ANALYTICAL STUDY OF THE PANTOGRAPH EQUATION USING JACOBI THETA FUNCTIONS

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Abstract. The aim of this paper is to use the analytic theory of linear q-difference equations for the study of the functional-differential equation $y'(x) = ay(qx)+by(x)$, where a and b are two non-zero real or complex numbers. When $0 < q < 1$ and $y(0) = 1$, the associated Cauchy problem admits a unique power series solution, $\sum_{n\geq 0} \frac{(-a/b;q)_n}{n!}(bx)^n$, that converges in the whole complex x-plane. The principal result obtained in the paper explains how to express this entire function solution into a linear combination of solutions at infinity with the help of integral representations involving Jacobi theta functions. As a by-product, this connection formula between zero and infinity allows one to rediscover the classic theorem of Kato and McLeod on the asymptotic behavior of the solutions over the real axis.

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Introduction

Let's denote two non-zero real or complex numbers a and b. The following functional differential equation

$$y'(x) = ay(qx) + by(x) \tag{0.1}$$

has been first studied in detail in [6, 11] in the seventies of the last century. Since these works, many authors have worked on several extensions to systems or even to some nonlinear or variable-coefficients equations; while not claiming to be exhaustive or complete, we are content, for instance, to mention [3, 4, 8, 12] and the references therein. According to [6], the functional equation (0.1) with real coefficients a and b arises as a mathematical model of an industrial problem involving wave motion in the overhead supply line to an electrified railway system, so Eq. (0.1) is often called the pantograph equation.

As is often the case in the theory of q-difference equations [20,21], the situations are usually very different and may remain opposed to each other when $0 < q < 1$ is replaced by $q > 1$. This important fact has been developed in [11] by stating that the equation (0.1) with the boundary condition $y(0) = 1$ is well-posed only if $0 < q < 1$. In what follows, we shall treat only the case $0 < q < 1$; see §7 for some of the remarkable results due to Kato and McLeod [11] for this case.

In addition, the above-mentioned work [11] has been done within the real analysis' framework, while our present paper is devoted to an analysis of the functional

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differential equation (0.1) in the complex plane. This is the reason why the boundary problem studied in [11] will be transformed into a Cauchy problem. Accordingly, the real asymptotic behavior will be replaced with the asymptotic expansion over open sectors of the complex plane.

However, one can also notice that the analytic theory of linear functional q-difference equations is intimately linked with the theory of elliptic functions; see [2, 17]. It may probably be natural to make use of elliptic functions to study the functional differential equation (0.1), and that is exactly what we shall do in the present paper. More precisely, all analytic solutions of (0.1) will be represented, as often as possible, with the help of Jacobi's theta function, and it will be shown that almost all results given in [11] under the hypothesis $0 < q < 1$ can be obtained entirely from our analysis. It is also worth noticing that a family of non-lacunary power series having a natural boundary is deduced from the present work; see [25] and also §1.3 and §7, below.

At the same time, as Eq. (0.1) is a functional equation involving both the differential and q-difference operators, it will be shown that its power series solutions are made up of hypergeometric-type terms and their q-analogs, such as $(\alpha)_n$ and $(\alpha;q)_n$. Therefore, our present work may be seen as a preliminary step towards a theory of special functions including both usual and basic hypergeometric series; see (0.7) for the notations $(\alpha)_n$ and $(\alpha;q)_n$, and see our paper [20] for a general description of power series satisfying an analytic differential-q-difference equation.

The present paper is entirely devoted to the only case of $0 < q < 1$. We will treat the case of $q > 1$ in a future paper. The most important change in this case is that the functional differential equation (0.1) does not, in general, possess any analytic solution at the neighborhood of the origin or at infinity. Indeed, all power series solutions are divergent everywhere, and their coefficients have a growth such as $q^{n^2/2}$ or $n!$ as the index n tends to infinity. In order to obtain analytic solutions in sectors of the complex plane, one could apply some very different summation processes, that is Borel-Laplace summation and some of its q-analogs; see [15, 16, 21–23].

Organization of the paper. Throughout the whole paper, we will always suppose that $b \neq 0$ in the functional differential equation (0.1); by considering $y(-bx)$ instead of $y(x)$ in (0.1), one can suppose that $b = -1$, and this is what we shall do. Therefore, we are led to consider the equation

$$y'(x) = \alpha y(qx) - y(x), \quad \alpha = q^{\mu} \in \mathbb{C}^*, \tag{0.2}$$

where $\mu \in \mathbb{C}$.

We will start by establishing the fact that every $\mathcal{C}^1$-solution of (0.2) given on an interval $[0, r)$, $r \in \mathbb{C}^*$, can be continued into an entire function in the whole complex plane; see Proposition 1.1 in §1.1. Therefore, our study focuses on the analytic solution of (0.2) satisfying $y(0) = 1$, and this leads us to the power series $F(\mu; q, x)$ defined as being a combination of usual and basic-hypergeometric types of series; see (0.3). Concerning the point at infinity, one can find a system of solutions of the form $x^{-\mu_k} G_k(\frac{1}{x})$, where $k \in \mathbb{Z}$, $\mu_k = \mu - \frac{2\pi k i}{\ln q}$, and where $G_k(z)$ denotes some analytic function at $z = 0$ given also as a mix of two types of series; see (1.4).

Understanding the analytic structure of all solutions of (0.2) constitutes the main object of the present paper. For doing that, we shall express $F(\mu; q, x)$ in terms of all members of the infinite system $\{x^{-\mu_k} G_k(\frac{1}{x})\}_{k\in\mathbb{Z}}$, as given in Theorem 1.1.

This allows us to obtain, at infinity, an asymptotic expansion of F possessing as coefficients a family of q-periodic functions. Surprisingly, these functions can be represented by explicit non-lacunary power series and have a natural boundary; see Theorem 1.2 and Remark 1.2.

The principal steps we shall follow for proving Theorems 1.1 and 1.2 are inspired by the following observations:

(1) written as Dirichlet series, the following power series ($q^\mu = \alpha$)

$$F(\mu; q, x) = \sum_{n\geq 0} \frac{(q^\mu; q)_n}{n!}(-x)^n \tag{0.3}$$

can be viewed as a Laplace integral in the sense of the theory of q-integrals of Jackson; see Proposition 2.1 in §2.1, and Proposition 3.1 in §3.1;

(2) applying Laplace transform to functional equation (0.2) yields a first order homogeneous q-difference equation which admits a Fuchsian singularity at the origin and an irregular singularity at infinity; see §3.2;

(3) by considering the classical θ-modular relation as a connection formula for the functional equation $xy(qx) = y(x)$, that relates the analytic solution $\theta(x)$ on $\mathbb{C}^*$ and the ramified solution $e^{-\log^2 x/(2\ln q)}$, the q-difference equation obtained in (2) admits two types of solutions, and they are related via q-periodic functions; see Theorem 4.1 in §4.2.

Accordingly, we are led to consider, in Section 5, two families of Laplace integrals, one of which represents the solution F and the other, the functions G_k. Thanks to the well-known θ-modular formula, we can finish the proof of Theorems 1.1 and 1.2 in Section 6.

Finally, it may be interesting in this context to note that somewhat similar analysis was done in our previous works [24] and [9], establishing, respectively, connection formulae for q-Bessel functions and Ramanujan's entire function.

