



# Estimation of multiplicative noise operator statistics for reduced data assimilation in fluid mechanics

V. Resseguier, M. Ladvig,  
A. M. Picard & D. Heitz

**INRAE**

 **SCALIAN**

# Context

## Observer for wind turbine application

*Application:* Real-time estimation and prediction of 3D fluid flow  
using strongly-limited computational resources & few sensors

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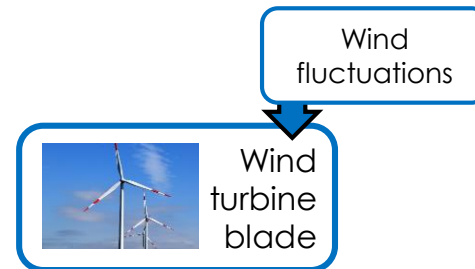


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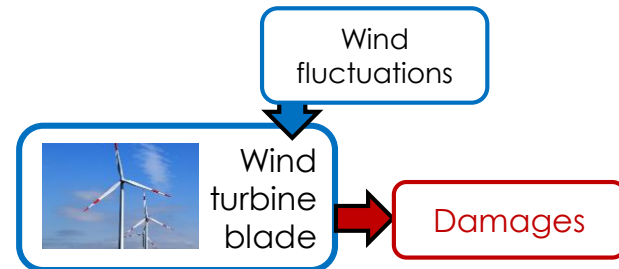


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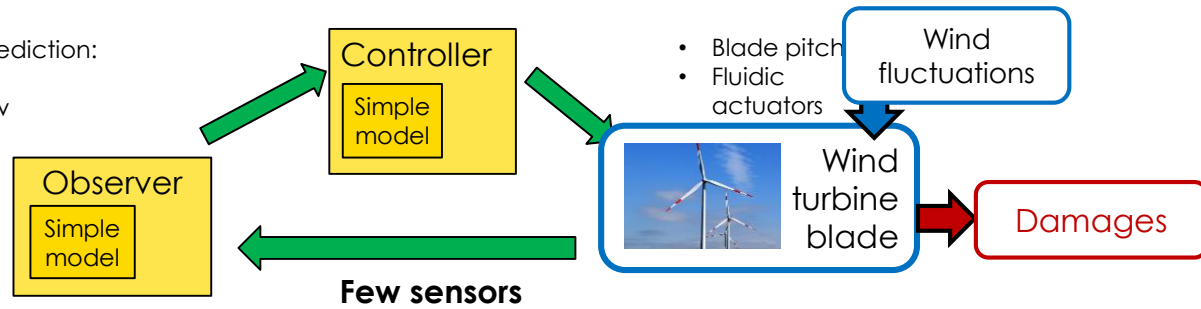
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Estimation and prediction:

- Air flow
- Lift, drag, inflow
- ...



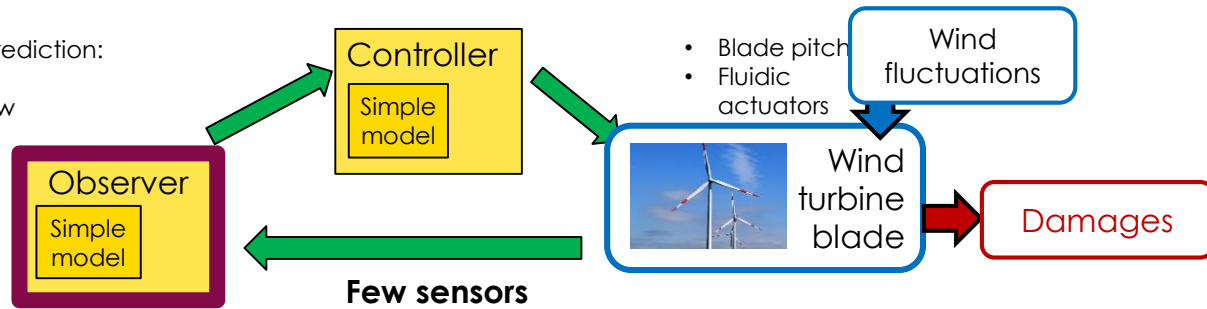
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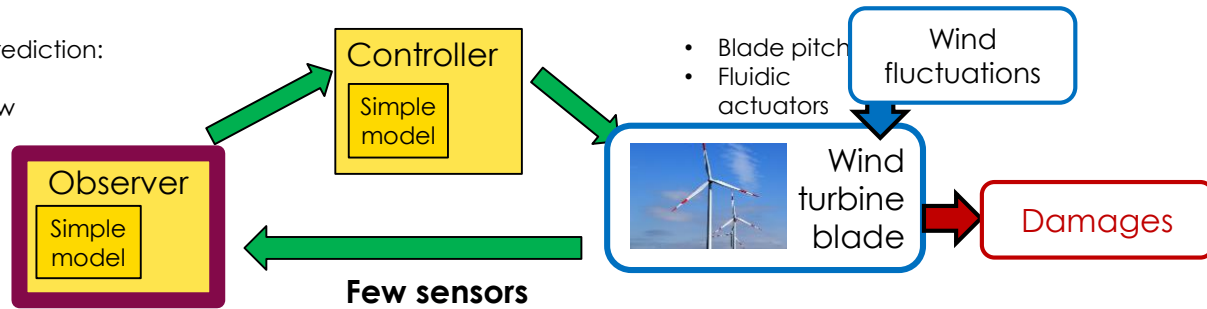
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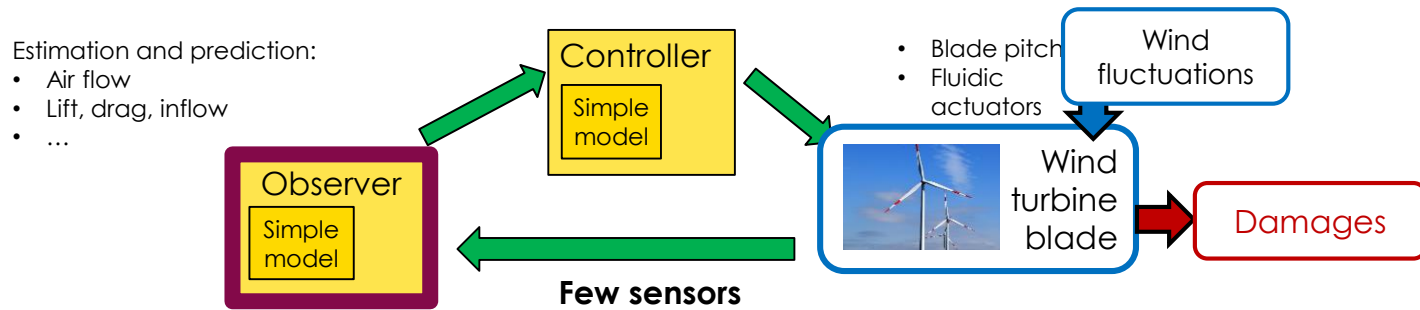


Which simple model?    How to combine model & measurements?

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Which simple model?    How to combine model & measurements?

**Scientific problem :**

**Simulation & data assimilation under severe dimensional reduction**

typically,  $10^7 \rightarrow O(10)$  degrees of freedom



# Content

- I. State of the art
  - a. Intrusive reduced order model (ROM)
  - b. Data assimilation
- II. Reduced location uncertainty models
  - a. Multiscale modeling
  - b. Location uncertainty models (LUM)
  - c. Reduced LUM
- III. Numerical results

# PART I

## STATE OF THE ART

- a. Intrusive reduced order model (ROM)
- b. Data assimilation


# Reduced order model (ROM)

Solution of an PDE with the form:

$$v(x, t) \approx \sum_{i=0}^n b_i(t) \phi_i(x)$$

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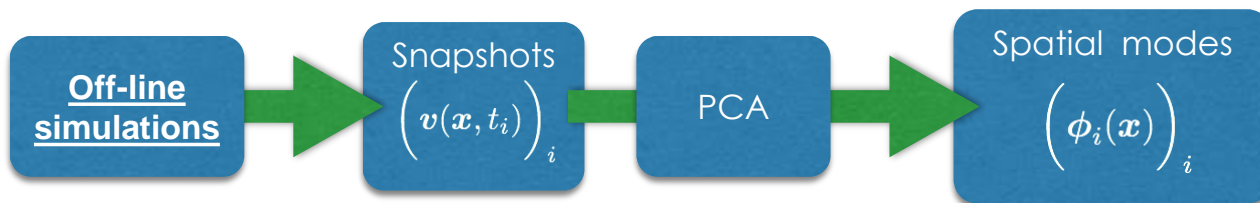
Order of magnitude examples in CFD

	Full space	Reduced space
Solution coordinates	$(v_q(x_i, t))_{q,i}$	$(b_i(t))_i$
Dimension	$M \times d \sim 10^7$	$n \sim 10 - 100$

# Intrusive reduced order model

## Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:



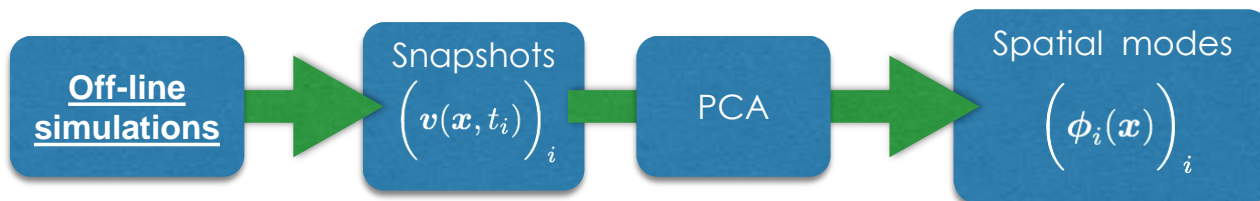
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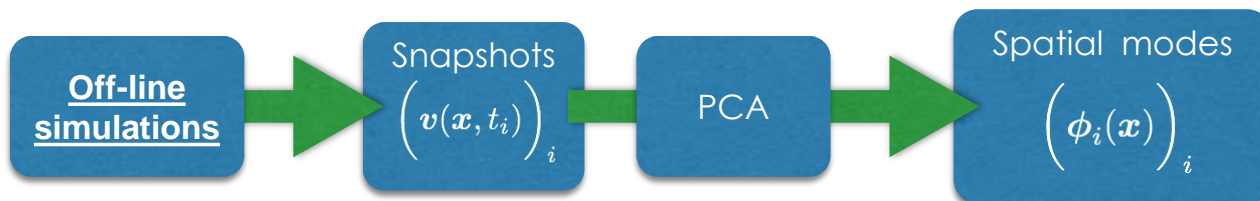
A yellow arrow points from the "Spatial modes" box in the diagram above to the  $\phi_i(x)$  term in the equation.



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- Projection of the “*physics*” onto the spatial modes (POD-Galerkin)

$$\int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation (e.g. Navier-Stokes)})$$

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→ ROM for very fast simulation of temporal modes

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The term  $b_i(t)$  is enclosed in a blue rounded rectangle labeled 'Resolved modes'.

Don't work in extrapolation!

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$$\int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation})$$

+ fitted correction

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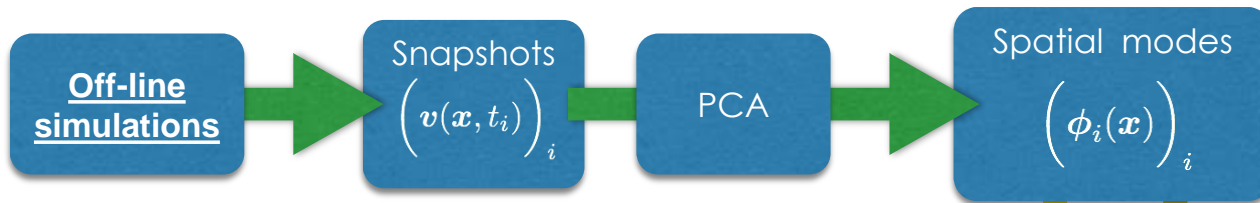
$$\int_{\Omega} dx \phi_i(x) \cdot ( \textit{Physical equation} + \textit{fitted correction} + \textit{additive noise} )$$

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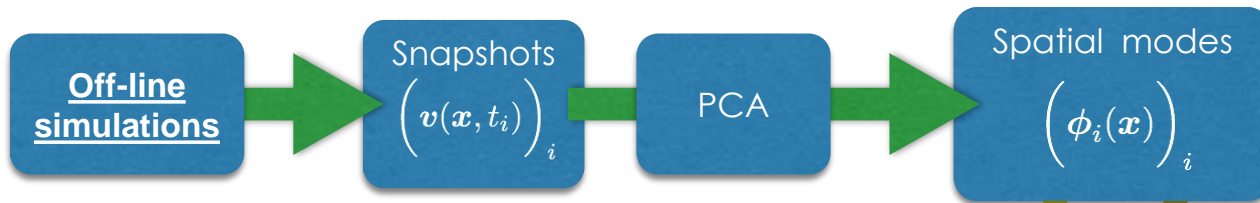
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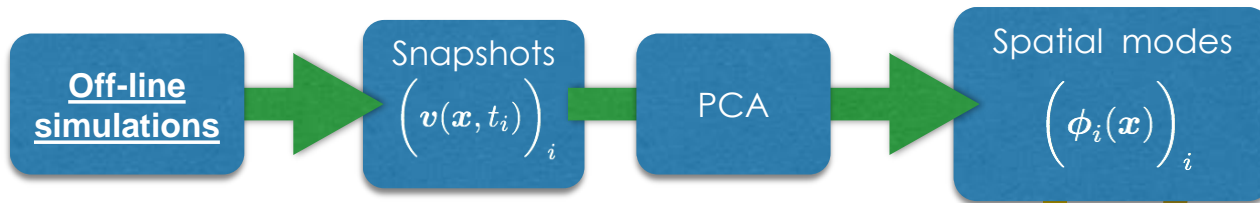
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SPOILER

- Projection of the “*physics*” onto the spatial modes (POD-Galerkin)

$$\int_{\Omega} dx \phi_i(x) \cdot \text{(Randomized Navier-Stokes)}$$

→ ROM for very fast simulation of temporal modes

# Data assimilation = Coupling simulations and measurements $y$

**Numerical  
Simulation  
(ROM)**

→ erroneous

**On-line  
measurements**

→ incomplete  
→ possibly noisy

Velocity



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**Numerical  
Simulation  
(ROM)**

