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Homogenisation for waves in 1D periodic media: Effective models, boundary and transmission correctors

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Theme 1:

Homogenisation for waves in 1D periodic media

Effective models, boundary and transmission correctors

Rémi Cornaggia¹

Joint work with Bojan B. Guzina² and Bruno Lombard³

CISM Course C2305: *Wave Motion in Heterogeneous Media: Analysis, Modeling and Design*
June 19-23, 2023



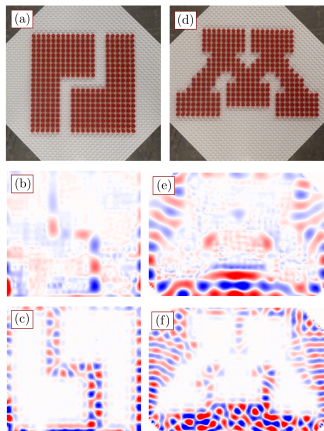
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Control of wave propagation using periodic structures

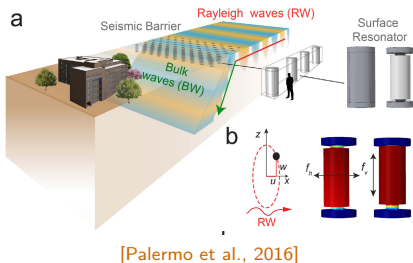


[Celli and Gonella, 2015]

... to design waveguides

- with resonators for long wavelengths / low frequencies
- with Bragg effects for medium wavelengths / frequencies

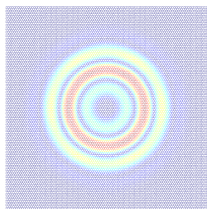
... to design seismic protections (with resonators)



[Palermo et al., 2016]

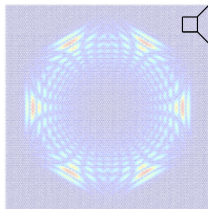
... and more (negative effective properties, cloaking ...)

A motivation : dispersion and anisotropy in 2D elasticity [Rosi and Auffray, 2019]

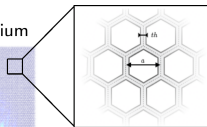


$$\frac{\ell}{\lambda_c} \ll 1$$

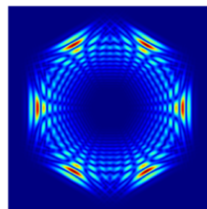
Microstructured medium



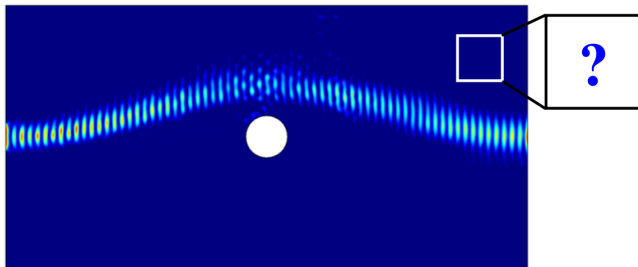
$$\frac{\ell}{\lambda_c} \approx \frac{1}{6}$$



Continuum model



Effective (strain-gradient) model.



Optimized effective model \Rightarrow "de-homogenization" ?

Homogenisation and interface conditions

How to model the wave propagation in periodic structures ?

- Accounting for all interactions \rightarrow costly (*dedicated numerical methods* out-of scope)
- *Floquet-Bloch* theory \implies See other lectures.
- *Homogenization* (/averaging/effective) methods and models
- **Both lectures:** *asymptotic two-scale homogenization*
 - ▶ Formal procedure to obtain effective models
 - ▶ Accuracy can be improved by pushing the expansion to higher orders
 - ▶ Extensions to *high/finite frequencies* [Craster et al., 2010, Guzina et al., 2019]
 \implies R. Craster and B. Guzina's lectures

How to account for boundaries and interfaces ?

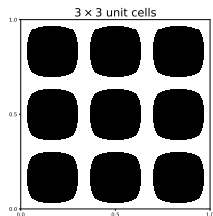
- In 2-3D, *boundary layers* must be accounted for at boundaries and interfaces [Dumontet, 1986, Moskow and Vogelius, 1997, Allaire and Amar, 1999, Gérard-Varet and Masmoudi, 2012, Armstrong et al., 2017, Cakoni et al., 2019] ...
- *Equivalent boundary or transmission conditions* preserving the asymptotics are difficult to build [Vinales, 2016, Marigo and Maurel, 2017, Maurel and Marigo, 2018, Beneteau, 2021]
- **This lecture:** 1D waves, “interfaces” are *points*.

Optimization

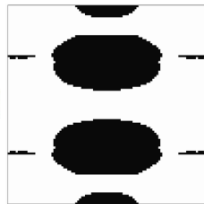
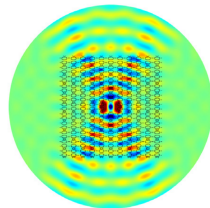
How to obtain desired properties ?

- Known design (cylinders, spheres, cones, LEGO bricks ...) \implies *parameter optimization* [Huang et al., 2016, Palermo et al., 2016] ...
- Given materials, unknown design \implies *topological optimization* [Vondřejc et al., 2017, Kook and Jensen, 2017, Allaire and Yamada, 2018] ...

Next lectures: topological optimization of microstructured 2D media and interfaces (long wavelengths, no resonances)

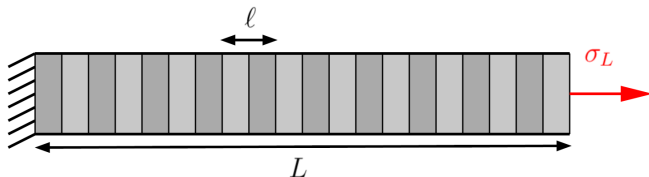


what we will see



[Noguchi et al., 2018] high frequency, hyperbolic propagation, see R. Craster's lecture

A preliminary example : 1D statics



- **Two-scale problem** : L (macro) and ℓ (micro)
- **Scale factor** $\varepsilon = \ell/L$
- ℓ -periodic properties $E_\ell(X) = E(X/\ell) = E(x/\varepsilon)$: E is 1-periodic
- Normalized coordinate $x = X/L$

$$\left\{ \begin{array}{ll} \partial_x \sigma(x) = 0 & x \in [0, 1] \quad \text{equilibrium} \\ \sigma(x) = E(x/\varepsilon) \partial_x u & x \in [0, 1] \quad \text{constitutive law} \\ u(0) = 0 & \text{displacement BC} \\ \sigma(1) = \sigma_L & \text{stress BC} \end{array} \right.$$

See dedicated Jupyter Notebook.

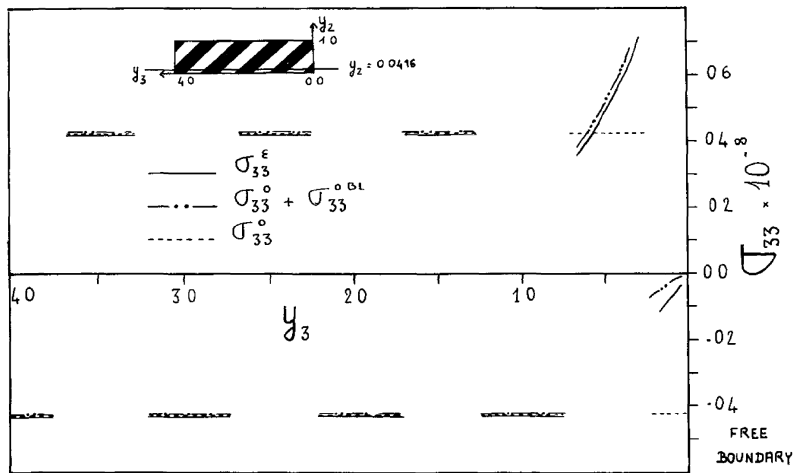
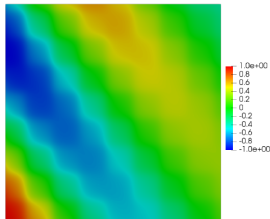


Figure 4. — The $(3, 3)$ components of the real stresses σ^E , of the micro-stresses σ^0 and of the boundary layer stresses σ^{0BL} added to σ^0 , plotted against y_3 , at $y_2 = 0.0416$ fixed.

