



# Markovian & non-Markovian closure for wave-turbulence interaction

Bertrand Chapron, Valentin Resseguier,  
Erwan Hascoet



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1. ERC STUOD
2. Wave-turbulence interactions
3. Non-Markovian closure  
with current multi-scale in space & in time
4. Markovian closure  
with current multi-scale in space & **white** in time  
- Simple cases and analytic solutions -

1.





Imperial College London

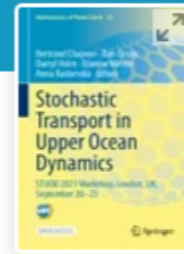


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# Stochastic Transport in Upper Ocean Dynamics

STUOD 2021 Workshop, London, UK, September 20–23

Home > Conference proceedings

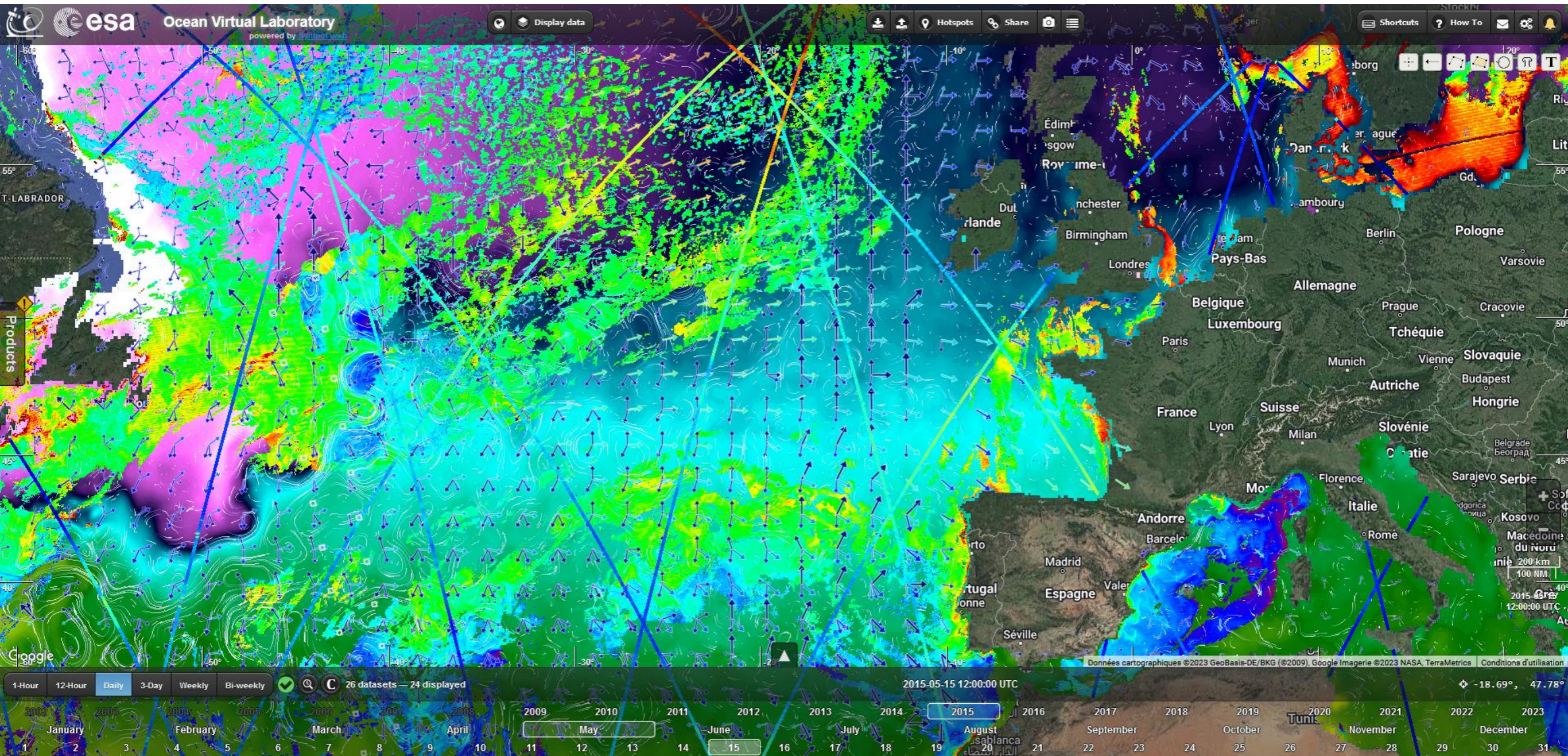
**Editors:** [Bertrand Chapron](#), [Dan Crisan](#), [Darryl Holm](#), [Etienne Mémin](#), [Anna Radomska](#)



Some STUOD objectives:

- Identify and quantify small-scale displacement fluctuations
- Include these fluctuations in a mathematical framework for the reduction of the equations of motion, solved at lower resolution

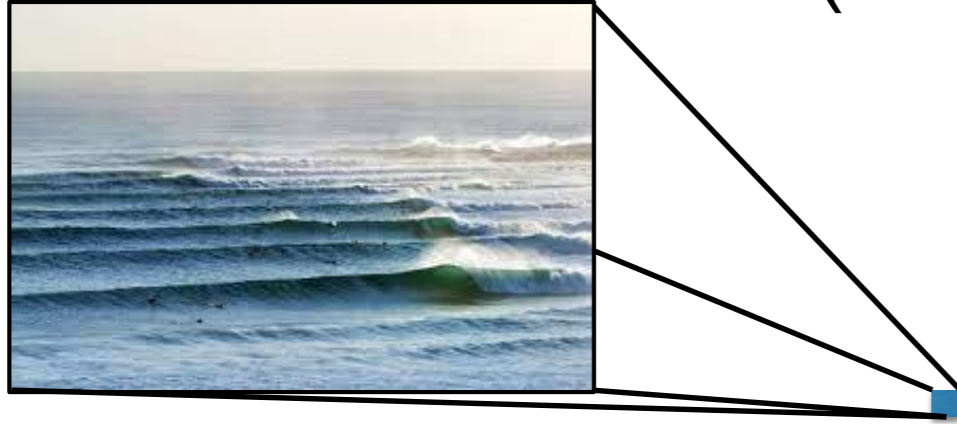
# PLENTY OF DATA & SPDEs LOOKING FOR STATISTICIANS !



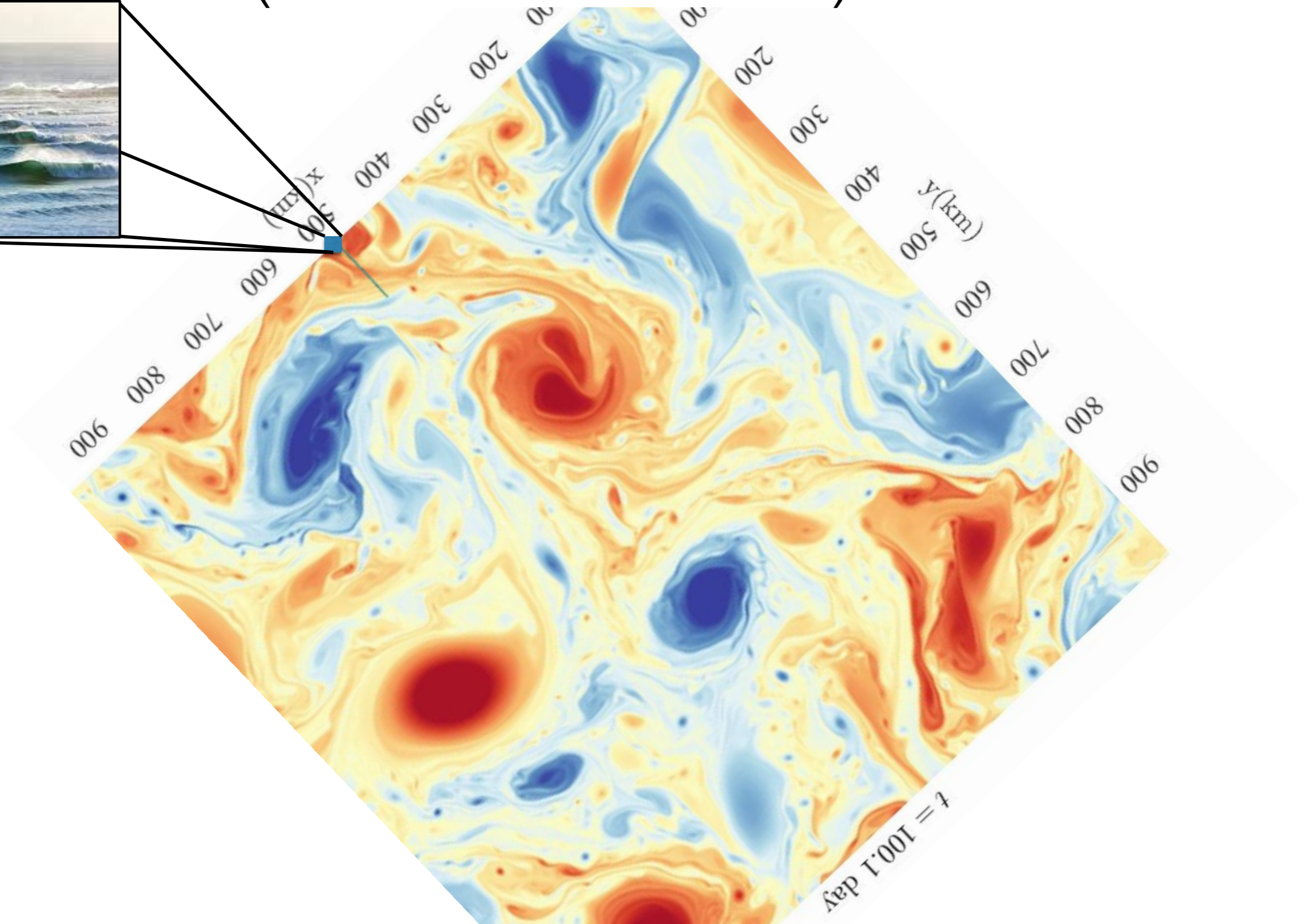
2.

# Wave-turbulence interactions

# Numerical example of wave (swell) traveling inside turbulence (oceanic currents)



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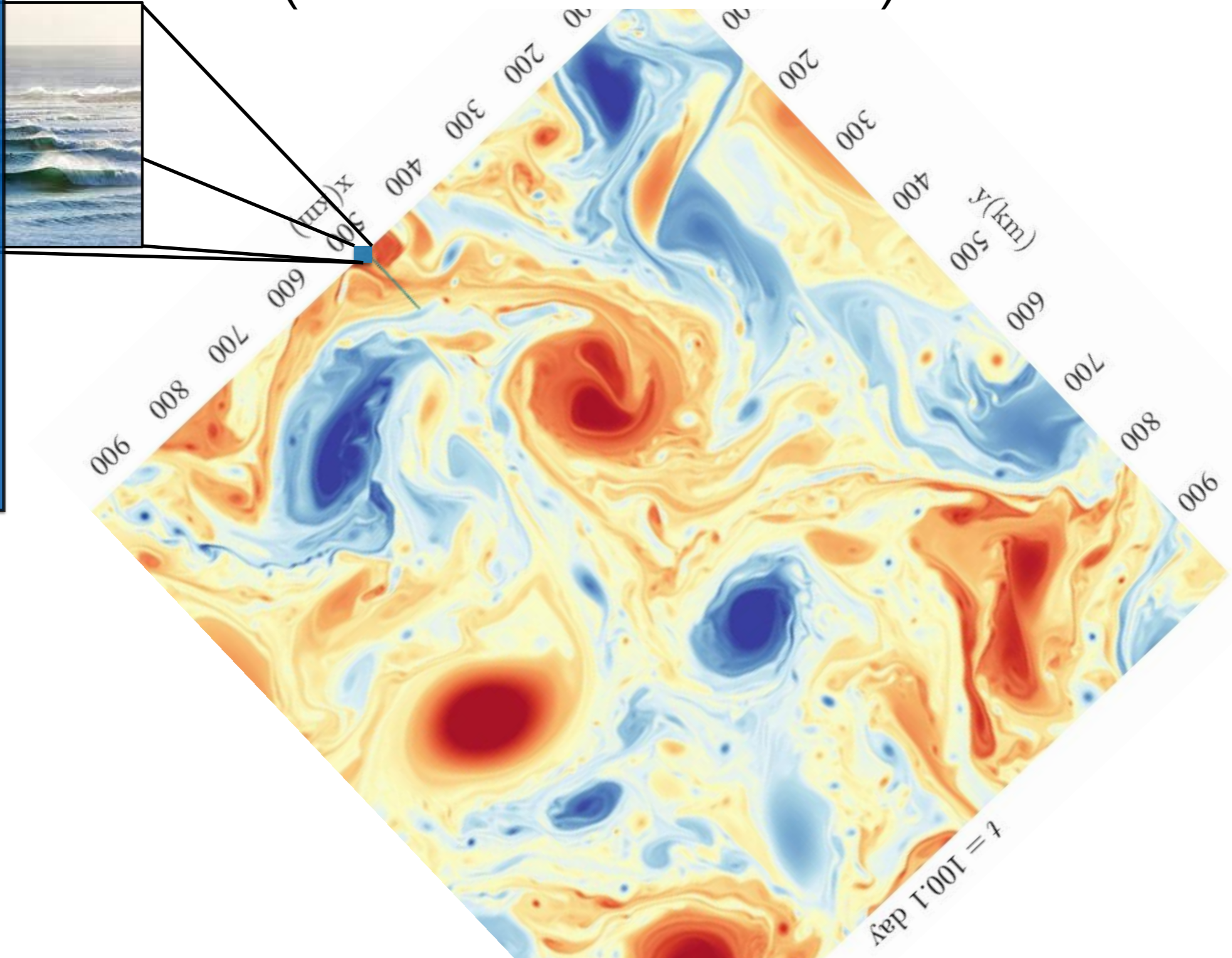




# Numerical example of wave (swell) traveling inside turbulence (oceanic currents)

Wave:

$$ae^{\frac{i}{\epsilon}\phi(\epsilon t, \epsilon x)}$$



Current  
velocity:  
 $v$

# Dispersion ratio

At the first order in steepness ( $=\|\nabla\phi\|a$ )

Wave:  
 $a e^{i\phi(\epsilon t, \epsilon x)}$

$k = \nabla\phi$

$$\frac{\partial\phi}{\partial t} + v \cdot \underbrace{\nabla\phi}_{=k} = \underbrace{\sqrt{g\|\nabla\phi\|}}_{=\omega_0(k)}$$

$$\frac{\partial\phi}{\partial t} = \omega(k) = -v \cdot k + \omega_0(k)$$

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Wave phase transport  
with a current velocity  $v$

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Wave phase transport  
with a current velocity  $v$

$$\frac{\partial\phi}{\partial t} + v \cdot \underbrace{\nabla\phi}_{=k}$$

Frequency without currents

$$\underbrace{\sqrt{g\|\nabla\phi\|}}_{=\omega_0(k)}$$

$$\frac{\partial\phi}{\partial t} = \omega(k) = -v \cdot k + \omega_0(k)$$

Total group velocity:

$$v + C_g^0$$

Wave:

$$ae^{\frac{i}{\epsilon}\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

# Method of characteristic (ray tracing)

$$\dot{X}_r = \nabla_k \omega = C_g^0 + v \quad \Bigg| \quad \text{Ray}$$

Refraction & contraction/dilatation

$$\dot{k} = \nabla_x \omega = -\nabla v^T k \quad \Bigg| \quad \text{Wave-vector}$$

Conservation of action

$$\dot{A} = -\nabla \cdot u_g A \quad \Bigg| \quad A = \frac{a^2}{\omega_0} \quad \Bigg| \quad \text{Amplitude}$$

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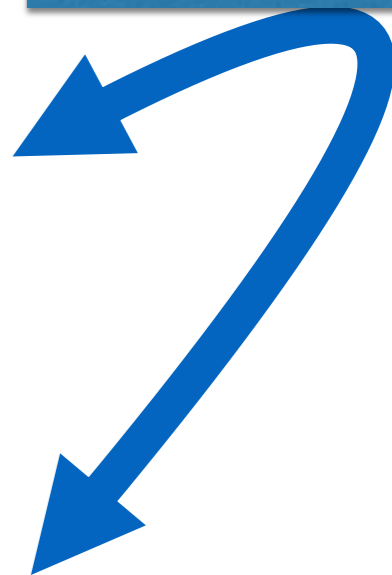
$$\dot{k} = \nabla_x \omega = -\nabla v^T k \quad | \quad \text{Wave-vector}$$

Fonction  
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Conservation of action

