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# Homogenisation for waves in 1D periodic media: Effective models, boundary and transmission correctors

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# Theme 1:

## Homogenisation for waves in 1D periodic media

### Effective models, boundary and transmission correctors

Rémi Cornaggia<sup>1</sup>

Joint work with Bojan B. Guzina<sup>2</sup> and Bruno Lombard<sup>3</sup>

CISM Course C2305: *Wave Motion in Heterogeneous Media: Analysis, Modeling and Design*  
June 19-23, 2023



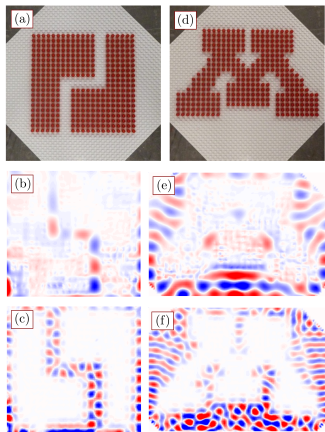
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# Control of wave propagation using periodic structures

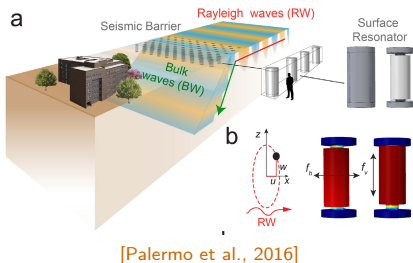


[Celli and Gonella, 2015]

... to design waveguides

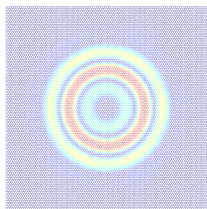
- with resonators for long wavelengths / low frequencies
- with Bragg effects for medium wavelengths / frequencies

... to design seismic protections (with resonators)



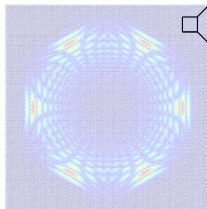
... and more (negative effective properties, cloaking ...)

# A motivation : dispersion and anisotropy in 2D elasticity [Rosi and Auffray, 2019]

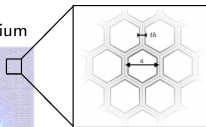


$$\frac{\ell}{\lambda_c} \ll 1$$

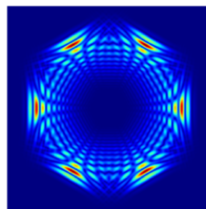
Microstructured medium



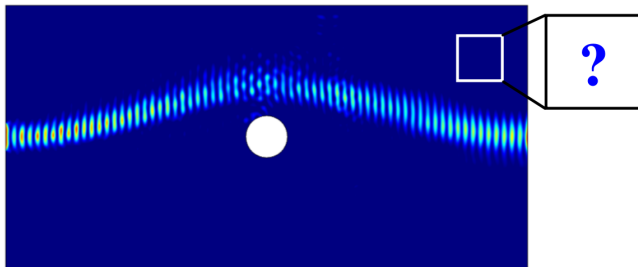
$$\frac{\ell}{\lambda_c} \approx \frac{1}{6}$$



Continuum model



Effective (strain-gradient) model.



Optimized effective model  $\Rightarrow$  "de-homogenization" ?

# Homogenisation and interface conditions

## How to model the wave propagation in periodic structures ?

- Accounting for all interactions  $\rightarrow$  costly (*dedicated numerical methods* out-of scope)
- *Floquet-Bloch* theory  $\implies$  [See other lectures](#).
- *Homogenization* (/averaging/effective) methods and models
- **Both lectures:** *asymptotic two-scale homogenization*
  - ▶ Formal procedure to obtain effective models
  - ▶ Accuracy can be improved by pushing the expansion to higher orders
  - ▶ Extensions to *high/finite frequencies* [[Craster et al., 2010](#), [Guzina et al., 2019](#)]  
 $\implies$  [R. Craster](#) and [B. Guzina's lectures](#)

## How to account for boundaries and interfaces ?

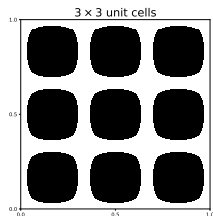
- In 2-3D, *boundary layers* must be accounted for at boundaries and interfaces [[Dumontet, 1986](#), [Moskow and Vogelius, 1997](#), [Allaire and Amar, 1999](#), [Gérard-Varet and Masmoudi, 2012](#), [Armstrong et al., 2017](#), [Cakoni et al., 2019](#)] ...
- *Equivalent boundary or transmission conditions* preserving the asymptotics are difficult to build [[Vinoles, 2016](#), [Marigo and Maurel, 2017](#), [Maurel and Marigo, 2018](#), [Beneteau, 2021](#)]
- **This lecture:** 1D waves, “interfaces” are *points*.

# Optimization

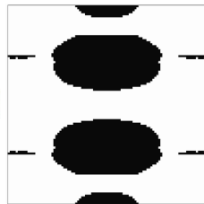
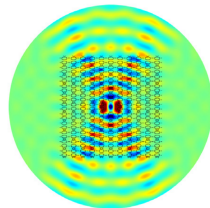
## How to obtain desired properties ?

- Known design (cylinders, spheres, cones, LEGO bricks ...)  $\implies$  *parameter optimization* [Huang et al., 2016, Palermo et al., 2016] ...
- Given materials, unknown design  $\implies$  *topological optimization* [Vondřejc et al., 2017, Kook and Jensen, 2017, Allaire and Yamada, 2018] ...

**Next lectures:** topological optimization of microstructured 2D media and interfaces (long wavelengths, no resonances)

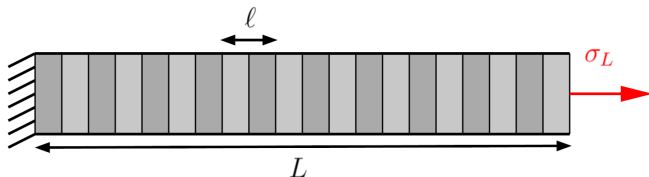


what we will see



[Noguchi et al., 2018] high frequency, hyperbolic propagation, see R. Craster's lecture

## A preliminary example : 1D statics



- **Two-scale problem** :  $L$  (macro) and  $\ell$  (micro)
- **Scale factor**  $\varepsilon = \ell/L$
- $\ell$ -periodic properties  $E_\ell(X) = E(X/\ell) = E(x/\varepsilon)$  :  $E$  is 1-periodic
- Normalized coordinate  $x = X/L$

$$\left\{ \begin{array}{lll} \partial_x \sigma(x) = 0 & x \in [0, 1] & \text{equilibrium} \\ \sigma(x) = E(x/\varepsilon) \partial_x u & x \in [0, 1] & \text{constitutive law} \\ u(0) = 0 & & \text{displacement BC} \\ \sigma(1) = \sigma_L & & \text{stress BC} \end{array} \right.$$

See dedicated Jupyter Notebook.

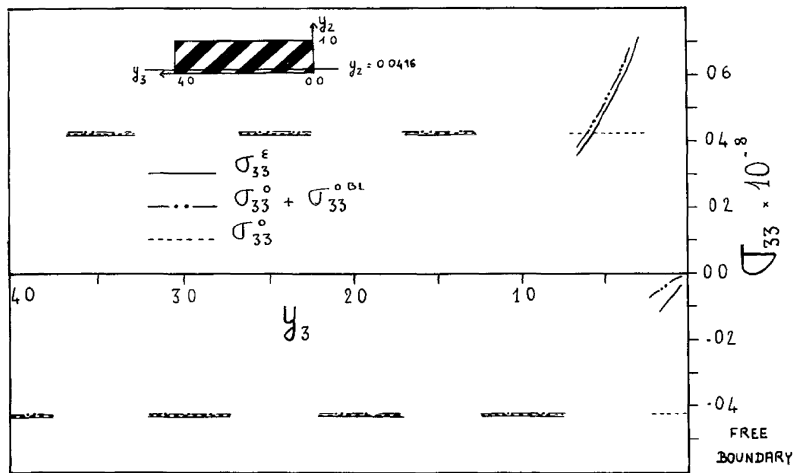


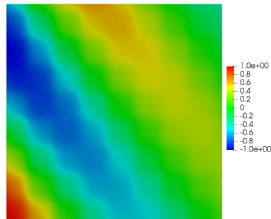
Figure 4. — The  $(3, 3)$  components of the real stresses  $\sigma^e$ , of the micro-stresses  $\sigma^0$  and of the boundary layer stresses  $\sigma^{0BL}$  added to  $\sigma^0$ , plotted against  $y_3$ , at  $y_2 = 0.0416$  fixed.