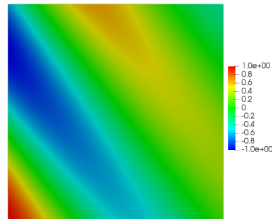
Examples: 2D time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

$$\varepsilon = \ell/\lambda = 0.125$$

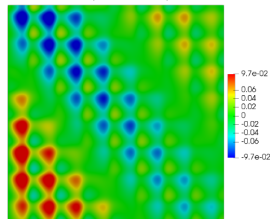
u (exact)



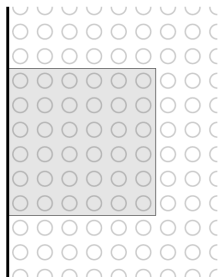
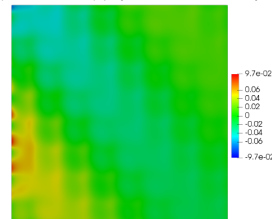
U_0



$|u - U_0|$

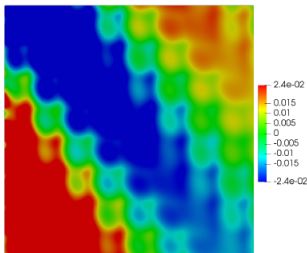


$|u - (U_0 + \varepsilon u_1)|$ (bulk corrector)

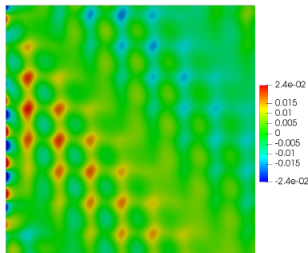


Examples: 2D time-harmonic dynamics [Beneteau, 2021] with S. Fliss, X. Claeys

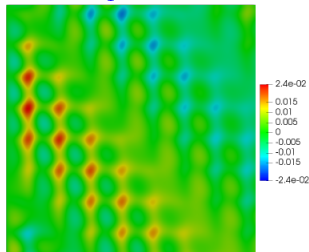
$|u - (U_0 + \varepsilon u_1)|$ (bulk corrector)



$|u - (U_0 + \varepsilon[U_1 + u_1])|$ (bulk and mean correctors)

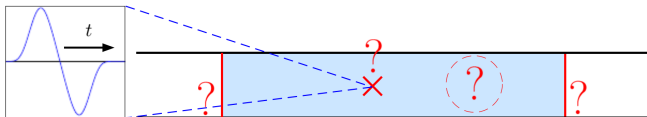
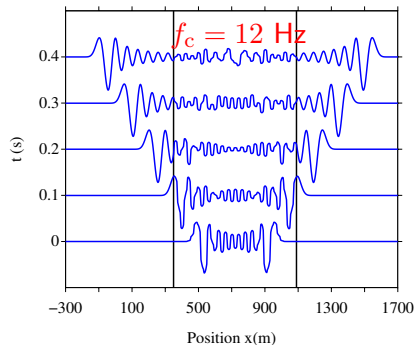
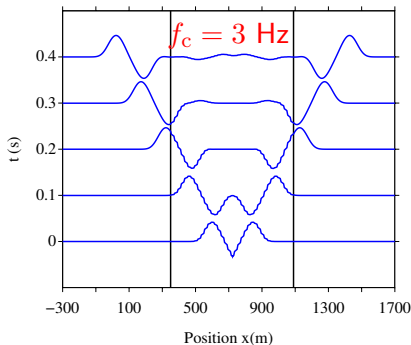
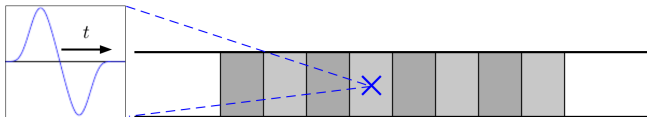


$|u - (U_0 + \varepsilon[U_1 + u_1 + u_1^{BL}])|$ (bulk, mean and boundary layer correctors)



(and then look for 2nd-order correctors ...)

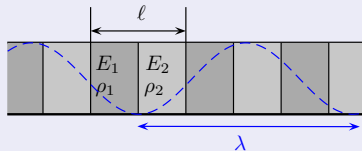
Challenges for 1D waves : dispersion, interfaces and source points



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- 1 Introduction
- 2 Second-order asymptotic homogenization for waves in 1D domains
 - Second-order models
 - Corrector functions and reciprocity identities
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 - Transmission conditions and first-order “hybrid” model
 - Application to point sources
- 5 Conclusions and perspectives

Two-scale asymptotic homogenization in a 1D periodic medium



- Wave equation in a ℓ -periodic medium

$$\rho \left(\frac{X}{\ell} \right) \frac{\partial^2 u_\ell}{\partial t^2} - \frac{\partial}{\partial X} \left[E \left(\frac{X}{\ell} \right) \frac{\partial u_\ell}{\partial X} \right] = 0$$

- Reference wavelength λ

[Sanchez-Palencia, 1974, Bensoussan et al., 1978, Cioranescu and Donato, 1999] ...

- **Scale separation** $\varepsilon = \ell/\lambda \ll 1$
- **Slow variable** $x = X/\lambda$ and **fast variable** $y = X/\ell = x/\varepsilon$
- Look for u_ℓ as a **function of both variables**: $u_\ell(X, t) = \hat{u}(x, y, t)$, **1-periodic in y**
- $\frac{\partial}{\partial X} \rightarrow \frac{1}{\lambda} \left[\frac{\partial}{\partial x} + \varepsilon^{-1} \frac{\partial}{\partial y} \right]$
- Ansatz: $\hat{u}(x, y, t) = \sum_{j \geq 0} \varepsilon^j u_j(x, y, t) \implies$ **cascade of equations** for the u_j
- **Separated variable solution** featuring **mean fields** $U_j(x, t)$ and **cell functions** $P_j(y)$:

$$u_0(x, y, t) = U_0(x, t)$$

$$u_1(x, y, t) = U_1(x, t) + U_{0,x}(x, t)P_1(y)$$

$$u_2(x, y, t) = U_2(x, t) + U_{1,x}(x, t)P_1(y) + U_{0,xx}(x, t)P_2(y)$$

Effective wave equations

- Equations satisfied by the mean fields:

$$\varrho_0 U_{0,tt} - \mathcal{E}_0 U_{0,xx} = 0 \quad \left| \begin{array}{l} \varrho_0 = \langle \rho \rangle \\ \mathcal{E}_0 = \langle E(1 + P_{1,y}) \rangle \end{array} \right.$$

$$\varrho_0 U_{1,tt} - \mathcal{E}_0 U_{1,xx} = 0 \quad \mathcal{E}_1 = \frac{\varrho_1}{\varrho_0} \mathcal{E}_0 \Rightarrow \text{no first-order contribution}$$

$$\varrho_0 U_{2,tt} - \mathcal{E}_0 U_{2,xx} + \varrho_2 U_{0,xtt} - \mathcal{E}_2 U_{0,xxx} = 0 \quad \left| \begin{array}{l} \varrho_2 = \langle \rho P_2 \rangle \\ \mathcal{E}_2 = \langle E(P_2 + P_{3,y}) \rangle \end{array} \right.$$

- Second-order **total mean field** $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$:

$$\varrho_0 U_{,tt} - \mathcal{E}_0 U_{,xx} + \varepsilon^2 (\varrho_2 U_{0,xtt} - \mathcal{E}_2 U_{0,xxx}) = 0$$

- “Boussinesq trick”: using (i) $\varrho_0 U_{0,tt} = \mathcal{E}_0 U_{0,xx}$ and (ii) $U_0 = U + o(1)$:

$$\frac{1}{c_0^2} U_{,tt} - U_{,xx} - \varepsilon^2 \left(\beta_x U_{,xxxx} - \frac{\beta_m}{c_0^2} U_{,xtt} - \frac{\beta_t}{c_0^4} U_{,tttt} \right) = 0, \text{ with } \beta_x - \beta_m - \beta_t = \beta$$

[Fish et al., 2002, Lamacz, 2011, Wautier and Guzina, 2015]

- Effective properties:** $c_0^2 = \frac{\mathcal{E}_0}{\varrho_0}$ and $\beta := \left[\frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\varrho_2}{\varrho_0} \right]$
- 2 degrees of freedom** to choose $(\beta_x, \beta_m, \beta_t)$.

Cell functions: displacement/velocity and stress correctors

Velocity-stress system:

$$\begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = 0 \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 \end{cases}$$

with $v_\ell = \partial_t u_\ell$
and $\sigma_\ell = E_\ell \partial_x u_\ell$.

