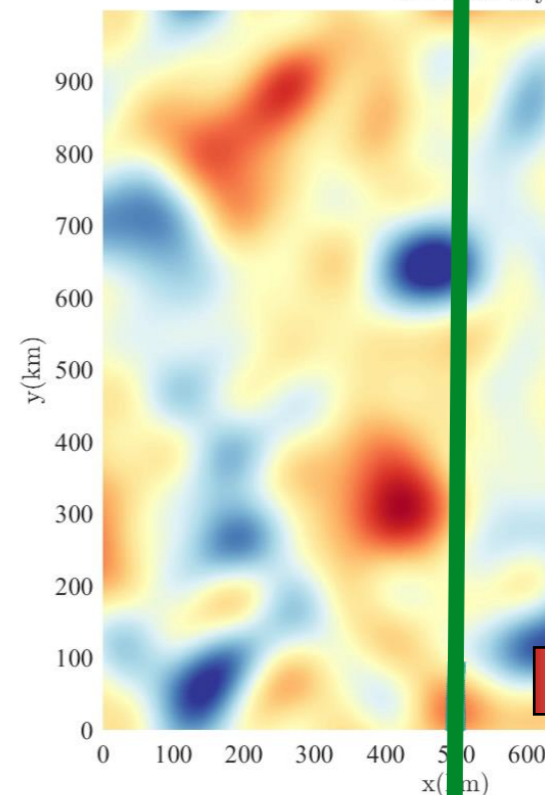


Deterministic
benchmark
wave groups
smoothed flow \bar{v}

10

$$v = \bar{v} + v'$$

$t = 4.1$ day



Our random
mode
wave group
smoothed flow \bar{v}
+ time-correlation

Time

Uncorrelated

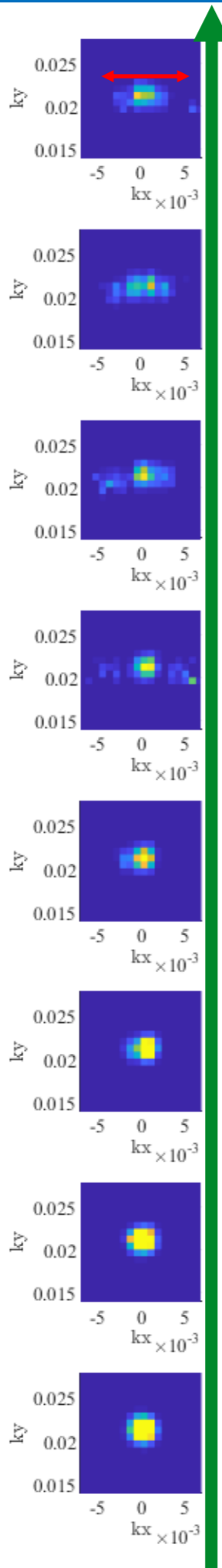
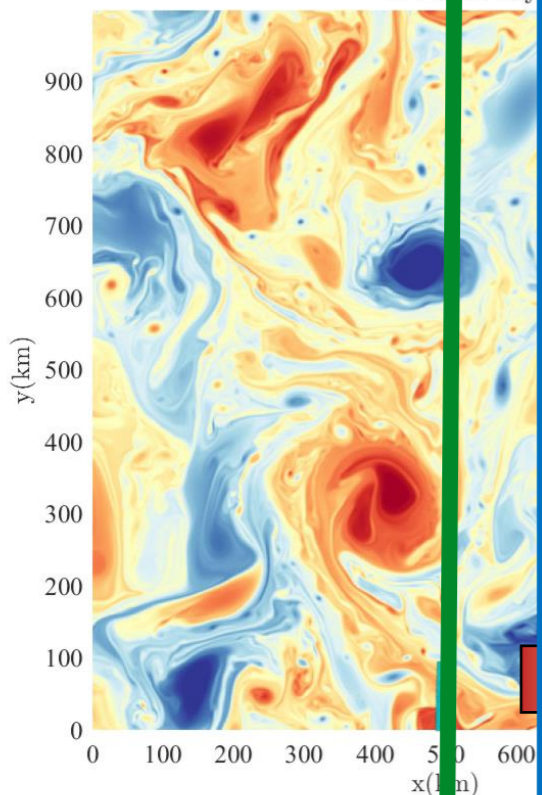
Model for v' :

Method of characteristics

Stochastic

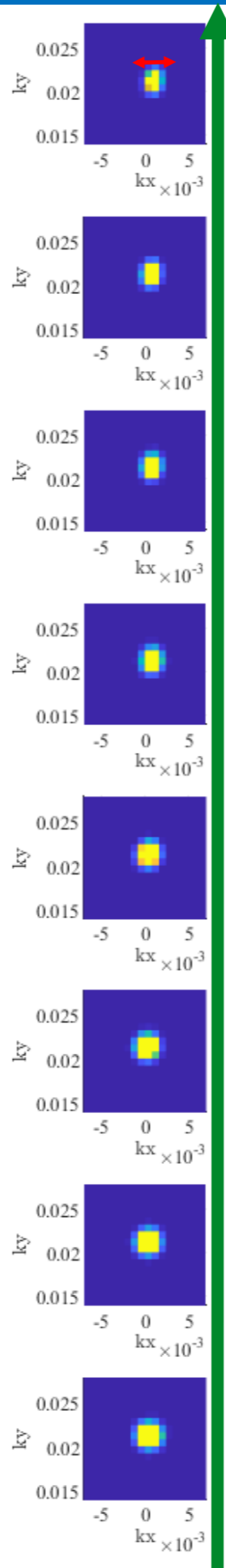
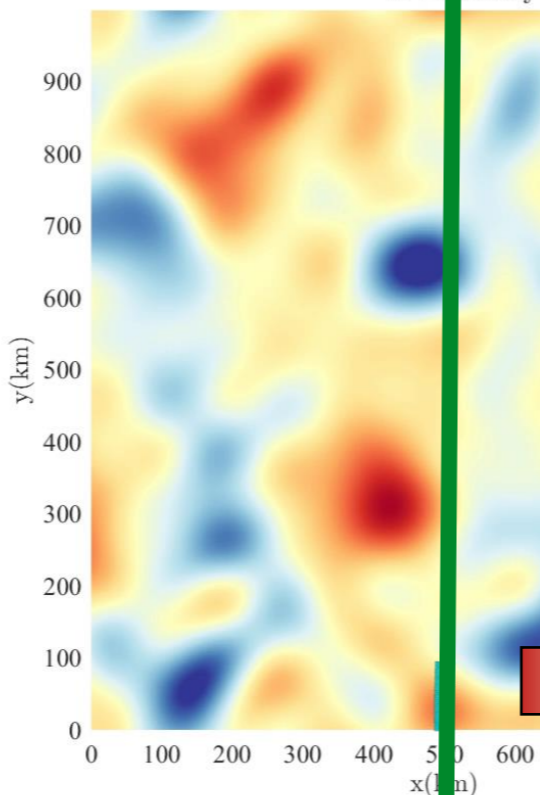
$$v = \bar{v} + v'$$

$t = 4.01$ day



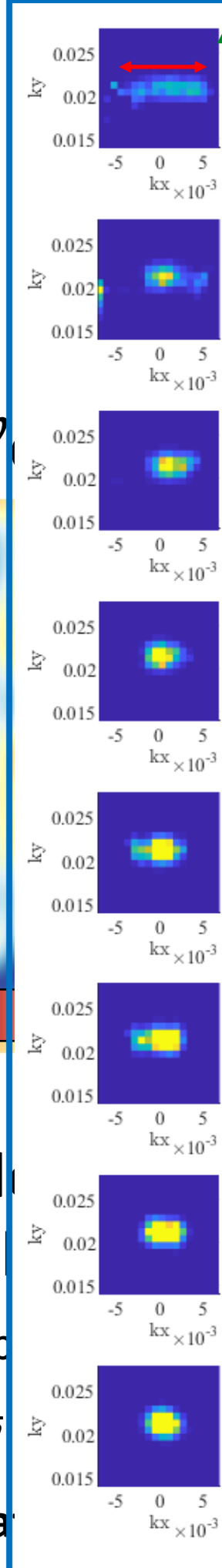
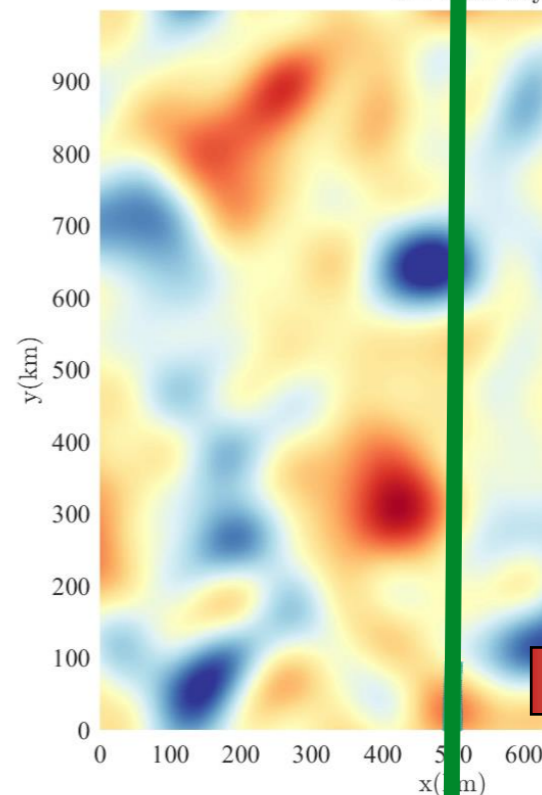
$$v = \bar{v}$$

$t = 4.01$ day



$$v = \bar{v} + v'$$

$t = 4.01$ day



Deterministic
reference
wave group
512 x 512 fl

Deterministic
benchmark
wave groups
smoothed flow \bar{v} (

Our random
mode
wave group
smoothed flow \bar{v}
+ time-correla

3.

Markovian closure with
current multi-scale in space &
white in time

- Simple cases & analytic solutions -

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

$$v = \bar{v} + v'$$

Large scale
group velocity:

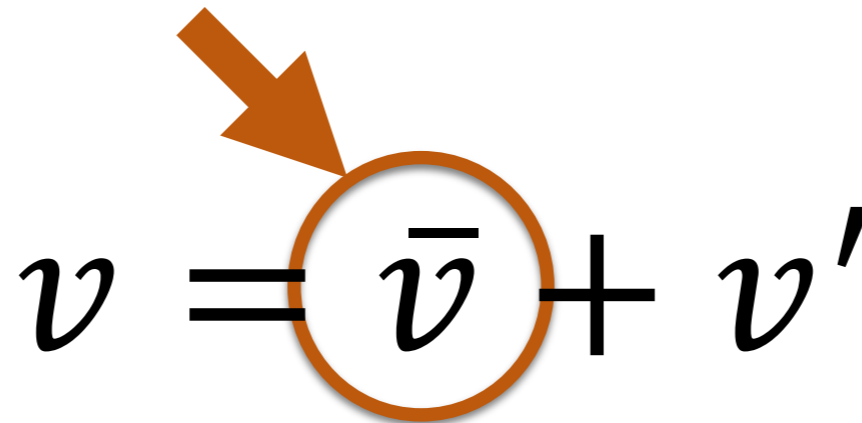
$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Resolved
large scales


$$v = \bar{v} + v'$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

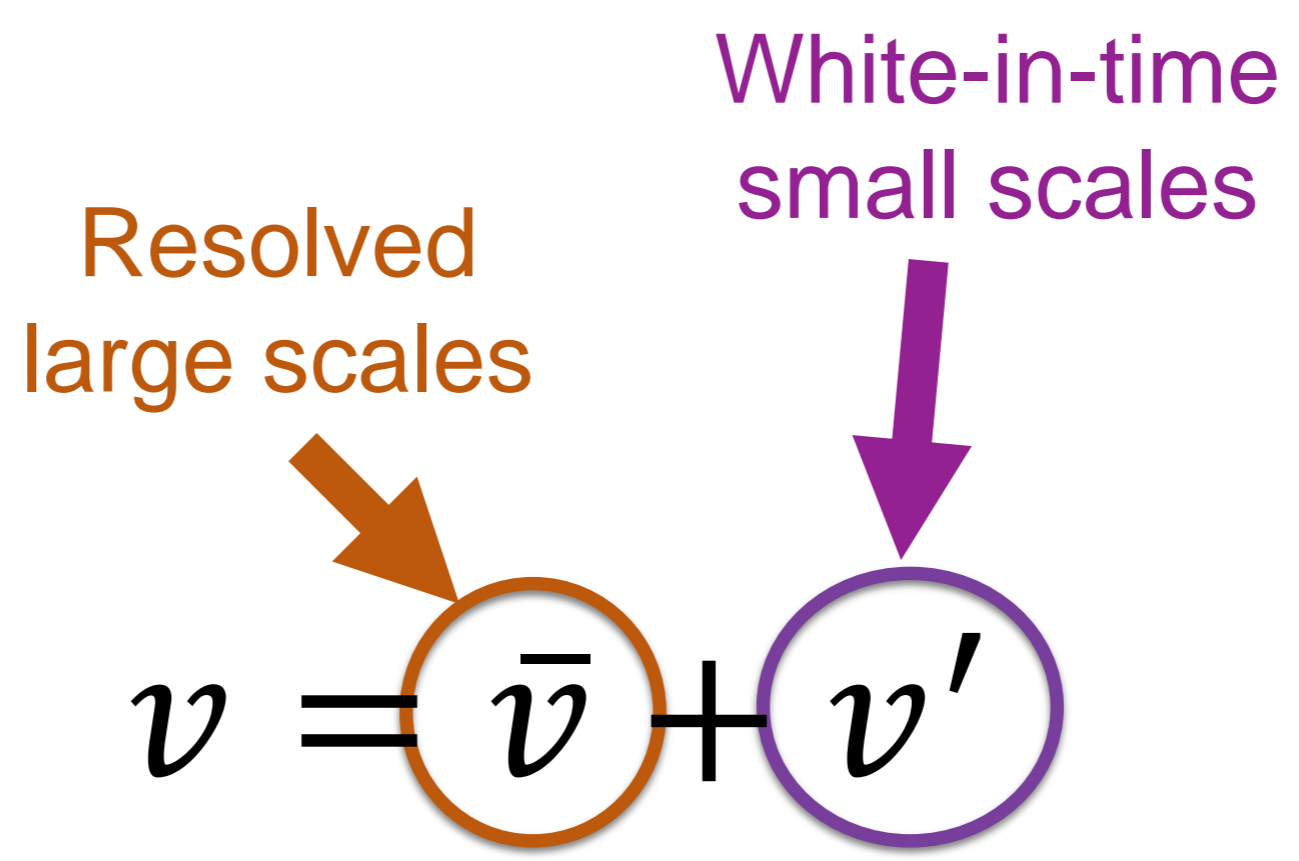
Resolved
large scales

White-in-time
small scales

$$v = \bar{v} + v'$$

Large scale group velocity:
 $\bar{v} + C_g^0$

Small scale group velocity:
 $v' = \sigma dB_t/dt$
 (Q Wiener process)



References : Mikulevicius & Rozovskii, 2004
 Flandoli, 2011

Location Uncertainty (LU)
Memin, 2014
 Resseguier et al. 2017 a, b, c
 Cai et al. 2017
 Chapron et al. 2018
 Yang & Memin 2019
 ...

Stoch. Adv. by Lie Transport (SALT)
Holm, 2015
 Holm and Tyranowski, 2016
 Arnaudon et al. 2017
 Crisan et al., 2017
 Gay-Balmaz & Holm 2017
 Cotter and al. 2018 a, b
 Cotter and al. 2019
 ...

Cotter and al. 2017 Resseguier et al. 2020, 2021, ...

Large scale group velocity:

$$\bar{v} + C_g^0$$

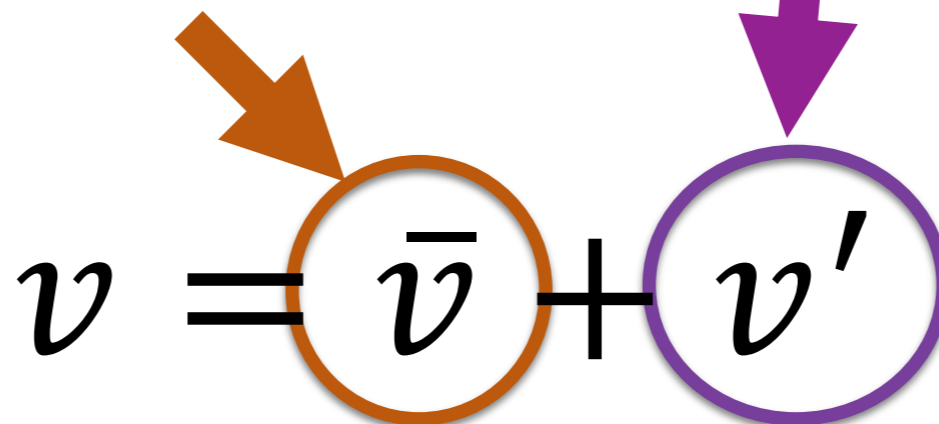
Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Resolved large scales

White-in-time small scales



Initially for uncertainty quantification & data assimilation

References : Mikulevicius & Rozovskii, 2004
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Location Uncertainty (LU)

Memin, 2014
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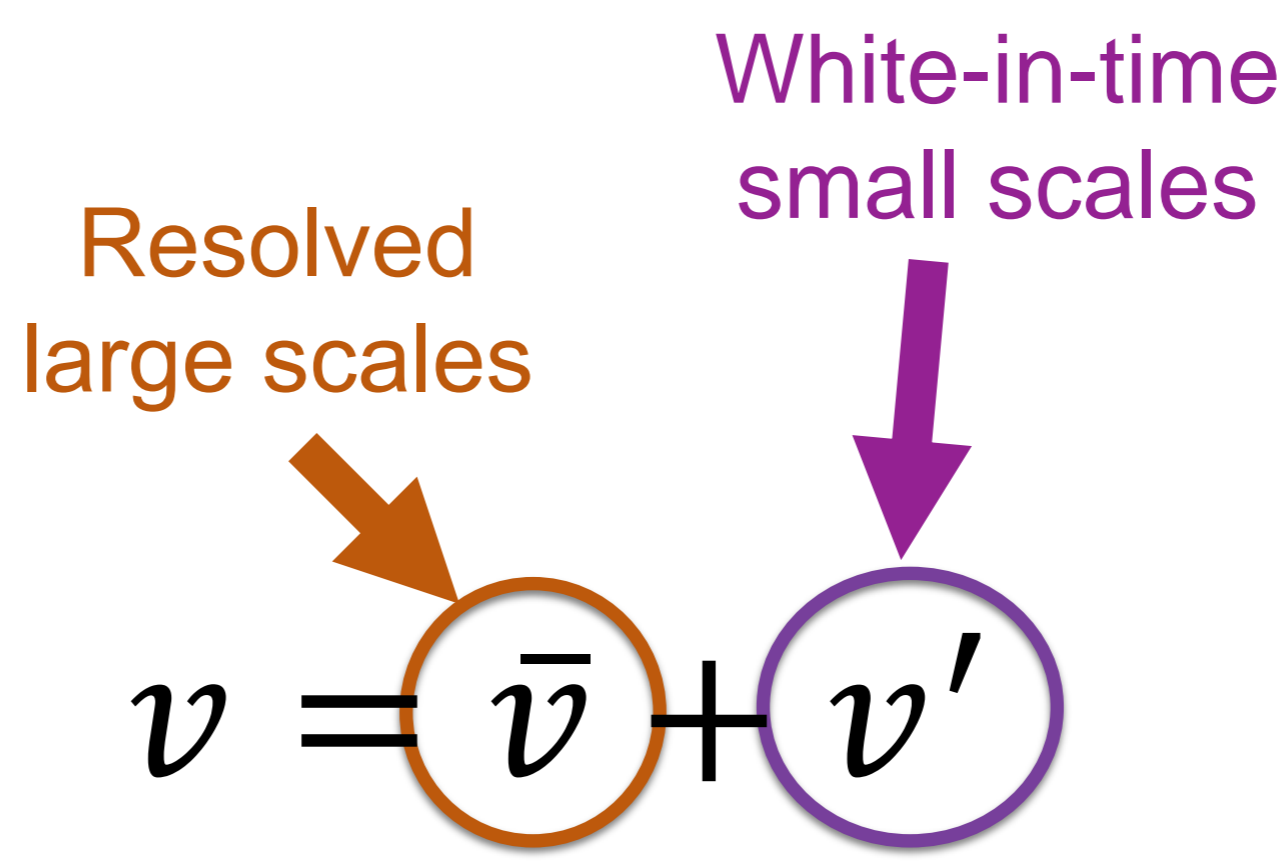
Stoch. Adv. by Lie Transport (SALT)

Holm, 2015
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Cotter and al. 2019

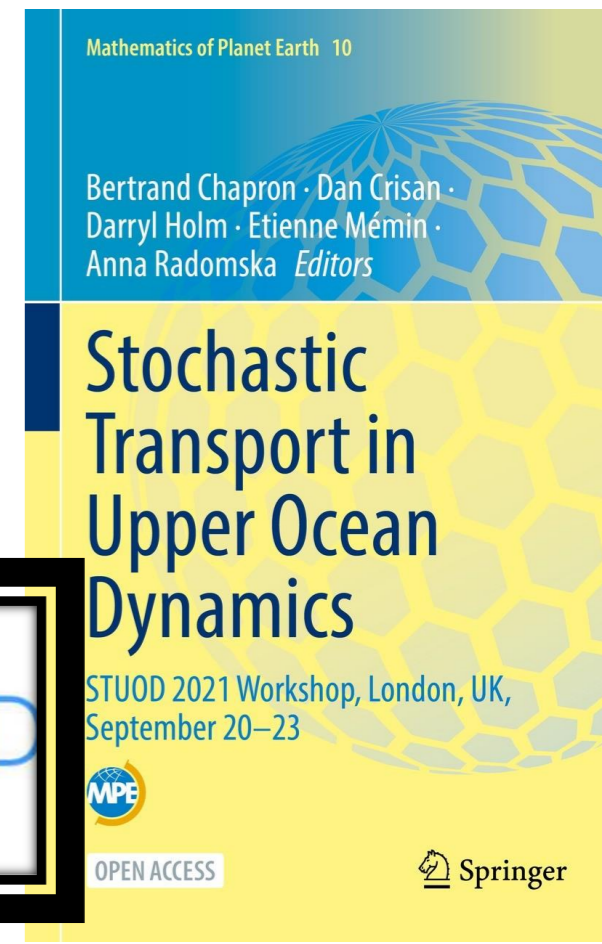
Cotter and al. 2017 Resseguier et al. 2020, 2021, ...