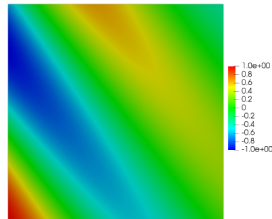
# Examples: 2D time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

$$\varepsilon = \ell/\lambda = 0.125$$

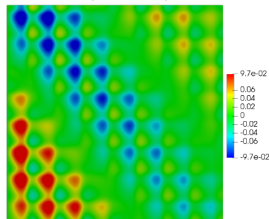
$u$  (exact)



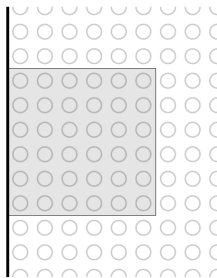
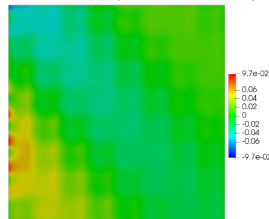
$U_0$



$|u - U_0|$

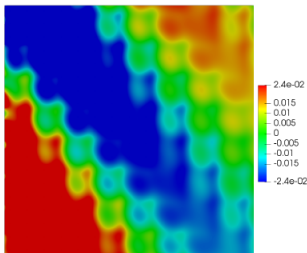


$|u - (U_0 + \varepsilon u_1)|$  (bulk corrector)

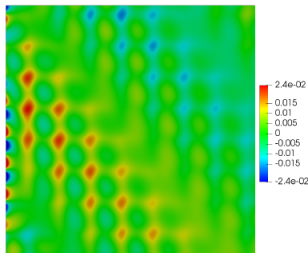


# Examples: 2D time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

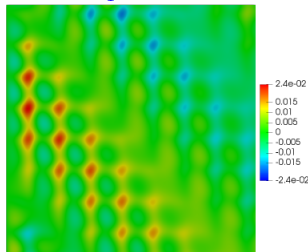
$|u - (U_0 + \varepsilon u_1)|$  (bulk corrector)



$|u - (U_0 + \varepsilon[U_1 + u_1])|$  (bulk and mean correctors)

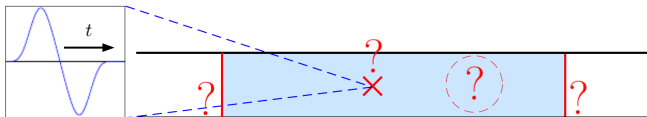
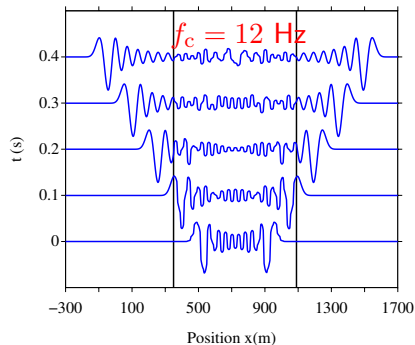
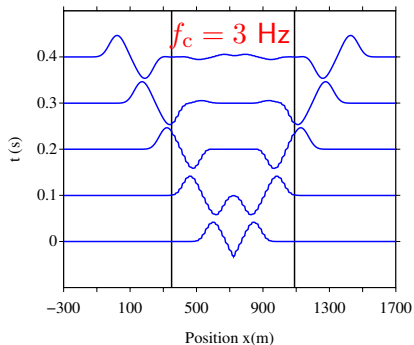
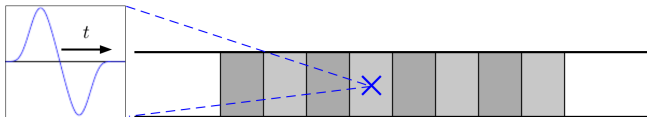


$|u - (U_0 + \varepsilon[U_1 + u_1 + u_1^{BL}])|$  (bulk, mean and boundary layer correctors)



(and then look for 2nd-order correctors ...)

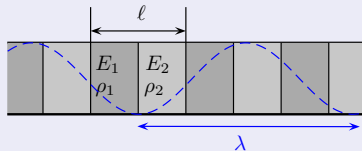
# Challenges for 1D waves : dispersion, interfaces and source points



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- 1 Introduction
- 2 Second-order asymptotic homogenization for waves in 1D domains
  - Second-order models
  - Corrector functions and reciprocity identities
  - Choosing a model
- 3 Boundary and transmission correctors for time-harmonic motion
- 4 Extension to transient waves
  - Hyperbolic system
  - Transmission conditions and first-order “hybrid” model
  - Application to point sources
- 5 Conclusions and perspectives

# Two-scale asymptotic homogenization in a 1D periodic medium



- Wave equation in a  $\ell$ -periodic medium

$$\rho \left( \frac{X}{\ell} \right) \frac{\partial^2 u_\ell}{\partial t^2} - \frac{\partial}{\partial X} \left[ E \left( \frac{X}{\ell} \right) \frac{\partial u_\ell}{\partial X} \right] = 0$$

- Reference wavelength  $\lambda$

[Sanchez-Palencia, 1974, Bensoussan et al., 1978, Cioranescu and Donato, 1999] ...

- **Scale separation**  $\varepsilon = \ell/\lambda \ll 1$
- **Slow variable**  $x = X/\lambda$  and **fast variable**  $y = X/\ell = x/\varepsilon$
- Look for  $u_\ell$  as a **function of both variables**:  $u_\ell(X, t) = \hat{u}(x, y, t)$ , **1-periodic in  $y$**
- $\frac{\partial}{\partial X} \rightarrow \frac{1}{\lambda} \left[ \frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y} \right]$
- Ansatz:  $\hat{u}(x, y, t) = \sum_{j \geq 0} \varepsilon^j u_j(x, y, t) \implies$  **cascade of equations** for the  $u_j$
- **Separated variable solution** featuring **mean fields**  $U_j(x, t)$  and **cell functions**  $P_j(y)$ :

$$u_0(x, y, t) = U_0(x, t)$$

$$u_1(x, y, t) = U_1(x, t) + U_{0,x}(x, t)P_1(y)$$

$$u_2(x, y, t) = U_2(x, t) + U_{1,x}(x, t)P_1(y) + U_{0,xx}(x, t)P_2(y)$$

## Effective wave equations

- Equations satisfied by the mean fields:

$$\varrho_0 U_{0,tt} - \mathcal{E}_0 U_{0,xx} = 0 \quad \left| \begin{array}{l} \varrho_0 = \langle \rho \rangle \\ \mathcal{E}_0 = \langle E(1 + P_{1,y}) \rangle \end{array} \right.$$

$$\varrho_0 U_{1,tt} - \mathcal{E}_0 U_{1,xx} = 0 \quad \mathcal{E}_1 = \frac{\varrho_1}{\varrho_0} \mathcal{E}_0 \Rightarrow \text{no first-order contribution}$$

$$\varrho_0 U_{2,tt} - \mathcal{E}_0 U_{2,xx} + \varrho_2 U_{0,xtt} - \mathcal{E}_2 U_{0,xxx} = 0 \quad \left| \begin{array}{l} \varrho_2 = \langle \rho P_2 \rangle \\ \mathcal{E}_2 = \langle E(P_2 + P_{3,y}) \rangle \end{array} \right.$$

- Second-order **total mean field**  $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$ :

$$\varrho_0 U_{,tt} - \mathcal{E}_0 U_{,xx} + \varepsilon^2 (\varrho_2 U_{0,xtt} - \mathcal{E}_2 U_{0,xxx}) = 0$$

- “Boussinesq trick”: using (i)  $\varrho_0 U_{0,tt} = \mathcal{E}_0 U_{0,xx}$  and (ii)  $U_0 = U + o(1)$  :

$$\frac{1}{c_0^2} U_{,tt} - U_{,xx} - \varepsilon^2 \left( \beta_x U_{,xxxx} - \frac{\beta_m}{c_0^2} U_{,xtt} - \frac{\beta_t}{c_0^4} U_{,tttt} \right) = 0, \text{ with } \beta_x - \beta_m - \beta_t = \beta$$

[Fish et al., 2002, Lamacz, 2011, Wautier and Guzina, 2015]

- Effective properties:**  $c_0^2 = \frac{\mathcal{E}_0}{\varrho_0}$  and  $\beta := \left[ \frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\varrho_2}{\varrho_0} \right]$
- 2 degrees of freedom** to choose  $(\beta_x, \beta_m, \beta_t)$ .

# Cell functions: displacement/velocity and stress correctors

## Velocity-stress system:

$$\begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = 0 \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 \end{cases}$$

with  $v_\ell = \partial_t u_\ell$   
and  $\sigma_\ell = E_\ell \partial_x u_\ell$ .

- **Displacement/velocity correctors** ( $P_1, P_2, P_3$ ) using  $V = \partial_t U$ :

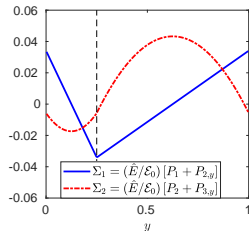
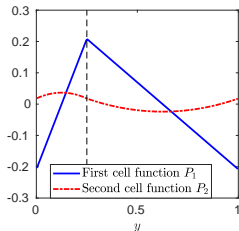
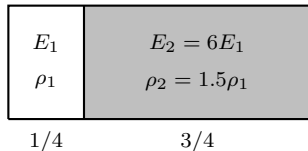
$$\begin{cases} u_\ell(x, t) \approx U(x, t) + \varepsilon U_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 U_{,xx}(x, t) P_2(x/\varepsilon) \\ v_\ell(x, t) \approx V(x, t) + \varepsilon V_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 V_{,xx}(x, t) P_2(x/\varepsilon) \end{cases}$$

- **Stress correctors** ( $\Sigma_1, \Sigma_2$ ) using **macroscopic stress**  $S := \varepsilon_0 \partial_x U$ :

$$\sigma_\ell(x, t) \approx S(x, t) + \varepsilon S_{,x}(x, t) \Sigma_1(x/\varepsilon) + \varepsilon^2 S_{,xx}(x, t) \Sigma_2(x/\varepsilon)$$

**Static equilibrium problems** posed in the unit cell  $Y = [0, 1]$  for  $(P_j, \Sigma_j)$

$$\begin{cases} \partial_y \Sigma_j = f_j[E, \rho, P_{j-1}, \Sigma_{j-1}] \\ \varepsilon_0 \Sigma_j = E (P_j + \partial_y P_{j+1}) \end{cases} \quad (P_j, \Sigma_j) \text{ are 1-periodic, } \langle P_j \rangle = 0$$



## Reciprocity identities on effective coefficients

- Classical expression of effective coefficients:

$$\rho_j = \langle \rho P_j \rangle, \quad \mathcal{E}_j = \langle E(P_j + P_{j+1,y}) \rangle = \mathcal{E}_0 \langle \Sigma_j \rangle, \quad \beta := \left[ \frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\rho_2}{\rho_0} \right]$$

$\Rightarrow$  *a priori* needs  $P_j$ ,  $j = 1, 2, 3$  for second-order effective properties.