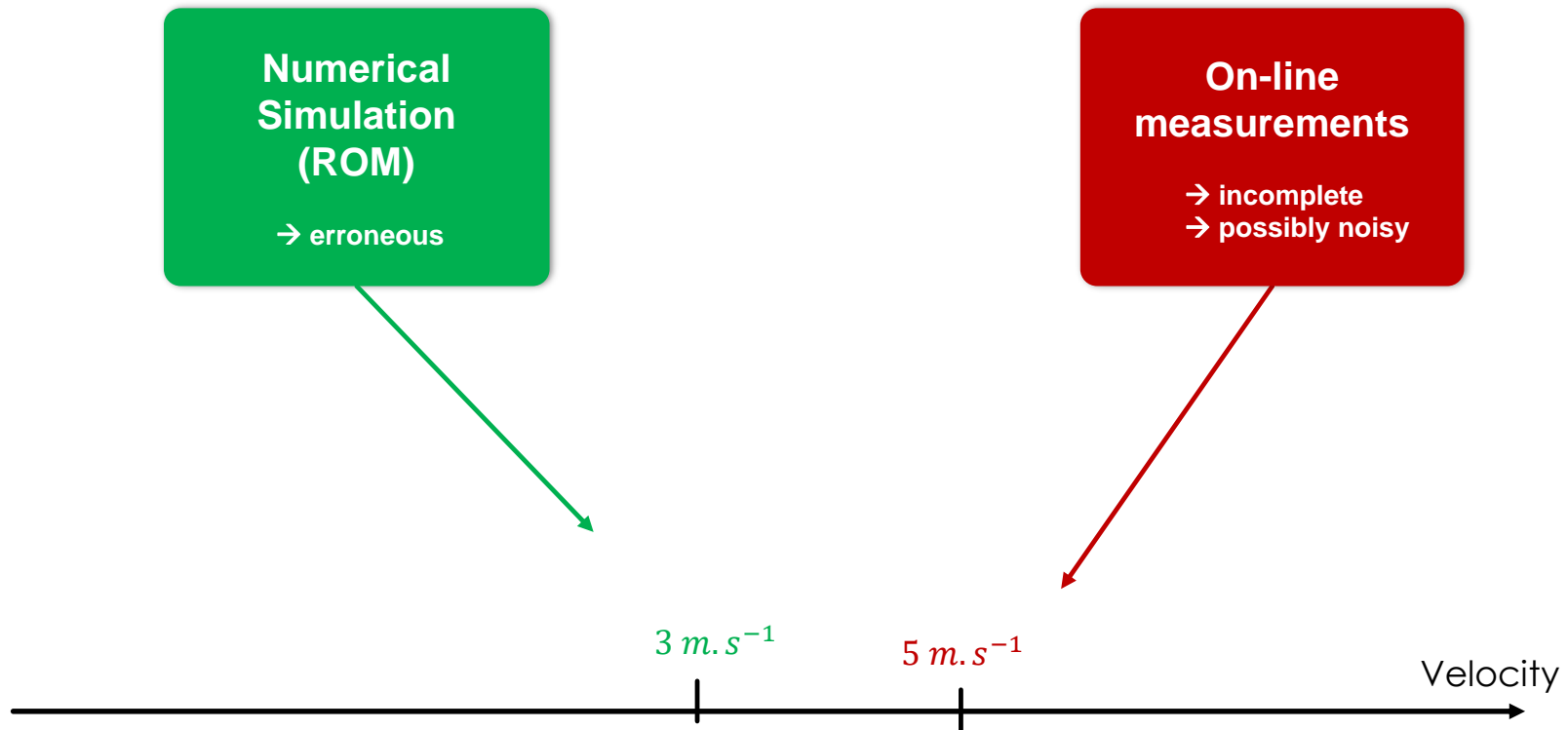
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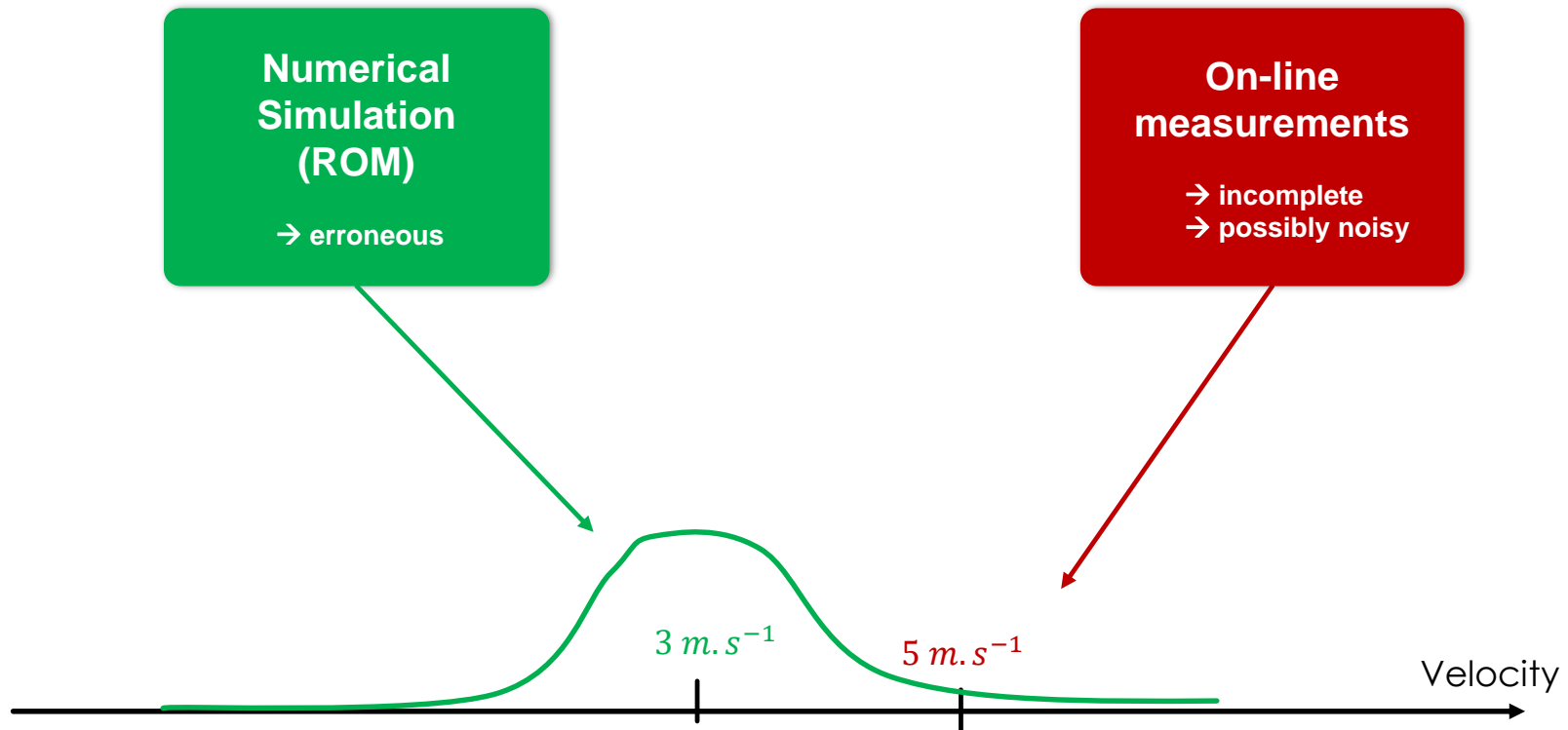
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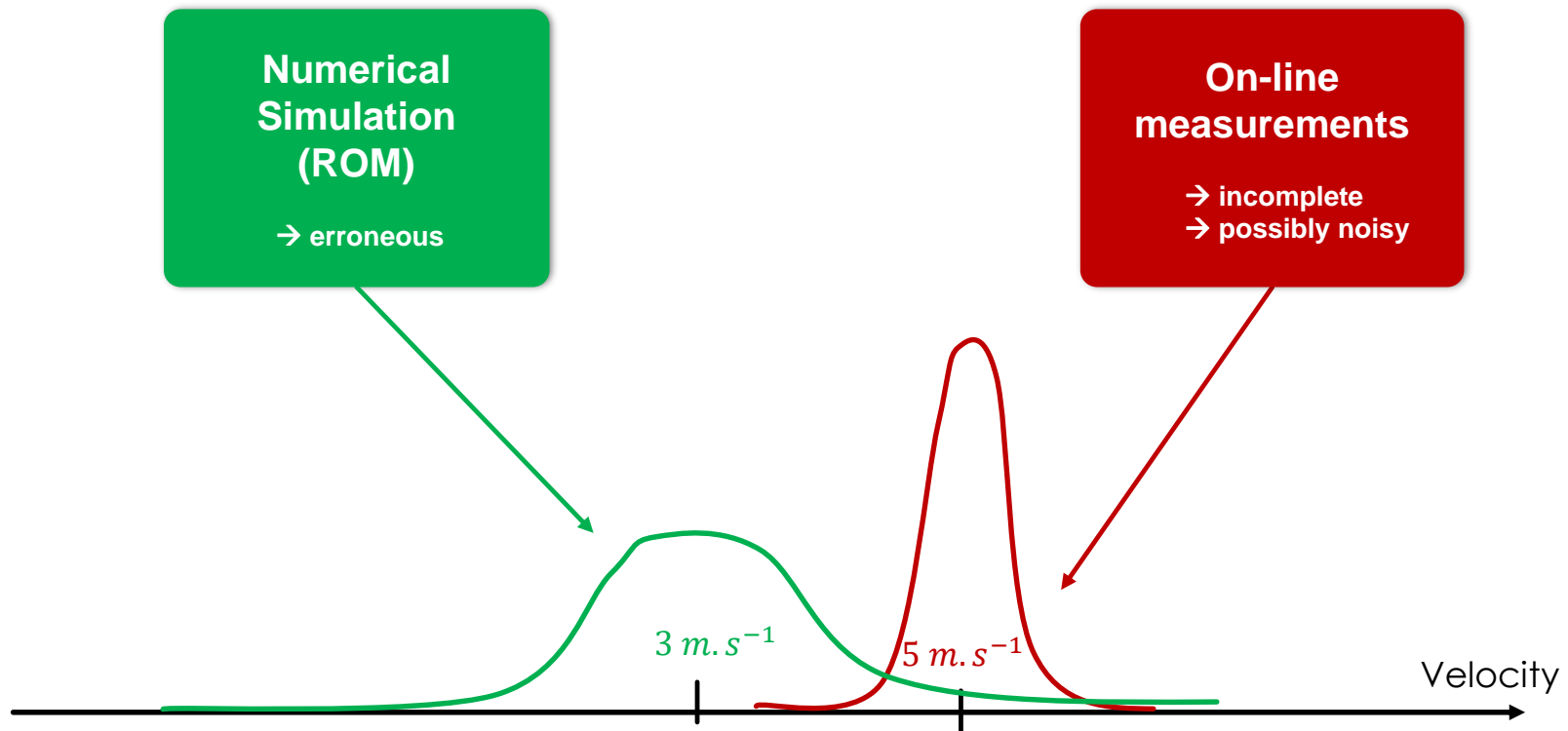
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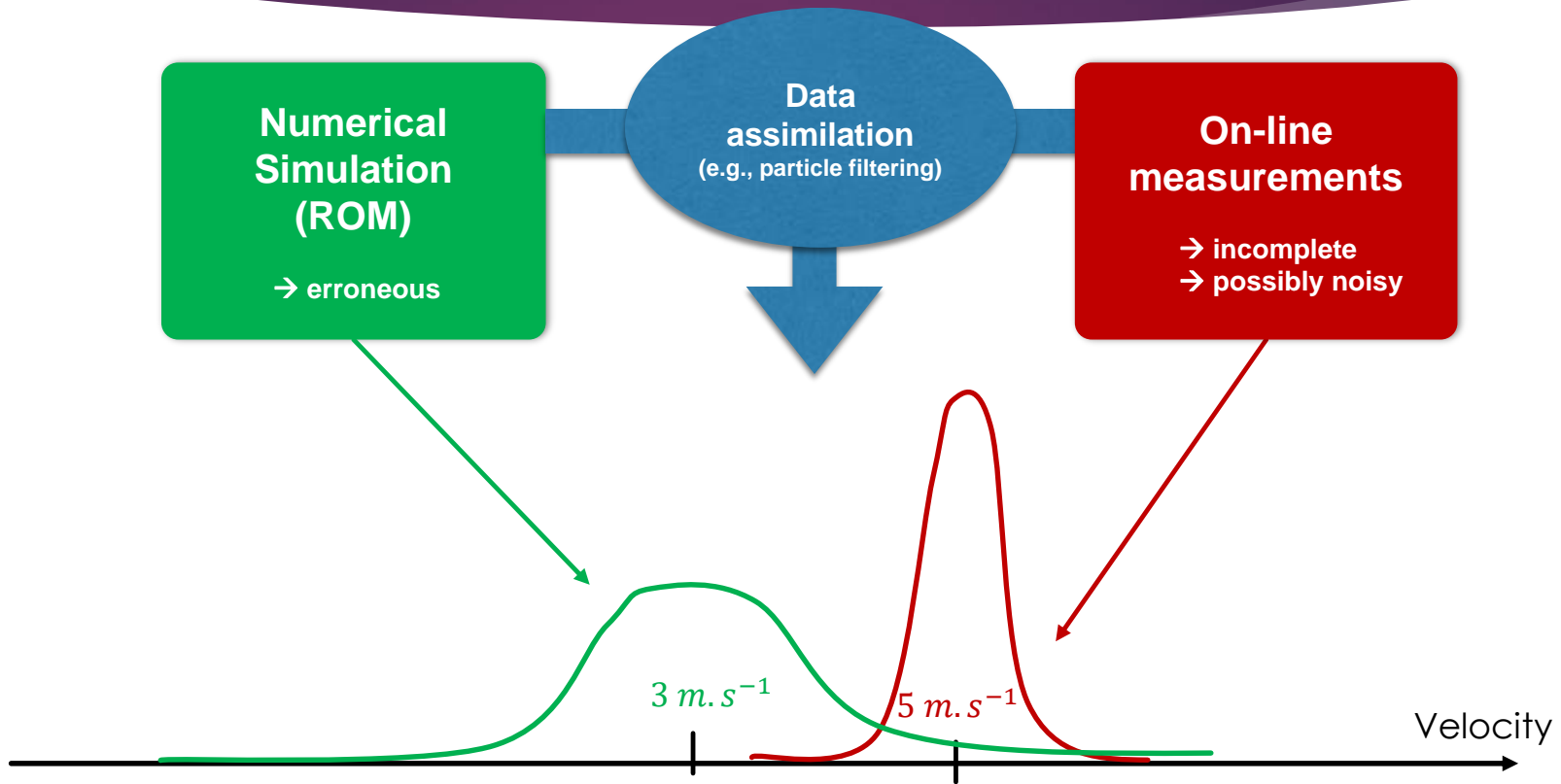
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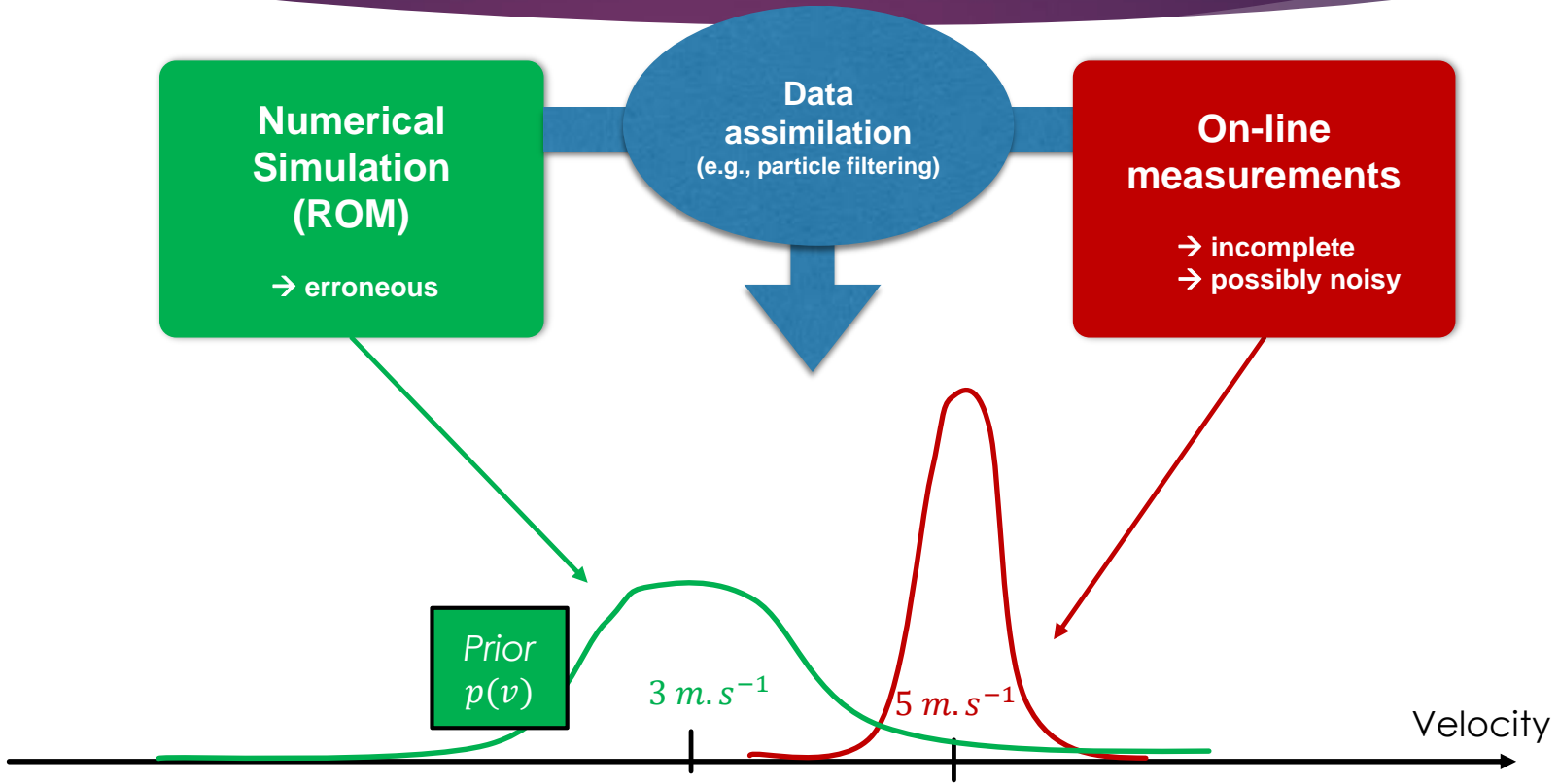
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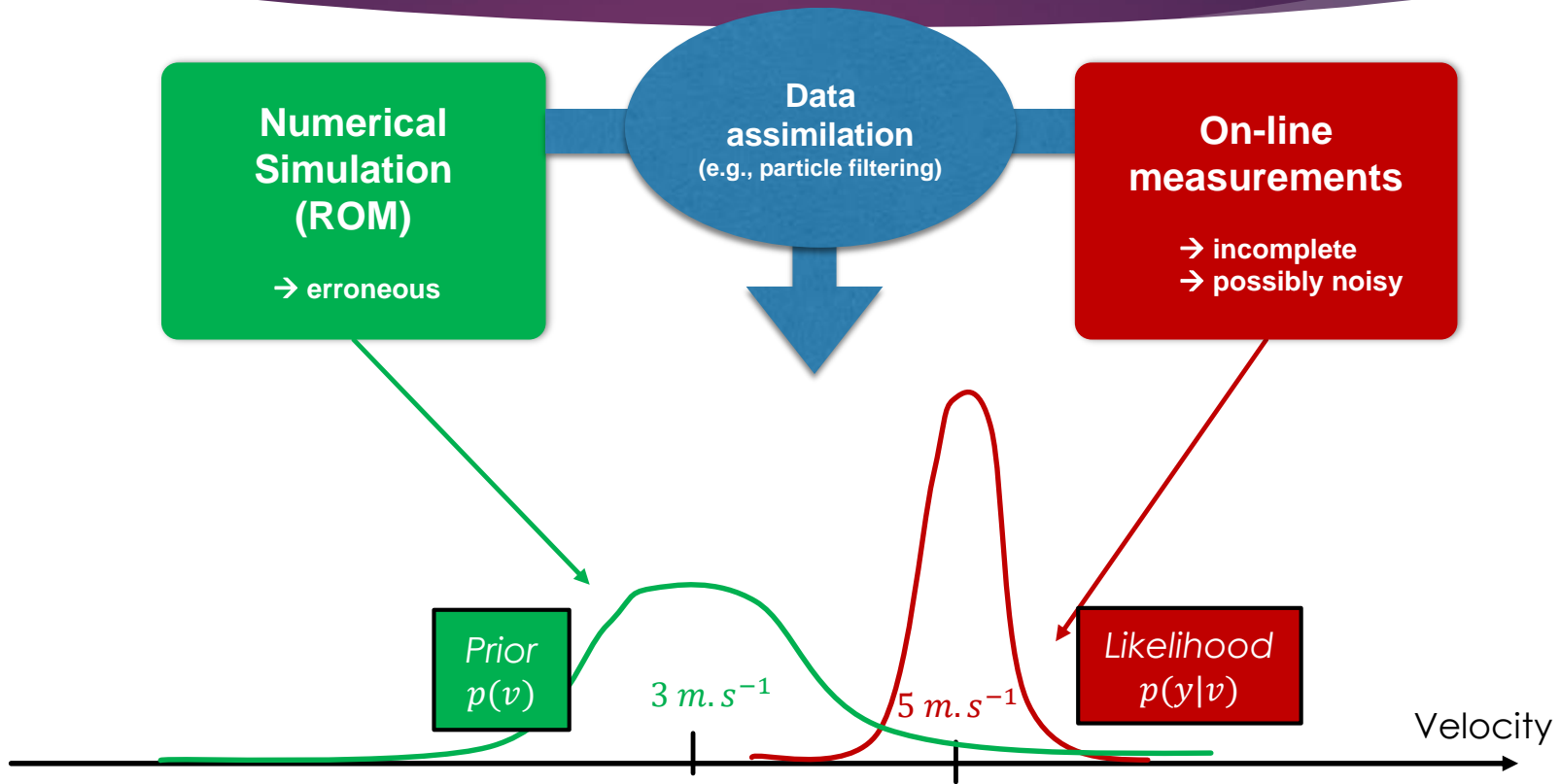