- **Displacement/velocity correctors** (P_1, P_2, P_3) using $V = \partial_t U$:

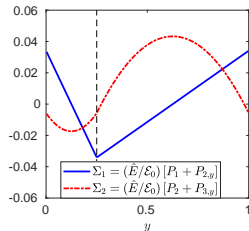
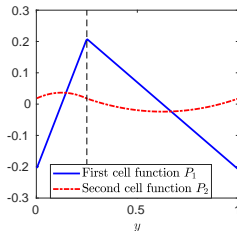
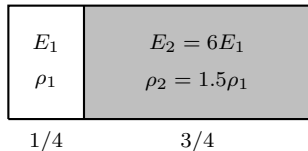
$$\begin{cases} u_\ell(x, t) \approx U(x, t) + \varepsilon U_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 U_{,xx}(x, t) P_2(x/\varepsilon) \\ v_\ell(x, t) \approx V(x, t) + \varepsilon V_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 V_{,xx}(x, t) P_2(x/\varepsilon) \end{cases}$$

- **Stress correctors** (Σ_1, Σ_2) using **macroscopic stress** $S := \varepsilon_0 \partial_x U$:

$$\sigma_\ell(x, t) \approx S(x, t) + \varepsilon S_{,x}(x, t) \Sigma_1(x/\varepsilon) + \varepsilon^2 S_{,xx}(x, t) \Sigma_2(x/\varepsilon)$$

Static equilibrium problems posed in the unit cell $Y = [0, 1]$ for (P_j, Σ_j)

$$\begin{cases} \partial_y \Sigma_j = f_j[E, \rho, P_{j-1}, \Sigma_{j-1}] \\ \varepsilon_0 \Sigma_j = E(P_j + \partial_y P_{j+1}) \end{cases} \quad (P_j, \Sigma_j) \text{ are 1-periodic, } \langle P_j \rangle = 0$$



Reciprocity identities on effective coefficients

- Classical expression of effective coefficients:

$$\rho_j = \langle \rho P_j \rangle, \quad \mathcal{E}_j = \langle E(P_j + P_{j+1,y}) \rangle = \mathcal{E}_0 \langle \Sigma_j \rangle, \quad \beta := \left[\frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\rho_2}{\rho_0} \right]$$

\Rightarrow *a priori* needs P_j , $j = 1, 2, 3$ for second-order effective properties.

- Weak form of cell problems:

$$\text{Find } P_j \in H_{\#0}^1, \quad \int_0^1 E(\partial_y P_j)(\partial_y w) \, dy = F_j(w) \quad \forall w \in H_{\#}^1, \quad j \in \{1, 2, 3\},$$

Reciprocity identities are obtained by “testing” one cell problem with another.

[Fish et al., 2002, Abdulle and Pouchon, 2020] ...

First reciprocity identity: using $F_1(P_2) = F_2(P_1)$,

$$\frac{\mathcal{E}_1}{\mathcal{E}_0} - \frac{\rho_1}{\rho_0} = 0 \quad \Rightarrow \quad \begin{cases} \text{No need for } P_2 \text{ to compute } \mathcal{E}_1. \\ \text{Cancels first-order contribution in effective wave equation.} \end{cases}$$

Second reciprocity identity: using $F_1(P_3) = F_3(P_1)$ (and after some work):

$$\beta = \frac{\mathcal{E}_2}{\mathcal{E}_0} - \frac{\rho_2}{\rho_0} = \frac{1}{2} \left\langle [P_1 - \Sigma_1]^2 \left(\frac{\mathcal{E}_0}{E} + \frac{\rho}{\rho_0} \right) \right\rangle \quad \Rightarrow \quad \begin{cases} \text{No need for } P_3 \text{ to compute } \beta \text{ (or } \mathcal{E}_2). \\ \text{Important positivity property: } \beta \geq 0. \end{cases}$$

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Choosing a model : link with gradient elasticity models

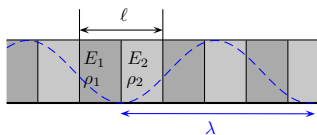
$$\frac{1}{c_0^2} \left[1 + \varepsilon^2 \beta_m \partial_{xx} + \varepsilon^2 \frac{\beta_t}{c_0^2} \partial_{tt} \right] U_{,tt} = \left[1 + \varepsilon^2 \beta_x \partial_{xx} \right] U_{,xx}, \quad \text{with } \beta_x - \beta_m - \beta_t = \beta > 0$$

	Gradient elasticity (see [Askes and Aifantis, 2011])	Two-scale homogenization	Properties
(x)	[Mindlin, 1964]	“natural” result when $\rho = 1$ [Fish and Chen, 2001]	unstable if $\beta_x > 0$ [Allaire et al., 2016]
(m)		[Fish et al., 2002, Lamacz, 2011]	stable if $\beta_m < 0$
(t)	?	?	?
(xm)	[Mindlin, 1964]	“natural” result for $\rho \neq 1$ Used to stabilize (x)	
(mt)	[Pichugin et al., 2008] [Forest and Sab, 2017]	[Cornaggia and Guzina, 2020] [Cornaggia and Lombard, 2023]	stable if $\beta_m < 0$ and $\beta_t > 0$
(xmt)	[Dontsov et al., 2013] [Schwan et al., 2021]	[Wautier and Guzina, 2015] [Schwan et al., 2021]	

$\beta_x \neq 0 \implies$ additional boundary conditions for bounded domains

- Well-posedness of the problem ? [Askes et al., 2008, Kaplunov and Pichugin, 2009]
- Difficult to deal with in homogenization theory (proofs when $\varepsilon \rightarrow 0$?)
- Not very useful to fit dispersion curves [Schwan et al., 2021].

Choosing a (mt) model from dispersion relations in layered media



- **Bloch wave** $u_\ell(X, t) = \tilde{u}(x)e^{i(\kappa x - \omega t)} \implies$ dispersion relation $\omega = f(\kappa)$.
About $(\omega, \kappa) = (0, 0)$ (on the acoustic branch):

$$\frac{\omega}{c_0} = \kappa \left[1 - \frac{\beta}{2}(\kappa\varepsilon)^2 + \frac{\beta(2 - 27\beta - 8\bar{\beta})}{40}(\kappa\varepsilon)^4 + O(\varepsilon^6) \right],$$

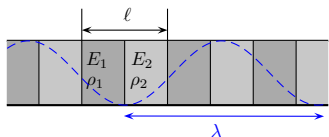
($\bar{\beta}$ known for layered media)

- **Plane wave** $U(x, t) = \tilde{U}e^{i(\kappa x - \omega t)}$ in the (mt) homogenized model:

$$\frac{\omega}{c_0} = \kappa \left[1 + \frac{\beta_m + \beta_t}{2}(\kappa\varepsilon)^2 + \frac{(\beta_m + \beta_t)(3\beta_m + 7\beta_t)}{8}(\kappa\varepsilon)^4 + O(\varepsilon^6) \right].$$

- *Second-order* approximation of ω/c_0 obtained for any (β_m, β_t) satisfying $-\beta_m - \beta_t = \beta$. Also true in 2-3D [Allaire et al., 2016].
- *Fourth-order* approximation for $\beta_m = \frac{-1 - 4\beta + 4\bar{\beta}}{10}$ and $\beta_t = \frac{1 - 6\beta - 4\bar{\beta}}{10}$
Similar approximation for spring-mass lattice in [Pichugin et al., 2008]

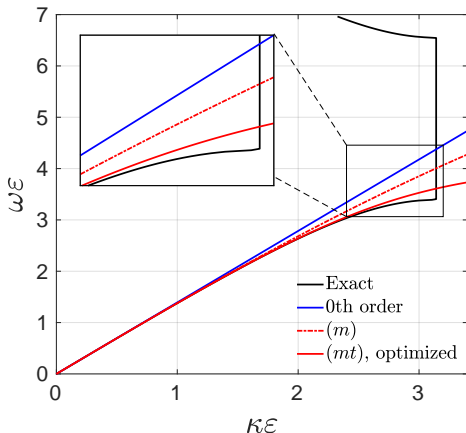
Choosing a (mt) model by fitting dispersion curves



Material contrasts: $\frac{E_2}{E_1} = 6, \quad \frac{\rho_2}{\rho_1} = 1.5$

Phase ratio: $\frac{\ell_2}{\ell_1} = 3$

(\approx maximal dispersion [Santosa and Symes, 1991])



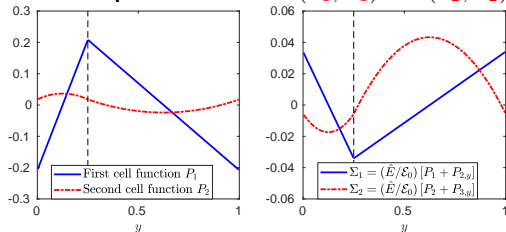
- Bloch wave: $u(x, y, t) = \phi(y)e^{i(\kappa x - \omega t)}$
 \implies Dispersion relation $\kappa = f(\omega)$
- 0th-order: $U_{,tt} - c_0^2 U_{,xx} = 0$
 \implies non-dispersive
- 2nd-order (m) :
 $\implies O((\kappa \varepsilon)^2)$ approximation
- Optimized (mt) :
 $\implies O((\kappa \varepsilon)^4)$ approximation
 [Pichugin et al., 2008,
 Cornaggia and Guzina, 2020]

Recap: second-order homogenization in one slide

From a given microstructure ...

