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Modou Kebe [(1)], El-Hadji Deme [(1,*)],
Yousri Slaoui [(2)] and Solym M. Manou-Abi [(3)]

(1) LERSTAD, UFR SAT, Université Gaston Berger, BP 234, Saint-Louis, Sénégal.

(2) LaMMASD, Département de Mathématiques, Université de Kara, BP 404, Kara, Togo

(3) Laboratoire de Mathématiques et Applications, UMR CNRS 7348 ,Université de Poitiers, France.

Abstract. The probability of ruin of an insurance company is one of the main risk measures considered in risk theory, and the problems of its calculation and approximation have attracted much attention. Statistical estimations have been developed on the ruin probability in infinite time for insurance loses from heavy-tailed distributions. However, these estimation suffer heavily from under-coverage or have a robustness problem, particularly when losses are contaminated by large variations in the arrival of claims. We therefore need another method for estimating the probability of ruin in infinite time for heavy-tailed losses. This is why, in this paper, we introduce a robust estimator of the infinite-time probability of ruin for such distributions. Our methodology is based on extreme value theory, which offers adequate statistical results for such distributions. Our approach is based on a sensitive distribution known as the t-Hill estimator (t-score or score moment estimation) for the index of any tail distribution and introduced in [Fabián and Stehlík (2009)]. We establish their asymptotic normality, and through a simulation study, illustrate their behavior in terms of absolute bias and mean squared error. The simulation results clearly show that our estimators perform well and that they are fairly robust to outliers.

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(1) Corresponding author: `elhadji.deme@ugb.edu.sn`

1 Introduction

The effect of insurance operations is the total or partial transfer of the financial consequences of the risk incurred by the insured to an insurance company. to an insurance company. The expenses covered by the company may correspond either to indemnities to be paid to third parties in respect of the insured's liability (civil, professional or other), or to compensation for damage suffered by the insured. But what happens to these insurance companies when they themselves are exposed to risk? One approach to this problem is based on the use of ruin theory (see, e.g., [Panjer and Willmot (1992)]).

In the field of insurance, risk is defined as the probability that an insurance company's reserve, i.e. the difference between the premium amount and the insurance amount, will be reduced of an insurance company, which is the difference between the total premiums received and the total amount of claims paid, becomes negative at some point. At that point, ruin is said to occur, due to a miscalculation of the policyholders' contribution rate, or claims that are too large to cover. Indeed, the probability of such an event is seen as a means of controlling risk behavior. It is also a useful way of controlling the insurer's funds in long-term planning. Let's recall the definition of a standard mathematical model for insurance risk (see, for example, [Čižek *et al* (2005)], p. 345).

The insurance company's initial capital is denoted u. The number of claims over the period $(0, t]$, denoted M_t, is described by a Poisson process with a fixed intensity (rate) $\lambda > 0$. We also define a sequence of non-negative random variables $\{X_i\}_{i=1}^{\infty}$ independent and identically distributed (i.i.d.) with a loss distribution function $F_1(x) := \mathbb{P}(X_i \leq x)$ representing loss severity, with an unknown finite mean $\mu_1 = \int_0^{\infty}(1-F_1(x))dx$. Knowledge of this mean value has been of great interest to insurance companies, since it is one of the most commonly used premium calculation principles, known as *the net premium*, and corresponds to the expected amount of claims for a given insurance period.

Let's then assume that $X_i^{'}s$ are independent of M_t and that the insurer collects a premium at a constant rate c per unit time and that the net profit condition is met, i.e. $c/\lambda > \mu_1$. The classical risk process $\{R_t\}_{t>0}$ is given by:

$$R_t := u + ct - \sum_{i=1}^{M_t} X_i, \quad t > 0.$$

The corresponding claim surplus process is defined by

$$S_t := u - R_t = ct - \sum_{i=1}^{M_t} X_i, \quad t > 0.$$

First of all, we're interested in the probability that S_t exceeds an initial reserve u at a time t before or at a horizon T. Explicitly, this probability can be written as follows:

$$\psi_T(u) := \mathbb{P}\left\{\sup_{0<t\leq T} S_t > u\right\}.$$

The ruin probability in infinite time is defined by,

$$\phi(u) := \lim_{T\to\infty} \psi_T(u). \tag{1.1}$$

In the actuarial field, the costs of large claims require the modeling of rare events, i.e. events with a low probability of occurrence, but with large claims amounts and disastrous effects. The analysis of these

extreme events can be carried out using the extreme value methodology, whose distribution functions F_1 are heavy-tailed and mainly characterized by their index, which indicates the size and frequency of certain extreme phenomena within a given probability distribution (see for example [Vandewalle and Beirlant (2006)]).