- Weak form of cell problems:

$$\text{Find } P_j \in H_{\#0}^1, \quad \int_0^1 E(\partial_y P_j)(\partial_y w) \, dy = F_j(w) \quad \forall w \in H_{\#}^1, \quad j \in \{1, 2, 3\},$$

**Reciprocity identities** are obtained by “testing” one cell problem with another.

[Fish et al., 2002, Abdulle and Pouchon, 2020] ...

**First reciprocity identity:** using  $F_1(P_2) = F_2(P_1)$ ,

$$\frac{\mathcal{E}_1}{\mathcal{E}_0} - \frac{\rho_1}{\rho_0} = 0 \quad \Rightarrow \quad \begin{cases} \text{No need for } P_2 \text{ to compute } \mathcal{E}_1. \\ \text{Cancels first-order contribution in effective wave equation.} \end{cases}$$

**Second reciprocity identity:** using  $F_1(P_3) = F_3(P_1)$  (and after some work):

$$\beta = \frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\rho_2}{\rho_0} = \frac{1}{2} \left\langle [P_1 - \Sigma_1]^2 \left( \frac{\mathcal{E}_0}{E} + \frac{\rho}{\rho_0} \right) \right\rangle \quad \Rightarrow \quad \begin{cases} \text{No need for } P_3 \text{ to compute } \beta \text{ (or } \mathcal{E}_2). \\ \text{Important positivity property: } \beta \geq 0. \end{cases}$$



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## Choosing a model : link with gradient elasticity models

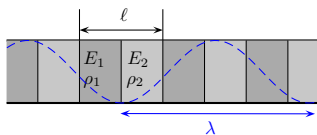
$$\frac{1}{c_0^2} \left[ 1 + \varepsilon^2 \beta_m \partial_{xx} + \varepsilon^2 \frac{\beta_t}{c_0^2} \partial_{tt} \right] U_{,tt} = \left[ 1 + \varepsilon^2 \beta_x \partial_{xx} \right] U_{,xx}, \quad \text{with } \beta_x - \beta_m - \beta_t = \beta > 0$$

	Gradient elasticity (see [Askes and Aifantis, 2011])	Two-scale homogenization	Properties
(x)	[Mindlin, 1964]	“natural” result when $\rho = 1$ [Fish and Chen, 2001]	unstable if $\beta_x > 0$ [Allaire et al., 2016]
(m)		[Fish et al., 2002, Lamacz, 2011]	stable if $\beta_m < 0$
(t)	?	?	?
(xm)	[Mindlin, 1964]	“natural” result for $\rho \neq 1$ Used to stabilize (x)	
(mt)	[Pichugin et al., 2008] [Forest and Sab, 2017]	[Cornaggia and Guzina, 2020] [Cornaggia and Lombard, 2023]	stable if $\beta_m < 0$ and $\beta_t > 0$
(xmt)	[Dontsov et al., 2013] [Schwan et al., 2021]	[Wautier and Guzina, 2015] [Schwan et al., 2021]	

$\beta_x \neq 0 \implies$  additional boundary conditions for bounded domains

- Well-posedness of the problem ? [Askes et al., 2008, Kaplunov and Pichugin, 2009]
- Difficult to deal with in homogenization theory (proofs when  $\varepsilon \rightarrow 0$  ?)
- Not very useful to fit dispersion curves [Schwan et al., 2021].

## Choosing a $(mt)$ model from dispersion relations in layered media



- **Bloch wave**  $u_\ell(X, t) = \tilde{u}(x)e^{i(\kappa x - \omega t)} \implies$  dispersion relation  $\omega = f(\kappa)$ .  
About  $(\omega, \kappa) = (0, 0)$  (on the acoustic branch):

$$\frac{\omega}{c_0} = \kappa \left[ 1 - \frac{\beta}{2}(\kappa\varepsilon)^2 + \frac{\beta(2 - 27\beta - 8\bar{\beta})}{40}(\kappa\varepsilon)^4 + O(\varepsilon^6) \right],$$

( $\bar{\beta}$  known for layered media)

- **Plane wave**  $U(x, t) = \tilde{U}e^{i(\kappa x - \omega t)}$  in the  $(mt)$  homogenized model:

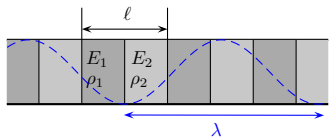
$$\frac{\omega}{c_0} = \kappa \left[ 1 + \frac{\beta_m + \beta_t}{2}(\kappa\varepsilon)^2 + \frac{(\beta_m + \beta_t)(3\beta_m + 7\beta_t)}{8}(\kappa\varepsilon)^4 + O(\varepsilon^6) \right].$$

- *Second-order approximation of  $\omega/c_0$  obtained for any  $(\beta_m, \beta_t)$  satisfying  $-\beta_m - \beta_t = \beta$ . Also true in 2-3D [Allaire et al., 2016].*

- *Fourth-order approximation for  $\beta_m = \frac{-1 - 4\beta + 4\bar{\beta}}{10}$  and  $\beta_t = \frac{1 - 6\beta - 4\bar{\beta}}{10}$*

Similar approximation for spring-mass lattice in [Pichugin et al., 2008]

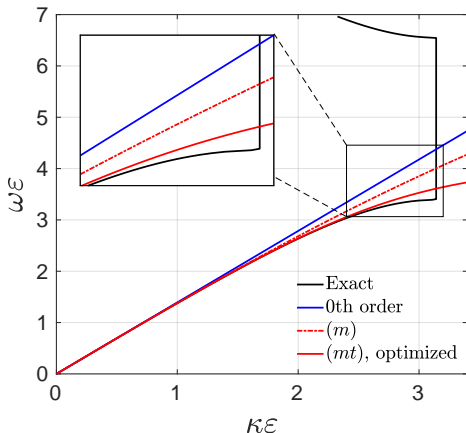
## Choosing a $(mt)$ model by fitting dispersion curves



Material contrasts:  $\frac{E_2}{E_1} = 6$ ,  $\frac{\rho_2}{\rho_1} = 1.5$

Phase ratio:  $\frac{\ell_2}{\ell_1} = 3$

( $\approx$  maximal dispersion [Santosa and Symes, 1991])



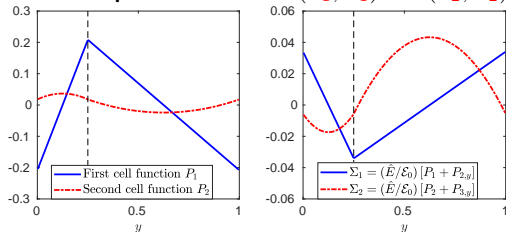
- Bloch wave:  $u(x, y, t) = \phi(y)e^{i(\kappa x - \omega t)}$   
 $\implies$  Dispersion relation  $\kappa = f(\omega)$
- 0th-order:  $U_{,tt} - c_0^2 U_{,xx} = 0$   
 $\implies$  non-dispersive
- 2nd-order  $(m)$ :  
 $\implies O((\kappa\mathcal{E})^2)$  approximation
- Optimized  $(mt)$ :  
 $\implies O((\kappa\mathcal{E})^4)$  approximation  
 [Pichugin et al., 2008,  
 Cornaggia and Guzina, 2020]

# Recap: second-order homogenization in one slide

From a given microstructure ...

$E_1$	$E_2 = 6E_1$
$\rho_1$	$\rho_2 = 1.5\rho_1$
1/4	3/4

... **1 - Compute cell solutions**  $(P_1, \Sigma_1)$  and  $(P_2, \Sigma_2)$



**2 - Choose a model** (e.g.  $(mt)$ ) and compute effective coefficients (e.g.  $c_0^2, \beta, \beta_m, \beta_t$ )

$$\frac{1}{c_0^2} U_{,tt} - U_{,xx} + \varepsilon^2 \left( \frac{\beta_m}{c_0^2} U_{,xxtt} + \frac{\beta_t}{c_0^4} U_{,tttt} \right) = 0,$$

**3 - Solve the effective wave equation** to find the macroscopic field  $U$

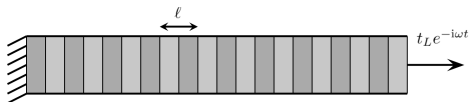
**4 - Add bulk correctors** to obtain approximations :

$$\begin{cases} u_\ell(x, t) \approx U(x, t) + \varepsilon U_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 U_{,xx}(x, t) P_2(x/\varepsilon) \\ \sigma_\ell(x, t) \approx S(x, t) + \varepsilon S_{,x}(x, t) \Sigma_1(x/\varepsilon) + \varepsilon^2 S_{,xx}(x, t) \Sigma_2(x/\varepsilon) \end{cases} \quad \text{with } S = \mathcal{E}_0 U_{,x}$$

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- 2 Second-order asymptotic homogenization for waves in 1D domains
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  - Choosing a model
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- 5 Conclusions and perspectives