E_1	$E_2 = 6E_1$
ρ_1	$\rho_2 = 1.5\rho_1$
1/4	3/4

... **1 - Compute cell solutions** (P_1, Σ_1) and (P_2, Σ_2)



2 - Choose a model (e.g. (mt)) and compute effective coefficients (e.g. $c_0^2, \beta, \beta_m, \beta_t$)

$$\frac{1}{c_0^2} U_{,tt} - U_{,xx} + \varepsilon^2 \left(\frac{\beta_m}{c_0^2} U_{,xxtt} + \frac{\beta_t}{c_0^4} U_{,tttt} \right) = 0,$$

3 - Solve the effective wave equation to find the macroscopic field U

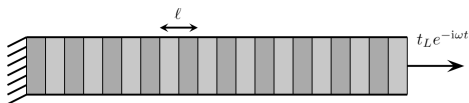
4 - Add bulk correctors to obtain approximations :

$$\begin{cases} u_\ell(x, t) \approx U(x, t) + \varepsilon U_{,x}(x, t) P_1(x/\varepsilon) + \varepsilon^2 U_{,xx}(x, t) P_2(x/\varepsilon) \\ \sigma_\ell(x, t) \approx S(x, t) + \varepsilon S_{,x}(x, t) \Sigma_1(x/\varepsilon) + \varepsilon^2 S_{,xx}(x, t) \Sigma_2(x/\varepsilon) \end{cases} \quad \text{with } S = \varepsilon_0 U_{,x}$$

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Model problem and leading-order approximation



- Time-harmonic model problem:

$$\begin{cases} [E(x/\varepsilon)u, x]_{,x} + \rho(x/\varepsilon)\omega^2 u = 0 & x \in Y_L :=]0, L[\\ u = 0 & x = 0 \\ \sigma = E(x)u, x = \sigma_L & x = L \end{cases}$$

- Leading-order homogenization: $(u, \sigma) \rightarrow (U, \mathcal{E}_0 U, x)$

$$\begin{cases} U, xx + k_0^2 U = 0 & x \in Y_L, \\ U = 0 & x = 0 \\ U, x = \sigma_L / \mathcal{E}_0 & x = L \end{cases} \quad k_0 := \omega / c_0$$

Boundary correctors using full-field approximations [Cornaggia and Guzina, 2020]

First-order approximations of displacement and stress (u, σ) :

$$\begin{cases} \tilde{u}^{(1)}(x) = U(x) + \varepsilon P_1 \left(\frac{x}{\varepsilon} \right) U_{,x}(x) \\ \tilde{\sigma}^{(1)}(x) = \mathcal{E}_0 \left[U_{,x}(x) + \varepsilon \Sigma_1 \left(\frac{x}{\varepsilon} \right) U_{,xx}(x) \right] \end{cases}$$

Using cell stress functions: $\Sigma_j := (E/\mathcal{E}_0) [P_j + P_{j+1,y}]$

Boundary-value problem for the mean field U :

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ \tilde{u}^{(1)} = 0 & x = 0 \\ \tilde{\sigma}^{(1)} = \sigma_L & x = L \end{cases}$$

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} + \varepsilon \Sigma_1(0) U_{,xx} = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$

$$\begin{cases} U_{,xx} + k_0^2 U = 0 & x \in Y_L \\ U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ U_{,x} - \varepsilon \Sigma_1(0) k_0^2 U = \sigma_L / \mathcal{E}_0 & x = L \end{cases}$$

Second-order approximation

- Approximations of the displacement and stress (u, σ) :

$$\left. \begin{aligned} \tilde{u}^{(2)}(x) &= U(x) + \varepsilon P_1\left(\frac{x}{\varepsilon}\right)U_{,x}(x) + \varepsilon^2 P_2\left(\frac{x}{\varepsilon}\right)U_{,xx}(x), \\ \tilde{\sigma}^{(2)}(x) &= \mathcal{E}_0 \left[U_{,x} + \varepsilon \Sigma_1\left(\frac{x}{\varepsilon}\right)U_{,xx}(x) + \varepsilon^2 \Sigma_2\left(\frac{x}{\varepsilon}\right)U_{,xxx}(x) \right] \end{aligned} \right\} U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2$$

- Boundary-value-problem for the second-order mean field U :

$$\begin{cases} U_{,xx} + k_{mt}^2 U = 0 & x \in Y_L \\ \left[1 - \varepsilon^2 k_{mt}^2 P_2(0) \right] U + \varepsilon P_1(0) U_{,x} = 0 & x = 0 \\ \left[1 - \varepsilon^2 k_{mt}^2 \Sigma_2(0) \right] U_{,x} - \varepsilon k_{mt}^2 \Sigma_1(0) U = \sigma_L / \mathcal{E}_0 & x = L. \end{cases}$$

using the (mt) dispersive model:

$$k_{mt}^2 := k_0^2 \frac{1 - \varepsilon^2 \beta_t k_0^2}{1 + \varepsilon^2 \beta_m k_0^2}$$

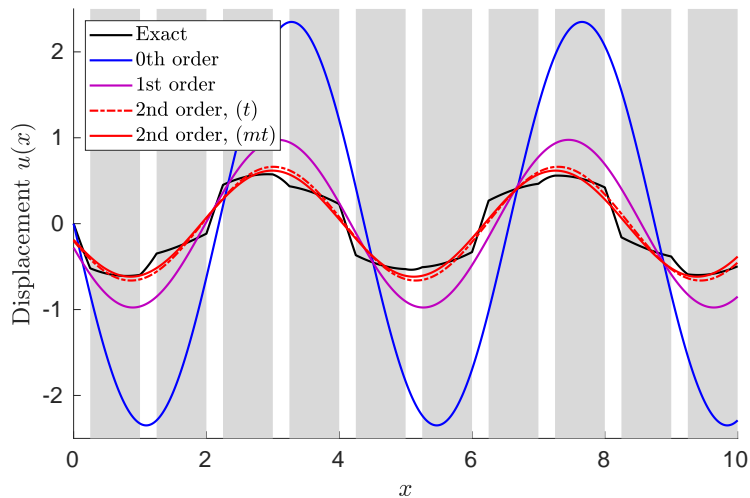
Lemma 2: $\exists C > 0 \quad \|u - \tilde{u}^{(2)}\|_{H^1} \leq C \varepsilon^2 \|U\|_{H^1}.$

Theorem 1: $\exists C > 0 \quad \|u - \tilde{u}^{(1)}\|_{L^2} \leq \|u - \tilde{u}^{(2)}\|_{L^2} + \|\tilde{u}^{(2)} - \tilde{u}^{(1)}\|_{L^2} \leq C \varepsilon^2 \|U\|_{H^2}.$

- ▶ No need for 2nd order BC to prove these results
- ▶ BCs surely needed for the next step : we hope that $\|u - \tilde{u}^{(2)}\|_{L^2} = O(\varepsilon^3)$