The heavy-tailed nature of claims requires particular attention to the analysis of tail distributions. Extreme value theory (EVT) therefore offers suitable statistical tools for modeling these distribution tails, see for example [Matthys *et al.* (2004)], [Vandewalle and Beirlant (2006)] [Reiss and Thomas (2007)], [Necir *et al.* (2010)], [Deme *et al.* (2013a)], [Deme *et al.* (2015)], [Deme *et al.* (2021)] etc... In addition, reinsurance companies have to calculate premiums to cover these excess claims, which are generally very high. EVT has become one of the leading theories in the development of statistical models for high insurance losses.

Now, suppose that F_1 is heavy-tailed, that is:

$$\lim_{x\to\infty} \exp(\delta x)(1-F_1) = \infty, \text{ for all } \delta > 0. \tag{1.2}$$

The class of regularly varying functions provides good examples of heavy-tailed models. We can cite the following models: Pareto, Burr, Student, Lévy-stable and log-gamma (see, for example, [Beirlant *et al* (2001)]). In the remainder of this paper, we restrict ourselves to this class of distributions. In other words, we assume that the survival function $1-F_1$ is smoothly varying at infinity with index $-1/\gamma < 0$, that

$$\lim_{t\to\infty} \frac{1-F_1(tx)}{1-F_1(t)} = x^{-1/\gamma}, \text{ for any } x > 0. \tag{1.3}$$

The parameter γ is the tail index and governs tail behavior, with higher values indicating heavier tails. For more details on these models, we can refer to [Bingham *et al.* (1987)], [Rolski *et al* (1999)] and [Reiss and Thomas (2007)]. It has been shown that for large initial reserve u, the ruin probability $\phi(u)$ can be approximated, under the assumption, by

$$\phi(u) := \left(\frac{c}{\lambda} - \mu_1\right)^{-1} \int_u^\infty (1-F_1(x))\,dx, \tag{1.4}$$

(see, e.g., [Asmussen (2000)]). This latter equation can be rewritten as

$$\phi(u) = \frac{\mu_2}{\omega - \mu_1}, \tag{1.5}$$

where $\mu_2 := \mathbb{E}(Y) < \infty$, with $Y := (X-u)_+ = \max(X-u, 0)$ and $\omega := c/\lambda$. Then, the distribution function of the stop-loss variable Y denoted by F_2 also satisfies the relationship (1.3) with the same index $-1/\gamma < 0$, i.e.

$$\lim_{t\to\infty} \frac{1-F_2(tx)}{1-F_2(t)} = x^{-1/\gamma} \text{ for any } x > 0. \tag{1.6}$$

Now let $Q_i,\ i = 1;2$ be the generalized inverse functions (or quantile functions) related to df $F_i,\ i = 1,2$ and defined as follows for all $s \in (0,1]$:

$$Q_i(s) := \inf\{x > 0 : F_i(x) \geq s\}.$$

From Corollary 1.2.10 (p. 23) in [**?**], we have for any $x > 0$

$$\lim_{s\downarrow 0} \frac{Q_i(1-sx)}{Q_i(1-s)} = x^{-\gamma}, \quad i = 1,2. \tag{1.7}$$