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Non-linear  
coupling





Total group velocity:

$$v + C_g^0$$

Wave:

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Group velocity without current

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# Method of characteristic (ray tracing)

Fonction of k

Main contribution

$$\dot{X}_r = \nabla_k \omega = C_g^0 + v \quad | \quad \text{Ray}$$

Refraction & contraction/dilatation

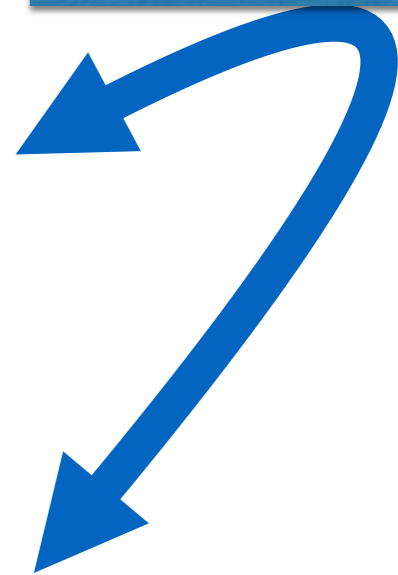
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Fonction of x

Conservation of action

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Non-linear coupling



Total group velocity:

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Wave:

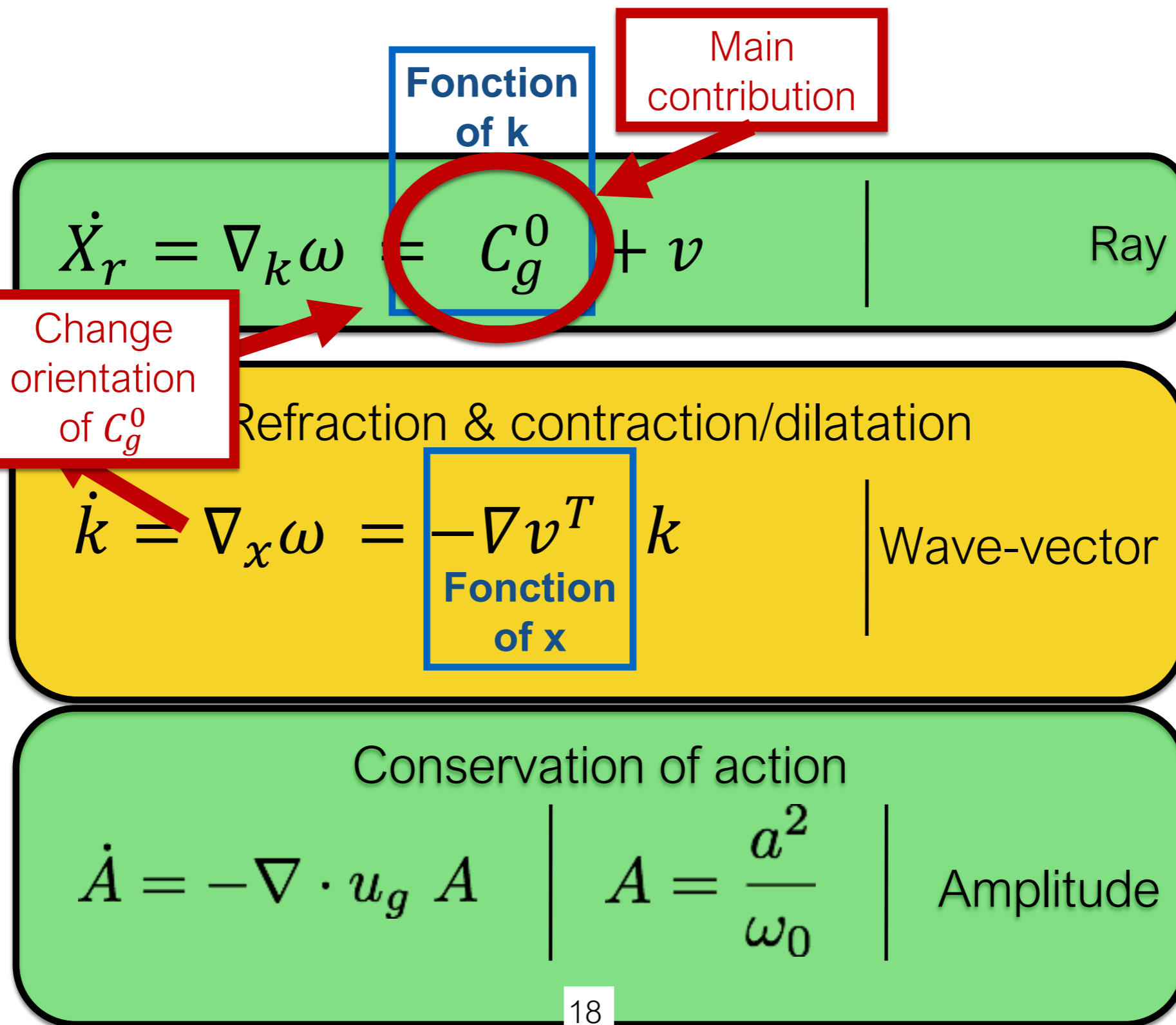
$$ae^{i\phi(\epsilon t, \epsilon x)}$$

Group velocity without current

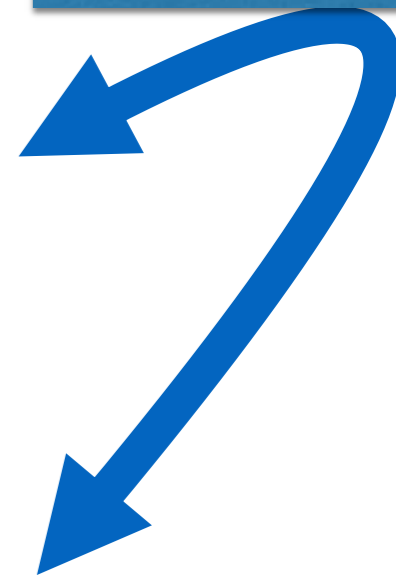
$$C_g^0 = \nabla_k \omega_0$$

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# Method of characteristic (ray tracing)



Non-linear coupling



Large scale  
group velocity:

$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v'$$

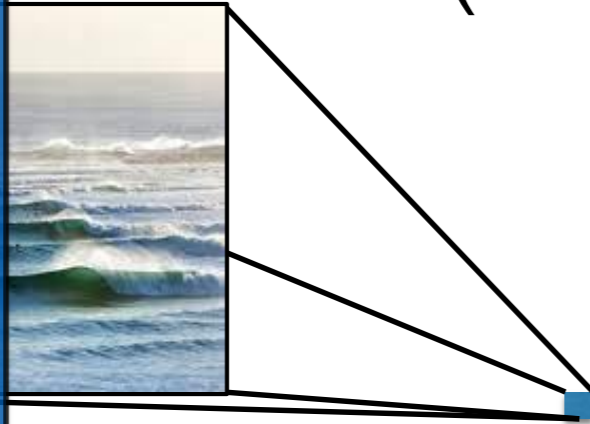
Wave:

$$ae^{\frac{i}{\epsilon}\phi(\epsilon t, \epsilon x)}$$

Group velocity  
without current

$$C_g^0 = \nabla_k \omega_0$$

# Numerical example of wave (swell) traveling inside **turbulence** (oceanic currents)



Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v'$$

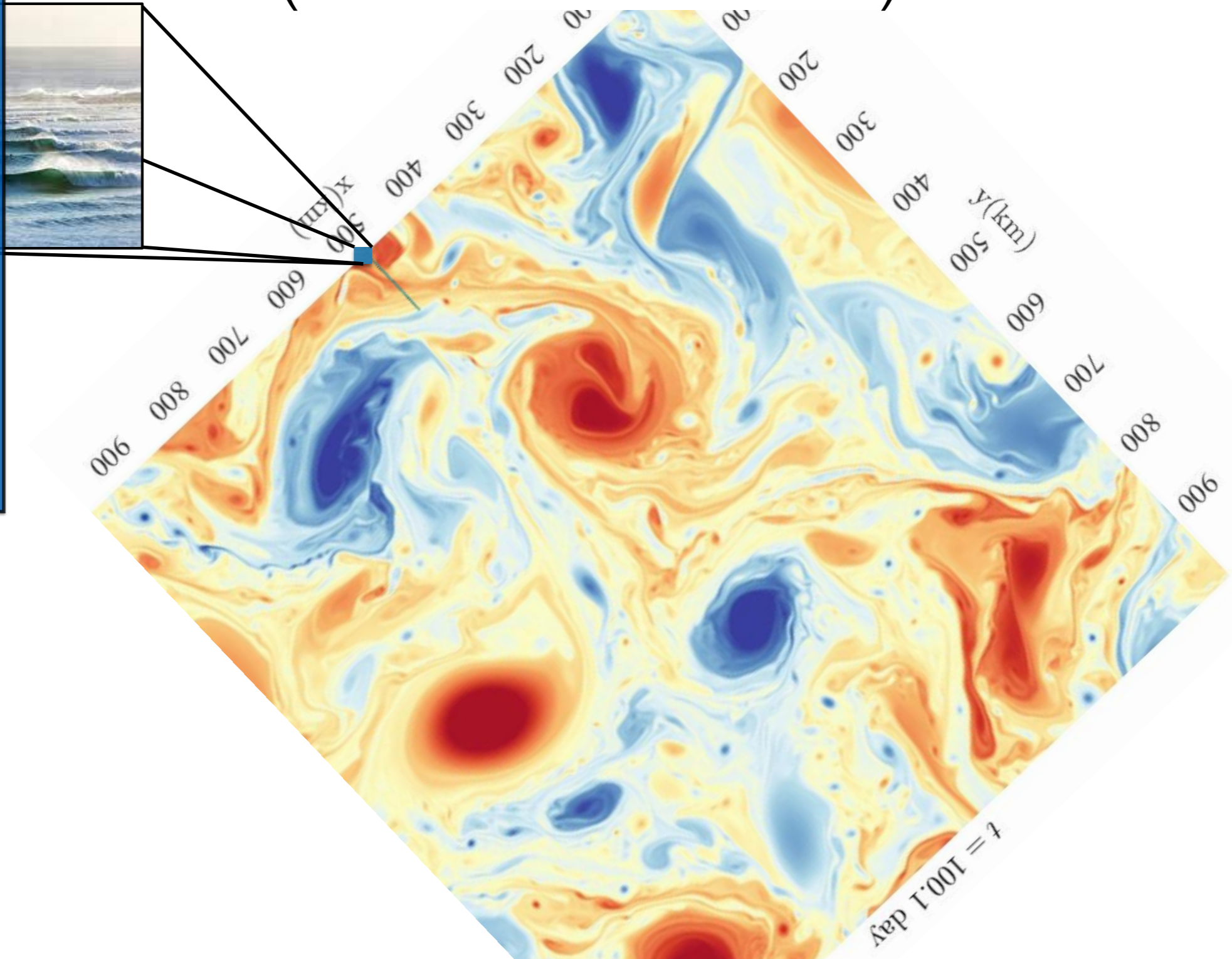
Wave:

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Group velocity without current

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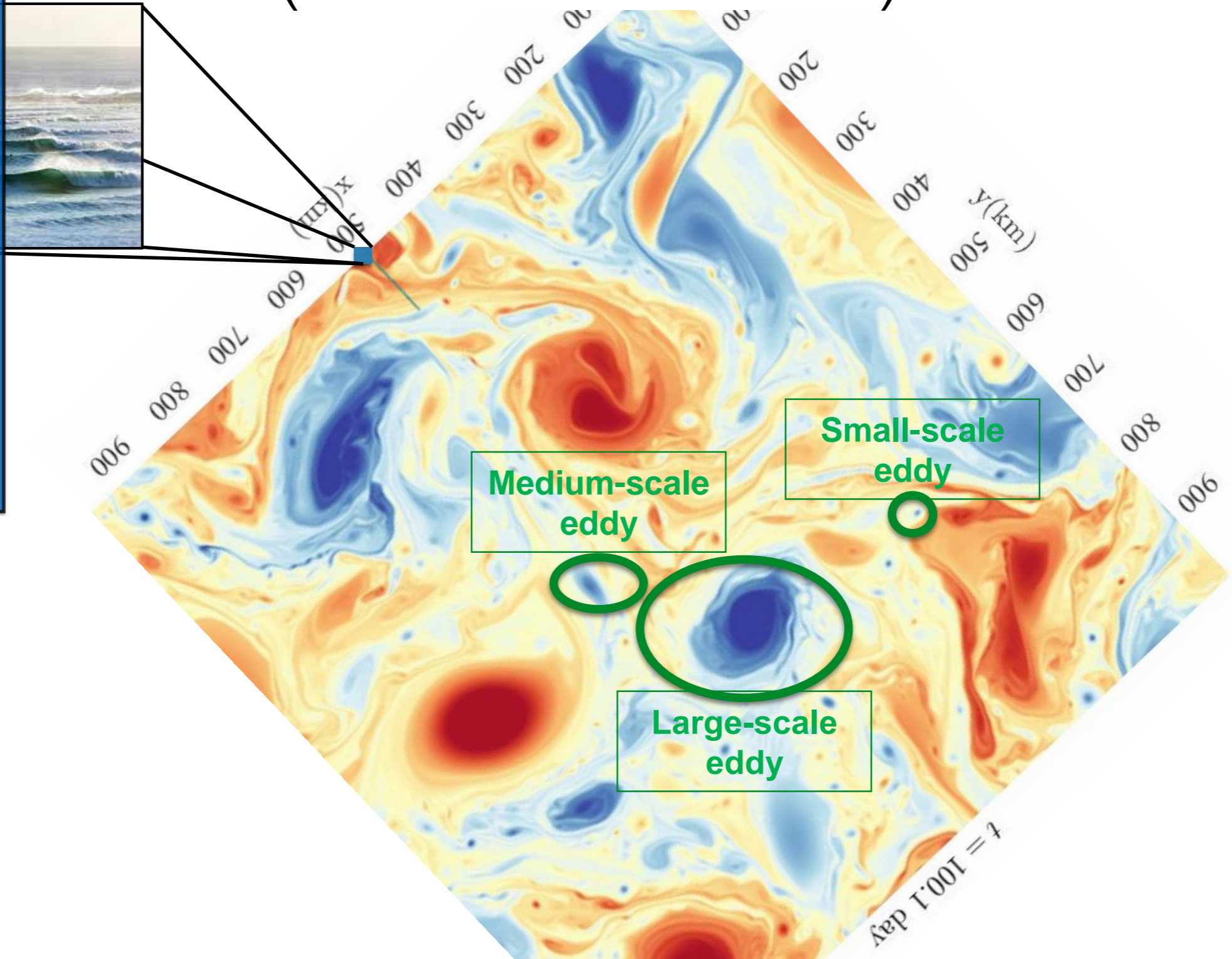
$$v'$$

Wave:

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Group velocity without current

