

RESOLUTION-AWARE PRIOR GENERATION FOR REDUCED DATA ASSIMILATION IN FLUID DYNAMICS

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AIM

Data assimilation of 3D fluid flow under severe dimensional reduction
typically, $10^7 \rightarrow O(10)$ degrees of freedom

Application:

Real-time estimation and prediction
using strongly-limited computational resources & few sensors
for monitoring or control

CONTENT

- I. State of the art
 - a. Intrusive reduced order model (ROM)
 - b. Data assimilation
- II. Reduced location uncertainty models
 - a. Location uncertainty models (LUM)
 - b. Reduced LUM
- III. Numerical results
 - a. Test cases
 - b. Data assimilation



PART I

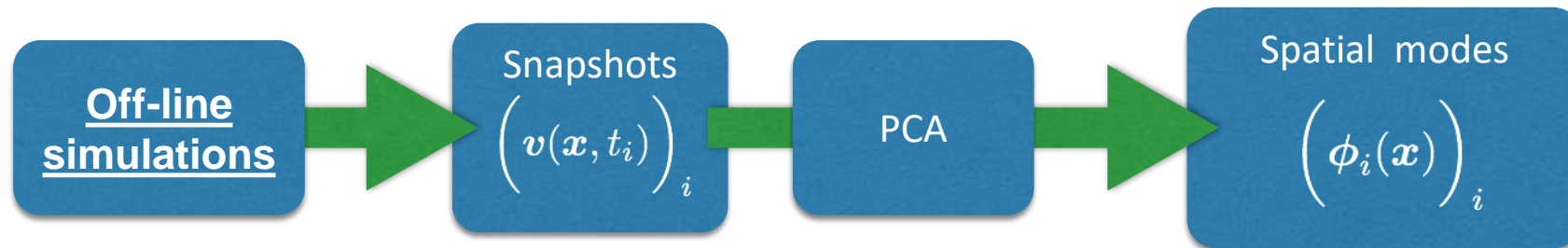
STATE OF THE ART

- a. Intrusive reduced order model (ROM)
- b. Data assimilation

INTRUSIVE REDUCED ORDER MODEL (ROM)

Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:



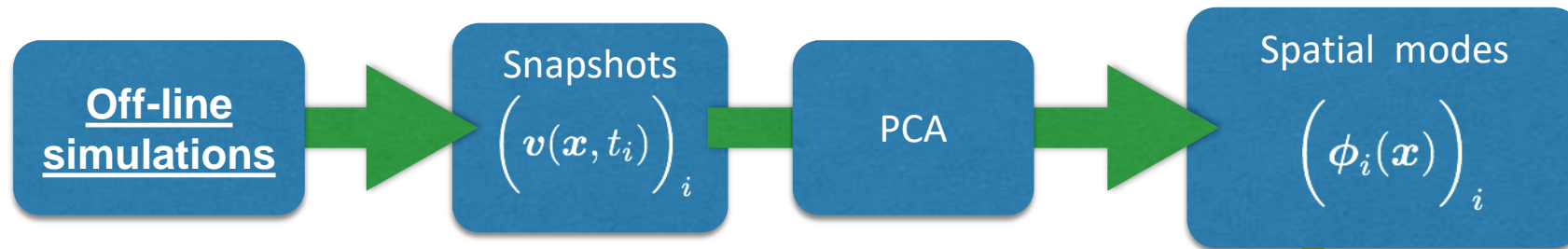
- Approximation:

$$v(x, t) \approx \sum_{i=0}^n b_i(t) \phi_i(x)$$

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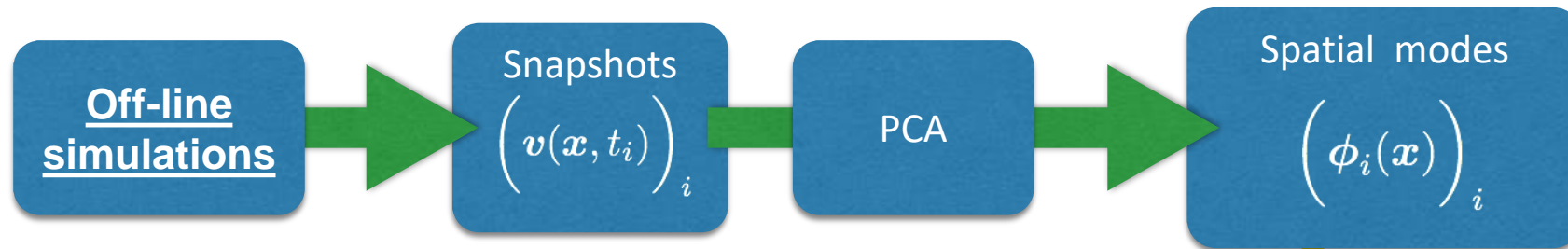
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A yellow arrow points from the $\phi_i(x)$ term in the equation to the "Spatial modes" box in the diagram above.

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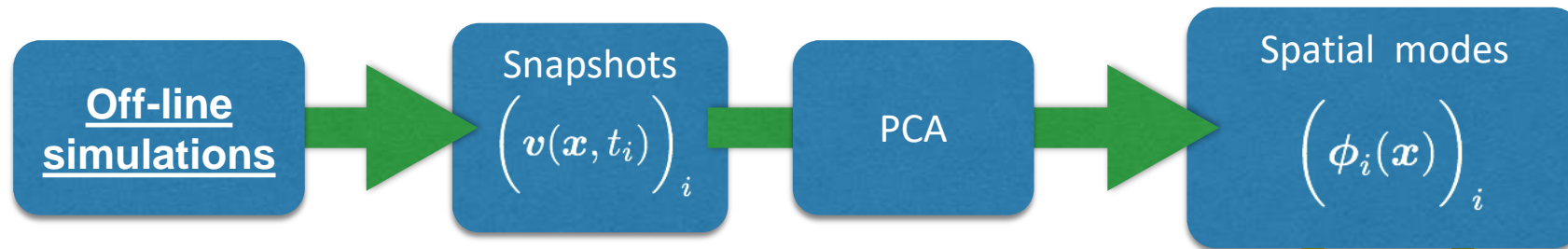
$$v(x, t) \approx \sum_{i=0}^n \underbrace{b_i(t)}_{\text{Resolved modes}} \phi_i(x)$$

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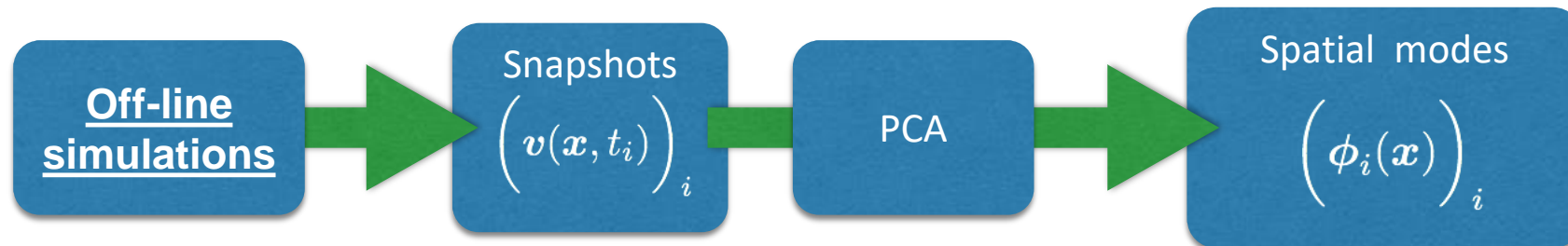
- Projection of the “*physics*”
onto the spatial modes
(POD-Galerkin)

$$\int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation (e.g. Navier-Stokes)})$$

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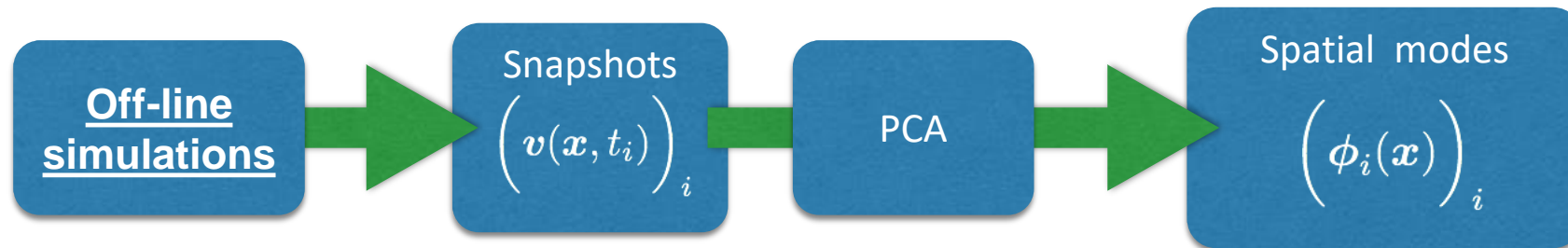
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→ ROM for very fast simulation of temporal modes

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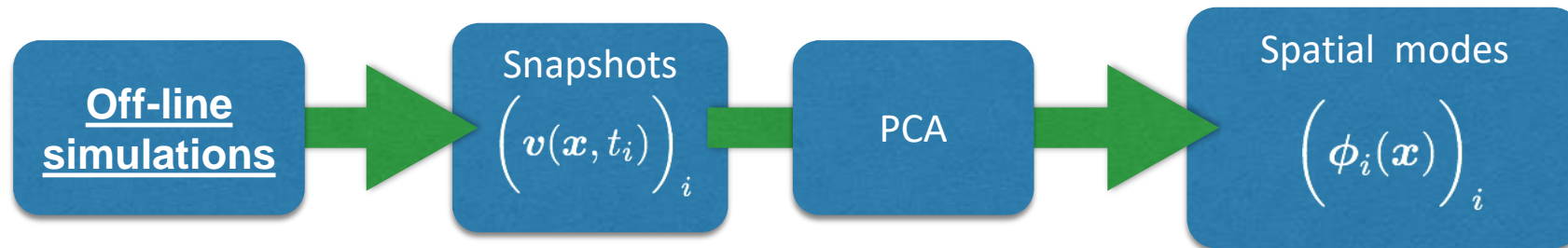
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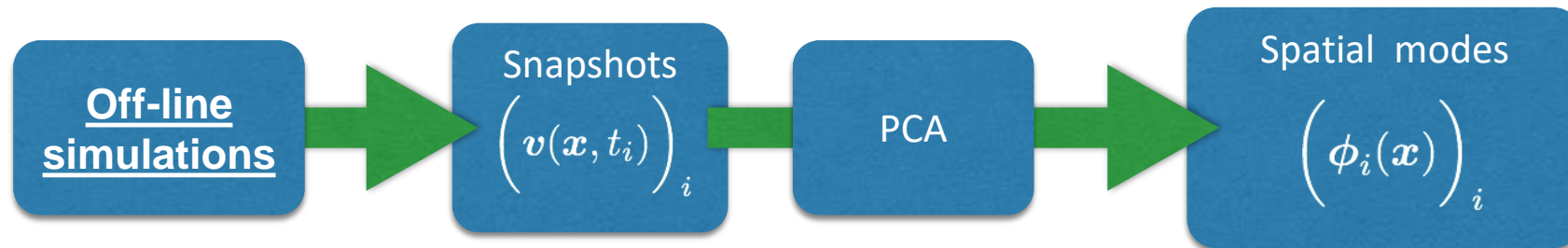
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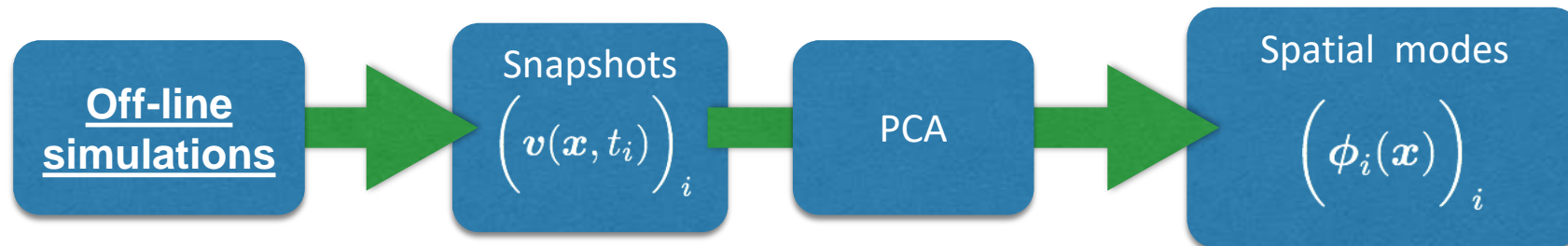
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SPOILER

- Projection of the “physics” onto the spatial modes (POD-Galerkin)

$$\int_{\Omega} dx \phi_i(x) \cdot \text{(Randomized Navier-Stokes)}$$

→ ROM for very fast simulation of temporal modes

DATA ASSIMILATION

= Coupling simulations and measurements y

**Numerical
Simulation
(ROM)**

→ erroneous

**On-line
measurements**

→ incomplete
→ possibly noisy

Velocity

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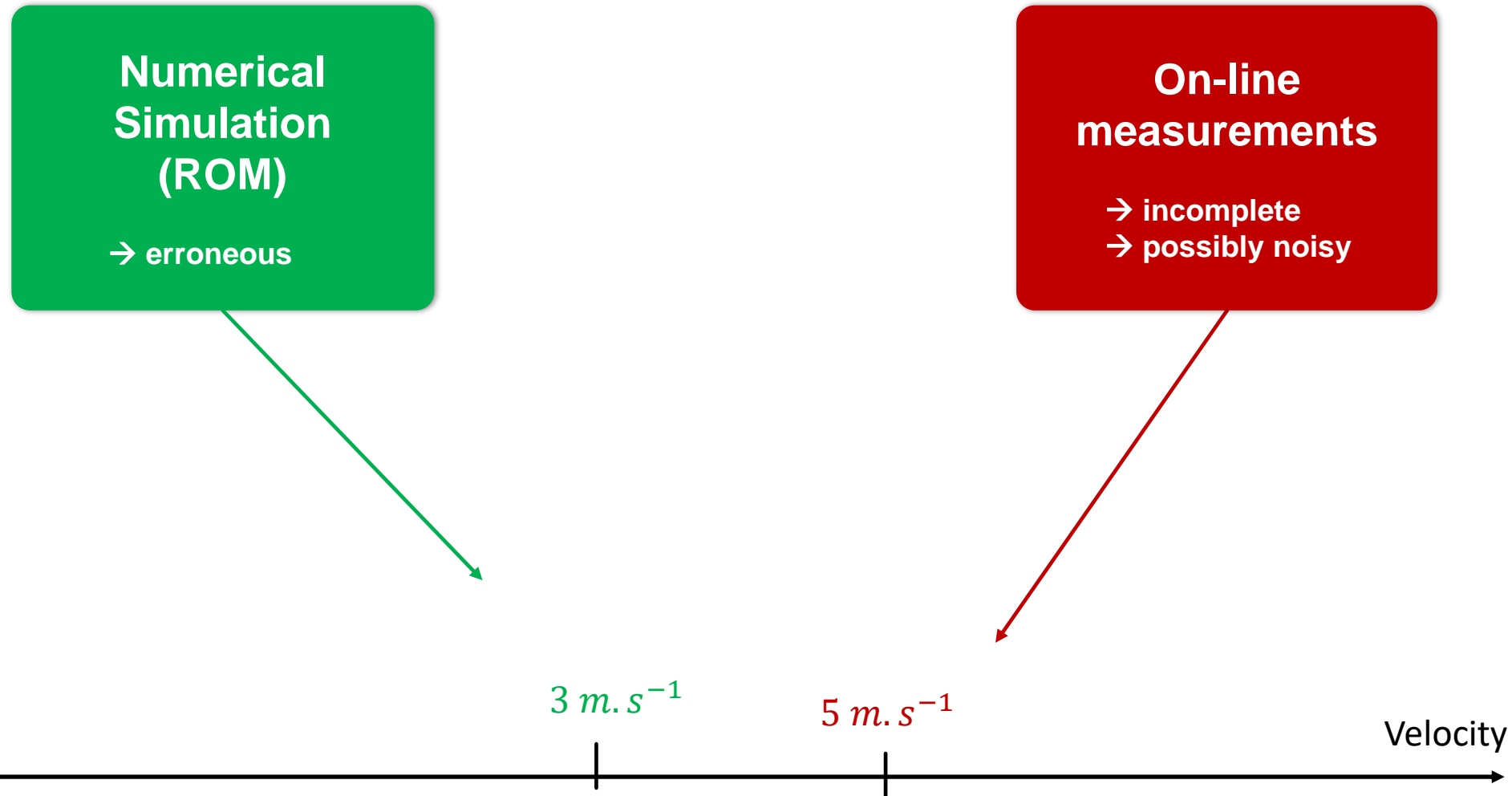
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$3 \text{ m} \cdot \text{s}^{-1}$

