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Homogenisation for waves in 1D periodic media: Effective models, boundary and transmission correctors

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► **To cite this version:**

Rémi Cornaggia. Homogenisation for waves in 1D periodic media: Effective models, boundary and transmission correctors. Doctoral. C2305: Wave Motion in Heterogeneous Media: Analysis, Modeling and Design, CISM, Udine, Italy. 2023. hal-04186509

HAL Id: hal-04186509

<https://hal.science/hal-04186509v1>

Submitted on 23 Aug 2023

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Theme 1:

Homogenisation for waves in 1D periodic media

Effective models, boundary and transmission correctors

Rémi Cornaggia¹

Joint work with Bojan B. Guzina² and Bruno Lombard³

CISM Course C2305: *Wave Motion in Heterogeneous Media: Analysis, Modeling and Design*
June 19-23, 2023

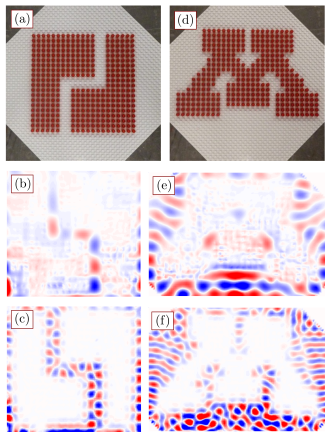


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Control of wave propagation using periodic structures

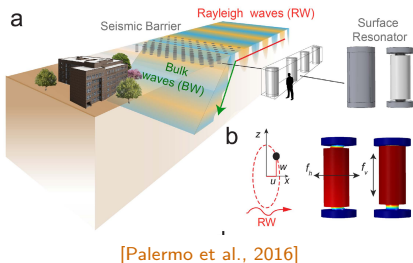


[Celli and Gonella, 2015]

... to design waveguides

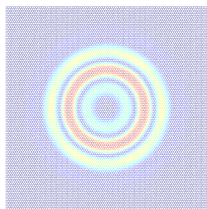
- with resonators for long wavelengths / low frequencies
- with Bragg effects for medium wavelengths / frequencies

... to design seismic protections (with resonators)



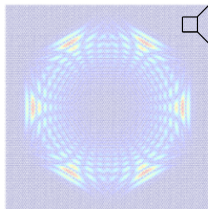
... and more (negative effective properties, cloaking ...)

A motivation : dispersion and anisotropy in 2D elasticity [Rosi and Auffray, 2019]

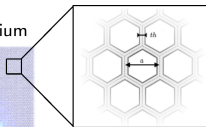


$$\frac{\ell}{\lambda_c} \ll 1$$

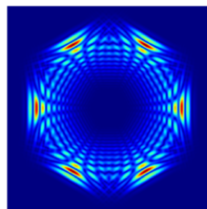
Microstructured medium



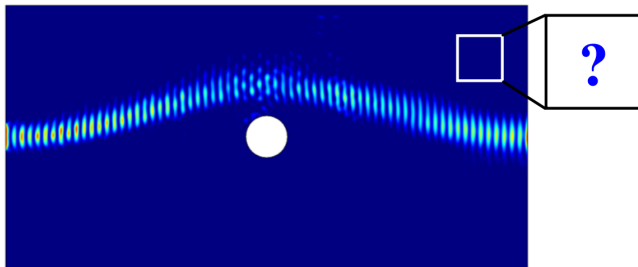
$$\frac{\ell}{\lambda_c} \approx \frac{1}{6}$$



Continuum model



Effective (strain-gradient) model.



Optimized effective model \Rightarrow "de-homogenization" ?

Homogenisation and interface conditions

How to model the wave propagation in periodic structures ?

- Accounting for all interactions \rightarrow costly (*dedicated numerical methods* out-of scope)
- *Floquet-Bloch* theory \implies See other lectures.
- *Homogenization* (/averaging/effective) methods and models
- **Both lectures:** *asymptotic two-scale homogenization*
 - ▶ Formal procedure to obtain effective models
 - ▶ Accuracy can be improved by pushing the expansion to higher orders
 - ▶ Extensions to *high/finite frequencies* [Craster et al., 2010, Guzina et al., 2019]
 \implies R. Craster and B. Guzina's lectures

How to account for boundaries and interfaces ?

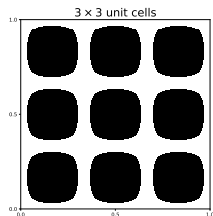
- In 2-3D, *boundary layers* must be accounted for at boundaries and interfaces [Dumontet, 1986, Moskow and Vogelius, 1997, Allaire and Amar, 1999, Gérard-Varet and Masmoudi, 2012, Armstrong et al., 2017, Cakoni et al., 2019] ...
- *Equivalent boundary or transmission conditions* preserving the asymptotics are difficult to build [Vinales, 2016, Marigo and Maurel, 2017, Maurel and Marigo, 2018, Beneteau, 2021]
- **This lecture:** 1D waves, “interfaces” are *points*.

Optimization

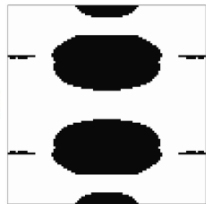
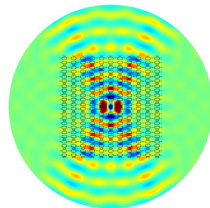
How to obtain desired properties ?

- Known design (cylinders, spheres, cones, LEGO bricks ...) \implies *parameter optimization* [Huang et al., 2016, Palermo et al., 2016] ...
- Given materials, unknown design \implies *topological optimization* [Vondřejc et al., 2017, Kook and Jensen, 2017, Allaire and Yamada, 2018] ...

Next lectures: topological optimization of microstructured 2D media and interfaces (long wavelengths, no resonances)

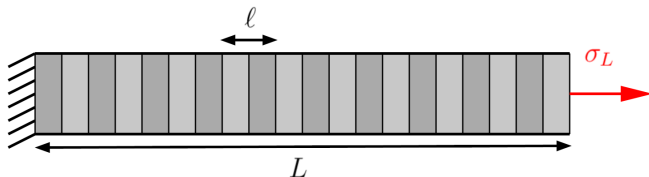


what we will see



[Noguchi et al., 2018] high frequency, hyperbolic propagation, see R. Craster's lecture

A preliminary example : 1D statics



- **Two-scale problem** : L (macro) and ℓ (micro)
- **Scale factor** $\varepsilon = \ell/L$
- ℓ -periodic properties $E_\ell(X) = E(X/\ell) = E(x/\varepsilon)$: E is 1-periodic
- Normalized coordinate $x = X/L$

$$\left\{ \begin{array}{lll} \partial_x \sigma(x) = 0 & x \in [0, 1] & \text{equilibrium} \\ \sigma(x) = E(x/\varepsilon) \partial_x u & x \in [0, 1] & \text{constitutive law} \\ u(0) = 0 & & \text{displacement BC} \\ \sigma(1) = \sigma_L & & \text{stress BC} \end{array} \right.$$

See dedicated Jupyter Notebook.

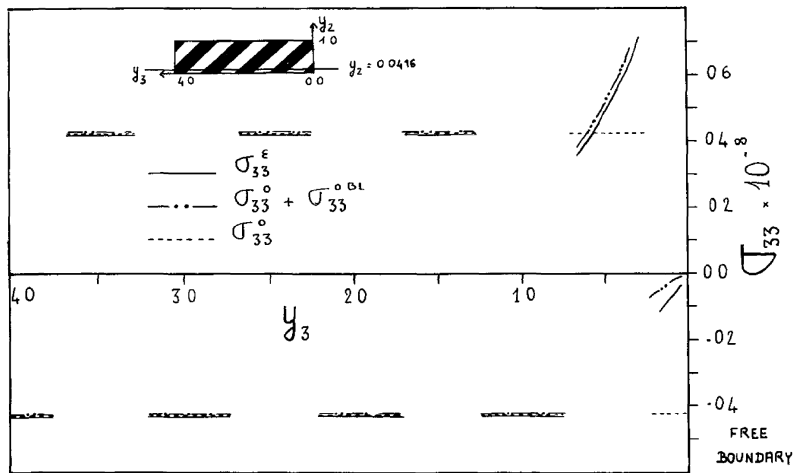
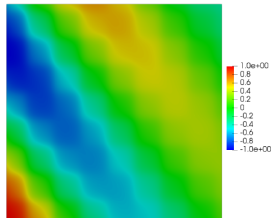


Figure 4. — The $(3, 3)$ components of the real stresses σ^E , of the micro-stresses σ^0 and of the boundary layer stresses σ^{0BL} added to σ^0 , plotted against y_3 , at $y_2 = 0.0416$ fixed.