Large scale group velocity:
 $\bar{v} + C_g^0$

Small scale group velocity:
 $v' = \sigma dB_t/dt$
 (Q Wiener process)



Initially for uncertainty quantification & data assimilation



References : Mikulevicius & Rozovskii, 2004
 Flandoli, 2011

Location Uncertainty (LU)		Stoch. Adv. by Lie Transport (SALT)	
Memin, 2014	Resseguier et al. 2017 a, b, c	Crisan et al., 2017	
Cai et al. 2017	Chapron et al. 2018	Gay-Balmaz & Holm 2017	
Yang & Memin 2019		Cotter and al. 2018 a, b	
...		Cotter and al. 2019	
	Cotter and al. 2017	Resseguier et al. 2020, 2021,

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

$$ae^{i\epsilon\phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

Time decorrelation assumption for v'

- **Advantages :**

- Simpler / analytic formula
⇒ physical comprehension & lighter CPU for simulations
- No precise knowledge needed about v' time dependency :
Markovian closure

- **Validity :**

$$\epsilon = \frac{\text{(Along-ray } v' \text{ correlation time)}}{\text{(characteristic time of wave group properties evolution)}} = \frac{\left(\frac{l_{v'}}{\|C_g^0\|}\right)}{\left(\frac{1}{\|\nabla v'\|}\right)} \ll 1$$

- **Limitations :**

- Swells ($\|C_g^0\| \gg 1$)
- Small-scale currents ($l_{v'} \ll 1$)
- Moderate current gradients ($\|\nabla v'\| \ll 1$) (⇒ moderate $\|\nabla v'\| \nabla$)

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t/dt$$

(Q Wiener process)

Wave:

$$ae^{\frac{i}{\epsilon} \phi(\epsilon t, \epsilon x)}$$

Group velocity
without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

Simple linear case

$$dX_r = (C_g^0 + \bar{v})dt + \sigma dB_t \quad \Bigg| \quad \text{Ray}$$

Refraction & contraction/dilatation

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k \quad \Bigg| \quad \text{Wave-vector}$$

Conservation of action

$$dA = \nabla \cdot \left((C_g^0 + \bar{v})dt + \sigma dB_t \right) A \quad \Bigg| \quad \text{Amplitude}$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

$$k = \nabla \phi$$

$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

Simple linear case 1:
stationary deterministic,
divergence-free and
linear in x large-scale velocity

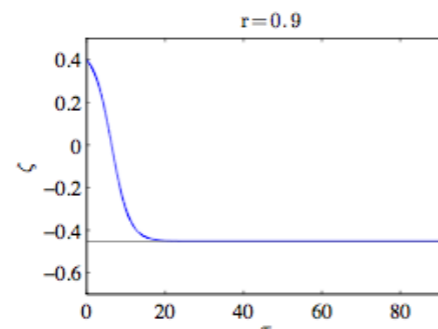
Stretching of phase

$$dk = -\nabla \bar{v}^T k dt$$

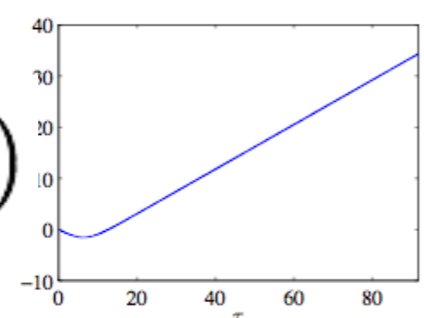
with $\nabla \bar{v}^T = \sigma_w \begin{bmatrix} 1 & r \\ -r & -1 \end{bmatrix}$

$$|r| < 1$$

Wave shortening

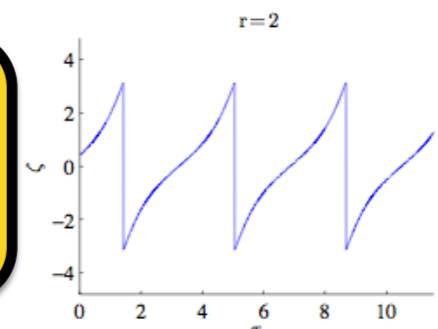


$\ln(\|k\|)$

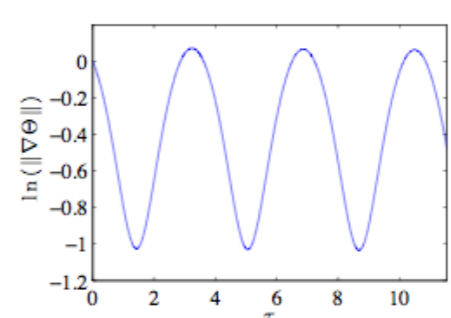


$$|r| > 1$$

Wave rotation



$\ln(\|k\|)$



$$\frac{d\zeta}{dt} = r - \cos(\zeta) = -\frac{\partial V}{\partial \zeta}(\zeta)$$

with $V(\zeta) = r\zeta - \sin \zeta$

$$\frac{d \ln(\kappa)}{dt} = -\sin(\zeta)$$

Large scale
group velocity:

$$+C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

$$k = \nabla\phi$$
$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$
$$\zeta = 2\theta - \frac{\pi}{2}$$

Simple linear case 2:

no large-scale current
+ isotropic and divergence-free
small-scale velocity

Stochastic stretching of phase

$$dk = -\nabla(\sigma dB_t)^T k$$

Log-normal wave-number

$$\begin{cases} \ln \kappa(t) = \alpha^2 t + \alpha B_t^{(1)} \\ \zeta(t) = (\sqrt{12}\alpha) B_t^{(2)} \end{cases}$$

with $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt.$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$a e^{i \frac{\phi}{\epsilon}}$$

$$k = \nabla \phi$$
$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$
$$\zeta = 2\theta - \frac{\pi}{2}$$

Simple linear case 3

divergence-free and linear-in-x large-scale current
+ isotropic and divergence-free
small-scale velocity

Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with $V(\zeta) = r \zeta - \sin \zeta$ and $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$.

We can solve the stationary Fokker-Planck for ζ ,
and then get the stationary distribution of k

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

$$k = \nabla \phi$$
$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$
$$\zeta = 2\theta - \frac{\pi}{2}$$

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divergence-free and linear-in-x large-scale current
+ isotropic and divergence-free
small-scale velocity

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phase

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with $V(\zeta) = r \zeta - \sin \zeta$ and $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$.

Large scale group velocity:

$$\bar{v} = C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

Simple linear case 3

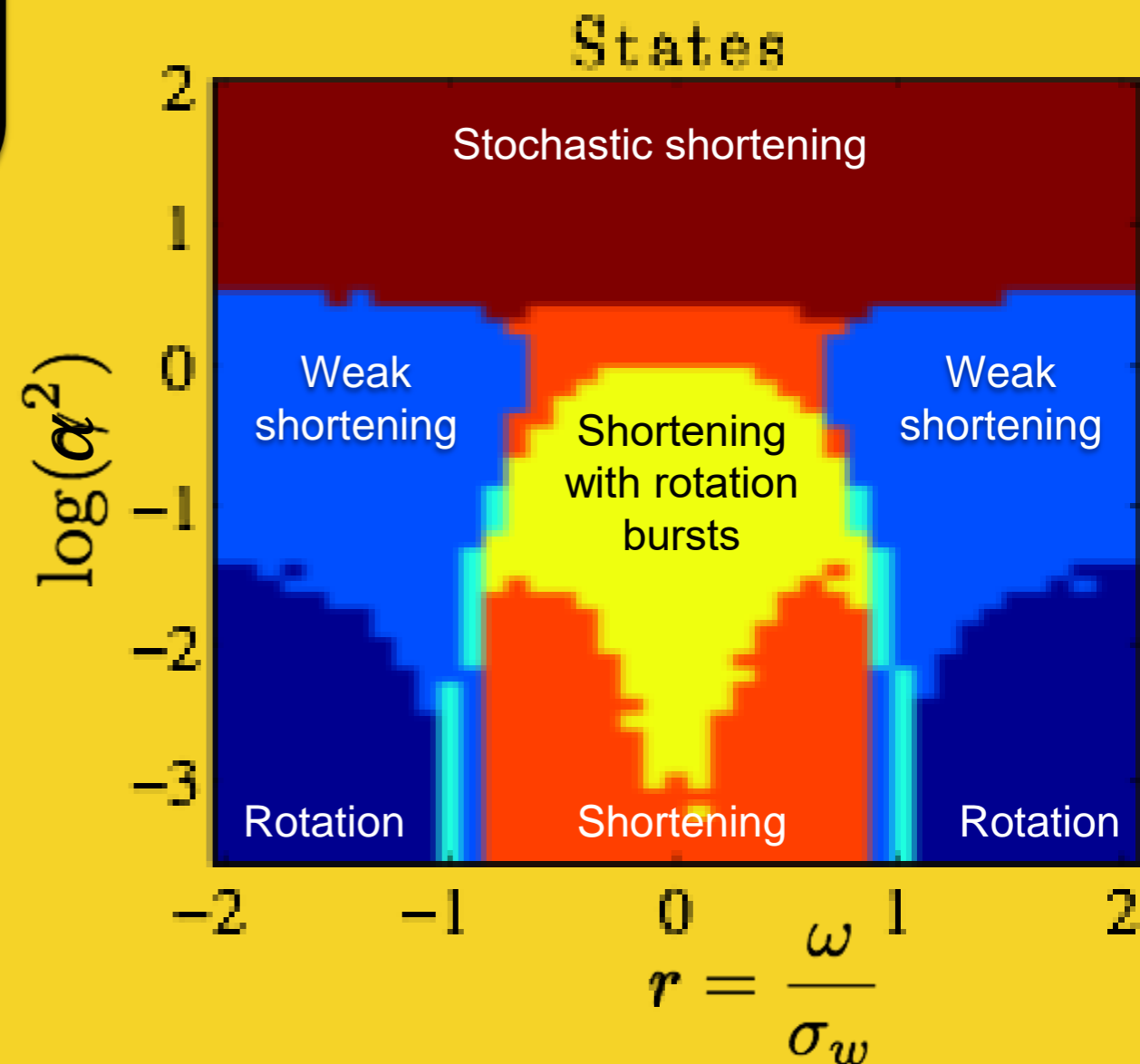
divergence-free and linear-in-x large-scale current
+ isotropic and divergence-free small-scale velocity

Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with $V(\zeta) = r\zeta - \sin\zeta$ and $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$.



Large scale group velocity:

$$\bar{v} = C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

Simple linear case 3

divergence-free and linear-in-x large-scale current
+ isotropic and divergence-free small-scale velocity

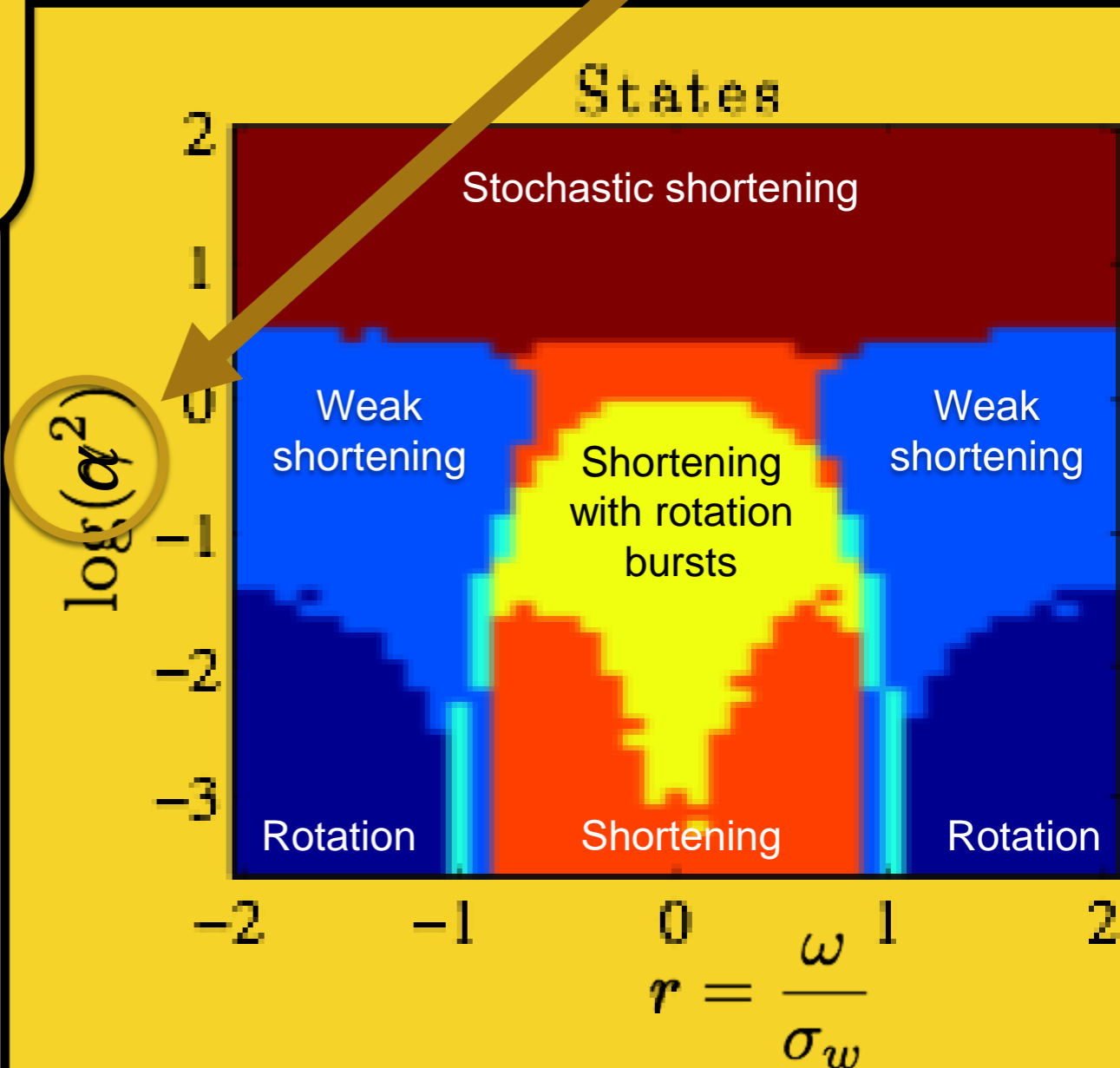
Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$

Depends on small scale's statistics

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with $V(\zeta) = r\zeta - \sin\zeta$ and $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$.



Large scale group velocity:

$$\bar{v} = C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

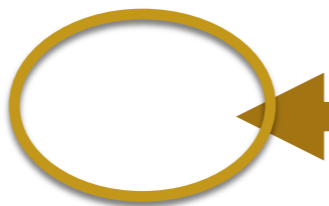
$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

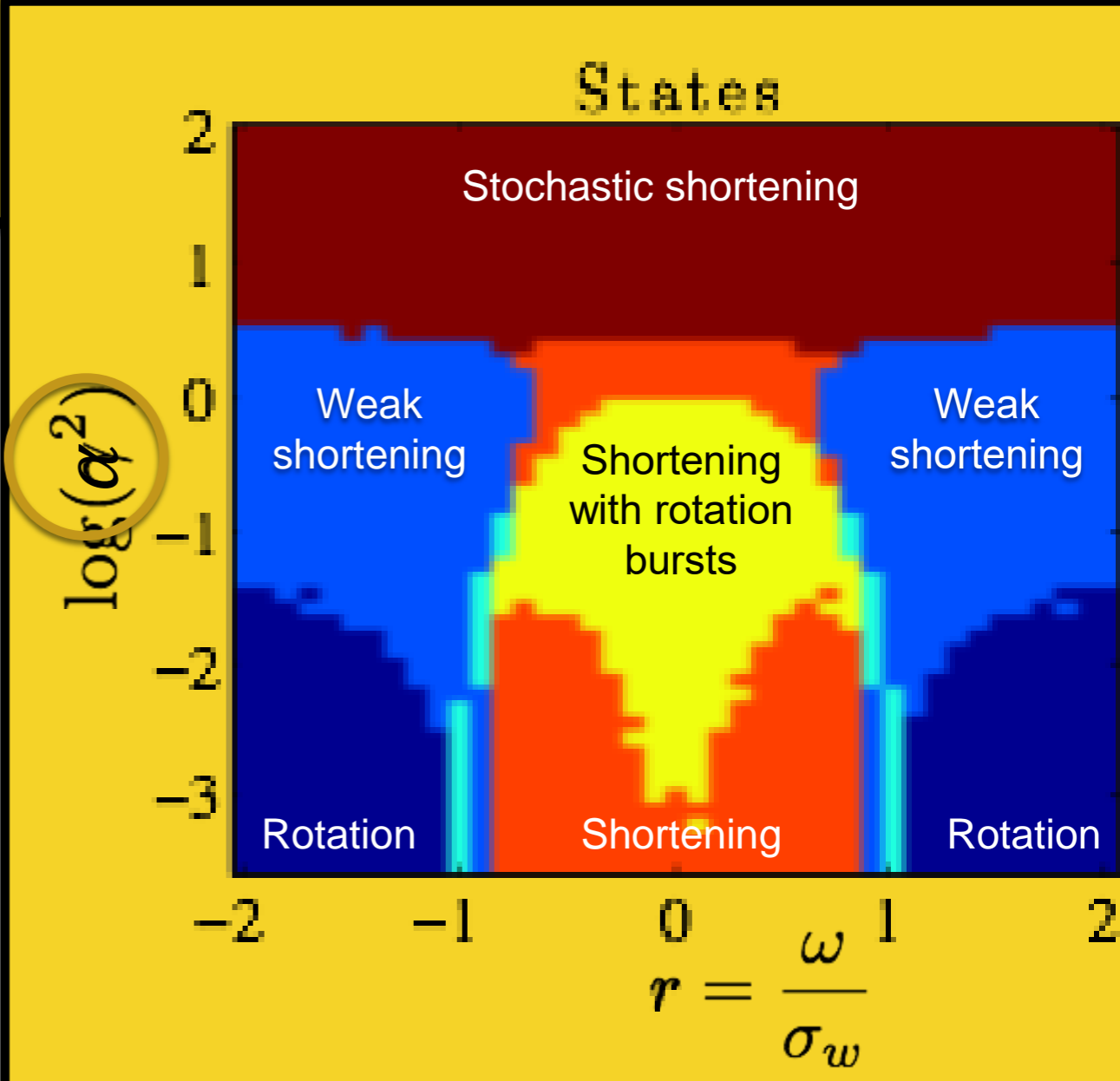
$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with $V(\zeta) = r\zeta - \sin\zeta$ and $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$.