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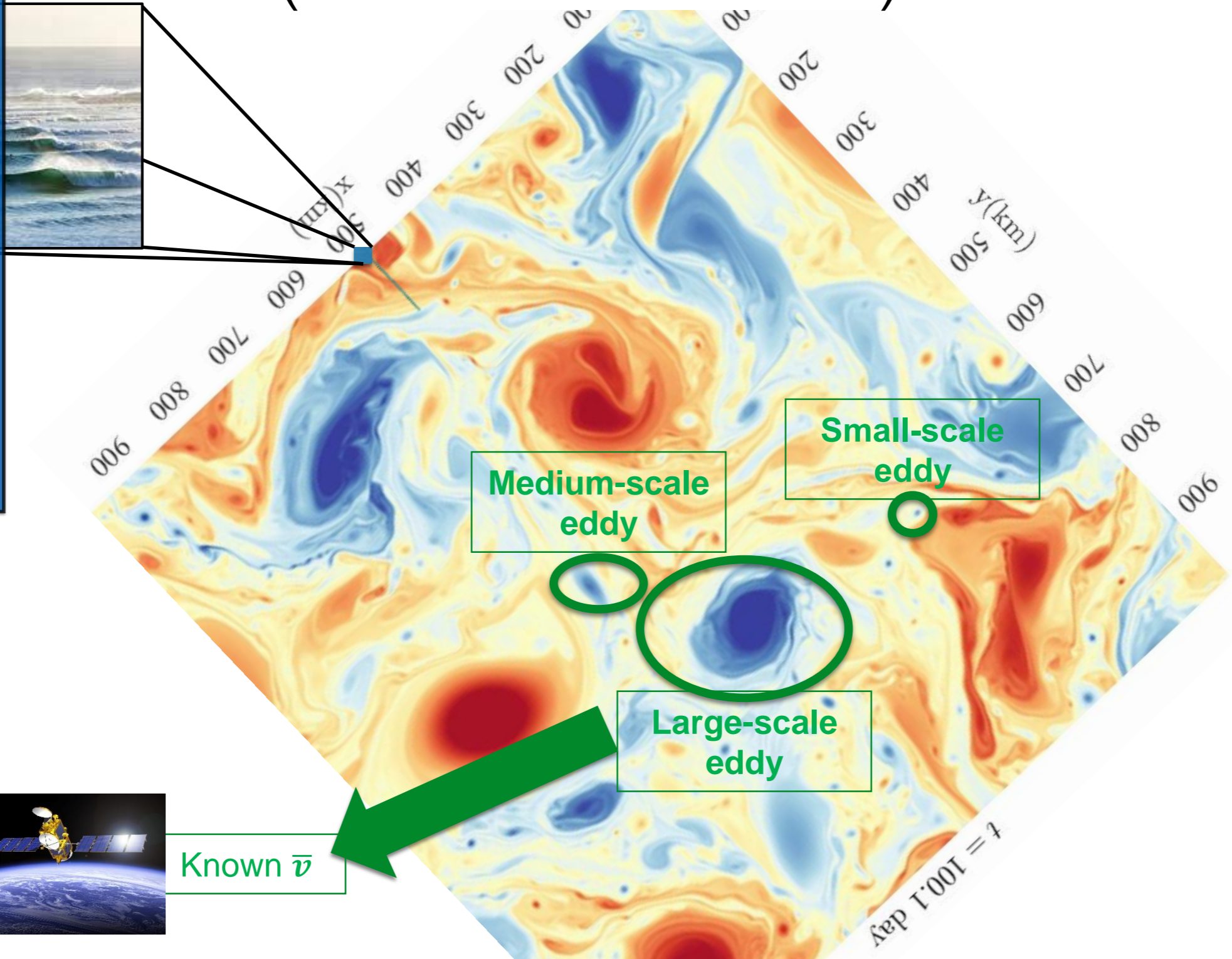
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Group velocity without current

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Known  $\bar{v}$

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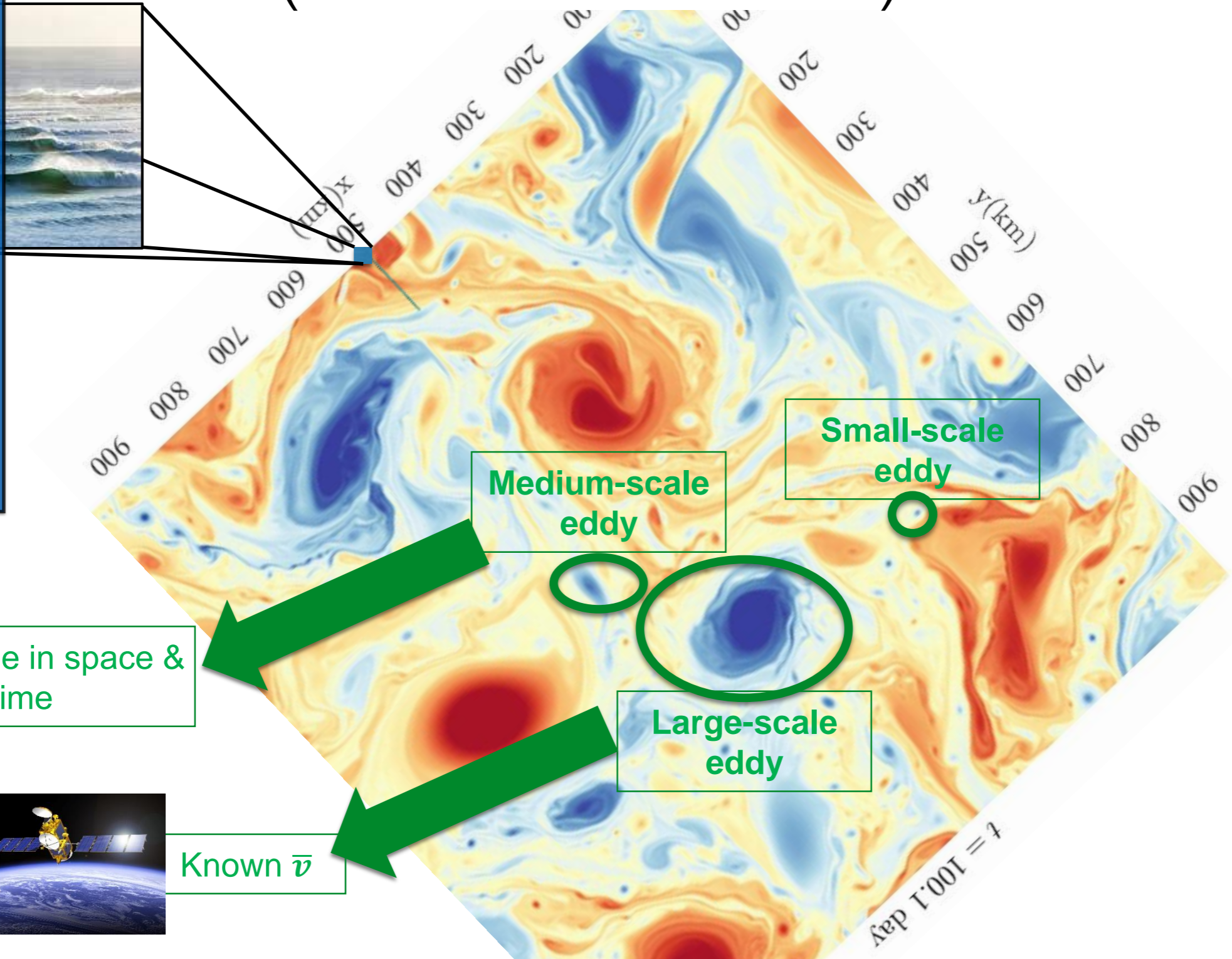
$$v'$$

Wave:

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Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$



3. Multi-scale in space & in time

Known  $\bar{v}$



# Numerical example of wave (swell) traveling inside turbulence (oceanic currents)

Large scale group velocity:

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Small scale group velocity:

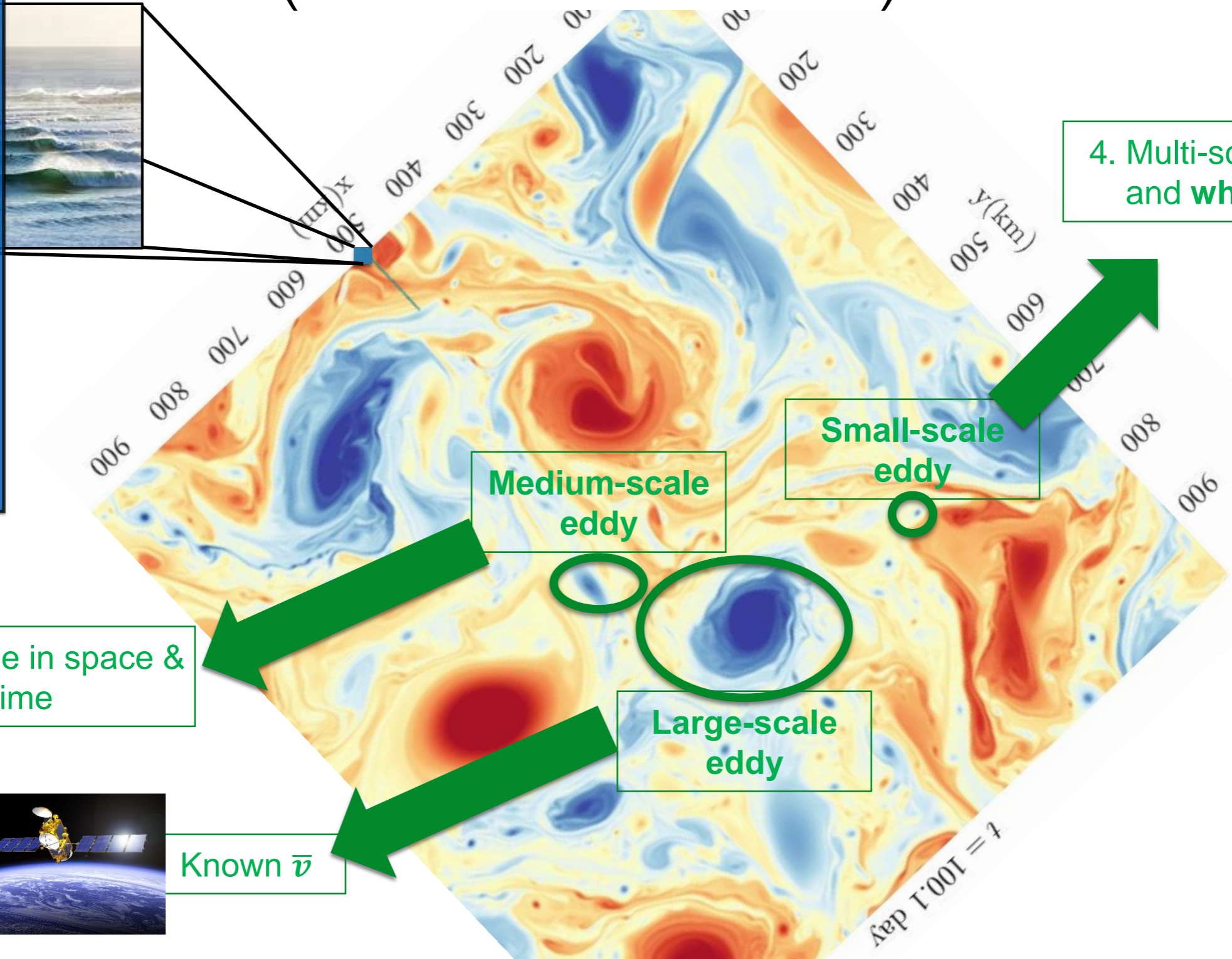
$$v'$$

Wave:

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Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$



4. Multi-scale in space and **white** in time

3. Multi-scale in space & in time

Known  $\bar{v}$





3.

Non-Markovian closure  
with current multi-scale in  
space & in time

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$v'$

Wave:

$$ae^{i\frac{z}{\epsilon}\phi}$$

# Time-correlated model for $v'$ :

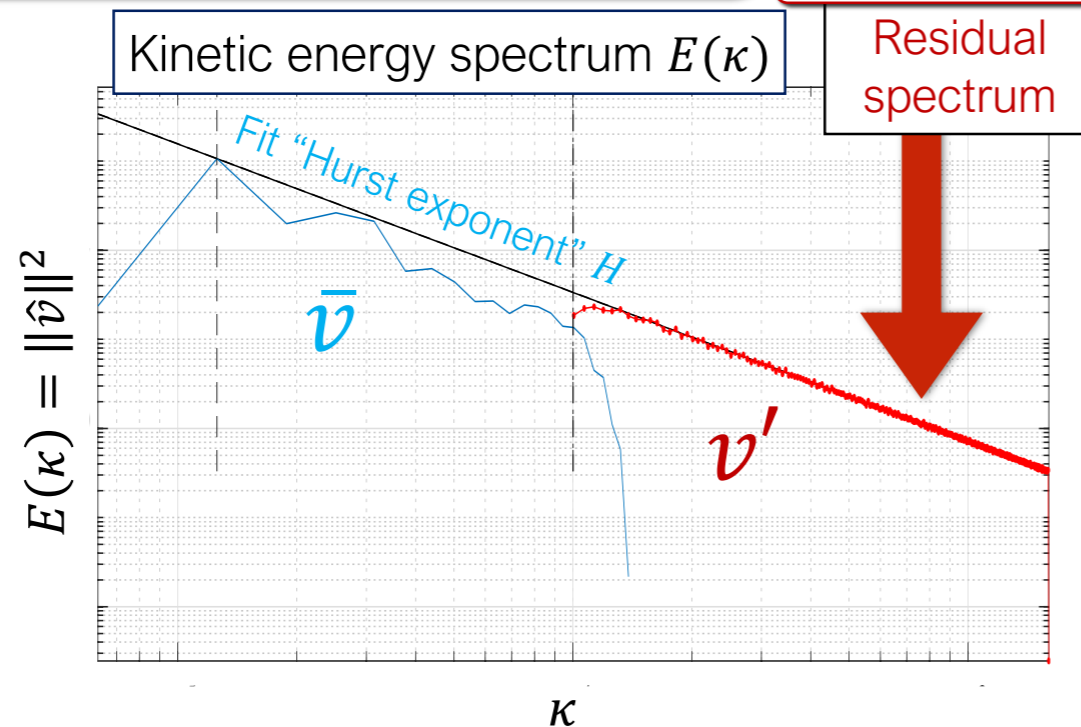
## Calibration

Physical scale symmetry

$$\text{On-line fit : } A \|k\|^{-H}$$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$



Kinetic energy spectrum  $E(k)$

Residual spectrum

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$v'$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

# Time-correlated model for $v'$ :

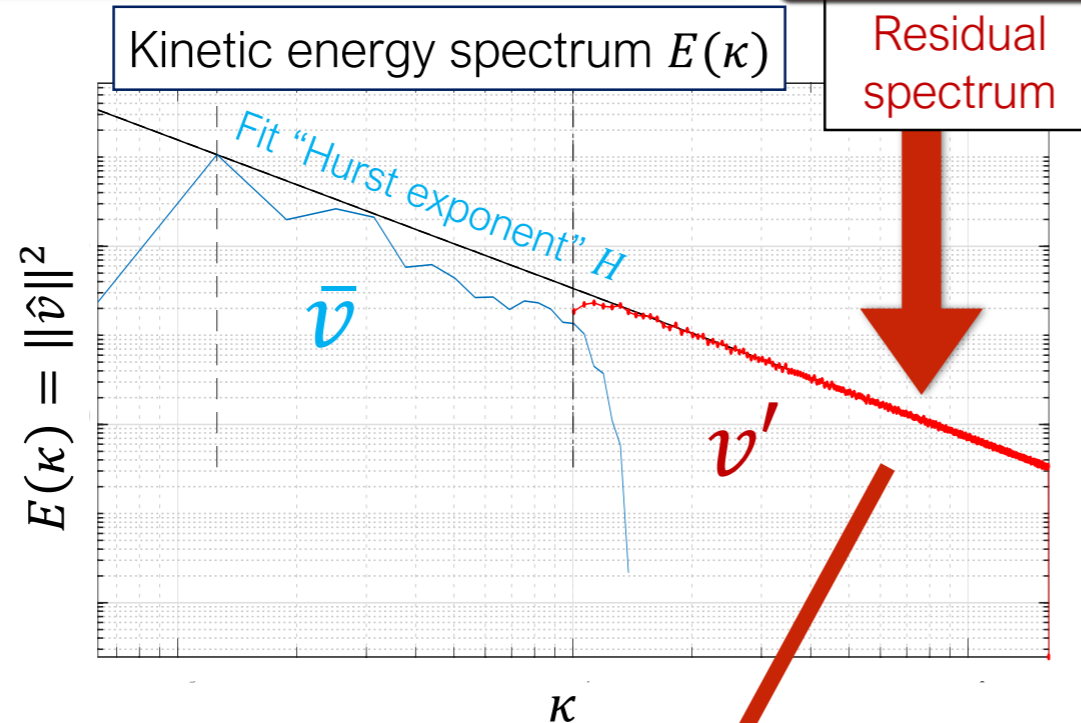
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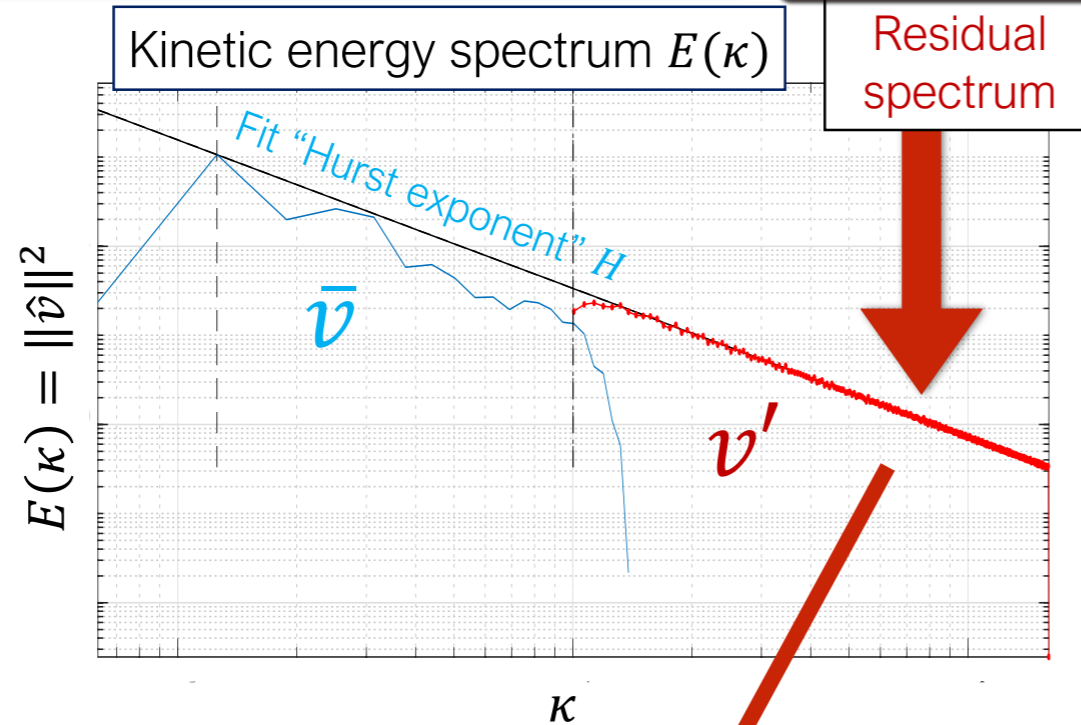
## Calibration