## Model problem and leading-order approximation



- Time-harmonic model problem:

$$\begin{cases} [E(x/\varepsilon)u, x]_{,x} + \rho(x/\varepsilon)\omega^2 u = 0 & x \in Y_L := ]0, L[ \\ u = 0 & x = 0 \\ \sigma = E(x)u, x = \sigma_L & x = L \end{cases}$$

- Leading-order homogenization:  $(u, \sigma) \rightarrow (U, \mathcal{E}_0 U, x)$

$$\begin{cases} U, xx + k_0^2 U = 0 & x \in Y_L, \\ U = 0 & x = 0 \\ U, x = \sigma_L / \mathcal{E}_0 & x = L \end{cases} \quad k_0 := \omega / c_0$$

## Boundary correctors using full-field approximations [Cornaggia and Guzina, 2020]

First-order approximations of displacement and stress  $(u, \sigma)$ :

$$\begin{cases} \tilde{u}^{(1)}(x) = U(x) + \varepsilon P_1 \left( \frac{x}{\varepsilon} \right) U_{,x}(x) \\ \tilde{\sigma}^{(1)}(x) = \mathcal{E}_0 \left[ U_{,x}(x) + \varepsilon \Sigma_1 \left( \frac{x}{\varepsilon} \right) U_{,xx}(x) \right] \end{cases}$$

Using cell stress functions:  $\Sigma_j := (E/\mathcal{E}_0) [P_j + P_{j+1,y}]$

Boundary-value problem for the mean field  $U$ :

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ \tilde{u}^{(1)} = 0 & x = 0 \\ \tilde{\sigma}^{(1)} = \sigma_L & x = L \end{cases}$$

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} + \varepsilon \Sigma_1(0) U_{,xx} = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} - \varepsilon \Sigma_1(0) k_0^2 U = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$



## Second-order approximation

- Approximations of the displacement and stress  $(u, \sigma)$ :

$$\left. \begin{aligned} \tilde{u}^{(2)}(x) &= U(x) + \varepsilon P_1\left(\frac{x}{\varepsilon}\right)U_{,x}(x) + \varepsilon^2 P_2\left(\frac{x}{\varepsilon}\right)U_{,xx}(x), \\ \tilde{\sigma}^{(2)}(x) &= \mathcal{E}_0 \left[ U_{,x} + \varepsilon \Sigma_1\left(\frac{x}{\varepsilon}\right)U_{,xx}(x) + \varepsilon^2 \Sigma_2\left(\frac{x}{\varepsilon}\right)U_{,xxx}(x) \right] \end{aligned} \right\} U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$$

- Boundary-value-problem for the second-order mean field  $U$ :

$$\begin{cases} U_{,xx} + k_{mt}^2 U = 0 & x \in Y_L \\ \left[ 1 - \varepsilon^2 k_{mt}^2 P_2(0) \right] U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ \left[ 1 - \varepsilon^2 k_{mt}^2 \Sigma_2(0) \right] U_{,x} - \varepsilon k_{mt}^2 \Sigma_1(0) U = \sigma_L / \mathcal{E}_0 & x = L. \end{cases}$$

using the  $(mt)$  dispersive model:

$$k_{mt}^2 := k_0^2 \frac{1 - \varepsilon^2 \beta_t k_0^2}{1 + \varepsilon^2 \beta_m k_0^2}$$

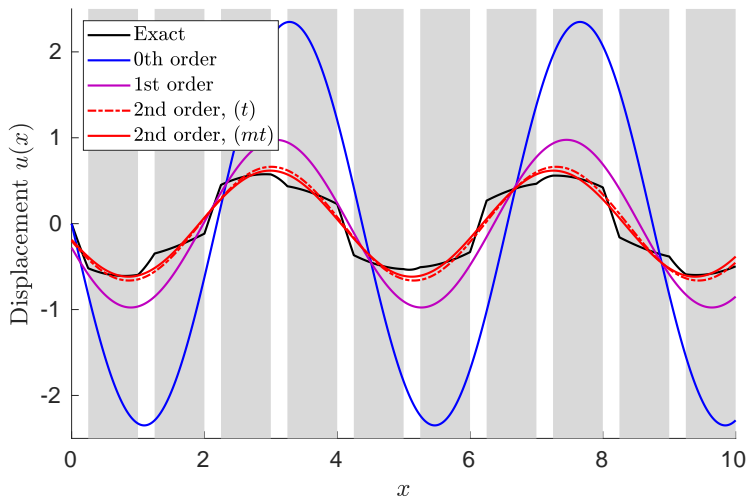
**Lemma 2:**  $\exists C > 0 \quad \|u - \tilde{u}^{(2)}\|_{H^1} \leq C \varepsilon^2 \|U\|_{H^1}.$

**Theorem 1:**  $\exists C > 0 \quad \|u - \tilde{u}^{(1)}\|_{L^2} \leq \|u - \tilde{u}^{(2)}\|_{L^2} + \|\tilde{u}^{(2)} - \tilde{u}^{(1)}\|_{L^2} \leq C \varepsilon^2 \|U\|_{H^2}.$

- ▶ No need for 2nd order BC to prove these results
- ▶ BCs surely needed for the next step : we hope that  $\|u - \tilde{u}^{(2)}\|_{L^2} = O(\varepsilon^3)$

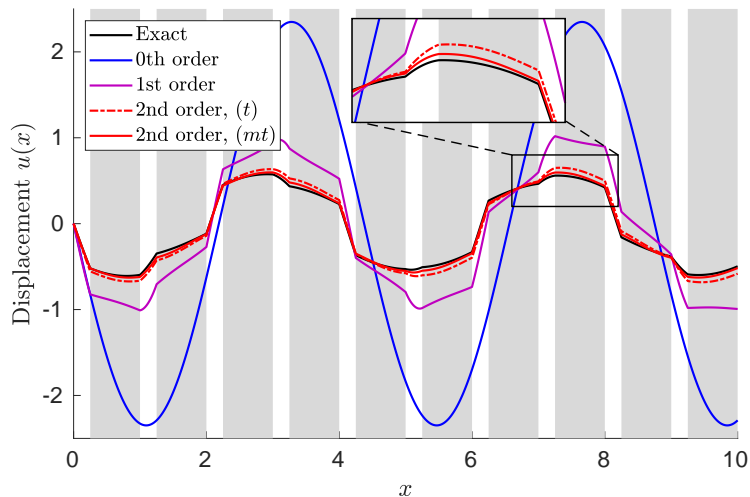
## Example for a layered material - mean fields $U$

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



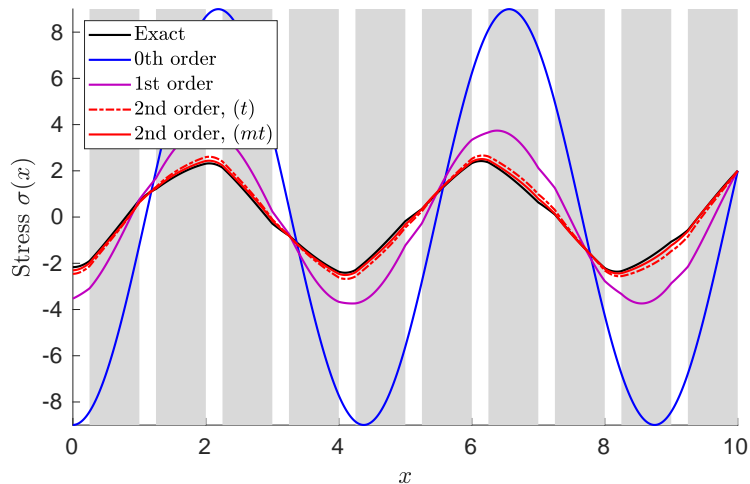
## Example for a layered material - total displacement $u$

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$

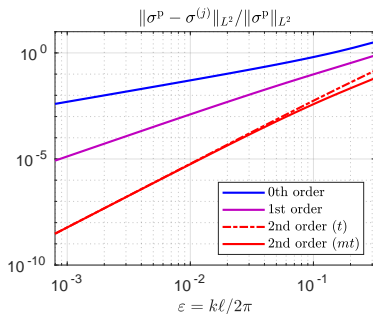
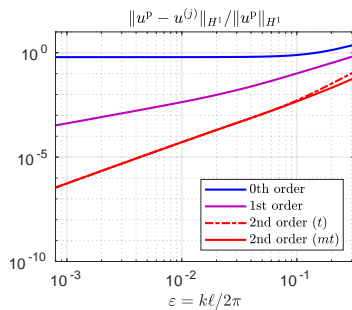
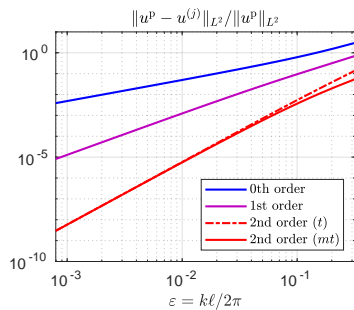