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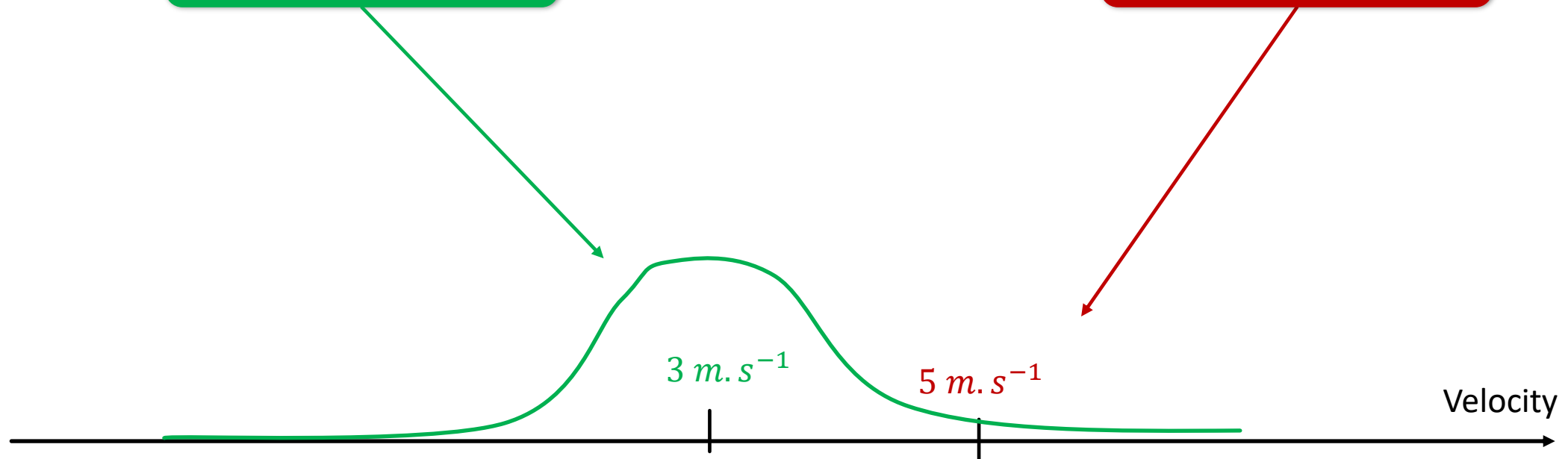
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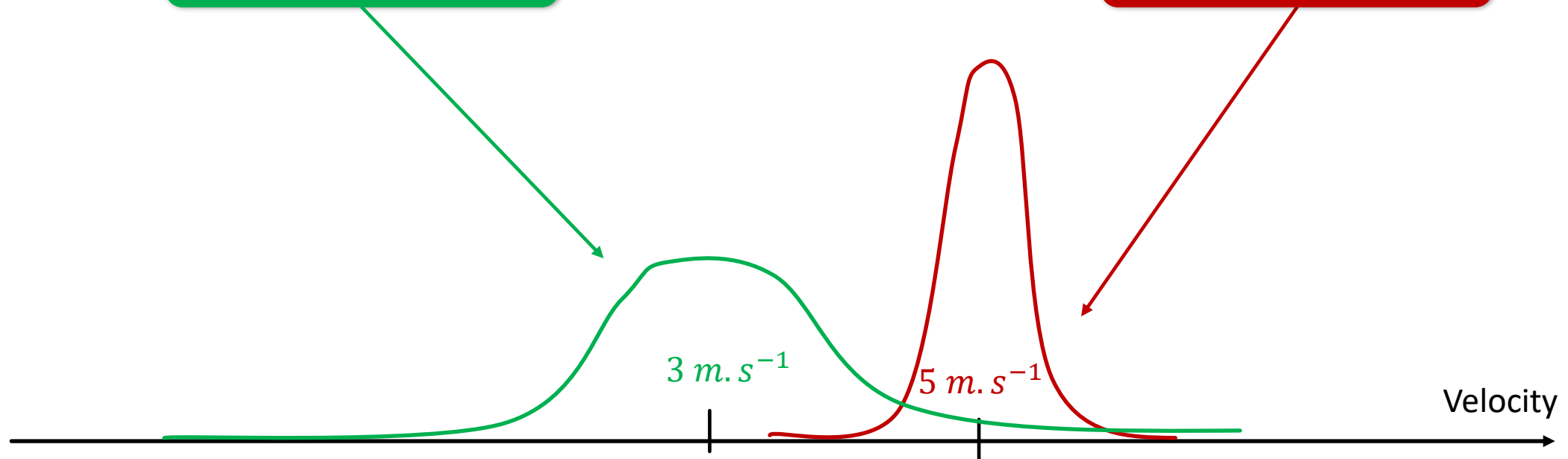
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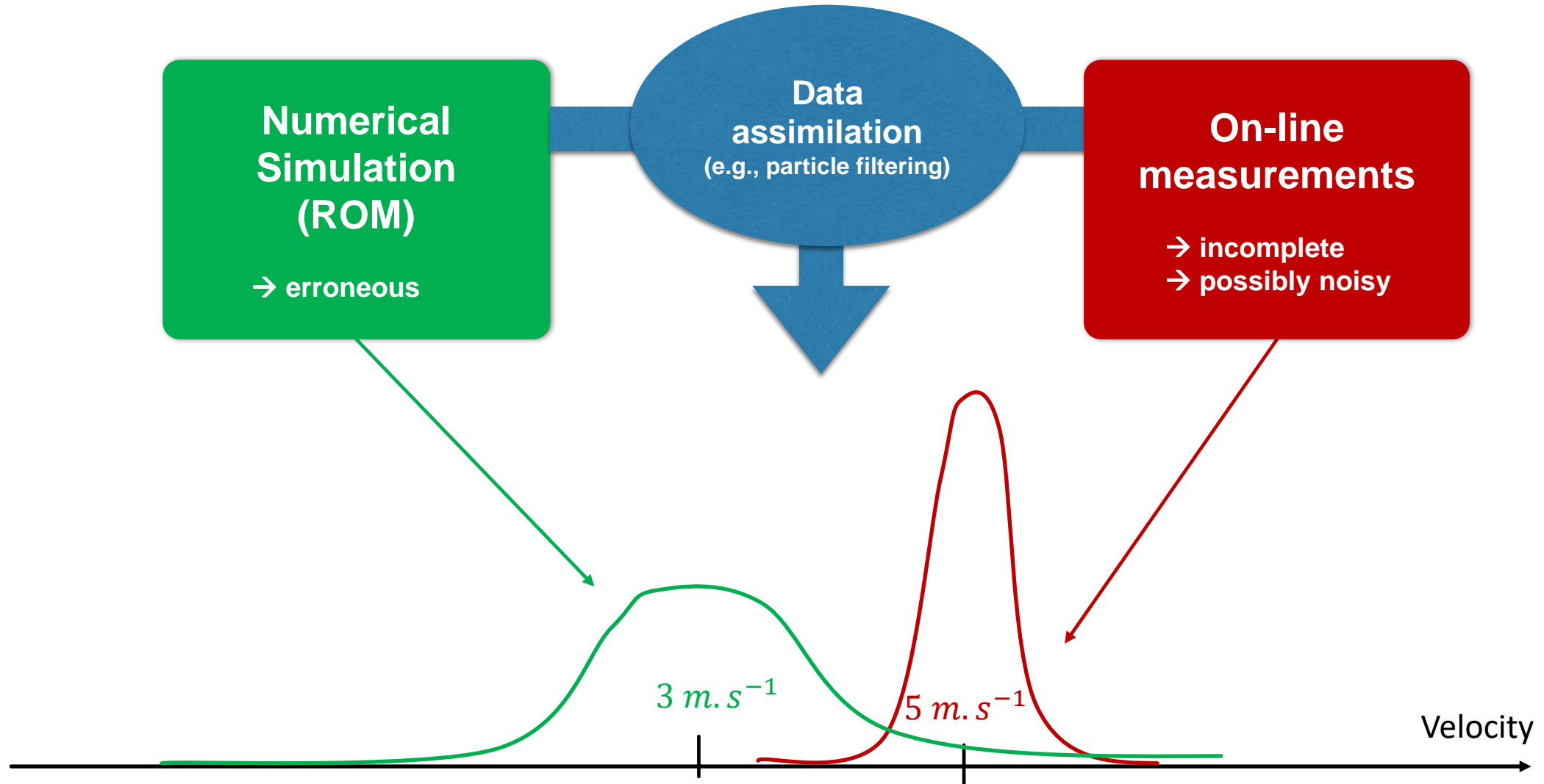
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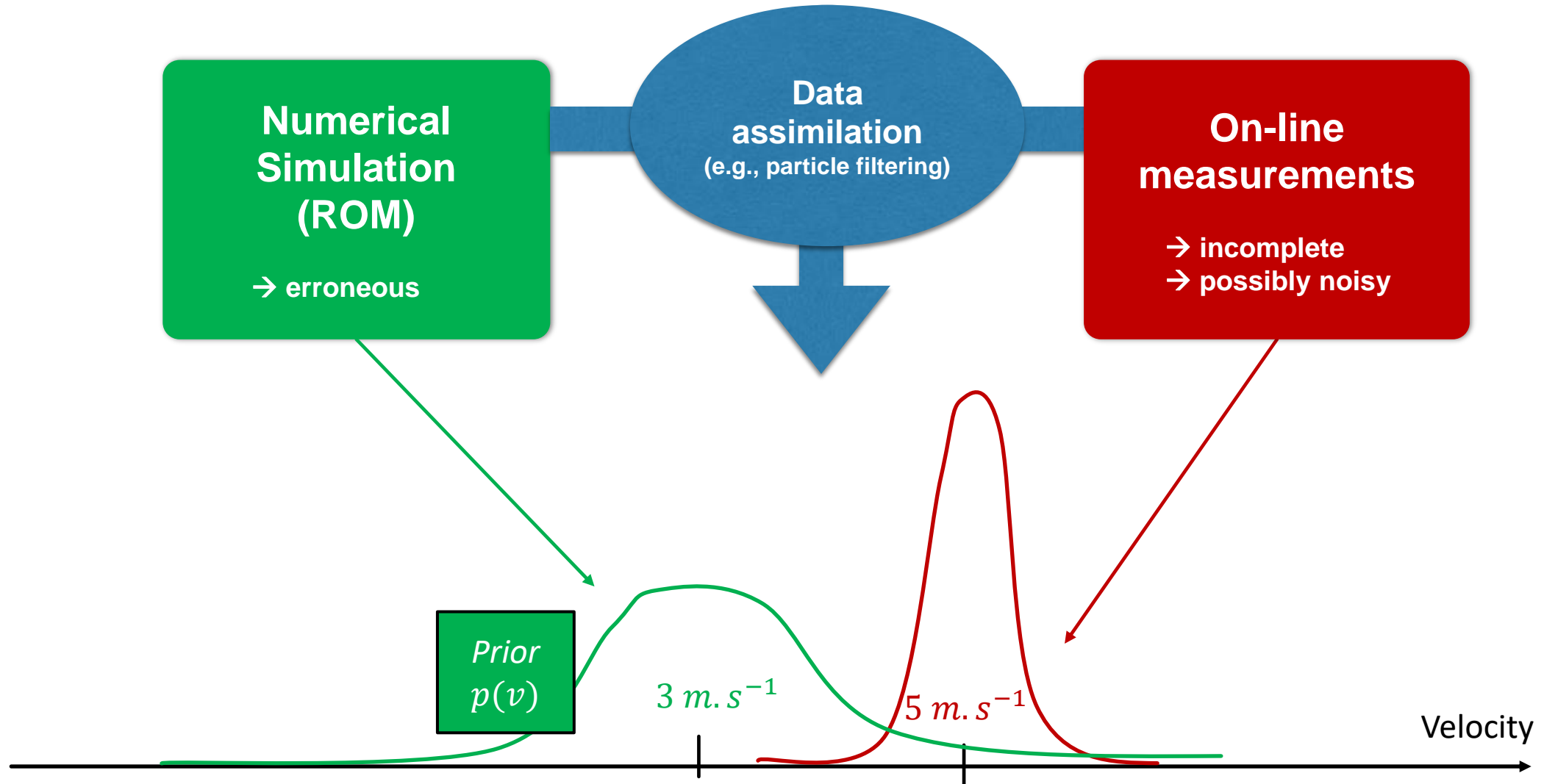
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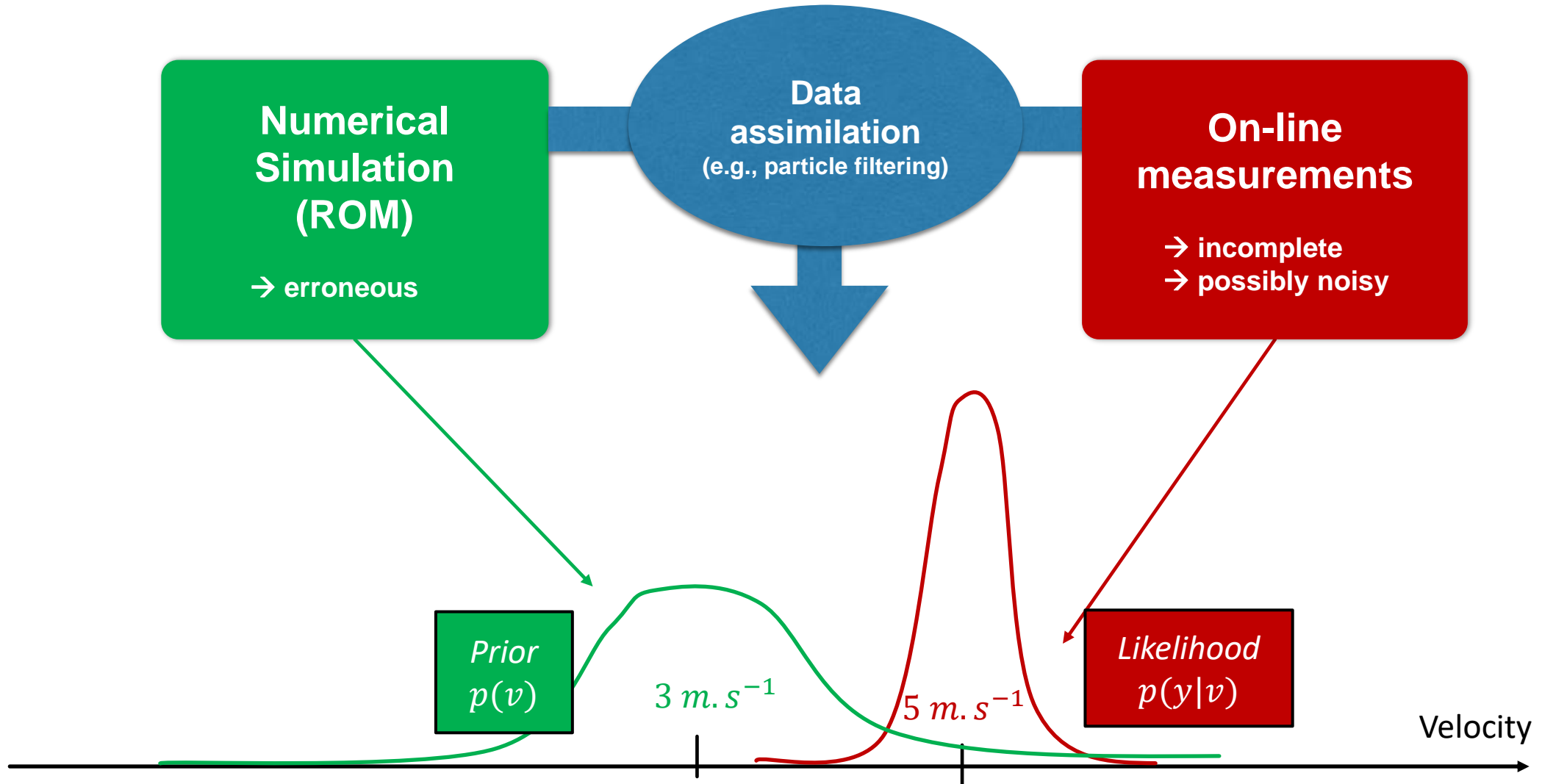
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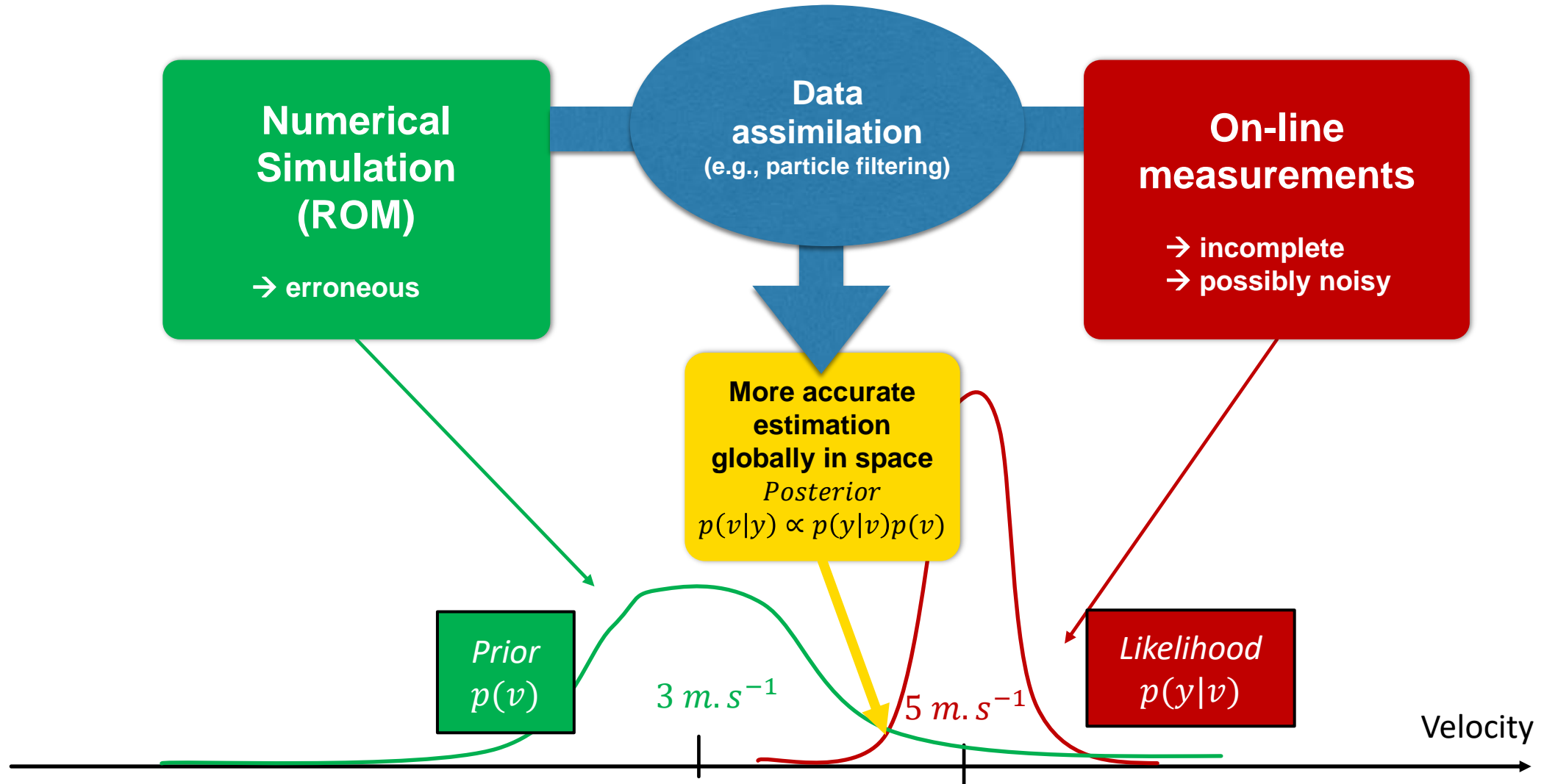
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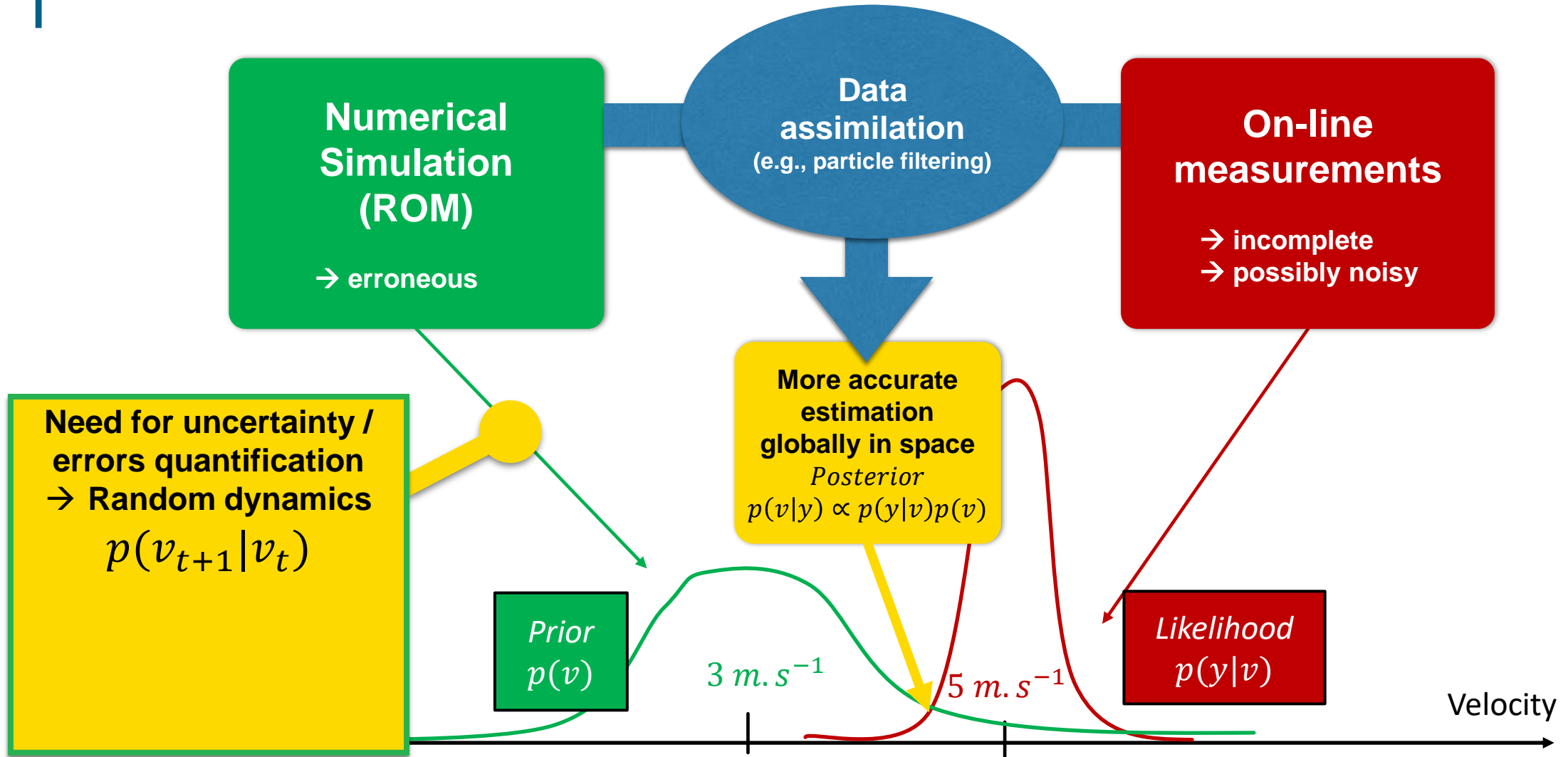
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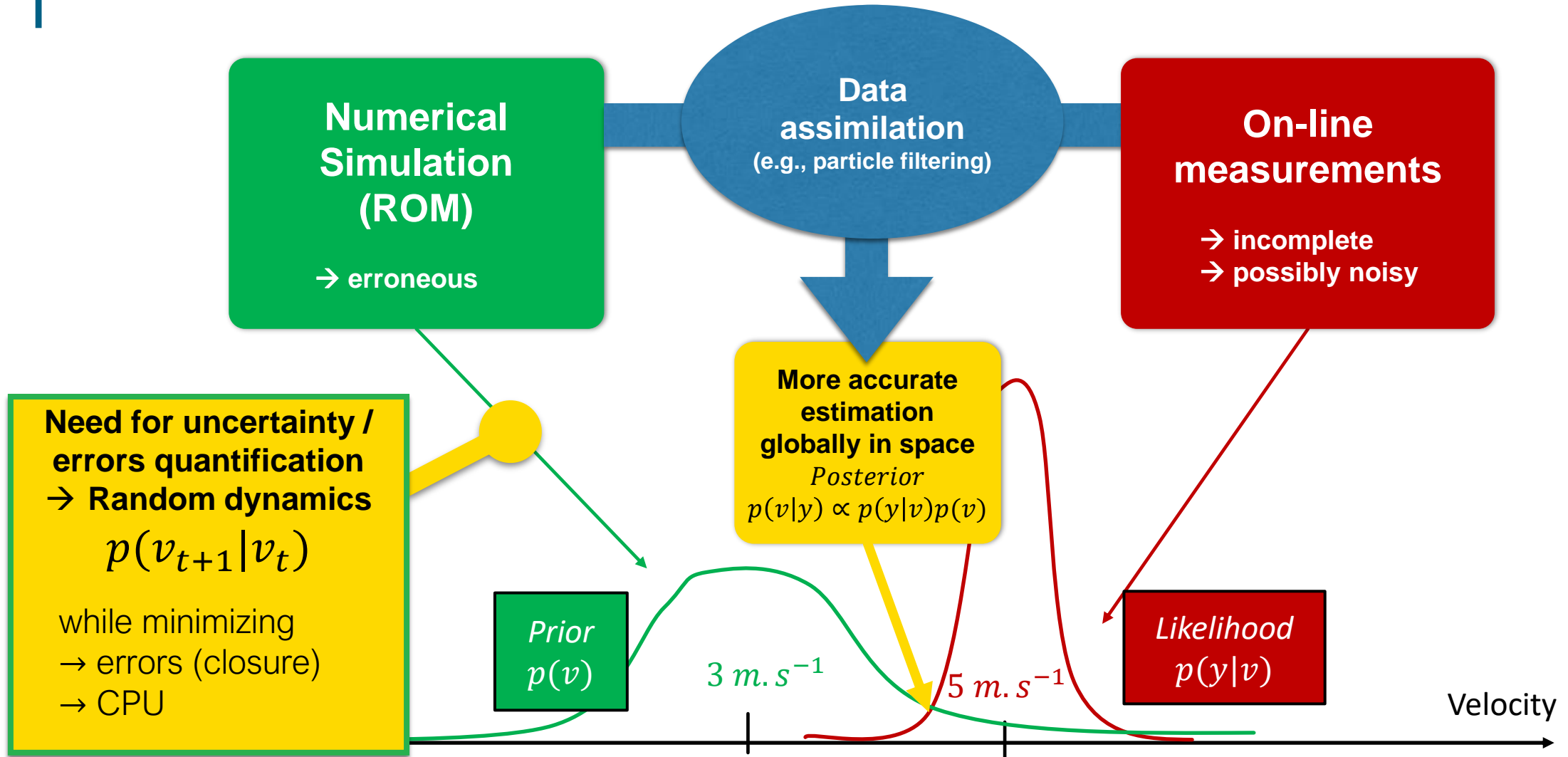
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PART II