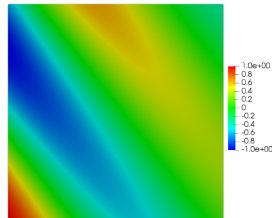
Examples: 2D time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

$$\varepsilon = \ell/\lambda = 0.125$$

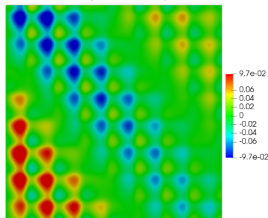
u (exact)



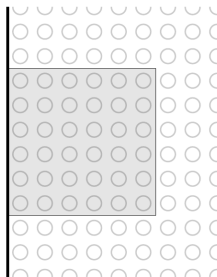
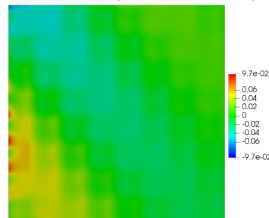
U_0



$|u - U_0|$

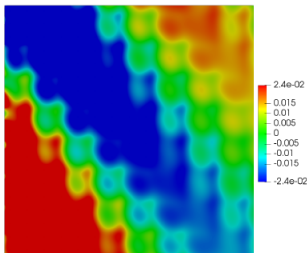


$|u - (U_0 + \varepsilon u_1)|$ (bulk corrector)

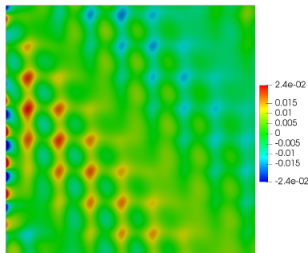


Examples: 2D time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

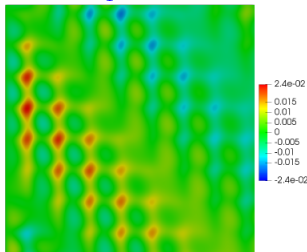
$|u - (U_0 + \varepsilon u_1)|$ (bulk corrector)



$|u - (U_0 + \varepsilon[U_1 + u_1])|$ (bulk and mean correctors)

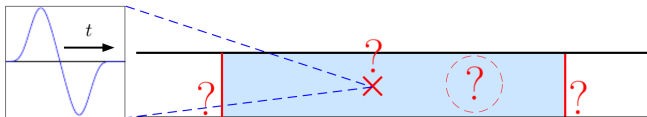
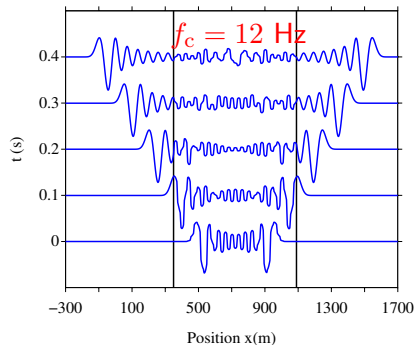
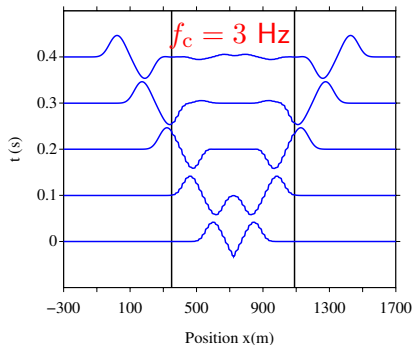
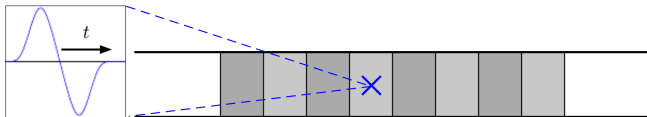


$|u - (U_0 + \varepsilon[U_1 + u_1 + u_1^{BL}])|$ (bulk, mean and boundary layer correctors)



(and then look for 2nd-order correctors ...)

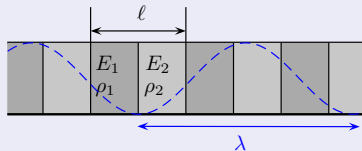
Challenges for 1D waves : dispersion, interfaces and source points



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- 1 Introduction
- 2 Second-order asymptotic homogenization for waves in 1D domains
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 - Transmission conditions and first-order “hybrid” model
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Two-scale asymptotic homogenization in a 1D periodic medium



- Wave equation in a ℓ -periodic medium

$$\rho \left(\frac{X}{\ell} \right) \frac{\partial^2 u_\ell}{\partial t^2} - \frac{\partial}{\partial X} \left[E \left(\frac{X}{\ell} \right) \frac{\partial u_\ell}{\partial X} \right] = 0$$

- Reference wavelength λ

[Sanchez-Palencia, 1974, Bensoussan et al., 1978, Cioranescu and Donato, 1999] ...

- **Scale separation** $\varepsilon = \ell/\lambda \ll 1$
- **Slow variable** $x = X/\lambda$ and **fast variable** $y = X/\ell = x/\varepsilon$
- Look for u_ℓ as a **function of both variables**: $u_\ell(X, t) = \hat{u}(x, y, t)$, **1-periodic in y**
- $\frac{\partial}{\partial X} \rightarrow \frac{1}{\lambda} \left[\frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y} \right]$
- Ansatz: $\hat{u}(x, y, t) = \sum_{j \geq 0} \varepsilon^j u_j(x, y, t) \implies$ **cascade of equations** for the u_j
- **Separated variable solution** featuring **mean fields** $U_j(x, t)$ and **cell functions** $P_j(y)$:

$$u_0(x, y, t) = U_0(x, t)$$

$$u_1(x, y, t) = U_1(x, t) + U_{0,x}(x, t)P_1(y)$$

$$u_2(x, y, t) = U_2(x, t) + U_{1,x}(x, t)P_1(y) + U_{0,xx}(x, t)P_2(y)$$

Effective wave equations

- Equations satisfied by the mean fields:

$$\varrho_0 U_{0,tt} - \mathcal{E}_0 U_{0,xx} = 0 \quad \left| \begin{array}{l} \varrho_0 = \langle \rho \rangle \\ \mathcal{E}_0 = \langle E(1 + P_{1,y}) \rangle \end{array} \right.$$

$$\varrho_0 U_{1,tt} - \mathcal{E}_0 U_{1,xx} = 0 \quad \mathcal{E}_1 = \frac{\varrho_1}{\varrho_0} \mathcal{E}_0 \Rightarrow \text{no first-order contribution}$$

$$\varrho_0 U_{2,tt} - \mathcal{E}_0 U_{2,xx} + \varrho_2 U_{0,xtt} - \mathcal{E}_2 U_{0,xxx} = 0 \quad \left| \begin{array}{l} \varrho_2 = \langle \rho P_2 \rangle \\ \mathcal{E}_2 = \langle E(P_2 + P_{3,y}) \rangle \end{array} \right.$$

- Second-order **total mean field** $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$:

$$\varrho_0 U_{,tt} - \mathcal{E}_0 U_{,xx} + \varepsilon^2 (\varrho_2 U_{0,xtt} - \mathcal{E}_2 U_{0,xxx}) = 0$$

- “Boussinesq trick”: using (i) $\varrho_0 U_{0,tt} = \mathcal{E}_0 U_{0,xx}$ and (ii) $U_0 = U + o(1)$:

$$\frac{1}{c_0^2} U_{,tt} - U_{,xx} - \varepsilon^2 \left(\beta_x U_{,xxxx} - \frac{\beta_m}{c_0^2} U_{,xtt} - \frac{\beta_t}{c_0^4} U_{,tttt} \right) = 0, \text{ with } \beta_x - \beta_m - \beta_t = \beta$$

[Fish et al., 2002, Lamacz, 2011, Wautier and Guzina, 2015]

- Effective properties:** $c_0^2 = \frac{\mathcal{E}_0}{\varrho_0}$ and $\beta := \left[\begin{array}{c} \mathcal{E}_2 \\ \mathcal{E}_0 \end{array} - \frac{\varrho_2}{\varrho_0} \right]$
- 2 degrees of freedom** to choose $(\beta_x, \beta_m, \beta_t)$.

Cell functions: displacement/velocity and stress correctors

Velocity-stress system:

$$\begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = 0 \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 \end{cases}$$

with $v_\ell = \partial_t u_\ell$
and $\sigma_\ell = E_\ell \partial_x u_\ell$.

- **Displacement/velocity correctors** (P_1, P_2, P_3) using $V = \partial_t U$:

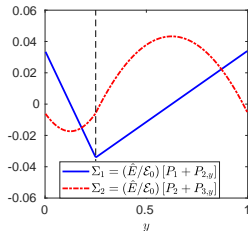
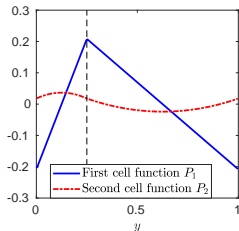
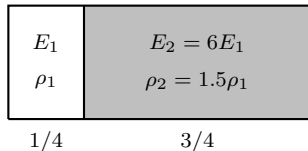
$$\begin{cases} u_\ell(x, t) \approx U(x, t) + \varepsilon U_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 U_{,xx}(x, t) P_2(x/\varepsilon) \\ v_\ell(x, t) \approx V(x, t) + \varepsilon V_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 V_{,xx}(x, t) P_2(x/\varepsilon) \end{cases}$$

- **Stress correctors** (Σ_1, Σ_2) using **macroscopic stress** $S := \varepsilon_0 \partial_x U$:

$$\sigma_\ell(x, t) \approx S(x, t) + \varepsilon S_{,x}(x, t) \Sigma_1(x/\varepsilon) + \varepsilon^2 S_{,xx}(x, t) \Sigma_2(x/\varepsilon)$$

Static equilibrium problems posed in the unit cell $Y = [0, 1]$ for (P_j, Σ_j)

$$\begin{cases} \partial_y \Sigma_j = f_j[E, \rho, P_{j-1}, \Sigma_{j-1}] \\ \varepsilon_0 \Sigma_j = E(P_j + \partial_y P_{j+1}) \end{cases} \quad (P_j, \Sigma_j) \text{ are 1-periodic, } \langle P_j \rangle = 0$$



Reciprocity identities on effective coefficients

- Classical expression of effective coefficients:

$$\rho_j = \langle \rho P_j \rangle, \quad \mathcal{E}_j = \langle E(P_j + P_{j+1,y}) \rangle = \mathcal{E}_0 \langle \Sigma_j \rangle, \quad \beta := \left[\frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\rho_2}{\rho_0} \right]$$

\Rightarrow *a priori* needs P_j , $j = 1, 2, 3$ for second-order effective properties.