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Filter  $h$

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Small scale group velocity:

$v'$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

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Physical scale symmetry

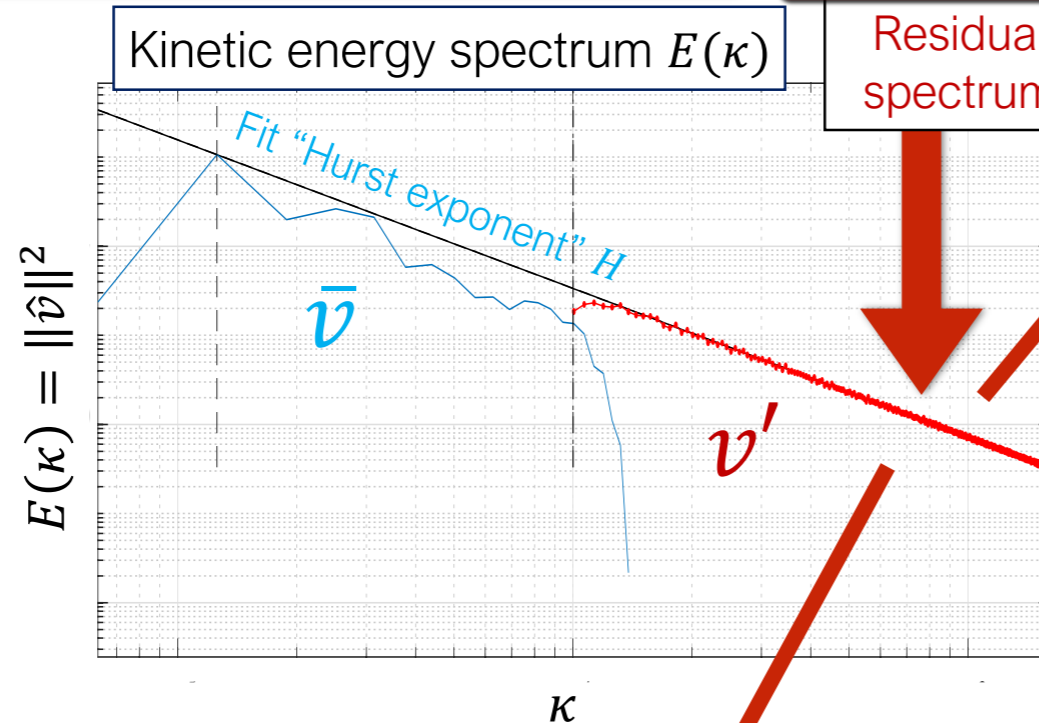
$$\text{On-line fit : } A \|k\|^{-H}$$

(Similar to fractional Brownian motion in space)

$$\hat{h}^2 = Ak^{-\alpha} - |\hat{v}|^2$$

Current correlation time

$$\tau(\kappa) = \tau_{cur}(\kappa) = \frac{1/\kappa}{v_k} = \frac{1/\kappa}{\sqrt{\kappa E(\kappa)}}$$



Filter  $h$

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

Large scale group velocity:

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Small scale group velocity:

$v'$

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

# Time-correlated model for $v'$ :

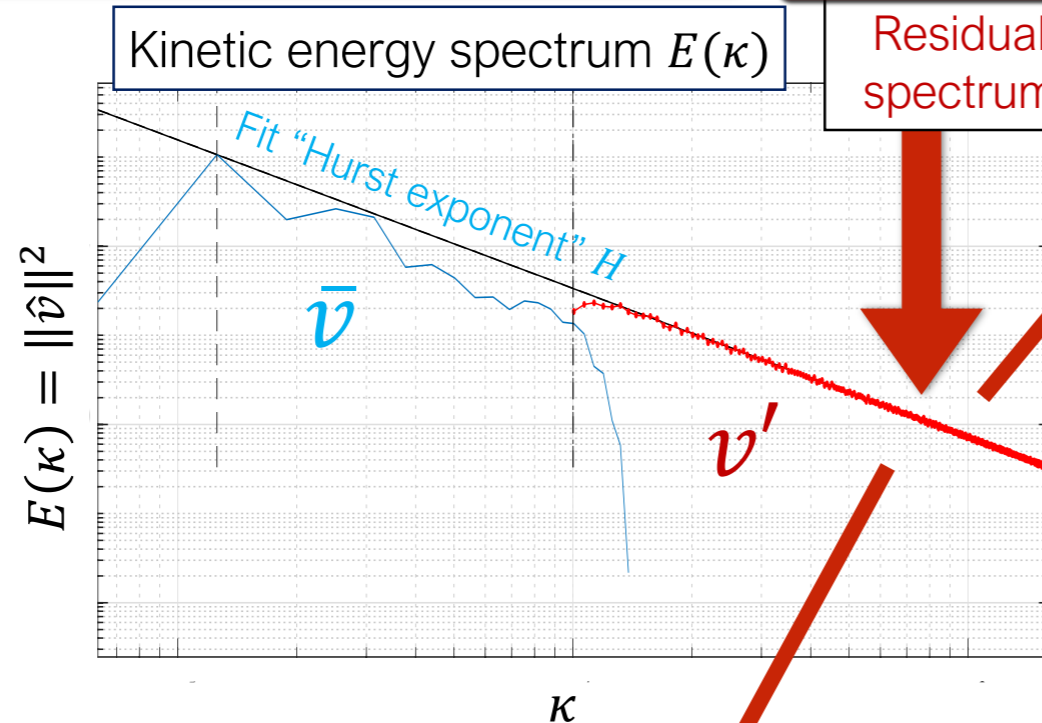
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Current correlation time

$$\tau(k) = \tau_{cur}(k) = \frac{1/k}{v_k} = \frac{1/k}{\sqrt{k E(k)}}$$

Multiscale noise  $B_t^c$

From  $B_t^c(x) = \iint \widehat{B}_t^c(k) e^{ik \cdot x} dx$ ,  
Ornstein-Uhlenbeck process in time,  
with correlation time  $\tau(k)$  at the spatial scale  $k$ :

$$d\widehat{B}_t^c(k) = -\frac{1}{\tau(k)} \widehat{B}_t^c(k) dt + \sqrt{\frac{2}{\tau(k)}} (dB_t)(k)$$

and  $dB_t/dt$  "spatiotemporal white noise"  
(cylindrical Wiener Process)

$$v' = (\text{filter}) * (\text{time-correl. noise})$$

# Time-correlated model for $v'$ :

## Method of characteristic

$$v = \bar{v} + v'$$

wave group amplitude

buoyancy

Deterministic  
reference:  
wave groups in  
512 x 512 flow

$$v = \bar{v}$$

wave group amplitude

buoyancy

Deterministic  
benchmark:  
wave groups in  
smoothed flow  $\bar{v}$  (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

wave group amplitude

buoyancy

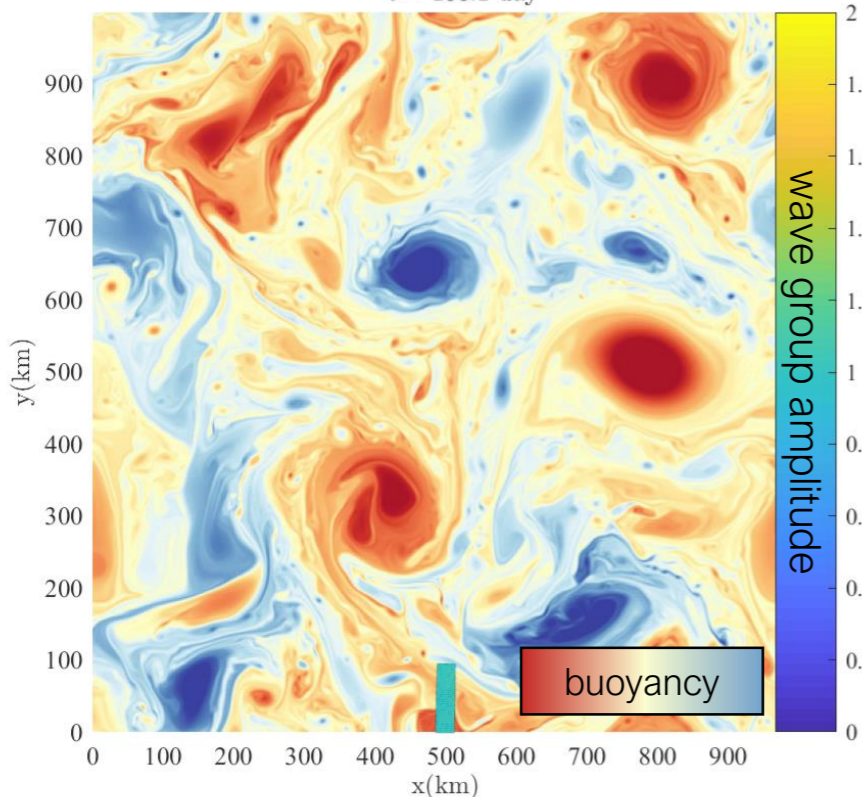
Our random  
model  
wave groups in  
smoothed flow  $\bar{v}$  (16 x 16)  
+ time-correlated  $v'$

# Time-correlated model for $v'$ :

## Method of characteristic

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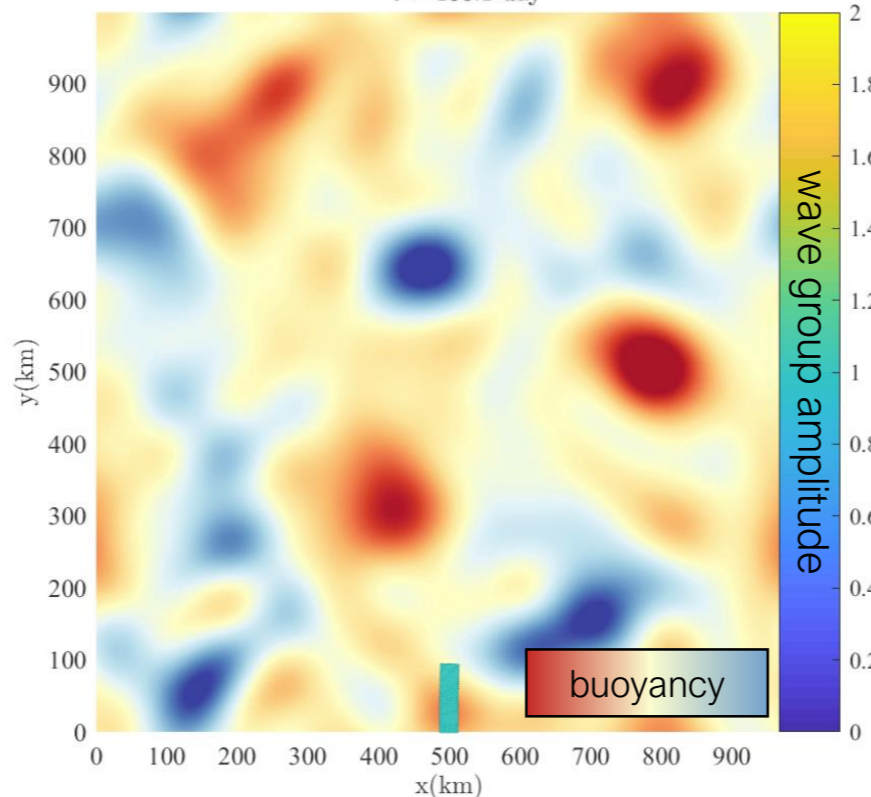
$t = 100.1$  day



Deterministic  
reference:  
wave groups in  
512 x 512 flow

$$v = \bar{v}$$

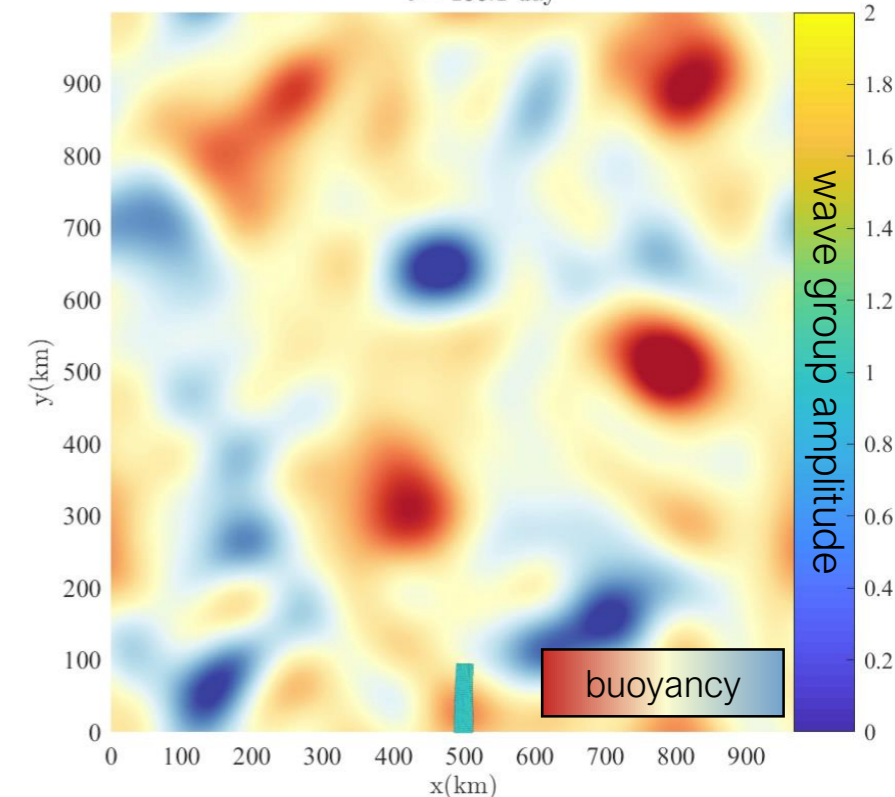
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smoothed flow  $\bar{v}$  (16 x 16)

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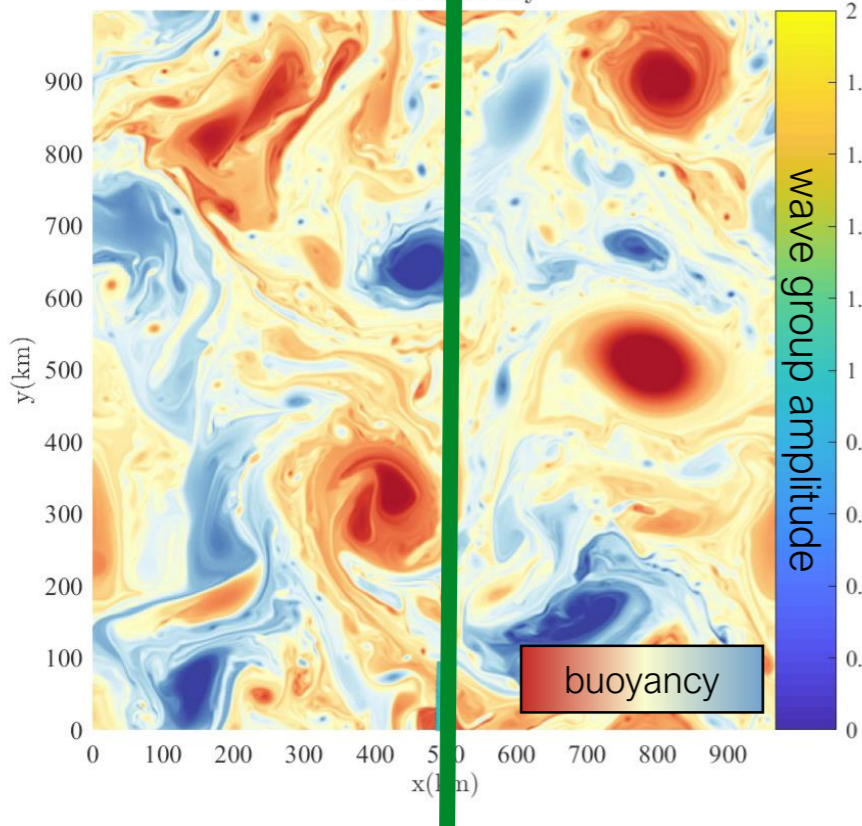