REDUCED LOCATION UNCERTAINTY MODELS

- a. Location uncertainty models (LUM)
- b. Reduced LUM (Red LUM)

LOCATION UNCERTAINTY MODELS (LUM)

$$v = w + v'$$

Resolved fluid velocity:

$$w = \sum_{i=0}^n b_i \phi_i$$

Unresolved fluid velocity:

$$v' = \sigma \dot{B}$$

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

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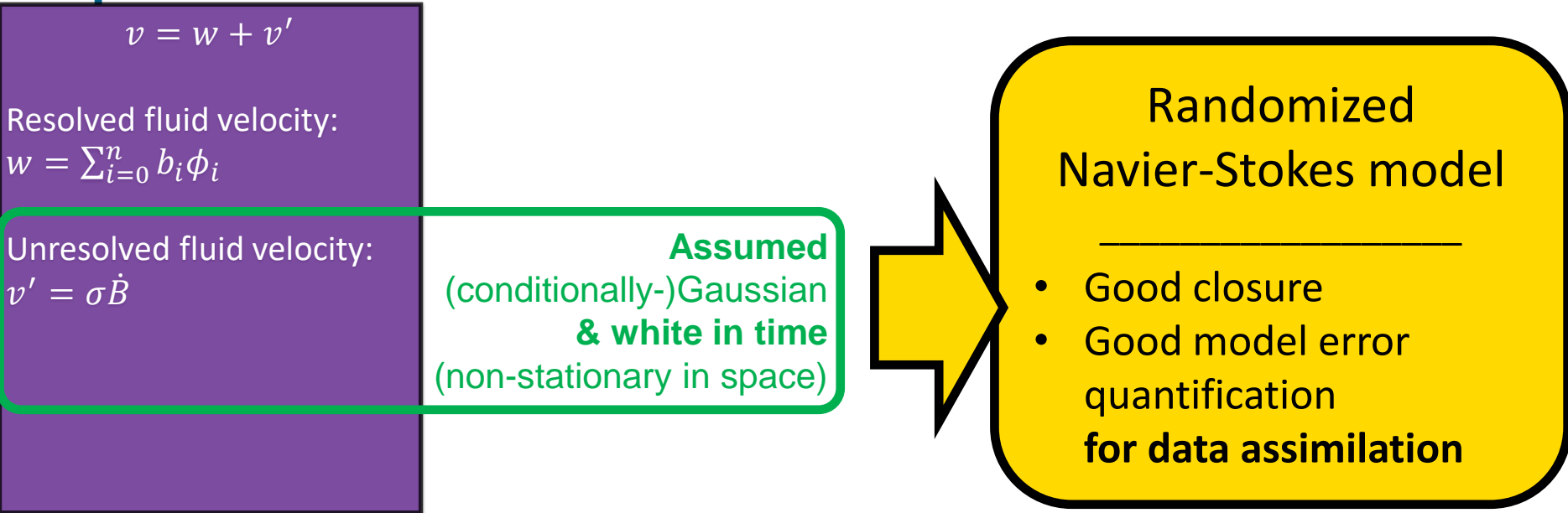
$$v' = \sigma \dot{B}$$

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Randomized
Navier-Stokes model

- Good closure
- Good model error quantification
for data assimilation

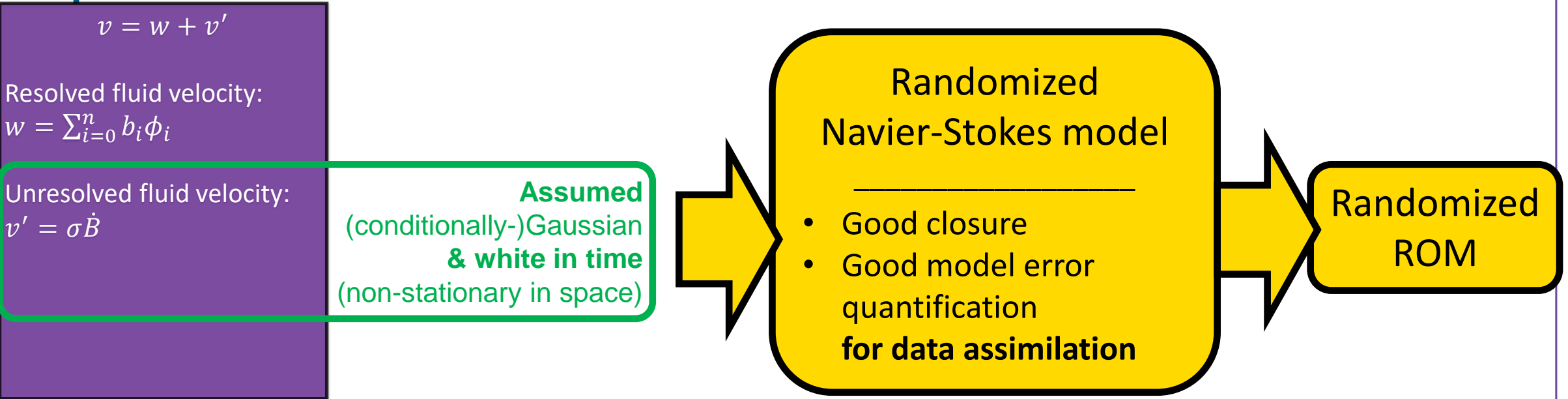
LOCATION UNCERTAINTY MODELS (LUM)



References :

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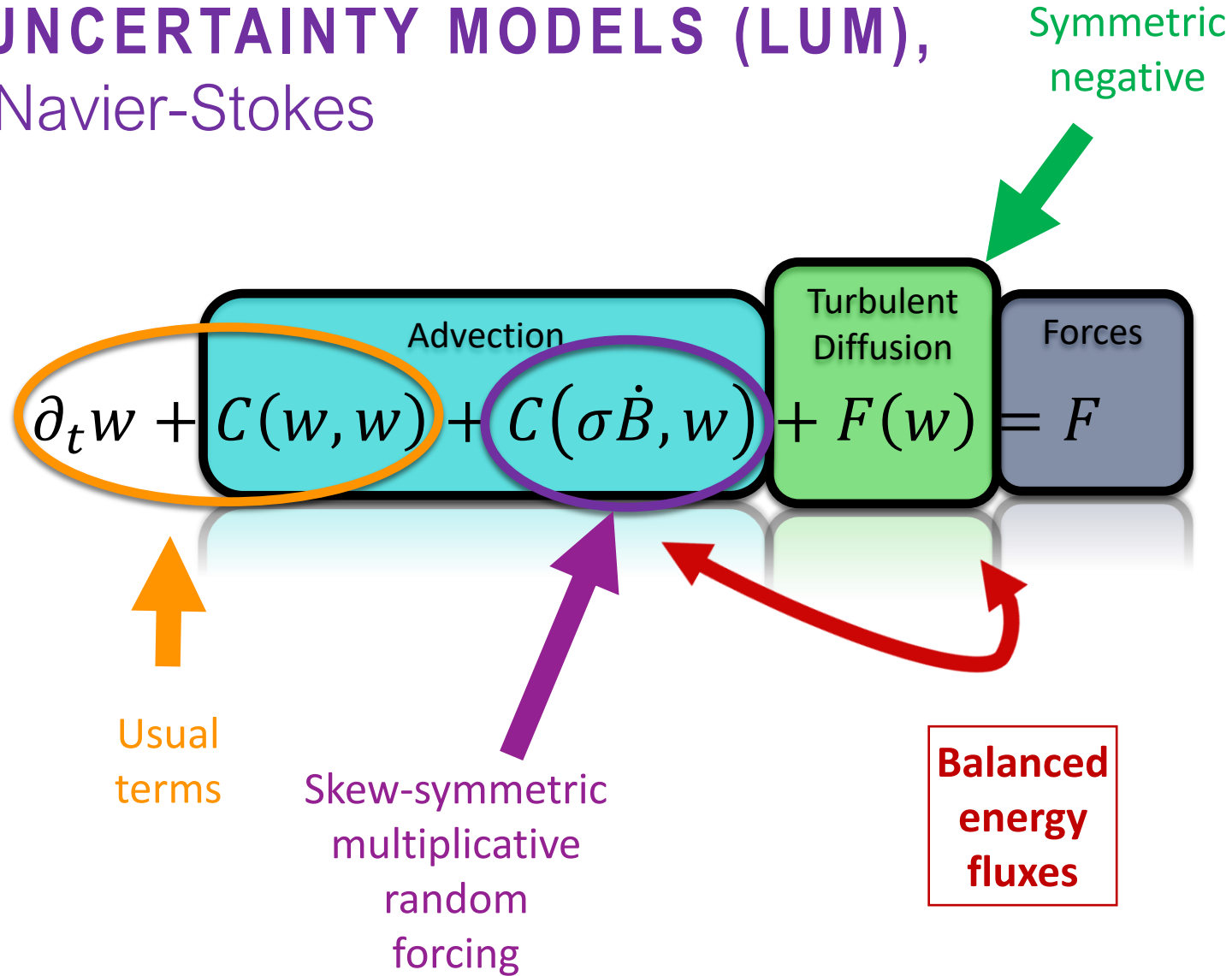
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LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

$v = w + v'$

Resolved fluid velocity:
 w

Unresolved fluid velocity:
 $v' = \sigma \dot{B}$ (Gaussian, white wrt t)



From Ito-Wentzell formula (Kunita 1990) with Ito notations

REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

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$$\int_{\Omega} \phi_i(x) \cdot (\partial_t w + C(w, w) + F(w) + C(\sigma \dot{B}, w) = F) dx$$

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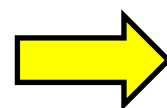
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$$\frac{d}{dt} b(t) = H(b(t)) + K(\sigma \dot{B}) b(t)$$

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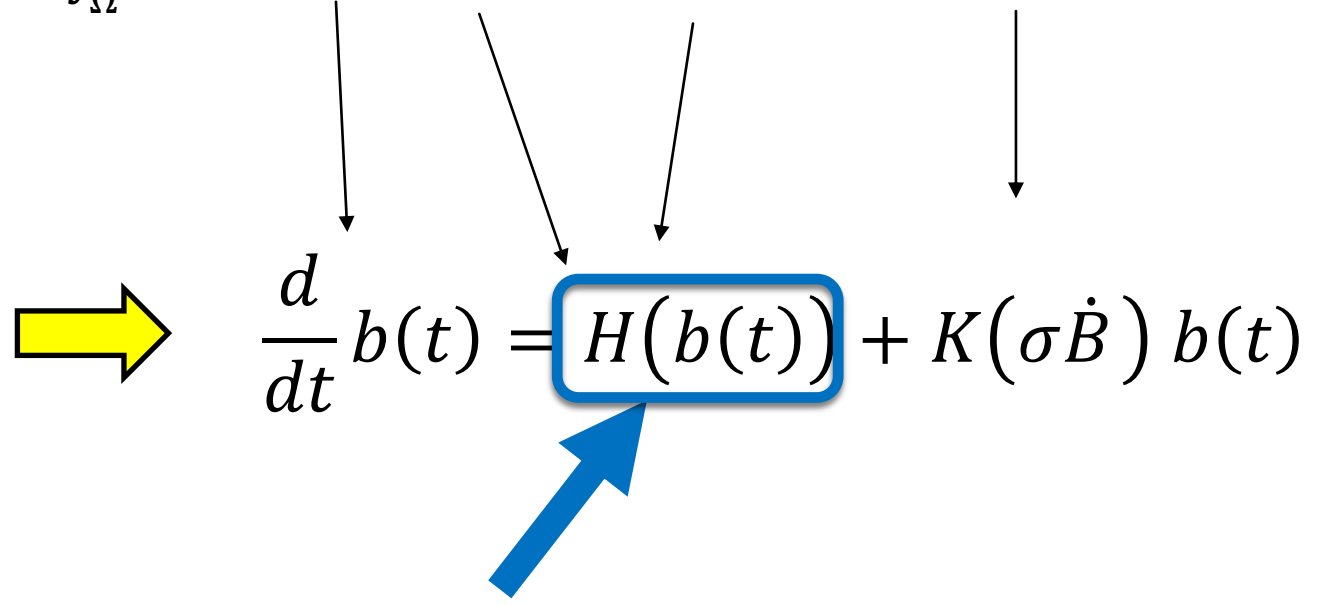
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2nd order polynomial