- Weak form of cell problems:

$$\text{Find } P_j \in H_{\#0}^1, \quad \int_0^1 E(\partial_y P_j)(\partial_y w) \, dy = F_j(w) \quad \forall w \in H_{\#}^1, \quad j \in \{1, 2, 3\},$$

Reciprocity identities are obtained by “testing” one cell problem with another.

[Fish et al., 2002, Abdulle and Pouchon, 2020] ...

First reciprocity identity: using $F_1(P_2) = F_2(P_1)$,

$$\frac{\mathcal{E}_1}{\mathcal{E}_0} - \frac{\rho_1}{\rho_0} = 0 \quad \Rightarrow \quad \begin{cases} \text{No need for } P_2 \text{ to compute } \mathcal{E}_1. \\ \text{Cancels first-order contribution in effective wave equation.} \end{cases}$$

Second reciprocity identity: using $F_1(P_3) = F_3(P_1)$ (and after some work):

$$\beta = \frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\rho_2}{\rho_0} = \frac{1}{2} \left\langle [P_1 - \Sigma_1]^2 \left(\frac{\mathcal{E}_0}{E} + \frac{\rho}{\rho_0} \right) \right\rangle \quad \Rightarrow \quad \begin{cases} \text{No need for } P_3 \text{ to compute } \beta \text{ (or } \mathcal{E}_2). \\ \text{Important positivity property: } \beta \geq 0. \end{cases}$$

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Choosing a model : link with gradient elasticity models

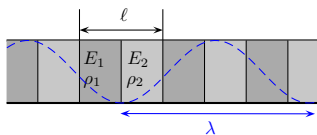
$$\frac{1}{c_0^2} \left[1 + \varepsilon^2 \beta_m \partial_{xx} + \varepsilon^2 \frac{\beta_t}{c_0^2} \partial_{tt} \right] U_{,tt} = \left[1 + \varepsilon^2 \beta_x \partial_{xx} \right] U_{,xx}, \quad \text{with } \beta_x - \beta_m - \beta_t = \beta > 0$$

	Gradient elasticity (see [Askes and Aifantis, 2011])	Two-scale homogenization	Properties
(x)	[Mindlin, 1964]	“natural” result when $\rho = 1$ [Fish and Chen, 2001]	unstable if $\beta_x > 0$ [Allaire et al., 2016]
(m)		[Fish et al., 2002, Lamacz, 2011]	stable if $\beta_m < 0$
(t)	?	?	?
(xm)	[Mindlin, 1964]	“natural” result for $\rho \neq 1$ Used to stabilize (x)	
(mt)	[Pichugin et al., 2008] [Forest and Sab, 2017]	[Cornaggia and Guzina, 2020] [Cornaggia and Lombard, 2023]	stable if $\beta_m < 0$ and $\beta_t > 0$
(xmt)	[Dontsov et al., 2013] [Schwan et al., 2021]	[Wautier and Guzina, 2015] [Schwan et al., 2021]	

$\beta_x \neq 0 \implies$ additional boundary conditions for bounded domains

- Well-posedness of the problem ? [Askes et al., 2008, Kaplunov and Pichugin, 2009]
- Difficult to deal with in homogenization theory (proofs when $\varepsilon \rightarrow 0$?)
- Not very useful to fit dispersion curves [Schwan et al., 2021].

Choosing a (mt) model from dispersion relations in layered media



- **Bloch wave** $u_\ell(X, t) = \tilde{u}(x)e^{i(\kappa x - \omega t)} \implies$ dispersion relation $\omega = f(\kappa)$.
About $(\omega, \kappa) = (0, 0)$ (on the acoustic branch):

$$\frac{\omega}{c_0} = \kappa \left[1 - \frac{\beta}{2}(\kappa\varepsilon)^2 + \frac{\beta(2 - 27\beta - 8\bar{\beta})}{40}(\kappa\varepsilon)^4 + O(\varepsilon^6) \right],$$

($\bar{\beta}$ known for layered media)

- **Plane wave** $U(x, t) = \tilde{U}e^{i(\kappa x - \omega t)}$ in the (mt) homogenized model:

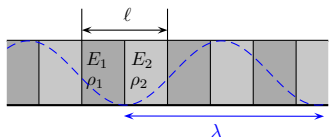
$$\frac{\omega}{c_0} = \kappa \left[1 + \frac{\beta_m + \beta_t}{2}(\kappa\varepsilon)^2 + \frac{(\beta_m + \beta_t)(3\beta_m + 7\beta_t)}{8}(\kappa\varepsilon)^4 + O(\varepsilon^6) \right].$$

- *Second-order approximation* of ω/c_0 obtained for any (β_m, β_t) satisfying $-\beta_m - \beta_t = \beta$. Also true in 2-3D [Allaire et al., 2016].

- *Fourth-order approximation* for $\beta_m = \frac{-1 - 4\beta + 4\bar{\beta}}{10}$ and $\beta_t = \frac{1 - 6\beta - 4\bar{\beta}}{10}$

Similar approximation for spring-mass lattice in [Pichugin et al., 2008]

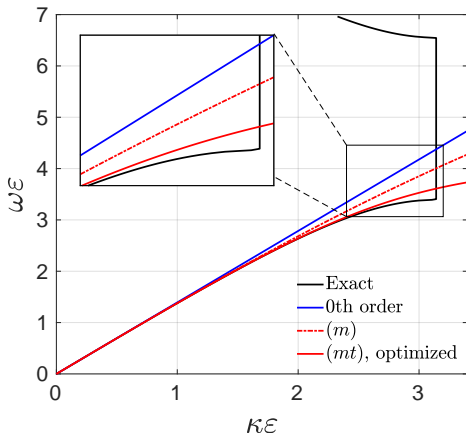
Choosing a (mt) model by fitting dispersion curves



Material contrasts: $\frac{E_2}{E_1} = 6$, $\frac{\rho_2}{\rho_1} = 1.5$

Phase ratio: $\frac{\ell_2}{\ell_1} = 3$

(\approx maximal dispersion [Santosa and Symes, 1991])



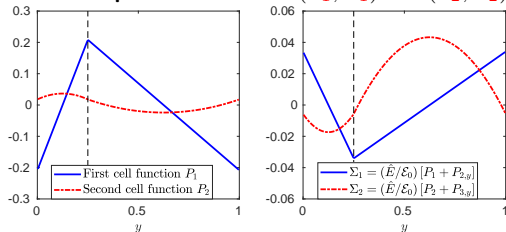
- Bloch wave: $u(x, y, t) = \phi(y)e^{i(\kappa x - \omega t)}$
 \implies Dispersion relation $\kappa = f(\omega)$
- 0th-order: $U_{,tt} - c_0^2 U_{,xx} = 0$
 \implies non-dispersive
- 2nd-order (m) :
 $\implies O((\kappa\varepsilon)^2)$ approximation
- Optimized (mt) :
 $\implies O((\kappa\varepsilon)^4)$ approximation
 [Pichugin et al., 2008,
 Cornaggia and Guzina, 2020]

Recap: second-order homogenization in one slide

From a given microstructure ...

E_1	$E_2 = 6E_1$
ρ_1	$\rho_2 = 1.5\rho_1$
1/4	3/4

... **1 - Compute cell solutions** (P_1, Σ_1) and (P_2, Σ_2)



2 - Choose a model (e.g. (mt)) and compute effective coefficients (e.g. $c_0^2, \beta, \beta_m, \beta_t$)

$$\frac{1}{c_0^2} U_{,tt} - U_{,xx} + \varepsilon^2 \left(\frac{\beta_m}{c_0^2} U_{,xxtt} + \frac{\beta_t}{c_0^4} U_{,tttt} \right) = 0,$$

3 - Solve the effective wave equation to find the macroscopic field U

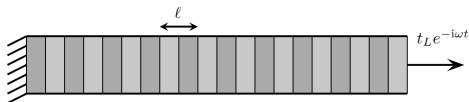
4 - Add bulk correctors to obtain approximations :

$$\begin{cases} u_\ell(x, t) \approx U(x, t) + \varepsilon U_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 U_{,xx}(x, t) P_2(x/\varepsilon) \\ \sigma_\ell(x, t) \approx S(x, t) + \varepsilon S_{,x}(x, t) \Sigma_1(x/\varepsilon) + \varepsilon^2 S_{,xx}(x, t) \Sigma_2(x/\varepsilon) \end{cases} \quad \text{with } S = \varepsilon_0 U_{,x}$$

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Model problem and leading-order approximation



- Time-harmonic model problem:

$$\begin{cases} [E(x/\varepsilon)u, x]_{,x} + \rho(x/\varepsilon)\omega^2 u = 0 & x \in Y_L :=]0, L[\\ u = 0 & x = 0 \\ \sigma = E(x)u, x = \sigma_L & x = L \end{cases}$$