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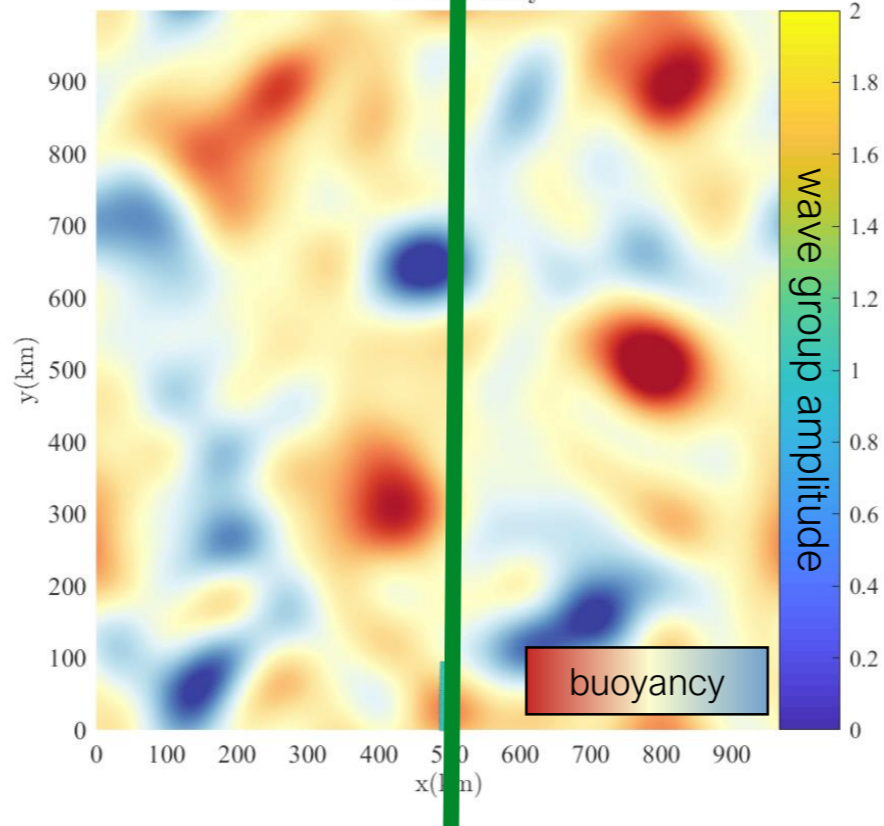
$t = 4.01$  day



Deterministic  
reference:  
wave groups in  
512 x 512 flow

$$v = \bar{v}$$

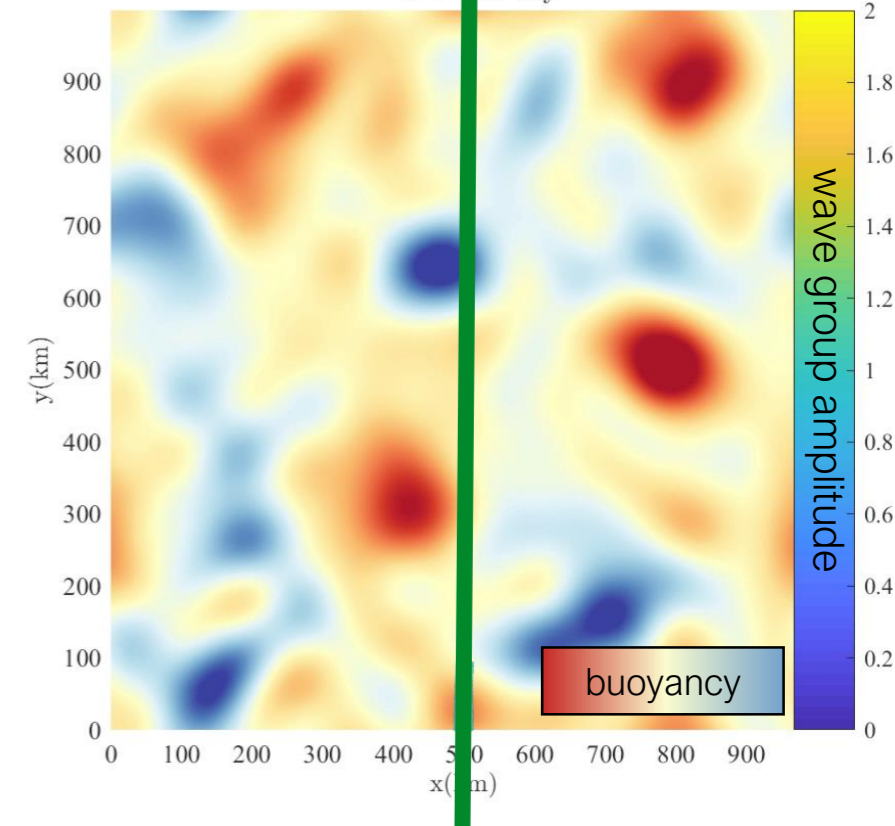
$t = 4.01$  day



Deterministic  
benchmark:  
wave groups in  
smoothed flow  $\bar{v}$  (16 x 16)

$$v = \bar{v} + v'_{\text{cor}}$$

$t = 4.01$  day



Our random  
model  
wave groups in  
smoothed flow  $\bar{v}$  (16 x 16)  
+ time-correlated  $v'$

Time

Uncorrelated

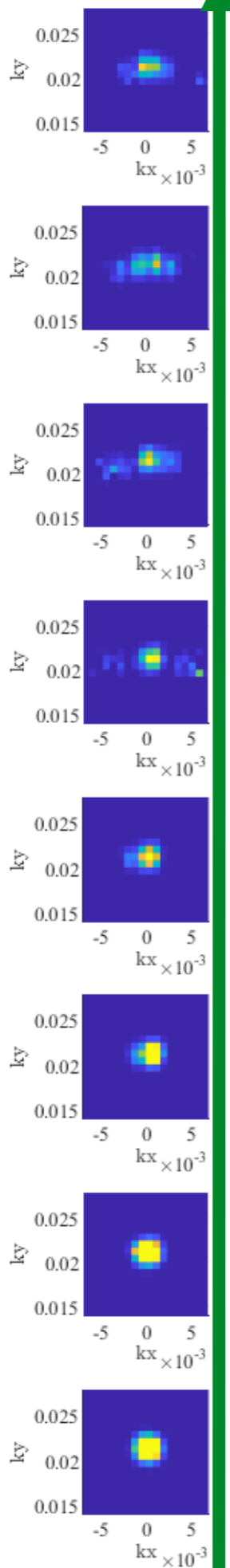
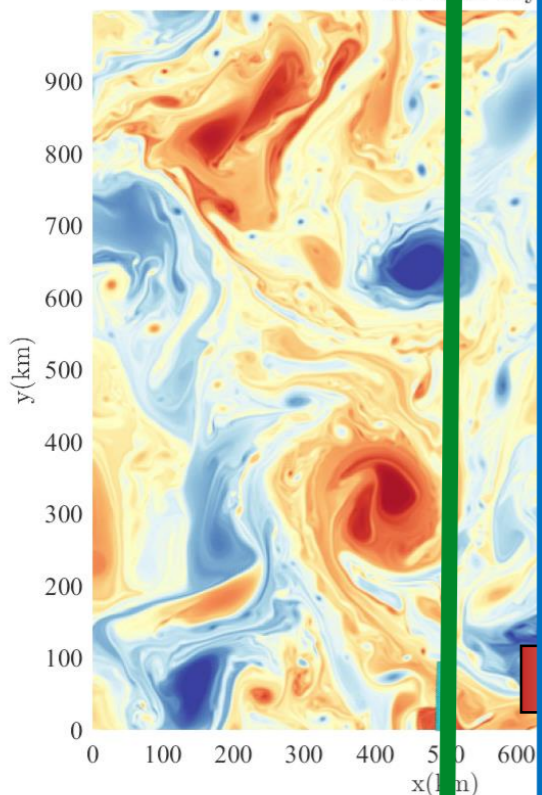
Model for  $v'$ :

Method of characteristics

Stochastic

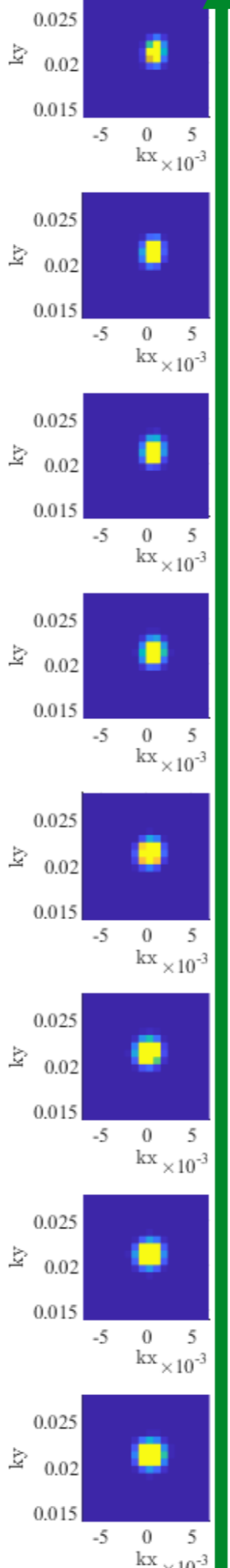
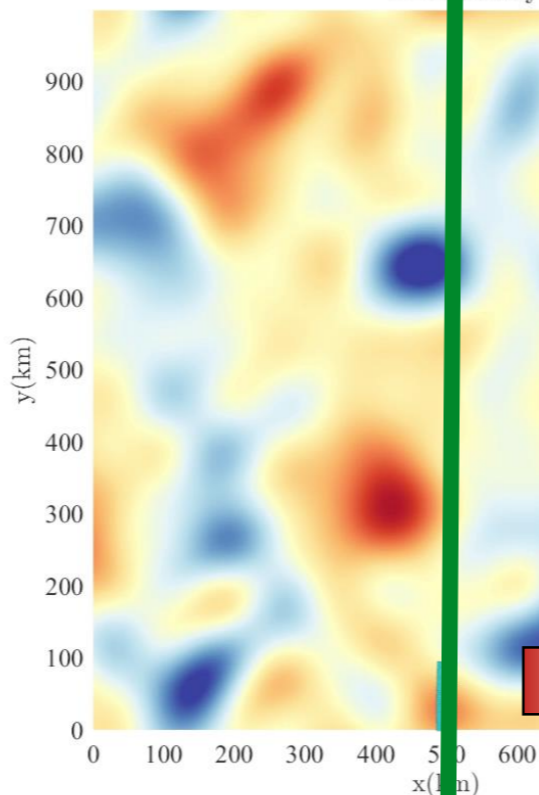
$$v = \bar{v} + v'$$

$t = 4.01$  day



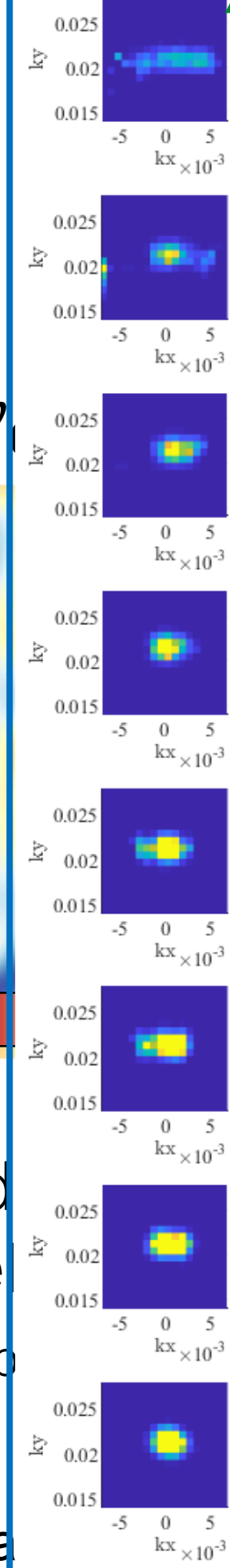
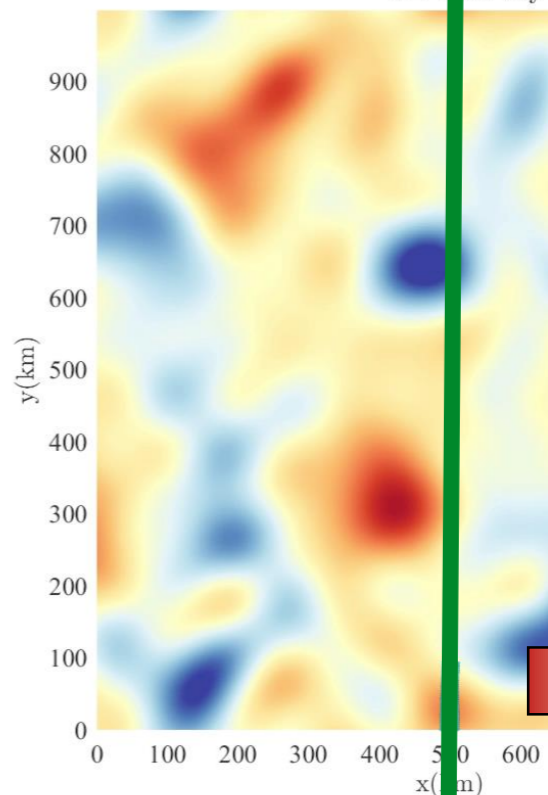
$$v = \bar{v}$$

$t = 4.01$  day



$$v = \bar{v} + v'$$

$t = 4.01$  day



Deterministic  
reference  
wave group  
512 x 512 ft

Deterministic  
benchmark  
wave groups  
smoothed flow  $\bar{v}$  (

Our random  
mode  
wave group  
smoothed flow  $\bar{v}$   
+ time-correla

Time

Uncorrelated

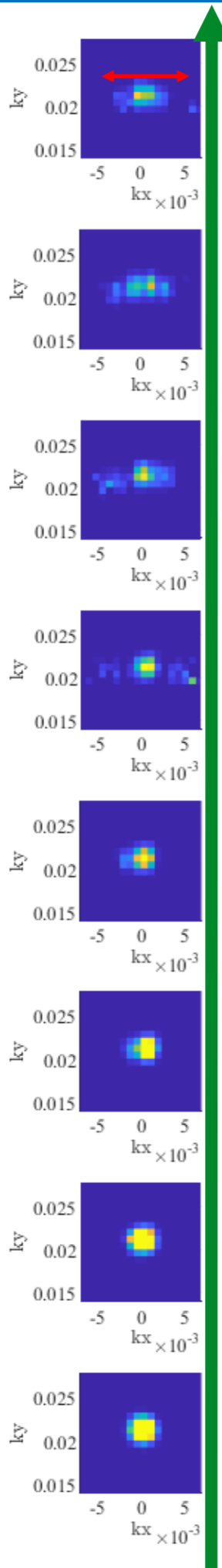
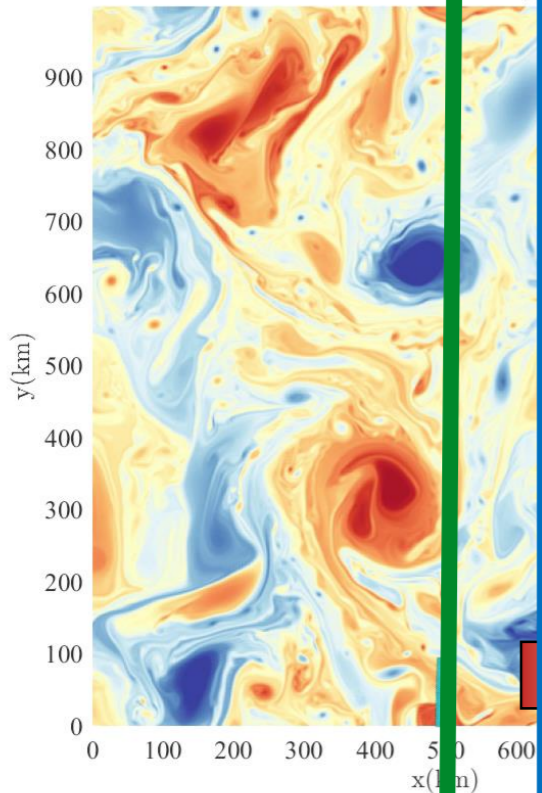
Model for  $v'$ :