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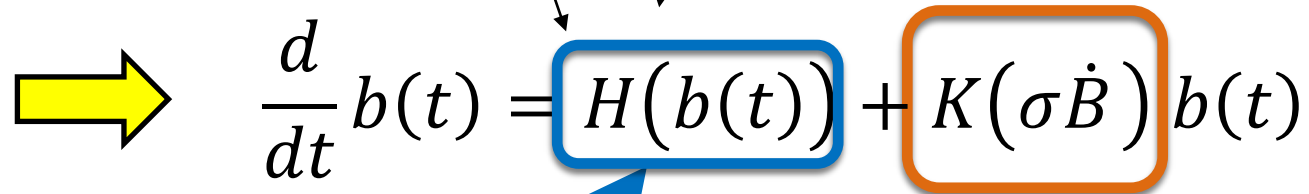
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Multiplicative skew-symmetric noise

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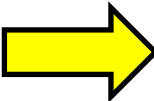
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$$K_{jq}[\xi] = - \int_{\Omega} \phi_j \cdot C(\xi, \phi_q)$$

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Variance tensor:

$$a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$$

$$\int_{\Omega} \phi_i(x) \cdot (\partial_t w + C(w, w) + F(w) + C(\sigma \dot{B}, w) = F) dx$$

$$\Rightarrow \frac{d}{dt} b(t) = H(b(t)) + K(\sigma \dot{B}) b(t)$$

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2nd order polynomial

Coefficients given by :

- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t \overline{v' (v')^T}$

$$\bar{f} = \frac{1}{T} \int_0^T f$$

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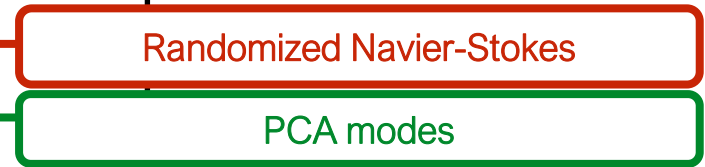
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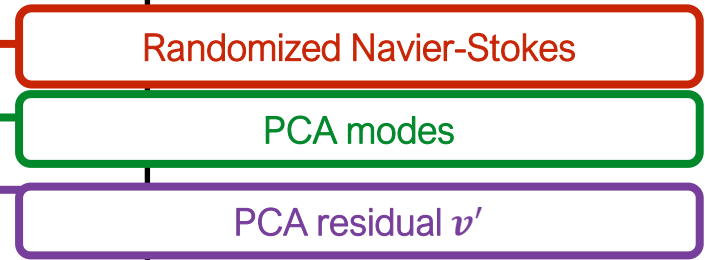
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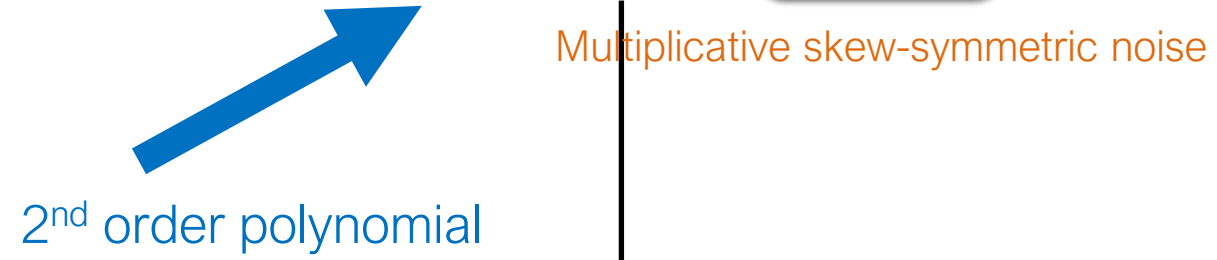
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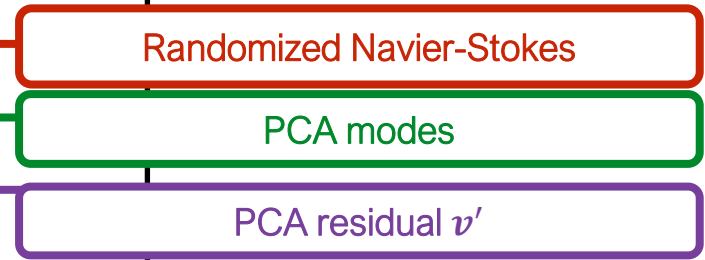
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$$K_{jq}[\xi] = - \int_{\Omega} \phi_j \cdot C(\xi, \phi_q)$$

$$\bar{f} = \frac{1}{T} \int_0^T f$$

REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

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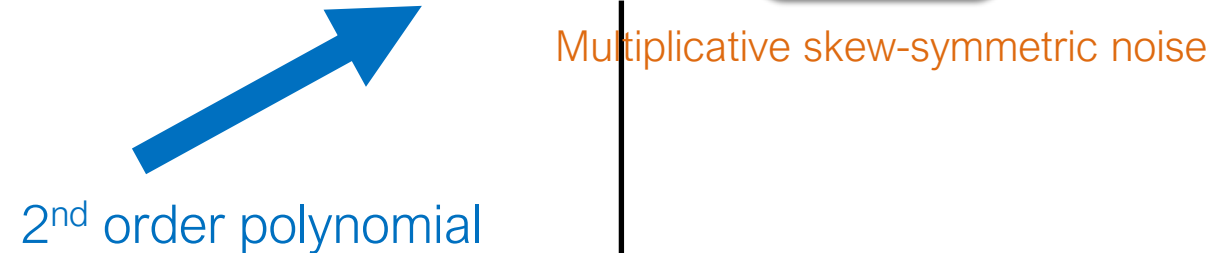
Variance tensor:
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$$\frac{d}{dt} b(t) = H(b(t)) + K(\sigma \dot{B}) b(t)$$

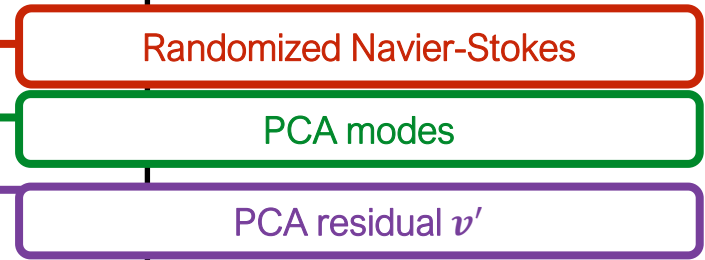
$H(b(t))$ +
 $K(\sigma \dot{B})$ $b(t)$

(n+1) x (n+1)
M x 1
n x 1



Coefficients given by :

- Randomized Navier-Stokes
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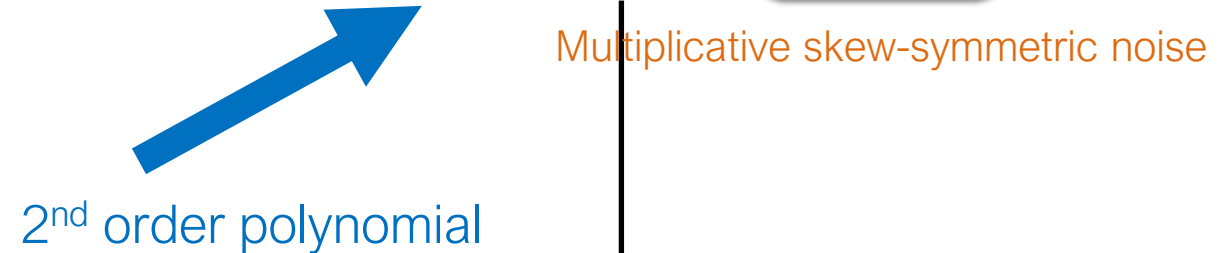
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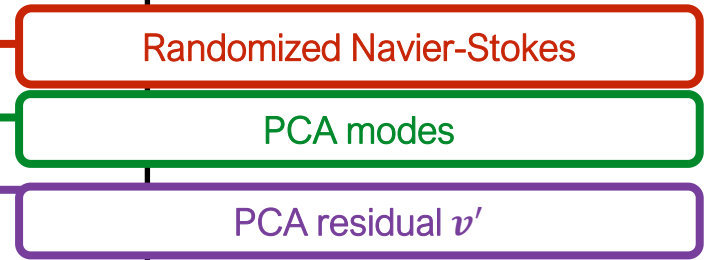
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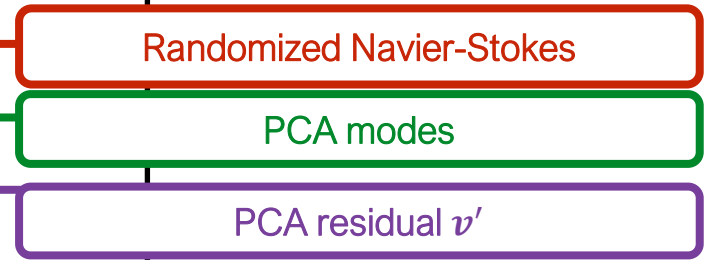
$$\Rightarrow \frac{d}{dt} b(t) = H(b(t)) + K(\sigma \dot{B}) b(t)$$

$(n+1) \times (n+1)$ $M \times 1$ $n \times 1$

Multiplicative skew-symmetric noise
Covariance to estimate
 2nd order polynomial

Coefficients given by :

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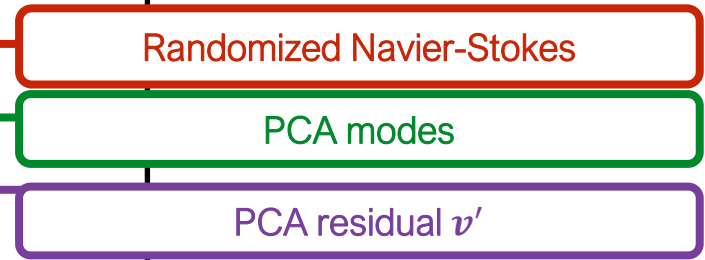
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$$\mathbb{E} \left(K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t K_{jq} \left[\frac{b_p}{b_p^2} \frac{\Delta b_i}{\Delta t} v' \right]$$

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2nd order polynomial

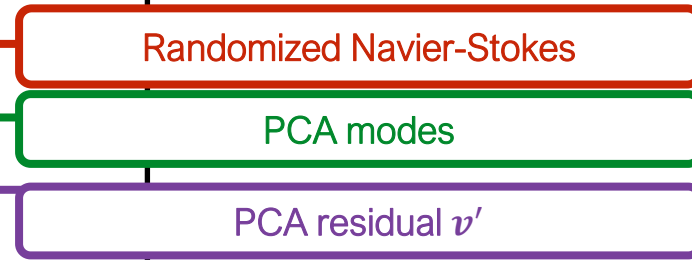
Multiplicative skew-symmetric noise

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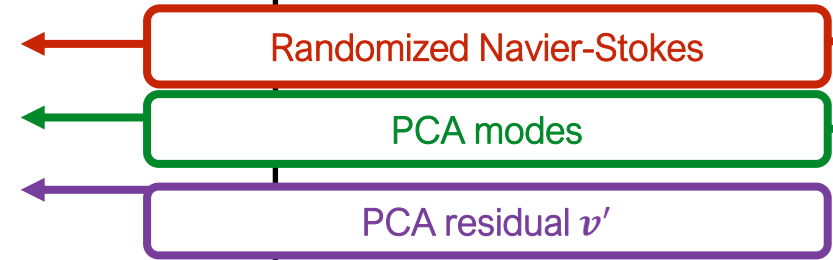
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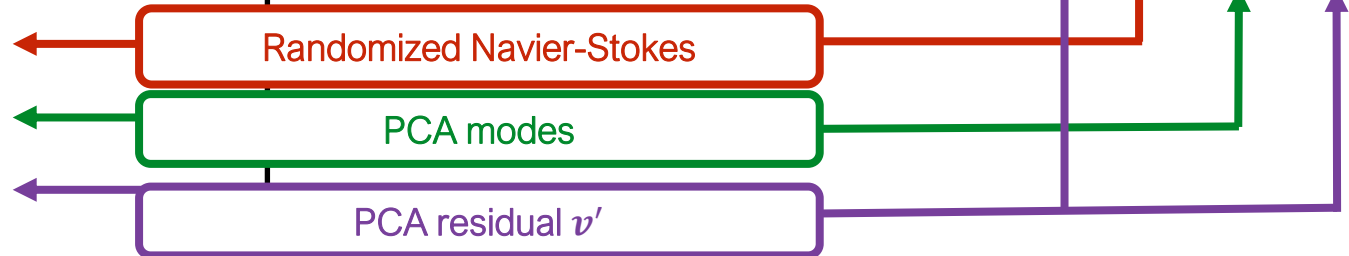
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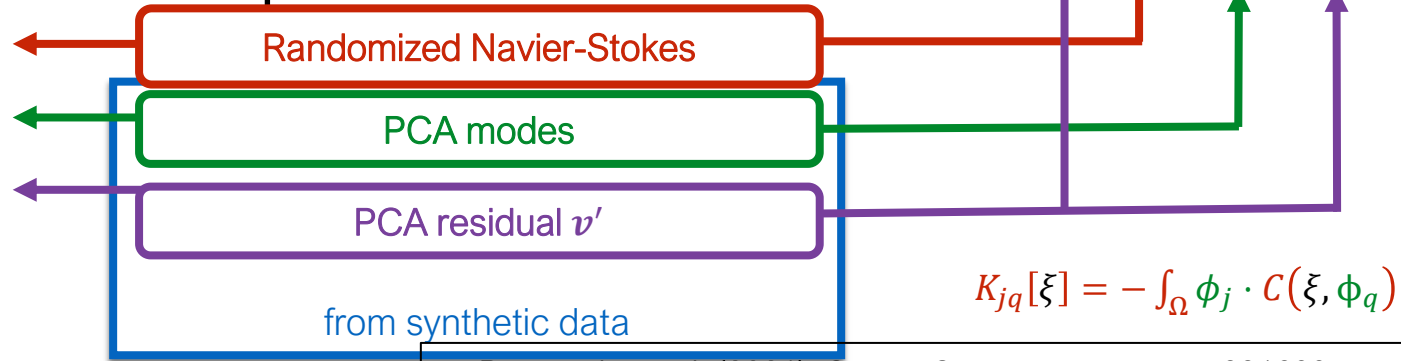
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New estimator

- Consistency proven ($\Delta t \rightarrow 0$)
- Numerically efficient
- Physically-based
- Robustness in extrapolation

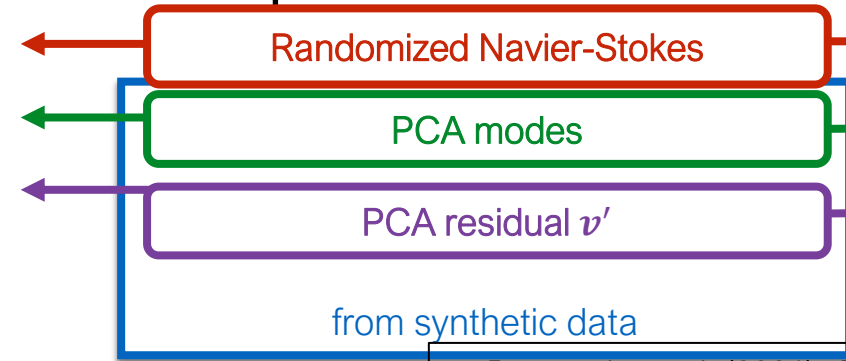
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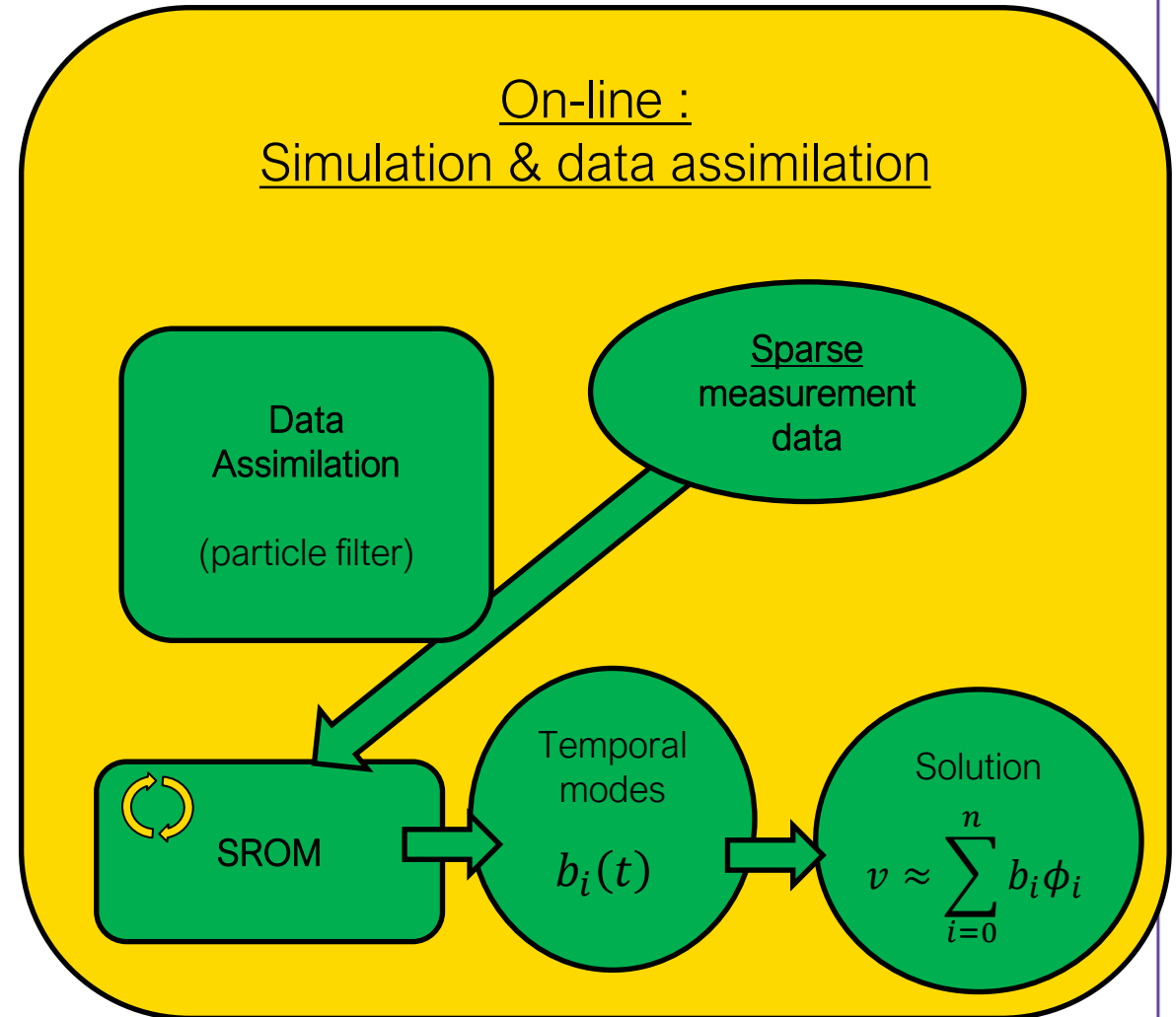
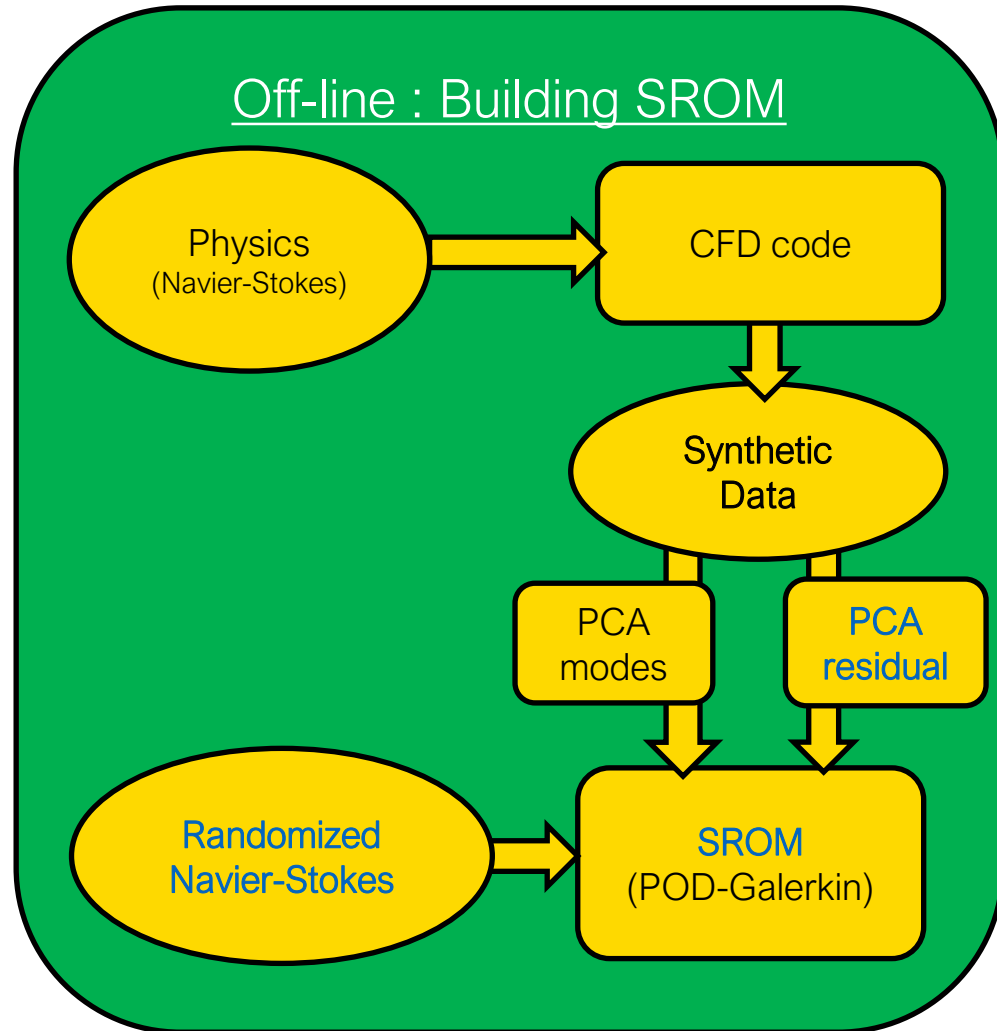