- Leading-order homogenization: $(u, \sigma) \rightarrow (U, \mathcal{E}_0 U, x)$

$$\begin{cases} U, xx + k_0^2 U = 0 & x \in Y_L, \\ U = 0 & x = 0 \\ U, x = \sigma_L / \mathcal{E}_0 & x = L \end{cases} \quad k_0 := \omega / c_0$$

Boundary correctors using full-field approximations [Cornaggia and Guzina, 2020]

First-order approximations of displacement and stress (u, σ) :

$$\begin{cases} \tilde{u}^{(1)}(x) = U(x) + \varepsilon P_1 \left(\frac{x}{\varepsilon} \right) U_{,x}(x) \\ \tilde{\sigma}^{(1)}(x) = \mathcal{E}_0 \left[U_{,x}(x) + \varepsilon \Sigma_1 \left(\frac{x}{\varepsilon} \right) U_{,xx}(x) \right] \end{cases}$$

Using cell stress functions: $\Sigma_j := (E/\mathcal{E}_0) [P_j + P_{j+1,y}]$

Boundary-value problem for the mean field U :

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ \tilde{u}^{(1)} = 0 & x = 0 \\ \tilde{\sigma}^{(1)} = \sigma_L & x = L \end{cases}$$

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} + \varepsilon \Sigma_1(0) U_{,xx} = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} - \varepsilon \Sigma_1(0) k_0^2 U = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$

Second-order approximation

- Approximations of the displacement and stress (u, σ) :

$$\left. \begin{aligned} \tilde{u}^{(2)}(x) &= U(x) + \varepsilon P_1\left(\frac{x}{\varepsilon}\right)U_{,x}(x) + \varepsilon^2 P_2\left(\frac{x}{\varepsilon}\right)U_{,xx}(x), \\ \tilde{\sigma}^{(2)}(x) &= \mathcal{E}_0 \left[U_{,x} + \varepsilon \Sigma_1\left(\frac{x}{\varepsilon}\right)U_{,xx}(x) + \varepsilon^2 \Sigma_2\left(\frac{x}{\varepsilon}\right)U_{,xxx}(x) \right] \end{aligned} \right\} U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$$

- Boundary-value-problem for the second-order mean field U :

$$\begin{cases} U_{,xx} + k_{mt}^2 U = 0 & x \in Y_L \\ \left[1 - \varepsilon^2 k_{mt}^2 P_2(0) \right] U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ \left[1 - \varepsilon^2 k_{mt}^2 \Sigma_2(0) \right] U_{,x} - \varepsilon k_{mt}^2 \Sigma_1(0) U = \sigma_L / \mathcal{E}_0 & x = L. \end{cases}$$

using the (mt) dispersive model:

$$k_{mt}^2 := k_0^2 \frac{1 - \varepsilon^2 \beta_t k_0^2}{1 + \varepsilon^2 \beta_m k_0^2}$$

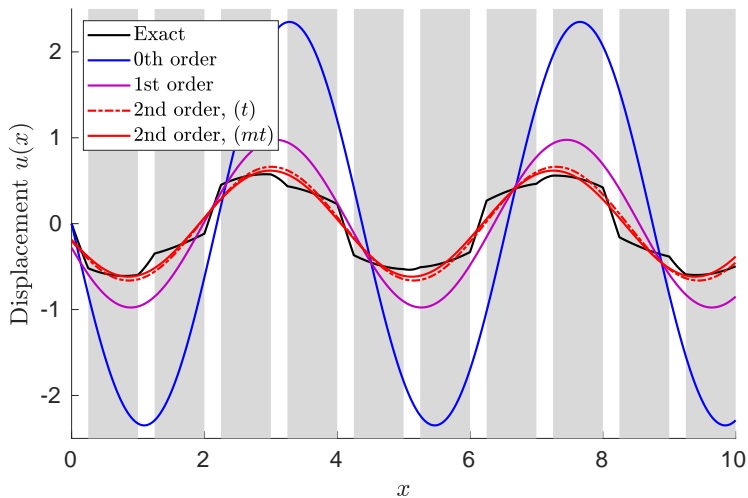
Lemma 2: $\exists C > 0 \quad \|u - \tilde{u}^{(2)}\|_{H^1} \leq C \varepsilon^2 \|U\|_{H^1}.$

Theorem 1: $\exists C > 0 \quad \|u - \tilde{u}^{(1)}\|_{L^2} \leq \|u - \tilde{u}^{(2)}\|_{L^2} + \|\tilde{u}^{(2)} - \tilde{u}^{(1)}\|_{L^2} \leq C \varepsilon^2 \|U\|_{H^2}.$

- ▶ No need for 2nd order BC to prove these results
- ▶ BCs surely needed for the next step : we hope that $\|u - \tilde{u}^{(2)}\|_{L^2} = O(\varepsilon^3)$

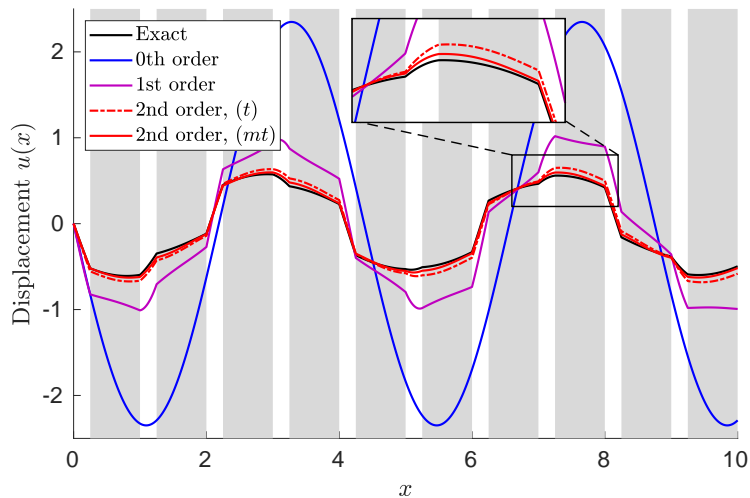
Example for a layered material - mean fields U

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



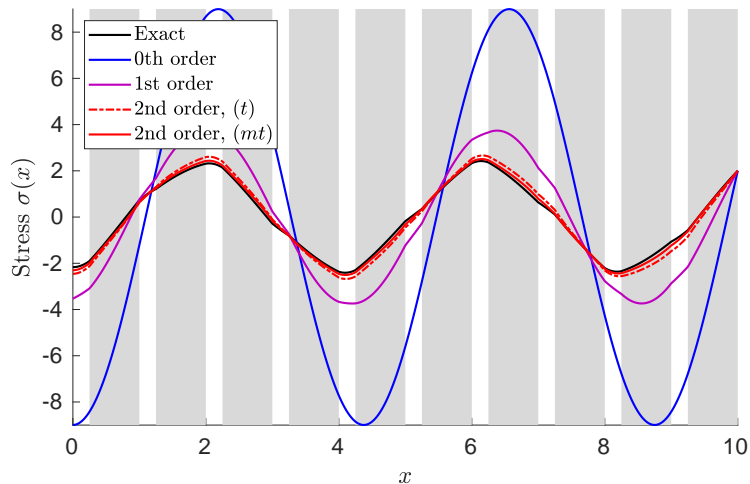
Example for a layered material - total displacement u

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$

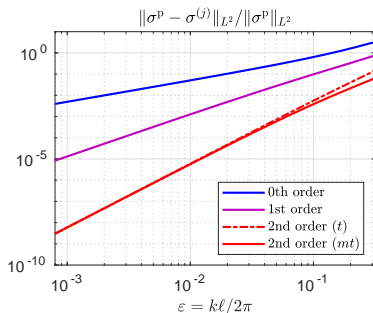
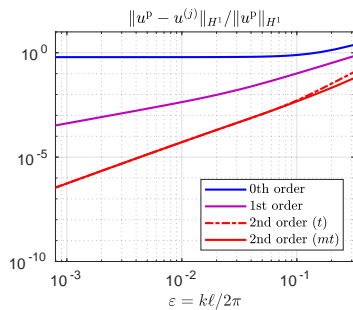
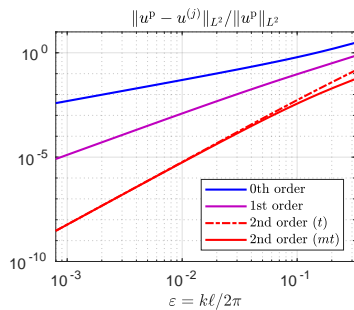