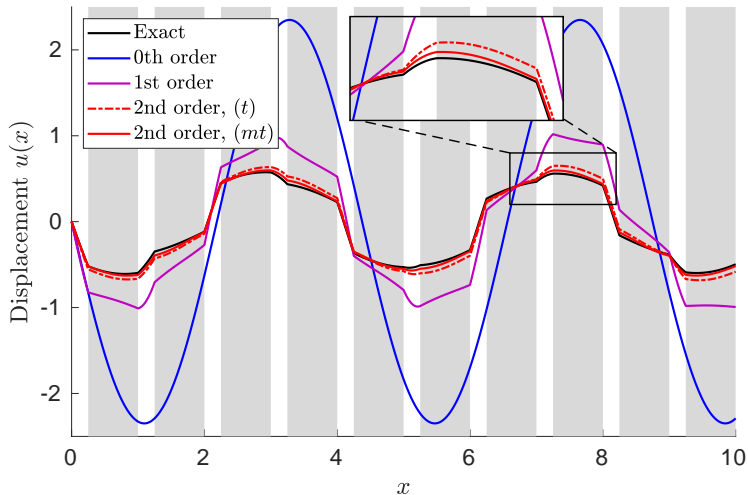
Example for a layered material - mean fields U

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



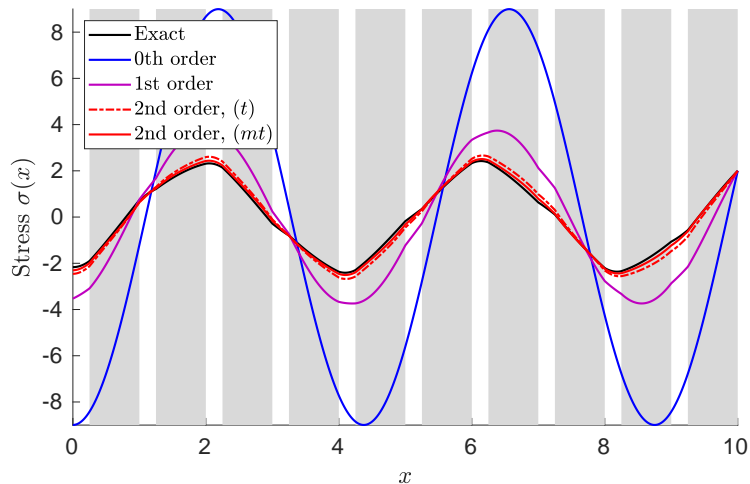
Example for a layered material - total displacement u

$$\varepsilon = \ell/\lambda_0 \approx 0.3$$

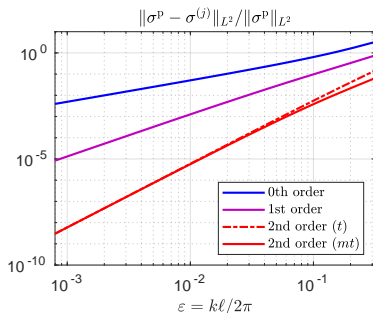
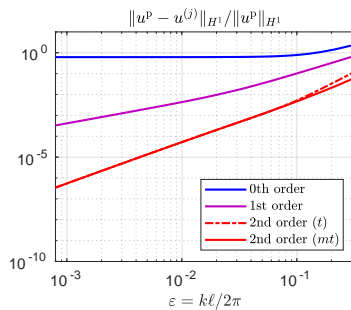
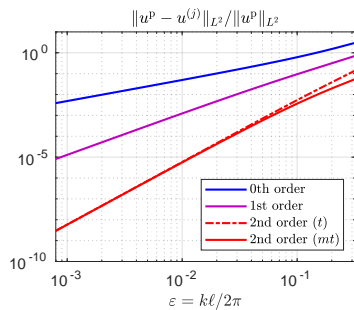


Example for a layered material - axial stress σ

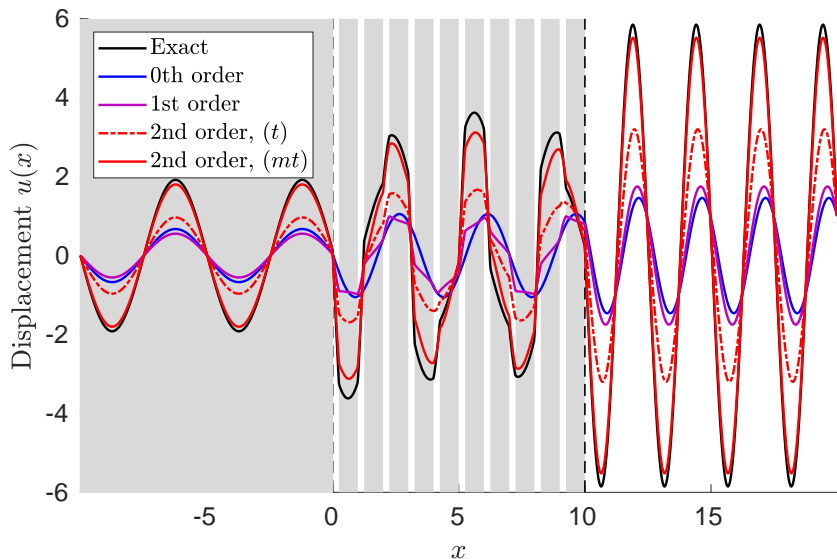
$$\varepsilon = \ell/\lambda_0 \approx 0.3$$



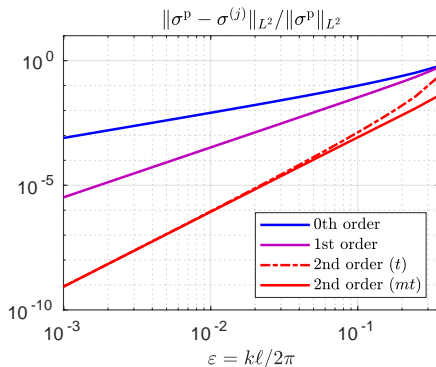
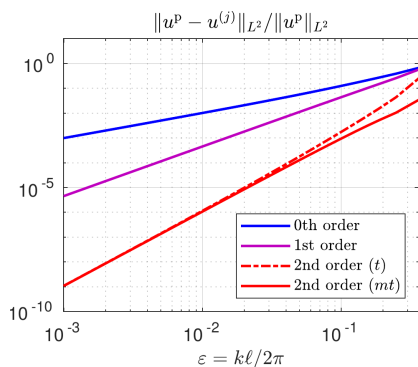
Errors for fixed ω and $\ell \rightarrow 0$



Application to a transmission problem ($\varepsilon \approx 0.3$)



Transmission problem - errors for fixed ω and $\ell \rightarrow 0$



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Equivalent hyperbolic system for the (mt) model

($\beta_x = 0$ because we don't want $U_{,xxxx}$ and additional BC/TC it brings)

Second-order homogenization, (mt) model:

$$\frac{1}{c_0^2} \left[1 + \varepsilon^2 \beta_m \partial_{xx} + \varepsilon^2 \frac{\beta_t}{c_0^2} \partial_{tt} \right] U_{,tt} = U_{,xx},$$

"Stress gradient" system [Forest and Sab, 2017]:

$$\begin{cases} \sigma = E \partial_x (u + \phi) \\ r = D \phi \end{cases} \quad \begin{cases} \rho \partial_{tt} u = \partial_x \sigma \\ \rho J \partial_{tt} \phi = \partial_x \sigma - r \end{cases}$$

So that:
$$\frac{\rho}{E} \left[1 - \frac{E(1+J)}{D} \partial_{xx} + \frac{\rho J}{D} \partial_{tt} \right] u_{,tt} = u_{,xx}$$

$$\Rightarrow \begin{cases} \partial_t V - \frac{a}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r \\ \partial_t S - \mathcal{E}_0 \partial_x V & = 0 \\ \partial_t \varphi - \frac{a-1}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r \\ \partial_t r & = \frac{\mathcal{E}_0}{\varepsilon^2 \beta} \varphi \end{cases}$$

- Macroscopic fields $V = \partial_t(u + \phi)$ and $S = \sigma$
- **Auxiliary fields** $\varphi = \partial_t \phi$ and r
- Parameter $a = -\beta_m / \beta_t$ to select a (mt) model.