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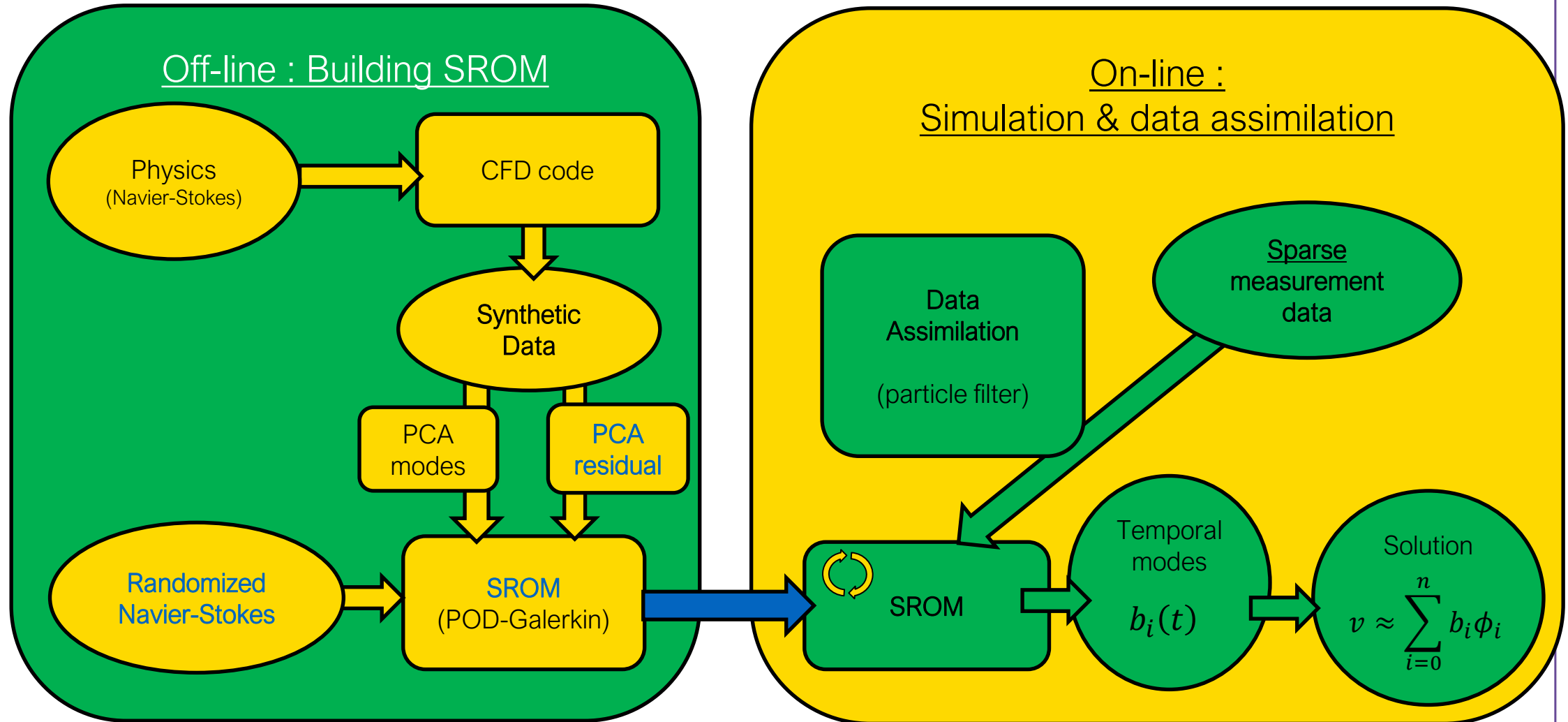
SUMMARY

Stochastic ROM + Data assimilation



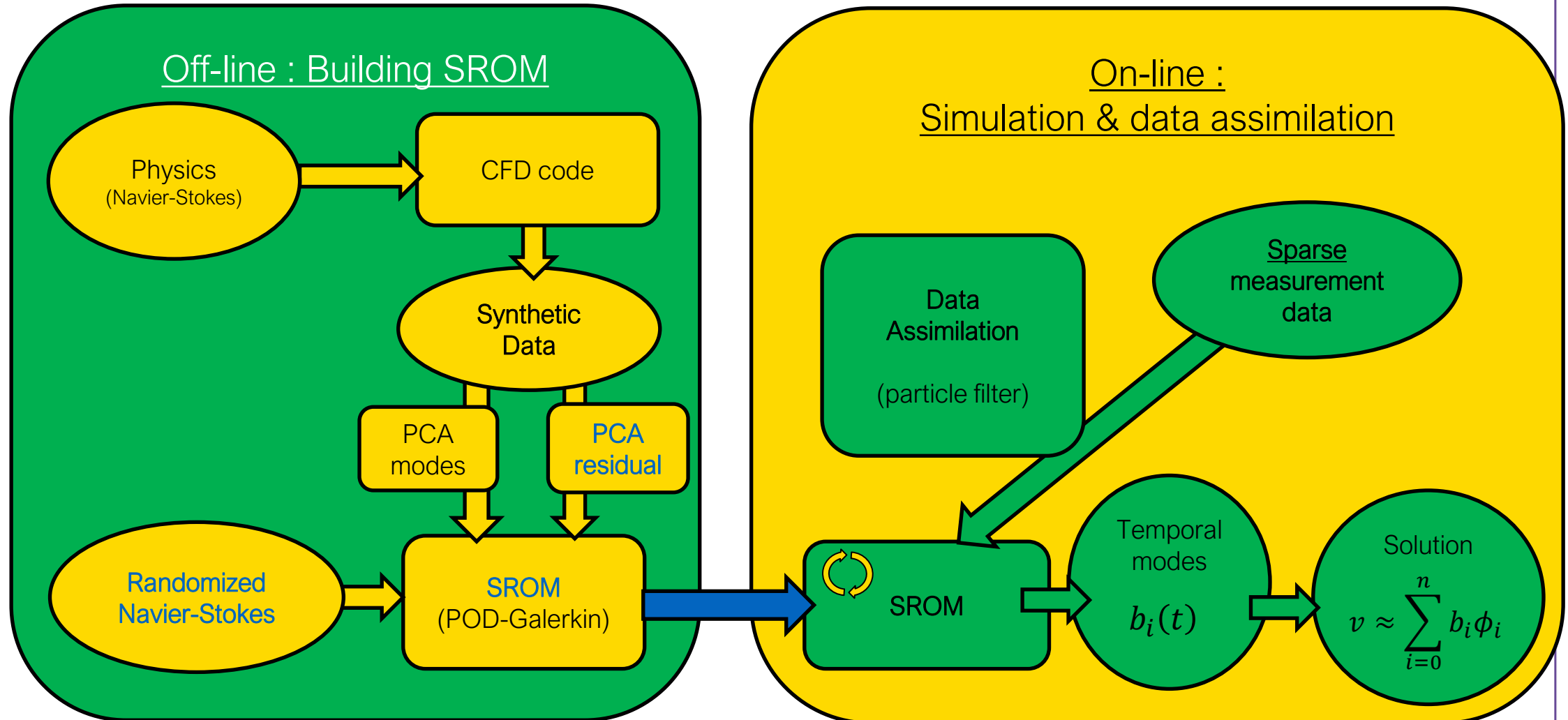
SUMMARY

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SUMMARY

Stochastic ROM + Data assimilation



PART III

NUMERICAL RESULTS

- a. Test cases
- b. Data assimilation

TEST CASES

From 10^7 to 8 degrees of freedom

Full-order reference

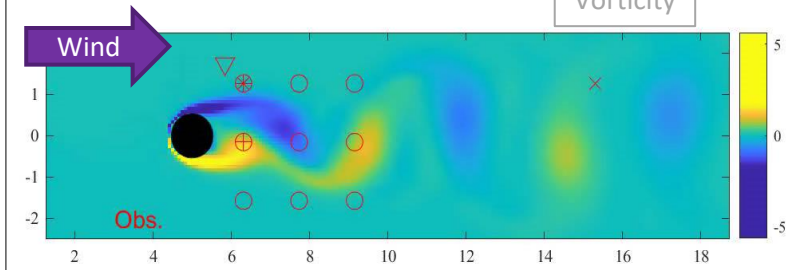
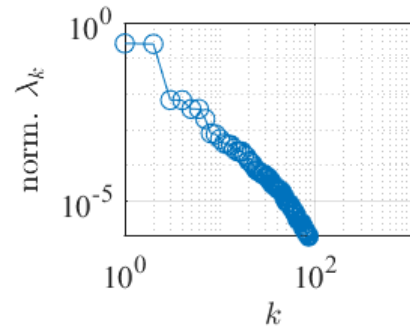
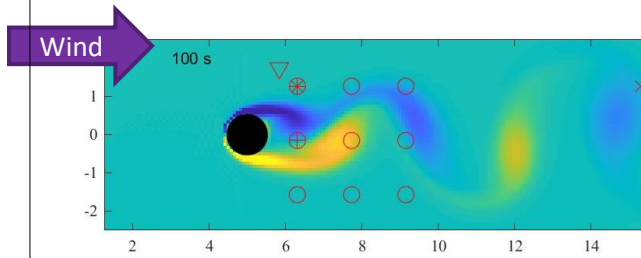
POD eigenvalues

Reduced-order reference

PCA-projection of the full-order simulation
(Optimal from 8-dof linear decomposition)

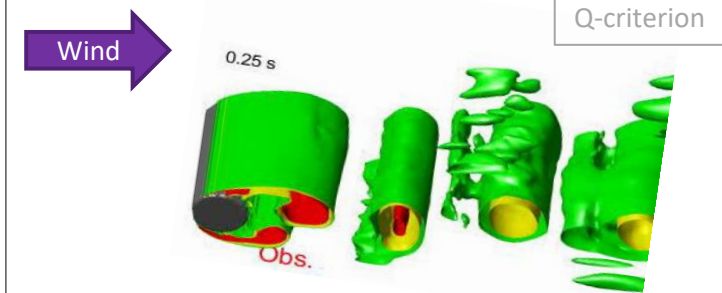
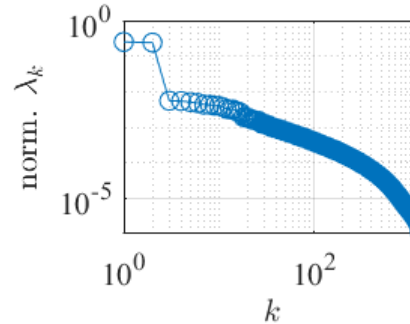
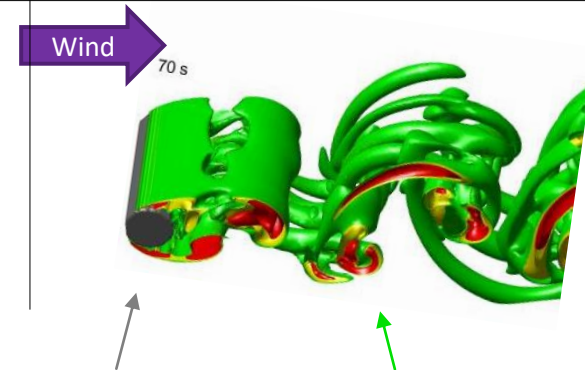
Easy case

Reynolds number
(Re) = 100
2D
(10^4 dof)



Difficult case

Reynolds number
(Re) = 300
3D
(10^7 dof)



(round) wind turbine blade

vortices

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

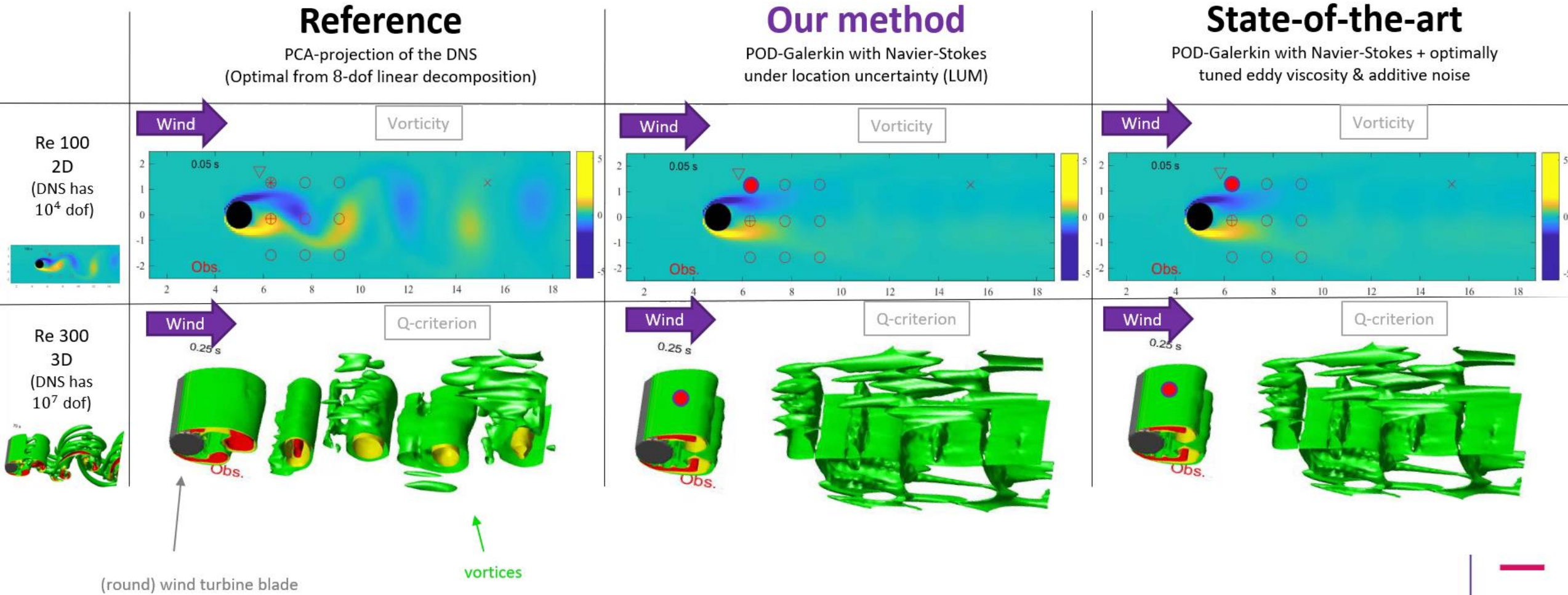
DATA ASSIMILATION

On-line estimation of the solution

DATA ASSIMILATION

On-line estimation of the solution


From 10^7 to 8 degrees of freedom
 ● Single measurement point (blurred & noisy velocity)