Example for a layered material - axial stress σ

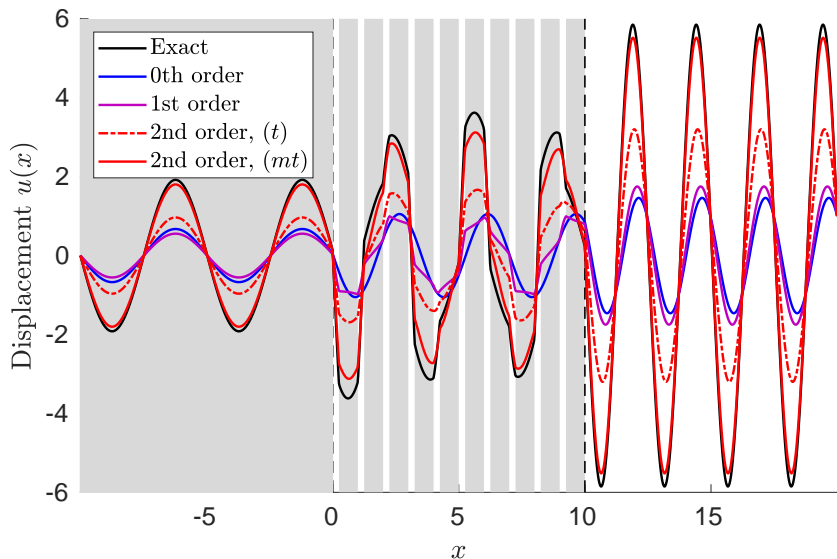
$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



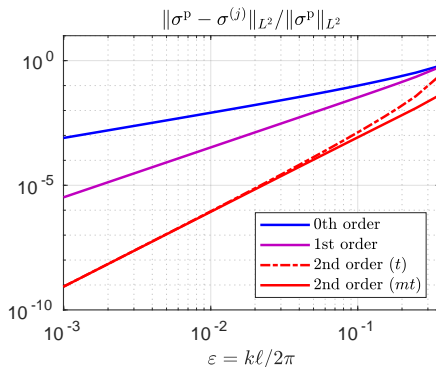
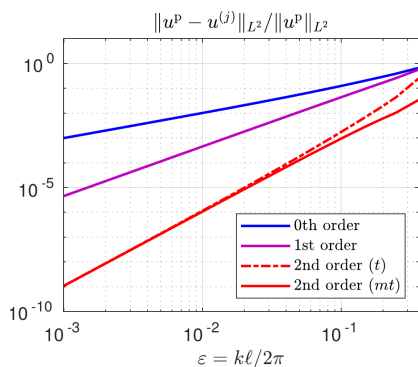
Errors for fixed ω and $\ell \rightarrow 0$



Application to a transmission problem ($\varepsilon \approx 0.3$)



Transmission problem - errors for fixed ω and $\ell \rightarrow 0$



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Equivalent hyperbolic system for the (mt) model

($\beta_x = 0$ because we don't want $U_{,xxxx}$ and additional BC/TC it brings)

Second-order homogenization, (mt) model:

$$\frac{1}{c_0^2} \left[1 + \varepsilon^2 \beta_m \partial_{xx} + \varepsilon^2 \frac{\beta_t}{c_0^2} \partial_{tt} \right] U_{,tt} = U_{,xx},$$

“Stress gradient” system [Forest and Sab, 2017]:

$$\begin{cases} \sigma = E \partial_x (u + \phi) \\ r = D \phi \end{cases} \quad \begin{cases} \rho \partial_{tt} u = \partial_x \sigma \\ \rho J \partial_{tt} \phi = \partial_x \sigma - r \end{cases}$$

So that: $\frac{\rho}{E} \left[1 - \frac{E(1+J)}{D} \partial_{xx} + \frac{\rho J}{D} \partial_{tt} \right] u_{,tt} = u_{,xx}$

$$\Rightarrow \begin{cases} \partial_t V - \frac{a}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r \\ \partial_t S - \mathcal{E}_0 \partial_x V & = 0 \\ \partial_t \varphi - \frac{a-1}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r \\ \partial_t r & = \frac{\mathcal{E}_0}{\varepsilon^2 \beta} \varphi \end{cases}$$

- Macroscopic fields $V = \partial_t(u + \phi)$ and $S = \sigma$
- **Auxiliary fields** $\varphi = \partial_t \phi$ and r
- Parameter $a = -\beta_m / \beta_t$ to select a (mt) model.

Well-posedness (hyperbolicity, stability, conserved positive energy) for:

$$\beta \geq 0 \quad (\checkmark), \quad \beta_m < 0 \quad \text{and} \quad \beta_t > 0$$

Hyperbolicity and stability (See B. Lombard's first lecture)

- Matricial form of system:

$$\partial_t \mathbf{U} + \mathbf{A} \cdot \partial_x \mathbf{U} = \mathbf{S} \cdot \mathbf{U}, \quad \text{where } \mathbf{U} := (V, S, \varphi, r)^T$$

$$\mathbf{A} = \begin{bmatrix} 0 & -a/\varrho_0 & 0 & 0 \\ -\mathcal{E}_0 & 0 & 0 & 0 \\ 0 & -(a-1)/\varrho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & -(a-1)/\varrho_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(a-1)/\varrho_0 \\ 0 & 0 & \mathcal{E}_0/(\varepsilon^2 \beta) & 0 \end{bmatrix}.$$

- Hyperbolicity : **real** eigenvalues of \mathbf{A}

$$\text{Sp}(\mathbf{A}) = \{0, 0, \pm c_0 \sqrt{a}\}$$

- Stability : **imaginary** eigenvalues of \mathbf{S}

$$\text{Sp}(\mathbf{S}) = \{0, 0, \pm c_0 / (\varepsilon \sqrt{-\beta_t})\}$$

$$\left\{ \begin{array}{l} \text{Hyperbolicity: } a > 0 \\ \text{Stability: } \beta_t > 0 \end{array} \right\} \implies \left\{ \begin{array}{l} -\frac{\beta_m}{\beta_t} > 0 \\ \beta_t > 0 \end{array} \right\} \implies \left\{ \begin{array}{l} \beta_m < 0, \\ \beta_t > 0. \end{array} \right.$$

Symmetrizer and energy conservation for $\partial_t U + A \cdot \partial_x U = S \cdot U$

- Symmetrizer matrix:

$$M = \begin{bmatrix} \rho_0 & 0 & -\rho_0 & 0 \\ 0 & 1/\varepsilon_0 & 0 & 0 \\ -\rho_0 & 0 & a\rho_0/(a-1) & 0 \\ 0 & 0 & 0 & \varepsilon^2\beta/\varepsilon_0 \end{bmatrix},$$

So that:

$$M \cdot A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M \cdot S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

- Multiplying the system by $U^T \cdot M$:

$$\partial_t \frac{1}{2} (U^T \cdot M \cdot U) + \frac{1}{2} \partial_x (U^T \cdot M \cdot A \cdot U) = 0.$$

Integrating over $\Omega \times [0, T]$, (large Ω so that $U(\cdot, T) = \mathbf{0}$ on $\partial\Omega$)

Energy conservation: $\mathcal{E}^{\text{vol}}(T) = \mathcal{E}^{\text{vol}}(0)$.

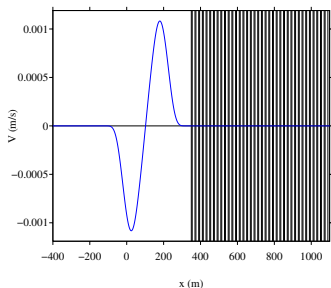
$$\mathcal{E}^{\text{vol}} = \frac{1}{2} \int_{\Omega} U^T \cdot M \cdot U \, dx = \frac{1}{2} \int_{\Omega} \rho_0 \left(v^2 + \frac{\beta_t}{\beta} \varphi^2 \right) + \frac{1}{\varepsilon_0} (\sigma^2 + \varepsilon^2 \beta r^2) \, dx \geq 0 \text{ if } \beta \geq 0$$

(with $v = V - \varphi$)

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First-order transmission conditions



- Fields of interest:

$$\begin{cases} (V, S) = (\partial_t U, E - \partial_x U) & x < 0 \\ (V, S, \varphi, r) & x > 0 \end{cases}$$

- Transmission conditions from **total fields continuity**:

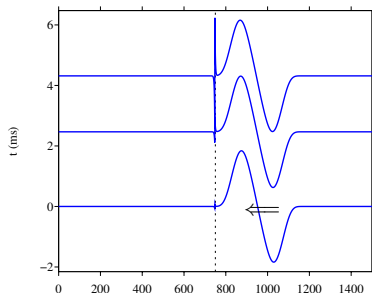
$$\begin{cases} V(0^-) = V(0^+) + \varepsilon P_1(0) V_{,x}(0^+) \\ S(0^-) = S(0^+) + \varepsilon \Sigma_1(0) S_{,x}(0^+) \end{cases}$$

- Energy analysis : **interface terms** when integrating $\partial_x(\mathbf{U}^T \cdot \mathbf{M} \cdot \mathbf{A} \cdot \mathbf{U})$ over $\Omega_- \cup \Omega_+$

$$\partial_t(\mathcal{E}_-^{\text{vol}} + \mathcal{E}_+^{\text{vol}}) + \mathcal{D}^{\text{int}} = 0,$$

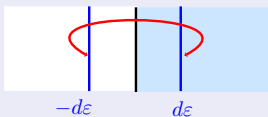
$$\begin{aligned} \mathcal{D}^{\text{int}} &= \llbracket V \rrbracket_0 \langle S \rangle_0 + \langle V \rangle_0 \llbracket S \rrbracket_0 \\ &= -\varepsilon [P_1(0)(\partial_x V)^+ \langle S \rangle_0 + \Sigma_1(0)(\partial_x S)^+ \langle V \rangle_0] \\ &= ??? \end{aligned}$$

- We would like $\mathcal{D}^{\text{int}} = \partial_t \mathcal{E}^{\text{int}}$ with $\mathcal{E}_1^{\text{int}} \geq 0$