Well-posedness (hyperbolicity, stability, conserved positive energy) for:

$$\beta \geq 0 \ (\checkmark), \quad \beta_m < 0 \quad \text{and} \quad \beta_t > 0$$

Hyperbolicity and stability (See B. Lombard's first lecture)

- Matricial form of system:

$$\partial_t \mathbf{U} + \mathbf{A} \cdot \partial_x \mathbf{U} = \mathbf{S} \cdot \mathbf{U}, \quad \text{where } \mathbf{U} := (V, S, \varphi, r)^T$$

$$\mathbf{A} = \begin{bmatrix} 0 & -a/\varrho_0 & 0 & 0 \\ -\mathcal{E}_0 & 0 & 0 & 0 \\ 0 & -(a-1)/\varrho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & -(a-1)/\varrho_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(a-1)/\varrho_0 \\ 0 & 0 & \mathcal{E}_0/(\varepsilon^2 \beta) & 0 \end{bmatrix}.$$

- Hyperbolicity : **real** eigenvalues of \mathbf{A}

$$\text{Sp}(\mathbf{A}) = \{0, 0, \pm c_0 \sqrt{a}\}$$

- Stability : **imaginary** eigenvalues of \mathbf{S}

$$\text{Sp}(\mathbf{S}) = \{0, 0, \pm c_0 / (\varepsilon \sqrt{-\beta_t})\}$$

$$\left\{ \begin{array}{l} \text{Hyperbolicity: } a > 0 \\ \text{Stability: } \beta_t > 0 \end{array} \right\} \implies \left\{ \begin{array}{l} -\frac{\beta_m}{\beta_t} > 0 \\ \beta_t > 0 \end{array} \right\} \implies \left\{ \begin{array}{l} \beta_m < 0, \\ \beta_t > 0. \end{array} \right.$$

Symmetrizer and energy conservation for $\partial_t U + A \cdot \partial_x U = S \cdot U$

- Symmetrizer matrix:

$$M = \begin{bmatrix} \rho_0 & 0 & -\rho_0 & 0 \\ 0 & 1/\varepsilon_0 & 0 & 0 \\ -\rho_0 & 0 & a\rho_0/(a-1) & 0 \\ 0 & 0 & 0 & \varepsilon^2\beta/\varepsilon_0 \end{bmatrix},$$

So that:

$$M \cdot A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M \cdot S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

- Multiplying the system by $U^T \cdot M$:

$$\partial_t \frac{1}{2} (U^T \cdot M \cdot U) + \frac{1}{2} \partial_x (U^T \cdot M \cdot A \cdot U) = 0.$$

Integrating over $\Omega \times [0, T]$, (large Ω so that $U(\cdot, T) = 0$ on $\partial\Omega$)

Energy conservation: $\mathcal{E}^{\text{vol}}(T) = \mathcal{E}^{\text{vol}}(0)$.

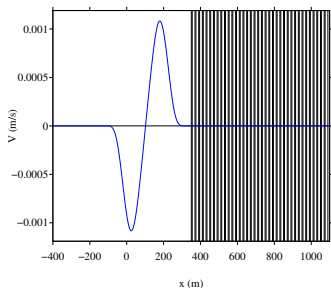
$$\mathcal{E}^{\text{vol}} = \frac{1}{2} \int_{\Omega} U^T \cdot M \cdot U \, dx = \frac{1}{2} \int_{\Omega} \rho_0 \left(v^2 + \frac{\beta_t}{\beta} \varphi^2 \right) + \frac{1}{\varepsilon_0} (\sigma^2 + \varepsilon^2 \beta r^2) \, dx \geq 0 \text{ if } \beta \geq 0$$

(with $v = V - \varphi$)

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First-order transmission conditions



- Fields of interest:

$$\begin{cases} (V, S) = (\partial_t U, E - \partial_x U) & x < 0 \\ (V, S, \varphi, r) & x > 0 \end{cases}$$

- Transmission conditions from **total fields continuity**:

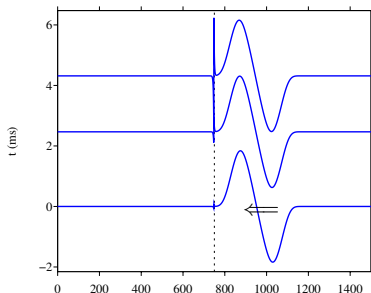
$$\begin{cases} V(0^-) = V(0^+) + \varepsilon P_1(0) V_{,x}(0^+) \\ S(0^-) = S(0^+) + \varepsilon \Sigma_1(0) S_{,x}(0^+) \end{cases}$$

- Energy analysis : **interface terms** when integrating $\partial_x(\mathbf{U}^T \cdot \mathbf{M} \cdot \mathbf{A} \cdot \mathbf{U})$ over $\Omega_- \cup \Omega_+$

$$\partial_t(\mathcal{E}_-^{\text{vol}} + \mathcal{E}_+^{\text{vol}}) + \mathcal{D}^{\text{int}} = 0,$$

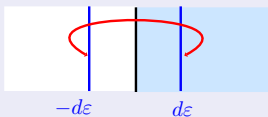
$$\begin{aligned} \mathcal{D}^{\text{int}} &= \llbracket V \rrbracket_0 \langle S \rangle_0 + \langle V \rangle_0 \llbracket S \rrbracket_0 \\ &= -\varepsilon [P_1(0)(\partial_x V)^+ \langle S \rangle_0 + \Sigma_1(0)(\partial_x S)^+ \langle V \rangle_0] \\ &= ??? \end{aligned}$$

- We would like $\mathcal{D}^{\text{int}} = \partial_t \mathcal{E}^{\text{int}}$ with $\mathcal{E}_1^{\text{int}} \geq 0$



First-order transmission conditions - stabilization

Spring-mass conditions across a thin interface $I_d = [-d\varepsilon, d\varepsilon]$ (additional $o(\varepsilon)$ error):



$$\begin{cases} \llbracket V \rrbracket_{I_d} = \varepsilon A_1 \partial_t \langle S \rangle_{I_d} & A_1 = -P_1(0) \mathcal{E}_0^{-1} + d(E_-^{-1} + \mathcal{E}_0^{-1}) \\ \llbracket S \rrbracket_{I_d} = \varepsilon B_1 \partial_t \langle V \rangle_{I_d} & B_1 = -\Sigma_1(0) \varrho_0 + d(\rho_- + \varrho_0) \end{cases}$$

- New expression of interface term:

$$\partial_t (\mathcal{E}_-^{\text{vol}} + \mathcal{E}_+^{\text{vol}}) + \mathcal{D}_d^{\text{int}} = 0,$$

$$\mathcal{D}_d^{\text{int}} = \partial_t \mathcal{E}_d^{\text{int}} \quad \text{with} \quad \mathcal{E}_d^{\text{int}} = \frac{\varepsilon}{2} \left[A_1 \langle S \rangle_{I_d}^2 + B_1 \langle V \rangle_{I_d}^2 \right]$$

- **Stability** of {Hyperbolic system + transmission conditions} for $A_1 \geq 0$ and $B_1 \geq 0$

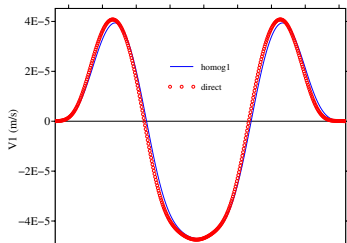
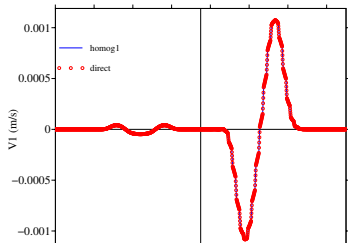
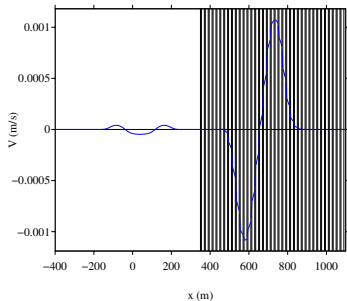
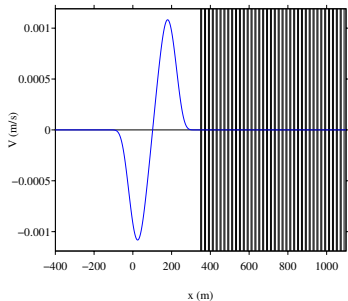
$$\Rightarrow \text{Choice of } d \geq d_{\min} := \max \left(\frac{E_- P_1(0)}{E_- + \mathcal{E}_0}, \frac{\varrho_0 \Sigma_1(0)}{\rho_- + \varrho_0}, 0 \right)$$

Well-posed “hybrid” model:

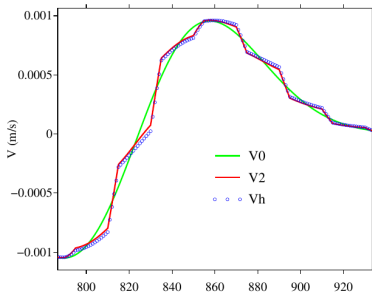
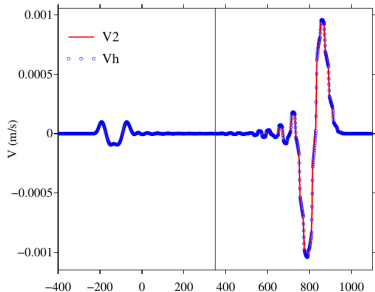
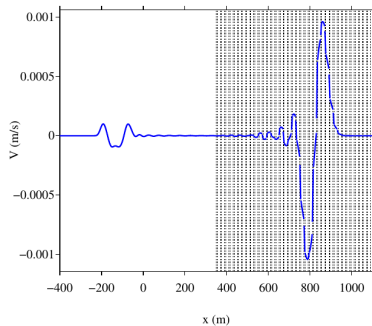
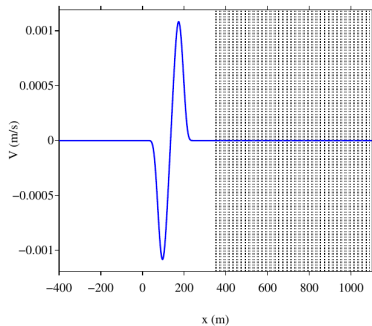
- **dispersive system** from 2nd-order bulk homogenization
- **transmission conditions** (1st-order for now)

Transmission conditions for $\varepsilon = \ell/\lambda_c \approx 0.04$ (Numerics by B. Lombard)

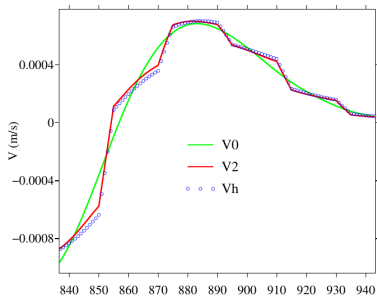
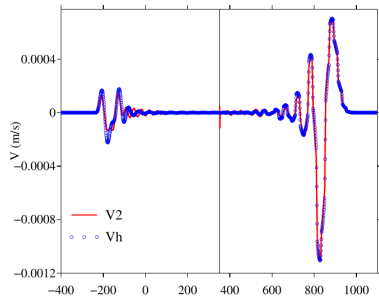
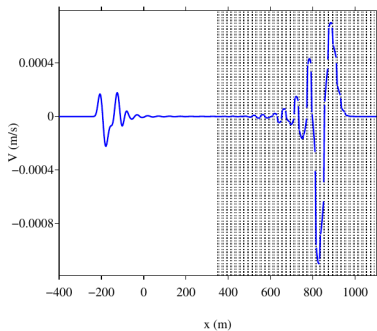
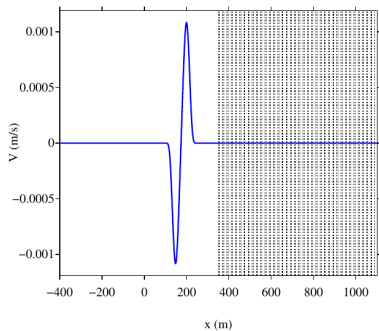
$E_{-\rho_-} = \mathcal{E}_0 \varrho_0 \implies$ reflected wave is a 1st-order effect



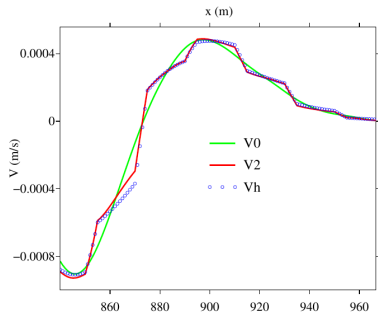
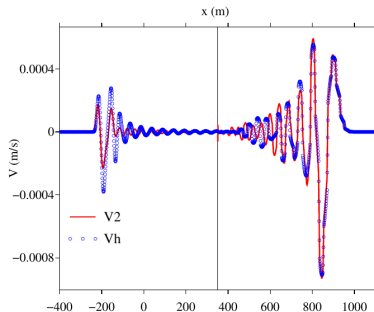
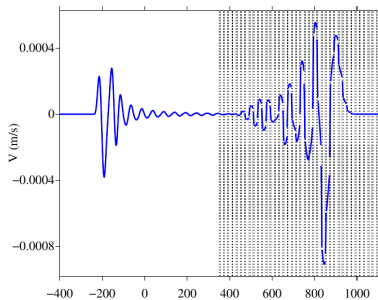
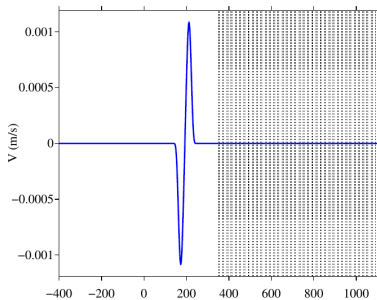
Transmission conditions, increasing central frequency, $\varepsilon \approx 0.09$



Transmission conditions, increasing central frequency, $\varepsilon \approx 0.13$



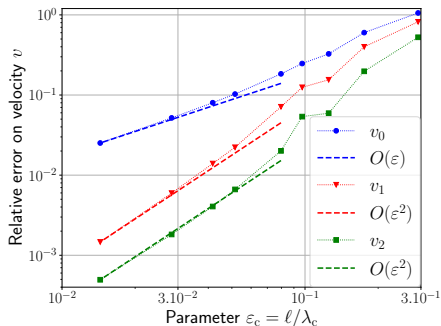
Transmission conditions, increasing central frequency, $\varepsilon \approx 0.17$



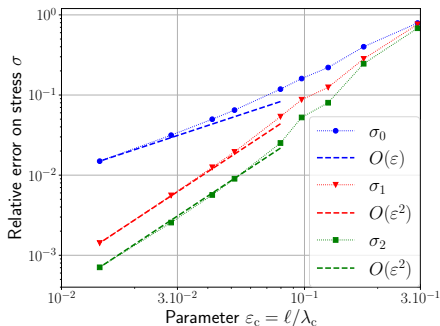
Transmission conditions, errors on velocity and stress fields

- Fields emitted at source $x_s < 0$, measured at receiver $x_r > 0$ during time T .
- Increasing frequency f , constant microstructure periodicity ℓ .
- Positions (x_s, x_r) and time T accordingly rescaled for each frequency.

$$\frac{\|(v_h - v_j)(x_r, \cdot)\|_{L^2(0,T)}}{\|v_h(x_r, \cdot)\|_{L^2(0,T)}}$$



$$\frac{\|(\sigma_h - \sigma_j)(x_r, \cdot)\|_{L^2(0,T)}}{\|\sigma_h(x_r, \cdot)\|_{L^2(0,T)}}$$



(v_1, σ_1) : Adding 1st-order bulk correctors (P_1, Σ_1) and 1st-order TC

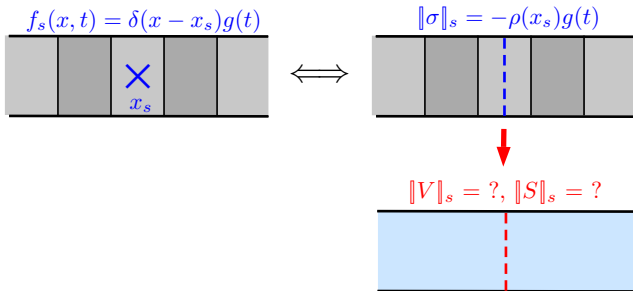
(v_2, σ_2) : “Hybrid” dispersive system (but still 1st-order TC)

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In 1D, source points \Leftrightarrow interfaces with stress jumps

$$\begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = \delta_s g(t) & x \in \mathbb{R} \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 & x \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} \partial_t v_\ell - \frac{1}{\rho_\ell} \partial_x \sigma_\ell = 0 & x < x_s \text{ and } x > x_s, \\ \partial_t \sigma_\ell - E_\ell \partial_x v_\ell = 0 & x < x_s \text{ and } x > x_s, \\ \llbracket v_\ell \rrbracket_s = 0 & x = x_s, \\ \llbracket \sigma_\ell \rrbracket_s = -\rho_s g & x = x_s, \end{cases}$$



First-order correction

- **Same tools** (and no need for an enlarged interface):