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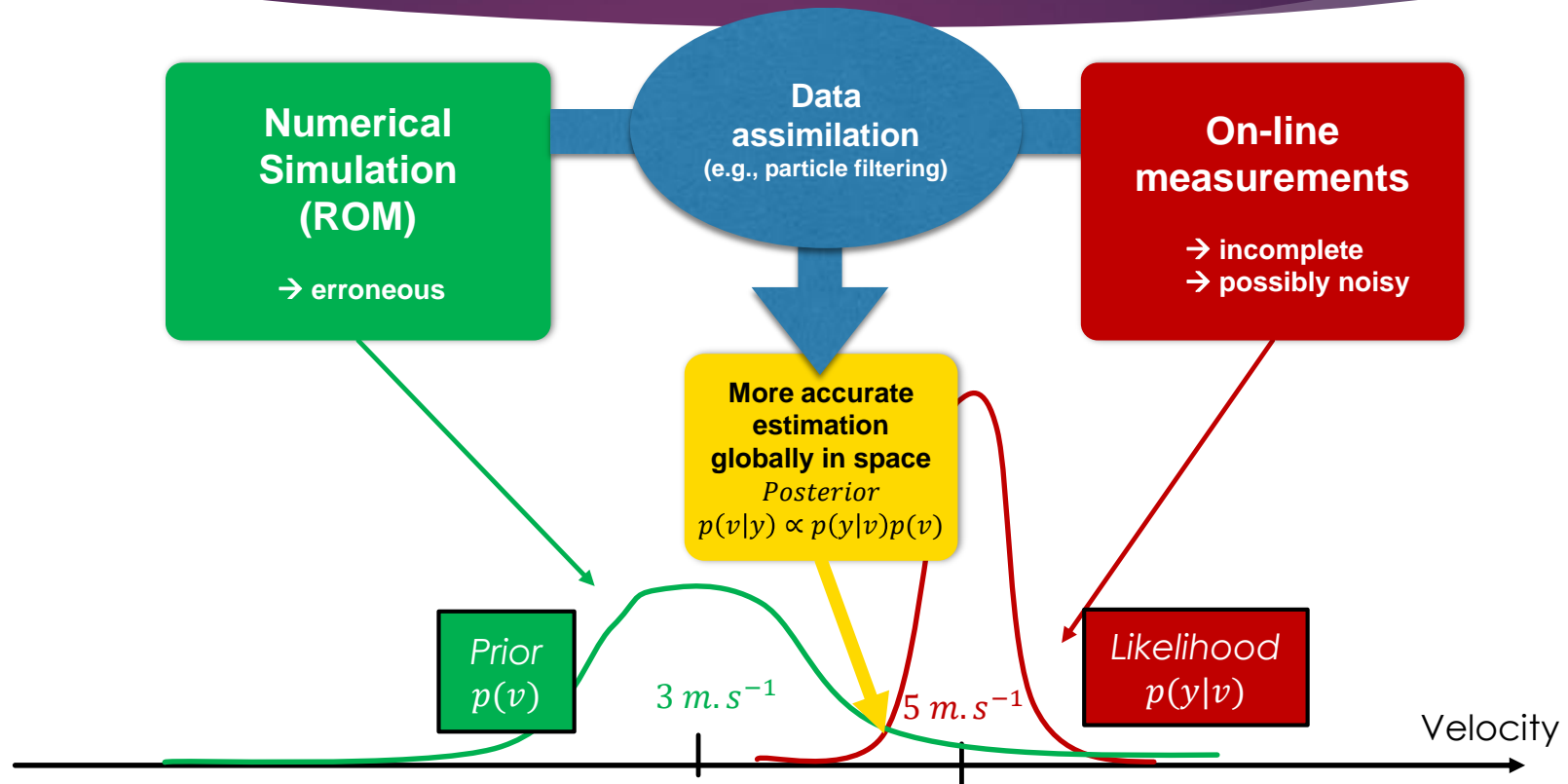