## Example for a layered material - axial stress $\sigma$

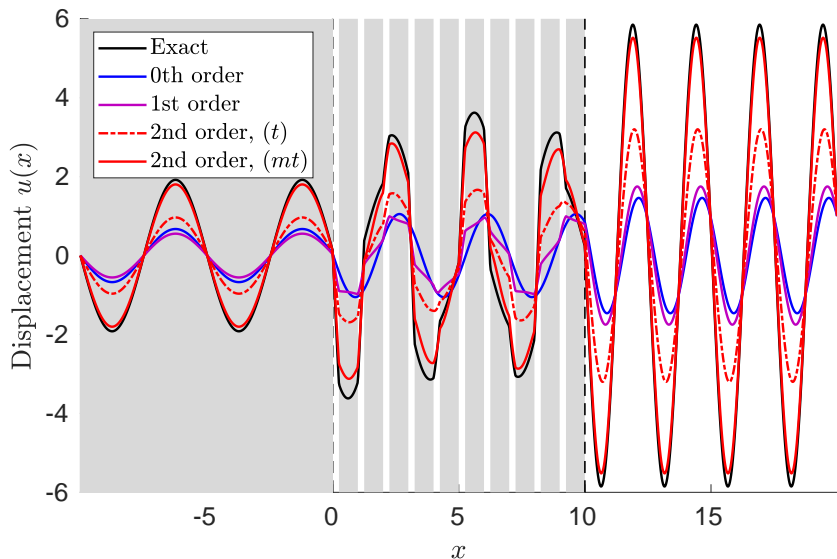
$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



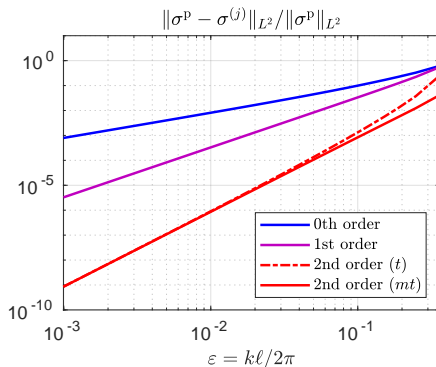
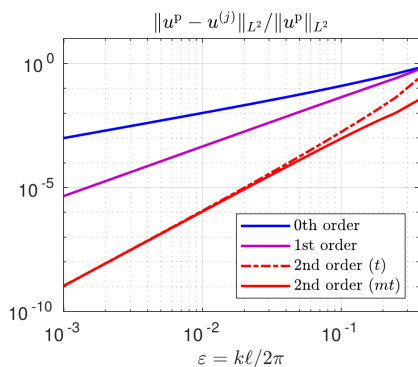
# Errors for fixed $\omega$ and $\ell \rightarrow 0$



## Application to a transmission problem ( $\varepsilon \approx 0.3$ )



# Transmission problem - errors for fixed $\omega$ and $\ell \rightarrow 0$



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## Equivalent hyperbolic system for the $(mt)$ model

( $\beta_x = 0$  because we don't want  $U_{,xxxx}$  and additional BC/TC it brings)

Second-order homogenization,  $(mt)$  model:

$$\frac{1}{c_0^2} \left[ 1 + \varepsilon^2 \beta_m \partial_{xx} + \varepsilon^2 \frac{\beta_t}{c_0^2} \partial_{tt} \right] U_{,tt} = U_{,xx},$$

“Stress gradient” system [Forest and Sab, 2017]:

$$\begin{cases} \sigma = E \partial_x (u + \phi) \\ r = D \phi \end{cases} \quad \begin{cases} \rho \partial_{tt} u = \partial_x \sigma \\ \rho J \partial_{tt} \phi = \partial_x \sigma - r \end{cases}$$

So that:  $\frac{\rho}{E} \left[ 1 - \frac{E(1+J)}{D} \partial_{xx} + \frac{\rho J}{D} \partial_{tt} \right] u_{,tt} = u_{,xx}$

$$\Rightarrow \begin{cases} \partial_t V - \frac{a}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r \\ \partial_t S - \mathcal{E}_0 \partial_x V & = 0 \\ \partial_t \varphi - \frac{a-1}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r \\ \partial_t r & = \frac{\mathcal{E}_0}{\varepsilon^2 \beta} \varphi \end{cases}$$

- Macroscopic fields  $V = \partial_t(u + \phi)$  and  $S = \sigma$
- **Auxiliary fields**  $\varphi = \partial_t \phi$  and  $r$
- Parameter  $a = -\beta_m / \beta_t$  to select a  $(mt)$  model.

**Well-posedness** (hyperbolicity, stability, conserved positive energy) for:

$$\beta \geq 0 \quad (\checkmark), \quad \beta_m < 0 \quad \text{and} \quad \beta_t > 0$$

## Hyperbolicity and stability (See B. Lombard's first lecture)

- Matricial form of system:

$$\partial_t \mathbf{U} + \mathbf{A} \cdot \partial_x \mathbf{U} = \mathbf{S} \cdot \mathbf{U}, \quad \text{where } \mathbf{U} := (V, S, \varphi, r)^T$$

$$\mathbf{A} = \begin{bmatrix} 0 & -a/\varrho_0 & 0 & 0 \\ -\mathcal{E}_0 & 0 & 0 & 0 \\ 0 & -(a-1)/\varrho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & -(a-1)/\varrho_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(a-1)/\varrho_0 \\ 0 & 0 & \mathcal{E}_0/(\varepsilon^2 \beta) & 0 \end{bmatrix}.$$

- Hyperbolicity : **real** eigenvalues of  $\mathbf{A}$

$$\text{Sp}(\mathbf{A}) = \{0, 0, \pm c_0 \sqrt{a}\}$$

- Stability : **imaginary** eigenvalues of  $\mathbf{S}$

$$\text{Sp}(\mathbf{S}) = \{0, 0, \pm c_0 / (\varepsilon \sqrt{-\beta_t})\}$$

$$\left\{ \begin{array}{l} \text{Hyperbolicity: } a > 0 \\ \text{Stability: } \beta_t > 0 \end{array} \right\} \implies \left\{ \begin{array}{l} -\frac{\beta_m}{\beta_t} > 0 \\ \beta_t > 0 \end{array} \right\} \implies \left\{ \begin{array}{l} \beta_m < 0, \\ \beta_t > 0. \end{array} \right.$$

# Symmetrizer and energy conservation for $\partial_t U + A \cdot \partial_x U = S \cdot U$

- Symmetrizer matrix:

$$M = \begin{bmatrix} \rho_0 & 0 & -\rho_0 & 0 \\ 0 & 1/\varepsilon_0 & 0 & 0 \\ -\rho_0 & 0 & a\rho_0/(a-1) & 0 \\ 0 & 0 & 0 & \varepsilon^2\beta/\varepsilon_0 \end{bmatrix},$$

So that:

$$M \cdot A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M \cdot S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

- Multiplying the system by  $U^T \cdot M$ :

$$\partial_t \frac{1}{2} (U^T \cdot M \cdot U) + \frac{1}{2} \partial_x (U^T \cdot M \cdot A \cdot U) = 0.$$

Integrating over  $\Omega \times [0, T]$ , (large  $\Omega$  so that  $U(\cdot, T) = \mathbf{0}$  on  $\partial\Omega$ )

**Energy conservation:**  $\mathcal{E}^{\text{vol}}(T) = \mathcal{E}^{\text{vol}}(0)$ .

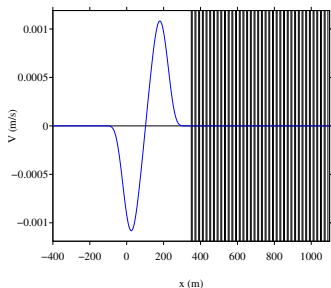
$$\mathcal{E}^{\text{vol}} = \frac{1}{2} \int_{\Omega} U^T \cdot M \cdot U \, dx = \frac{1}{2} \int_{\Omega} \rho_0 \left( v^2 + \frac{\beta_t}{\beta} \varphi^2 \right) + \frac{1}{\varepsilon_0} (\sigma^2 + \varepsilon^2 \beta r^2) \, dx \geq 0 \text{ if } \beta \geq 0$$

(with  $v = V - \varphi$ )

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# First-order transmission conditions



- Fields of interest:

$$\begin{cases} (V, S) = (\partial_t U, E - \partial_x U) & x < 0 \\ (V, S, \varphi, r) & x > 0 \end{cases}$$

- Transmission conditions from **total fields continuity**:

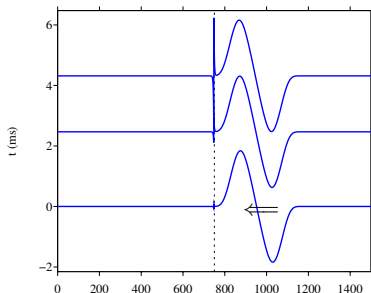
$$\begin{cases} V(0^-) = V(0^+) + \varepsilon P_1(0) V_{,x}(0^+) \\ S(0^-) = S(0^+) + \varepsilon \Sigma_1(0) S_{,x}(0^+) \end{cases}$$

- Energy analysis : **interface terms** when integrating  $\partial_x(\mathbf{U}^T \cdot \mathbf{M} \cdot \mathbf{A} \cdot \mathbf{U})$  over  $\Omega_- \cup \Omega_+$

$$\partial_t(\mathcal{E}_-^{\text{vol}} + \mathcal{E}_+^{\text{vol}}) + \mathcal{D}^{\text{int}} = 0,$$

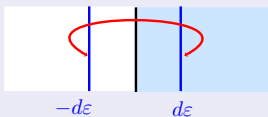
$$\begin{aligned} \mathcal{D}^{\text{int}} &= \llbracket V \rrbracket_0 \langle S \rangle_0 + \langle V \rangle_0 \llbracket S \rrbracket_0 \\ &= -\varepsilon [P_1(0)(\partial_x V)^+ \langle S \rangle_0 + \Sigma_1(0)(\partial_x S)^+ \langle V \rangle_0] \\ &= ??? \end{aligned}$$

- We would like  $\mathcal{D}^{\text{int}} = \partial_t \mathcal{E}^{\text{int}}$  with  $\mathcal{E}_1^{\text{int}} \geq 0$