Depends on small scale's statistics

Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$


Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

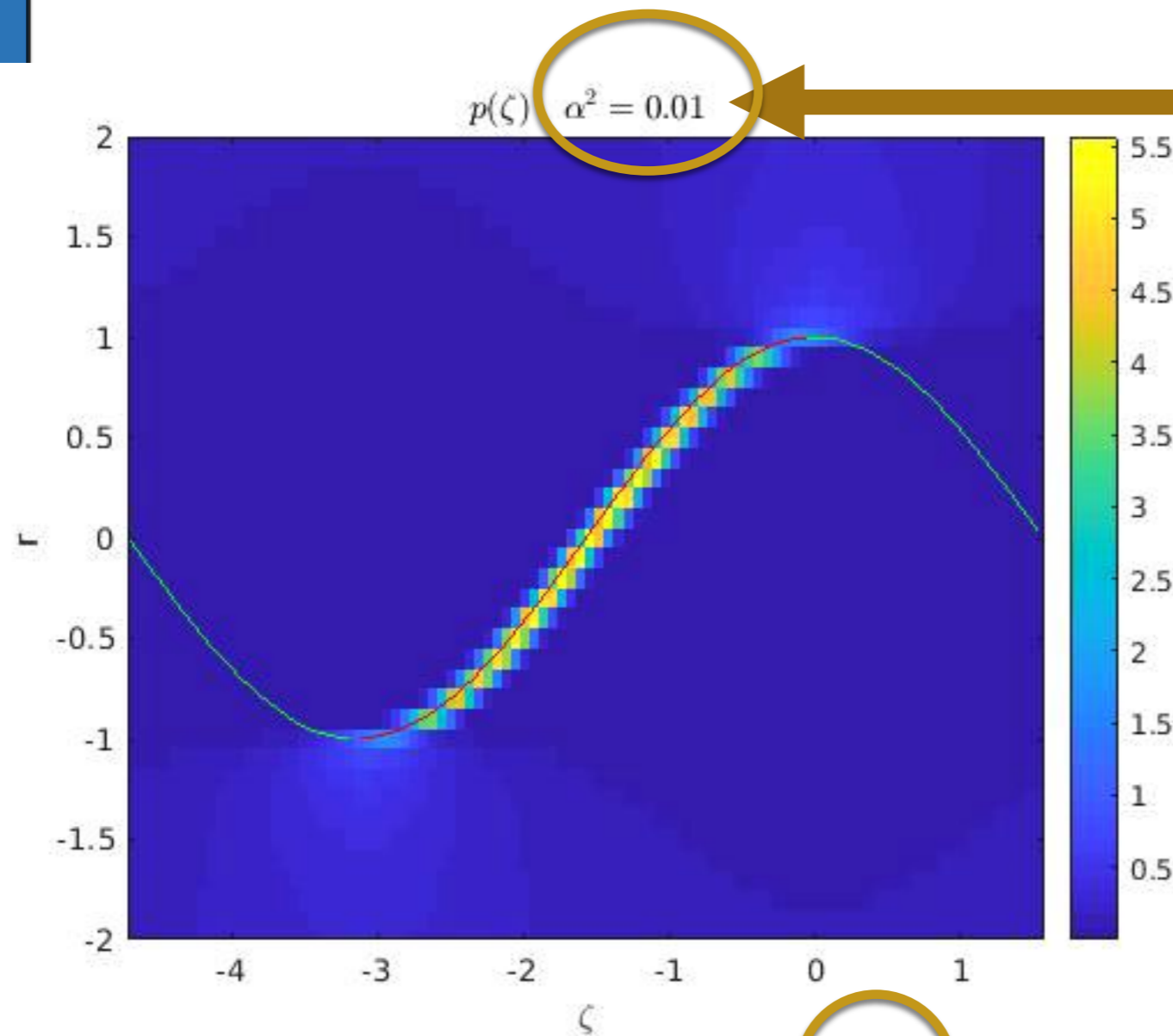
$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

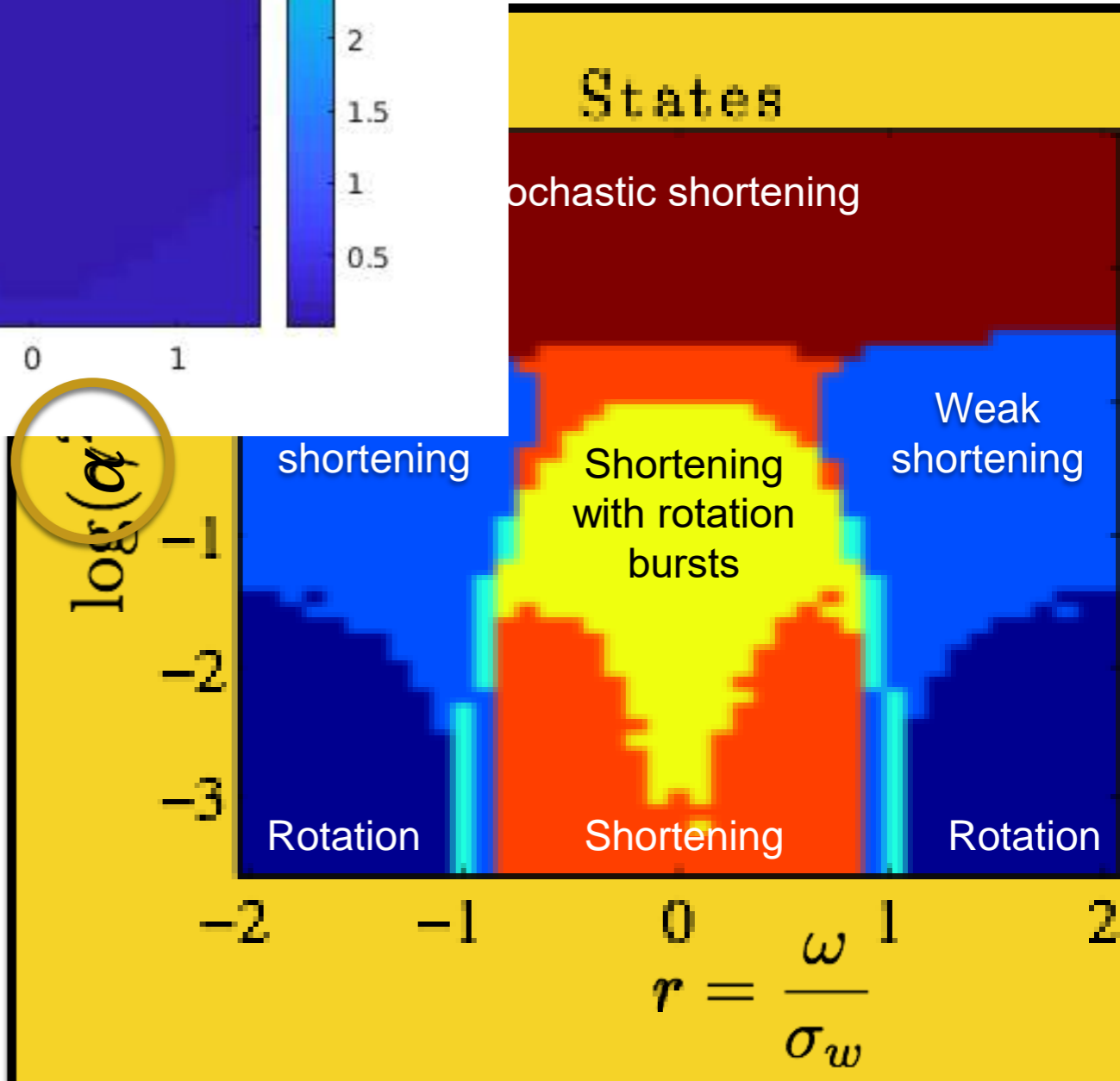
$$\zeta = 2\theta - \frac{\pi}{2}$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with $V(\zeta) = r\zeta - \sin\zeta$ and $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$.



Depends on small scale's statistics



Conclusion

Conclusion

We propose multi-scale closures for simulation of wave-turbulence interactions.

- At large scales,
the oceanic currents are known because measured by satellite.
- At intermediate scales,
new stochastic closure, which is non-Markovian, with stochastic currents multi-scale in space and time.
- For smaller scales,
Markovian approaches – e.g., LU & SALT [2,1,3,4,5] – applied (in the wave frame).
We found an analytical formula for the probability distribution of wave properties at long time.

In the future, we would like to use it as prior emulator of data assimilation.

References

1. Holm, D. D. (2015). Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176), 20140963.
2. Mémin, E. (2014). Fluid flow dynamics under location uncertainty. *Geophysical & Astrophysical Fluid Dynamics*, 108(2), 119-146.
3. Resseguier, V., Pan, W., & Fox-Kemper, B. (2020). Data-driven versus self-similar parameterizations for stochastic advection by Lie transport and location uncertainty. *Nonlinear Processes in Geophysics*, 27(2), 209-234.
4. Resseguier, V., Li, L., Jouan, G., Dérian, P., Mémin, E., & Chapron, B. (2021). New trends in ensemble forecast strategy: uncertainty quantification for coarse-grid computational fluid dynamics. *Archives of Computational Methods in Engineering*, 28(1), 215-261.
5. Resseguier, V., Hascoët, E., & Chapron, B. (2023). Random ocean swell-rays: a stochastic framework. In *Stochastic Transport in Upper Ocean Dynamics Annual Workshop* (pp. 259-271). Springer, Cham.

Bonus slides

Large scale
group velocity:

$$\bar{v} = c_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

$$ae^{i\epsilon\phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

Time-uncorrelated model for v'

$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Large scale
group velocity:

$$\bar{v} + C_g^0$$

Small scale
group velocity:

$$v' = \sigma dB_t/dt$$

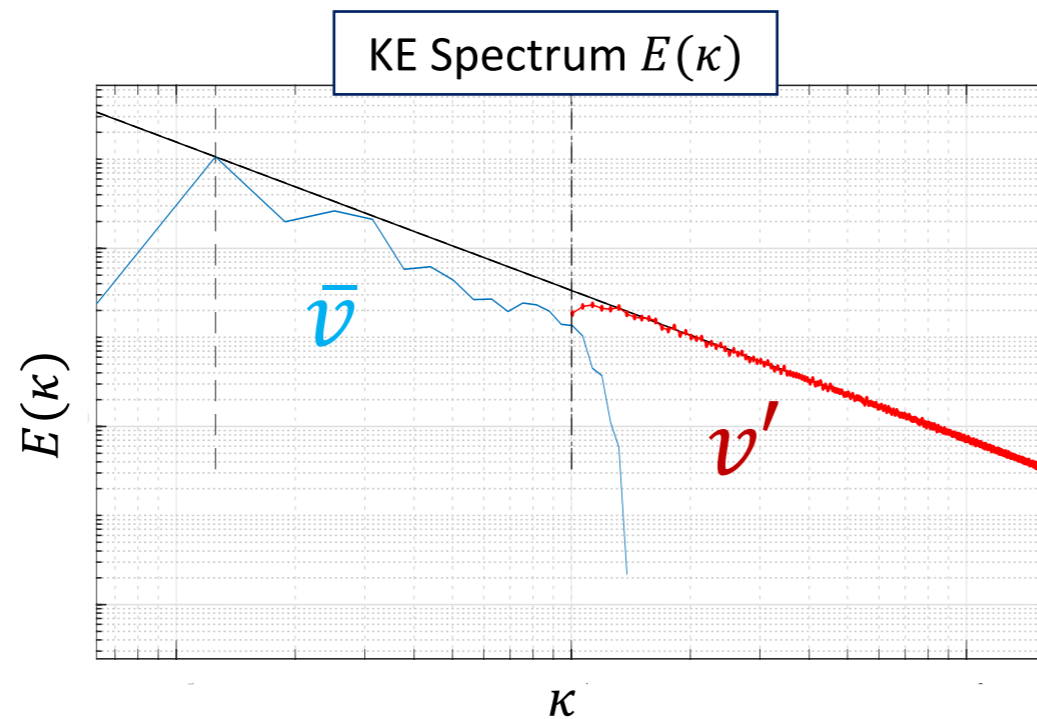
Wave:

$$ae^{i\epsilon\phi}$$

Doppler
frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

Time-uncorrelated model for v'



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

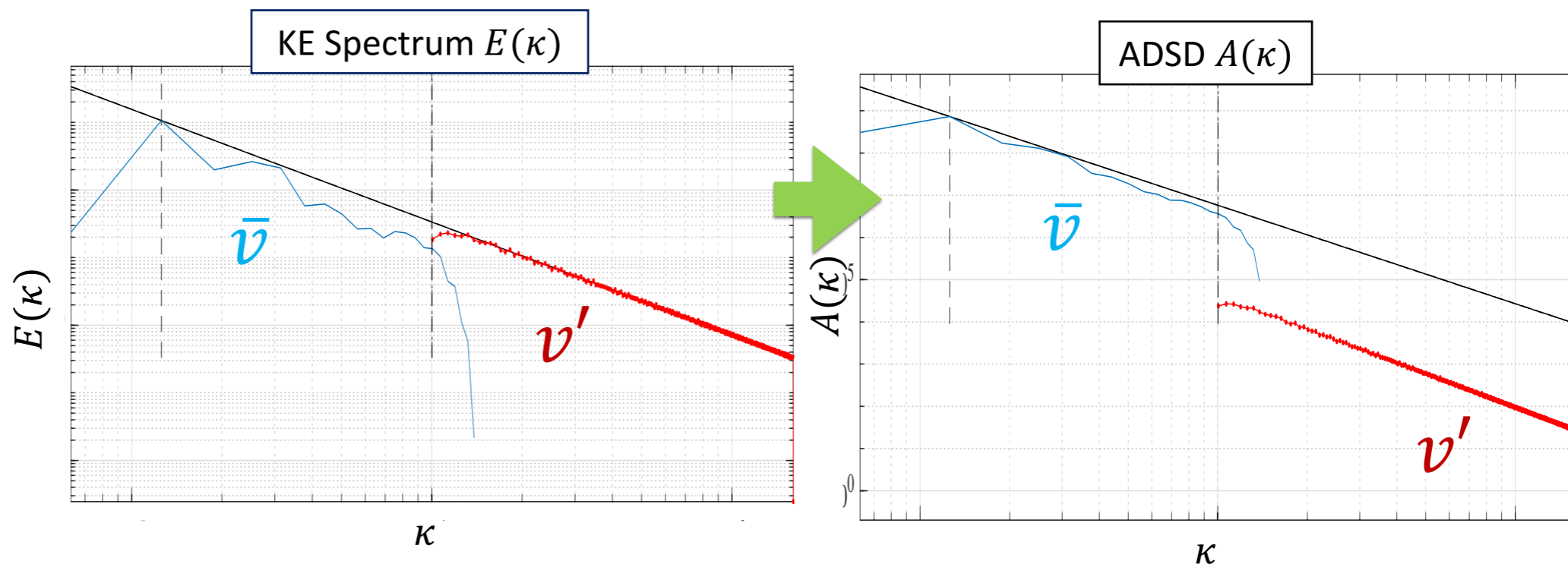
$$ae^{i\epsilon\phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

Time-uncorrelated model for v'

Absolute Diffusivity Spectral Density
 $A(\kappa) = E(\kappa) \tau(\kappa)$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:
[3] Resseguier, Pan & Fox-Kemper 2020

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

$$ae^{i\epsilon\phi}$$

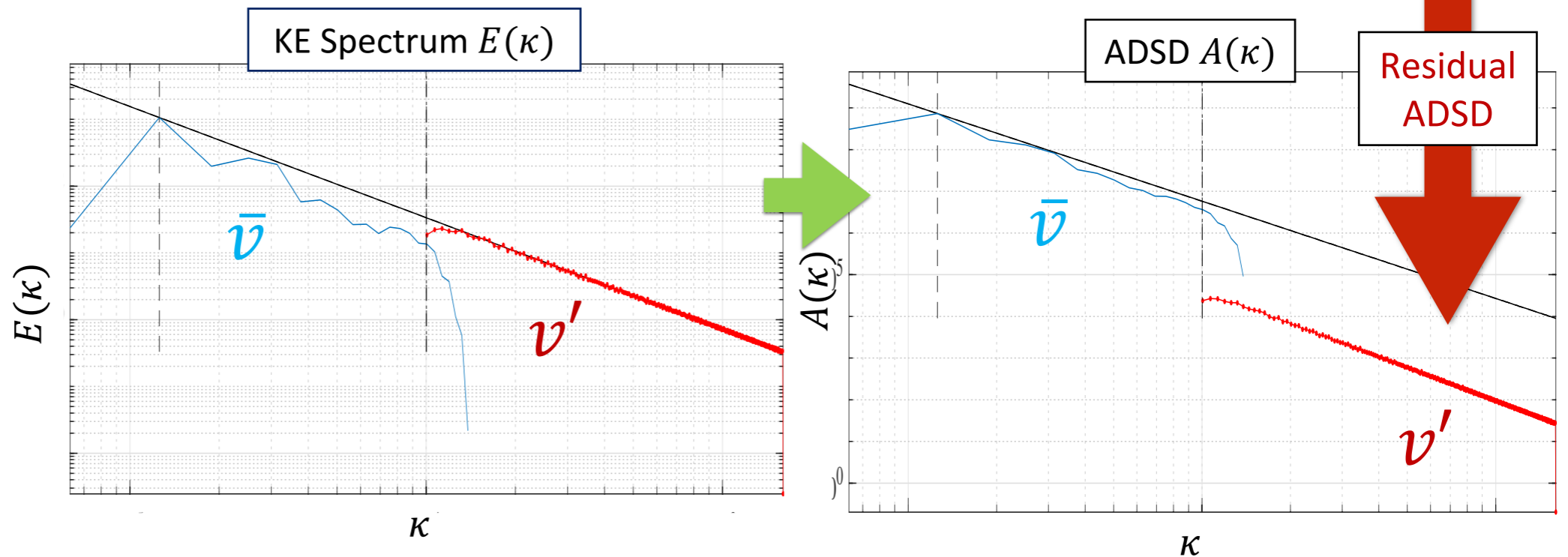
Doppler frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

Time-uncorrelated model for v'

Absolute Diffusivity Spectral Density
 $A(\kappa) = E(\kappa) \tau(\kappa)$

Correlation time along ray
 $\tau(\kappa) = \tau_{ray}(\kappa) = \frac{1/\kappa}{\|C_g^0\|}$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:
 [3] Resseguier, Pan & Fox-Kemper 2020

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

$$ae^{i\epsilon\phi}$$

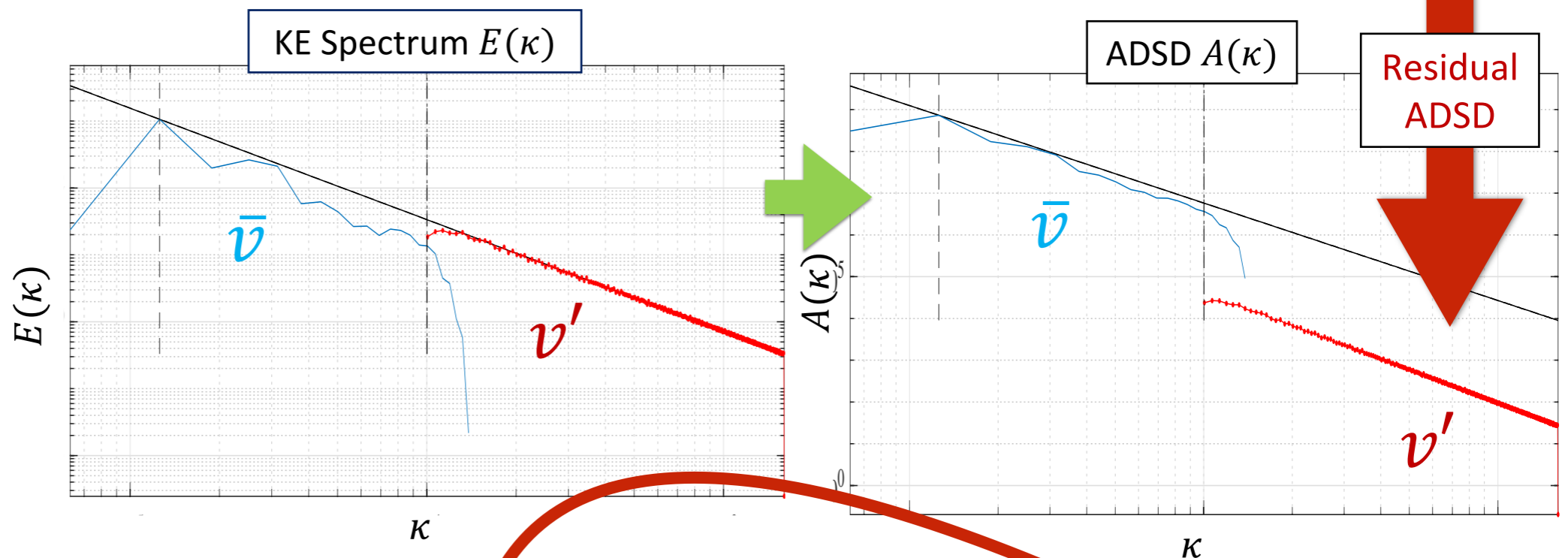
Doppler frequency:

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Time-uncorrelated model for v'

Absolute Diffusivity Spectral Density
 $A(\kappa) = E(\kappa) \tau(\kappa)$

Correlation time along ray
 $\tau(\kappa) = \tau_{ray}(\kappa) = \frac{1/\kappa}{\|C_g^0\|}$