Notation. In what follows, we will denote by log the complex logarithm function defined over its Riemann surface $\tilde{\mathbb{C}}^*$, and $x^\alpha = e^{\alpha \log x}$ for all $\alpha \in \mathbb{C}$ and $x \in \tilde{\mathbb{C}}^*$. As usual, the set $\tilde{\mathbb{C}}^*$ will be identified with the product set $]0, \infty[\times\mathbb{R}$ via the relation $x = |x|e^{i \arg x}$. Moreover, the following notation will be used:

- for all a, $b \in \mathbb{R}$ such that $a < b$, we denote by $S(a, b)$ the open sector of $\tilde{\mathbb{C}}^*$ given by

$$S(a, b) = \{x \in \tilde{\mathbb{C}}^* : a < \arg x < b\}; \tag{0.4}$$

  by convention, one will write the right half-plane as $\mathbb{C}^+ = S(-\frac{\pi}{2}, \frac{\pi}{2})$;
- we denote by $\theta(q, x)$ the Jacobi theta function

$$\theta(q, x) = \sum_{n\in\mathbb{Z}} q^{n(n-1)/2} x^n; \tag{0.5}$$

  we will write $\theta(x)$ instead of $\theta(q, x)$ if any confusion does not occur;
- we denote by κ_q or simply κ the positive number given by the relation

$$\kappa = \kappa_q = -\frac{2\pi}{\ln q} > 0; \tag{0.6}$$

- for any $\alpha \in \mathbb{C}$, let $(\alpha)_n$ and $(\alpha;q)_n$ be the sequences given as follows: $(\alpha;q)_0 = (\alpha)_0 = 1$, and for $n \geq 1$:

$$(\alpha;q)_n = \prod_{j=0}^{n-1}(1-\alpha q^j), \quad (\alpha)_n = \prod_{j=0}^{n-1}(\alpha+j). \tag{0.7}$$

It is clear to see that $(\alpha;q)_n$ can be extended to $(\alpha;q)_\infty$ by taking $n \to \infty$.

1. Preliminary remarks and statement of results

Consider the functional differential equation (0.2), recalled as follows:

$$y'(x) = \alpha y(qx) - y(x), \quad \alpha \in \mathbb{C}^*. \tag{0.2}$$

In what follows, we will fix a $\mu \in \mathbb{C}$ such that $\alpha = q^\mu$. Moreover, if $k \in \mathbb{Z}$, we will write

$$\mu_k = \mu + k\kappa i\,, \tag{1.1}$$

where κ is as given in (0.6) in the above.

1.1. $\mathcal{C}^\infty$ **or analytic solutions.** Let Ω be a non-empty open set of $\mathbb{C}$ or $\mathbb{R}$ such that $q\Omega \subset \Omega$. For any $x_0 \in \Omega$, it follows that $q^n x_0 \in \Omega$ for all positive integer n, so that $0 \in \bar{\Omega}$. Moreover, if y denotes any given $\mathcal{C}^1$-solution of (0.2) on Ω, one may notice that y belongs necessarily to the set $\mathcal{C}^\infty(\Omega,\mathbb{C})$, by taking into account the following relations deduced from (0.2) by iteration:

$$y^{(n+1)}(x) = \alpha\, q^n\, y^{(n)}(qx) - y^{(n)}(x), \quad \forall n \in \mathbb{N}. \tag{1.2}$$

Lemma 1.1. *Let Ω be a connected open set of $\mathbb{C}$ such that $q\Omega \subset \Omega \neq \emptyset$, and let $y \in \mathcal{C}^\infty(\Omega;\mathbb{C})$. If y is a solution of (0.2) such that $\sup_{x\in\Omega,|x|<R}|y(x)| < \infty$ for some $R > 0$, then y can be analytically continued into an entire function.*

Proof. Firstly, one can notice that the following relation holds for any positive integer n :

$$y^{(n)}(x) = \sum_{k=0}^{n}(-1)^{n-k}\alpha^k\, q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q y(q^k x)\,, \tag{1.3}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}\,.$$

This can be easily checked by making use of (1.2), and we left the details to the interested reader.

Let $x_0 \in \Omega$ be such that $|x_0| < R$. By hypothesis, it follows that $|y(q^n x_0)| \leq K < \infty$ for all integer $n \geq 0$, so that, from (1.3) and [1, p. 484, (10.0.9)], one obtains easily that

$$|y^{(n)}(x_0)| \leq K\prod_{k=0}^{n}(1+|\alpha|q^k) \leq K(-|\alpha|;q)_\infty\,.$$

Accordingly, we find that y has a Taylor series expansion whose radius of convergence equals to infinity; in other words, y can be analytically continued over the whole complex-plane $\mathbb{C}$. □

By almost the same way as what done for Lemma 1.1, one can find the following result:

Proposition 1.1. *Let $d \in \mathbb{R} \bmod 2\pi\mathbb{Z}$ and let $R > 0$. Every given $\mathcal{C}^\infty$ solution of (0.2) on $[0, Re^{id})$ can be analytically continued to be an entire function.*

Accordingly, Eq. (0.2) has no nontrivial $\mathcal{C}^\infty$-solution on $[0, Re^{id})$ such that $y(0) = 0$.

Proof. Let y be a $\mathcal{C}^\infty$ solution of (0.2) on $[0, Re^{id})$. For any $x_0 \in (0, Re^{id})$, by considering the relation (1.3) with $x = x_0$ and by noticing $y(q^n x_0) \to y(0)$ for $n \to \infty$, one finds that y admits a Taylor expansion that converges on the whole plane. This implies that y can be extended into an entire function on $\mathbb{C}$.

Moreover, when $y(0) = 0$, the above-obtained Taylor expansion is zero and represents the trivial solution of (0.2). □

Consequently, we shall only consider the analytic solutions of (0.2) on $\mathbb{C}$.

1.2. **Connection formula between power series-type solutions.** Let y be an analytic function solution to (0.2) in a neighborhood of $x = 0$ in $\mathbb{C}$. If $a_n = y^{(n)}(0)$ for all integer $n \geq 0$, then putting $x = 0$ into Eq. (1.2) gives rise to the relation

$$a_{n+1} = -(1 - \alpha q^n)\, a_n\,.$$

Therefore, we are led to the power series solution $F(\mu; q, x)$ defined in (0.3). Since $(q^\mu; q)_n$ admits a finite limit as $n \to \infty$, one finds that $F(\mu; q, x)$ defines an entire function.

Proposition 1.2. *Let $\alpha = q^\mu$ as before, $d \in \mathbb{R} \bmod 2\pi\mathbb{Z}$, and $R > 0$. Then the entire function $F(\mu; q, x)$ represents the unique $\mathcal{C}^\infty$ solution of (0.2) on $[0, Re^{id})$ such that $y(0) = 1$.*

Proof. It follows on immediately from Proposition 1.1. □

In order to study the asymptotic behavior of $F(\mu; q, x)$ at infinity, we introduce the following power series:

$$G(a; q, x) = \sum_{n \geq 0} \frac{(a)_n q^{n(n+1)/2}}{(q; q)_n} (-x)^n\,, \tag{1.4}$$

which obviously defines an entire function.

One main result that we shall establish in the paper is the following

Theorem 1.1. *Let $\mu \in \mathbb{C}$, $\alpha = q^\mu \in \mathbb{C}^*$, and let μ_k as in (1.1). Let F and G be as in (0.3) and (1.4), respectively. Then the following properties hold:*

1. *the functions $F(\mu; q, x)$ and $x^{-\mu_k} G(\mu_k; q, \frac{1}{x})$, $k \in \mathbb{Z}$, all satisfy the functional differential equation (0.2);*
2. *moreover, if $\mu \in \mathbb{C} \setminus (\mathbb{Z}_{\leq 0} \oplus \kappa\mathbb{Z}i)$, then for all $x \in \mathbb{C}^+$, we have*

$$F(\mu; q, x) = \frac{\kappa\,(q^\mu; q)_\infty}{2\pi\,(q; q)_\infty} \sum_{k \in \mathbb{Z}} \Gamma(\mu_k) x^{-\mu_k} G(\mu_k; q, \frac{1}{x})\,, \tag{1.5}$$

where Γ denotes the usual Euler Gamma function.