# First-order transmission conditions - stabilization

Spring-mass conditions across a thin interface  $I_d = [-d\varepsilon, d\varepsilon]$  (additional  $o(\varepsilon)$  error):



$$\begin{cases} \llbracket V \rrbracket_{I_d} = \varepsilon A_1 \partial_t \langle S \rangle_{I_d} & A_1 = -P_1(0) \mathcal{E}_0^{-1} + d(E_-^{-1} + \mathcal{E}_0^{-1}) \\ \llbracket S \rrbracket_{I_d} = \varepsilon B_1 \partial_t \langle V \rangle_{I_d} & B_1 = -\Sigma_1(0) \varrho_0 + d(\rho_- + \varrho_0) \end{cases}$$

- New expression of interface term:

$$\partial_t (\mathcal{E}_-^{\text{vol}} + \mathcal{E}_+^{\text{vol}}) + \mathcal{D}_d^{\text{int}} = 0,$$

$$\mathcal{D}_d^{\text{int}} = \partial_t \mathcal{E}_d^{\text{int}} \quad \text{with} \quad \mathcal{E}_d^{\text{int}} = \frac{\varepsilon}{2} \left[ A_1 \langle S \rangle_{I_d}^2 + B_1 \langle V \rangle_{I_d}^2 \right]$$

- **Stability** of {Hyperbolic system + transmission conditions} for  $A_1 \geq 0$  and  $B_1 \geq 0$

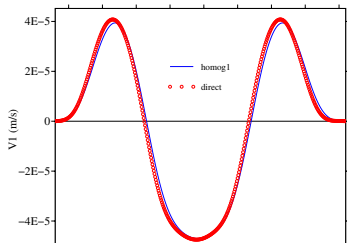
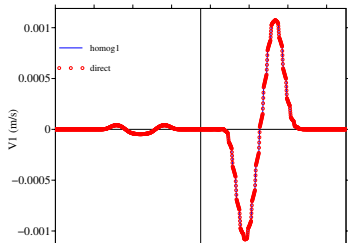
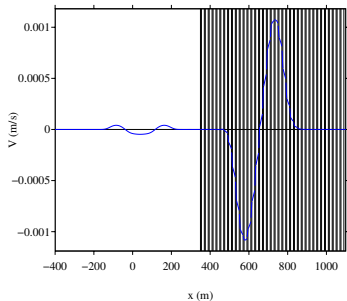
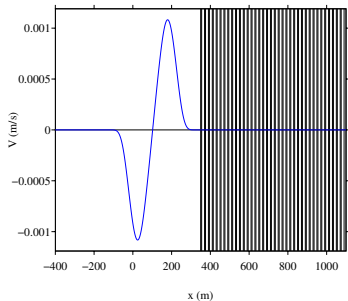
$$\Rightarrow \text{Choice of } d \geq d_{\min} := \max \left( \frac{E_- P_1(0)}{E_- + \mathcal{E}_0}, \frac{\varrho_0 \Sigma_1(0)}{\rho_- + \varrho_0}, 0 \right)$$

**Well-posed “hybrid” model:**

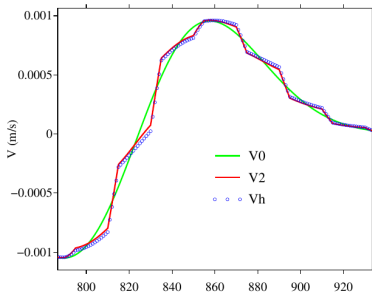
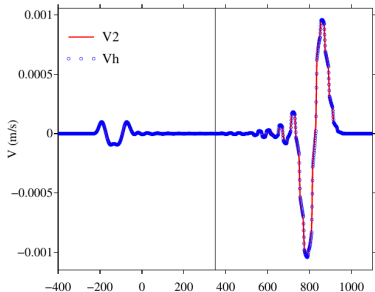
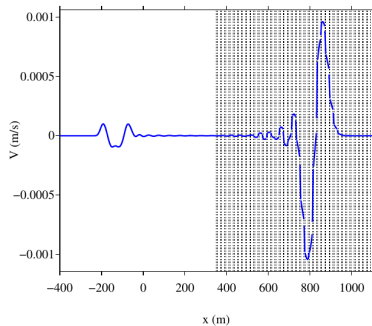
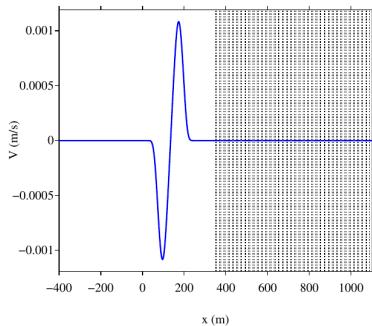
- **dispersive system** from 2nd-order bulk homogenization
- **transmission conditions** (1st-order for now)

# Transmission conditions for $\varepsilon = \ell/\lambda_c \approx 0.04$ (Numerics by B. Lombard)

$E_{-\rho_-} = \mathcal{E}_0 e_0 \implies$  reflected wave is a 1st-order effect

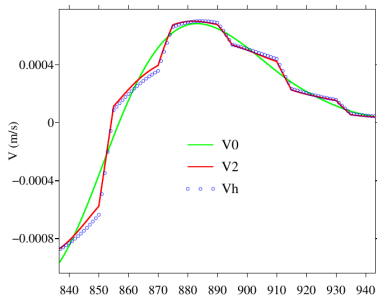
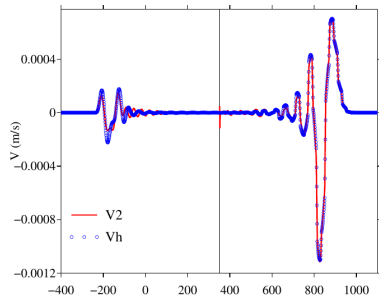
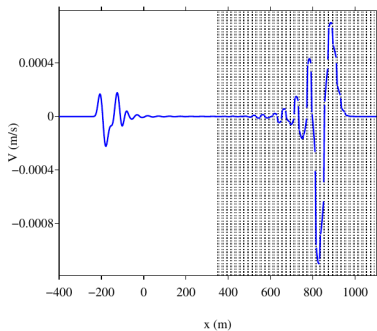
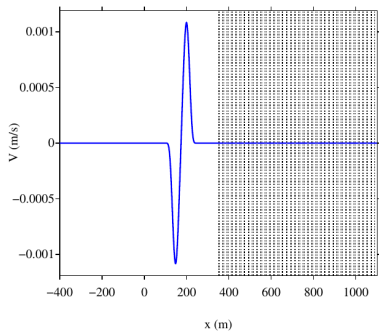


# Transmission conditions, increasing central frequency, $\varepsilon \approx 0.09$

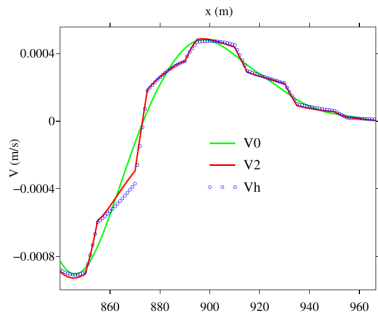
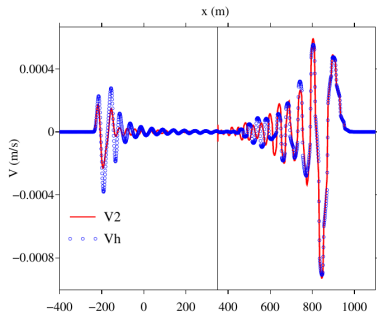
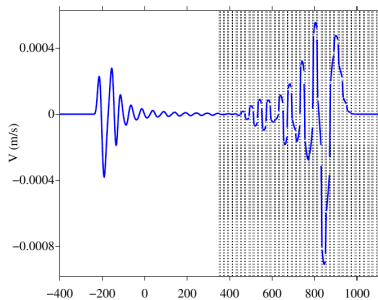
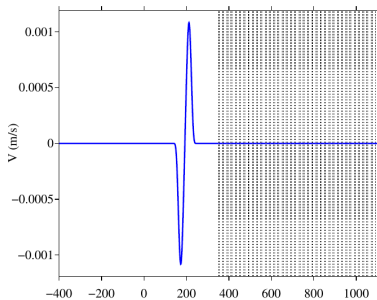




# Transmission conditions, increasing central frequency, $\varepsilon \approx 0.13$



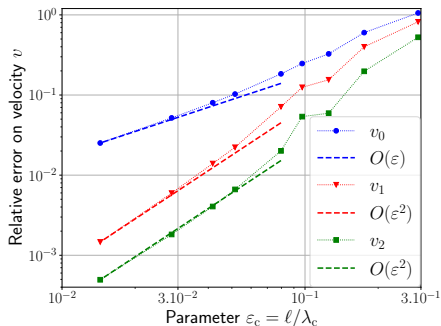
# Transmission conditions, increasing central frequency, $\varepsilon \approx 0.17$



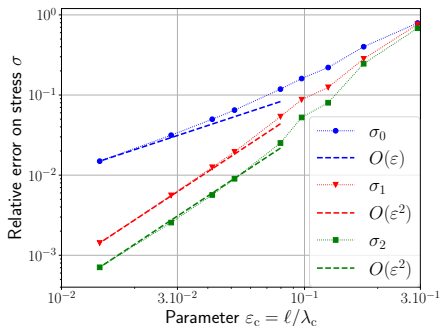
## Transmission conditions, errors on velocity and stress fields