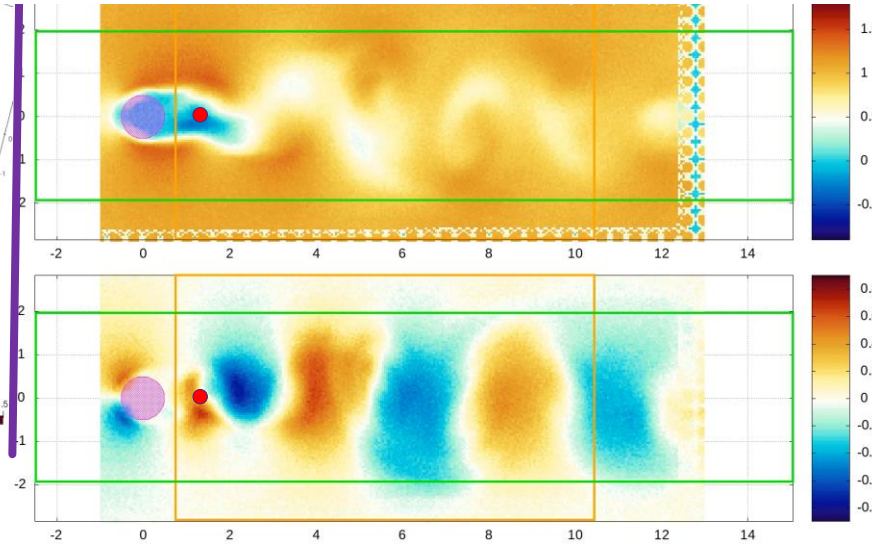
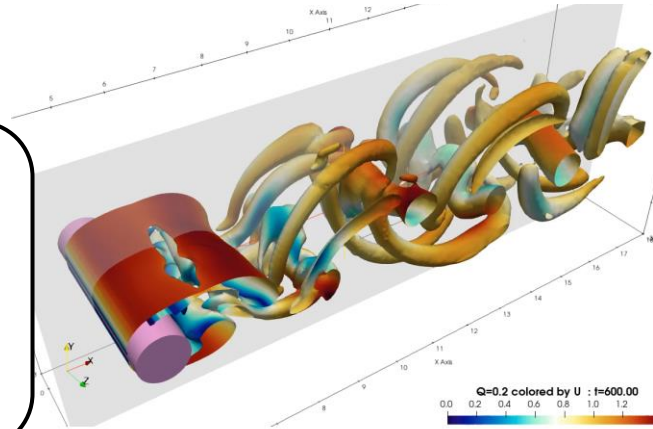
DATA ASSIMILATION

On-line estimation of the solution

From 10^7 to 4 degrees of freedom

 Single measurement point (blurred & noisy velocity)

DNS
“true” simulation
(Re 300, 10^7 degrees
of freedom)



Synthetic PIV
blurred & noisy
2D velocity

Reference :
PCA-projection of the
“true” simulation
(DNS)
(Optimal from 4-
degrees of freedom
linear decomposition)

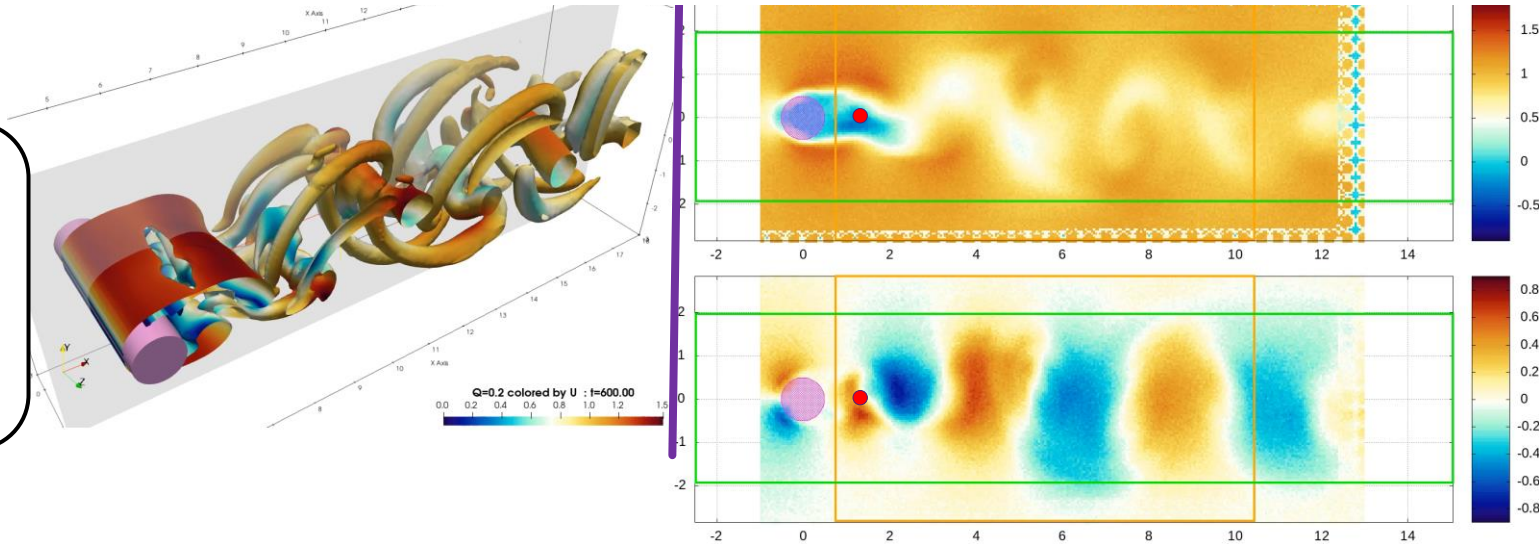
Our method :
Data assimilation
with
POD-Galerkin of
randomized
Navier-Stokes
(LUM)
(4 degrees of freedom)

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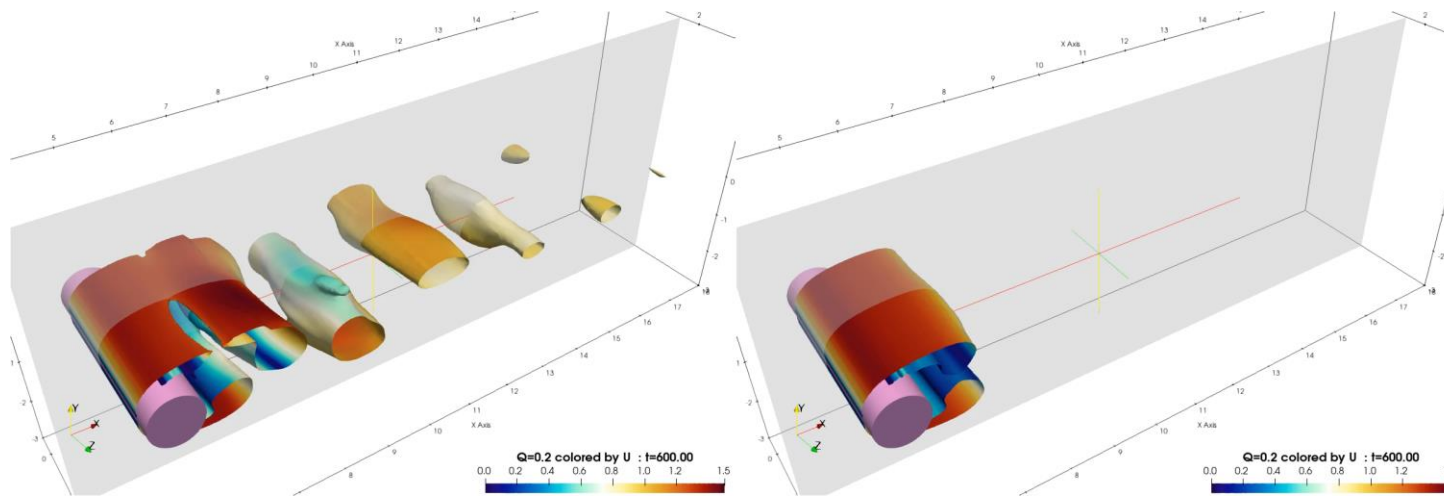
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CONCLUSION

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- ▶ Intrusive ROM : for very fast and robust CFD ($10^7 \rightarrow 8$ degrees of freedom.)
 - Closure problem handled by LUM
 - Efficient estimator for the multiplicative noise
 - Efficient generation of prior / Model error quantification
 - Now implemented in ITHACA-FV
- ▶ Data assimilation (Bayesian inverse problem) :
to correct the fast simulation on-line by incomplete/noisy measurements
- ▶ First results
 - Optimal unsteady flow estimation/prediction in the whole spatial domain (large-scale structures)
 - Robust far outside the training set



NEXT STEPS

- ▶ Real measurements
- ▶ Parametric ROM (unknown inflow)
- ▶ Increasing Reynolds
(ROM of (non-polynomial) turbulence models)



BONUS SLIDES

CONTEXT

Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow
using strongly-limited computational resources & few sensors

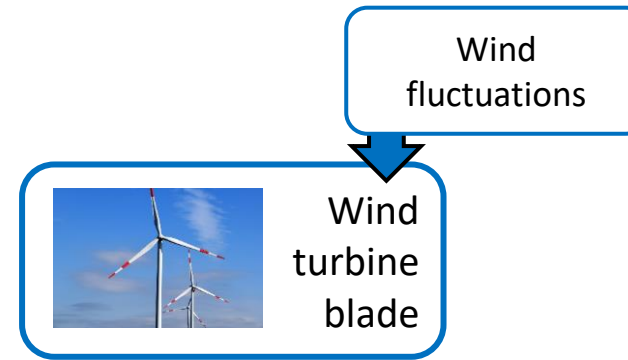


Wind
turbine
blade

CONTEXT

Observer for wind turbine application

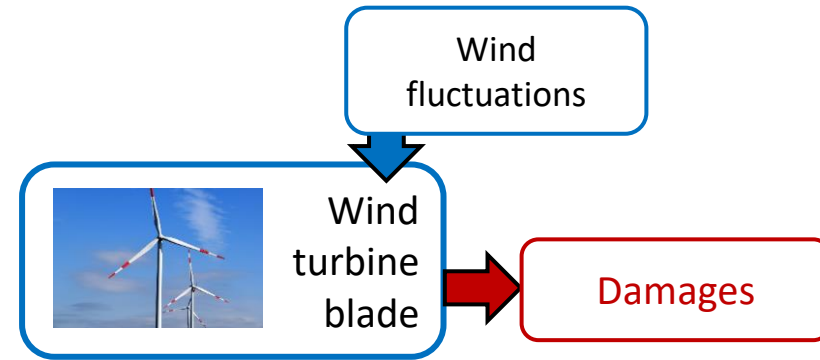
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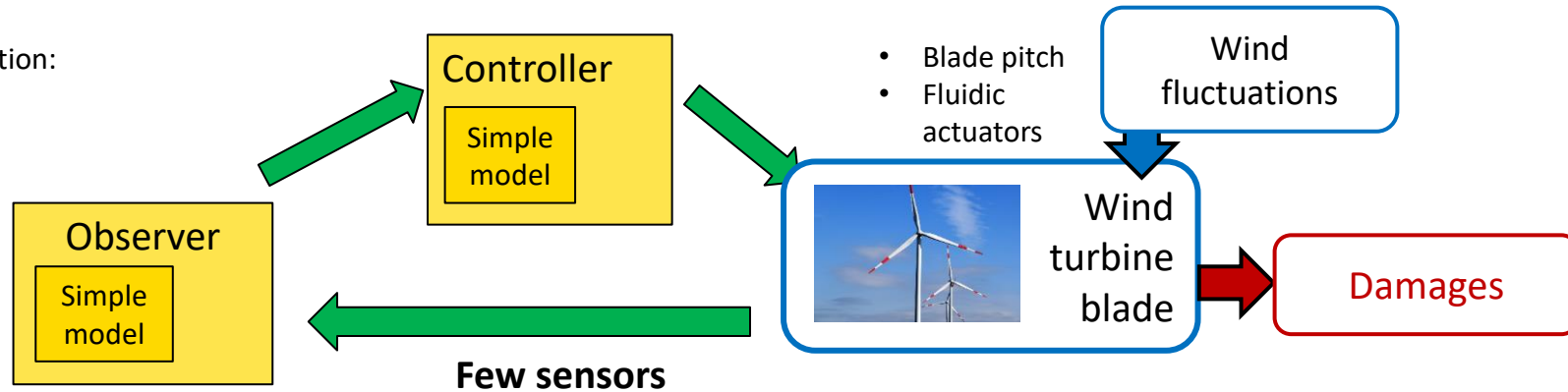
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Estimation and prediction:

- Air flow
- Lift, drag, inflow
- ...



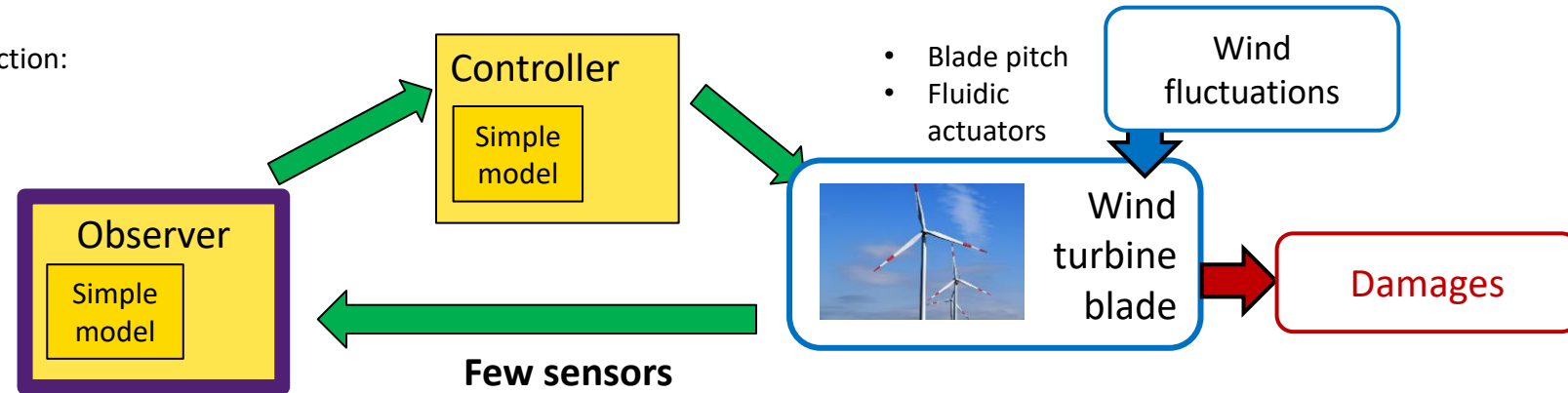
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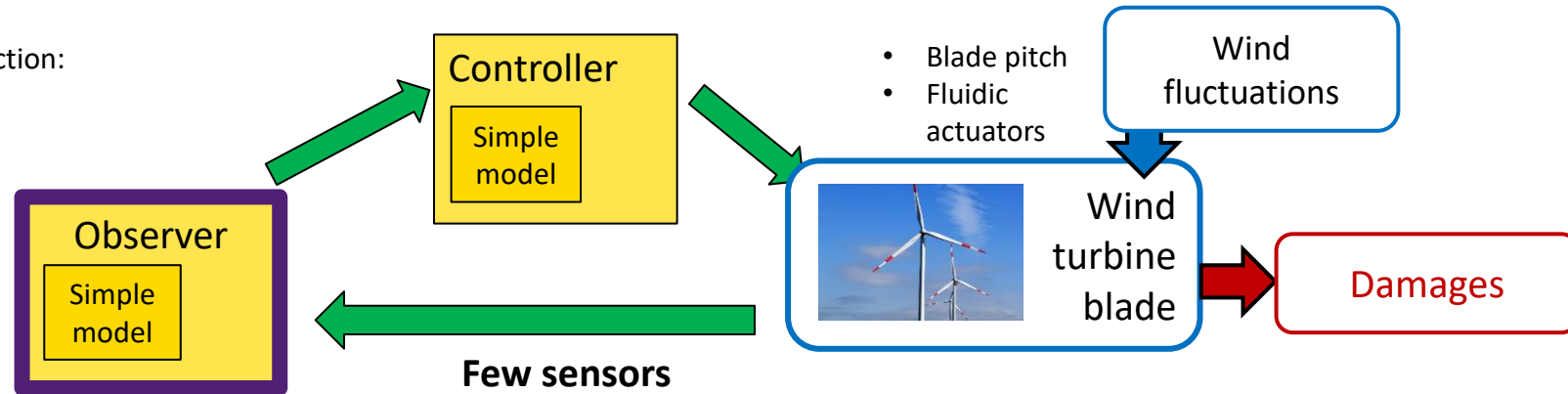
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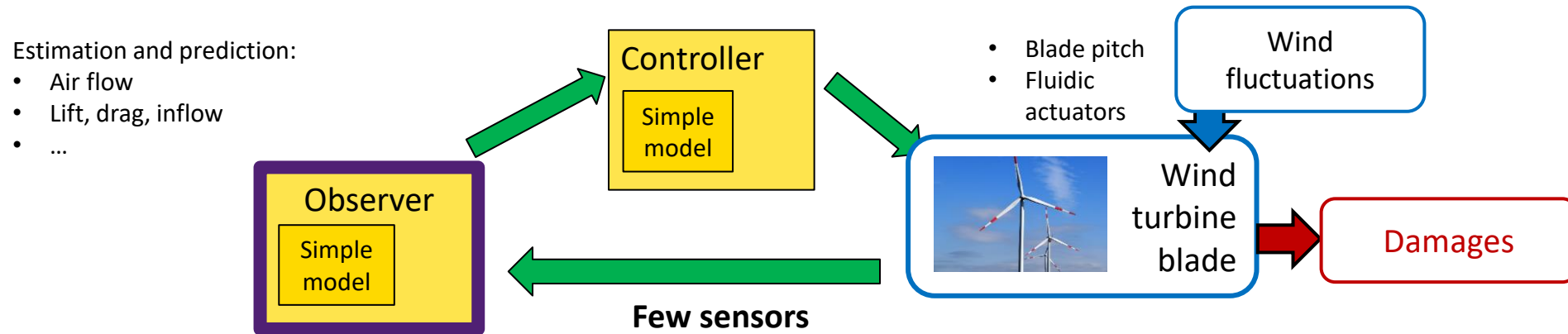


Which simple model? How to combine model & measurements?

CONTEXT

Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors



Which simple model? How to combine model & measurements?