First-order transmission conditions - stabilization

Spring-mass conditions across a thin interface $I_d = [-d\varepsilon, d\varepsilon]$ (additional $o(\varepsilon)$ error):



$$\begin{cases} \llbracket V \rrbracket_{I_d} = \varepsilon A_1 \partial_t \langle S \rangle_{I_d} & A_1 = -P_1(0) \mathcal{E}_0^{-1} + d(E_-^{-1} + \mathcal{E}_0^{-1}) \\ \llbracket S \rrbracket_{I_d} = \varepsilon B_1 \partial_t \langle V \rangle_{I_d} & B_1 = -\Sigma_1(0) \varrho_0 + d(\rho_- + \varrho_0) \end{cases}$$

- New expression of interface term:

$$\partial_t (\mathcal{E}_-^{\text{vol}} + \mathcal{E}_+^{\text{vol}}) + \mathcal{D}_d^{\text{int}} = 0,$$

$$\mathcal{D}_d^{\text{int}} = \partial_t \mathcal{E}_d^{\text{int}} \quad \text{with} \quad \mathcal{E}_d^{\text{int}} = \frac{\varepsilon}{2} \left[A_1 \langle S \rangle_{I_d}^2 + B_1 \langle V \rangle_{I_d}^2 \right]$$

- **Stability** of {Hyperbolic system + transmission conditions} for $A_1 \geq 0$ and $B_1 \geq 0$

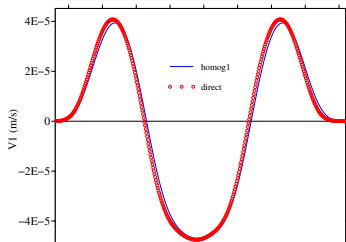
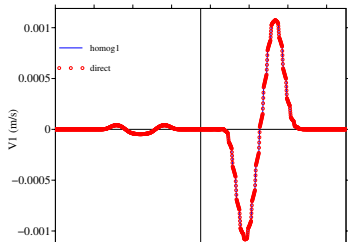
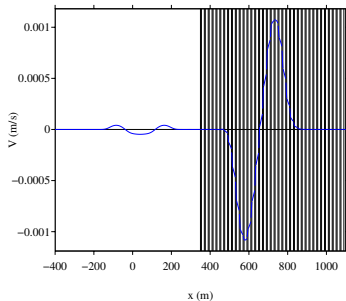
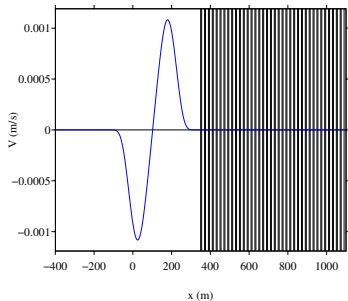
$$\Rightarrow \text{Choice of } d \geq d_{\min} := \max \left(\frac{E_- P_1(0)}{E_- + \mathcal{E}_0}, \frac{\varrho_0 \Sigma_1(0)}{\rho_- + \varrho_0}, 0 \right)$$

Well-posed “hybrid” model:

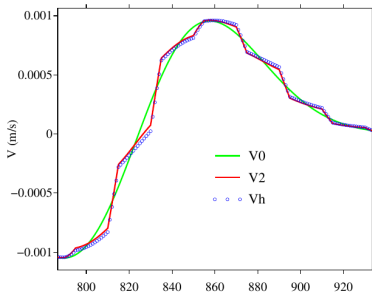
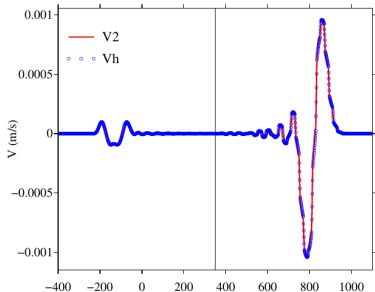
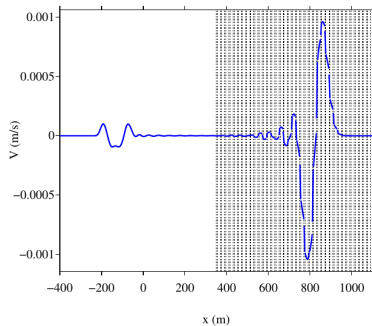
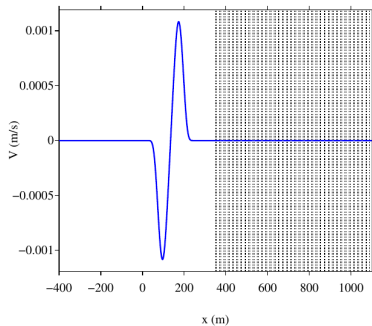
- **dispersive system** from 2nd-order bulk homogenization
- **transmission conditions** (1st-order for now)

Transmission conditions for $\varepsilon = \ell/\lambda_c \approx 0.04$ (Numerics by B. Lombard)

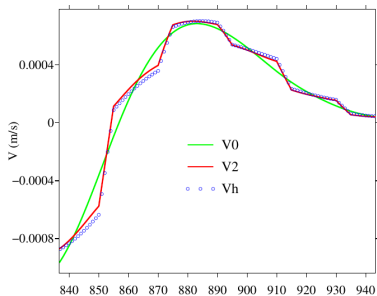
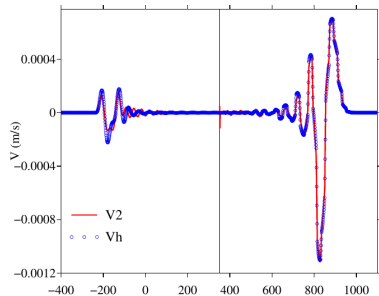
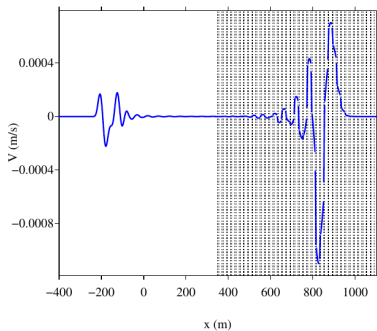
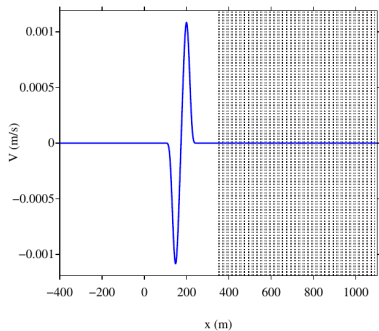
$E_{-\rho_-} = \mathcal{E}_0 e_0 \implies$ reflected wave is a 1st-order effect



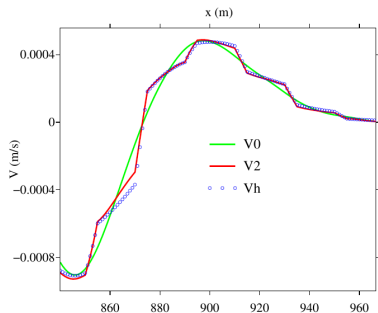
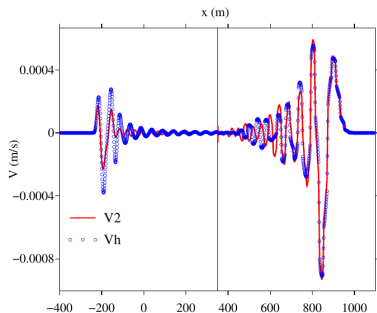
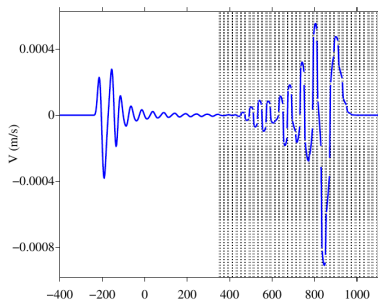
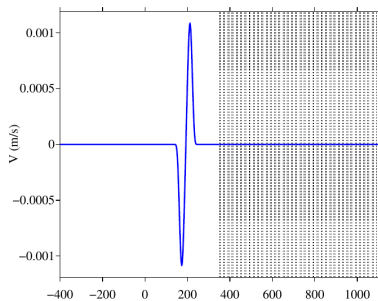
Transmission conditions, increasing central frequency, $\varepsilon \approx 0.09$



Transmission conditions, increasing central frequency, $\varepsilon \approx 0.13$



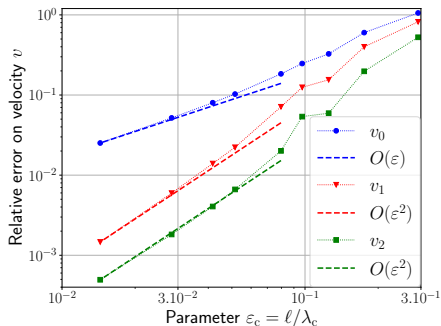
Transmission conditions, increasing central frequency, $\varepsilon \approx 0.17$



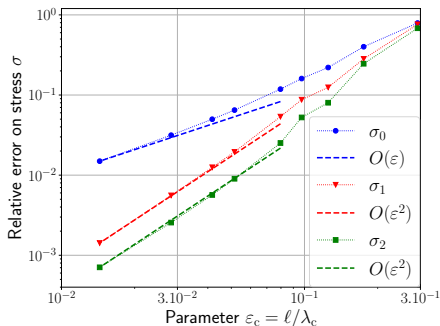
Transmission conditions, errors on velocity and stress fields