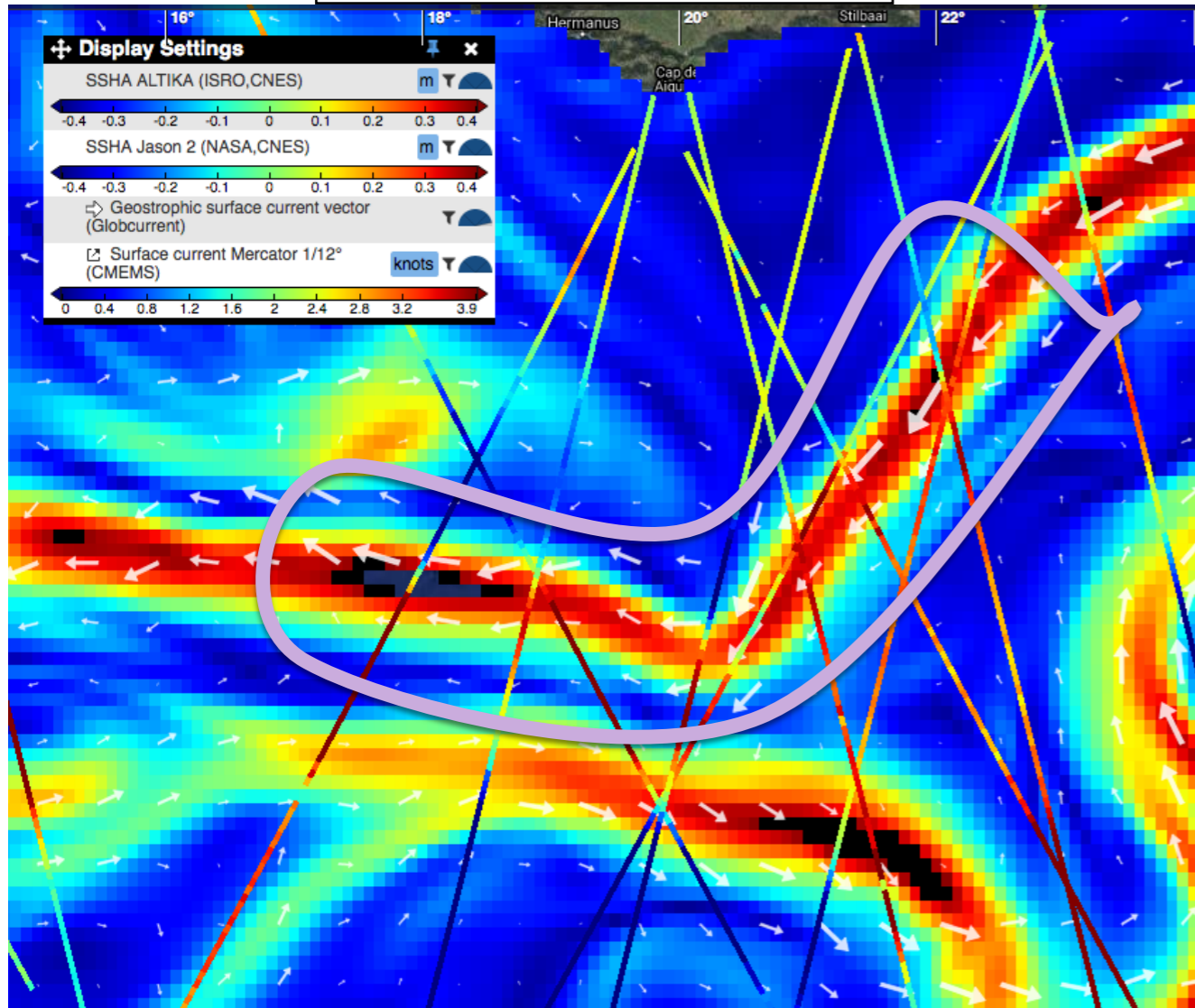
$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$

Reference:
[3] Resseguier, Pan & Fox-Kemper 2020

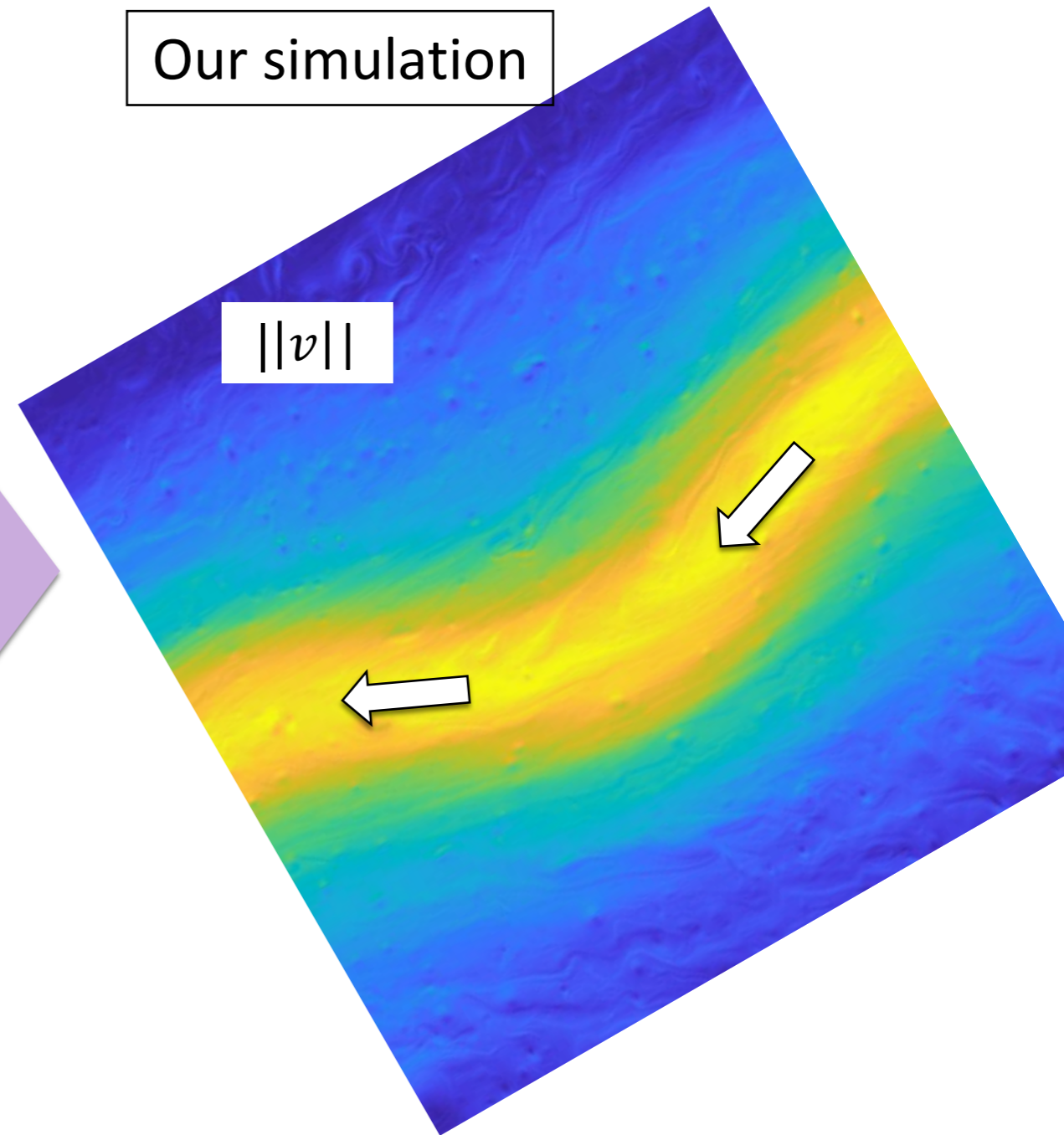
Numeric simulations with heterogenous turbulence

Heterogenous flow : the jet example

Situation to mimic:



Our simulation



Agulhas current
Mercator current (01/01/2016)
+ SSHA weekly

Time-correlated model for v'

Forward ray tracing

$$v = \bar{v} + v'$$

$$v = \bar{v}$$

$$v = \bar{v} + v'_{\text{cor}}$$

wave group amplitude

wave group amplitude

wave group amplitude

Vorticity

Vorticity

Vorticity

Deterministic
reference:

wave groups in
512 x 512 **SQG** flow

Deterministic
benchmark:

wave groups in
smoothed flow \bar{v} (**16 x 16**)

Our random
model

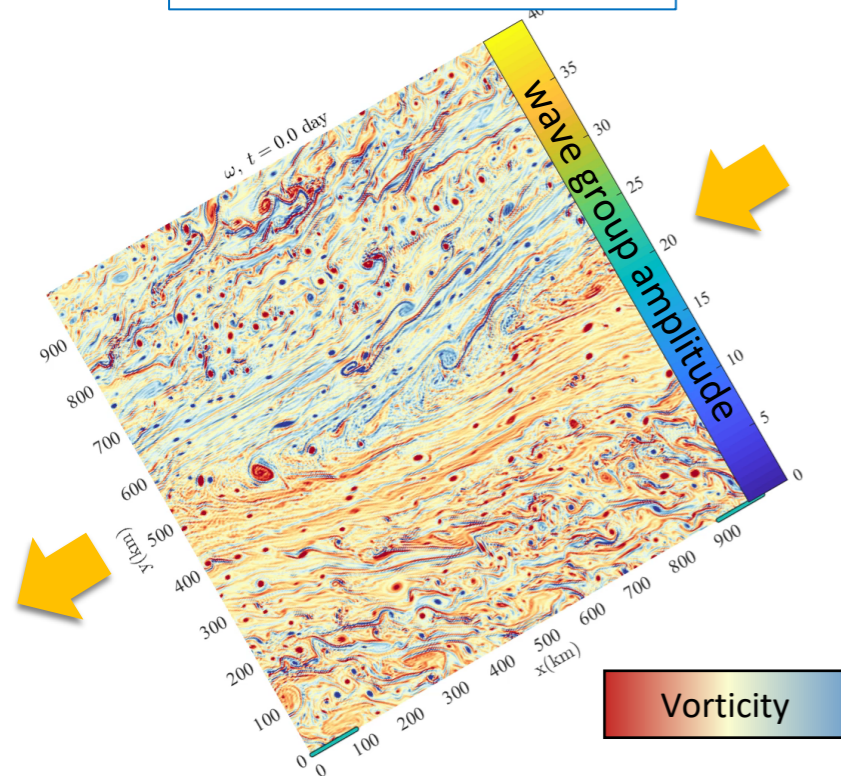
wave groups in
smoothed flow \bar{v} (**16 x 16**)

+ time-correlated v'

Time-correlated model for v'

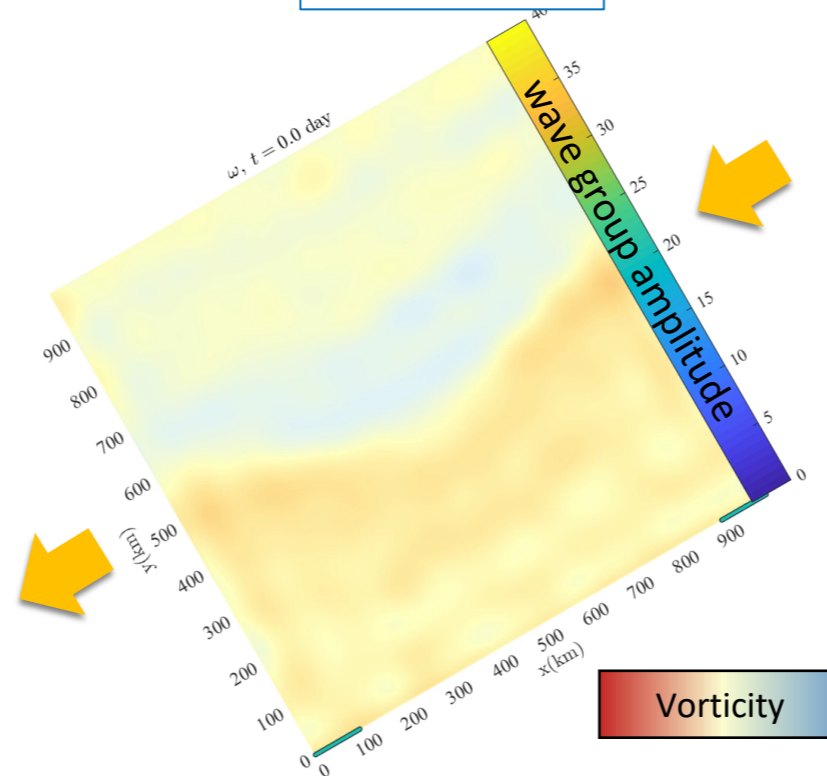
Forward ray tracing

$$v = \bar{v} + v'$$



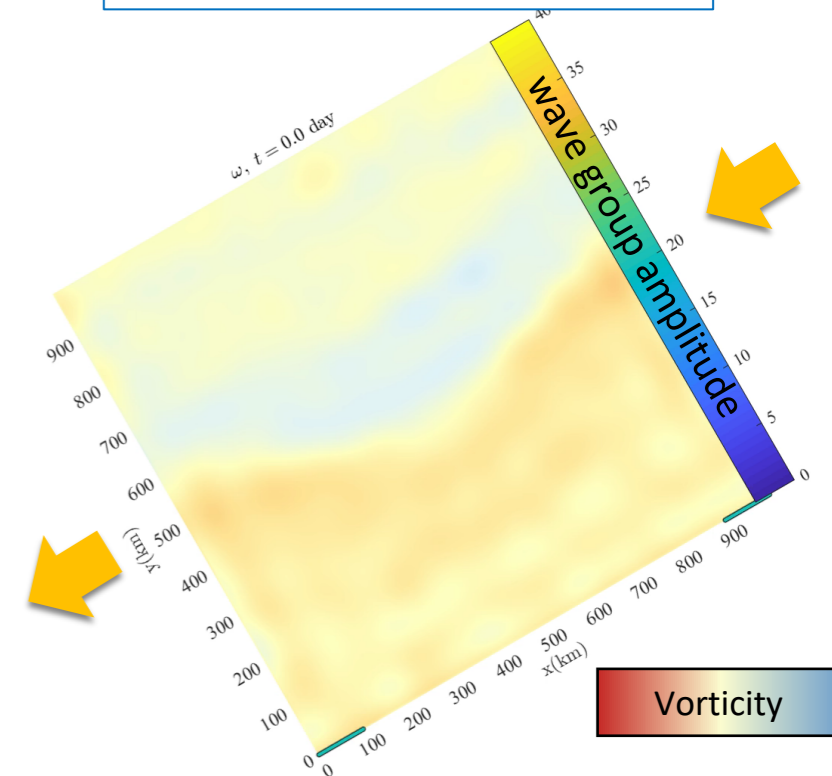
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

$$v = \bar{v} + v'_{\text{cor}}$$

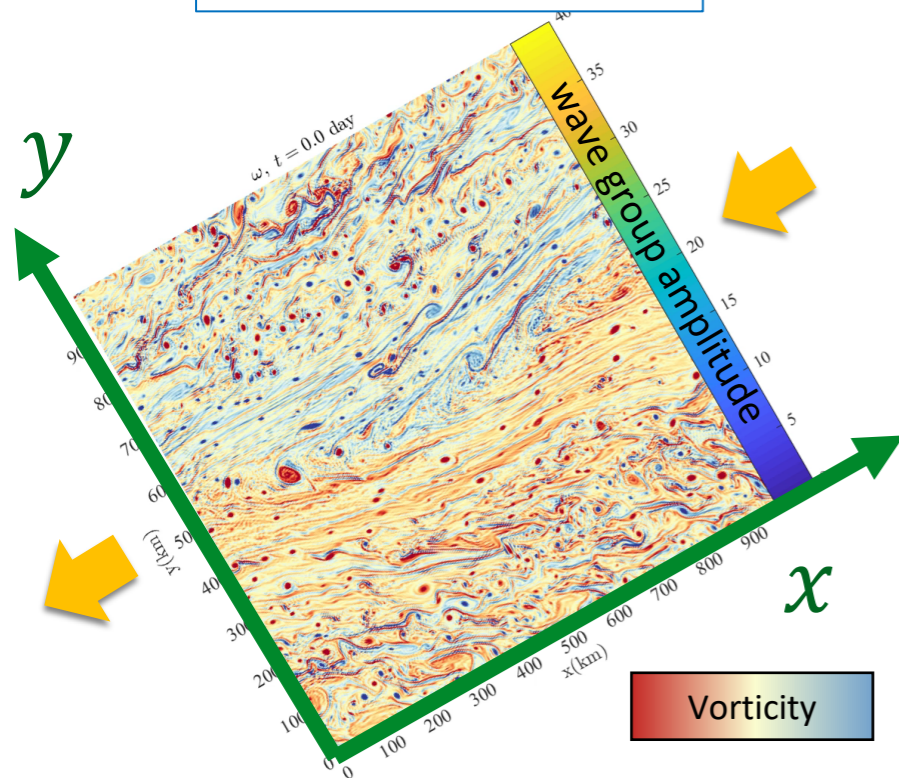


Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v'

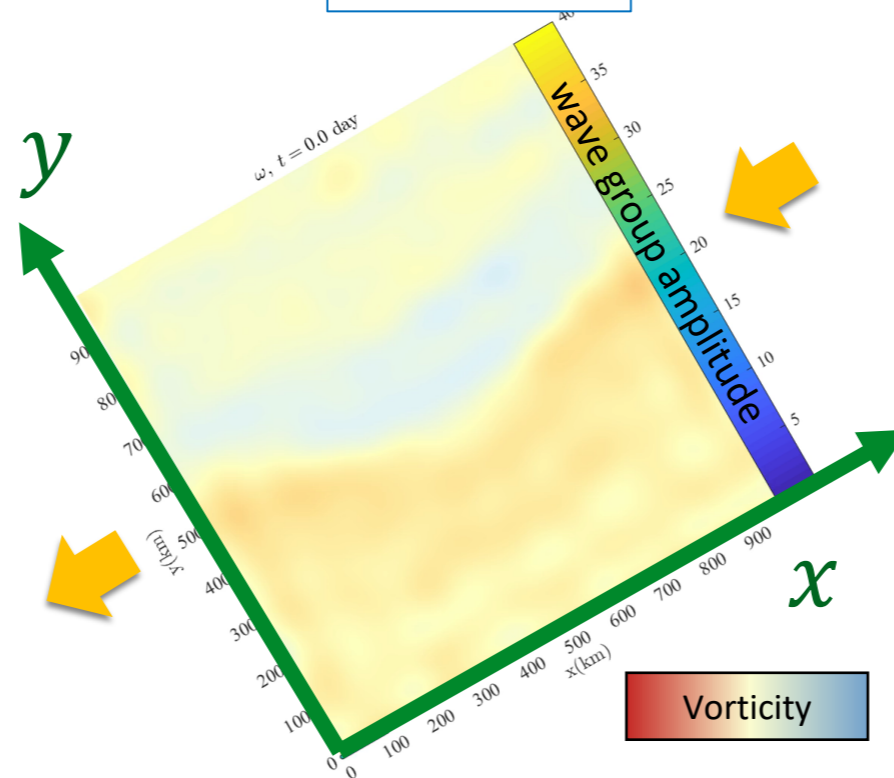
Forward ray tracing

$$v = \bar{v} + v'$$



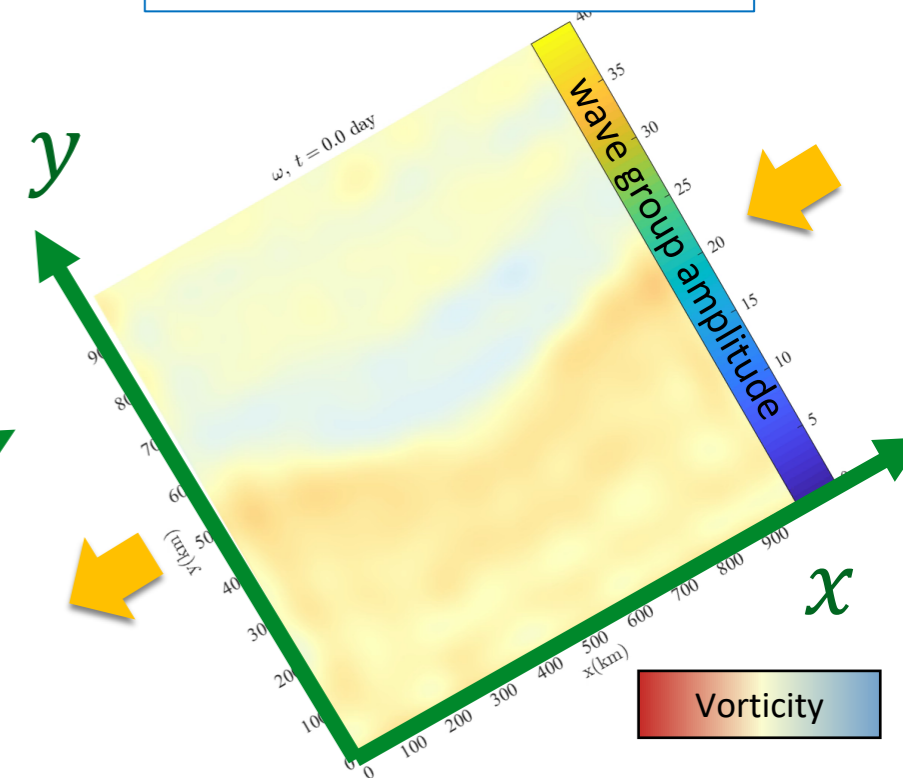
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Deterministic
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$$v = \bar{v} + v'_{\text{cor}}$$