Scientific problem :

Simulation & data assimilation under severe dimensional reduction

typically, $10^7 \rightarrow O(10)$ degrees of freedom

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

$$v = w + v'$$

Resolved fluid velocity:
 w

Unresolved fluid velocity:
 $v' = \frac{\sigma dB_t}{dt}$ (Gaussian, white wrt t)

(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Momentum conservation

$$\frac{d}{dt} \left(w(t, X_t) \right) = F \text{ (Forces)}$$

Positions of fluid parcels X_t :

$$\frac{d}{dt} X_t = w(t, X_t) + \underbrace{\sigma(t, X_t) \frac{dB_t}{dt}}_{\text{Gaussian process white in time}}$$

Gaussian
process
white in time

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

$$v = w + v'$$

Resolved fluid velocity:

w

Unresolved fluid velocity:

$$v' = \sigma \dot{B} \text{ (Gaussian, white wrt } t)$$

(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Variance tensor:

$$a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$$

$$\partial_t w + w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) = F$$

From Ito-Wentzell
formula (Kunita 1990)
with Ito notations

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Variance tensor:

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Advection

$$\partial_t w + w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) = F$$

From Ito-Wentzell
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$$\partial_t w + \underbrace{w^* \cdot \nabla w + \sigma \dot{B} \cdot \nabla w}_{\text{Advection}} - \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla w \right)}_{\text{"Turbulent" diffusion}} = F$$

From Ito-Wentzell
formula (Kunita 1990)
with Ito notations

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

From Ito-Wentzell
formula (Kunita 1990)
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LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

From Ito-Wentzell formula (Kunita 1990) with Ito notations

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Advection
"Turbulent" diffusion
Forces

↑
Usual terms

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

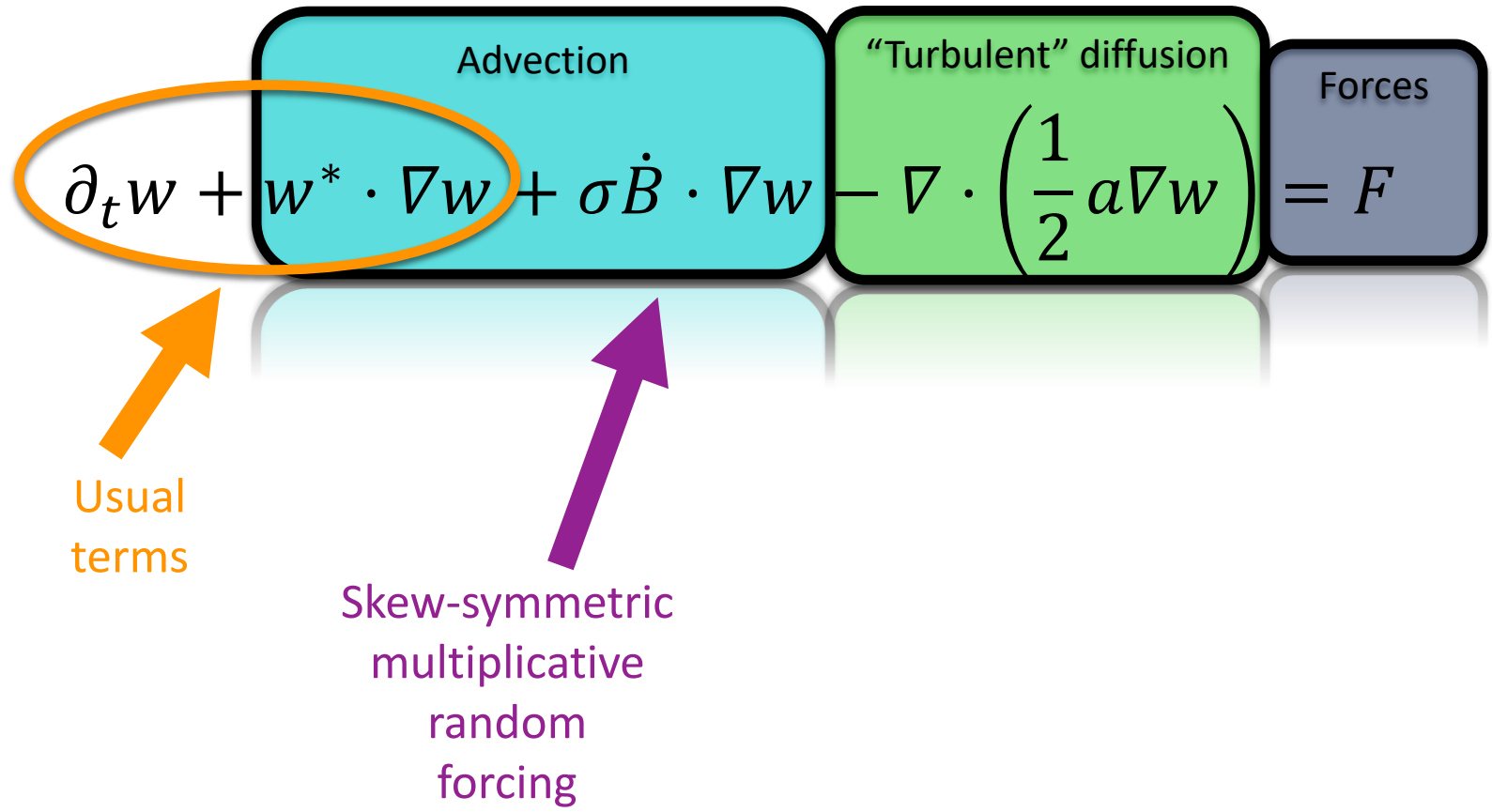
From Ito-Wentzell formula (Kunita 1990) with Ito notations

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LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

From Ito-Wentzell formula (Kunita 1990) with Ito notations

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 w

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Advection

"Turbulent" diffusion

Forces

Usual terms

Skew-symmetric multiplicative random forcing

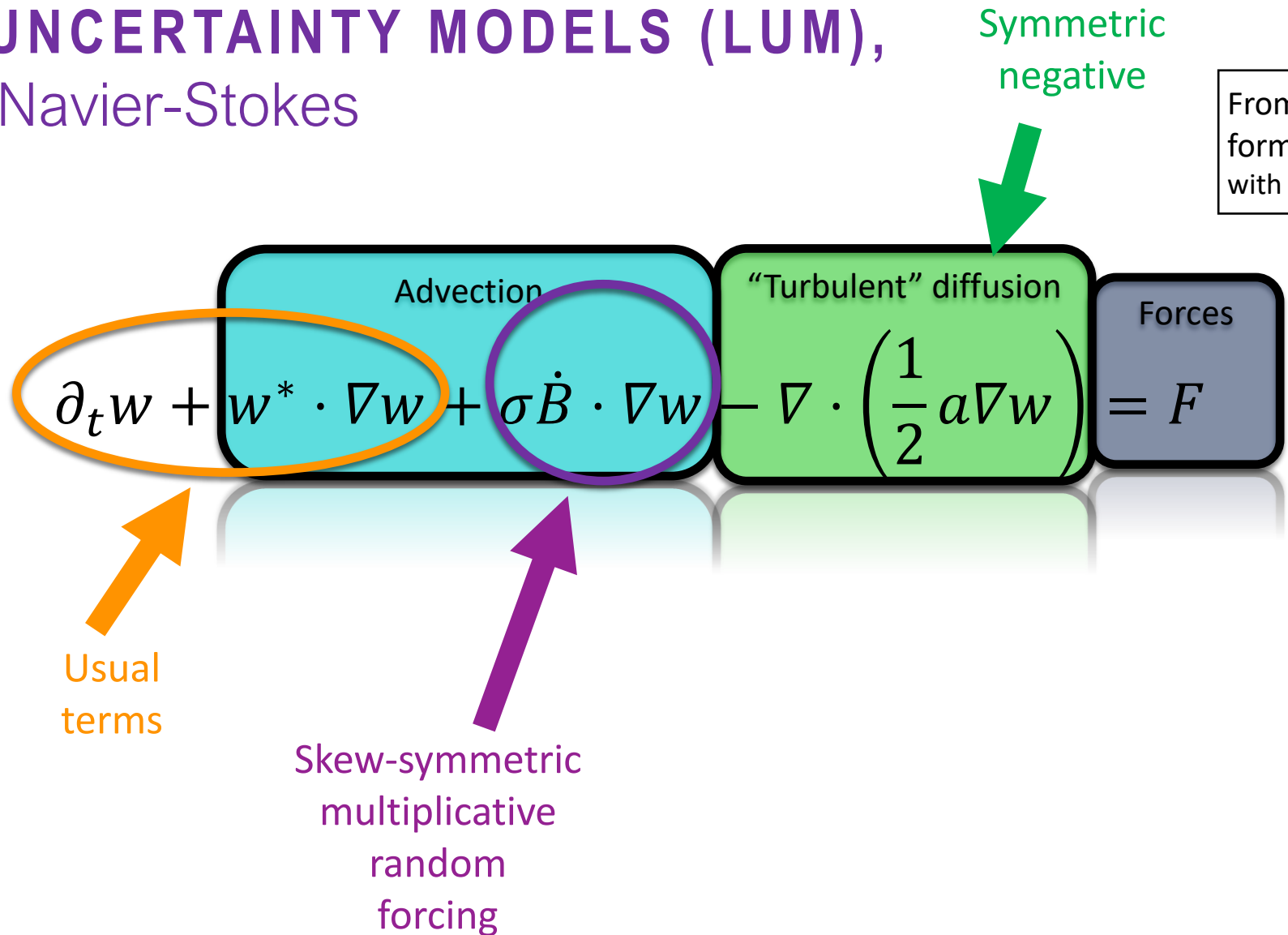
LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

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From Ito-Wentzell formula (Kunita 1990) with Ito notations

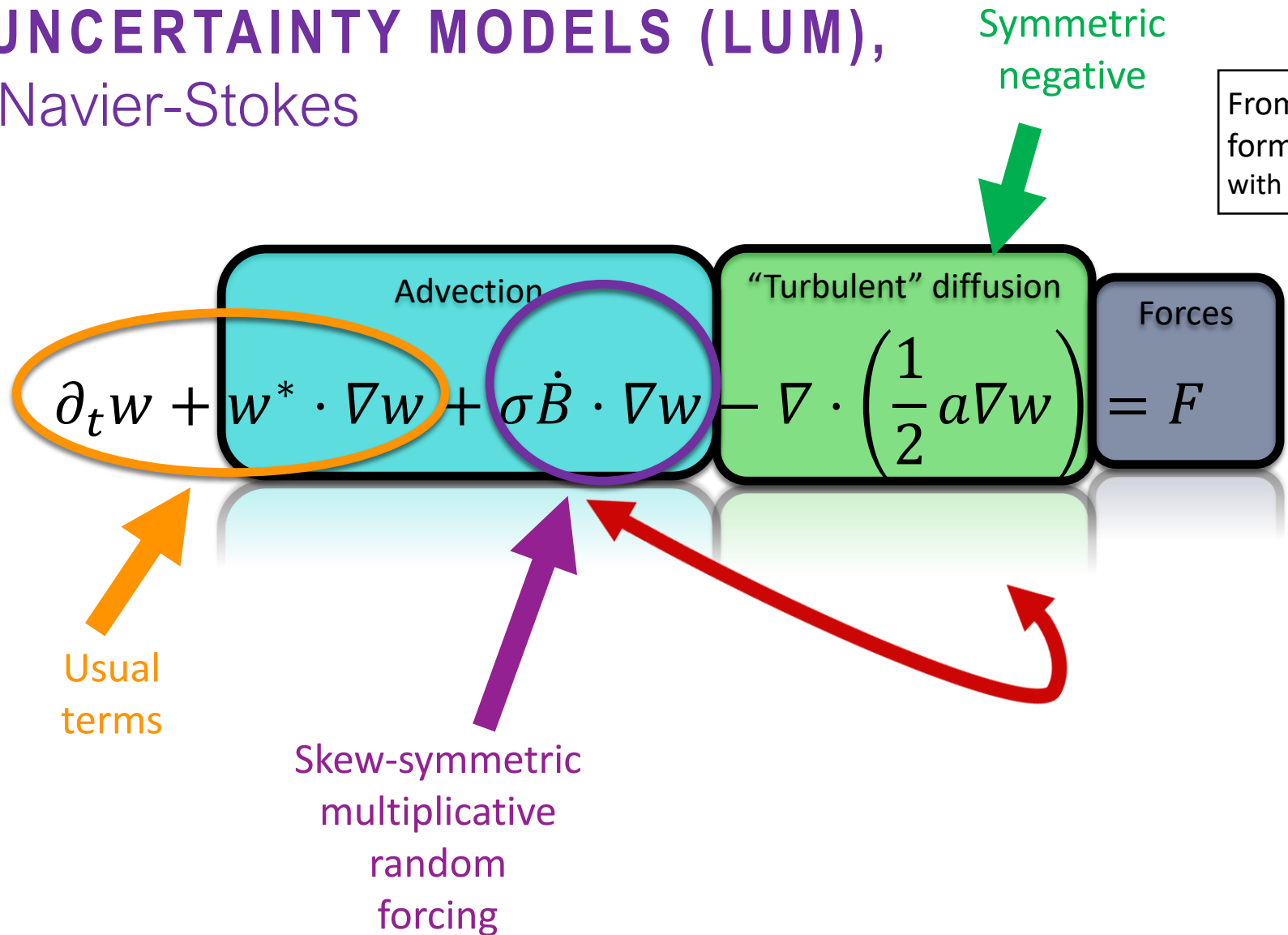
LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

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From Ito-Wentzell formula (Kunita 1990) with Ito notations

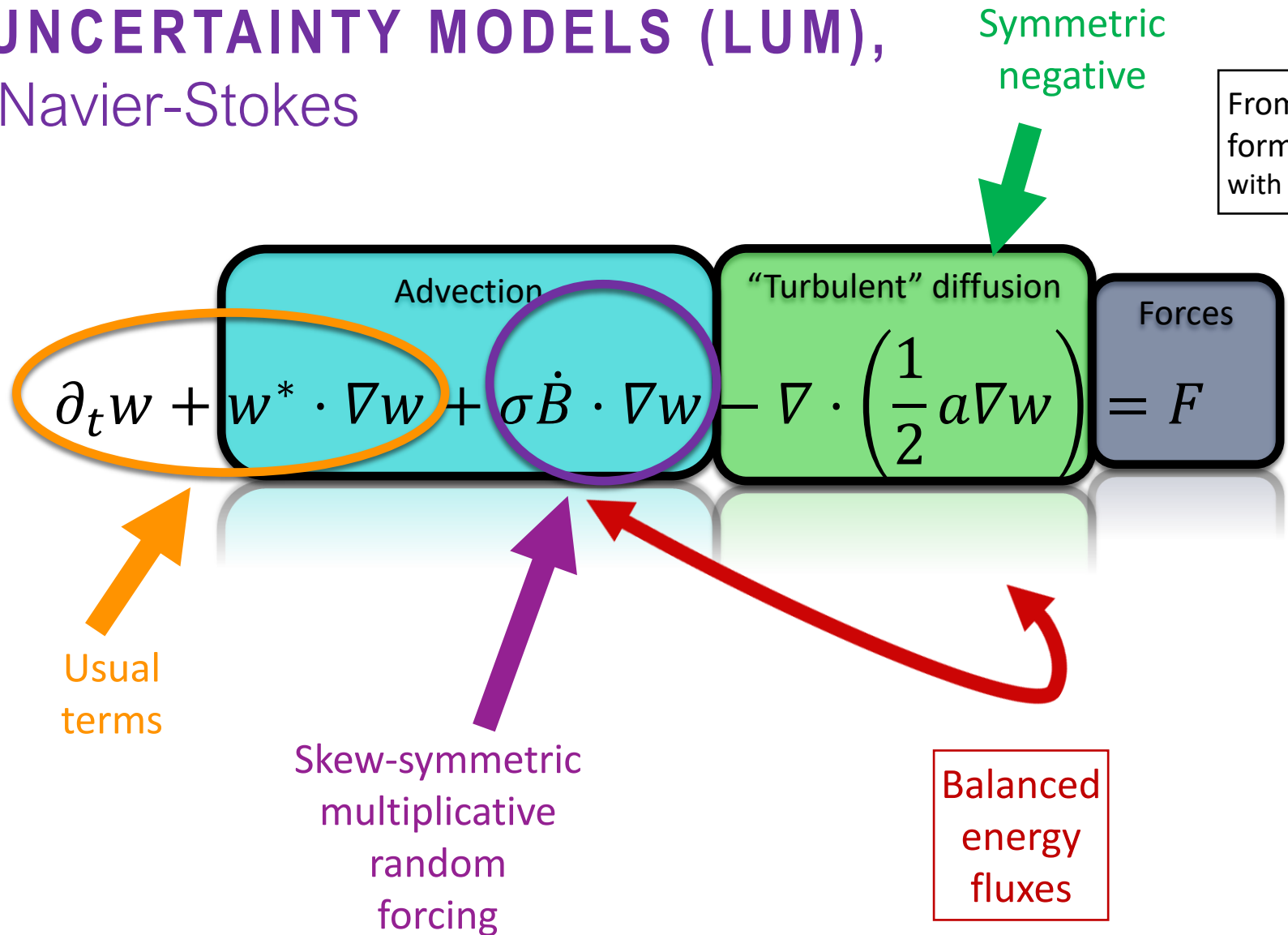
LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

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 w

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From Ito-Wentzell formula (Kunita 1990) with Ito notations

REDUCED LUM (RED LUM)

Multiplicative noise covariance

Full order (~ nb spatial grid points): $M \sim 10^7$
 Reduced order : $n \sim 10$
 Number of time steps : $N \sim 10^4$

$$v = w + v'$$

Resolved fluid velocity:

$$w(x, t) = \sum_{i=0}^n b_i(t) \phi_i(x)$$

Unresolved fluid velocity:

$$v' = \frac{\sigma dB_t}{dt} \text{ (Gaussian, white wrt } t)$$

$$db(t) = H(b(t)) dt + K(\sigma dB_t) b(t) \text{ with } K_{jq}[\xi] = - \int_{\Omega} \phi_j \cdot C(\xi, \phi_q)$$

$M \times 1$
 $(n+1) \times (n+1)$

► **Curse of dimensionality**

- Since σdB_t is white in time,
 $\Sigma_{jq, ip} = \mathbb{E} \left(K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t \overline{K_{jq}(v') K_{ip}(v')}$
- K is a matrix of integro-differential operators → cannot be evaluated on $v'(x, t)$ at every time t
- Covariance of $\sigma dB_t \approx \Delta t^2 \overline{(v'(x, t))(v'(y, t))^T} : M \times M \sim 10^{13}$ coefficients → intractable

New estimator

- Consistency proven ($\Delta t \rightarrow 0$)
- Numerically efficient
- Physically-based
 → Robustness in extrapolation

Randomized Navier-Stokes

PCA modes

PCA residual v'

from synthetic data

$$\bar{f} = \frac{1}{T} \int_0^T f$$

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 requires only $O(n^2 M)$ correlation estimations and $O(n^2)$ evaluations of K

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► **Consistency of our estimator** (convergence in probability for $\Delta t \rightarrow 0$, using stochastic calculus and continuity of K)

$$\Delta t K_{jq} \left[\overline{\frac{b_p}{b_p^2} \frac{\Delta b_i}{\Delta t} v'} \right] = \Delta t \overline{b_p \frac{\Delta b_i}{\Delta t} K_{jq} [v']} \approx \frac{1}{T} \int_0^T b_p d \langle b_i, K_{jq}(\sigma B) \rangle = \frac{1}{T} \int_0^T b_p \sum_{r=0}^n b_r d \langle K_{ir}(\sigma B), K_{jq}(\sigma B) \rangle = \sum_{r=0}^n \Sigma_{jq,ir} \overline{b_p b_r} = \Sigma_{jq,ip} \overline{b_p^2} \text{ (orthogonality from PCA)}$$

New estimator

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Randomized Navier-Stokes

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► Optimal time subsampling at Δt needed to meet the white assumption