- Fields emitted at source $x_s < 0$, measured at receiver $x_r > 0$ during time T .
- Increasing frequency f , constant microstructure periodicity ℓ .
- Positions (x_s, x_r) and time T accordingly rescaled for each frequency.

$$\frac{\|(v_h - v_j)(x_r, \cdot)\|_{L^2(0,T)}}{\|v_h(x_r, \cdot)\|_{L^2(0,T)}}$$



$$\frac{\|(\sigma_h - \sigma_j)(x_r, \cdot)\|_{L^2(0,T)}}{\|\sigma_h(x_r, \cdot)\|_{L^2(0,T)}}$$



(v_1, σ_1) : Adding 1st-order bulk correctors (P_1, Σ_1) and 1st-order TC

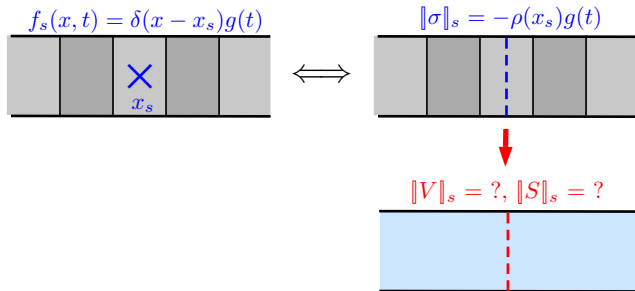
(v_2, σ_2) : “Hybrid” dispersive system (but still 1st-order TC)

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In 1D, source points \Leftrightarrow interfaces with stress jumps

$$\begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = \delta_s g(t) & x \in \mathbb{R} \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 & x \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = 0 & x < x_s \text{ and } x > x_s, \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 & x < x_s \text{ and } x > x_s, \\ \llbracket v_\ell \rrbracket_s = 0 & x = x_s, \\ \llbracket \sigma_\ell \rrbracket_s = -\rho_s g & x = x_s, \end{cases}$$



First-order correction

- **Same tools** (and no need for an enlarged interface):

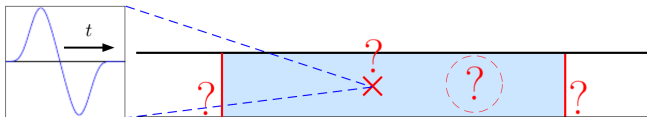
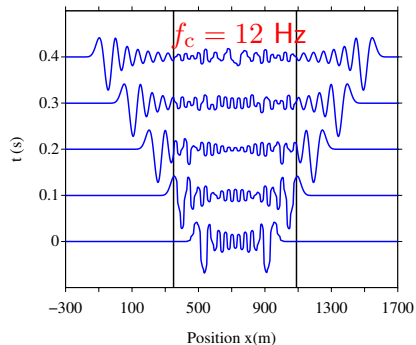
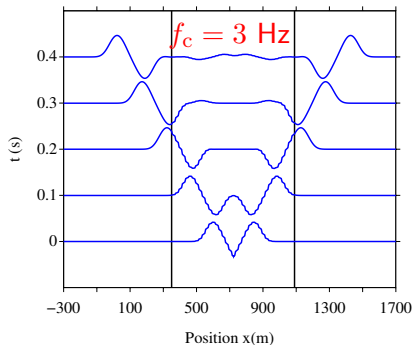
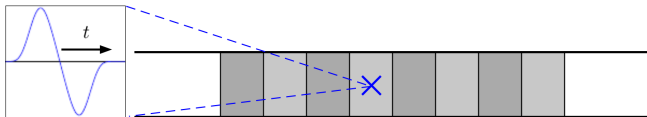
$$\begin{cases} \llbracket V \rrbracket_s = \varepsilon P_1(x_s/\varepsilon) \frac{\rho_s}{\varepsilon_0} \partial_t g, \\ \llbracket S \rrbracket_s = -\rho_s g, \end{cases}$$

- ▶ **First-order** corrector on V involving $\partial_t g$ and the local values ρ_s and $P_1(x_s/\varepsilon)$.
- ▶ Already given by [Capdeville et al., 2010] with another argument.

- Equivalent system with Dirac notation:

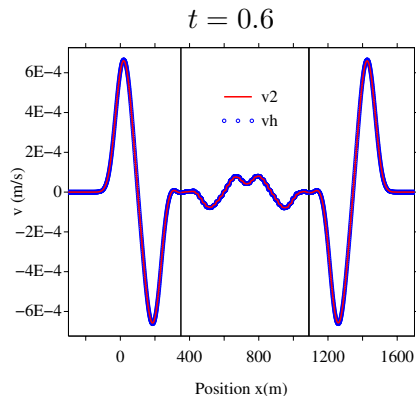
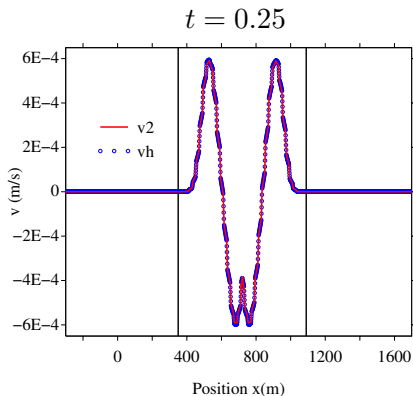
$$\begin{cases} \partial_t V - \frac{a}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r + \frac{a}{\varrho_0} \rho_s g \delta_s, \\ \partial_t S - \varepsilon_0 \partial_x V & = -\varepsilon \rho_s P_1(x_s/\varepsilon) \partial_t g \delta_s, \\ \partial_t \varphi - \frac{a-1}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r + \frac{a-1}{\varrho_0} \rho_s g \delta_s, \\ \partial_t r & = \frac{\varepsilon_0}{\ell^2 \beta} \varphi. \end{cases}$$

Back to the initial motivation



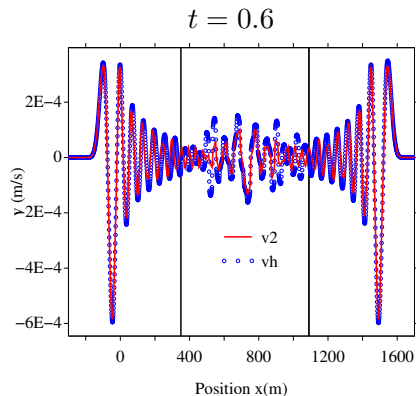
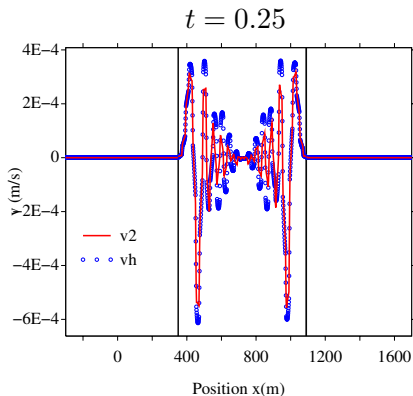
Velocity fields in a slab with a source point

$$f_c = 3 \text{ Hz}, \varepsilon_c = 0.04$$



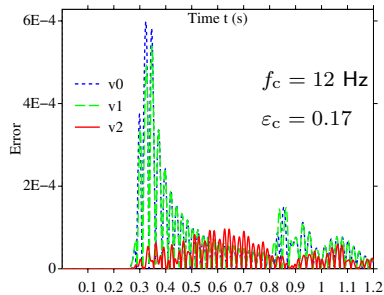
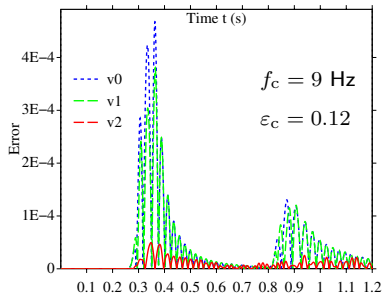
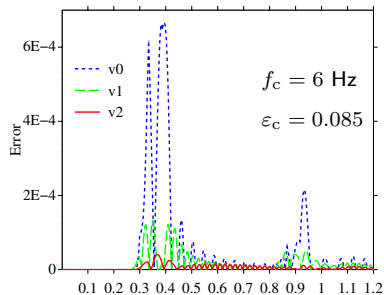
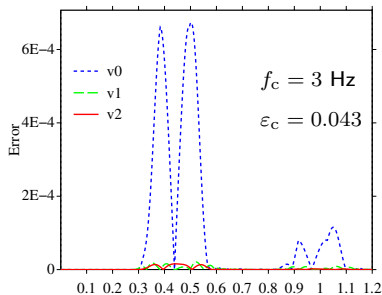
Velocity fields in a slab with a source point

$$f_c = 12 \text{ Hz}, \varepsilon_c = 0.17$$



Slab with a source point - errors at several frequencies

receiver in left homogeneous domain



Key ideas

- The second-order asymptotic homogenization accounts for dispersive effects.
- Reciprocity identities between cell problems are a big help.
- Boundary and transmission conditions can be designed to complement the inner expansion. (including in 2-3D [Vinoles, 2016, Maurel and Marigo, 2018, Cakoni et al., 2019, Beneteau, 2021] ...)
- For transient waves, **stability** can be addressed using (i) hyperbolic formalism and (ii) enlarged interfaces (in 1D for now).
- Source points can be addressed using the same tools (specific to 1D).