- Fields emitted at source  $x_s < 0$ , measured at receiver  $x_r > 0$  during time  $T$ .
- Increasing frequency  $f$ , constant microstructure periodicity  $\ell$ .
- Positions  $(x_s, x_r)$  and time  $T$  accordingly rescaled for each frequency.

$$\frac{\|(v_h - v_j)(x_r, \cdot)\|_{L^2(0,T)}}{\|v_h(x_r, \cdot)\|_{L^2(0,T)}}$$



$$\frac{\|(\sigma_h - \sigma_j)(x_r, \cdot)\|_{L^2(0,T)}}{\|\sigma_h(x_r, \cdot)\|_{L^2(0,T)}}$$



$(v_1, \sigma_1)$ : Adding 1st-order bulk correctors  $(P_1, \Sigma_1)$  and 1st-order TC

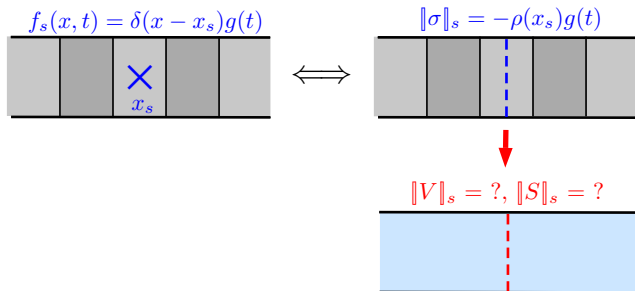
$(v_2, \sigma_2)$ : “Hybrid” dispersive system (but still 1st-order TC)

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In 1D, source points  $\Leftrightarrow$  interfaces with stress jumps

$$\begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = \delta_s g(t) & x \in \mathbb{R} \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 & x \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = 0 & x < x_s \text{ and } x > x_s, \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 & x < x_s \text{ and } x > x_s, \\ \llbracket v_\ell \rrbracket_s = 0 & x = x_s, \\ \llbracket \sigma_\ell \rrbracket_s = -\rho_s g & x = x_s, \end{cases}$$



## First-order correction

- **Same tools** (and no need for an enlarged interface):

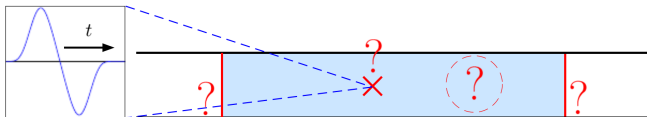
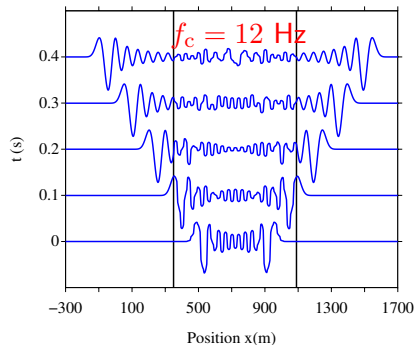
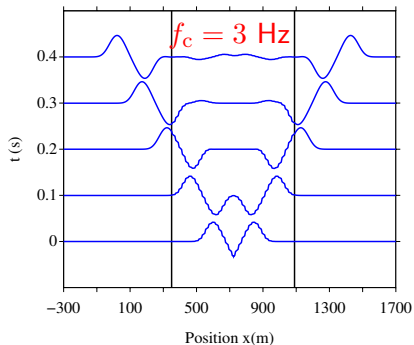
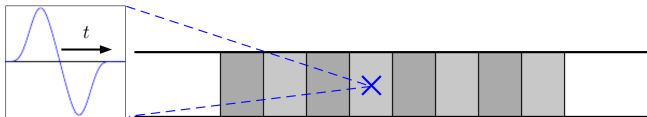
$$\begin{cases} \llbracket V \rrbracket_s = \varepsilon P_1(x_s/\varepsilon) \frac{\rho_s}{\mathcal{E}_0} \partial_t g, \\ \llbracket S \rrbracket_s = -\rho_s g, \end{cases}$$

- ▶ **First-order** corrector on  $V$  involving  $\partial_t g$  and the local values  $\rho_s$  and  $P_1(x_s/\varepsilon)$ .
- ▶ Already given by [Capdeville et al., 2010] with another argument.

- Equivalent system with Dirac notation:

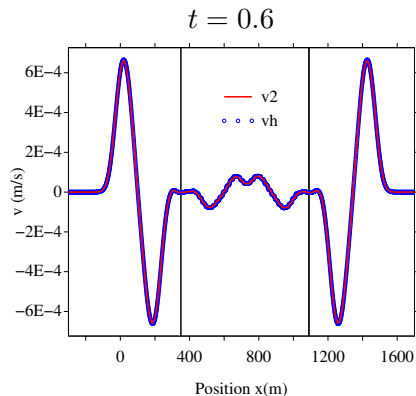
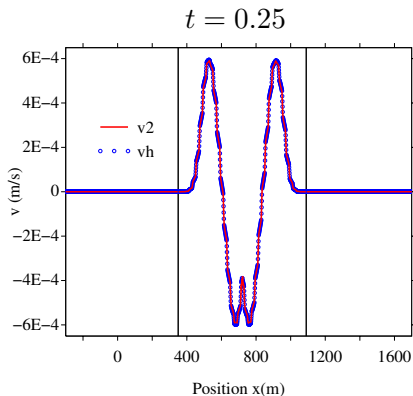
$$\begin{cases} \partial_t V - \frac{a}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r + \frac{a}{\varrho_0} \rho_s g \delta_s, \\ \partial_t S - \mathcal{E}_0 \partial_x V & = -\varepsilon \rho_s P_1(x_s/\varepsilon) \partial_t g \delta_s, \\ \partial_t \varphi - \frac{a-1}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r + \frac{a-1}{\varrho_0} \rho_s g \delta_s, \\ \partial_t r & = \frac{\mathcal{E}_0}{\ell^2 \beta} \varphi. \end{cases}$$

## Back to the initial motivation



# Velocity fields in a slab with a source point

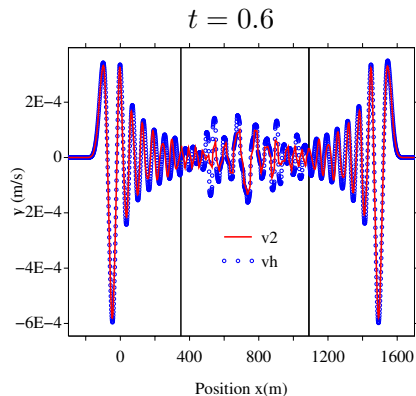
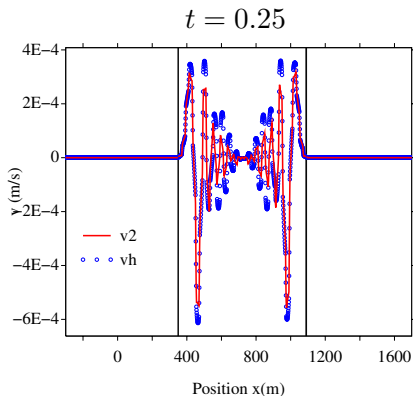
$$f_c = 3 \text{ Hz}, \varepsilon_c = 0.04$$





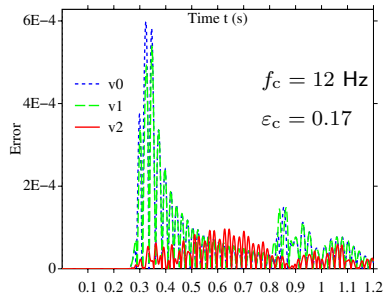
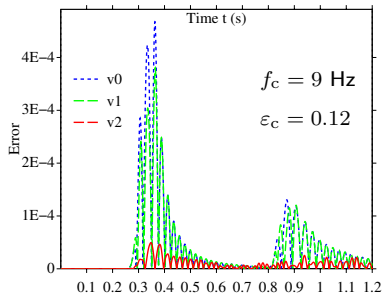
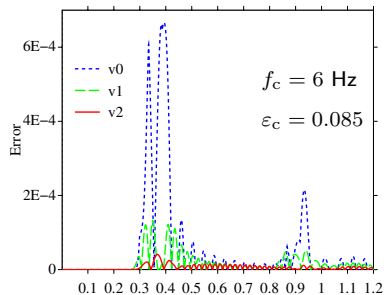
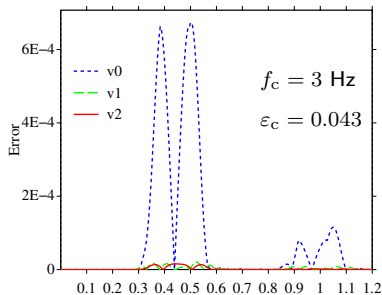
# Velocity fields in a slab with a source point

$$f_c = 12 \text{ Hz}, \varepsilon_c = 0.17$$



# Slab with a source point - errors at several frequencies

receiver in left homogeneous domain



## Key ideas

- The second-order asymptotic homogenization accounts for dispersive effects.
- Reciprocity identities between cell problems are a big help.
- Boundary and transmission conditions can be designed to complement the inner expansion. (including in 2-3D [Vinoles, 2016, Maurel and Marigo, 2018, Cakoni et al., 2019, Beneteau, 2021] ...)
- For transient waves, **stability** can be addressed using (i) hyperbolic formalism and (ii) enlarged interfaces (in 1D for now).
- Source points can be addressed using the same tools (specific to 1D).