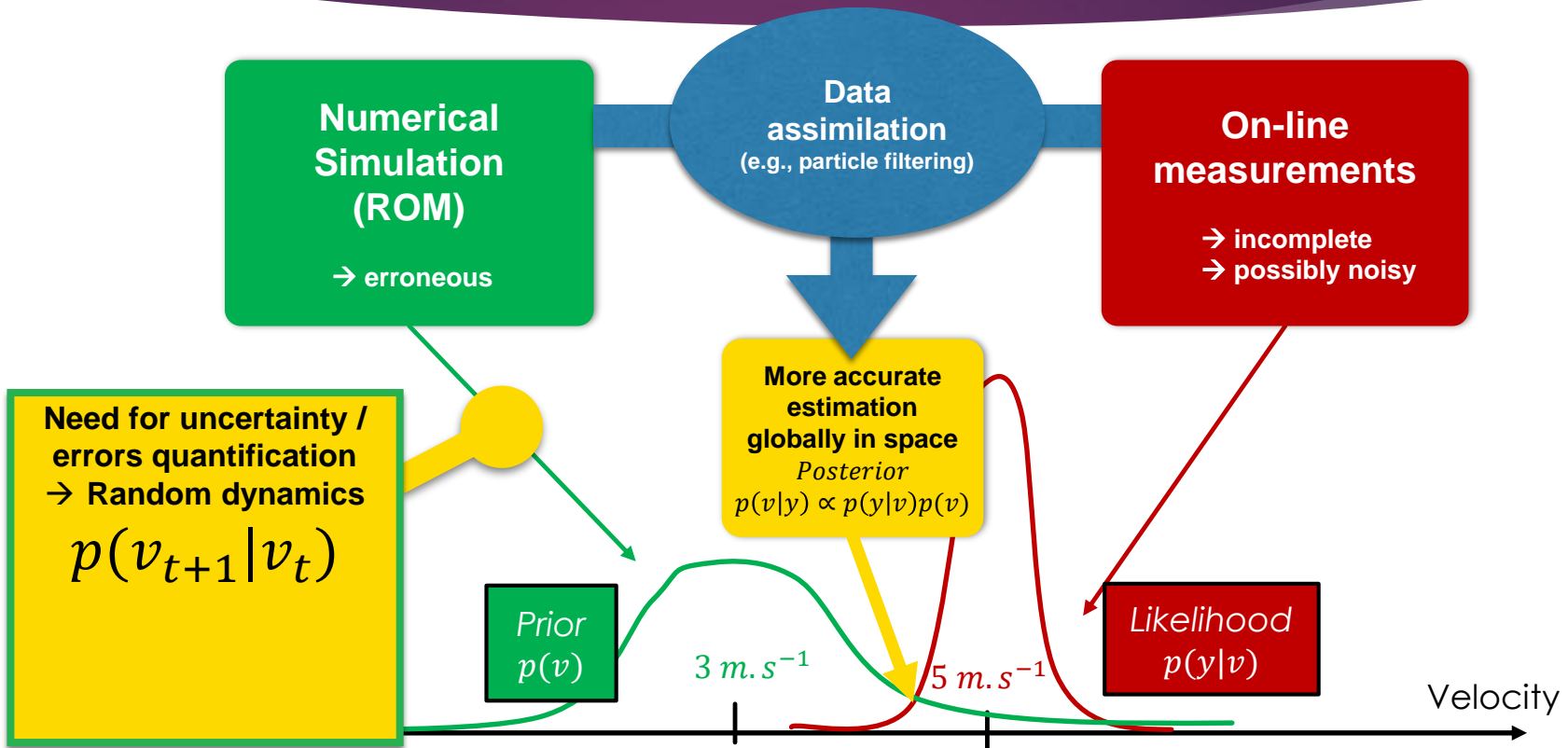
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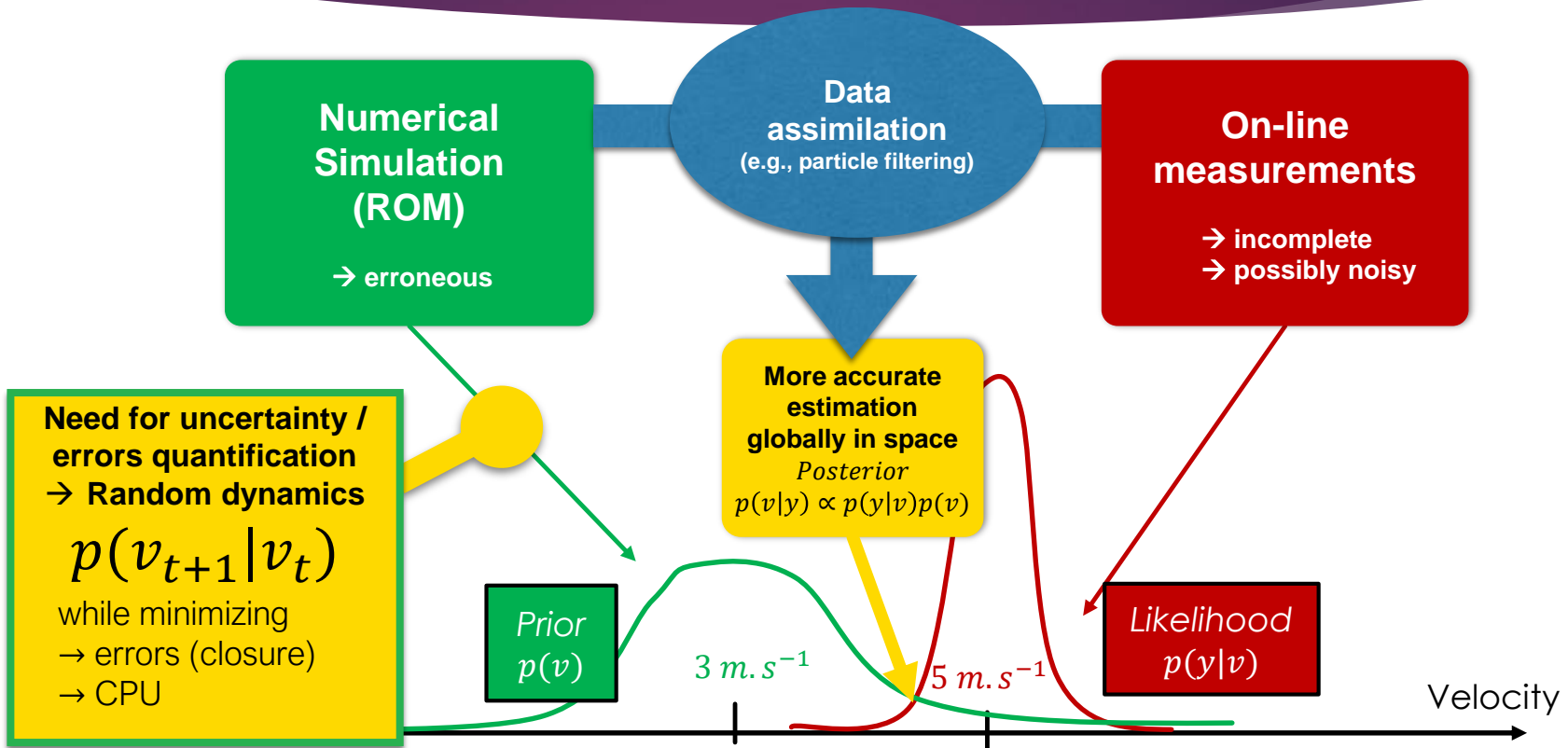
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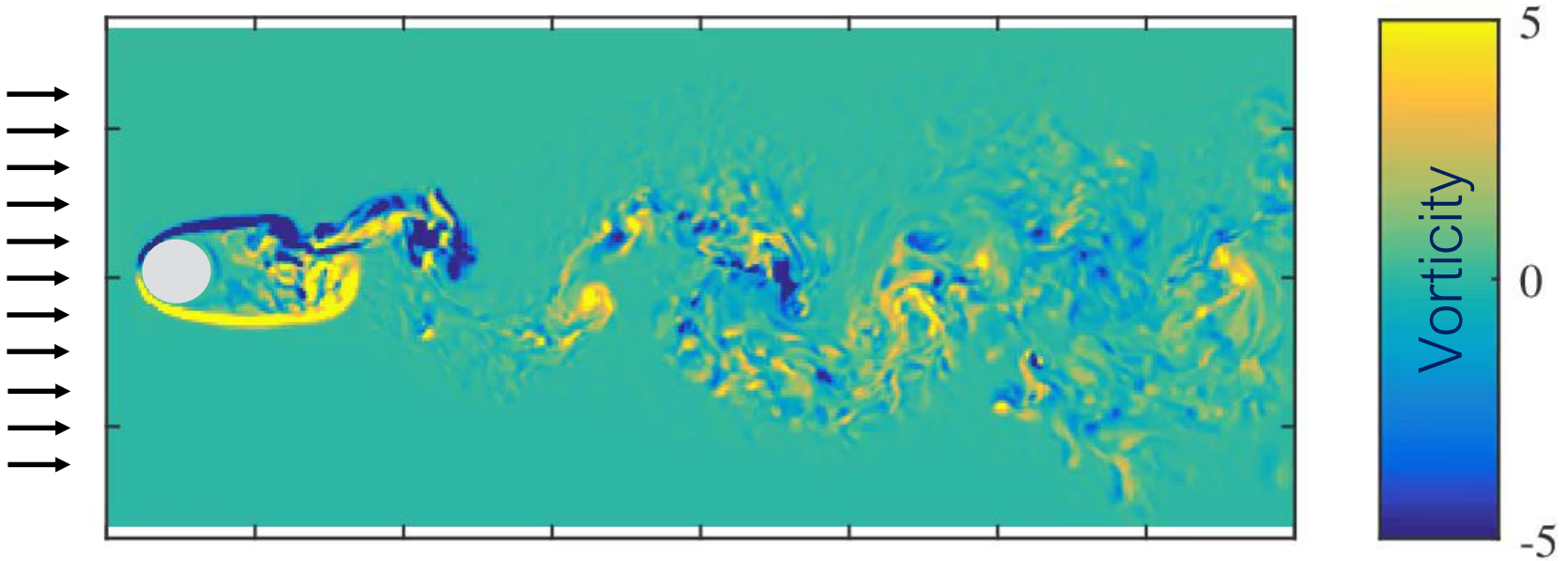


## Part II

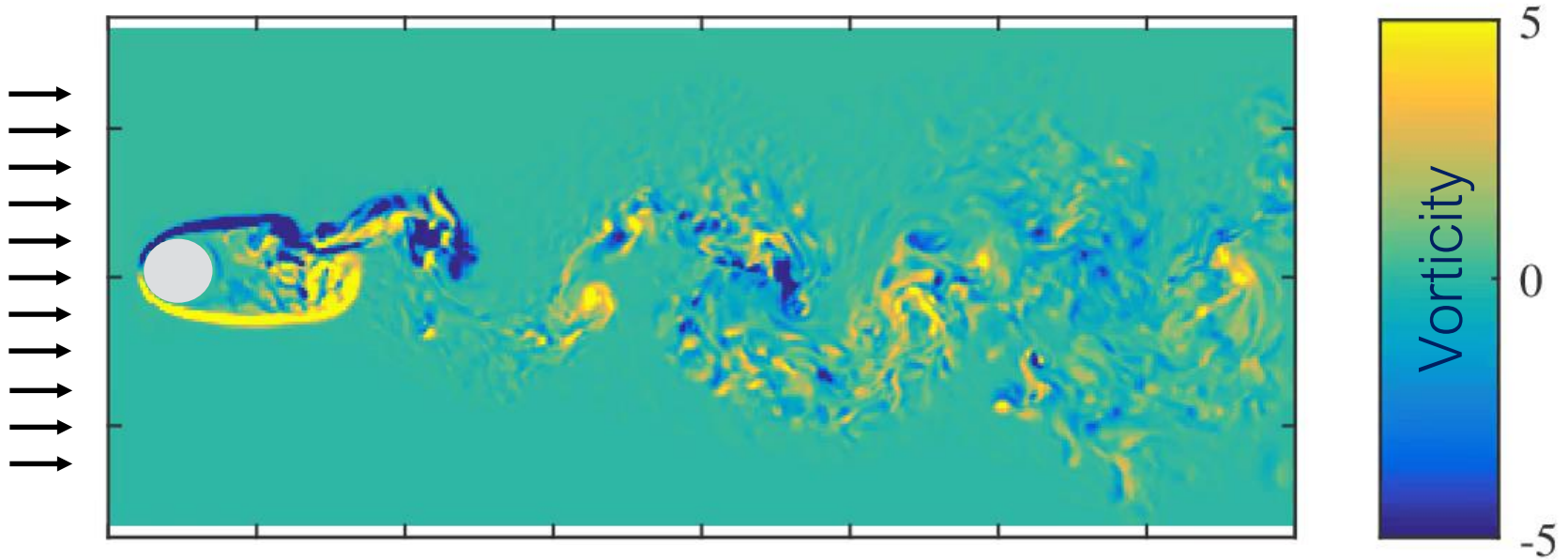
### REDUCED LOCATION UNCERTAINTY MODELS

- a. Multiscale modeling
- b. Location uncertainty models (LUM)
- c. Reduced LUM (Red LUM)

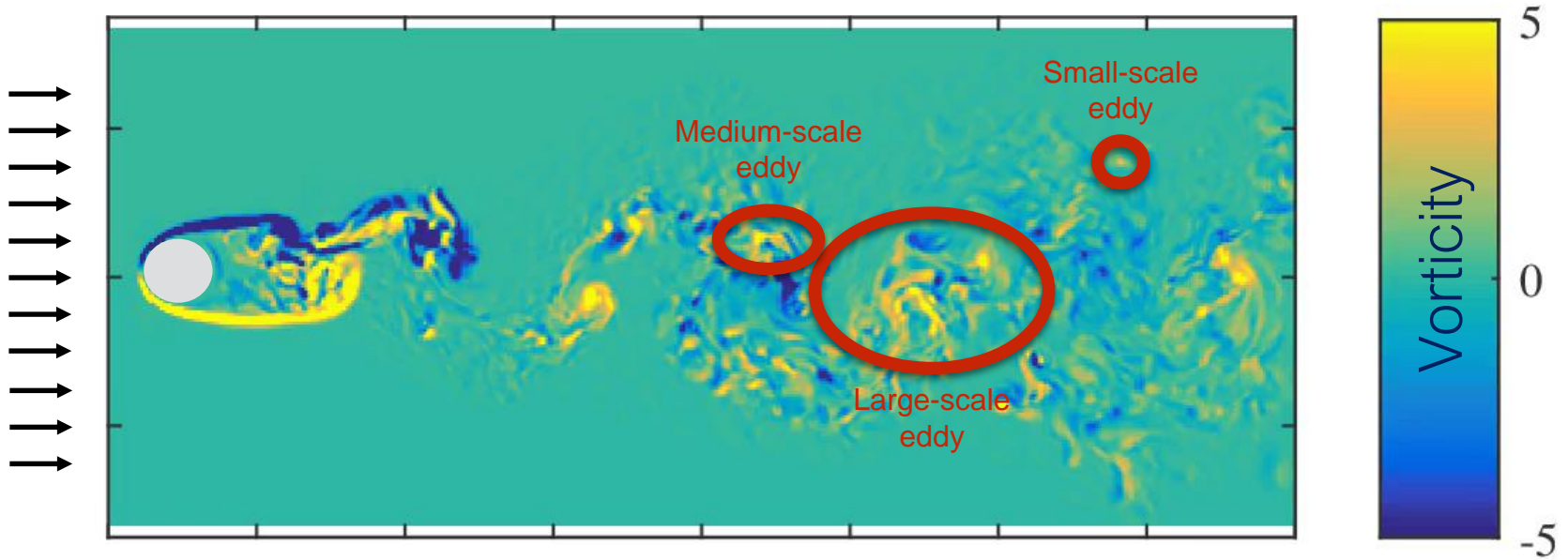
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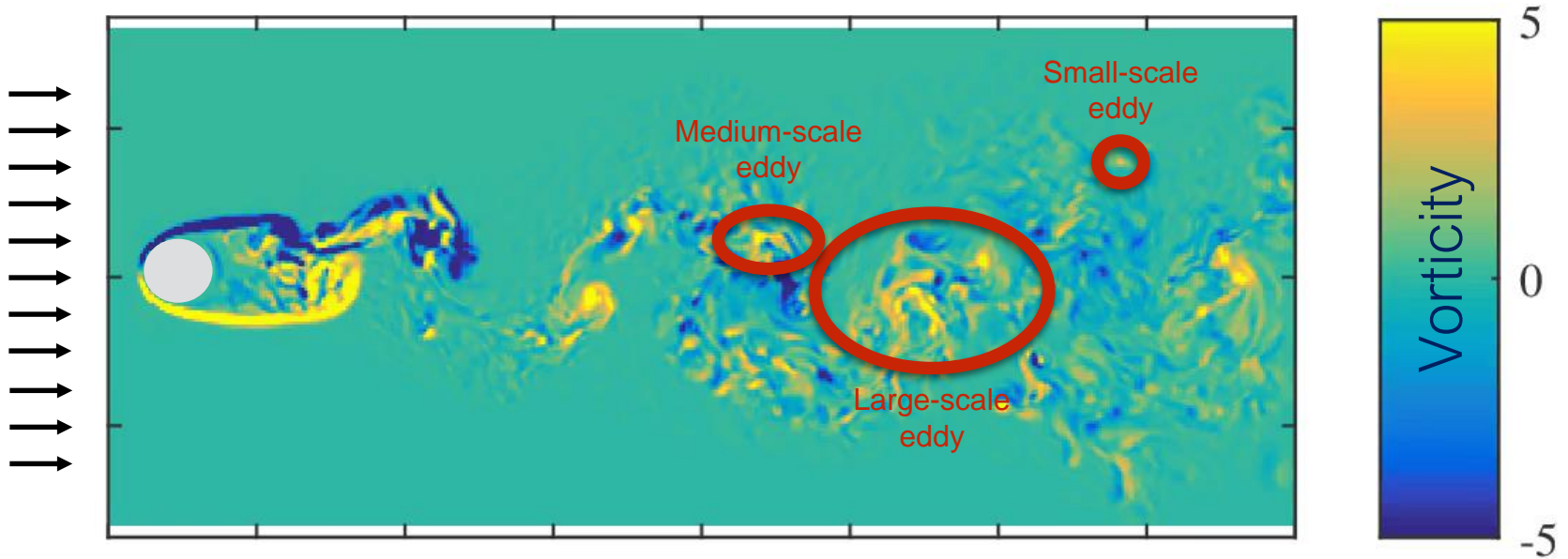


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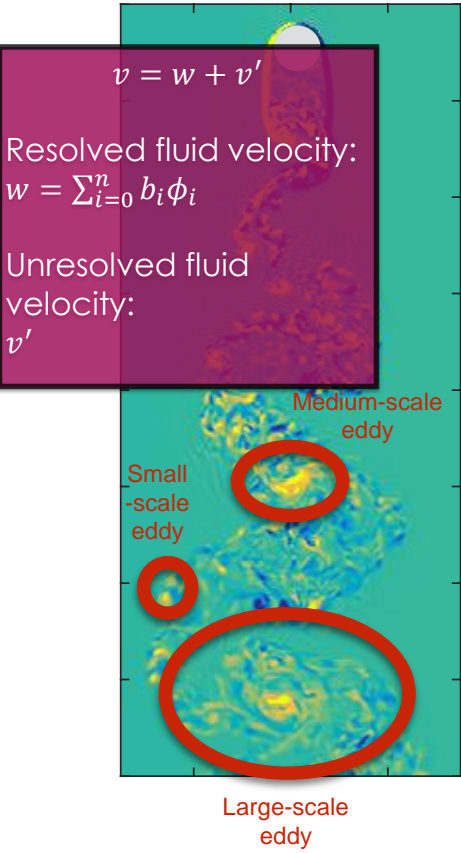


# Multiscale modeling



Fluids are multiscale  $\longrightarrow$  Many coupled degrees of freedom

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Fluids are multiscale  $\longrightarrow$  Many coupled degrees of freedom  
 We cannot simulate (or observe) every scales.