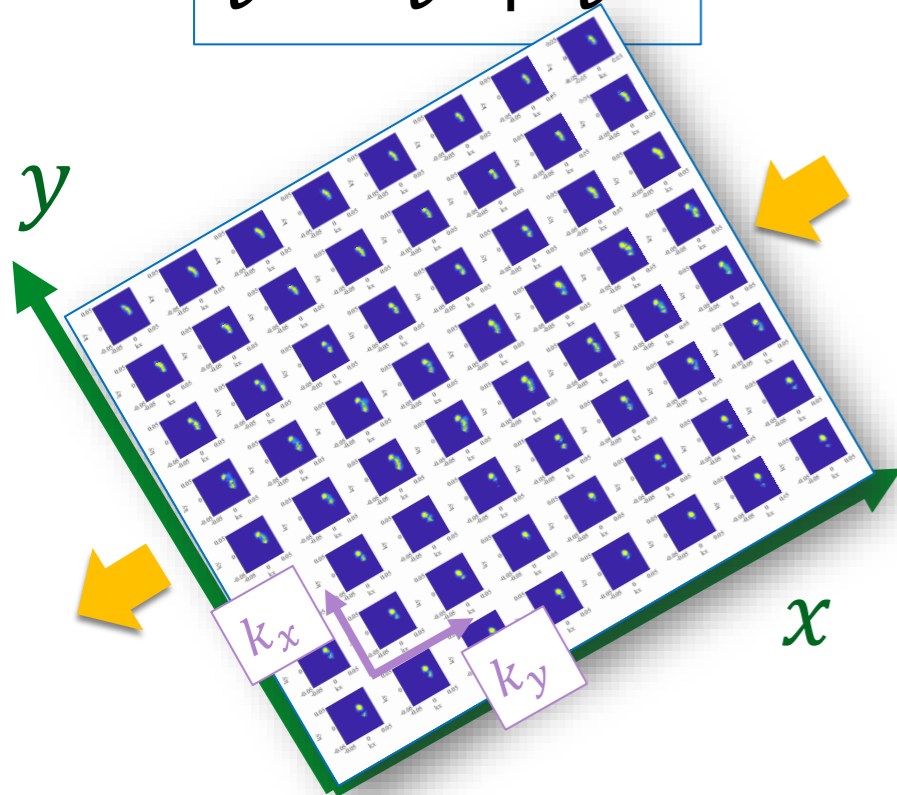


Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v'

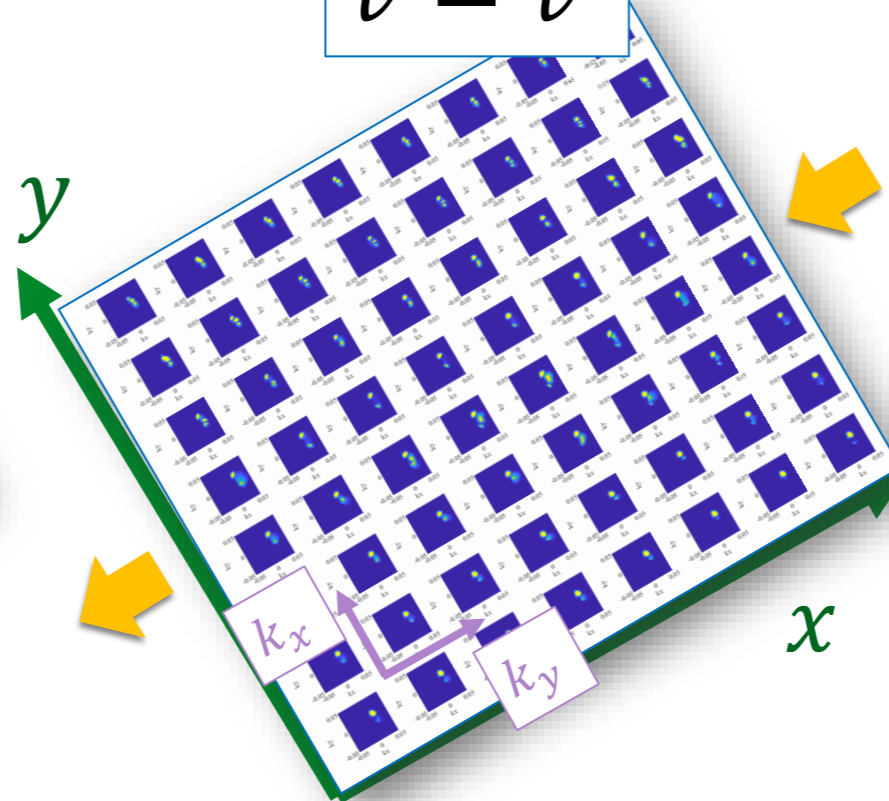
Wave spectra

$$v = \bar{v} + v'$$



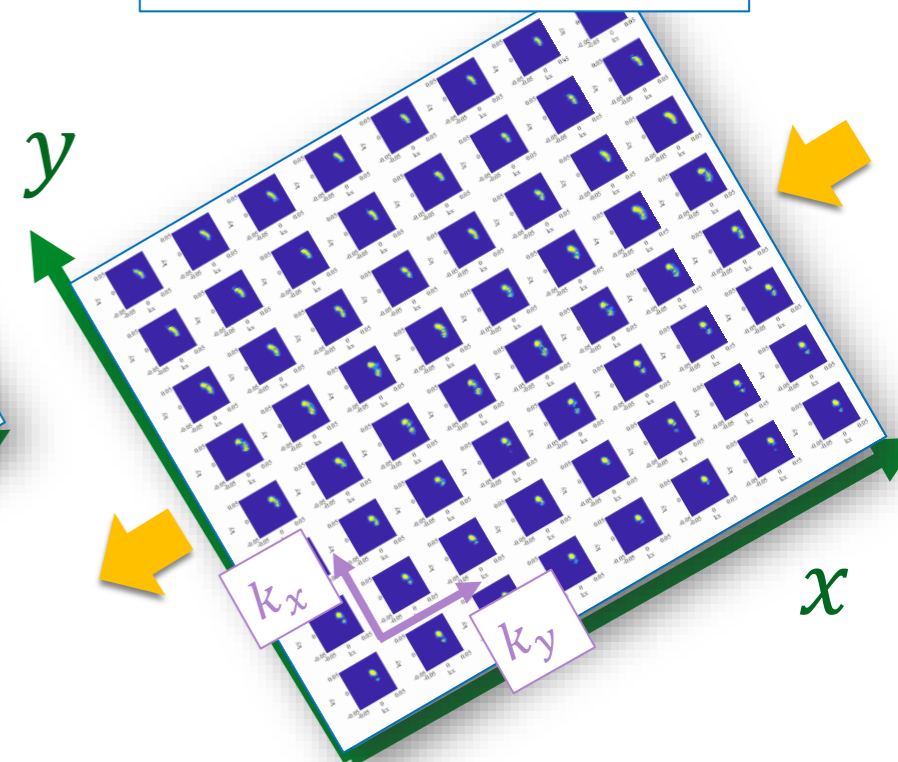
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

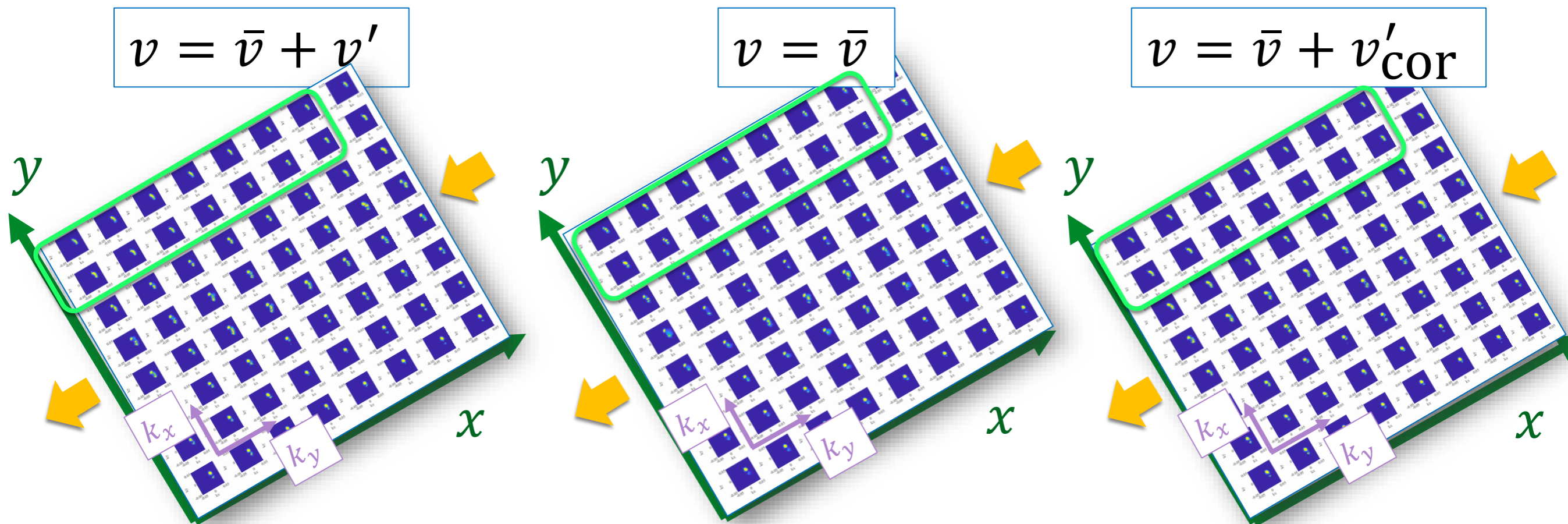
$$v = \bar{v} + v'_{\text{cor}}$$



Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v'

Wave spectra



Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

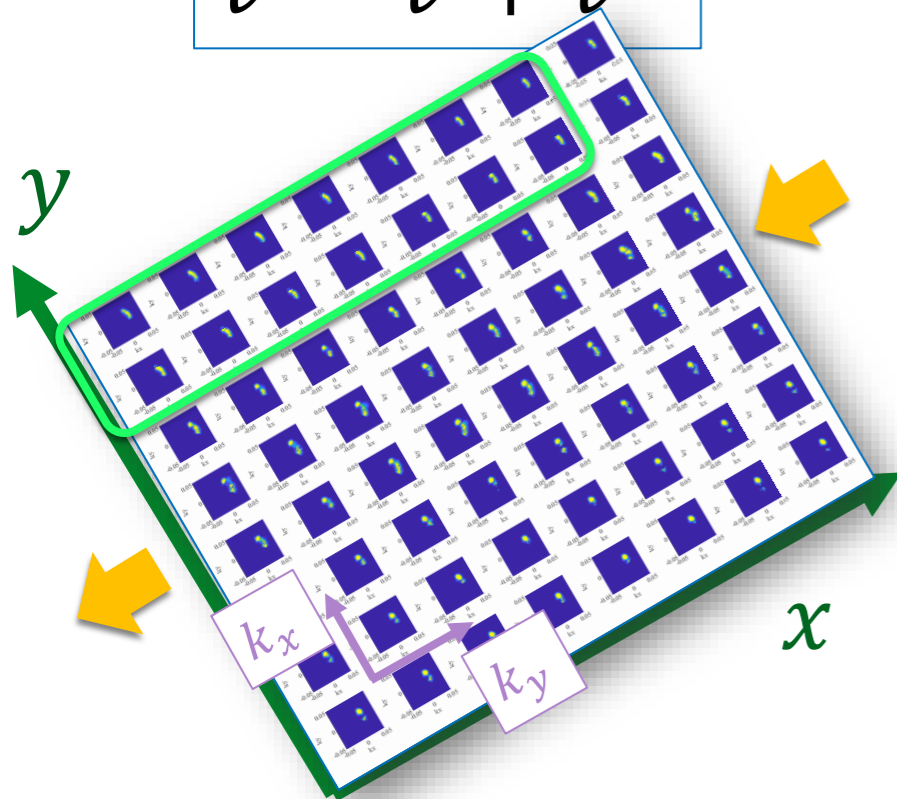
Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

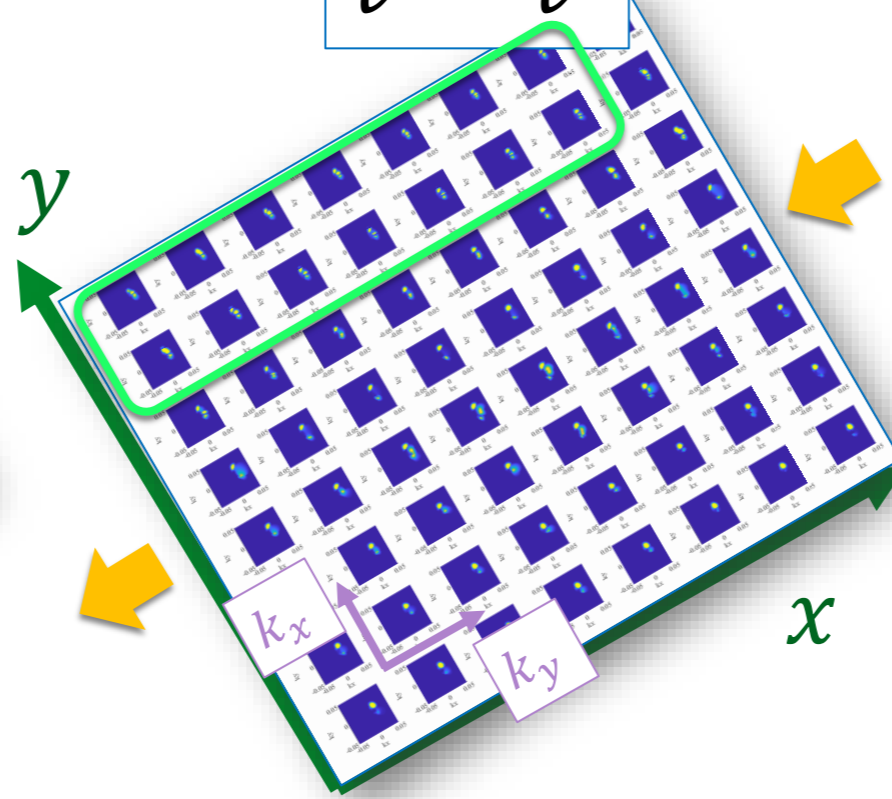
Time-correlated model for v'

Wave spectra

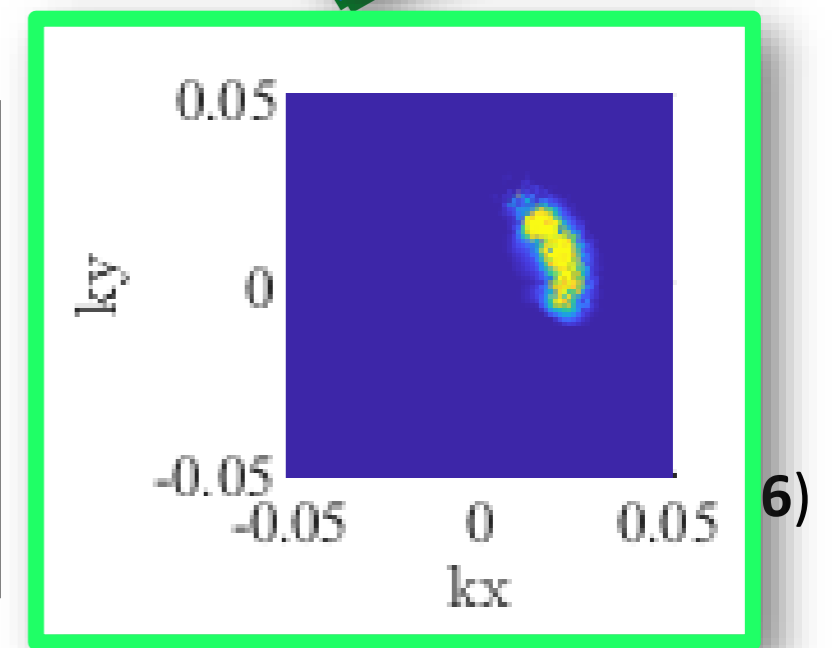
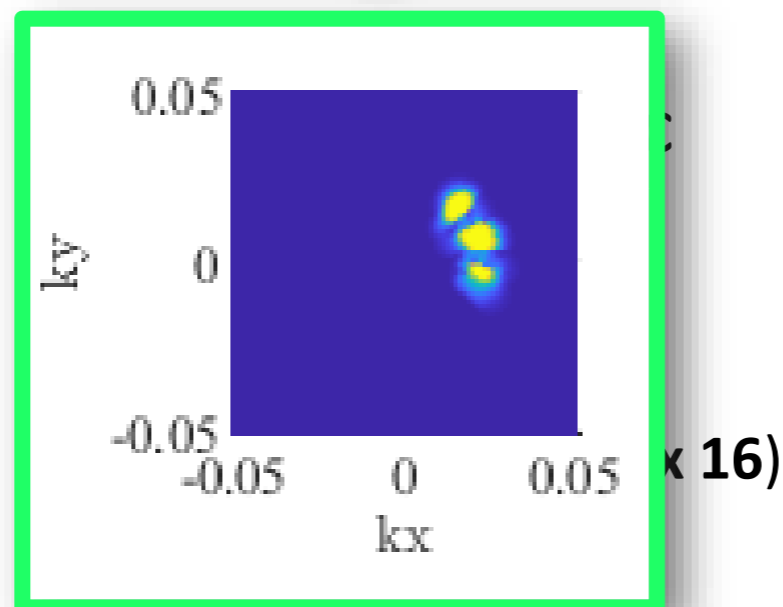
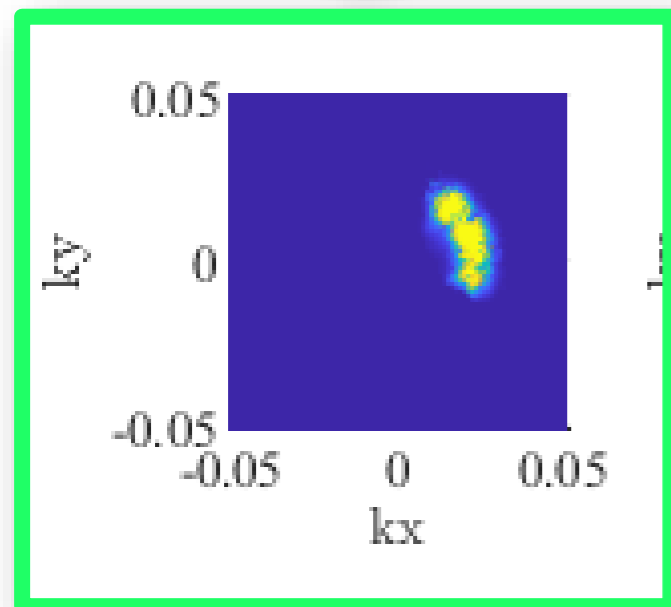
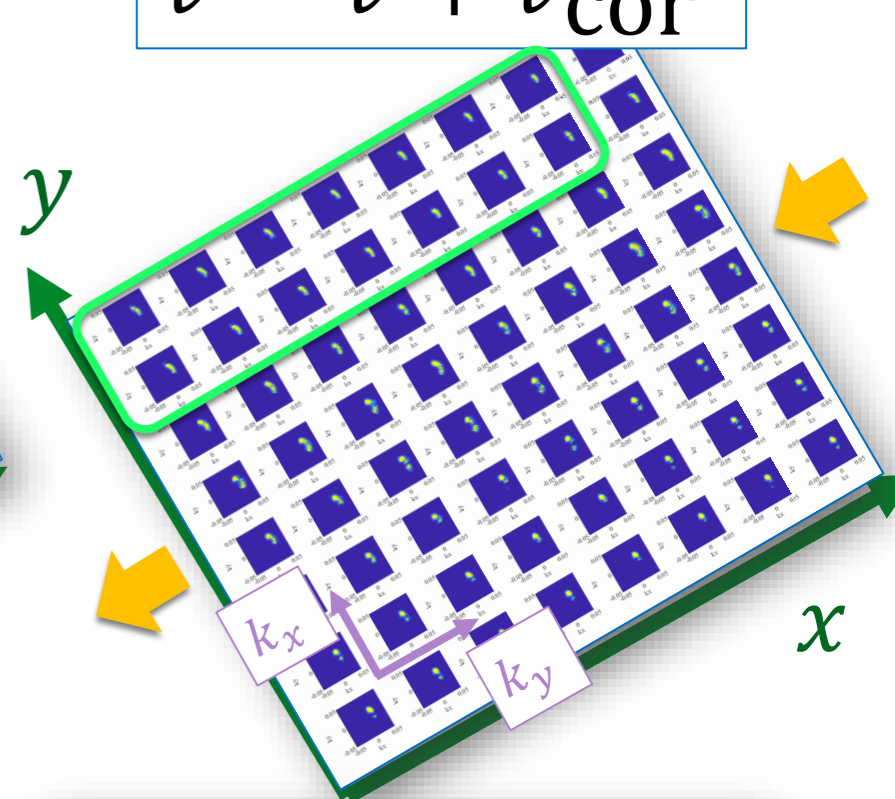
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