If $\mu = -m - \epsilon$ and $m \in \mathbb{N}$, one has $(q^\mu; q)_\infty = (q^{-m-\epsilon}; q)_m\,(1 - q^{-\epsilon})\,(q^{1-\epsilon}; q)_\infty$ and $\Gamma(\mu) = \frac{\pi}{\sin(\pi(-m-\epsilon))} \frac{1}{\Gamma(1+m+\epsilon)}$. This implies that

$$\lim_{\epsilon \to 0} (q^\mu; q)_\infty\, \Gamma(\mu) = (-1)^{m+1}\, \ln q\, (q^{-m}; q)_m\, (q; q)_\infty / m!\,,$$

so that we can observe the following

Remark 1.1. When $\mu \in \mathbb{Z}_{\leq 0} \oplus \kappa\mathbb{Z}i$, the relation (1.5) is reduced to the following one:

$$F(-m;q,x) = \frac{(q;q)_m}{m!}\, q^{-m(m+1)/2}\, x^m\, G(-m;q,\frac{1}{x}) \qquad (m \in \mathbb{N}),$$

that can be directly verified as the functions F and G become polynomial.

1.3. **Natural boundary for the analytic continuation in terms of modular variable.** As in [25], let $\Psi(u,v,z)$ be the Laurent series of x associated with $(u,v) \in \mathbb{C}\times\mathbb{R}$ such that $u \notin \mathbb{Z}_{\leq 0} \oplus \frac{2vi}{\pi}\mathbb{Z}$:

$$\Psi(u,v,z) = \sum_{n\in\mathbb{Z}} \Gamma(u + \frac{2ivn}{\pi})\, z^n\,. \tag{1.6}$$

By the Stirling's formula on Γ (see [1, §1.4]), it follows that $\Psi(u,v,z)$ is convergent over the annulus $\mathcal{C}_\nu$, where

$$\mathcal{C}_\nu = \{z \in \mathbb{C} : e^{-|v|} < |z| < e^{|v|}\}\,.$$

By [25, Théorème 1], the function $z \mapsto \Psi(u,v,z)$ cannot be analytically continued beyond the borders $\partial\mathcal{C}_\nu$ of $\mathcal{C}_\nu$. Thus, the relation (1.5) can be stated as follows.

Theorem 1.2. *Let μ be as in Theorem 1.1* (2)*. Then, for all $x \in \mathbb{C}^+$, it follows that*

$$F(\mu;q,x) = \frac{\kappa\,(q^\mu;q)_\infty}{2\pi\,(q;q)_\infty}\,\big(\frac{1}{x}\big)^\mu \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n}\,\Psi(\mu+n,\frac{\kappa\pi}{2},x^*)\,\big(-\frac{1}{x}\big)^n\,, \tag{1.7}$$

where we denote by x^* the modular variable defined as follows:

$$x^* = x^{-\kappa i} = e^{2\pi i \frac{\log x}{\ln q}}\,.$$

From the formulas (1.5) and (1.7), one finds that $F(\mu;q,x) = O(x^{-\mu})$ as $x \to \infty$ in the right half plane $\mathbb{C}^+$. On the other hand, it will be seen that $F(\mu;q,x)$ is exponentially large if $\Re(x) \to -\infty$; see Theorems 2.1 and 2.2 in Section 2.

Remark 1.2. The formulas (1.5) and (1.7) are only valid for $\Re(x) > 0$, and this explains why **each function $\Psi(n+\mu,\frac{\kappa\pi}{2},x^*)$ has a natural boundary on the imaginary axis $\Re(x) = 0$ or, equivalently, on the circles $|x^*| = e^{\pm\pi^2/\ln q}$.** This is exactly the subject of [25, Théorème 1], which is proved by making use of lacunary Dirichlet series.

For the proof of Theorems 1.1 and 1.2, see §6.2.

2. Power series-type solutions and Dirichlet series representation

The Dirichlet series expansion techniques are often used for the investigations of pantograph equations; see [6–8, 14]. In what follows, the entire function $F(\mu;q,x)$ will be expanded as a Dirichlet series from the point of view of the q-series. This expansion will be used for the study of the asymptotic behavior of $F(\mu;q,x)$ while $x \to \infty$ inside the left half plane $\Re(x) < 0$.

2.1. **A Dirichlet series representation of $F(\mu;q,x)$.** The following expression of $F(\mu;q,x)$ may be known to the researchers of pantograph-type equations, but unfortunately the author has not found a precise reference about it. However, the proof we shall give seems somewhat interesting, combining the Hadamard product with a Heine formula for the q-series.

Proposition 2.1. *If $q^\mu = \alpha$ and $\Re(\mu) > 0$, then the following relation holds for all $x \in \mathbb{C}$:*

$$F(\mu;q,x) = (\alpha;q)_\infty \sum_{n\geq 0} \frac{\alpha^n e^{-q^n x}}{(q;q)_n}\,. \tag{2.1}$$

Proof. By considering $F(\mu;q,x)$ as the Hadamard product of the following power series:

$$\sum_{n\geq 0} (\alpha;q)_n x^n, \qquad \sum_{n\geq 0} \frac{(-1)^n}{n!} x^n,$$

one can find that

$$F(\mu;q,x) = \frac{(\alpha;q)_\infty}{2\pi i} \sum_{n\geq 0} \int_{|t|=r<1} \frac{1}{(q;q)_n}\, \frac{\alpha^n}{1-\frac{q^n x}{t}}\, e^{-t}\, \frac{dt}{t}\,,$$

where the integral is taken over the circle in the anti-clockwise sense. Moreover, by hypothesis, $|\alpha| < 1$; setting $a = q$, $b = \alpha$ and $c = 0$ in [1, p. 521, Theorem 10.9.1] implies that

$$\sum_{n\geq 0} (\alpha;q)_n x^n = (\alpha;q)_\infty \sum_{n\geq 0} \frac{1}{(q;q)_n}\, \frac{\alpha^n}{1-q^n x}\,.$$

The wanted relation (2.1) is thus obtained by Cauchy's formula. □

The right-hand side of (2.1) represents a *Dirichlet series* in the sense of [18, Chapter IX, §8, p. 432-440]. An alternative proof of Proposition 2.1 can be done by checking merely that this Dirichlet series converges really to an analytic solution in the complex x-plane of the Cauchy problem of (0.2) with the initial condition $y(0) = 1$; indeed, such an analytic solution is unique.

Remark 2.1. The power series $F(\mu;q,x)$ becomes a polynomial of x if, and only if, $(q^\mu;q)_\infty = 0$, which means exactly that $\mu \in \mathbb{Z}_{\leq 0} \oplus \kappa i\mathbb{Z}$; see also Remark 1.1.

2.2. **An auxiliary functional equation on $F(\mu;q,x)$.** In order to remove the condition $\Re(\mu) > 0$ from Proposition 2.1, one shall make use of the following functional relation:

$$\partial_x F(\mu;q,x) = (\alpha - 1)\, F(\mu+1;q,x)\,. \tag{2.2}$$

Indeed, one can obtain the last formula from direct computation by taking into account the identity

$$(\alpha;q)_{n+1} = (1-\alpha)(q\alpha;q)_n\,, \qquad \forall n \in \mathbb{N}.$$

Proposition 2.2. *For any positive integer k, it follows that*

$$\partial_x^k F(\mu;q,x) = (-1)^k (\alpha;q)_k\, F(\mu+k;q,x)\,. \tag{2.3}$$

Proof. Direct calculation by induction on k. □

The functional relation (2.2) gives rise to a characterization of the function $F(\mu;q,x)$, in view of the following:

Proposition 2.3. *Consider an analytic function $f(\alpha, x)$ in $\mathbb{C} \times \mathbb{C}$. If there exists an entire function $\alpha \mapsto u(\alpha)$ such that*

$$\partial_x f(\alpha, x) = u(\alpha) f(q\alpha, x),$$

then f is uniquely determined by its values taken at the complex line $x = 0$ in $\mathbb{C}^2$. More precisely, if we set $f_0(\alpha) = f(\alpha, 0)$, then f can be expanded in the following manner:

$$f(\alpha, x) = \sum_{n=0}^{\infty} \frac{f_n(\alpha)}{n!}\, x^n\,, \tag{2.4}$$

where, for all positive integer n,

$$f_n(\alpha) = u(\alpha) \cdots u(q^{n-1}\alpha)\, f_0(q^n \alpha)\,.$$

Proof. One may easily notice that f_n's satisfy the recurrent relation

$$f_n(\alpha) = u(\alpha)\, f_{n-1}(q\alpha)\,,$$

which allows us to conclude the proof. □

In the case of $f(\alpha, x) = F(\mu; q, x)$, relation (2.2) implies that $u(\alpha) = \alpha - 1$, with $f_0(\alpha) = 1$.