Generally, authors

- simulate large scales  $w$ ,
- model *the effect of* small scales  $v'$  in the equations (closure).

Here, we

- model the small scales  $v'$  through *stochastic processes*, calibrated from data and/or from physical scale symmetries.
- inject those in *physical* equations for physical understanding, simulations & data assimilation.

# location uncertainty models (LUM)

$$v = w + v'$$

Resolved fluid velocity:

$$w = \sum_{i=0}^n b_i \phi_i$$

Unresolved fluid velocity:

$$v' = \frac{\sigma dB_t}{dt}$$

**Assumed**  
(conditionally-)Gaussian  
& white in time  
(non-stationary in space)

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Randomized  
Navier-Stokes  
model

- Good closure
- Good model error quantification  
**for data assimilation**

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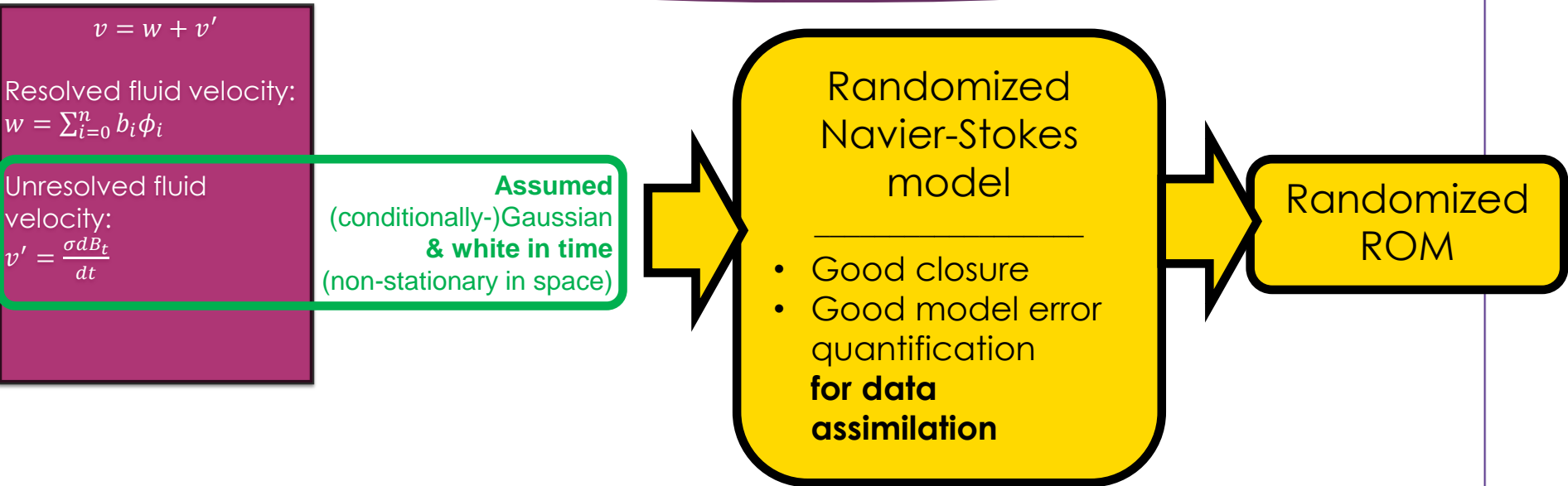
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wrt  $t$ )

(assuming  $\nabla \cdot w = 0$  and  $\nabla \cdot v' = 0$ )

Momentum conservation

$$d(w(t, X_t)) = dF \quad (\text{Forces})$$

Positions of fluid parcels  $X_t$  :

$$dX_t = w(t, X_t)dt + \underbrace{\sigma(t, X_t)dB_t}_{\text{Gaussian process white in time}}$$

Gaussian  
process  
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Variance tensor:  
 $a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

From Ito-Wentzell  
formula (Kunita  
1990)  
with Ito notations

$$d_t w + w^* dt \cdot \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt = dF$$

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Usual terms



Skew-symmetric multiplicative random forcing

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Usual  
terms



Skew-symmetric  
multiplicative  
random  
forcing

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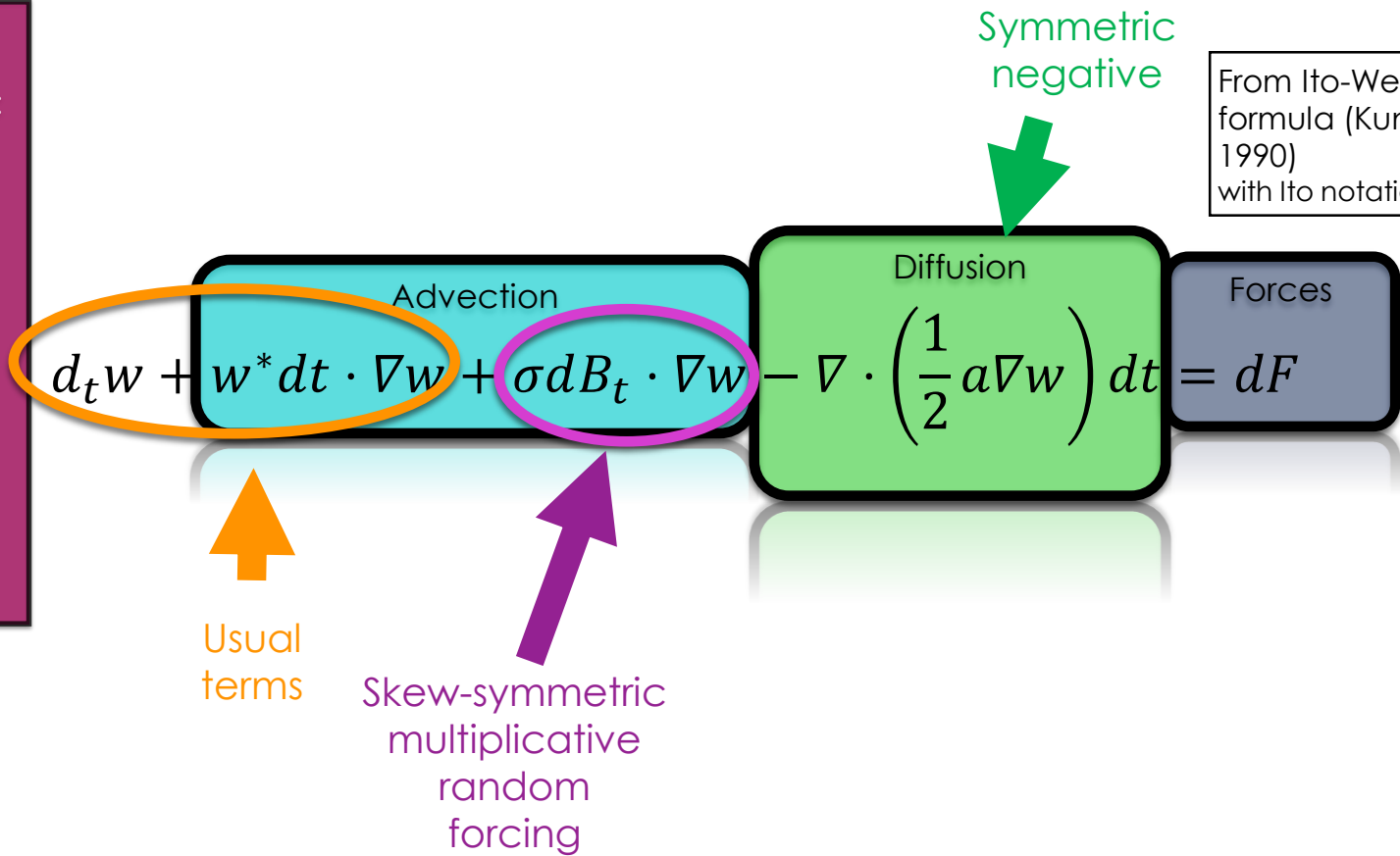
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Unresolved fluid velocity:  
 $v' = \frac{\sigma dB_t}{dt}$  (Gaussian, white wrt  $t$ )

(assuming  $\nabla \cdot w = 0$  and  $\nabla \cdot v' = 0$ )

Variance tensor:  
 $a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$





# location uncertainty models (LUM), Randomized Navier-Stokes

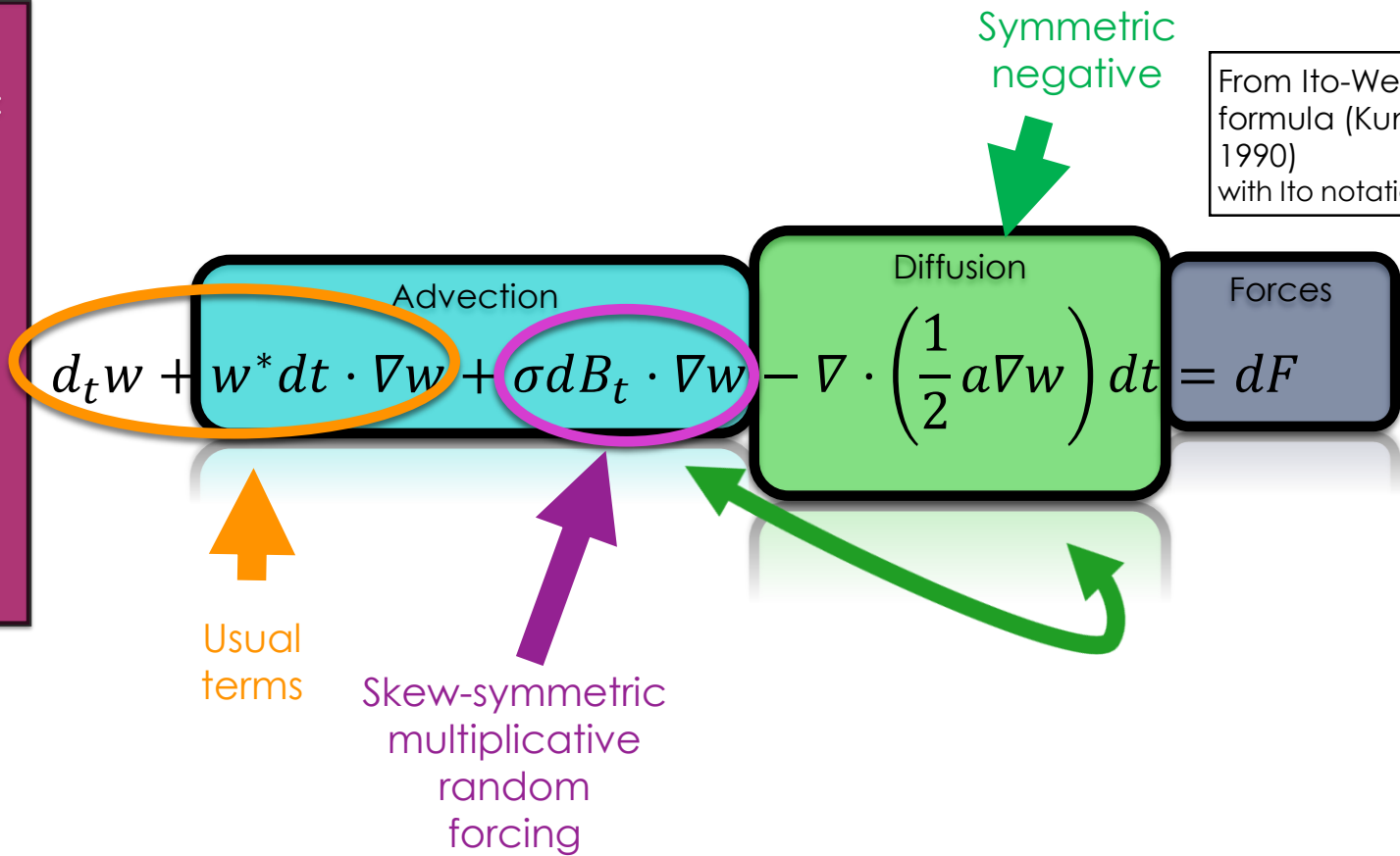
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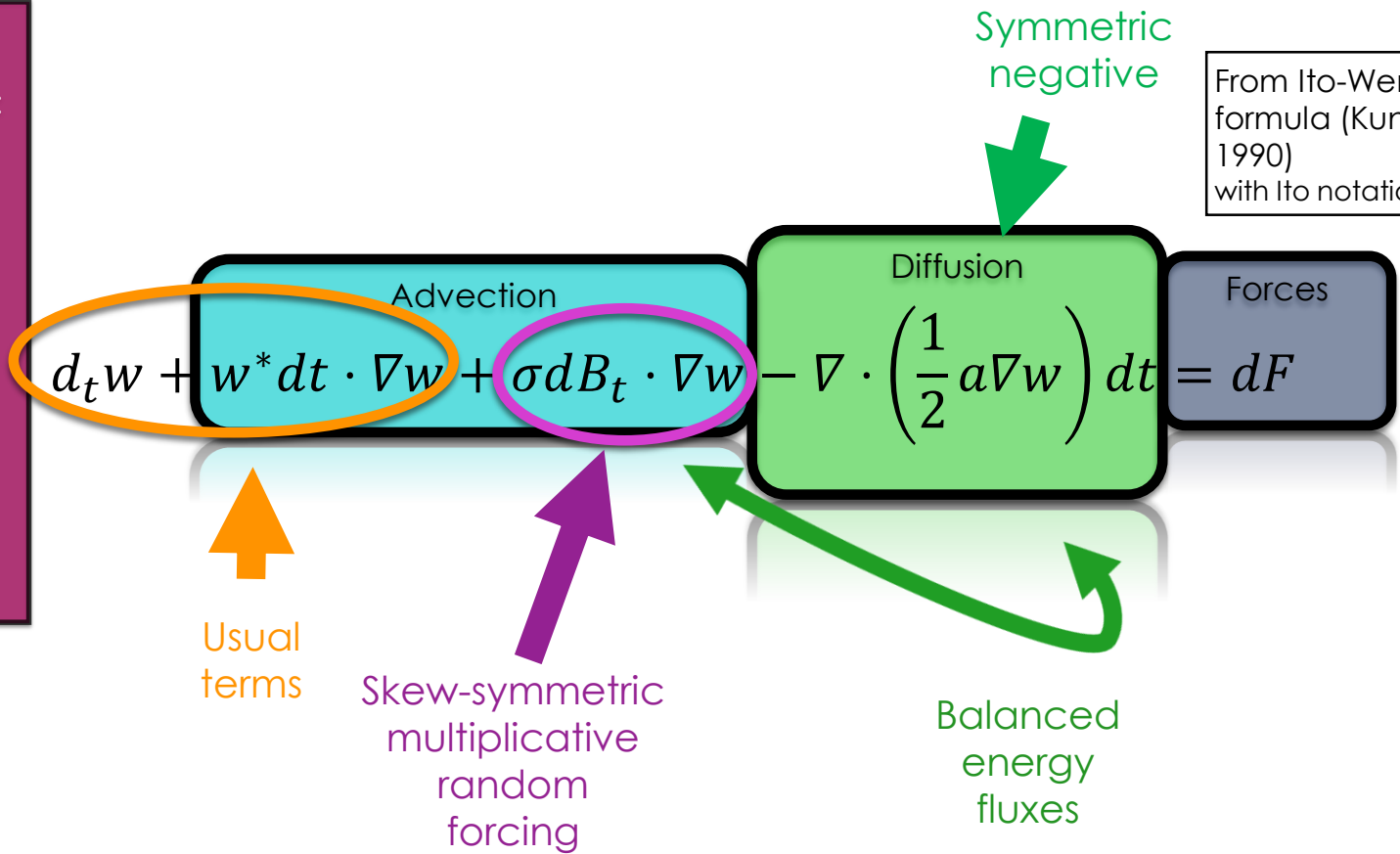
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Usual terms