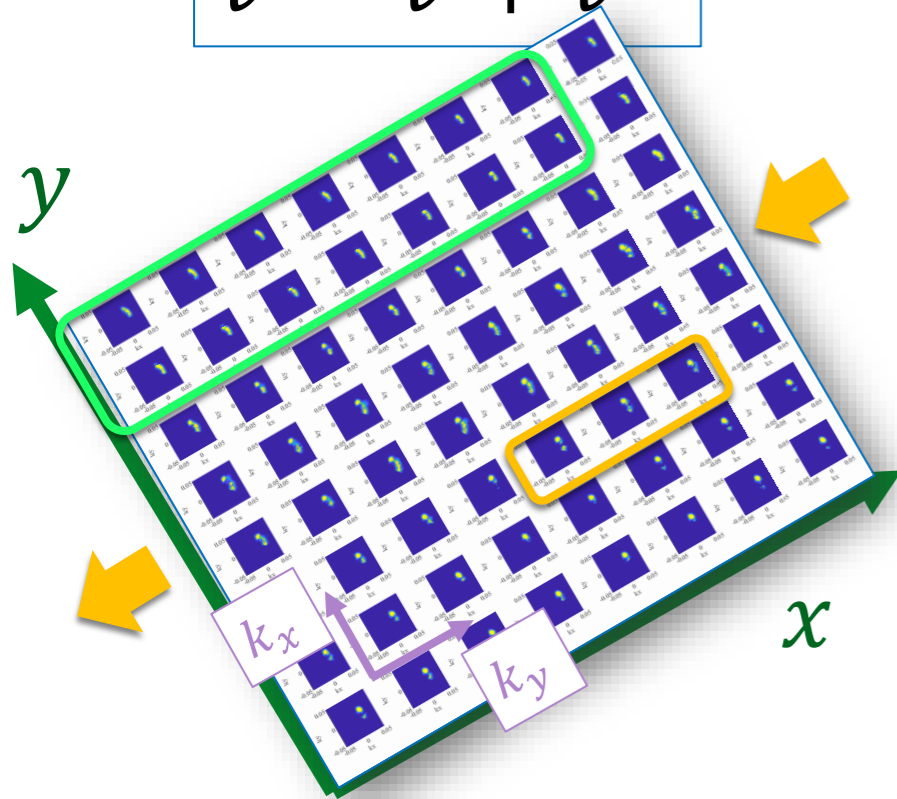
$$v = \bar{v} + v'_{\text{cor}}$$



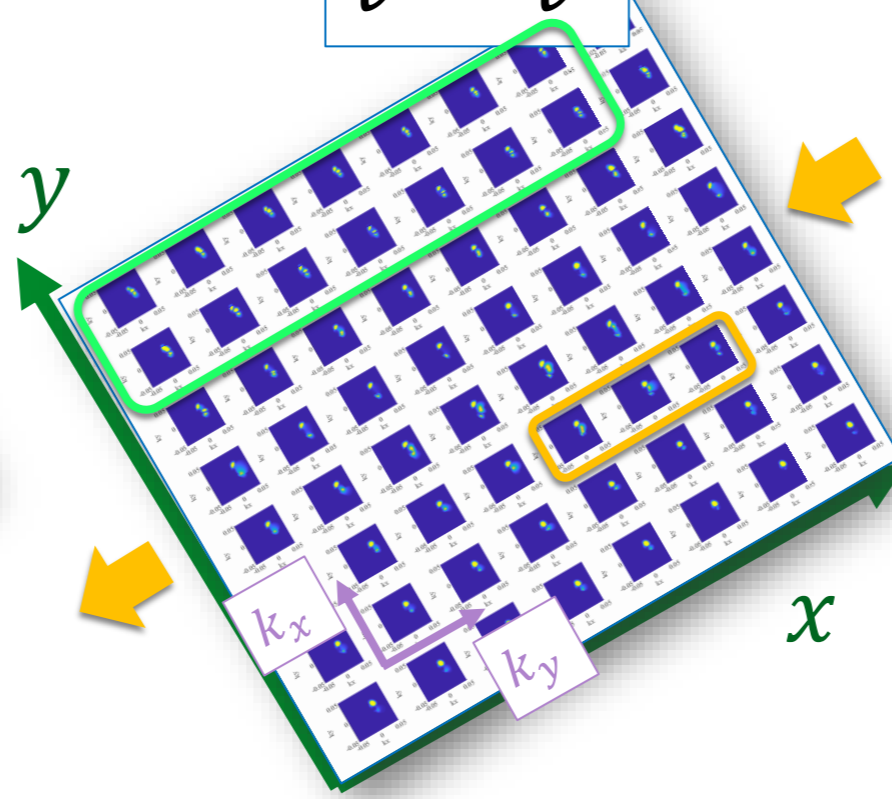
Time-correlated model for v'

Wave spectra

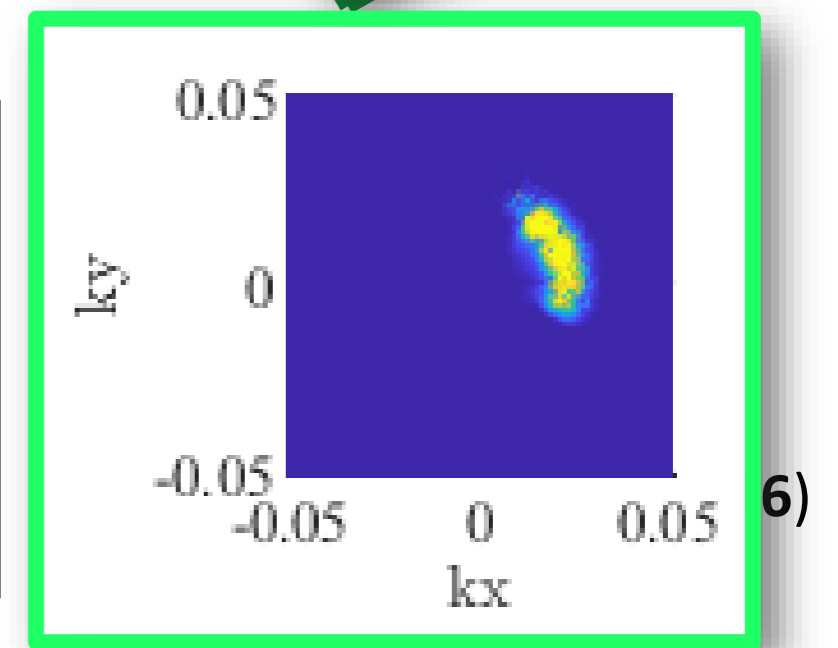
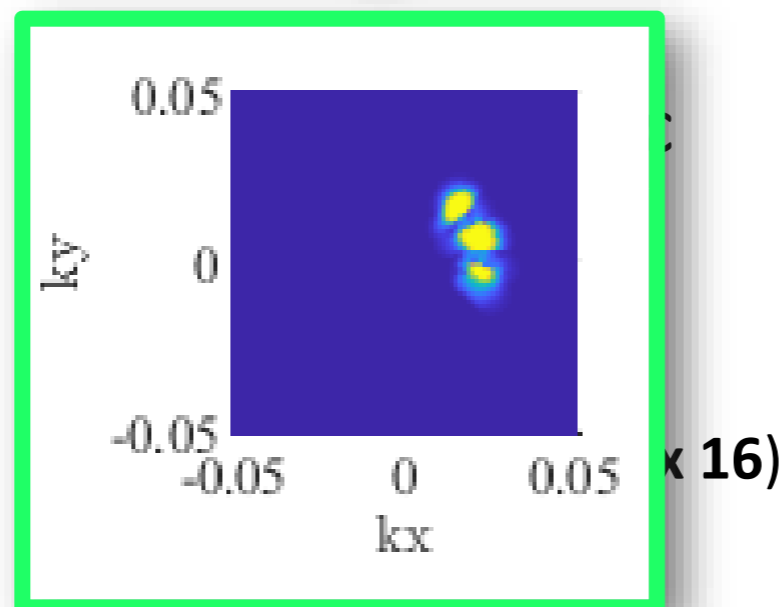
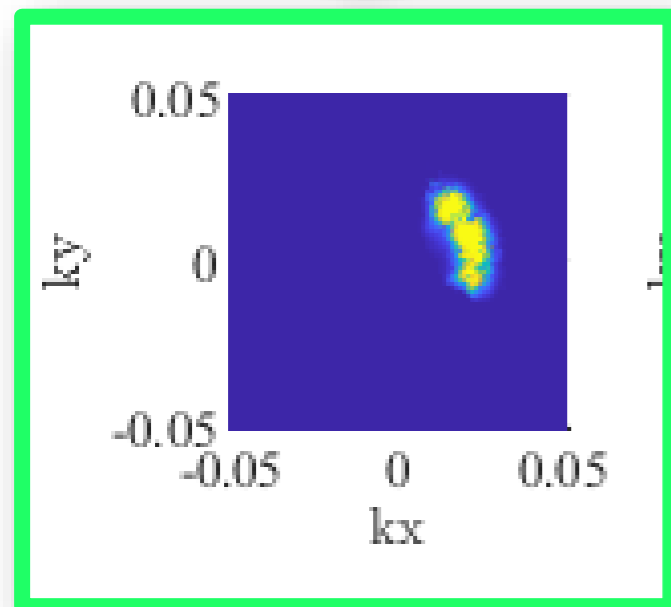
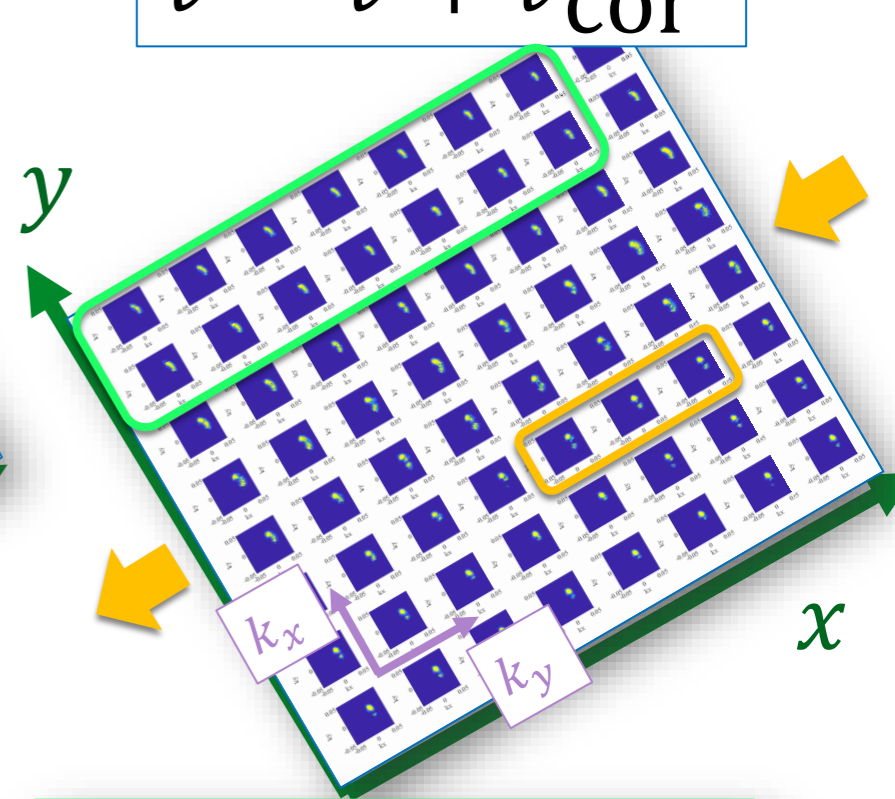
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



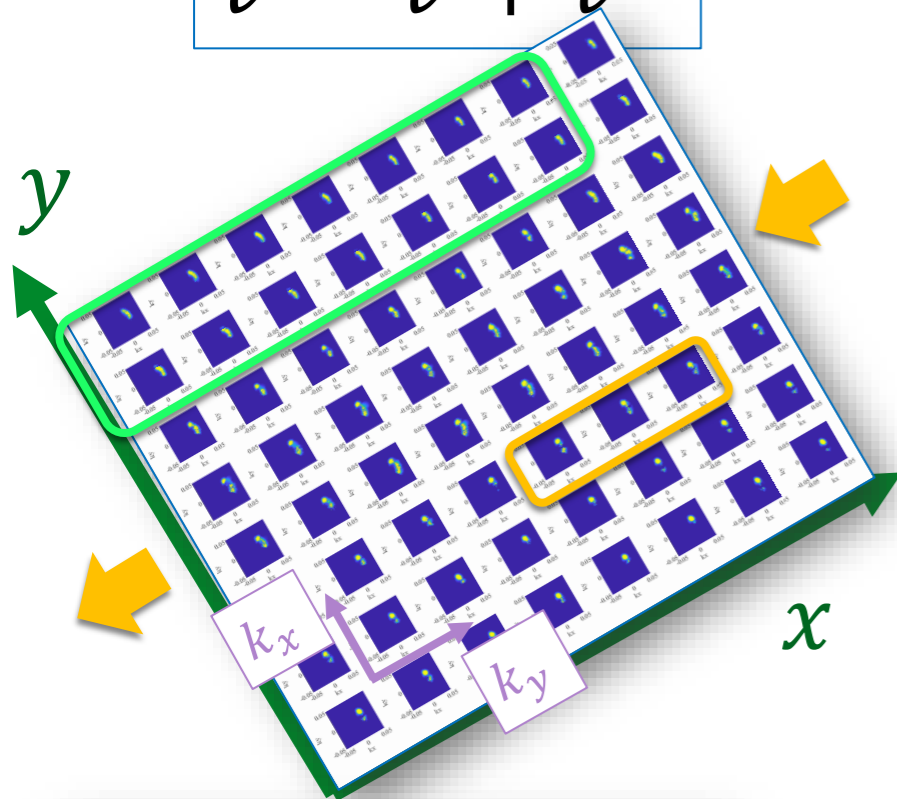
$$v = \bar{v} + v'_{\text{cor}}$$



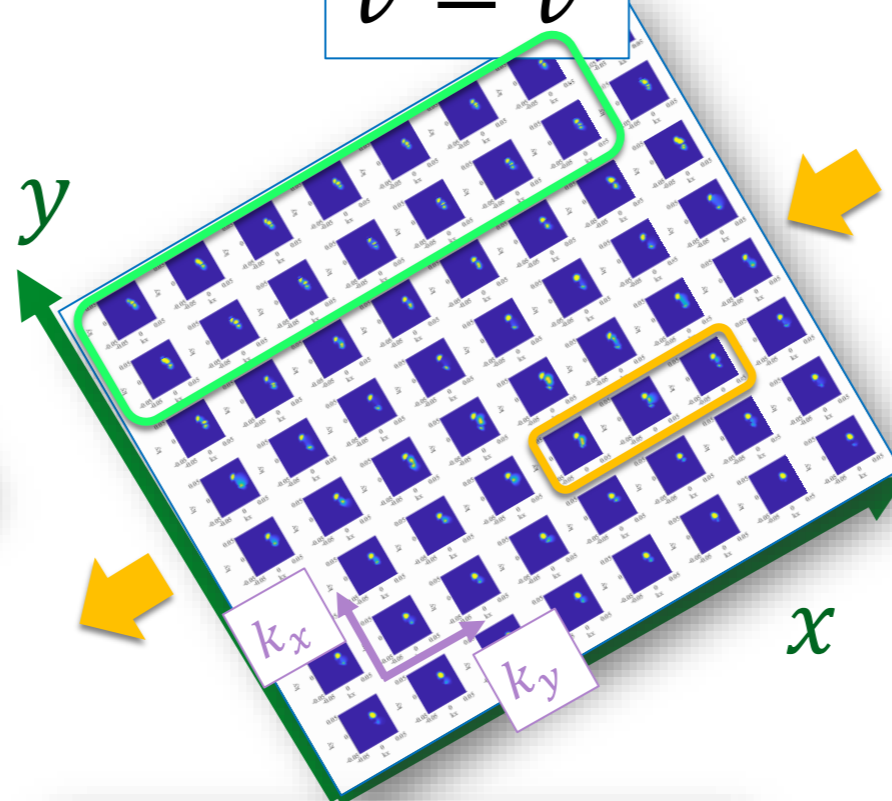
Time-correlated model for v'

Wave spectra

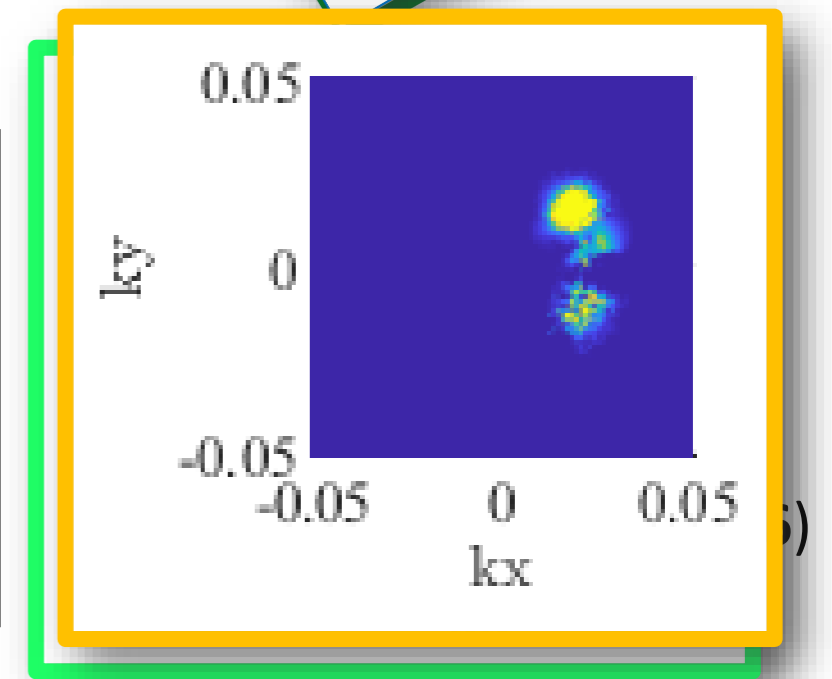
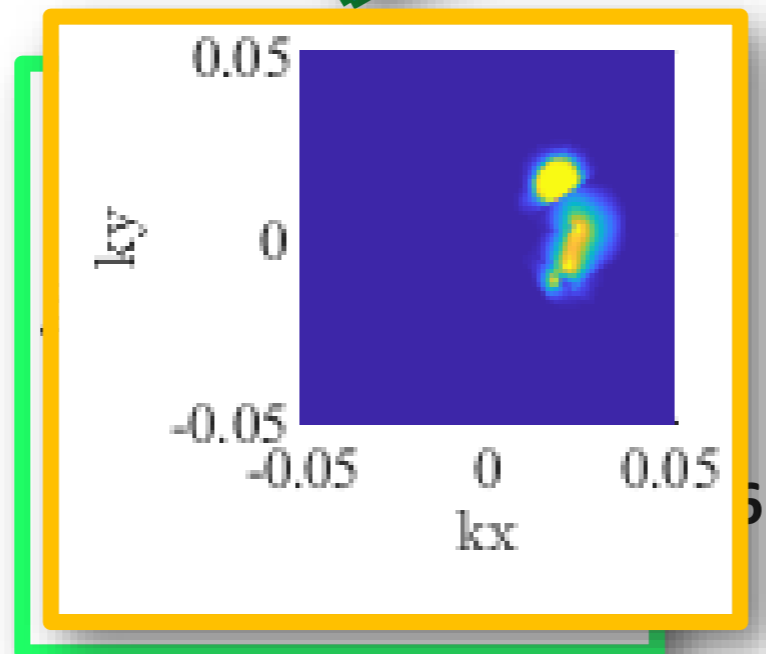
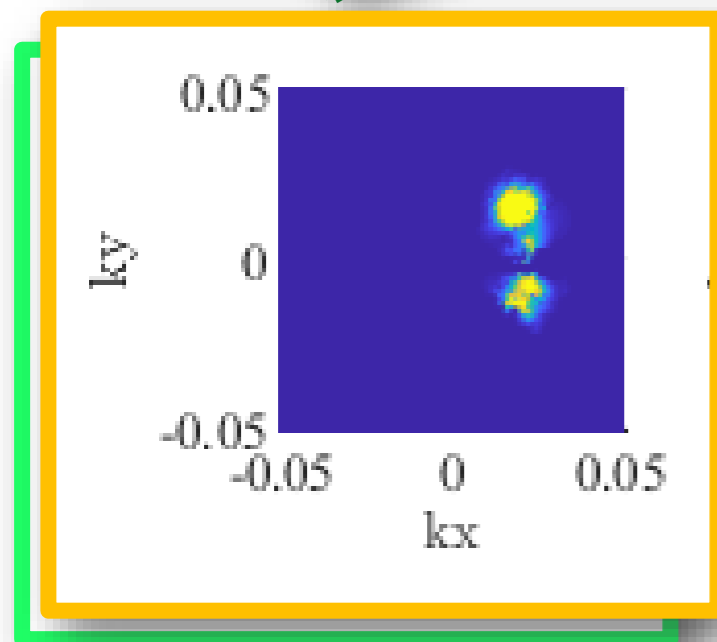
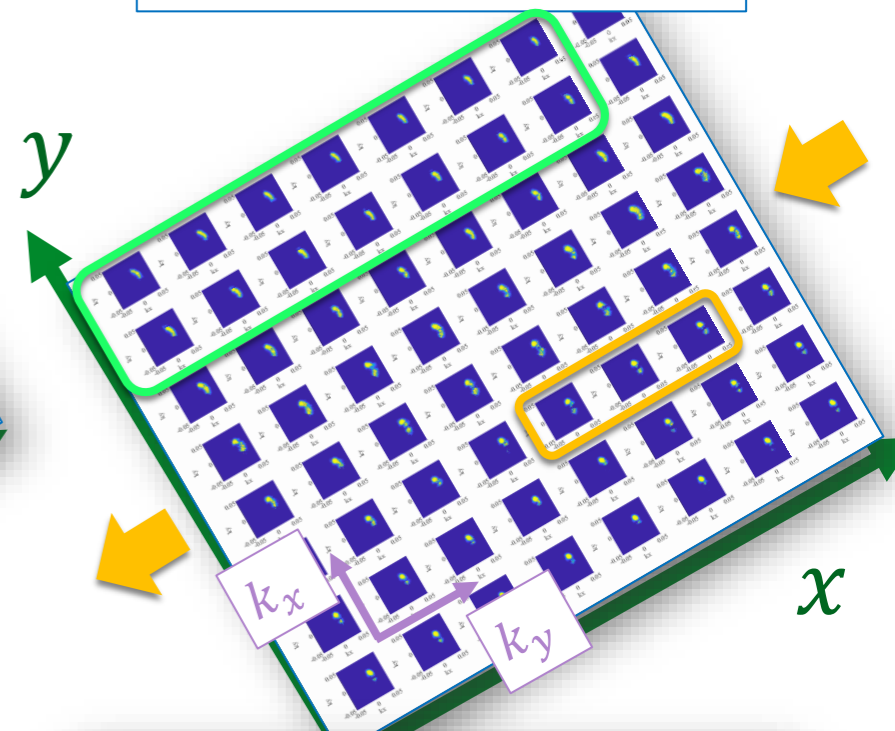
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