2.3. **Asymptotic behavior of $F(\mu; q, x)$ in the left half-plane.** Under the condition $\Re(\mu) > 0$, the formula (2.1) implies that the first term $(\alpha; q)_\infty\, e^{-x}$ is a dominating term of $F(\mu; q, x)$ as $\Re(x) \to -\infty$. The general case can be treated with the help of Proposition 2.2, as shown in the following

Theorem 2.1. *Let $\alpha = q^\mu \in \mathbb{C}^*$. The following limit holds as $x \to \infty$ in the left half-plane $\mathbb{C}^-$:*

$$\lim_{\Re(x) \to -\infty} e^x F(\mu; q, x) = (\alpha; q)_\infty\,. \tag{2.5}$$

More precisely, for any given open sector $V = S(a, b)$ with $\frac{\pi}{2} < a < b < \frac{3\pi}{2}$, there exists a positive constant C_V such that the following inequality holds for all $x \in V$:

$$\left| F(\mu; q, x) - (\alpha; q)_\infty\, e^{-x} \right| < C_V\, e^{-q\Re(x)}\,. \tag{2.6}$$

Proof. Let V be an open sector as given in Theorem 2.1. From Proposition 2.1, we obtain the expected relation (2.6) while the condition $\Re(\mu) > 0$ is assumed. For an arbitrary complex number μ, choose an enough large positive integer m such that $m + \Re(\mu) > 0$, and set $\alpha' = \alpha q^m = q^{\mu'}$ with $\mu' = \mu + m$. Therefore, one can write

$$F(\mu'; q, x) = (\alpha'; q)_\infty\, e^{-x} + h(x)\, e^{-qx}\,,$$

where h denotes a bounded analytic function over V.

Let $\beta = \alpha q^{m-1} = q^\nu$, with $\nu = \mu' + 1$. From (2.2), it follows that

$$F(\nu; q, x) = (\beta - 1) \int_0^x F(\mu'; q, t) dt\,,$$

where the integral is taken over the segment going from the point at origin to the point of affix x in V. An elementary calculation shows that

$$F(\nu; q, x) = (\beta; q)_\infty\, e^{-x} + H(x)\, e^{-qx}\,,$$

where

$$H(x) = -(\beta; q)_\infty\, e^{qx} + (\beta - 1) \int_0^x h(x - t)\, e^{qt}\, dt\,.$$

Thus, one finds easily that $F(\nu;q,x)$ satisfies the relation (2.6) while replacing μ by ν; therefore, the function $H(x)$ possesses the same property as $h(x)$ for $F(\mu';q,x)$. Consequently, one can continue this analysis and finally obtain the relation (2.6) for all $\mu \in \mathbb{C}$.

The relation (2.5) is an evident consequence of (2.6). □

From Remark 2.1, if $(q^\mu;q)_\infty = 0$, then $F(\mu;q,x)$ becomes a polynomial in x, thus one obtains the following

Remark 2.2. The function $F(\mu;q,x)$ is exponentially large for $x \in \mathbb{C}^-$ if and only if $\mu \notin \mathbb{Z}_{\leq 0} \oplus \kappa i\mathbb{Z}$.

On the other hand, the relation (2.6) can be improved to any order N as follows.

Theorem 2.2. *Let $\mu \in \mathbb{C}$, $\alpha = q^\mu$, and let $V = S(a,b)$ be an open sector such that $\frac{\pi}{2} < a < b < \frac{3\pi}{2}$. Then there exists a positive constant $C = C_V$ such that the following estimates hold for all integers $N \geq 1$ and all $x \in V$:*

$$\left|F(\mu;q,x) - (\alpha;q)_\infty \sum_{n=0}^{N-1} \frac{\alpha^n}{(q;q)_n} e^{-q^n x}\right| < C^N\, e^{-q^N \Re(x)}\,. \tag{2.7}$$

Proof. We omit the proof, which can be done in a similar manner as what proposed for the proof of Theorem 2.1. □

2.4. **Power series solutions at infinity involving $G(\mu;q,\frac{1}{x})$.** Replacing respectively α and y by q^μ and $x^{-\mu}\,(1+\sum_{n\geq 1} a_n x^{-n})$ in (0.2) leads us to the following relations:

$$a_{n+1} = \frac{\mu+n}{1-q^{-n-1}}\, a_n\,, \tag{2.8}$$

where $n \geq 0$. Thus one finds the following

Proposition 2.4. *For any $k \in \mathbb{Z}$, let μ_k be as in (1.1). Then $x^{-\mu_k}\, G(\mu_k;q,\frac{1}{x})$ is an analytic solution of (0.2) in the Riemann surface $\tilde{\mathbb{C}}^*$ of the logarithm.*

Proof. By replacing μ with any μ_k in the second relation of (2.8), it follows that

$$a_n = -\frac{(\mu_k+n-1)}{1-q^n}\,(q^n)\,a_{n-1} = \ldots = (-1)^n\,\frac{(\mu_k)_n}{(q;q)_n}\,q^{n(n+1)/2}$$

for all $n \in \mathbb{N}$. One gets thus the expression (1.4) for the definition of $G(\mu_k;q,\frac{1}{x})$.

It is obvious that $G(\mu_k;q,x)$ converges for all $x \in \mathbb{C}$, so that $x^{-\mu_k}\, G(\mu_k;q,\frac{1}{x})$ is analytic on the whole surface $\tilde{\mathbb{C}}^*$. □

Remark 2.3. The first assertion of Theorem 1.1 follows from Propositions 1.2 and 2.4.

3. Solving (0.2) by Laplace integrals

In [13], Mahler made use of an integral of the type $\displaystyle\int_{\mathbb{R}} u(t)e^{xq^{it}}dt$ to find a special solution for the functional equation

$$y(x+\omega) - y(x) = \omega f(qx), \quad \omega \neq 0. \tag{3.1}$$

Indeed, this integral permits to transform (3.1) into a simple difference equation as follows:

$$u(t+i) = \frac{e^{\omega q^{it}} - 1}{\omega}\, u(t)\,;$$

which is clearly equivalent to a first order q-difference equation if one writes $s = q^{it}$ and $U(s) = u(t)$.

Almost in the same way, a Laplace-type integral will be applied to the functional differential equation (0.2), which will be transformed into a first-order q-difference equation.

3.1. $F(\mu; q, x)$ **is viewed as q-analogue of the Laplace integral.** In the work [10] of F. H. Jackson (see also [1, §10.1], [5, §1]), the q-integral of a suitable function $f(t)$ over $[0, 1]$ is defined as follows:

$$\int_0^1 f(t)\, d_q t = (1-q) \sum_{n\geq 0} f(q^n) q^n.$$

By means of this discrete integral, we can express $F(\alpha; q, x)$ as a q-integral of Laplace type.