Skew-symmetric multiplicative random forcing



Symmetric negative



Balanced energy fluxes

From Ito-Wentzell formula (Kunita 1990) with Ito notations

# Reduced Lum (Red LUM)

## POD-Galerkin gives SDEs for resolved modes

$$v = w + v'$$

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 $w(x, t) = \sum_{i=0}^n b_i(t) \phi_i(x)$

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Full order :  $M \sim 10^7$   
 Reduced order :  $n \sim 10$

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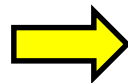
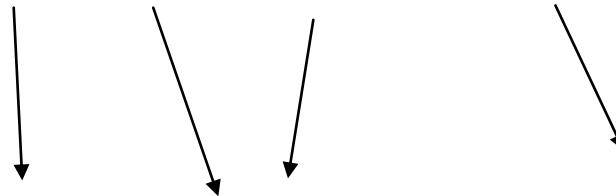
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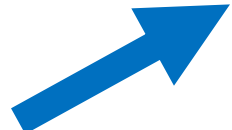
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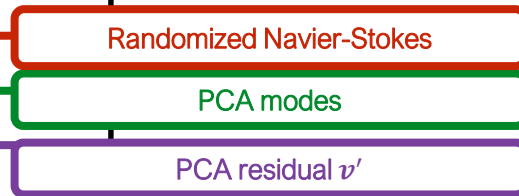
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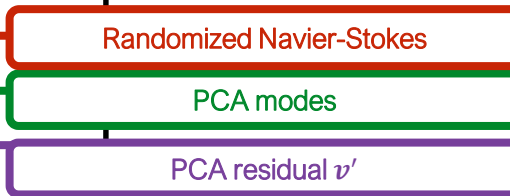
$$\Rightarrow db(t) = H(b(t)) dt + K(\sigma dB_t) b(t)$$

2<sup>nd</sup> order polynomial

Multiplicative skew-symmetric noise

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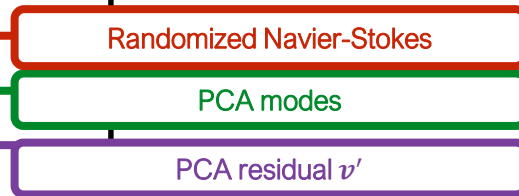
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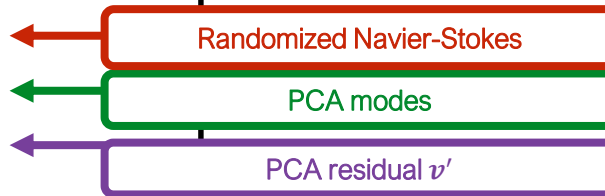
2<sup>nd</sup> order polynomial

Multiplicative skew-symmetric noise

**Covariance to estimate**

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$(n+1) \times (n+1)$        $(n+1) \times 1$   
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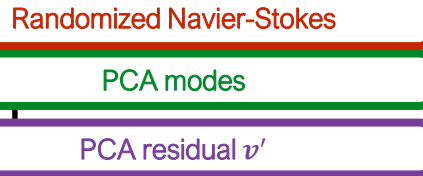
Multiplicative skew-symmetric noise

**Covariance to estimate**

$$\mathbb{E} (K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t)) / dt \approx \Delta t K_{jq} \left[ \frac{b_p}{b_p^2} \frac{\Delta b_i}{\Delta t} v' \right]$$

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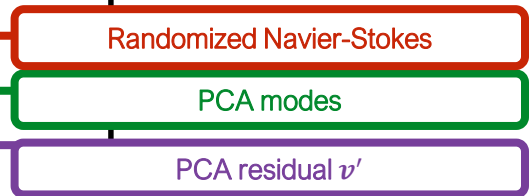
Multiplicative skew-symmetric noise

**Covariance to estimate**

$$\mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t K_{jq} \left[ \begin{matrix} \frac{b_p}{b_p^2} & \frac{\Delta b_i}{\Delta t} \\ & v' \end{matrix} \right]$$

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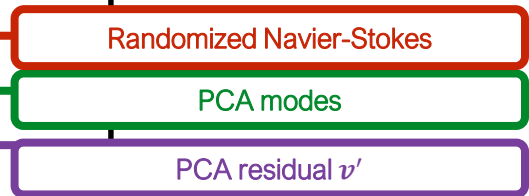
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  $K(\sigma dB_t)$     
  $b(t)$

2<sup>nd</sup> order polynomial

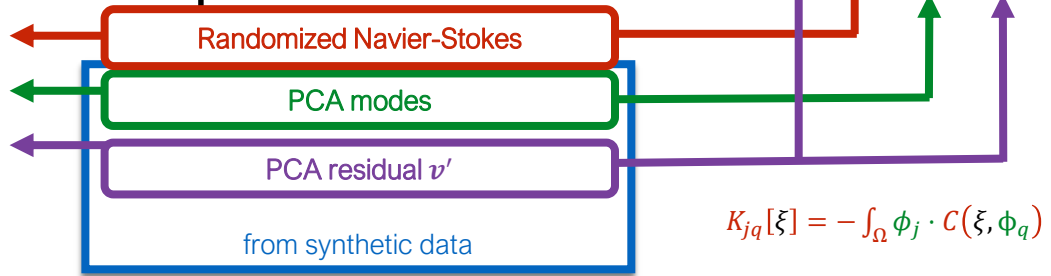
Multiplicative skew-symmetric noise

**Covariance to estimate**

$$\mathbb{E} (K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t)) / dt \approx \Delta t K_{jq} \left[ \frac{b_p}{b_p^2} \frac{\Delta b_i}{\Delta t} v' \right]$$

Coefficients given by :

- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t \overline{v' (v')^T}$



$$\bar{f} = \frac{1}{T} \int_0^T f$$

# Reduced Lum (Red LUM) POD-Galerkin gives SDEs for resolved modes

$v = w + v'$

Resolved fluid velocity:  
 $w(x, t) = \sum_{i=0}^n b_i(t) \phi_i(x)$

Unresolved fluid velocity:  
 $v' = \frac{\sigma dB_t}{dt}$  (Gaussian, white wrt t)

Variance tensor:  
 $a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

Full order :  $M \sim 10^7$   
 Reduced order :  $n \sim 10$

$$\int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w)) = dF) dx$$

$$db(t) = H(b(t)) dt + K(\sigma dB_t) b(t)$$

$H(b(t))$     
  $K(\sigma dB_t)$     
  $b(t)$

**New estimator**

- Consistency proven ( $\Delta t \rightarrow 0$ )
- Numerically efficient
- Physically-based

→ Robustness in extrapolation

2<sup>nd</sup> order polynomial

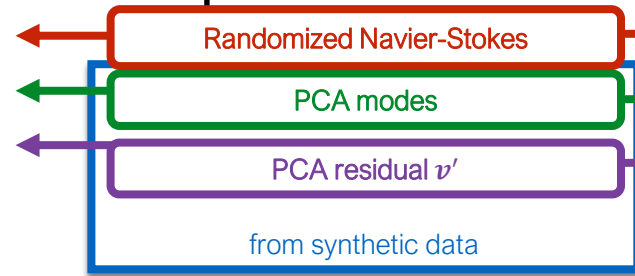
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Coefficients given by :

- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t v' (v')^T$



$$K_{jq}[\xi] = - \int_{\Omega} \phi_j \cdot C(\xi, \phi_q)$$

$$\bar{f} = \frac{1}{T} \int_0^T f$$

# Reduced Lum (Red LUM) Multiplicative noise covariance

$$v = w + v'$$

Resolved fluid velocity:  
 $w(x, t) = \sum_{i=0}^n b_i(t) \phi_i(x)$

Unresolved fluid velocity:  
 $v' = \frac{\sigma dB_t}{dt}$  (Gaussian, white wrt  $t$ )

Full order ( $\sim$  nb spatial grid points):  $M \sim 10^7$

Reduced order :  $n \sim 10$

Number of time steps :  $N \sim 10^4$

$$db(t) = H(b(t)) dt + \underbrace{K(\sigma dB_t)}_{(n+1) \times (n+1)} b(t) \quad \text{with} \quad K_{jq}[\xi] = -\int_{\Omega} \phi_j \cdot C(\xi, \phi_q)$$

► **Curse of dimensionality**

- Since  $\sigma dB_t$  is white in time,  
 $\Sigma_{jq,ip} = \mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t \overline{K_{jq}(v') K_{ip}(v')}$
- $K$  is a matrix of integro-differential operators  $\rightarrow$  cannot be evaluated on  $v'(x, t)$  at every time  $t$
- Covariance of  $\sigma dB_t \approx \Delta t^2 \overline{(v'(x, t))(v'(y, t))^T} : M \times M \sim 10^{13}$  coefficients  $\rightarrow$  intractable

**New estimator**

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- Numerically efficient
- Physically-based
- $\rightarrow$  Robustness in extrapolation

Randomized Navier-Stokes

PCA modes

PCA residual  $v'$

from synthetic data

$$\bar{f} = \frac{1}{T} \int_0^T f$$



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$$v = w + v'$$

Resolved fluid velocity:  
 $w(x, t) = \sum_{i=0}^n b_i(t) \phi_i(x)$

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Full order (~ nb spatial grid points):  $M \sim 10^7$

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 requires only  $O(n^2 M)$  correlation estimations and  $O(n^2)$  evaluations of  $K$

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$M \times 1$   
 $(n+1) \times (n+1)$

## New estimator

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### ► Consistency of our estimator (convergence in probability for $\Delta t \rightarrow 0$ , using stochastic calculus and continuity of $K$ )

$$\Delta t K_{jq} \left[ \frac{b_p}{b_p^2} \frac{\Delta b_i}{\Delta t} v' \right] = \Delta t b_p \frac{\Delta b_i}{\Delta t} K_{jq}[v'] \approx \frac{1}{T} \int_0^T b_p d < b_i, K_{jq}(\sigma B) > = \frac{1}{T} \int_0^T b_p \Sigma_{i=0}^n b_r d < K_{ir}(\sigma B), K_{jq}(\sigma B) > = \Sigma_{i=0}^n \Sigma_{jq,ir} \overline{b_p b_r} = \Sigma_{jq,ip} \overline{b_p^2} \quad (\text{orthogonality from PCA})$$

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► **Optimal time subsampling at  $\Delta t$  needed** to meet the white assumption

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► **Optimal time subsampling at  $\Delta t$  needed** to meet the white assumption

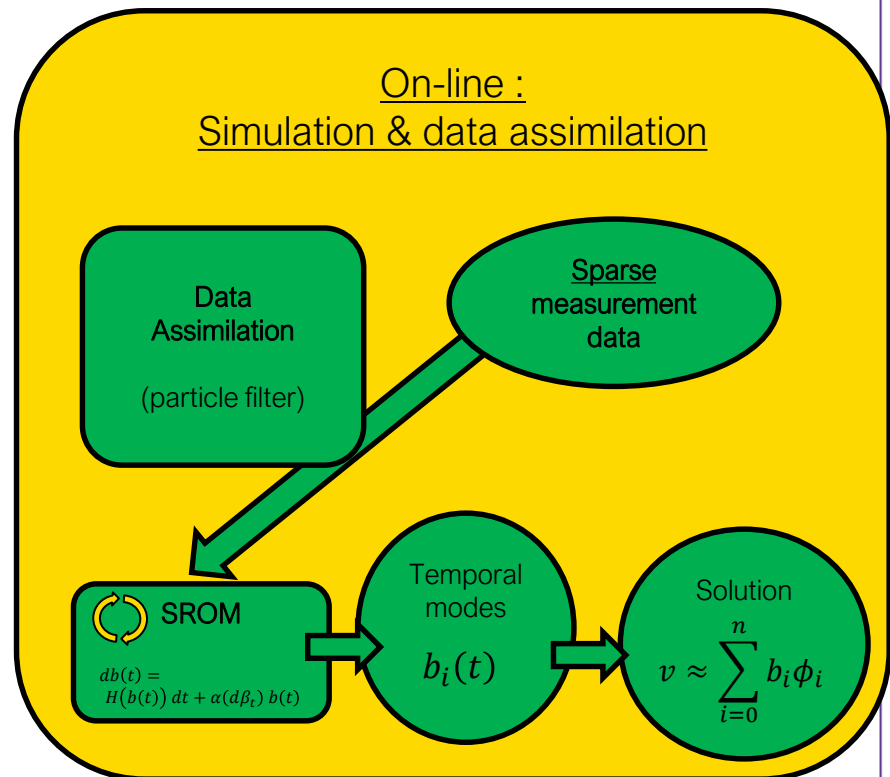
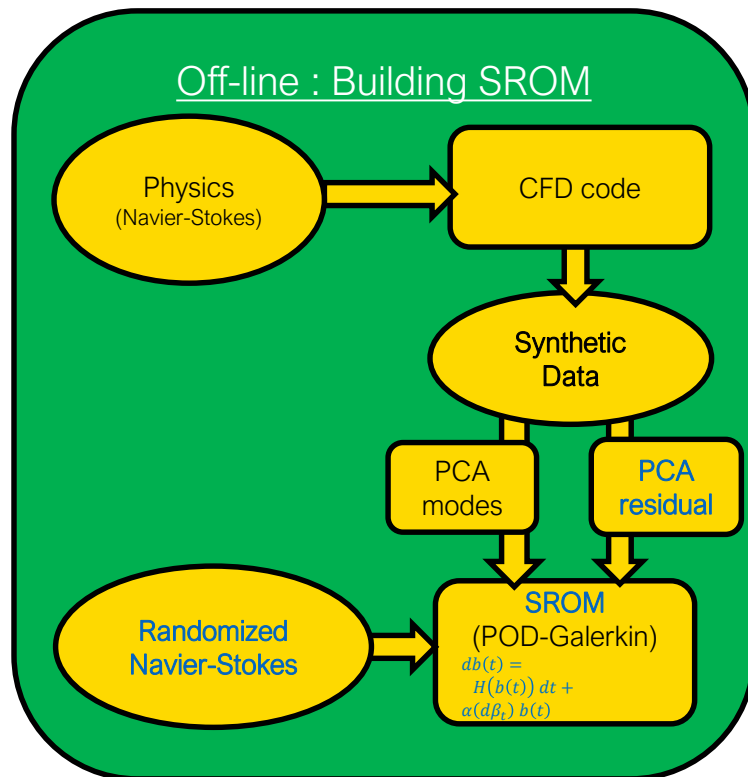
► **Additional reduction for efficient sampling :**