Method of char

stic

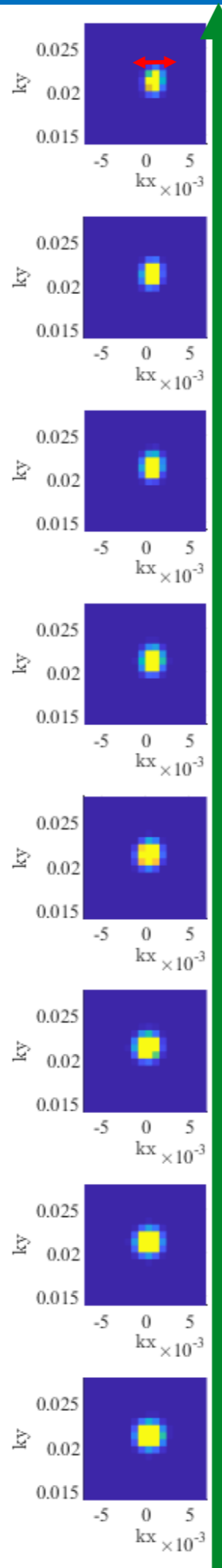
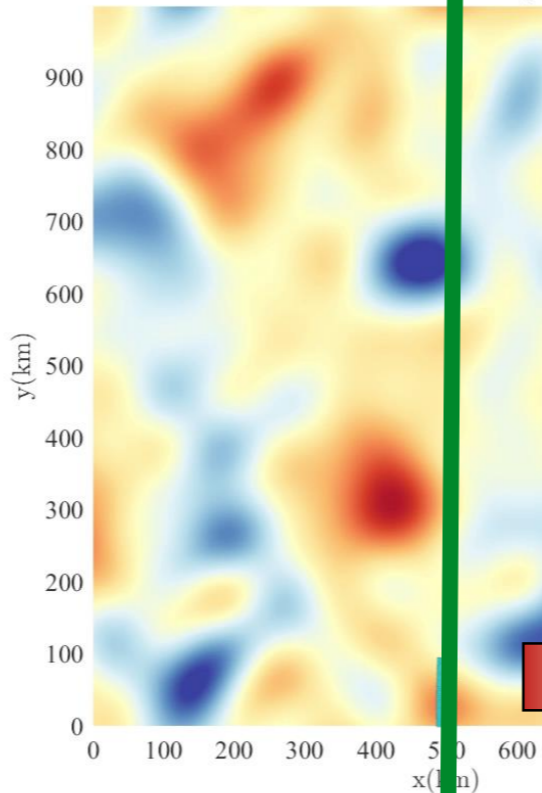
$$v = \bar{v} + v'$$

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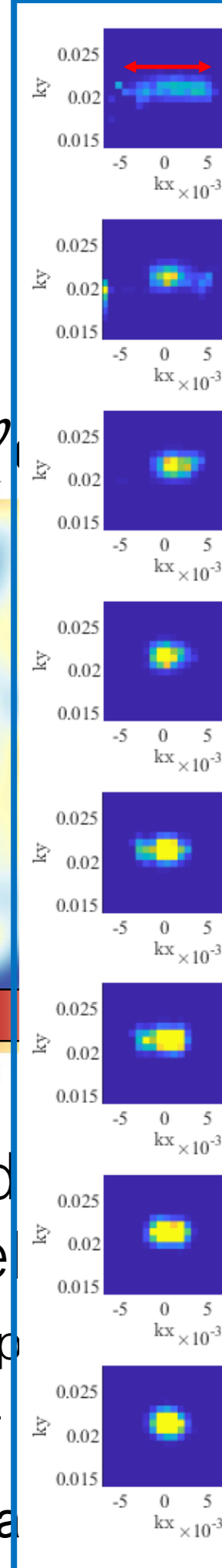
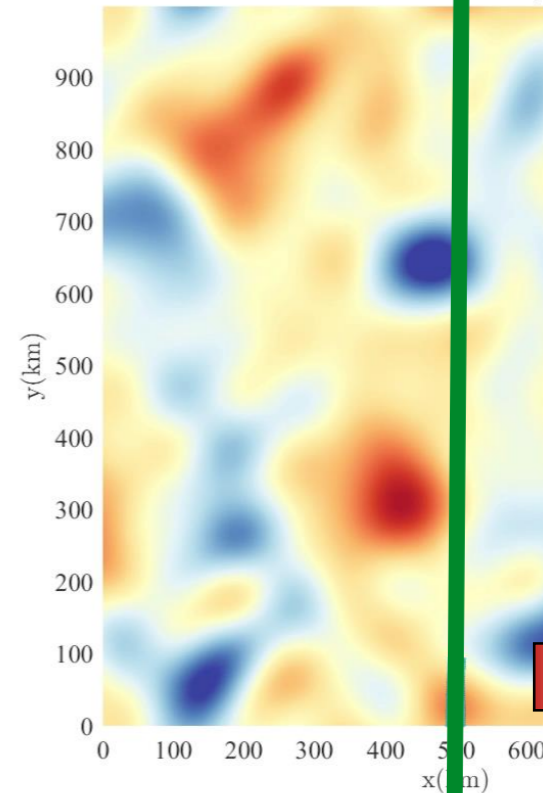
$$v = \bar{v}$$

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Deterministic  
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wave group  
512 x 512 ft

Deterministic  
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Our random  
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+ time-correla

3.

Markovian closure with  
current multi-scale in space  
& **white** in time

- Simple cases & analytic solutions -

Large scale  
group velocity:

$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t/dt$$

(Q Wiener process)

$$v = \bar{v} + v'$$

Large scale  
group velocity:

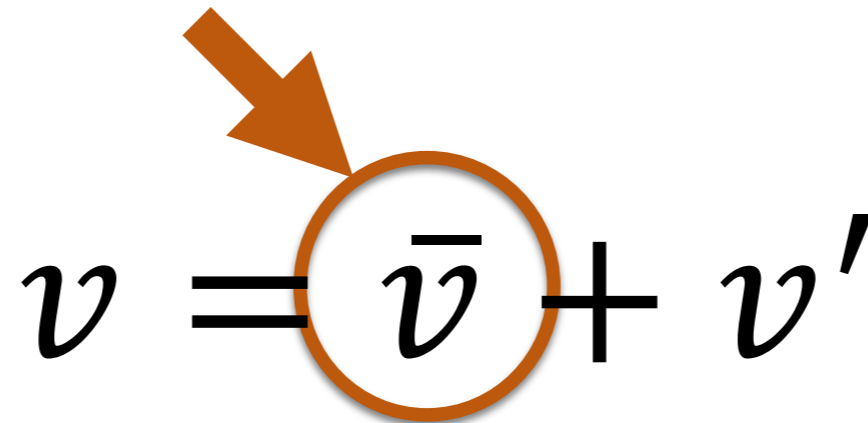
$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Resolved  
large scales


$$v = \bar{v} + v'$$

Large scale  
group velocity:

$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t/dt$$

(Q Wiener process)

Resolved  
large scales

White-in-time  
small scales

$$v = \bar{v} + v'$$

Large scale  
group velocity:

$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t/dt$$

(Q Wiener process)

Resolved  
large scales

White-in-time  
small scales

$$v = \bar{v} + v'$$

References : Mikulevicius & Rozovskii, 2004  
Flandoli, 2011

LU

**Memin**, 2014  
Resseguier et al. 2017 a, b, c  
Cai et al. 2017  
Chapron et al. 2018  
Yang & Memin 2019

...

SALT

**Holm**, 2015  
Holm and Tyranowski, 2016  
Arnaudon et al. 2017  
Crisan et al., 2017  
Gay-Balmaz & Holm 2017  
Cotter and al. 2018 a, b  
Cotter and al. 2019

...

Cotter and al. 2017 Resseguier et al. 2020, 2021, ...



Large scale group velocity:  
 $\bar{v} + C_g^0$

Small scale group velocity:  
 $v' = \sigma dB_t/dt$   
 (Q Wiener process)

Resolved large scales

White-in-time small scales

$$v = \bar{v} + v'$$

Initially for uncertainty quantification & data assimilation

References : Mikulevicius & Rozovskii, 2004  
 Flandoli, 2011

LU	SALT
<b>Memin</b> , 2014 Resseguier et al. 2017 a, b, c Cai et al. 2017 Chapron et al. 2018 Yang & Memin 2019 ...	Crisan et al., 2017 Gay-Balmaz & Holm 2017 <b>Holm</b> , 2015 Holm and Tyranowski, 2016 Arnaudon et al. 2017 Cotter and al. 2018 a, b Cotter and al. 2019 ...
Cotter and al. 2017 Resseguier et al. 2020, 2021, ...	

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

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References : Mikulevicius & Rozovskii, 2004  
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<p>Cotter and al. 2017</p>	<p>Resseguier et al. 2020, 2021, ...</p>

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$a e^{i\epsilon \phi(\epsilon t, \epsilon x)}$$

Group velocity without current

$$C_g^0 = \nabla_k \omega_0$$

$$\nabla = \nabla_x$$

## Simple linear case

$$dX_r = (C_g^0 + \bar{v})dt + \sigma dB_t \quad \Bigg| \quad \text{Ray}$$

Refraction & contraction/dilatation

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k \quad \Bigg| \quad \text{Wave-vector}$$

Conservation of action

$$dA = \nabla \cdot \left( (C_g^0 + \bar{v})dt + \sigma dB_t \right) A \quad \Bigg| \quad \text{Amplitude}$$

Large scale group velocity:

$$\bar{v} + C_g^0$$

Wave:

$$a e^{\frac{i}{\epsilon} \phi}$$

$$k = \nabla \phi$$

$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

Simple linear case 1:  
stationary deterministic,  
divergence-free and  
linear in x large-scale velocity

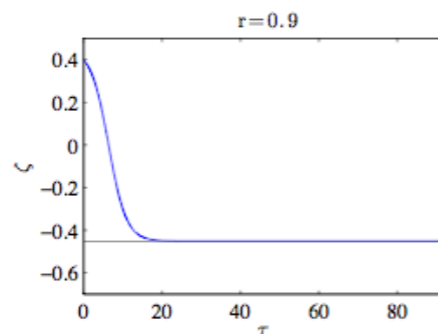
Stretching of phase

$$dk = -\nabla \bar{v}^T k dt$$

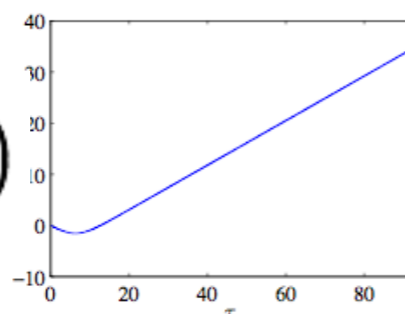
with  $\nabla \bar{v}^T = \sigma_w \begin{bmatrix} 1 & r \\ -r & -1 \end{bmatrix}$

$$|r| < 1$$

Wave shortening

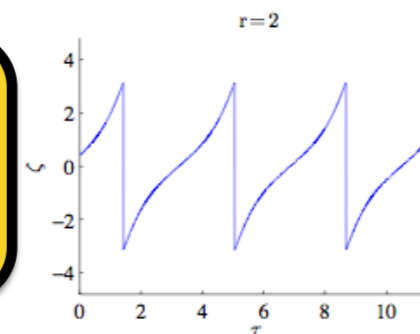


$\ln(\|k\|)$

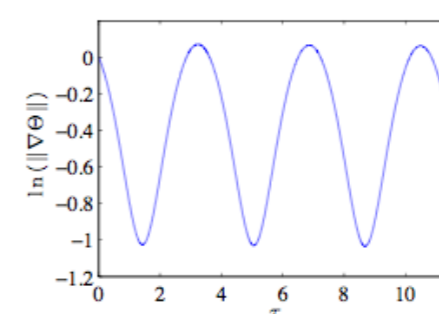


$$|r| > 1$$

Wave rotation



$\ln(\|k\|)$



$$\frac{d\zeta}{dt} = r - \cos(\zeta) = -\frac{\partial V}{\partial \zeta}(\zeta)$$

with  $V(\zeta) = r\zeta - \sin \zeta$

$$\frac{d \ln(\kappa)}{dt} = -\sin(\zeta)$$

Large scale  
group velocity:

$$+C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{\frac{i}{\epsilon}\phi}$$

$$k = \nabla\phi$$
$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$\zeta = 2\theta - \frac{\pi}{2}$$

## Simple linear case 2:

no large-scale current

+ stationary (in space) and divergence-free  
small-scale velocity

Stochastic stretching of phase

$$dk = -\nabla(\sigma dB_t)^T k$$

Log-normal wave-number

$$\begin{cases} \ln \kappa(t) = \alpha^2 t + \alpha B_t^{(1)} \\ \zeta(t) = (\sqrt{12}\alpha) B_t^{(2)} \end{cases}$$

with  $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt.$

Large scale  
group velocity:

$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$a e^{i \frac{\phi}{\epsilon}}$$

$$k = \nabla \phi$$

$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

## Simple linear case 3

divergence-free and linear-in-x large-scale current  
+ stationary (in space) and divergence-free  
small-scale velocity

Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with  $V(\zeta) = r \zeta - \sin \zeta$  and  $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$ .

We can solve the stationary Fokker-Planck for  $\zeta$ ,  
and then get the stationary distribution of  $k$

Large scale  
group velocity:

$$\bar{v} + C_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t/dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

$$k = \nabla\phi$$
$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$\zeta = 2\theta - \frac{\pi}{2}$$

## Simple linear case 3

divergence-free and linear-in-x large-scale current  
+ stationary (in space) and divergence-free  
small-scale velocity

Stochastic stretching of  
phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with  $V(\zeta) = r\zeta - \sin\zeta$  and  $\alpha^2 = \frac{3}{2}\mathbb{E}\|\nabla(\sigma dB_t)^T\|^2/dt$ .

Large scale group velocity:

$$\bar{v} = C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$a e^{i \frac{\omega}{\epsilon} \phi}$$

$$k = \nabla \phi$$

$$= \kappa \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

# Simple linear case 3

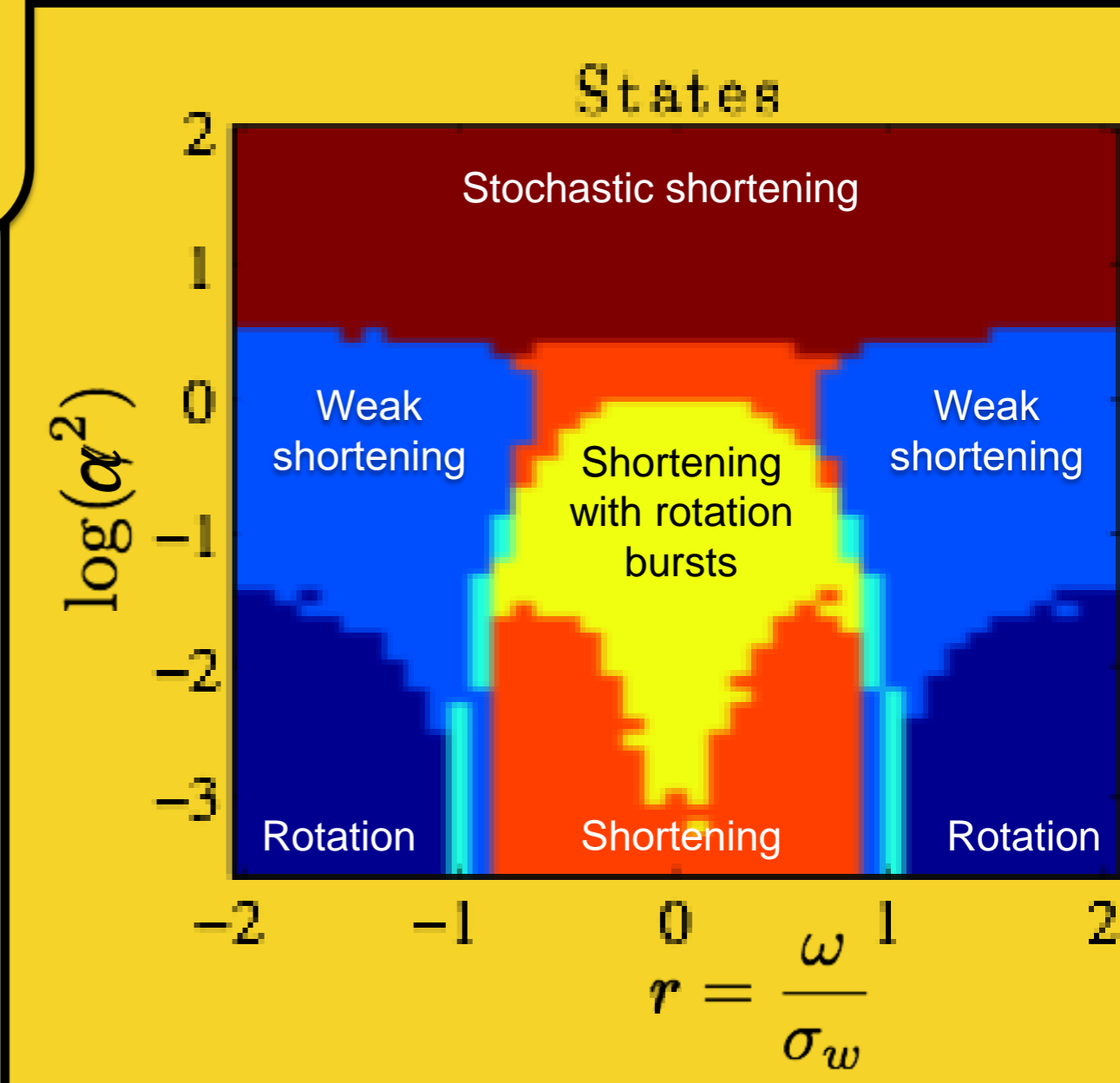
divergence-free and linear-in-x large-scale current  
 + stationary (in space) and divergence-free  
 small-scale velocity

Stochastic stretching of phase

$$dk = -\nabla(\bar{v} dt + \sigma dB_t)^T k$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t)) dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t)) dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with  $V(\zeta) = r \zeta - \sin \zeta$  and  $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$ .