Proposition 3.1. *If $\alpha = q^\mu$ and $\Re(\mu) > 0$, then the following relation holds for all $x \in \mathbb{C}$:*

$$F(\mu; q, x) = \frac{(\alpha; q)_\infty}{(1-q)\,(q; q)_\infty} \int_0^1 (qt; q)_\infty\, e^{-tx}\, t^\mu\, \frac{d_q t}{t}\,. \tag{3.2}$$

Proof. Under the assumption, it follows that $|\alpha| < 1$, so that one can express $F(\alpha; q, x)$ by the Dirichlet series (2.1). Thus, putting together $\alpha^n = q^{n\mu}$ and

$$(q; q)_n = \frac{(q; q)_\infty}{(q \cdot q^n; q)_\infty}$$

in the expansion (2.1) permits to get the wanted q-integral representation (3.2). □

3.2. **From** (0.2) **to a q-difference equation via Laplace transform.** Let L be a smooth loop in the complex t-plane and let $qL = \{qt : t \in L\}$ be the loop obtained as the image of L for the operator $t \mapsto qt$. Consider the following Laplace integral:

$$y(x) = \int_L f(t) e^{-tx} \frac{dt}{t}\,, \tag{3.3}$$

where f denotes a unknown function. If we suppose L and f to be chosen such that

$$\int_{qL} f(t) e^{-tx} \frac{dt}{t} = \int_L f(t) e^{-tx} \frac{dt}{t}, \tag{3.4}$$

then the equation (0.2) will be transformed as follows:

$$-tf(t) = \alpha f(\frac{t}{q}) - f(t),$$

or, equivalently,

$$(1-qt) f(qt) = \alpha f(t)\,. \tag{3.5}$$

Equation (3.5) is Fuchsian at $t = 0$ and admits an irregular singular point at $t = \infty$; see [2, 19, 21]. If we write $f = gh$, we may decompose (3.5) into two q-difference equations:

$$(1-qt) g(qt) = g(t) \tag{3.6}$$

and

$$h(qt) = \alpha h(t). \tag{3.7}$$

By iterating (3.6), one obtains easily the following solution:

$$g(t) = (qt;q)_\infty = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n}(-t)^n, \tag{3.8}$$

which is an entire function with respect to the variable t. The power series expansion in (3.8) is due to Euler; see [1, p. 490, Corollary 10.2.2 (b)].

On the other hand, we may make use of several solutions of (3.7) and, by this way, we will get different solutions of (3.5). The choices we will consider are the following:

$$h(t) = t^\nu, \quad q^\nu = \alpha, \tag{3.9}$$

or

$$h(t) = \frac{\theta(\lambda t)}{\theta(\mu t)}, \quad \frac{\mu}{\lambda} = \alpha. \tag{3.10}$$

In Section 4, we shall consider links between the two solutions of (3.7), and in Section 5, two types of Laplace integrals will be studied.

4. Remarks on character functions

By [19], any Fuchsian type linear q-difference equation whose coefficients are analytic functions at $x = 0$ has a fundamental system of solutions made up of analytic functions in a whole neighborhood of the origin except over some q-spirals. Here "fundamental" means that the q-Wronskian of the system is not identically null. One main idea consists of making use of the character function $x \mapsto \dfrac{\theta(\mu x)}{\theta(\lambda x)}$ instead of the multi-valued function x^ν, the latter being traditionally used in this domain since Birkhoff [2]. Indeed, if $q^\nu = \frac{\lambda}{\mu} = a$ and $\sigma_q f(x) = f(qx)$, it follows:

$$\frac{\sigma_q x^\nu}{x^\nu} = \frac{\sigma_q \frac{\theta(\mu x)}{\theta(\lambda x)}}{\frac{\theta(\mu x)}{\theta(\lambda x)}} = a\,.$$

In this case, one finds that $x^\nu \dfrac{\theta(\lambda x)}{\theta(\mu x)}$ is σ_q-invariant or is called to be q-periodic.

In what follows, we shall make use of the θ-modular relation to find the Fourier expansion of such q-periodic functions.

4.1. **Character functions expressed by means of θ-modular relation.** Let κ be as given in (0.6). For any $x \in \tilde{\mathbb{C}}^*$, let

$$e(q,x) = e(x) = e^{-\frac{\log^2 \frac{x}{\sqrt{q}}}{2\ln q}}. \tag{4.1}$$

It is easy to see that both $\theta(x)$ and $e(x)$ satisfy the functional q-difference equation $xy(qx) = y(x)$. Furthermore, the well-known modular formula on $\theta(q,x)$ states that if we set

$$q^* = e^{-2\pi\kappa}, \quad x^* = \iota_q(x) = x^{-\kappa i},$$

then the following relation holds [1, p. 498, (10.4.2)]:

$$\theta(q, \sqrt{q}\,x) = \sqrt{\kappa}\, e(q, \sqrt{q}\,x)\, \theta(q^*, \sqrt{q^*}\,x^*)$$

or, equivalently,

$$\theta(q,-x)=\sqrt{\kappa}\,e(q,-x)\,\theta(q^*,-x^*), \tag{4.2}$$

where $-x=e^{i\pi}x$ in $e(q,-x)$; see [26] for a point of view of q-series.

Lemma 4.1. *The following identity holds for all $\mu\in\mathbb{C}^*\subset\tilde{\mathbb{C}}^*$:*

$$\frac{\theta(q,-q^{\mu}x)}{\theta(q,-x)}=q^{-\mu(\mu-1)/2}\,(e^{\pi i}x)^{-\mu}\,\frac{\theta(q^*,-e^{2\pi i\mu}x^*)}{\theta(q^*,-x^*)}\,. \tag{4.3}$$

Proof. It follows directly from (4.2). □

Remark that if $\mu=n\in\mathbb{Z}$, the relation (4.3) can be read as follows:

$$\theta(q^n x)=q^{-n(n-1)/2}\,x^{-n}\,\theta(x)\,. \tag{4.4}$$

4.2. **Decomposition of character functions into Laurent series.** The Jacobi triple product formula says that

$$\theta(x)=(q,-x,-\frac{q}{x};q)_{\infty}\,. \tag{4.5}$$

Therefore, one finds that for any given $\lambda\in\mathbb{C}^*\setminus q^{\mathbb{Z}}$, the function $x\mapsto\dfrac{\theta(-\lambda x)}{\theta(-x)}$ is analytic over $\mathbb{C}^*\setminus q^{\mathbb{Z}}$.

Lemma 4.2. *Let $\lambda\in\mathbb{C}^*\setminus q^{\mathbb{Z}}$ and let $m\in\mathbb{Z}$. If $q^m<|x|<q^{m-1}$, then the following Laurent series expansion holds:*

$$\frac{\theta(-\lambda x)}{\theta(-x)}=\frac{\lambda^{1-m}\,\theta(-\lambda)}{(q;q)_{\infty}^3}\sum_{\ell\in\mathbb{Z}}\frac{(q^{1-m}x)^{\ell}}{1-\lambda q^{\ell}}\,. \tag{4.6}$$

Proof. This can be seen as a special case of Ramanujan's ${}_1\psi_1$-summation formula. Indeed, putting $a=\lambda$, $b=q\lambda$ and replacing x by $q^{1-m}x$ in [1, p. 502, (10.5.3)] yields our wanted formula. □

4.3. **Fourier series expansion of character functions.** We shall conclude this section by proving the following

Theorem 4.1. *Let $\mu\in\mathbb{C}\setminus\mathbb{Z}$ and let $m\in\mathbb{Z}$. Then, the following relation holds for all $x\in S(-2m\pi,2(1-m)\pi)\subset\tilde{\mathbb{C}}^*$:*

$$\frac{\theta(-q^{\mu}x)}{\theta(-x)}=C(q,m,\mu)\,x^{-\mu}\sum_{\ell\in\mathbb{Z}}\frac{e^{2\pi(m-1)\kappa\ell}}{1-e^{2\pi i(\mu+\kappa i\ell)}}\,x^{-\kappa i\ell}, \tag{4.7}$$

where κ is given as in (0.6) *and where*

$$C(q,m,\mu)=\frac{\kappa\,(q^{\mu},q^{1-\mu};q)_{\infty}}{i\,(q,q;q)_{\infty}}\,e^{2(1-m)\pi i\mu}\,. \tag{4.8}$$

Proof. For any $x\in S(-2m\pi,2(1-m)\pi)$, it follows that

$$\Im(\log x)=\arg x\in(-2m\pi,2(1-m)\pi),$$

so that the following relation holds:

$$|x^*|=e^{\kappa\,\arg(x)}\in(q^{*m},q^{*m-1}),$$

where $q^* = e^{4\pi^2/\ln q} = e^{-2\pi\kappa}$. Thus, by Lemmas 4.1 and 4.2, if one writes

$$C(q,m,\mu) = \frac{q^{-\mu(\mu-1)/2}\, e^{(1-2m)\pi i\mu}\, \theta(q^*, -e^{2\pi i\mu})}{(q^*;q^*)_\infty^3}\,,$$

then one gets the following identity:

$$\frac{\theta(-q^\mu x)}{\theta(-x)} = C(q,m,\mu)\, x^{-\mu} \sum_{\ell\in\mathbb{Z}} \frac{q^{*(1-m)\ell}}{1-e^{2\pi i\mu}\, q^{*\ell}}\, x^{*\ell}\,.$$