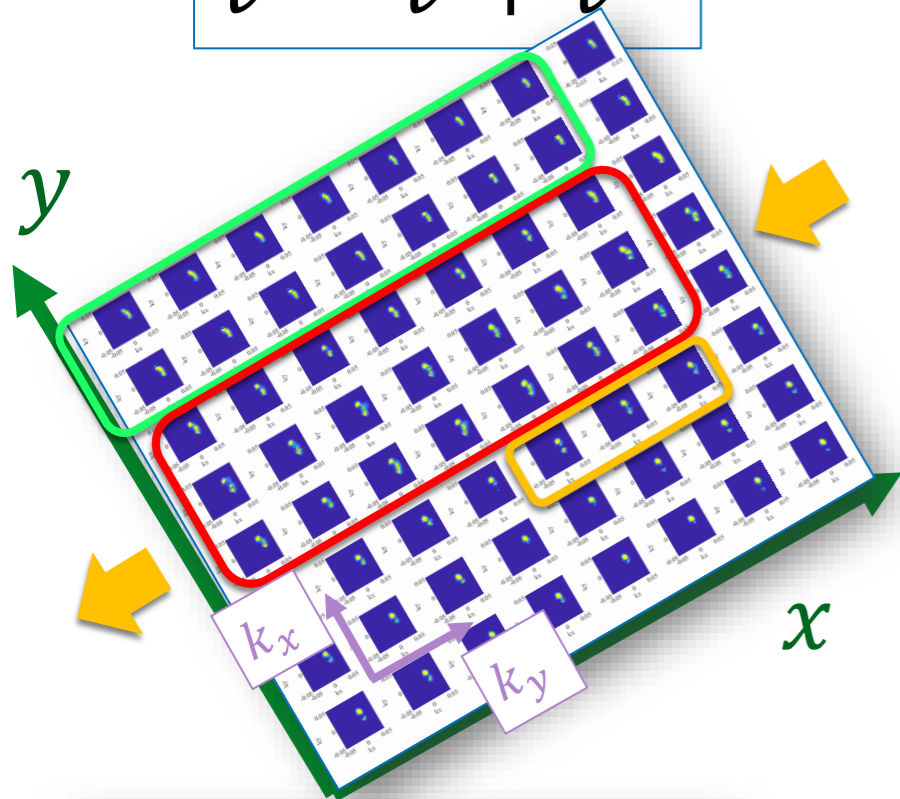
$$v = \bar{v} + v'_{\text{cor}}$$



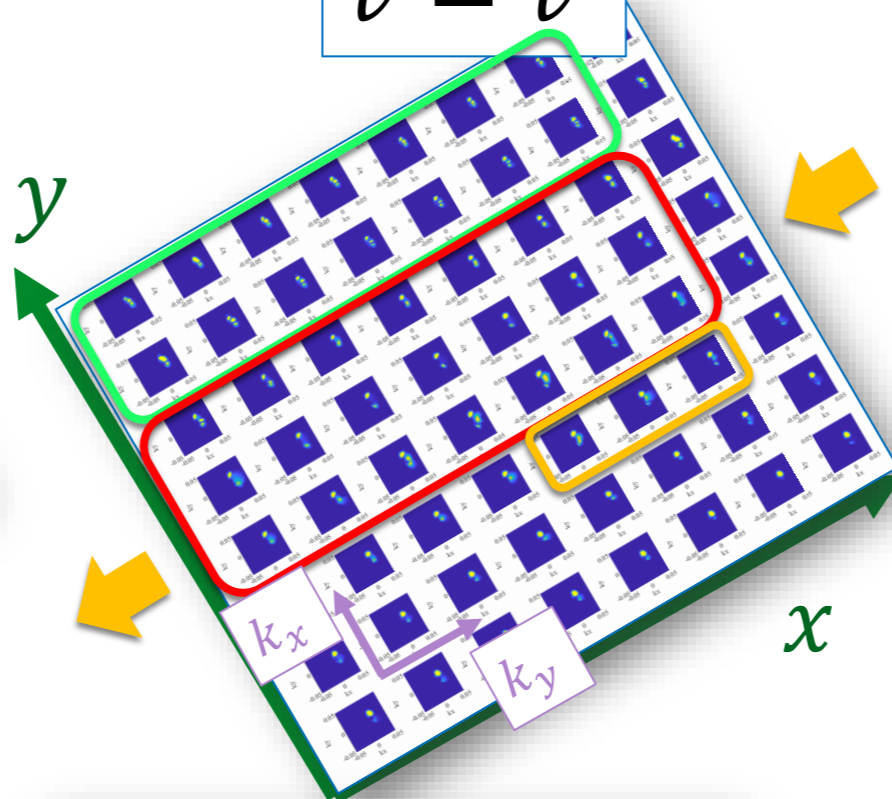
Time-correlated model for v'

Wave spectra

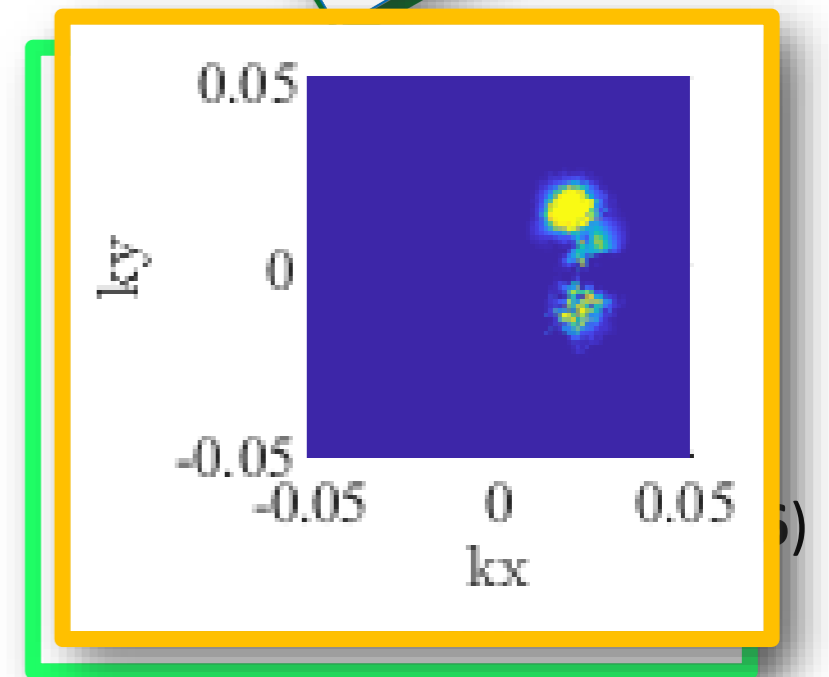
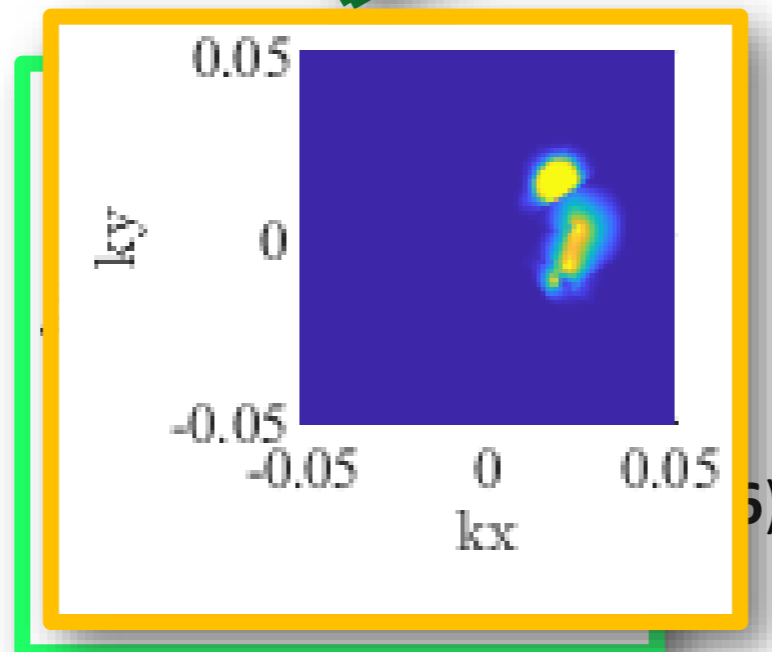
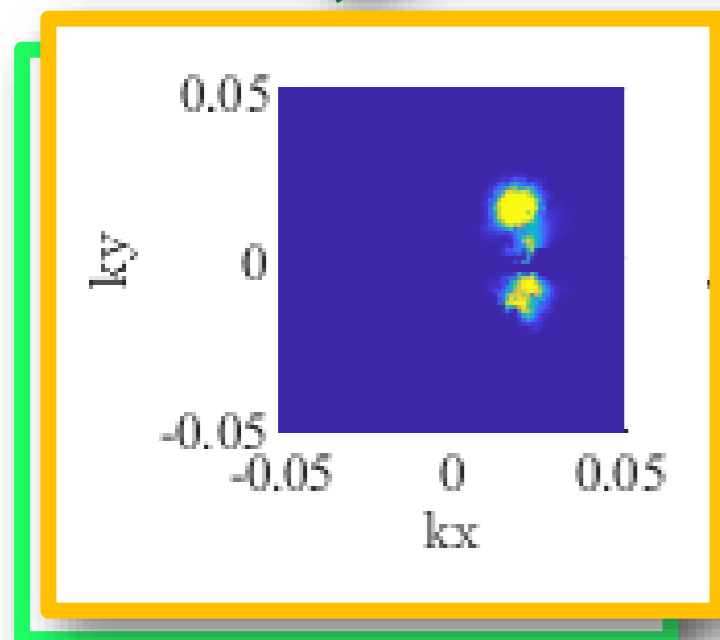
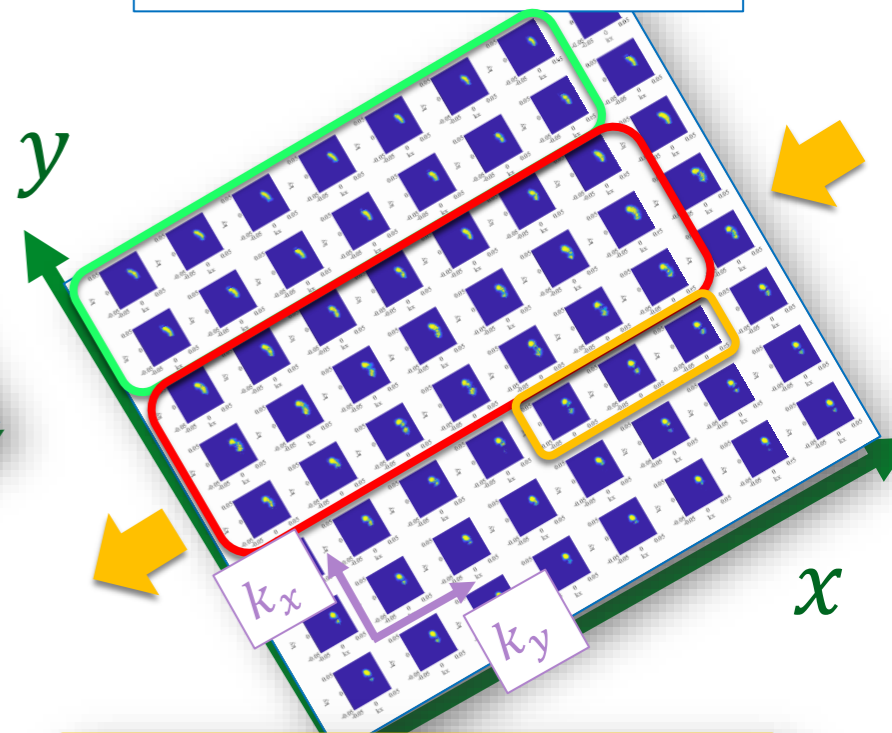
$$v = \bar{v} + v'$$



$$v = \bar{v}$$



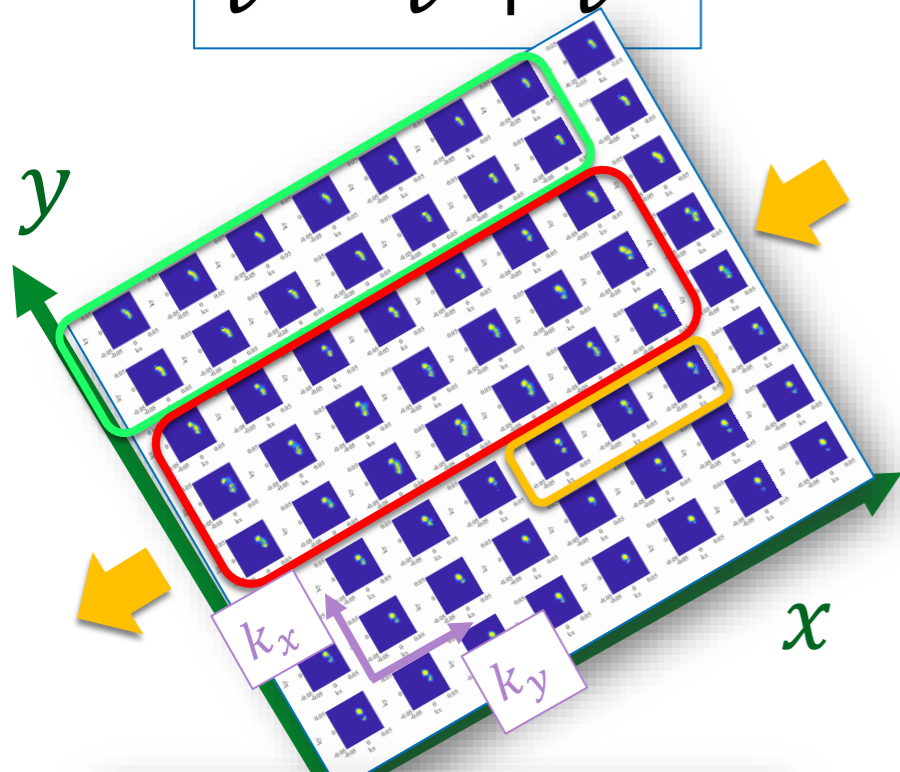
$$v = \bar{v} + v'_{\text{cor}}$$



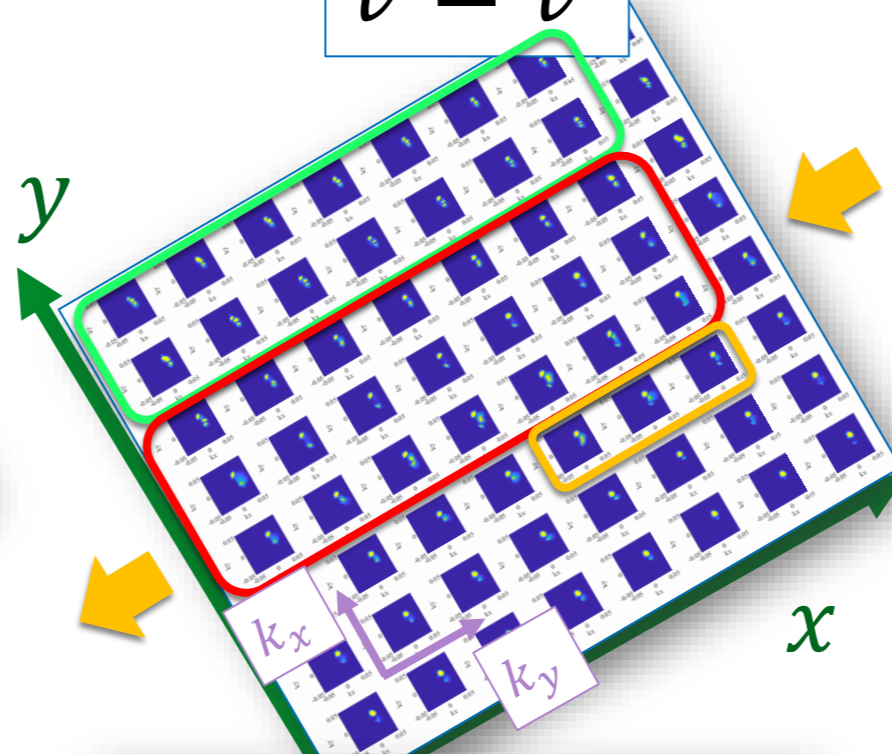
Time-correlated model for v'

Wave spectra

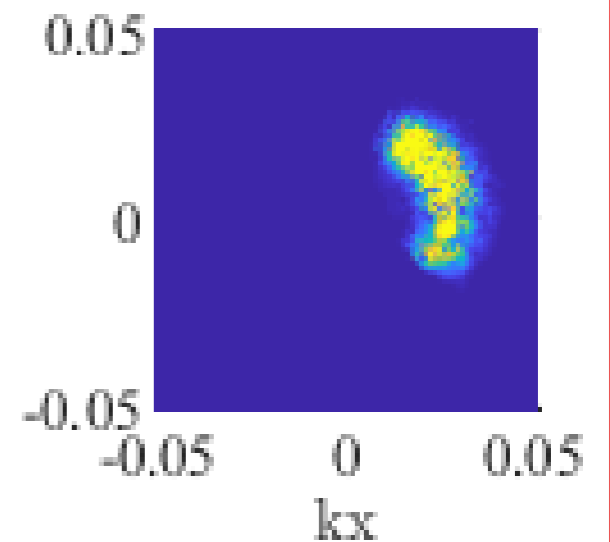
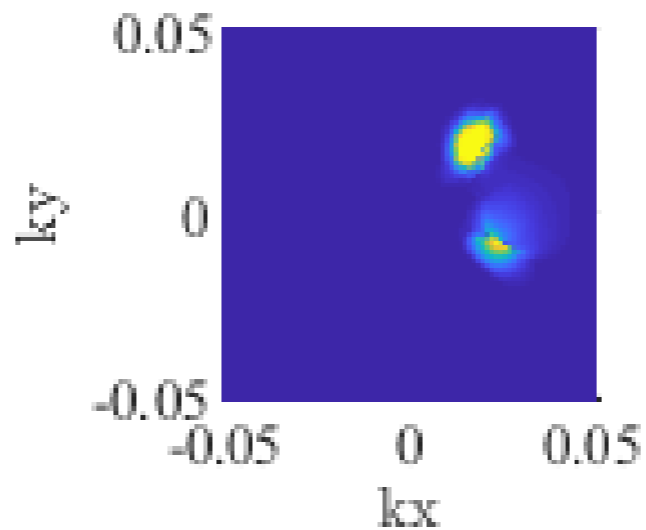
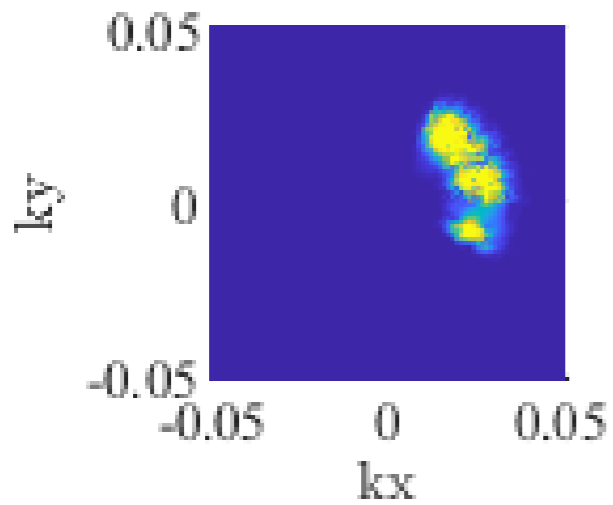
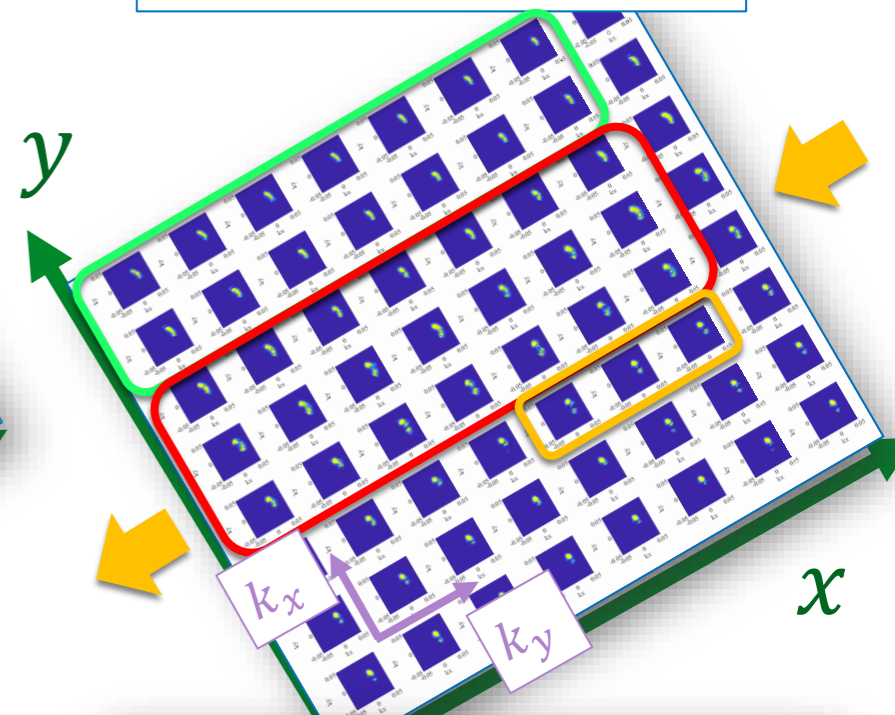
$$v = \bar{v} + v'$$