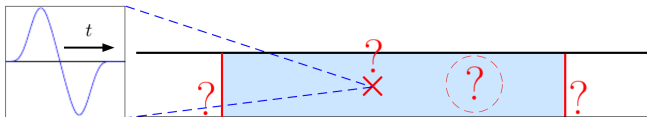
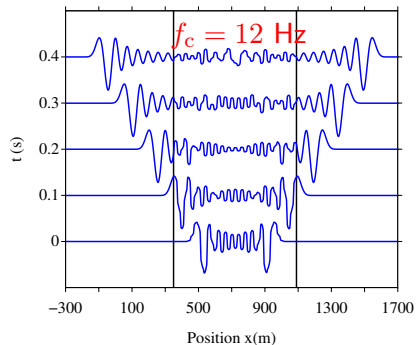
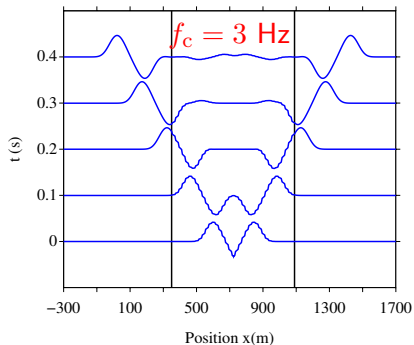
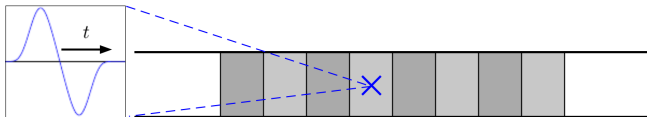
$$\begin{cases} \llbracket V \rrbracket_s = \varepsilon P_1(x_s/\varepsilon) \frac{\rho_s}{\mathcal{E}_0} \partial_t g, \\ \llbracket S \rrbracket_s = -\rho_s g, \end{cases}$$

- ▶ **First-order** corrector on V involving $\partial_t g$ and the local values ρ_s and $P_1(x_s/\varepsilon)$.
- ▶ Already given by [Capdeville et al., 2010] with another argument.

- Equivalent system with Dirac notation:

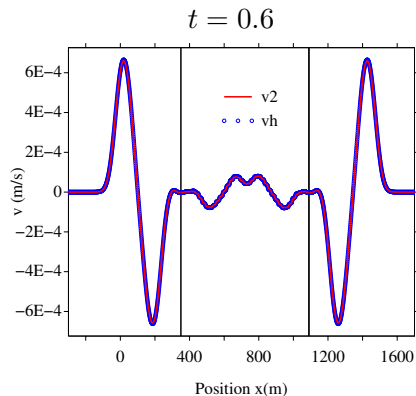
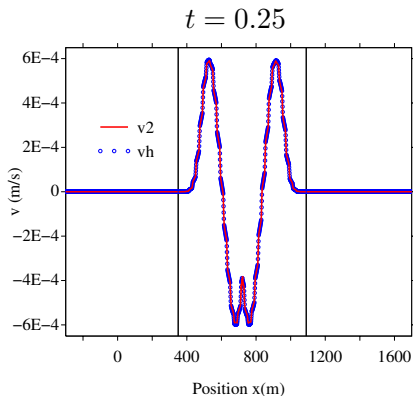
$$\begin{cases} \partial_t V - \frac{a}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r + \frac{a}{\varrho_0} \rho_s g \delta_s, \\ \partial_t S - \mathcal{E}_0 \partial_x V & = -\varepsilon \rho_s P_1(x_s/\varepsilon) \partial_t g \delta_s, \\ \partial_t \varphi - \frac{a-1}{\varrho_0} \partial_x S & = -\frac{a-1}{\varrho_0} r + \frac{a-1}{\varrho_0} \rho_s g \delta_s, \\ \partial_t r & = \frac{\mathcal{E}_0}{\ell^2 \beta} \varphi. \end{cases}$$

Back to the initial motivation



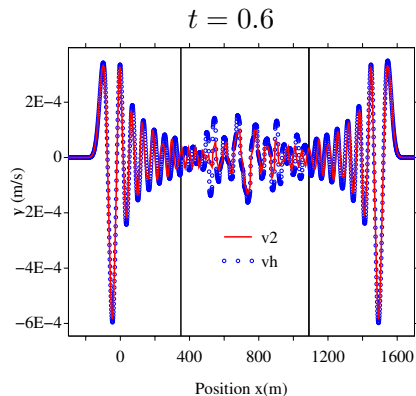
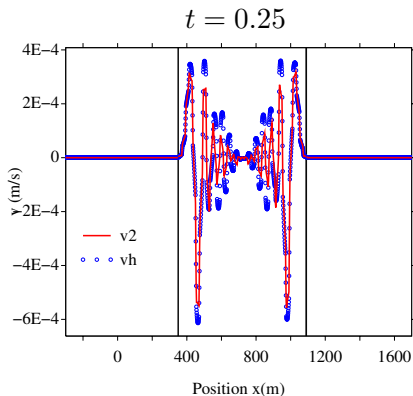
Velocity fields in a slab with a source point

$$f_c = 3 \text{ Hz}, \varepsilon_c = 0.04$$



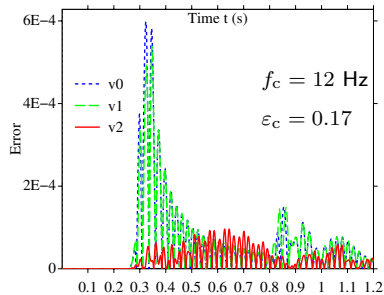
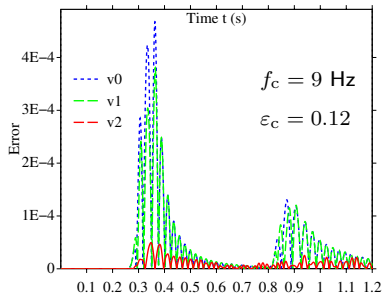
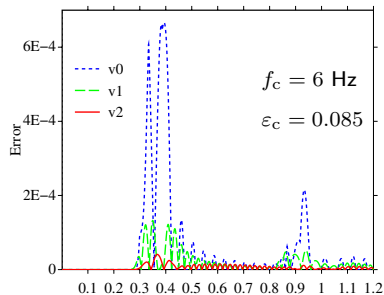
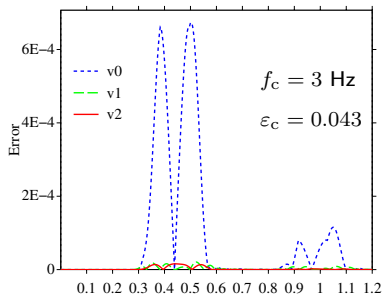
Velocity fields in a slab with a source point

$$f_c = 12 \text{ Hz}, \varepsilon_c = 0.17$$



Slab with a source point - errors at several frequencies

receiver in left homogeneous domain



Key ideas

- The second-order asymptotic homogenization accounts for dispersive effects.
- Reciprocity identities between cell problems are a big help.
- Boundary and transmission conditions can be designed to complement the inner expansion. (including in 2-3D [Vinoles, 2016, Maurel and Marigo, 2018, Cakoni et al., 2019, Beneteau, 2021] ...)
- For transient waves, **stability** can be addressed using (i) hyperbolic formalism and (ii) enlarged interfaces (in 1D for now).
- Source points can be addressed using the same tools (specific to 1D).

Perspectives

- Pursue the transient case up to *full second-order model* (as done in the time-harmonic case [Cornaggia and Guzina, 2020])
- Apply the built framework to other 1D configurations:
 - ▶ High-frequency homogenization [Craster et al., 2010, Guzina et al., 2019, Assier et al., 2020]
 - ▶ Solids with inner imperfect interfaces [Assier et al., 2020]
 - ▶ Non-linearities [Bellis et al., 2021]
- Reciprocity identities for higher dimensions and **elasticity** [Boutin and Auriault, 1993] (*ongoing*).
- Hyperbolic formalism for higher dimensions and other models ? (2D acoustics, elasticity ...)

Thanks for your attention !

*Second-order homogenization of boundary and transmission conditions
for one-dimensional waves in periodic media*

Rémi Cornaggia, Bojan B. Guzina

International Journal of Solids and Structures, 2020

*An homogenized model accounting for dispersion, interfaces and source points
for transient waves in 1D periodic media*

Rémi Cornaggia, Bruno Lombard

ESAIM: Mathematical Modelling and Numerical Analysis (M2AN), 2023

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