diagonalization of  $\Sigma \rightarrow K(\sigma dB_t) \approx \alpha(d\beta_t)$  with a  $n$ -dimensional (instead of  $(n+1)^2$ -dimensional) Brownian motion  $\beta$

$$\bar{f} = \frac{1}{T} \int_0^T f$$

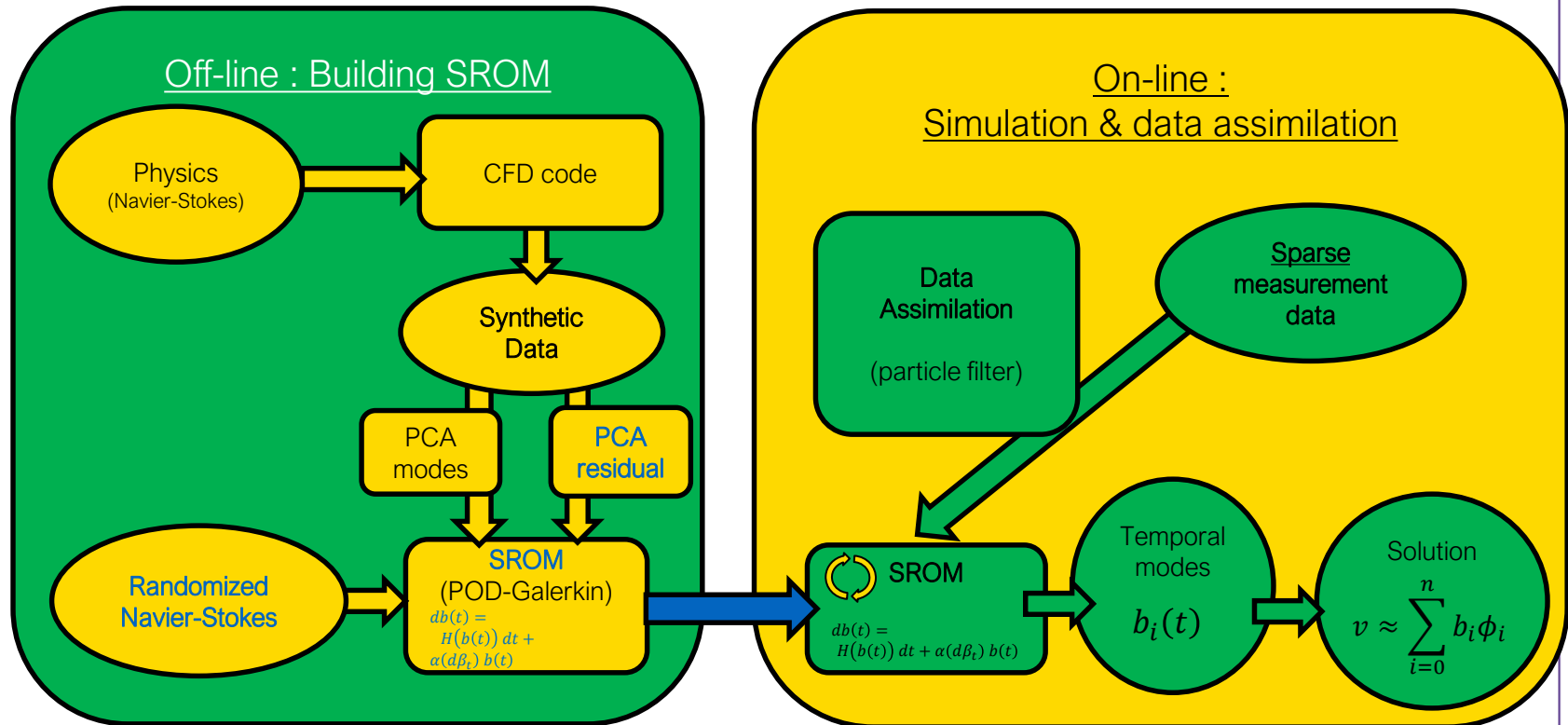
# Summary

## Stochastic ROM + Data assimilation



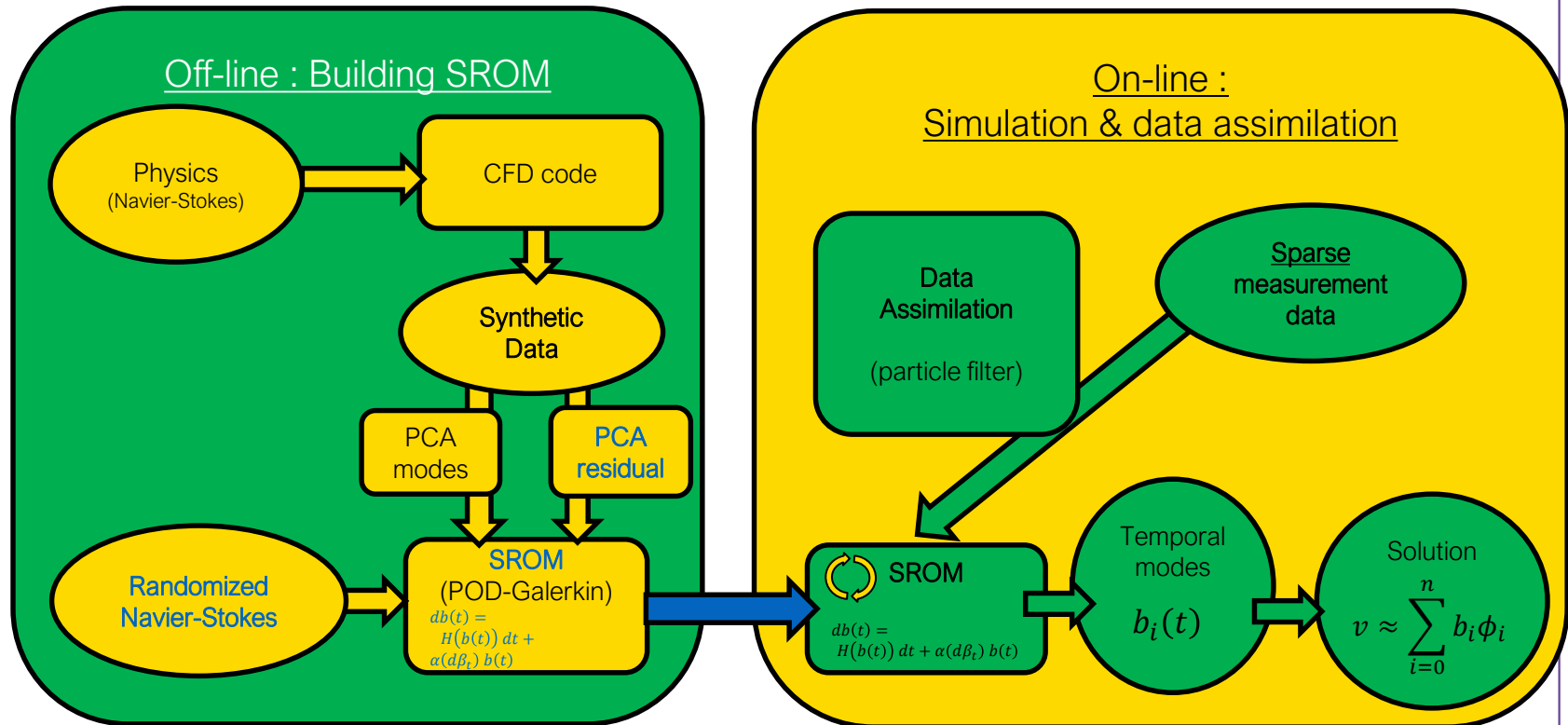
# Summary

## Stochastic ROM + Data assimilation



# Summary

## Stochastic ROM + Data assimilation



# Part III

## NUMERICAL RESULTS



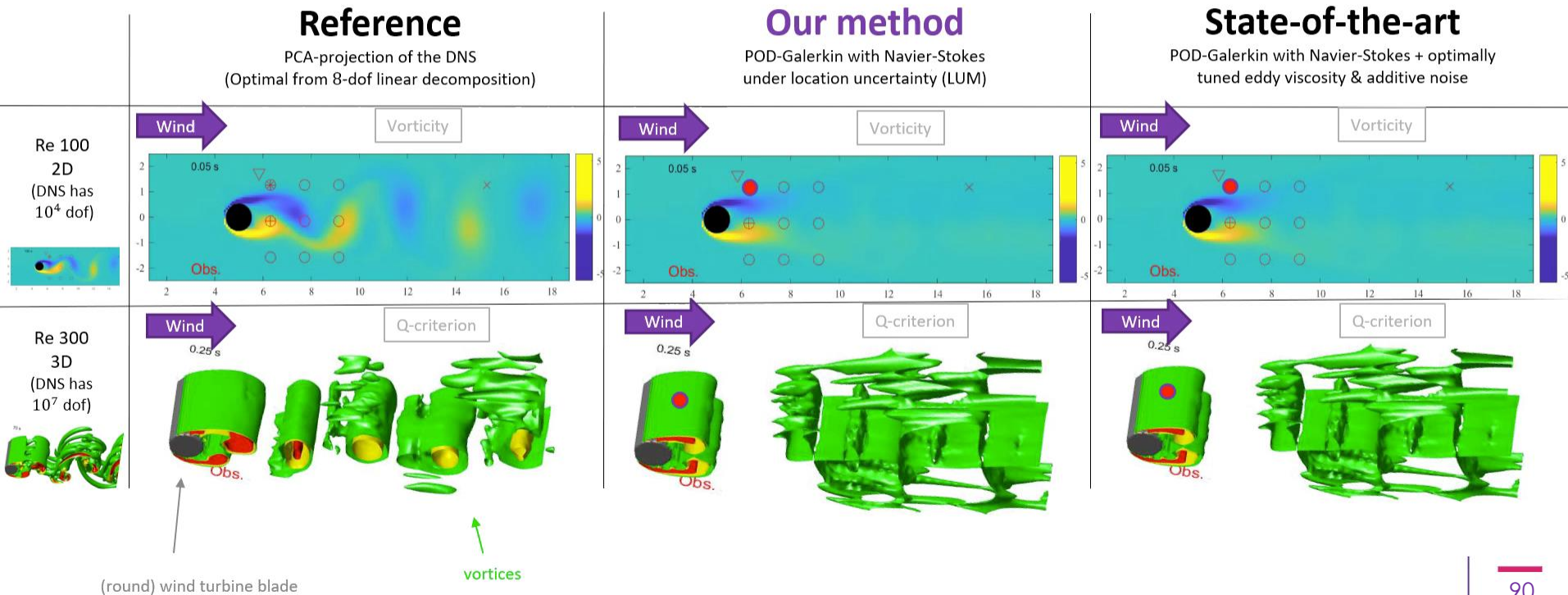
# Data assimilation (posterior)

## On-line estimation of the solution

# Data assimilation (posterior) On-line estimation of the solution

## FIRST RED LUM NUMERICAL RESULTS FAST OBSERVER OF THE WHOLE 3D FLOW

From  $10^7$  to 8 degrees of freedom  
● Single measurement point (blurred & noisy velocity)



# CONCLUSION

[valentin.resseguier@scalian.com](mailto:valentin.resseguier@scalian.com)

# Conclusion

- ▶ Reduced order model (ROM) : for very fast and robust CFD ( $10^7 \rightarrow 8$  degrees of freedom.)
  - ▶ Combine data & physics (built off-line)
  - ▶ Closure problem handled by LUM
  - ▶ Efficient estimator for the multiplicative noise
  - ▶ Efficient generation of prior / Model error quantification
- ▶ Data assimilation (Bayesian inverse problem) :  
to correct the fast simulation on-line by incomplete/noisy measurements
- ▶ First results
  - ▶ Optimal unsteady flow estimation/prediction in the whole spatial domain (large-scale structures)
  - ▶ Robust far outside the training set

## NEXT STEPS

- ▶ Real measurements
- ▶ Better stochastic closure
- ▶ Parametric ROM (unknown inflow)
- ▶ More turbulent flows

# THANK YOU

- ▶ Resseguier, V., Mémin, E., & Chapron, B. (2017). Geophysical flows under location uncertainty, Part I Random transport and general models. *Geophysical & Astrophysical Fluid Dynamics*, 111(3), 149-176.
- ▶ Resseguier, V., Picard, A. M., Mémin, E., & Chapron, B. (2021). Quantifying truncation-related uncertainties in unsteady fluid dynamics reduced order models. *SIAM/ASA Journal on Uncertainty Quantification*, 9(3), 1152-1183.
- ▶ Resseguier, V., Ladvig, M., & Heitz, D. (2022). Real-time estimation and prediction of unsteady flows using reduced-order models coupled with few measurements. *Journal of Computational Physics*, 471, 111631.

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Bonus slides

## Data assimilation

Example : the Particle Filter (PF) generates an ensemble  $\sim p(v|y)$

- ▶ Initialization  
 $v_{t=0}^{(j)} \sim \mathcal{N}(0, \Sigma)$

- ▶ Loop over time  $t$

*Importance sampling*

- ▶  $v_t^{(j)} = M(v_{t-1}^{(j)}, \text{noise}(t-1))$       Forecast ("Prior" or "background")

- ▶ If an observation  $y_t$  is available at the current time  $t$

- ▶  $W_j(t) \propto p(y_t | v_t^{(j)})$       Likelihood evaluation, up to a constant

- ▶  $\mathbf{W}_j(t) = \frac{W_j(t)}{\sum_{k=1}^{N_p} W_k(t)}$       Normalization

*Resampling*

- ▶ Each new  $v_t^{(j)}$  is replaced by one of the old particles  $v_t^{(1)}, \dots, v_t^{(N_p)}$  with probability  $\mathbf{W}_1(t), \dots, \mathbf{W}_{N_p}(t)$ , respectively.