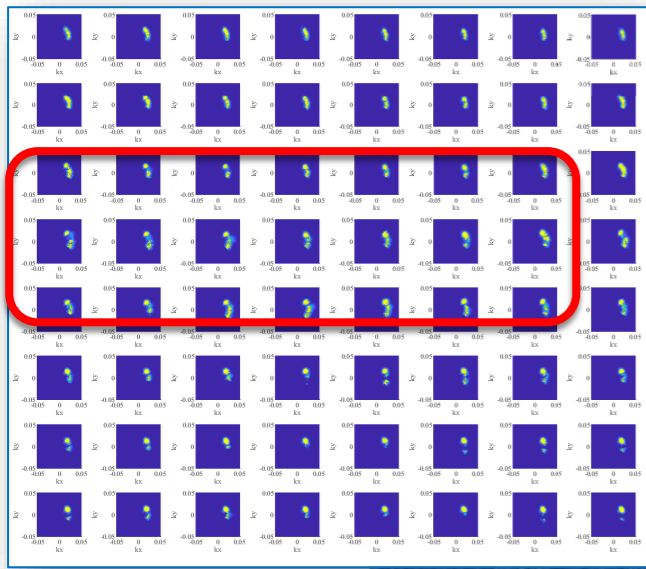
$$v = \bar{v}$$



$$v = \bar{v} + v'_{\text{cor}}$$



Sweel trapped in a jet

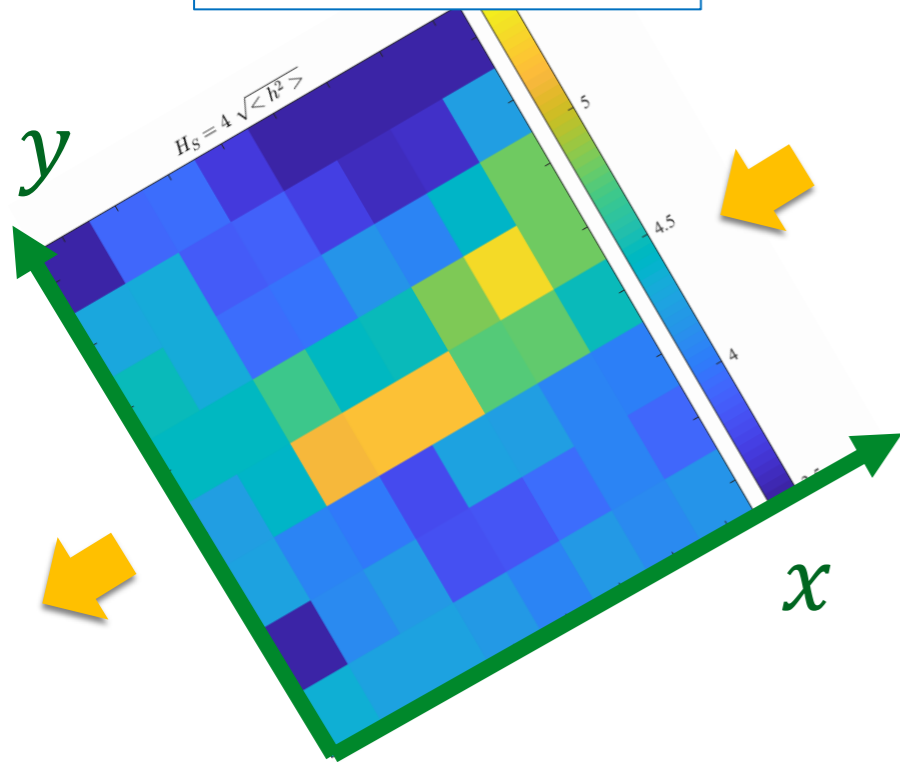


Sentinel 2

Time-correlated model for v'

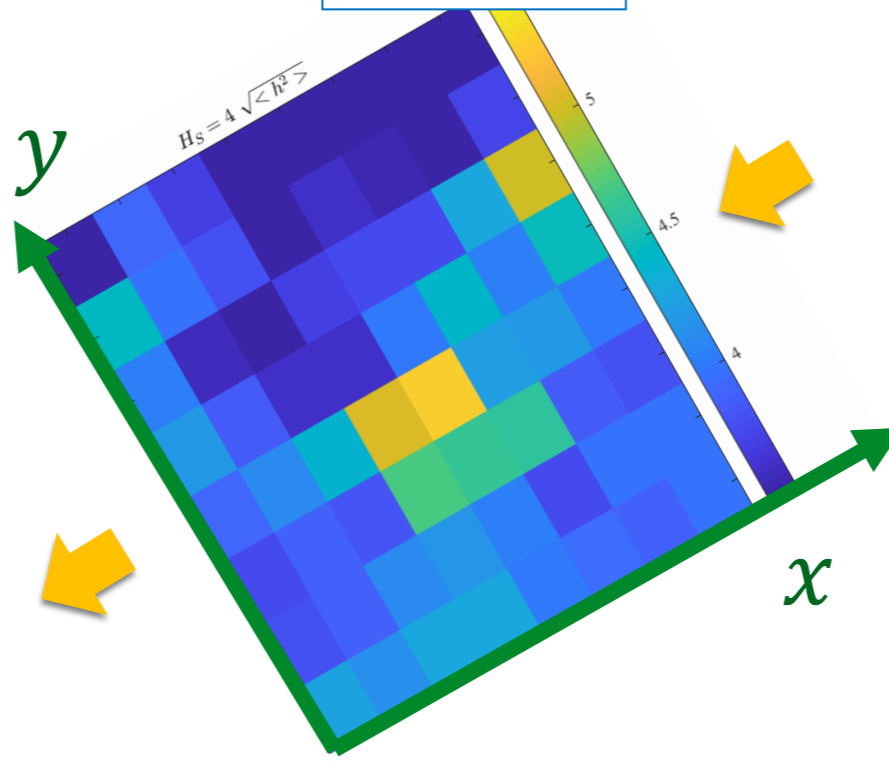
Large-scale significant wave height H_s

$$v = \bar{v} + v'$$



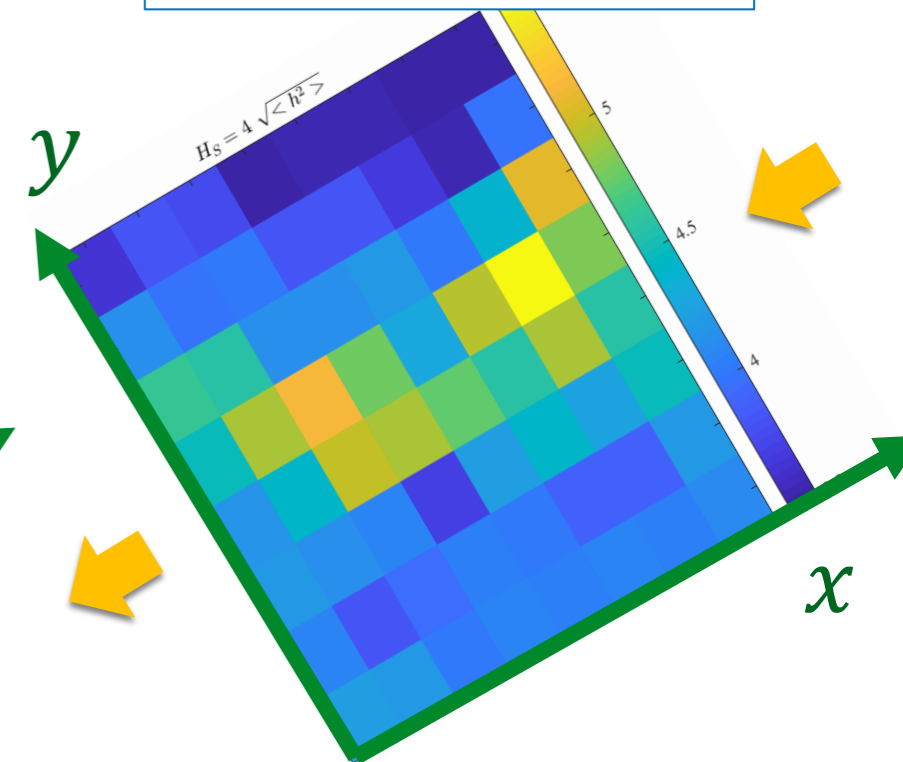
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

$$v = \bar{v} + v'_{\text{cor}}$$

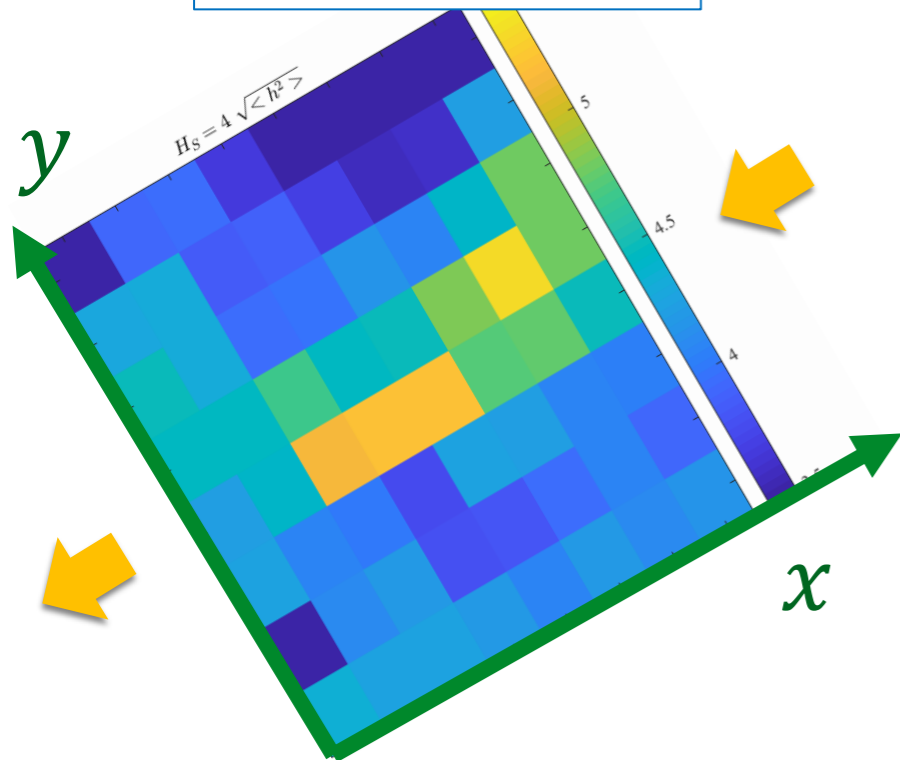


Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v'

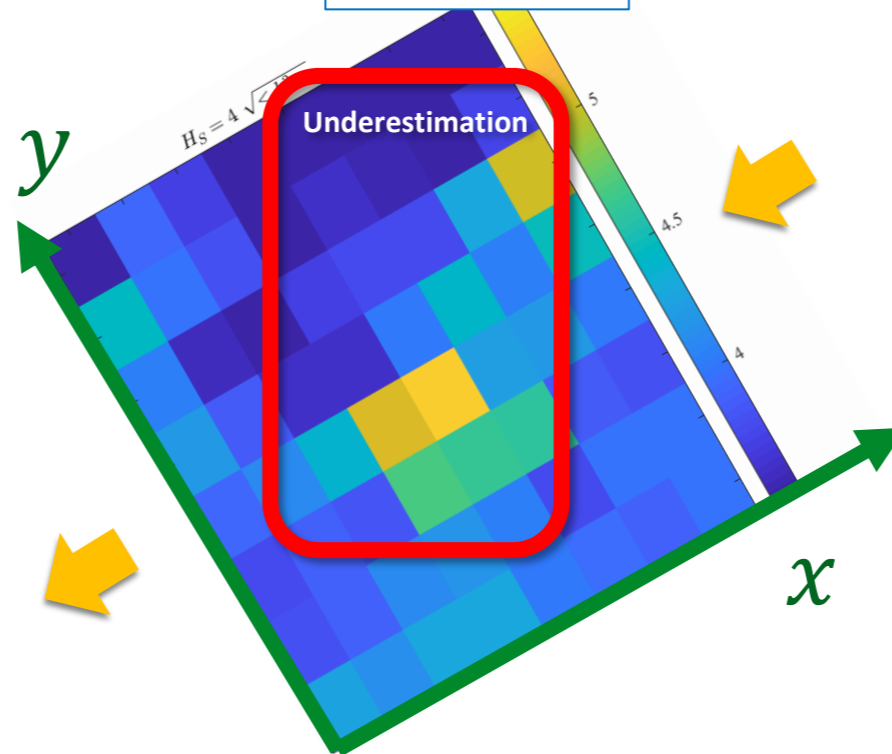
Large-scale significant wave height H_s

$$v = \bar{v} + v'$$



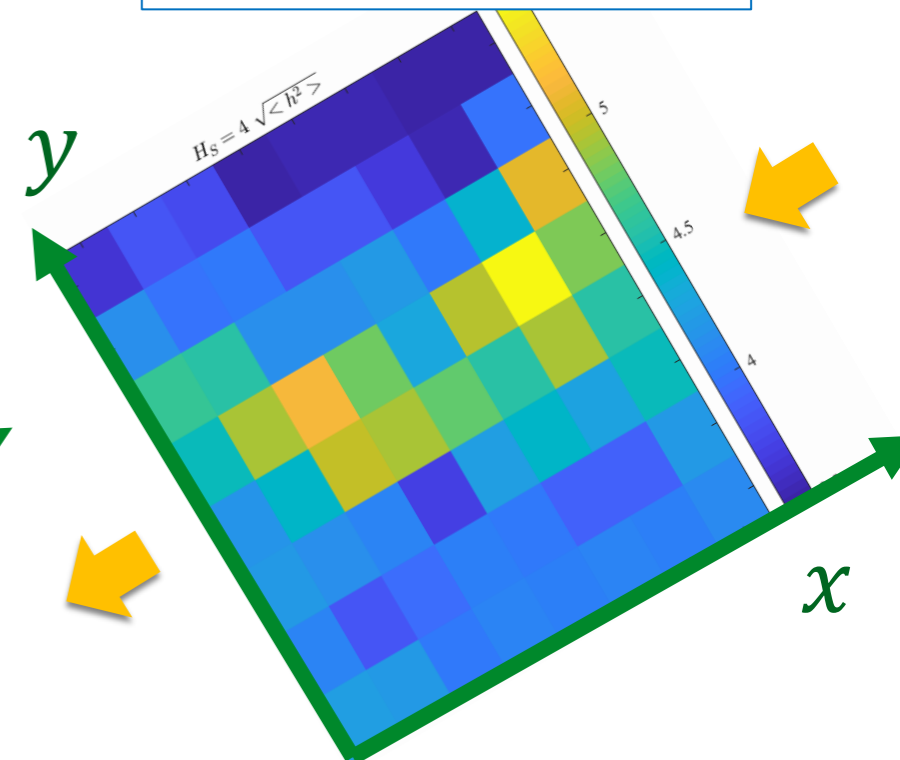
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

$$v = \bar{v}$$



Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

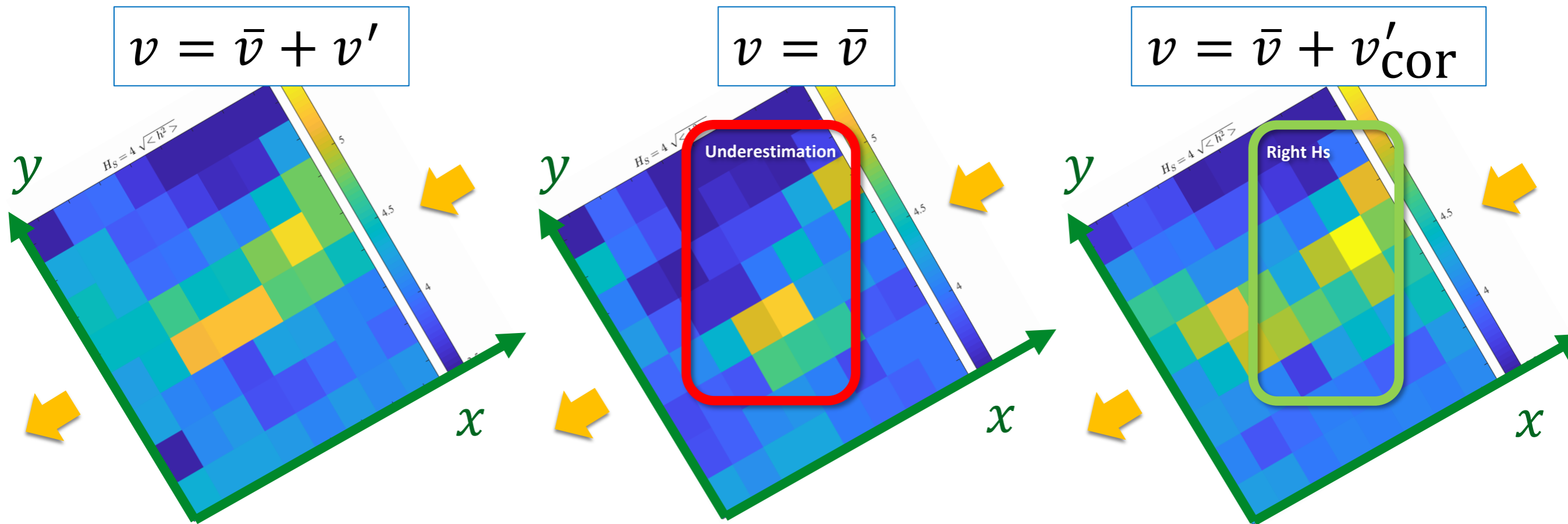
$$v = \bar{v} + v'_{\text{cor}}$$



Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v'

Large-scale significant wave height H_s



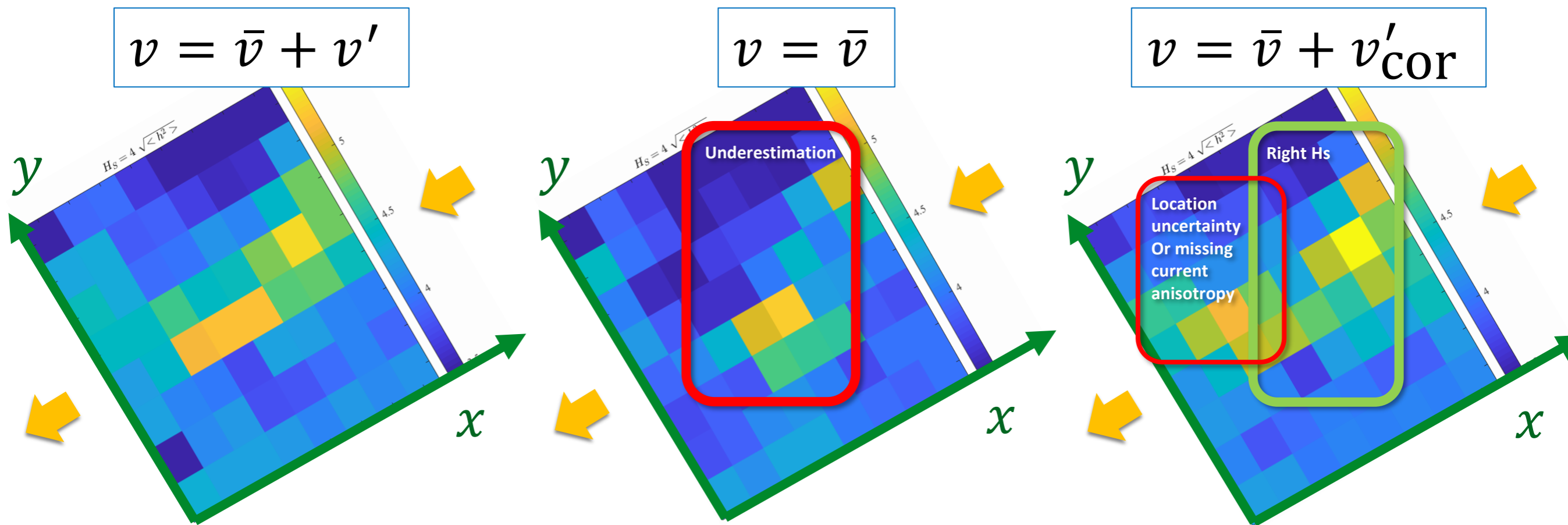
Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'

Time-correlated model for v'

Large-scale significant wave height H_s



Deterministic
reference:
wave groups in
512 x 512 **SQG** flow

Deterministic
benchmark:
wave groups in
smoothed flow \bar{v} (**16 x 16**)

Our random
model
wave groups in
smoothed flow \bar{v} (**16 x 16**)
+ time-correlated v'