New estimator

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- Numerically efficient
- Physically-based
 \rightarrow Robustness in extrapolation

Randomized Navier-Stokes

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► Optimal time subsampling at Δt needed to meet the white assumption

► Additional reduction for efficient sampling :

diagonalization of $\Sigma \rightarrow K(\sigma dB_t) \approx \alpha(d\beta_t)$ with a n -dimensional (instead of $(n+1)^2$ -dimensional) Brownian motion β

New estimator

- Consistency proven ($\Delta t \rightarrow 0$)
- Numerically efficient
- Physically-based
 \rightarrow Robustness in extrapolation

Randomized Navier-Stokes

PCA modes

PCA residual v'

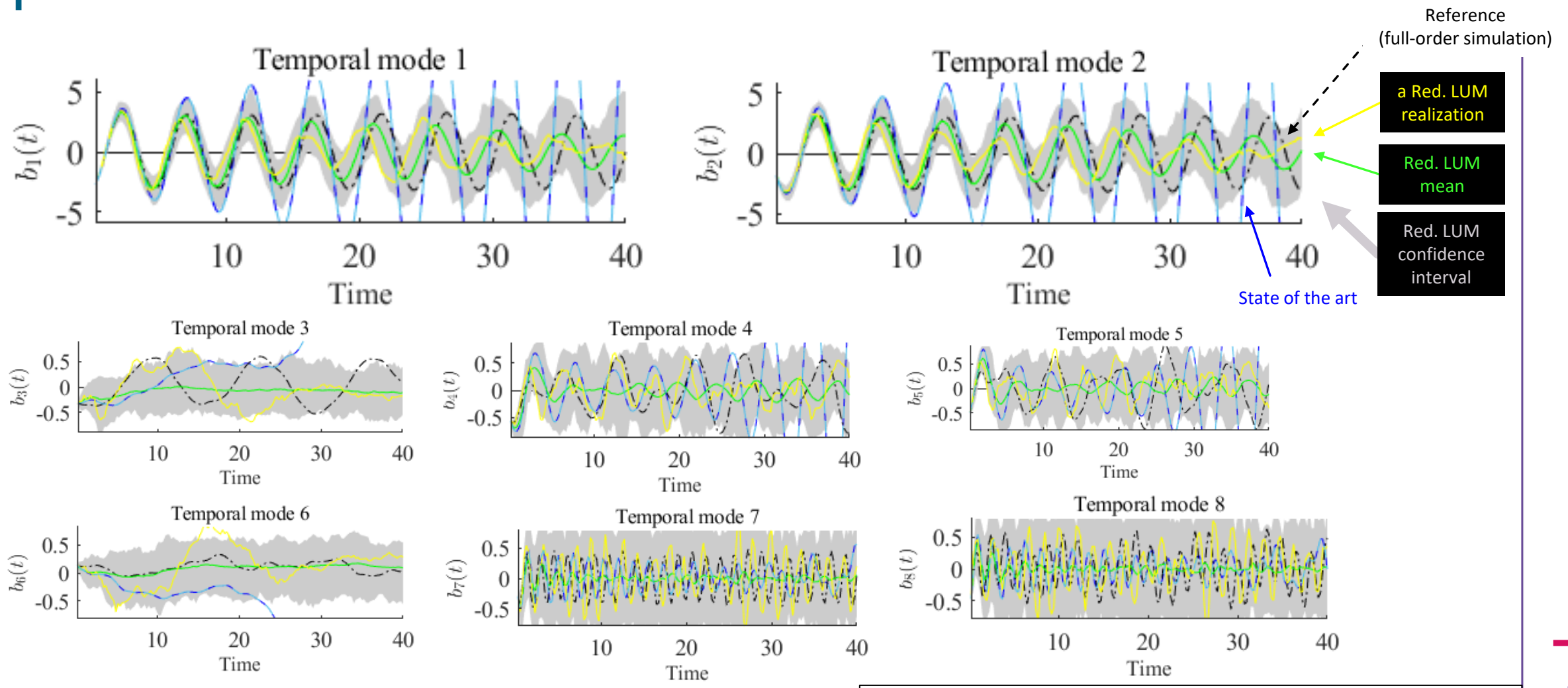
from synthetic data

$$\bar{f} = \frac{1}{T} \int_0^T f$$

UNCERTAINTY QUANTIFICATION (PRIOR)

$b_i(t)$ VS reference

From 10^7 to 8 degrees of freedom
 No data assimilation
 Known initial conditions $b(t = 0)$



UNCERTAINTY QUANTIFICATION (PRIOR)

Error on the reduced solution w

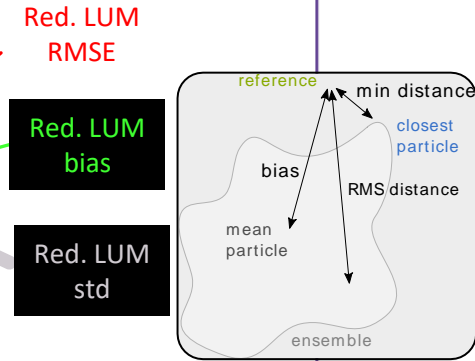
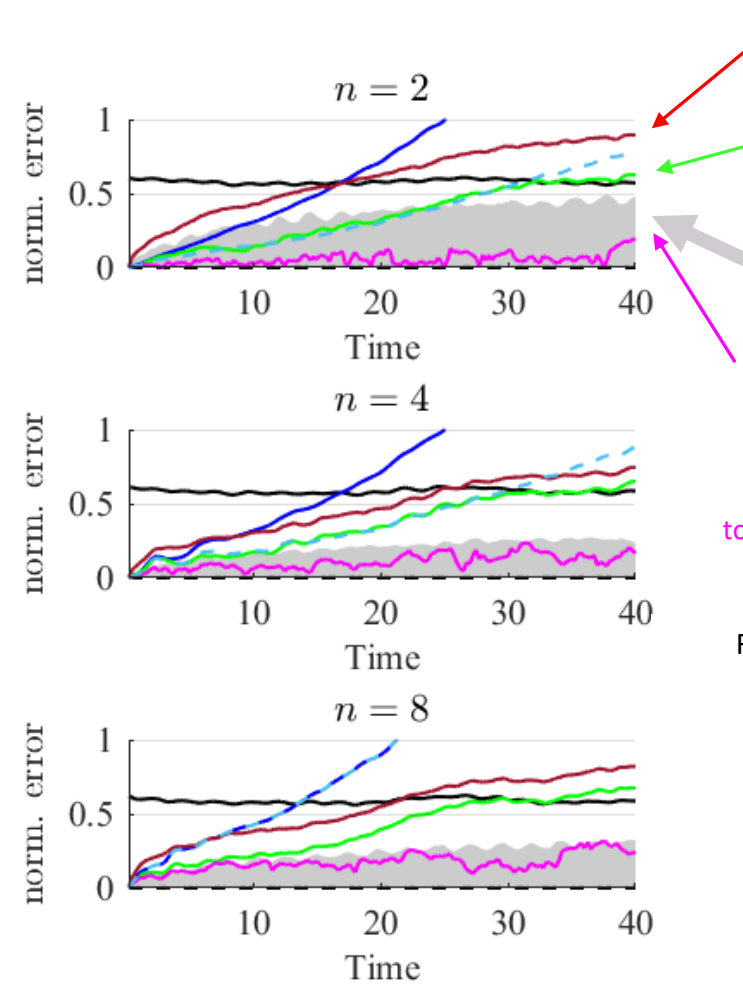
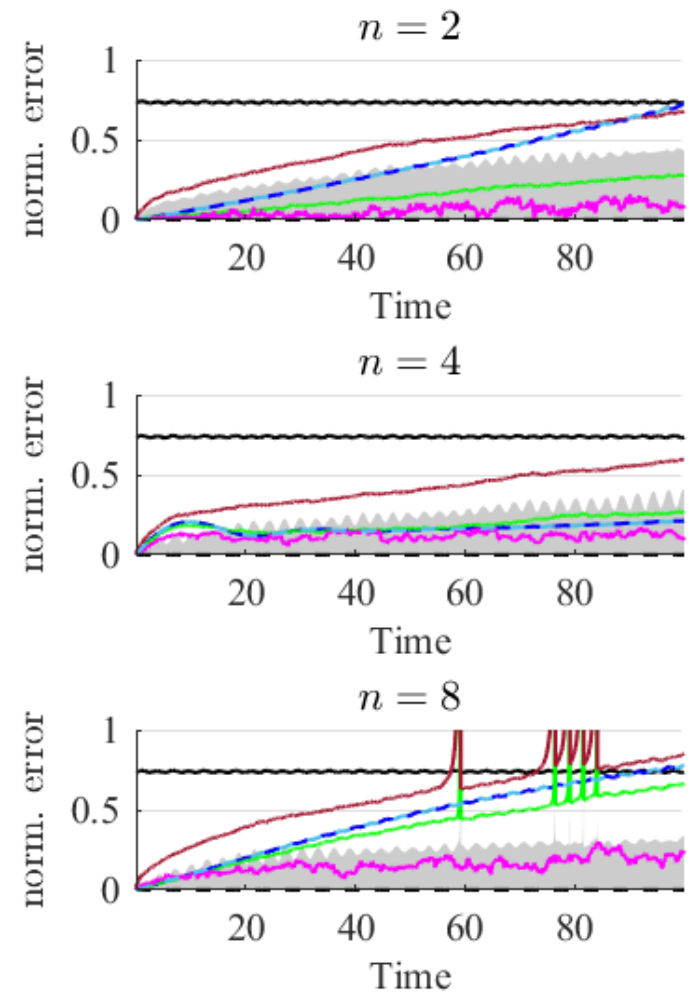
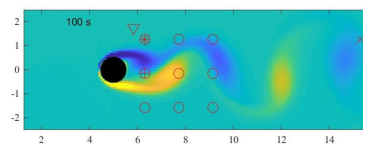
From 10^7 to 8 degrees of freedom
 No data assimilation
 Known initial conditions $b(t = 0)$

$v = w + v'$

Resolved fluid velocity:
 $w = \sum_{i=0}^n b_i \phi_i$

Unresolved fluid velocity:
 v'

Reynolds number (Re) = 100 / 2D
 (full-order simulation has 10^4 dof)



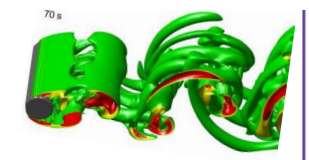
Red. LUM RMSE

Red. LUM bias

Red. LUM std

Red. LUM ensemble minimal distance to the reference

Reynolds number (Re) = 300 3D
 (full-order simulation has 10^7 dof)



UNCERTAINTY QUANTIFICATION (PRIOR)

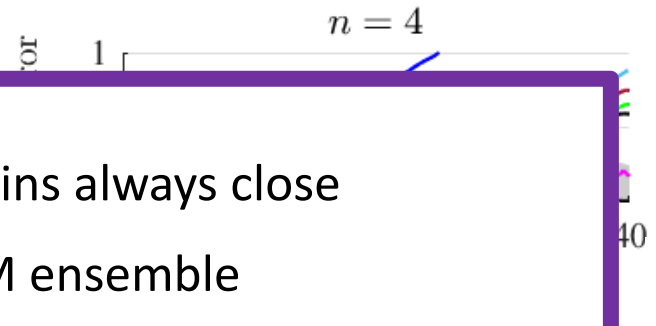
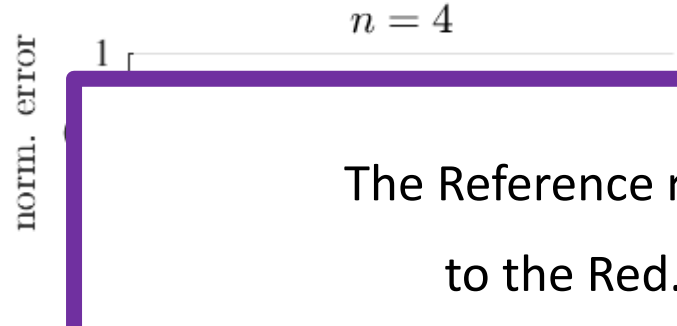
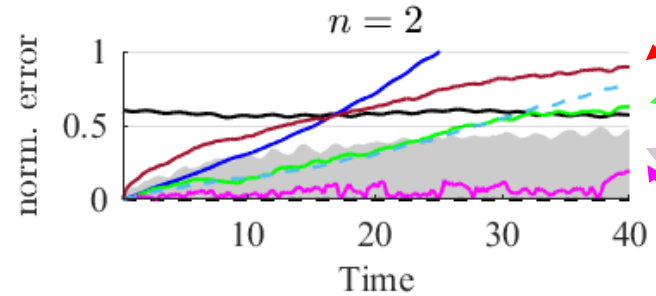
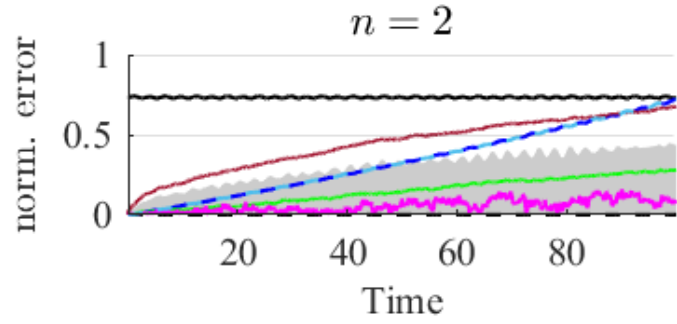
Error on the reduced solution w

From 10^7 to 8 degrees of freedom
 No data assimilation
 Known initial conditions $b(t = 0)$

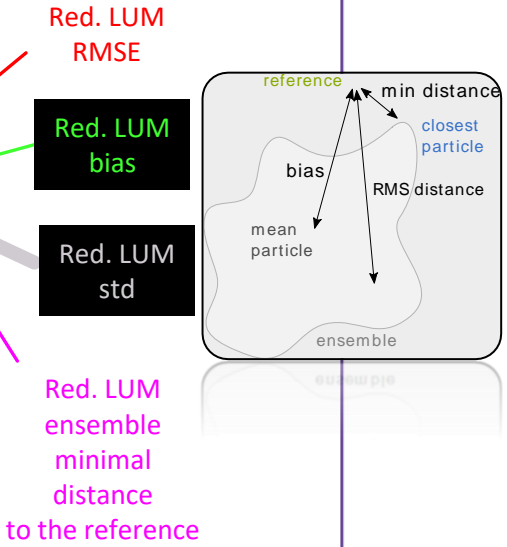
$v = w + v'$

Resolved fluid velocity:
 $w = \sum_{i=0}^n b_i \phi_i$

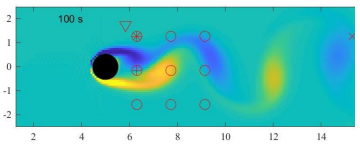
Unresolved fluid velocity:
 v'



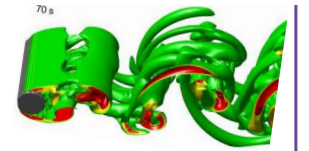
The Reference remains always close to the Red. LUM ensemble



Reynolds number (Re) = 100 / 2D
 (full-order simulation has 10^4 dof)



Reynolds number (Re) = 300 3D
 (full-order simulation has 10^7 dof)



DATA ASSIMILATION

Error on the solution estimation

$$v = w + v'$$

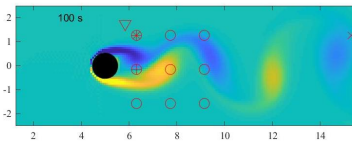
Resolved fluid velocity:

$$w = \sum_{i=0}^n b_i \phi_i$$

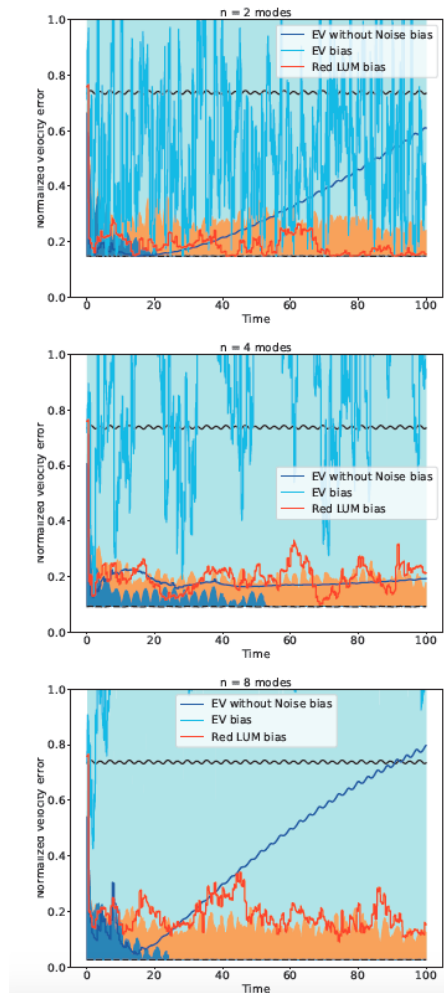
Unresolved fluid velocity:

$$v'$$

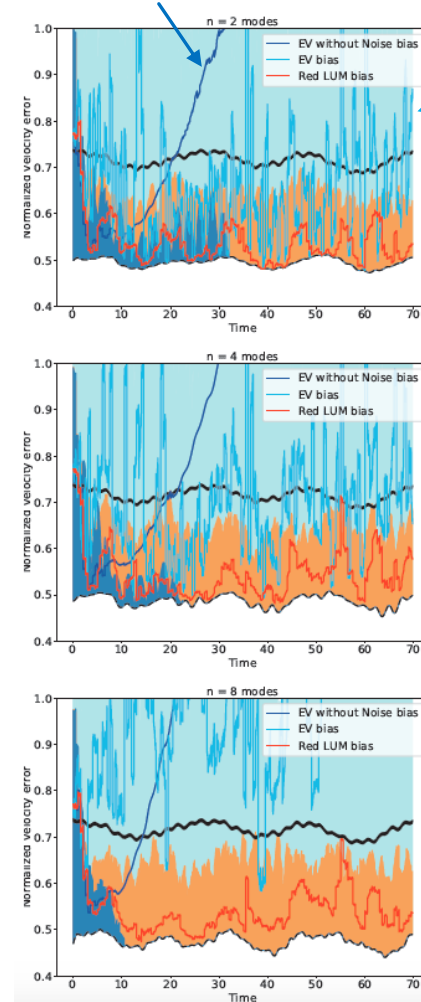
Easy case



Reynolds number
(Re) = 100
2D
(10⁴ dof)



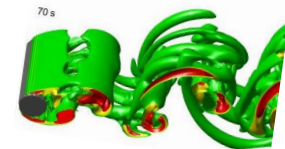
State of the art



State of the art

Red. LUM std
Red. LUM bias

Difficult case



Reynolds number
(Re) = 300
3D
(10⁷ dof)