- ▶ Final posterior distribution

$$p(v_t | y_{t_1}, \dots, y_{t_K}) \approx \sum_{k=1}^{N_p} \frac{1}{N_p} \delta(v_t - v_t^{(k)})$$

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# UNCERTAINTY QUANTIFICATION (Prior)

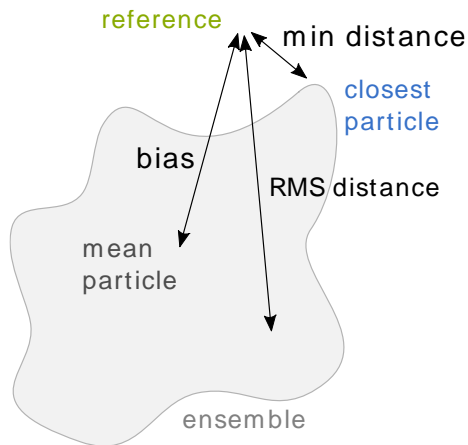
## Known initial conditions $b(t = 0)$

From  $10^7$  to 8 degrees of freedom  
 No data assimilation  
 Known initial conditions  $b(t = 0)$

$$db(t) = H(b(t)) dt + \alpha(d\beta_t) b(t)$$

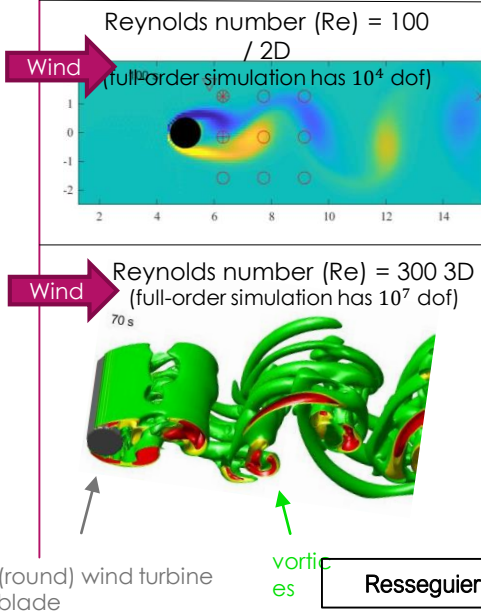
### Metrics choice

- $b_i(t)$  VS reference
- Error metrics



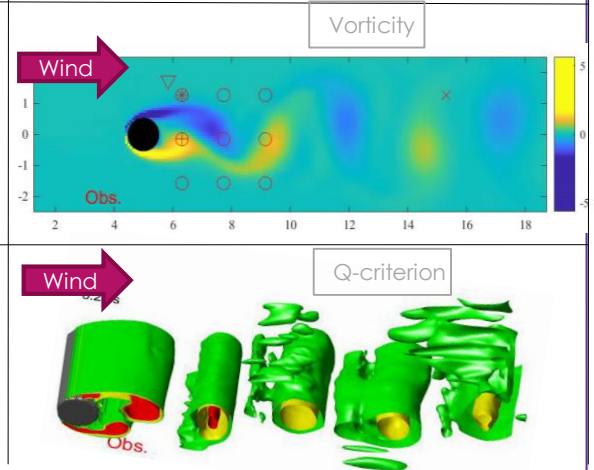
### Test cases

#### Full-order reference



#### Reduced-order reference

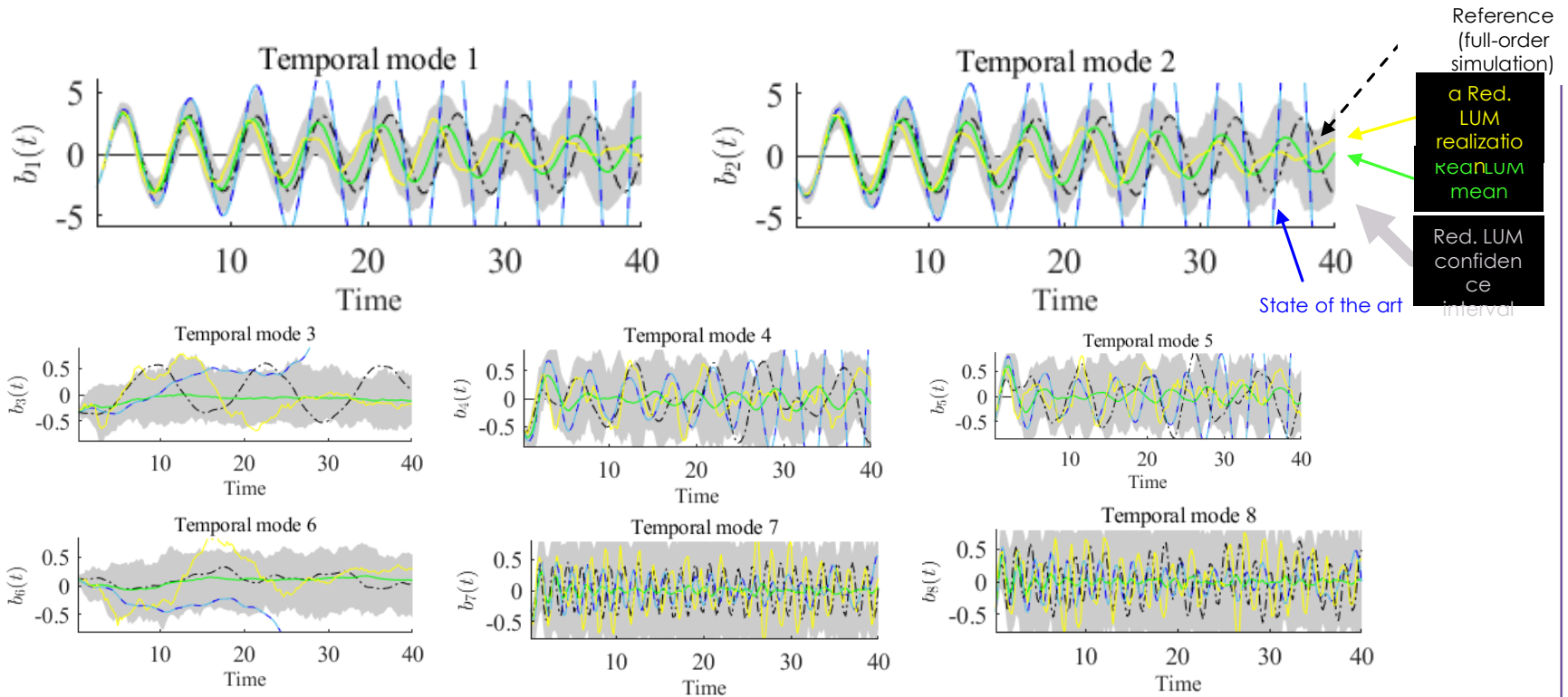
PCA-projection of the full-order simulation  
 (Optimal from 8-dof linear decomposition)



# UNCERTAINTY QUANTIFICATION (Prior)

## $b_i(t)$ VS reference

From  $10^7$  to 8 degrees of freedom  
 No data assimilation  
 Known initial conditions  $b(t = 0)$



# UNCERTAINTY QUANTIFICATION (Prior)

## Error on the reduced solution $w$

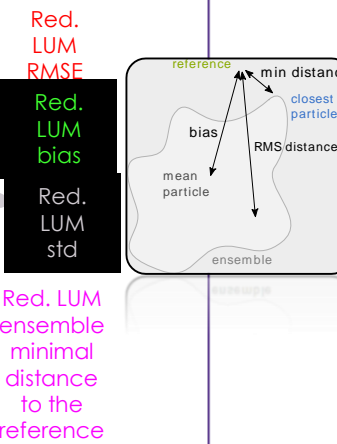
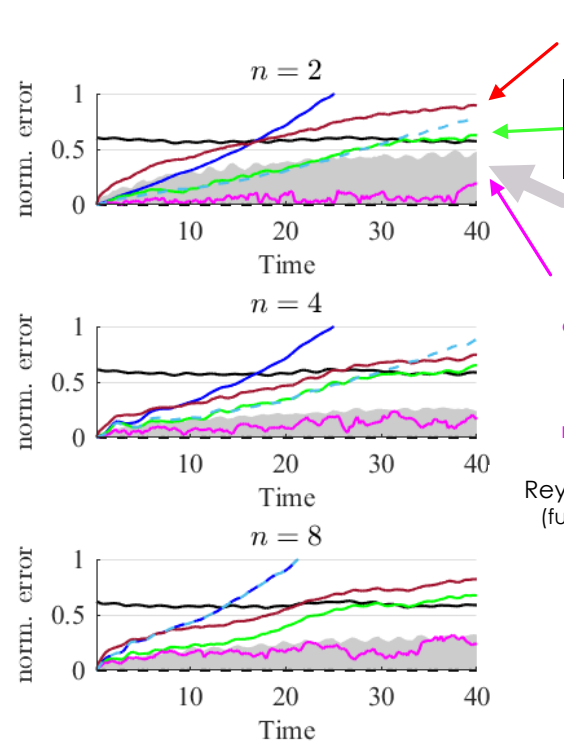
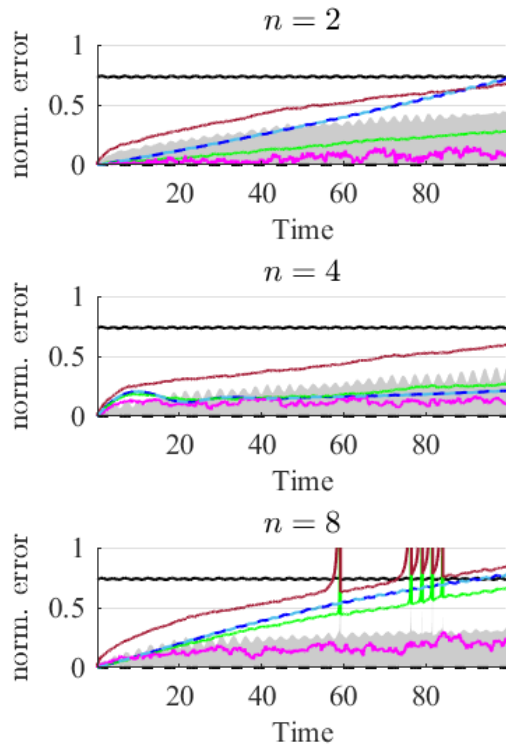
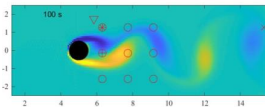
From  $10^7$  to 8 degrees of freedom  
 No data assimilation  
 Known initial conditions  $b(t = 0)$

$v = w + v'$

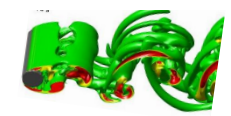
Resolved fluid velocity:  
 $w = \sum_{i=0}^n b_i \phi_i$

Unresolved fluid velocity:  
 $v'$

Reynolds number (Re) = 100 / 2D  
 (full-order simulation has  $10^4$  dof)



Reynolds number (Re) = 300 3D  
 (full-order simulation has  $10^7$  dof)



# UNCERTAINTY QUANTIFICATION (Prior)

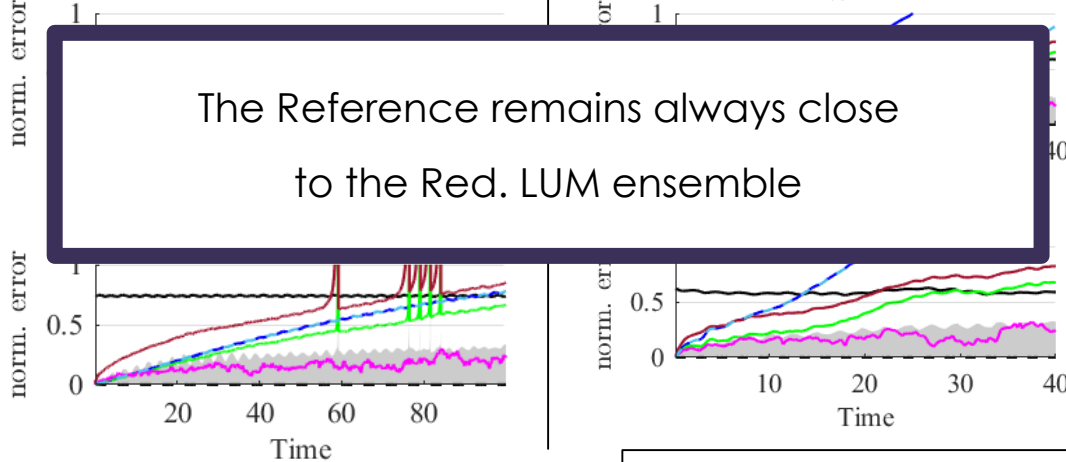
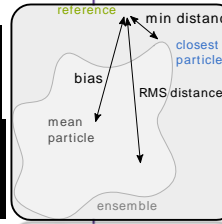
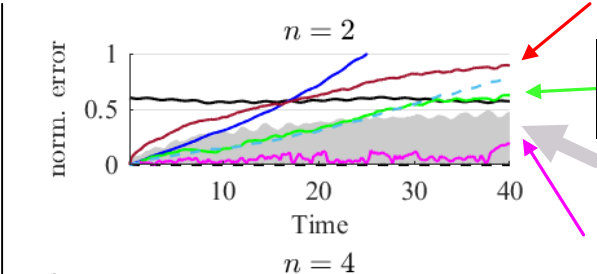
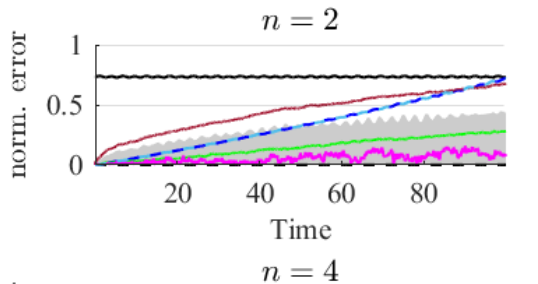
## Error on the reduced solution $w$

From  $10^7$  to 8 degrees of freedom  
 No data assimilation  
 Known initial conditions  $b(t = 0)$

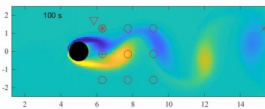
$v = w + v'$

Resolved fluid velocity:  
 $w = \sum_{i=0}^n b_i \phi_i$

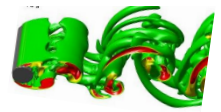
Unresolved fluid velocity:  
 $v'$



Reynolds number (Re) = 100 / 2D  
 (full-order simulation has  $10^4$  dof)



Reynolds number (Re) = 300 3D  
 (full-order simulation has  $10^7$  dof)



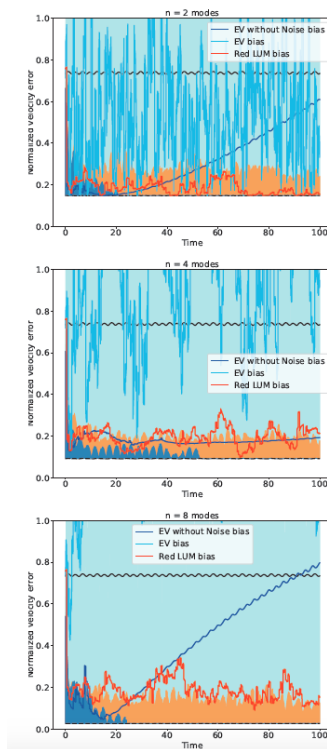
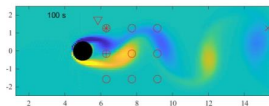
# Data assimilation (posterior) Error on the solution estimation

$$v = w + v'$$

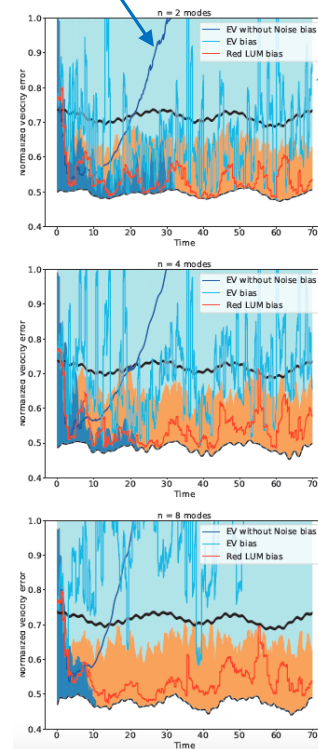
Resolved fluid velocity:  
 $w = \sum_{i=0}^n b_i \phi_i$

Unresolved fluid velocity:  
 $v'$

Reynolds number (Re) = 100  
/ 2D  
(full-order simulation has  $10^4$  dof)



State of the art



State of the art

Red.  
LUM  
std  
neu.  
LUM  
bias

Reynolds number (Re) = 300 3D  
(full-order simulation has  $10^7$  dof)