Large scale group velocity:

$$\bar{v} = C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

# Simple linear case 3

divergence-free and linear-in-x large-scale current  
 + stationary (in space) and divergence-free small-scale velocity

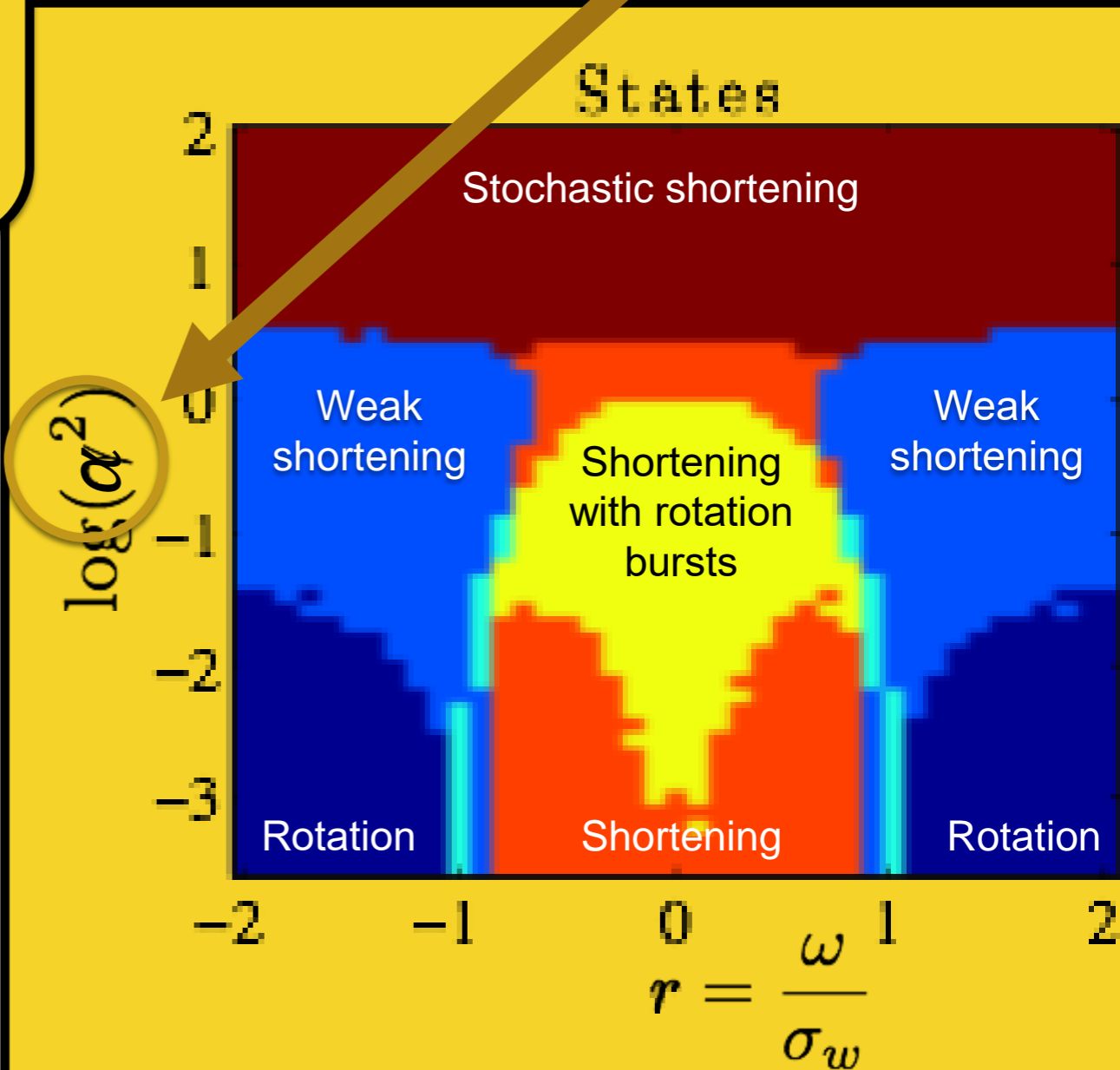
Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$

Depends on small scale's statistics

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with  $V(\zeta) = r\zeta - \sin\zeta$  and  $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$ .



Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

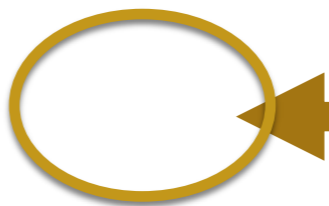
$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\zeta = 2\theta - \frac{\pi}{2}$$

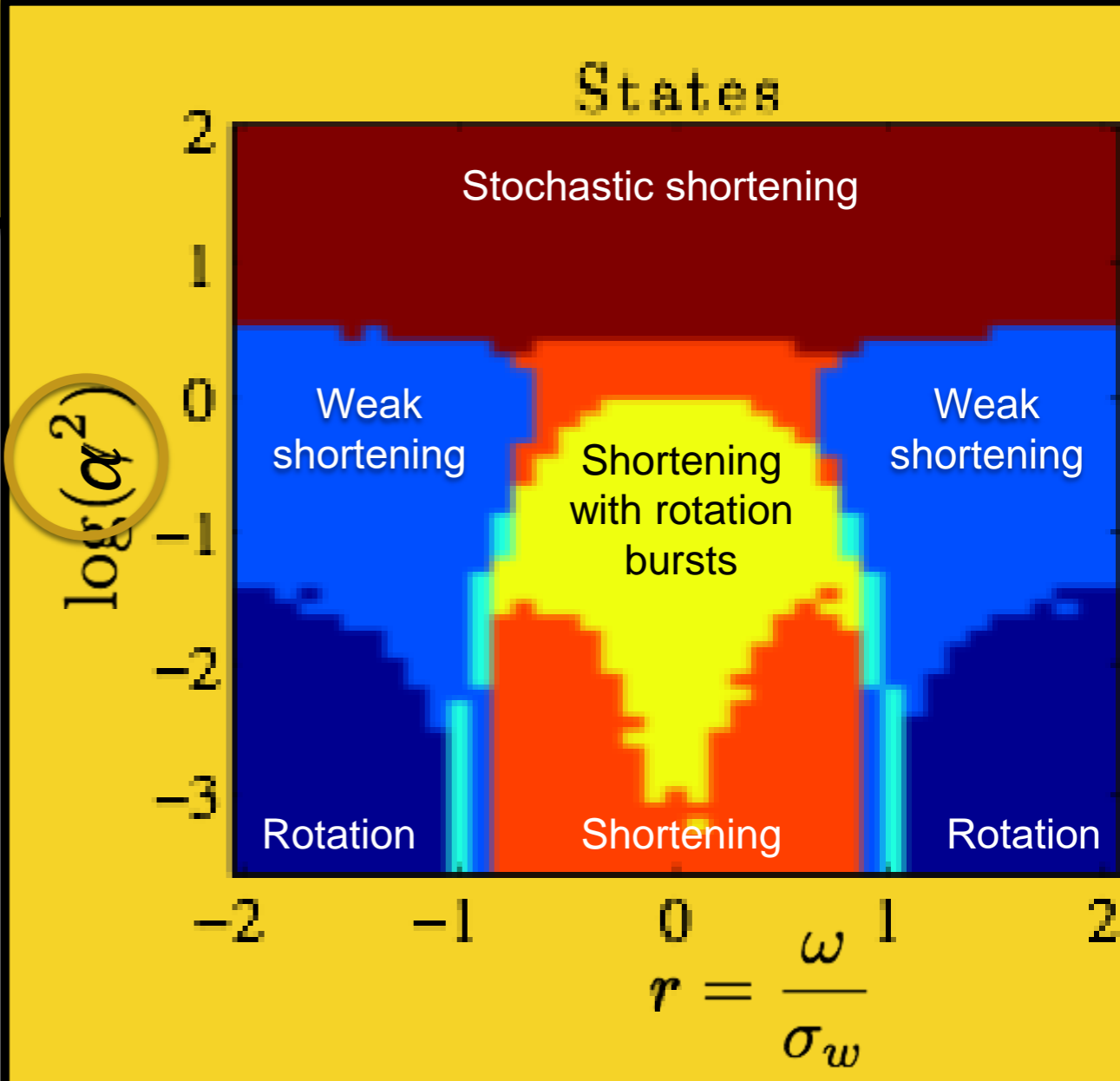
$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with  $V(\zeta) = r\zeta - \sin\zeta$  and  $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$ .



Depends on small scale's statistics

Stochastic stretching of phase

$$dk = -\nabla(\bar{v}dt + \sigma dB_t)^T k$$


Large scale group velocity:

$$\bar{v} = +C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t / dt$$

(Q Wiener process)

Wave:

$$ae^{i\epsilon\phi}$$

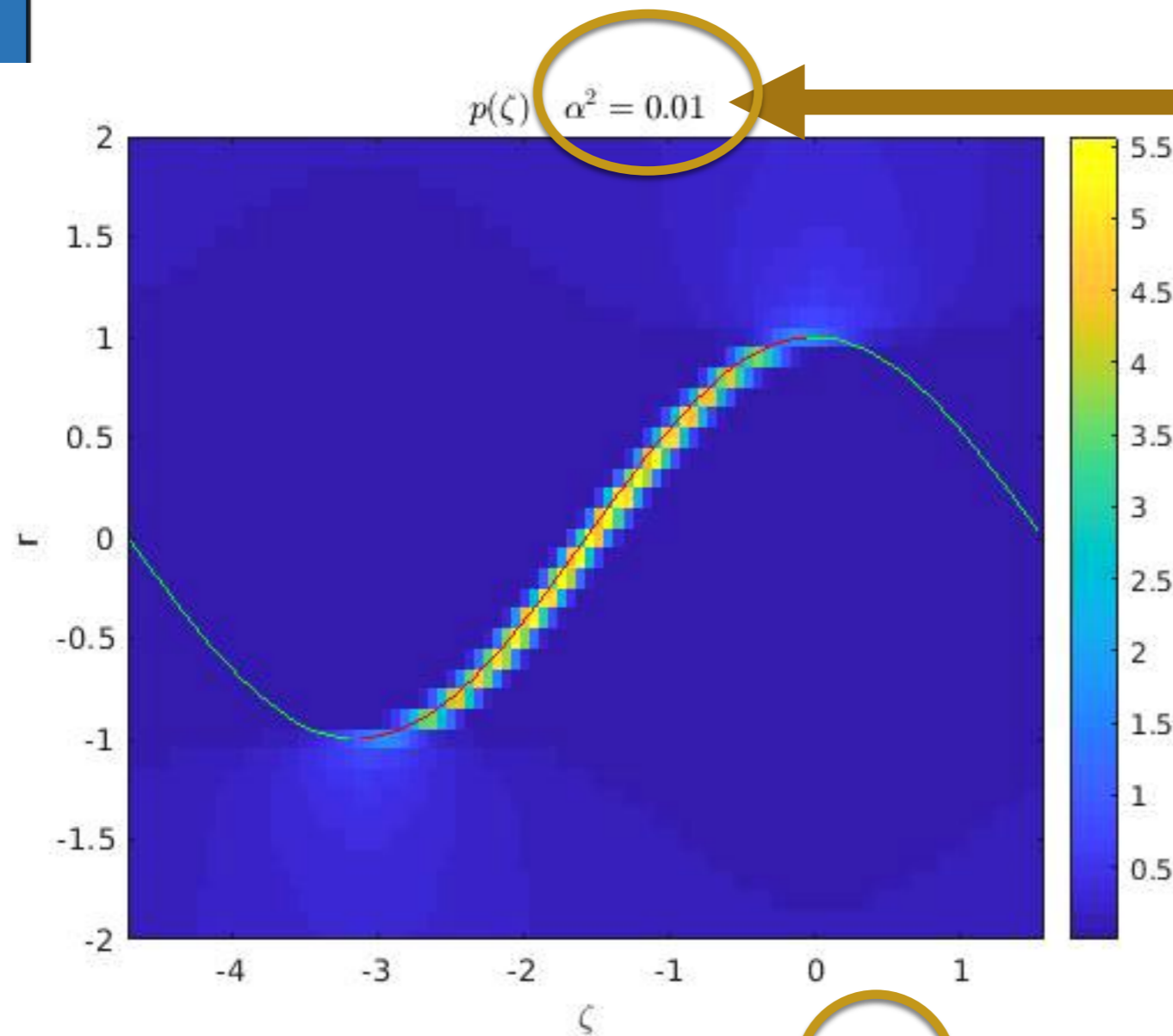
$$k = \nabla\phi$$

$$= \kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

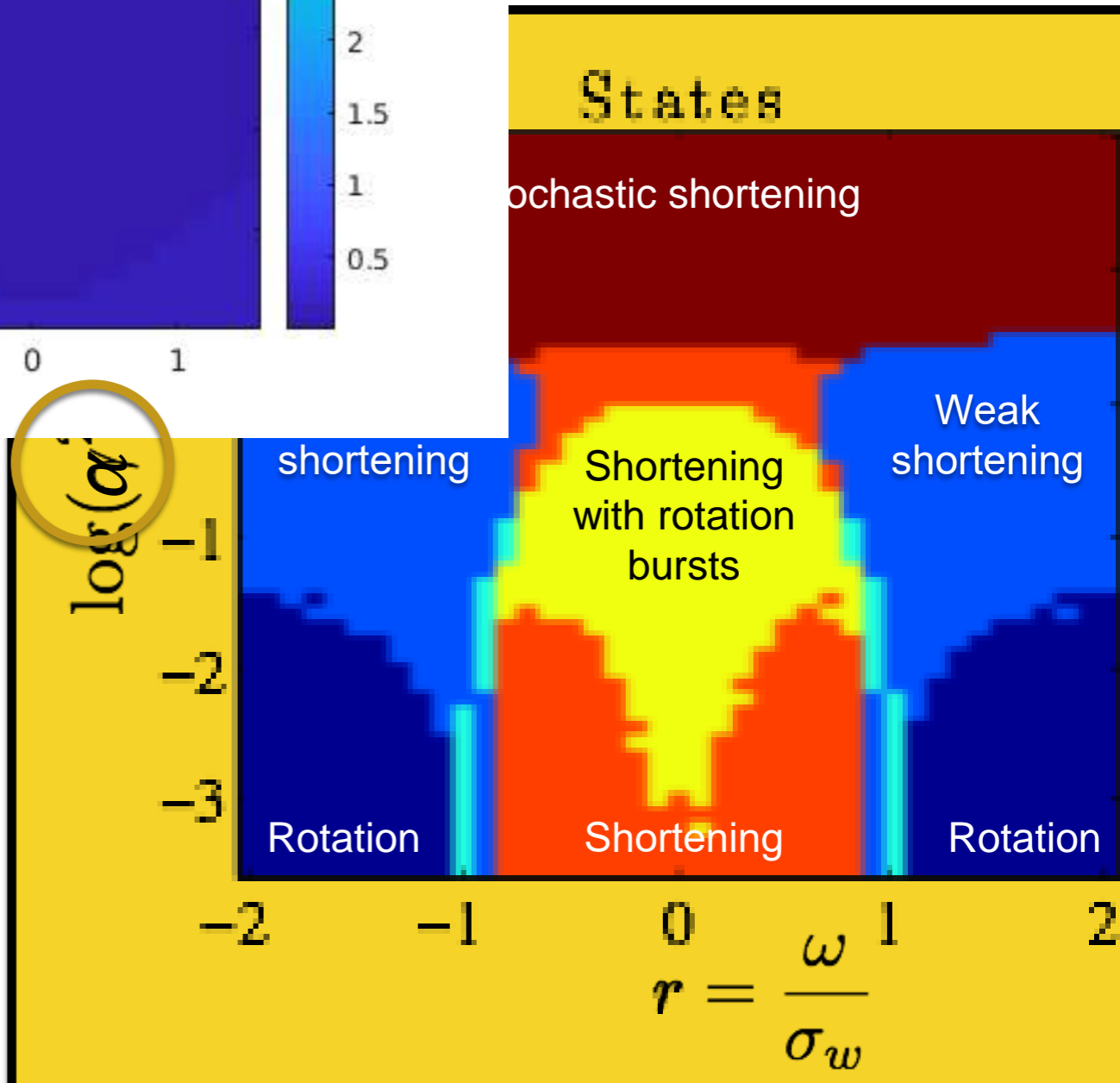
$$\zeta = 2\theta - \frac{\pi}{2}$$

$$\begin{cases} d \ln \kappa(t) = (\alpha^2 - \sin \zeta(t))dt + \alpha dB_t^{(1)} \\ d\zeta(t) = -\frac{\partial V}{\partial \zeta}(\zeta(t))dt + (\sqrt{12}\alpha) dB_t^{(2)} \end{cases}$$

with  $V(\zeta) = r\zeta - \sin\zeta$  and  $\alpha^2 = \frac{3}{2} \mathbb{E} \|\nabla(\sigma dB_t)^T\|^2 / dt$ .