Applying the θ-modular formula (4.2) to $\theta(q^*, -e^{2\pi i\mu})$ yields that

$$\theta(q^*, -e^{2\pi i\mu}) = \frac{q^{1/8}}{i\sqrt{\kappa}}\, e^{\mu\pi i + \kappa\pi/4}\, q^{\mu(\mu-1)/2}\, \theta(q, -q^\mu)\,.$$

Thus, by considering the η-modular relation [1, p. 538, Theorem 10.12.8]:

$$(q^*;q^*)_\infty = \frac{q^{1/24}}{\sqrt{\kappa}}\, e^{\kappa\pi/12}\, (q;q)_\infty\,,$$

one finds the given expression (4.8) for $C(q,m,\mu)$. This ends the proof of the expected relation (4.7). □

The series given in the right-hand side of (4.7) is a Fourier series expansion with respect to the variable $X = \frac{\log x}{\ln q}$, for which we have $e^{2\pi i X} = x^{-\kappa i} = x^*$.

Remark 4.1. When $\mu \to n \in \mathbb{Z}$, one can notice that

$$C(q,m,\mu) \sim (-1)^n \frac{\kappa}{i}\, q^{-n(n-1)/2}\, (1-q^{n-\mu})\,,$$

so that the relation (4.7) reduces to (4.4) up to replacing x with $-x$.

5. Two Laplace integrals

Let us come back to the Laplace integral (3.3) introduced in §3.2. The loop L will be chosen among two types of curves: closed curves near the point of origin, which will be denoted as $\mathcal{C}$, and half straight-lines starting from the point at origin.

5.1. **Function $I(\alpha;q,x)$.** Let $\alpha \in \mathbb{C}^*$. Let $\mathcal{C}$ be any smooth and anti-clockwise Jordan curve whose interior contains the set $q^{\mathbb{N}} = \{1, q, q^2, q^3, ...\}$. We consider the function $x \mapsto I(\alpha;q,x)$ defined by the following relation:

$$I(\alpha;q,x) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\theta(-\frac{\alpha}{t})}{(\frac{1}{t};q)_\infty}\, e^{-xt}\, \frac{dt}{t}\,.$$

From the analyticity of the function under the integral, one see easily that $I(\alpha;q,x)$ is independent of the choice of the curve $\mathcal{C}$.

Lemma 5.1. *Let $\alpha \in \mathbb{C}^*$. The function $x \mapsto I(\alpha;q,x)$ is the unique entire function solution of* (0.2) *such that $y(0) = (\frac{q}{\alpha};q)_\infty$.*

Proof. A direct computation shows that $x \mapsto I(\alpha;q,x)$ satisfies the given functional differential equation (0.2). Indeed, let

$$f(t) = \frac{\theta(-\frac{\alpha}{t})}{(\frac{1}{t};q)_\infty} = \frac{\theta(-\frac{qt}{\alpha})}{(\frac{1}{t};q)_\infty}, \quad h(t) = \frac{\theta(-\frac{\alpha}{t})}{\theta(-qt)} = \frac{\theta(-\frac{qt}{\alpha})}{\theta(-qt)}\,.$$

Thanks to Jacobi's triple product formula (4.5), we find that

$$f(t) = (q;q)_\infty\, g(t)\, h(t)\,,$$

where $g(t)$ denotes the function given by (3.8) and where h satisfies the q-difference equation (3.7). One may also notice that the conditions required for f and $L = \frac{1}{q}\mathcal{C}$ permit us to transform the functional equation (0.2) into (3.5).

In order to get the value taken at $x = 0$ for $I(\alpha; q, x)$, we may suppose that $|\alpha| < 1$, the general case resulting from a standard analytic continuation argument.

By applying the residues theorem to the integral $\displaystyle\int_{\mathcal{C}} f(t)\frac{dt}{t}$ $\left(= \displaystyle\int_{\frac{1}{q}\mathcal{C}} f(t)\frac{dt}{t}\right)$, we find the following relation:

$$I(\alpha; q, 0) = \sum_{n\geq 0} \frac{\theta(-\frac{\alpha}{q^n})}{(q^{-n}; q)_n (q; q)_\infty}.$$

From the relation (4.4) and the fact that

$$(q^{-m}; q)_m = (-1)^m \, (q; q)_m \, q^{-m(m+1)/2} \quad (m \in \mathbb{N})$$

one deduces that

$$I(\alpha; q, 0) = \frac{\theta(-\alpha)}{(q; q)_\infty} \sum_{n\geq 0} \frac{1}{(q; q)_n} \, (\alpha)^n.$$

By taking into account the following Euler's identity [1, p. 490, Corollary 10.2.2 (a)]:

$$\sum_{n\geq 0} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty} \quad (|x| < 1)\,, \tag{5.1}$$

one finally finds that

$$I(\alpha; q, 0) = \frac{\theta(-\alpha)}{(q, \alpha; q)_\infty}\,,$$

which, together with the Jacobi triple product formula (4.5), allows to complete the proof. □

Proposition 5.1. *The following relation holds for any non-zero complex number* $\alpha = q^\mu$*:*

$$I(\alpha; q, x) = (\frac{q}{\alpha}; q)_\infty \, F(\mu; q, x)\,. \tag{5.2}$$

In other words, if $\mu \notin \mathbb{N}^* \oplus \kappa i\mathbb{Z}$*, then*

$$F(\mu; q, x) = \frac{1}{(\frac{q}{\alpha}; q)_\infty} \, I(\alpha; q, x)\,. \tag{5.3}$$

Proof. By taking into account of Lemma 5.1, one needs only to notice that the function $x \mapsto F(\mu; q, x)$ is the unique function analytic over $\mathbb{C}$ that satisfies (0.2) with the condition initial $F(\mu; q, 0) = 1$; see Proposition 1.2. □

5.2. **Function** $I_\nu(\alpha; q, x)$**.** Let $\nu \in \mathbb{C}$, $\alpha = q^\mu \in \mathbb{C}^*$ be such that the following inequality holds:

$$\Re(\nu + \mu) > 0\,. \tag{5.4}$$

For any real number $d \in (0, 2\pi)$, we define

$$I_\nu^{[d]}(\alpha; q, x) = \int_0^{\infty e^{id}} \frac{\theta(-\frac{\alpha}{t})}{(\frac{1}{t}; q)_\infty} \, e^{-xt} \, t^\nu \, \frac{dt}{t}\,; \tag{5.5}$$

under the condition (5.4), the integral of (5.5) converges for all x belonging to the open sector $S(-d - \frac{\pi}{2}, -d + \frac{\pi}{2})$ of $\tilde{\mathbb{C}}^*$. Therefore, by the analytic continuation

process, we get an analytic function defined over the sector $S(-\frac{5\pi}{2}, \frac{\pi}{2})$; this function will be denoted by $I_\nu(\alpha; q, x)$.

By taking into account of the functional equation (4.4), one may remark that the following relation holds for any integer $k \in \mathbb{Z}$:

$$I_\nu(q^k\alpha; q, x) = \frac{q^{-k(k-1)/2}}{(-\alpha)^k}\, I_{\nu+k}(\alpha; q, x)\,. \tag{5.6}$$

In particular, when $\nu = 0$, the last formula can be read as follows:

$$I_k(\alpha; q, x) = (-\alpha)^k\, q^{k(k-1)/2}\, I_0(q^k\alpha; q, x)\,. \tag{5.7}$$

Lemma 5.2. *The function $x \mapsto I_\nu(\alpha; q, x)$ satisfies the following functional differential equation:*

$$y'(x) + y(x) - q^\nu \alpha y(qx) = 0. \tag{5.8}$$

Proof. The result may be proved by a direct computation, in the same spirit as the beginning of the proof of Lemma 5.1; see also §3.2. □

If we take the derivation with respect to x in the integral (5.5) of $I_\nu(\alpha; q, x)$, we find that for any positive integer k, the following identity holds:

$$\partial_x^k I_\nu(\alpha; q, x) = (-1)^k I_{\nu+k}(\alpha; q, x).$$

Thus, from (5.6) one deduces the following relation:

$$\partial_x^k I_\nu(\alpha; q, x) = \alpha^k\, q^{k(k-1)/2}\, I_\nu(q^k\alpha; q, x), \tag{5.9}$$

which is similar to that satisfied by $F(\mu; q, x)$; see (2.3).