## Perspectives

- Pursue the transient case up to *full second-order model* (as done in the time-harmonic case [Cornaggia and Guzina, 2020])
- Apply the built framework to other 1D configurations:
  - ▶ High-frequency homogenization [Craster et al., 2010, Guzina et al., 2019, Assier et al., 2020]
  - ▶ Solids with inner imperfect interfaces [Assier et al., 2020]
  - ▶ Non-linearities [Bellis et al., 2021]
- Reciprocity identities for higher dimensions and **elasticity** [Boutin and Auriault, 1993] (*ongoing*).
- Hyperbolic formalism for higher dimensions and other models ? (2D acoustics, elasticity ...)

Thanks for your attention !

*Second-order homogenization of boundary and transmission conditions  
for one-dimensional waves in periodic media*

Rémi Cornaggia, Bojan B. Guzina

International Journal of Solids and Structures, 2020

*An homogenized model accounting for dispersion, interfaces and source points  
for transient waves in 1D periodic media*

Rémi Cornaggia, Bruno Lombard

ESAIM: Mathematical Modelling and Numerical Analysis (M2AN), 2023

# References



Abdulle, A. and Pouchon, T. (2020).  
Effective models and numerical homogenization for wave propagation in heterogeneous media on arbitrary timescales.  
*Foundations of Computational Mathematics*, 20(6):1505–1547.



Allaire, G. and Amar, M. (1999).  
Boundary layer tails in periodic homogenization.  
*ESAIM: Control, Optimisation and Calculus of Variations*, 4:209–243.



Allaire, G., Briane, M., and Vanninathan, M. (2016).  
A comparison between two-scale asymptotic expansions and Bloch wave expansions for the homogenization of periodic structures.  
*SeMA Journal*, 73(3):237–259.



Allaire, G. and Yamada, T. (2018).  
Optimization of dispersive coefficients in the homogenization of the wave equation in periodic structures.  
*Numerische Mathematik*, 140(2):265–326.



Armstrong, S., Kuusi, T., Mourrat, J.-C., and Prange, C. (2017).  
Quantitative analysis of boundary layers in periodic homogenization.  
*Archive for Rational Mechanics and Analysis*, 226(2):695–741.



Askes, H. and Aifantis, E. C. (2011).  
Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results.  
*International Journal of Solids and Structures*, 48(13):1962–1990.



Askes, H., Metrikine, A., Pichugin, A., and Bennett, T. (2008).  
Four simplified gradient elasticity models for the simulation of dispersive wave propagation.

*Philosophical Magazine*, 88(28-29):3415–3443.



Assier, R. C., Touboul, M., Lombard, B., and Bellis, C. (2020).  
High-frequency homogenization in periodic media with imperfect interfaces.  
*Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 476(2244):20200402.



Bellis, C., Lombard, B., Touboul, M., and Assier, R. (2021).  
Effective dynamics for low-amplitude transient elastic waves in a 1D periodic array of non-linear interfaces.  
*Journal of the Mechanics and Physics of Solids*, 149:104321.



Beneteau, C. (2021).  
*Modèles homogénéisés enrichis en présence de bords : Analyse et traitement numérique*.  
Ph. D. thesis, Institut Polytechnique de Paris.



Bensoussan, A., Lions, J. L., and Papanicolau, G. (1978).  
*Asymptotic Analysis for Periodic Structures*.  
North-Holland, Amsterdam.



Boutin, C. and Auriault, J. (1993).  
Rayleigh scattering in elastic composite materials.  
*International Journal of Engineering Science*, 31(12):1669–1689.



Cakoni, F., Guzina, B. B., Moskow, S., and Pangburn, T. (2019).  
Scattering by a bounded highly oscillating periodic medium and the effect of boundary correctors.  
*SIAM Journal on Applied Mathematics*, 79(4):1448–1474.



Capdeville, Y., Guillot, L., and Marigo, J.-J. (2010).  
1-D non-periodic homogenization for the seismic wave equation.  
*Geophysical Journal International*.



Celli, P. and Gonella, S. (2015).  
Manipulating waves with lego bricks: A versatile experimental platform for metamaterial architectures.  
*Applied Physics Letters*, 107(8).



Cioranescu, D. and Donato, P. (1999).  
*An introduction to homogenization*, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*.  
Oxford University Press.



Cornaggia, R. and Guzina, B. B. (2020).  
Second-order homogenization of boundary and transmission conditions for one-dimensional waves in periodic media.  
*International Journal of Solids and Structures*, 188-189:88 – 102.



Cornaggia, R. and Lombard, B. (2023).  
A homogenized model accounting for dispersion, interfaces and source points for transient waves in 1d periodic media.  
*ESAIM: Mathematical Modelling and Numerical Analysis*, 57(3):1413–1444.



Craster, R. V., Kaplunov, J., and Pichugin, A. V. (2010).  
High-frequency homogenization for periodic media.  
*Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 466(2120):2341–2362.



Dontsov, E. V., Tokmashev, R. D., and Guzina, B. B. (2013).  
A physical perspective of the length scales in gradient elasticity through the prism of wave dispersion.  
*International Journal of Solids and Structures*, 50(22-23):3674 – 3684.



Dumontet, H. (1986).  
Study of a boundary layer problem in elastic composite materials.  
*ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 20(2):265–286.



Fish, J. and Chen, W. (2001).

Higher-order homogenization of initial/boundary-value problem.  
*Journal of Engineering Mechanics*, 127(12):1223–1230.



Fish, J., Chen, W., and Nagai, G. (2002).  
Non-local dispersive model for wave propagation in heterogeneous media: one-dimensional case.  
*International Journal for Numerical Methods in Engineering*, 54(3):331–346.



Forest, S. and Sab, K. (2017).  
Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models.  
*Mathematics and Mechanics of Solids*, page 108128651772084.



Gérard-Varet, D. and Masmoudi, N. (2012).  
Homogenization and boundary layers.  
*Acta Mathematica*, 209(1):133–178.



Guzina, B. B., Meng, S., and Oudghiri-Idrissi, O. (2019).  
A rational framework for dynamic homogenization at finite wavelengths and frequencies.  
*Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 475(2223):20180547.



Huang, Y., Liu, S., and Zhao, J. (2016).  
A gradient-based optimization method for the design of layered phononic band-gap materials.  
*Acta Mechanica Sinica*, 29(4):429 – 443.



Kaplunov, J. and Pichugin, A. (2009).  
On rational boundary conditions for higher-order long-wave models.  
In Borodich, F., editor, *IUTAM Symposium on Scaling in Solid Mechanics*, volume 10 of *IUTAM Bookseries*, pages 81–90. Springer Netherlands.



Kook, J. and Jensen, J. S. (2017).  
Topology optimization of periodic microstructures for enhanced loss factor using acoustic-structure interaction.



Lamacz, A. (2011).

Dispersive effective models for waves in heterogeneous media.

*Mathematical Models and Methods in Applied Sciences*, 21(09):1871–1899.



Marigo, J.-J. and Maurel, A. (2017).

Second order homogenization of subwavelength stratified media including finite size effect.

*SIAM Journal on Applied Mathematics*, 77(2):721–743.



Maurel, A. and Marigo, J.-J. (2018).

Sensitivity of a dielectric layered structure on a scale below the periodicity: A fully local homogenized model.

*Physical Review B*, 98(2).



Mindlin, R. (1964).

Micro-structure in linear elasticity.

*Archive for Rational Mechanics and Analysis*, 16:51–78.



Moskow, S. and Vogelius, M. (1997).

First-order corrections to the homogenized eigenvalues of a periodic composite medium: the case of Neumann boundary conditions.

*Preprint Rutgers University*.



Noguchi, Y., Yamada, T., Izui, K., and Nishiwaki, S. (2018).

Topology optimization for hyperbolic acoustic metamaterials using a high-frequency homogenization method.

*Computer Methods in Applied Mechanics and Engineering*, 335:419–471.



Palermo, A., Krödel, S., Marzani, A., and Daraio, C. (2016).

Engineered metabarrier as shield from seismic surface waves.

*Scientific Reports*, 6(1).



Pichugin, A., Askes, H., and Tyas, A. (2008).

Asymptotic equivalence of homogenisation procedures and fine-tuning of continuum theories.

*Journal of Sound and Vibration*, 313(3-5):858 – 874.



Rosi, G. and Auffray, N. (2019).

Continuum modelling of frequency dependent acoustic beam focusing and steering in hexagonal lattices.

*European Journal of Mechanics - A/Solids*, page 103803.



Sanchez-Palencia, E. (1974).

Comportements local et macroscopique d'un type de milieux physiques hétérogènes.

*International Journal of Engineering Science*, 12(4):331 – 351.



Santosa, F. and Symes, W. W. (1991).

A dispersive effective medium for wave propagation in periodic composites.

*SIAM Journal on Applied Mathematics*, 51(4):984–1005.



Schwan, L., Favrie, N., Cottereau, R., and Lombard, B. (2021).

Extended stress gradient elastodynamics: Wave dispersion and micro-macro identification of parameters.

*International Journal of Solids and Structures*, 219-220:34–50.



Vinoles, V. (2016).

*Interface problems with metamaterials : modelling, analysis and simulations*.

Ph. D. thesis, ENSTA, Université Paris-Saclay.



Vondřejc, J., Rohan, E., and Heczeko, J. (2017).

Shape optimization of phononic band gap structures using the homogenization approach.

*International Journal of Solids and Structures*, 113-114:147 – 168.



Wautier, A. and Guzina, B. B. (2015).

On the second-order homogenization of wave motion in periodic media and the sound of a chessboard.

*Journal of the Mechanics and Physics of Solids*, 78:382 – 414.