Depends on small scale's statistics



# Conclusion

# Conclusion

We propose multi-scale closures for simulation of wave-turbulence interactions.

- At large scales,  
the oceanic currents are known because measured by satellite.
- At intermediate scales,  
new stochastic closure, which is non-Markovian, with stochastic currents multi-scale in space and time.
- For smaller scales,  
Markovian approaches – e.g., LU & SALT [2,1,3,4,5] – applied (in the wave frame). We found an analytical formula for the probability distribution of wave properties at long time.

In the future, we would like to use it as prior generation of data assimilation.

# References

1. Holm, D. D. (2015). Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176), 20140963.
2. Mémin, E. (2014). Fluid flow dynamics under location uncertainty. *Geophysical & Astrophysical Fluid Dynamics*, 108(2), 119-146.
3. Resseguier, V., Pan, W., & Fox-Kemper, B. (2020). Data-driven versus self-similar parameterizations for stochastic advection by Lie transport and location uncertainty. *Nonlinear Processes in Geophysics*, 27(2), 209-234.
4. Resseguier, V., Li, L., Jouan, G., Dérian, P., Mémin, E., & Chapron, B. (2021). New trends in ensemble forecast strategy: uncertainty quantification for coarse-grid computational fluid dynamics. *Archives of Computational Methods in Engineering*, 28(1), 215-261.
5. Resseguier, V., Hascoët, E., & Chapron, B. (2023). Random ocean swell-rays: a stochastic framework. In *Stochastic Transport in Upper Ocean Dynamics Annual Workshop* (pp. 259-271). Springer, Cham.

# Bonus slides

Large scale  
group velocity:

$$\bar{v} + c_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t / dt$$

Wave:

$$ae^{i\epsilon\phi}$$

Doppler  
frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

# Time decorrelation assumption for $v'$

- **Advantages :**
  - Simpler / analytic formula  
 $\Rightarrow$  physical comprehension & lighter CPU for simulations
  - No precise knowledge needed about  $v'$  time dependency :  
**Markovian** closure

- **Validity :**

$$\epsilon = \frac{\text{(Along-ray } v' \text{ correlation time)}}{\text{(characteristic time of wave group properties evolution)}} = \frac{\left(\frac{l_{v'}}{\|c_g^0\|}\right)}{\left(\frac{1}{\|\nabla v\|}\right)} \ll 1$$

- **Limitations :**
  - Swells ( $\|c_g^0\| \gg 1$ )
  - Small-scale currents ( $l_{v'} \ll 1$ )
  - Moderate current gradients ( $\|\nabla v\| \ll 1$ ) ( $\Rightarrow$  moderate  $\|\nabla v'\|$  !)



Large scale  
group velocity:

$$\bar{v} = c_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

$$ae^{i\epsilon\phi}$$

Doppler  
frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

# Time-uncorrelated model for $v'$

$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Large scale  
group velocity:

$$\bar{v} + c_g^0$$

Small scale  
group velocity:

$$v' = \sigma dB_t/dt$$

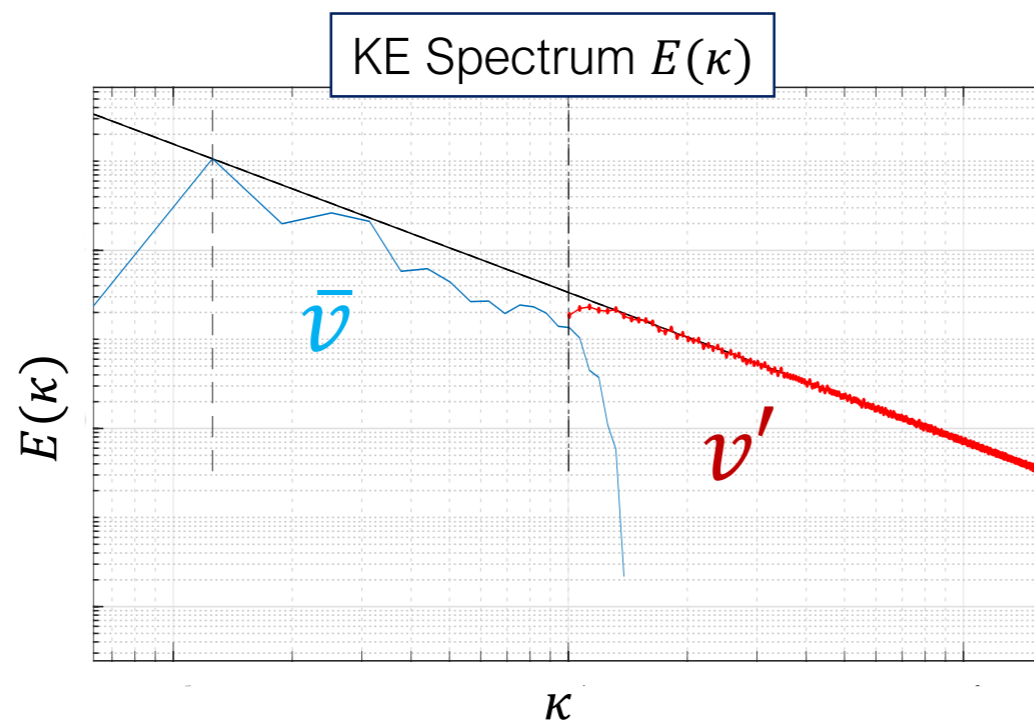
Wave:

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Doppler  
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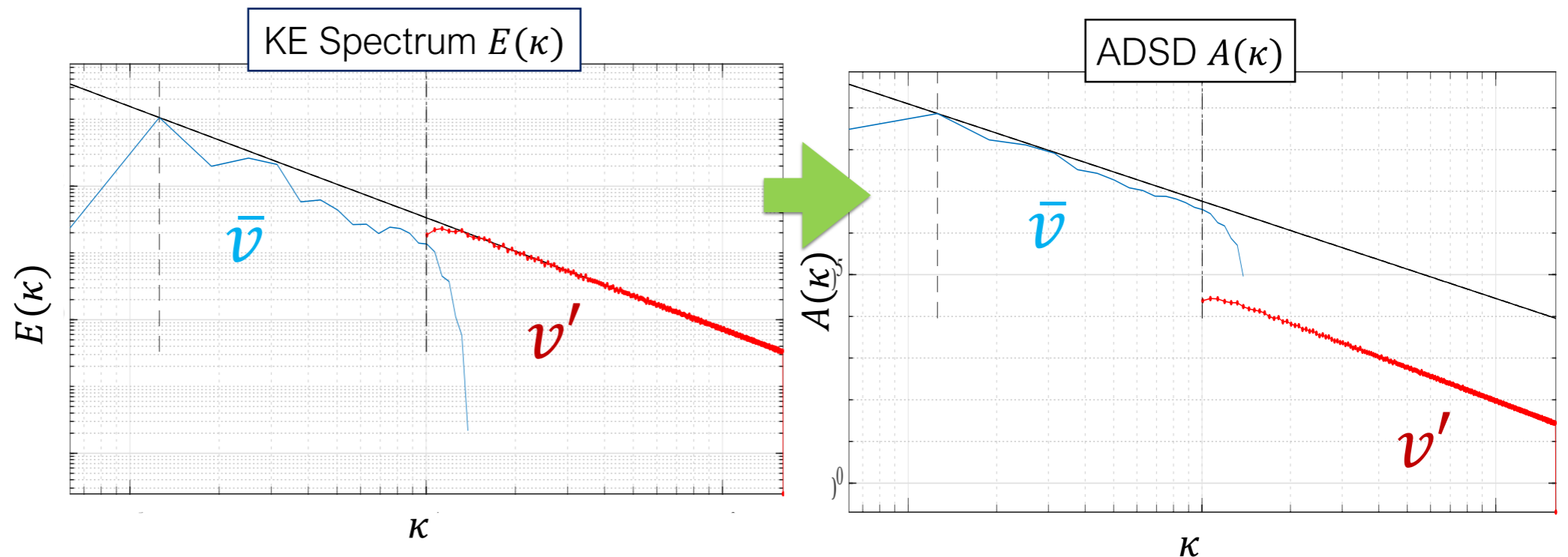
$$ae^{i\epsilon\phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

# Time-uncorrelated model for $v'$

Absolute Diffusivity Spectral Density  
 $A(\kappa) = E(\kappa) \tau(\kappa)$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:  
[3] Resseguier, Pan & Fox-Kemper 2020

Large scale group velocity:

$$\bar{v} + C_g^0$$

Small scale group velocity:

$$v' = \sigma dB_t/dt$$

Wave:

$$ae^{i\epsilon\phi}$$

Doppler frequency:

$$\omega_0 = \sqrt{g\|k\|}$$

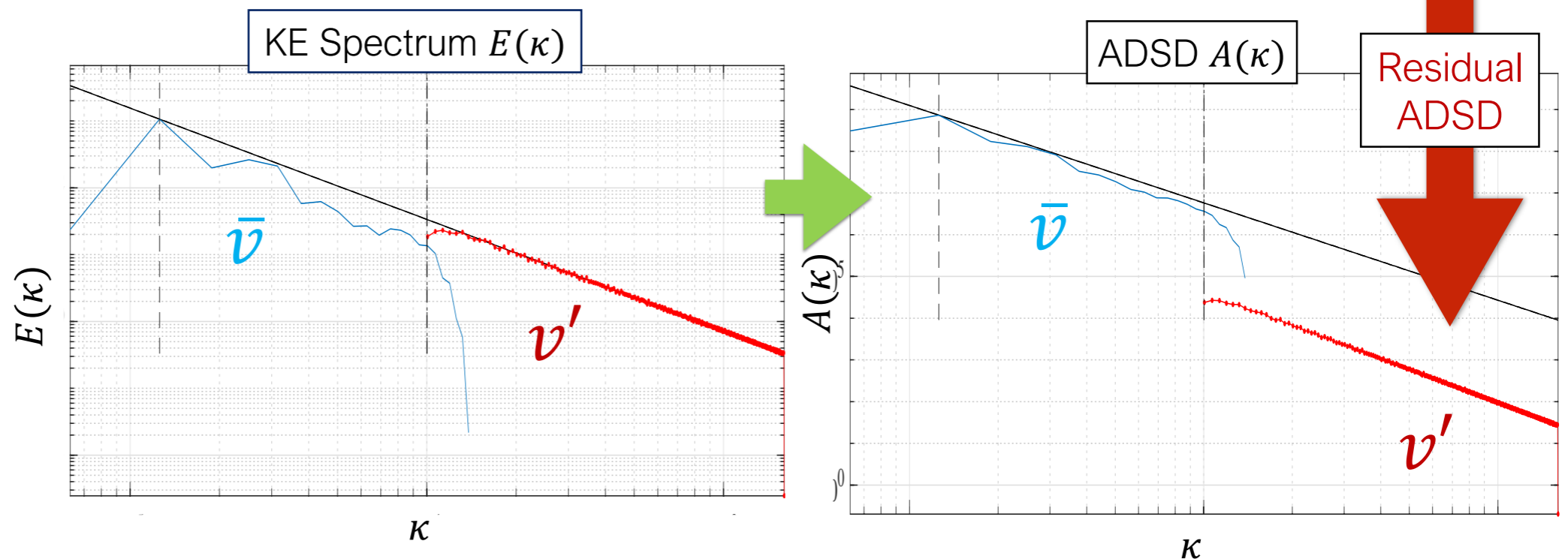
# Time-uncorrelated model for $v'$

Absolute Diffusivity Spectral Density

$$A(\kappa) = E(\kappa) \tau(\kappa)$$

Correlation time along ray

$$\tau(\kappa) = \tau_{ray}(\kappa) = \frac{1/\kappa}{\|C_g^0\|}$$



$$v' = \sigma \dot{B} = (\text{filter}) * (\text{white noise})$$

Reference:  
[3] Resseguier, Pan & Fox-Kemper 2020

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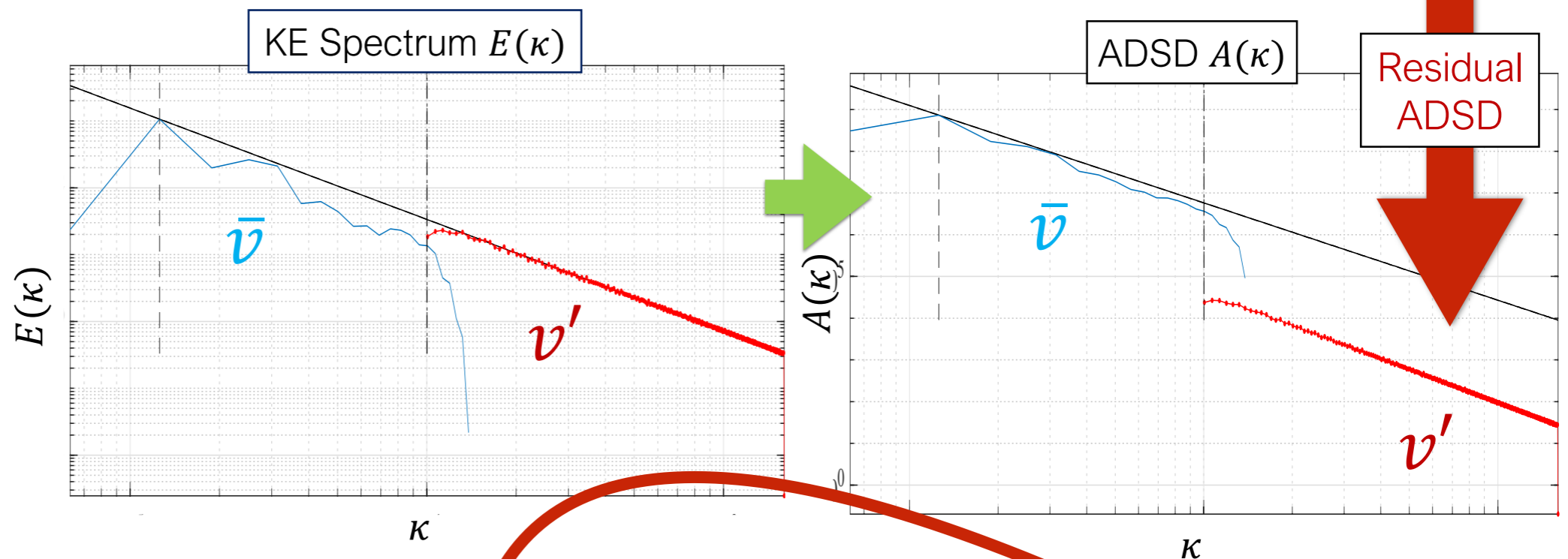
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