5.3. **Two special cases for $I_\nu(\alpha; q, x)$.** Let us consider two particular cases: (1) $\nu \in \mathbb{Z}$; (2) $\alpha \in q^{\mathbb{Z}}$. In particular, the first case contains the case of $\nu = 0$.

Proposition 5.2. *Let $\nu = k \in \mathbb{Z}$, $\alpha = q^\mu \in \mathbb{C}^*$ to be such that the condition* (5.4) *is satisfied. Then, the following relation holds for all $x \in \mathbb{C}^+ = S(-\frac{\pi}{2}, \frac{\pi}{2})$:*

$$I_k(\alpha; q, e^{-2\pi i}\, x) - I_k(\alpha; q, x) = C_k(\alpha)\, F(\mu + k; q, x)\,, \tag{5.10}$$

where

$$C_k(\alpha) = 2\pi i\, (-\alpha)^k\, (\frac{q^{1-k}}{\alpha}; q)_\infty\, q^{k(k-1)/2}\,.$$

Proof. Notice that when $x \in \mathbb{C}^+$, both x and $xe^{-2\pi i}$ belong to $S(-\frac{5\pi}{2}, \frac{\pi}{2})$, so the left-hand side of (5.10) is well-defined on $\mathbb{C}^+$. By using the relation (5.7), one can only consider the case of $k = 0$. Since, by Cauchy formula,

$$I_0(\alpha; q, x) - I_0(\alpha; q, xe^{-2\pi i}) = 2\pi i\, I(\alpha; q, x)\,,$$

one completes the proof, with the help of Proposition 5.1. □

Proposition 5.3. *Let $m \in \mathbb{Z}$ and $\nu \in \mathbb{C}$. If $\Re(\nu) + m > 0$, then the following relation holds in the sector $S(-\frac{5\pi}{2}, \frac{\pi}{2})$:*

$$I_\nu(q^m; q, x) = K_\nu(m)\, \big(\frac{1}{x}\big)^{m+\nu}\, G(m + \nu; q, \frac{1}{x})\,, \tag{5.11}$$

where

$$K_\nu(m) = (-1)^m\, (q; q)_\infty\, q^{-m(m-1)/2}\, \Gamma(m + \nu)\,.$$

Proof. Putting $\alpha = 1$ and $k = m$ in (5.6) implies that

$$I_\nu(q^m; q, x) = (-1)^m q^{-m(m-1)/2} I_{\nu+m}(1; q, x),$$

so that one needs only to prove (5.11) with $m = 0$ and $K_\nu(0) = (q;q)_\infty \, \Gamma(\nu)$. Thus, we shall suppose that $m = 0$ in the statement of Proposition 5.3.

By making use of the triple product formula (4.5) and the Euler's formula (5.1), we may write

$$\frac{\theta(-\frac{1}{t})}{(\frac{1}{t};q)_\infty} = (q;q)_\infty \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n} \, (-t)^n \, .$$

With the help of Fubini Theorem and the Euler's Gamma function, as $\sum \int = \int \sum$, one may obtain that

$$\int_0^\infty \frac{\theta(-\frac{1}{t})}{(\frac{1}{t};q)_\infty} \, e^{-tx} \, t^\nu \, \frac{dt}{t} = (q;q)_\infty \sum_{n\geq 0} (-1)^n \frac{\Gamma(\nu+n)}{(q;q)_n} \, q^{n(n+1)/2} \, x^{-n-\nu} \, .$$

The proof is thus completed. □

Remark 5.1. Putting $m = 0$ in Proposition 5.3 yields the following integral representation:

$$G(\nu; q, \frac{1}{x}) = \frac{x^\nu}{\Gamma(\nu)} \int_0^{e^{id}\infty} (qt;q)_\infty \, e^{-xt} \, t^\nu \frac{dt}{t} \, , \tag{5.12}$$

where $d \in \mathbb{R}$, $\Re(\nu) > 0$ and $x \in S(-\frac{\pi}{2} - d, \frac{\pi}{2} + d)$.

In order to get the asymptotic expansion of $I_\nu(\alpha; q, x)$ as $x \to \infty$, we will make use of the relation (4.3), which is reduced from the θ-modular formula (4.2).

6. End of the Proof of Theorems 1.1 and 1.2

For any $\delta \in \mathbb{R}$ and $q \in (0, 1)$, we denote by $D_{\delta,q}$ or simply D_δ the following annulus:

$$D_{\delta,q} = D_\delta := \{z \in \tilde{\mathbb{C}}^* : e^{-3\kappa\pi/2} < |ze^\delta| < e^{3\kappa\pi/2}\}, \tag{6.1}$$

where $\kappa = \kappa_q$.

For any $u \in \mathbb{C}^+$, $\mu \in \mathbb{C} \setminus \mathbb{Z}$, consider the function $z \mapsto \Phi(u, \mu; q, z)$ given by the following relation:

$$\Phi(u, \mu; q, z) = z^{-\frac{\mu i}{\kappa}} \sum_{\ell \in \mathbb{Z}} \frac{\Gamma(u + \mu + \kappa i \ell)}{1 - e^{2\pi i(\mu + \kappa i \ell)}} \, z^\ell \, . \tag{6.2}$$

By using the Stirling formula

$$\Gamma(u + \mu \pm \kappa i \ell) = O(\ell^{u+\mu-\frac{1}{2}} \, e^{-\frac{\kappa\pi}{2}\ell})$$

for $\ell \to +\infty$, the Laurent series of (6.2) converges in the domain $D_{\kappa\pi}$. Consequently, $\Phi(u, \mu; q, z)$ represents an analytic function on the annulus $D_{\kappa\pi}$ of $\tilde{\mathbb{C}}^*$. Consider the function $\Psi(u, v, x)$ given in (1.6), which is related with $\Phi(u, \mu; q, z)$ in the following manner:

Proposition 6.1. *The following relation holds for all $z \in D_{-\kappa\pi} \cap D_{\kappa\pi}$:*

$$\Phi(u, \mu; q, z) - \Phi(u, \mu; q, z \, e^{-2\kappa\pi}) = z^{-\frac{\pi i}{\kappa}} \, \Psi(u + \mu, \frac{\kappa\pi}{2}, z) \, . \tag{6.3}$$

Proof. It follows from a direct computation by making use of the definition (1.6) of Ψ. □

By considering the fact that $\Psi(u,v,x)$ has a natural boundary (see [25]), one may notice that $\Phi(u,\mu;q,z)$ can not be analytically continued beyond the boundaries of his convergence ring $D_{\kappa\pi}$ in the Riemann surface $\tilde{\mathbb{C}}^*$.