Perspectives

- Pursue the transient case up to *full second-order model* (as done in the time-harmonic case [Cornaggia and Guzina, 2020])
- Apply the built framework to other 1D configurations:
 - ▶ High-frequency homogenization [Craster et al., 2010, Guzina et al., 2019, Assier et al., 2020]
 - ▶ Solids with inner imperfect interfaces [Assier et al., 2020]
 - ▶ Non-linearities [Bellis et al., 2021]
- Reciprocity identities for higher dimensions and **elasticity** [Boutin and Auriault, 1993] (*ongoing*).
- Hyperbolic formalism for higher dimensions and other models ? (2D acoustics, elasticity ...)

Thanks for your attention !

*Second-order homogenization of boundary and transmission conditions
for one-dimensional waves in periodic media*

Rémi Cornaggia, Bojan B. Guzina

International Journal of Solids and Structures, 2020

*An homogenized model accounting for dispersion, interfaces and source points
for transient waves in 1D periodic media*

Rémi Cornaggia, Bruno Lombard

ESAIM: Mathematical Modelling and Numerical Analysis (M2AN), 2023

References



Abdulle, A. and Pouchon, T. (2020).
Effective models and numerical homogenization for wave propagation in heterogeneous media on arbitrary timescales.
Foundations of Computational Mathematics, 20(6):1505–1547.



Allaire, G. and Amar, M. (1999).
Boundary layer tails in periodic homogenization.
ESAIM: Control, Optimisation and Calculus of Variations, 4:209–243.



Allaire, G., Briane, M., and Vanninathan, M. (2016).
A comparison between two-scale asymptotic expansions and Bloch wave expansions for the homogenization of periodic structures.
SeMA Journal, 73(3):237–259.



Allaire, G. and Yamada, T. (2018).
Optimization of dispersive coefficients in the homogenization of the wave equation in periodic structures.
Numerische Mathematik, 140(2):265–326.



Armstrong, S., Kuusi, T., Mourrat, J.-C., and Prange, C. (2017).
Quantitative analysis of boundary layers in periodic homogenization.
Archive for Rational Mechanics and Analysis, 226(2):695–741.



Askes, H. and Aifantis, E. C. (2011).
Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results.
International Journal of Solids and Structures, 48(13):1962–1990.



Askes, H., Metrikine, A., Pichugin, A., and Bennett, T. (2008).
Four simplified gradient elasticity models for the simulation of dispersive wave propagation.

Philosophical Magazine, 88(28-29):3415–3443.



Assier, R. C., Touboul, M., Lombard, B., and Bellis, C. (2020).
High-frequency homogenization in periodic media with imperfect interfaces.
Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 476(2244):20200402.



Bellis, C., Lombard, B., Touboul, M., and Assier, R. (2021).
Effective dynamics for low-amplitude transient elastic waves in a 1D periodic array of non-linear interfaces.
Journal of the Mechanics and Physics of Solids, 149:104321.



Beneteau, C. (2021).
Modèles homogénéisés enrichis en présence de bords : Analyse et traitement numérique.
Ph. D. thesis, Institut Polytechnique de Paris.



Bensoussan, A., Lions, J. L., and Papanicolau, G. (1978).
Asymptotic Analysis for Periodic Structures.
North-Holland, Amsterdam.



Boutin, C. and Auriault, J. (1993).
Rayleigh scattering in elastic composite materials.
International Journal of Engineering Science, 31(12):1669–1689.



Cakoni, F., Guzina, B. B., Moskow, S., and Pangburn, T. (2019).
Scattering by a bounded highly oscillating periodic medium and the effect of boundary correctors.
SIAM Journal on Applied Mathematics, 79(4):1448–1474.



Capdeville, Y., Guillot, L., and Marigo, J.-J. (2010).
1-D non-periodic homogenization for the seismic wave equation.
Geophysical Journal International.



Celli, P. and Gonella, S. (2015).
Manipulating waves with lego bricks: A versatile experimental platform for metamaterial architectures.
Applied Physics Letters, 107(8).



Cioranescu, D. and Donato, P. (1999).
An introduction to homogenization, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*.
Oxford University Press.



Cornaggia, R. and Guzina, B. B. (2020).
Second-order homogenization of boundary and transmission conditions for one-dimensional waves in periodic media.
International Journal of Solids and Structures, 188-189:88 – 102.



Cornaggia, R. and Lombard, B. (2023).
A homogenized model accounting for dispersion, interfaces and source points for transient waves in 1d periodic media.
ESAIM: Mathematical Modelling and Numerical Analysis, 57(3):1413–1444.



Craster, R. V., Kaplunov, J., and Pichugin, A. V. (2010).
High-frequency homogenization for periodic media.
Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 466(2120):2341–2362.



Dontsov, E. V., Tokmashev, R. D., and Guzina, B. B. (2013).
A physical perspective of the length scales in gradient elasticity through the prism of wave dispersion.
International Journal of Solids and Structures, 50(22-23):3674 – 3684.



Dumontet, H. (1986).
Study of a boundary layer problem in elastic composite materials.
ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, 20(2):265–286.



Fish, J. and Chen, W. (2001).

Higher-order homogenization of initial/boundary-value problem.
Journal of Engineering Mechanics, 127(12):1223–1230.



Fish, J., Chen, W., and Nagai, G. (2002).
Non-local dispersive model for wave propagation in heterogeneous media: one-dimensional case.
International Journal for Numerical Methods in Engineering, 54(3):331–346.



Forest, S. and Sab, K. (2017).
Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models.
Mathematics and Mechanics of Solids, page 108128651772084.



Gérard-Varet, D. and Masmoudi, N. (2012).
Homogenization and boundary layers.
Acta Mathematica, 209(1):133–178.



Guzina, B. B., Meng, S., and Oudghiri-Idrissi, O. (2019).
A rational framework for dynamic homogenization at finite wavelengths and frequencies.
Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 475(2223):20180547.



Huang, Y., Liu, S., and Zhao, J. (2016).
A gradient-based optimization method for the design of layered phononic band-gap materials.
Acta Mechanica Sinica, 29(4):429 – 443.



Kaplunov, J. and Pichugin, A. (2009).
On rational boundary conditions for higher-order long-wave models.
In Borodich, F., editor, *IUTAM Symposium on Scaling in Solid Mechanics*, volume 10 of *IUTAM Bookseries*, pages 81–90. Springer Netherlands.



Kook, J. and Jensen, J. S. (2017).
Topology optimization of periodic microstructures for enhanced loss factor using acoustic-structure interaction.



Lamacz, A. (2011).

Dispersive effective models for waves in heterogeneous media.

Mathematical Models and Methods in Applied Sciences, 21(09):1871–1899.



Marigo, J.-J. and Maurel, A. (2017).

Second order homogenization of subwavelength stratified media including finite size effect.

SIAM Journal on Applied Mathematics, 77(2):721–743.



Maurel, A. and Marigo, J.-J. (2018).

Sensitivity of a dielectric layered structure on a scale below the periodicity: A fully local homogenized model.

Physical Review B, 98(2).



Mindlin, R. (1964).

Micro-structure in linear elasticity.

Archive for Rational Mechanics and Analysis, 16:51–78.



Moskow, S. and Vogelius, M. (1997).

First-order corrections to the homogenized eigenvalues of a periodic composite medium: the case of Neumann boundary conditions.

Preprint Rutgers University.



Noguchi, Y., Yamada, T., Izui, K., and Nishiwaki, S. (2018).

Topology optimization for hyperbolic acoustic metamaterials using a high-frequency homogenization method.

Computer Methods in Applied Mechanics and Engineering, 335:419–471.



Palermo, A., Krödel, S., Marzani, A., and Daraio, C. (2016).

Engineered metabarrier as shield from seismic surface waves.

Scientific Reports, 6(1).



Pichugin, A., Askes, H., and Tyas, A. (2008).

Asymptotic equivalence of homogenisation procedures and fine-tuning of continuum theories.

Journal of Sound and Vibration, 313(3-5):858 – 874.



Rosi, G. and Auffray, N. (2019).

Continuum modelling of frequency dependent acoustic beam focusing and steering in hexagonal lattices.

European Journal of Mechanics - A/Solids, page 103803.



Sanchez-Palencia, E. (1974).

Comportements local et macroscopique d'un type de milieux physiques hétérogènes.

International Journal of Engineering Science, 12(4):331 – 351.



Santosa, F. and Symes, W. W. (1991).

A dispersive effective medium for wave propagation in periodic composites.

SIAM Journal on Applied Mathematics, 51(4):984–1005.



Schwan, L., Favrie, N., Cottéreau, R., and Lombard, B. (2021).

Extended stress gradient elastodynamics: Wave dispersion and micro-macro identification of parameters.

International Journal of Solids and Structures, 219-220:34–50.



Vinoles, V. (2016).

Interface problems with metamaterials : modelling, analysis and simulations.

Ph. D. thesis, ENSTA, Université Paris-Saclay.



Vondřejc, J., Rohan, E., and Heczeko, J. (2017).

Shape optimization of phononic band gap structures using the homogenization approach.

International Journal of Solids and Structures, 113-114:147 – 168.



Wautier, A. and Guzina, B. B. (2015).

On the second-order homogenization of wave motion in periodic media and the sound of a chessboard.

Journal of the Mechanics and Physics of Solids, 78:382 – 414.