6.1. **Expansion of $I_\nu(\alpha;q,x)$ by means of $\Phi(u,\mu;q,z)$.** Let $\alpha = q^\mu$, with $\mu \in \mathbb{C}$; we will consider the behavior of $I_\nu(q^\mu;q,x)$ as $x \to \infty$. The main result of this section is the following

Theorem 6.1. *Let $\mu \in \mathbb{C}$ and $\nu \in \mathbb{C}$ be such that $\Re(\mu+\nu) > 0$; let $\Phi(u,\mu;q,z)$ be the function given in (6.2). If $q^\mu \notin q^{\mathbb{Z}}$, then the following relation holds for all x belonging to the open sector $S(-\frac{5\pi}{2},\frac{\pi}{2})$:*

$$I_\nu(q^\mu;q,x) = C(q,\mu)\,\big(\frac{1}{x}\big)^\nu \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n}\,\Phi(n+\nu,\mu;q,x^*)\,\big(-\frac{1}{x}\big)^n\,, \tag{6.4}$$

where $C(q,\mu)$ denotes the constant given in the following manner

$$C(q,\mu) = \frac{\kappa\,(q^\mu,q^{1-\mu};q)_\infty}{i\,(q;q)_\infty}\,,$$

(which is related to the constant $C(q,m,\mu)$ of Theorem 4.1)

Proof. Consider the integral (5.5) of $I_\nu(\alpha;q,x)$ and write

$$\frac{\theta(-\frac{q^\mu}{t})}{(\frac{1}{t};q)_\infty} = (q;q)_\infty\, g(t)\, h(t),$$

where, as in the proof of Lemma 5.1 ($\alpha = q^\mu$), we set

$$g(t) = (qt;q)_\infty, \quad h(t) = \frac{\theta(-\frac{q^\mu}{t})}{\theta(-\frac{1}{t})}\,.$$

We apply Theorem 4.1 to expand $h(t)$ into a Fourier series for $t \in S(0,2\pi)$: since $\frac{1}{t} \in S(-2\pi,0)$, putting $m=1$ in (4.7) allows us to obtain the following expression:

$$h(t) = \frac{\kappa\,(q^\mu,q^{1-\mu};q)_\infty}{i\,(q,q;q)_\infty} \sum_{\ell\in\mathbb{Z}} \frac{t^{\mu+\kappa i\ell}}{1-e^{2\pi i(\mu+\kappa i\ell)}}\,,$$

where q^* was replaced by $e^{-2\pi\kappa}$ and $\kappa = \kappa_q = -\frac{2\pi}{\ln q}$.

Therefore, from the Euler's relation (3.8) it follows that

$$\frac{\theta(-\frac{q^\mu}{t})}{(\frac{1}{t};q)_\infty} = C(q,\mu) \sum_{n\geq 0}\sum_{\ell\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n}\,\frac{t^{n+\mu+\kappa i\ell}}{1-e^{2\pi i(\mu+\kappa i\ell)}}\,, \tag{6.5}$$

where

$$C(q,\mu) = \frac{\kappa\,(q^\mu,q^{1-\mu};q)_\infty}{i\,(q;q)_\infty}\,.$$

In (6.5), the double series indexed by n and ℓ is normally convergent on any compact of $S(0,2\pi)$. At the same time, in view of the relation $x^* = e^{-\kappa i}$, one may notice that

$$\int_0^\infty t^{n+\mu+\nu+\kappa i\ell}\,e^{-tx}\,\frac{dt}{t} = \Gamma(n+\mu+\nu+\kappa i\ell)\,\big(x^*\big)^{\ell-\frac{\mu i}{\kappa}}\,\big(\frac{1}{x}\big)^{n+\nu}.$$

Hence, if one considers the expansion (6.5) in the integral (5.5) and makes use of the termwise integration for each $e^{-tx}\,t^\gamma$, one can finally obtain the formula (6.4), according to Lebesgue's dominated convergence theorem. The proof of Theorem 6.1 is thus completed. □

Theorem 6.1 states a remarkable fact in relation to the asymptotic behavior at infinity of the function $I_\nu(q^\mu;q,x)$: *it can be expanded as a power series of* $\frac{1}{x}$ *having q-periodic functions as coefficients.* This phenomenon will also occur for $F(\alpha;q,x)$ and other functions.

6.2. **End of the Proof of Theorems 1.1 and 1.2.** The assertion (1) of Theorem 1.1 can be easily checked; see Remark 2.3. The relations (1.5) and (1.7) can be obtained directly one from the other, so we shall make use of Proposition 5.2 and of Theorem 6.1 to conclude only the proof of Theorem 1.2.

Let $x\in S(-\frac{\pi}{2},\frac{\pi}{2})$ and let $\mu\in\mathbb{C}^+$ such that $q^\mu\notin q^{\mathbb{Z}}$. In light of the relation $(xe^{-2\pi i})^*=x^*\,q^*=x^*\,e^{-2\pi\kappa}$, combining the formulas (6.3) and (6.4) implies the following identity:

$$I_0(q^\mu;q,x)-I_0(q^\mu;q,xe^{-2\pi i})=C(q,\mu)\,\big(\frac{1}{x}\big)^\mu\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(q;q)_n}\,\Psi(n+\mu,\frac{\kappa\pi}{2},x^*)\,\big(-\frac{1}{x}\big)^n\,,$$

where $C(q,\mu)$ is the constant defined in Theorem 6.1

By letting $k=0$ and $\alpha=q^\mu$ in the relation (5.10), one finds finally that

$$F(q^\mu;q,x)=C_0(q,\mu)\,\big(\frac{1}{x}\big)^\mu\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(q;q)_n}\,\Psi(n+\mu,\frac{\kappa\pi}{2},x^*)\,\big(-\frac{1}{x}\big)^n\,,$$

where

$$C_0(q,\mu)=-\frac{C(q,\mu)}{C_0(q^\mu)}=\frac{\kappa\,(q^\mu;q)_\infty}{2\pi\,(q;q)_\infty}\,.$$

Remark that one can remove the restriction $\Re(\mu)>0$ from the above-done analysis (see (2.2)), by reasoning with a standard analytic continuation process. Thus the proof of Theorem 1.2, and therefore that of Theorem 1.1, are achieved.□

7. Revisit one Theorem due to Kato and McLeod

By Theorem 1.5, one can give more precision to the following result, which constitutes probably one of the most important steps for the investigations of the asymptotic behavior of solutions of the functional differential equation (0.1).

Theorem 7.1 (Theorem 3, [11])**.** *Consider the boundary problem associated with equation* (0.1) *for* $0\leq x<\infty$*, and the boundary condition* $y(0)=1$*, and suppose that* $0<q<1$*,* $a\in\mathbb{C}^*$ *and* $b<0$*. Let* μ *to be a complex number such that* $q^\mu=-a/b$*. Then the following assertions hold:*

1. *There exists no solution* $y(x)$ *such that* $y(x)=o(x^{-\Re(\mu)})$ *as* $x\to\infty$*.*
2. *Every solution* $y(x)$ *is* $O(x^{-\Re(\mu)})$ *at the infinity and may be written as follows:*

$$y(x)=x^{-\mu}\big\{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n}\,g_n(\log x)\,(-\frac{1}{bx})^n\big\}\,, \tag{7.1}$$

where $g_0=g$ *denotes some* $\mathcal{C}^\infty(\mathbb{R};\mathbb{C})$*-periodic function of period* $|\ln q|$ *verifying*

$$|g^{(n)}(s)|\leq K^n q^{-n^2/2},\quad \forall n\in\mathbb{N} \tag{7.2}$$

for some constant $K>0$*, and where all the functions* g_n *are recursively given by the following relation:*

$$g_{n+1}=g_n'-(\mu+n)g_n\,. \tag{7.3}$$

Indeed, if one writes $f(x) = y(-bx)$, then f will satisfies the boundary problem about the equation (0.2) with $\alpha = -a/b$, $0 \leq x < \infty$ and $f(0) = 1$. By Proposition 1.2, the function f is *unique* and is necessarily represented by $F(\mu; q, x)$. Let $s = \log x$ and write $x^* = e^{-i\kappa s}$ in relation (1.7) of Theorem 1.1; one finds that the functions g_n, $n \geq 0$, appearing in (7.1) can be defined as follows:

$$g_n(s) = (-1)^n \, \frac{\kappa \, (q^\mu; q)_\infty}{2\pi \, (q; q)_\infty} \, \Psi(n + \mu, \frac{\kappa\pi}{2}, e^{-i\kappa s}) \, .$$

Therefore, one can easily get the conditions (7.2) and (7.3) by the definition (1.6) of Ψ; see also [25, §1.2] for the functional equation (7.3).

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