By a change of variable, the expected values $\mu_i = \int_0^\infty (1 - F_i(x))dx$, $i = 1, 2$ can be rewritten in terms of quantile function Q_i as follows: $\mu_i = \int_0^1 Q_i(s)ds$, $i = 1, 2..$ Thus, the probability of ruin in infinite time can be rewritten as follows:

$$\phi(u) = \frac{\int_0^1 Q_2(s)ds}{\omega - \int_0^1 Q_1(s)ds}. \tag{1.8}$$

Notice that for fixed large u, we have

$$1 - F_1(x) \sim 1 - F_2(x) \quad \text{as} \quad x \to \infty,$$

and therefore

$$Q_1(1 - s) \sim Q_2(1 - s) \quad \text{as} \quad s \downarrow 0.$$

This paper is organized as follows. In section 2, we present some preliminaries on classical estimators of the probability of ruin $phi(u)$. As these estimates suffer greatly from under-coverage or have a robustness problem, especially when losses are contaminated by large variations in the arrival of claims, we introduce, in subsection 3.1, a robust infinite-time estimator of the probability of ruin for heavy-tailed insured losses. Using extreme value methodology, we establish its asymptotic distribution in subsection 3.2. In subsection 4.1, we perform a simulation study to illustrate the behavior of our robust estimator compared to the classical estimator in terms of absolute bias and mean squared error. In subsection 4.2, we present a contamination study in which the robustness of the estimator is evaluated. All proofs are reported in section 5.

2 Estimating the ruin probability $\phi(u)$

First, we set a large initial reserve u. Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ be two independent samples of risks X and Y respectively. The non-parametric estimators of the distribution functions F_1 and F_2 are respectively defined as follows $F_{1,n}(x) = n^{-1} \sum_{i=1}^n \mathbb{I}_{\{X_i \leq x\}}$ and $F_{2,n}(y) = n^{-1} \sum_{i=1}^n \mathbb{I}_{\{Y_i \leq y\}}$. Thus, their corresponding empirical quantile functions are expressed by $Q_{i,n}(s) = \inf\{x; F_{i,n(}x) \geq s\}$, $i = 1, 2$ where $\mathbb{I}_S$ is the indicator function of the set S. Let's denote by $X_{1,n} \leq \ldots \leq X_{n,n}$ and $Y_{1,n} \leq \ldots \leq Y_{n,n}$ the order statistics associated respectively with the samples $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$. Therefore, $Q_{1,n}(t) = X_{j,n}$ and $Q_{2,n}(t) = Y_{j,n}$ for all $t \in ((j-1)/n, j/n]$, and for all $j = 1, ..., n$.

To this end, a natural candidate for the empirical estimator of $\phi(u)$ is obtained by replacing in (1.8) the real quantiles $Q_1(\cdot)$ and $Q_2(\cdot)$ by their respective sample quantiles $Q_{1,n}(\cdot)$ and $Q_{2,n}(\cdot)$. We obtain the following "traditional" non-parametric estimator of the probability of ruin:

$$\overline{\phi}_n(u) = \frac{\overline{Y}}{\omega - \overline{X}}. \tag{2.9}$$

where $\overline{X} := \frac{1}{n} \sum_{j=1}^n X_j$ and $\overline{Y} := \frac{1}{n} \sum_{j=1}^n Y_j$ are the sample estimators the mean of μ_1 and μ_2 respectively.

Note that for $\gamma \geq 1$, the expected value of X (respectively Y) does not exist. Consequently, we will focus exclusively on distributions whose tail indices lie in the unit interval $0 < \gamma < 1$.

Next, the random variable $\sqrt{n}\left(\overline{\phi}_n(u) - \phi(u)\right)$ can be rewritten as:

$$\frac{1}{\left(\omega - \overline{X}\right)(\omega - \mu_1)} \sqrt{n} \left(\left(\overline{Y} - \mu_1\right)(\omega - \mu_1) + \left(\overline{X} - \mu_1\right)\mu_2\right).$$

According to the law of large numbers (LLN), the random variable $(\omega - \overline{X})$ converges in probability to $(\omega - \mu_1)$. Using asymptotic theory for L-statistics (e.g. [Shorack and Wellner (1986)]), and the underlying distributions with a with a sufficient number of finite moments, we obtain from [Jone and Zitikis(2007)] the following asymptotic normality result:

$$\sqrt{n}\left(\overline{\phi}_n(u) - \phi(u)\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_\phi^2\right), \text{ as } n \to \infty, \tag{2.10}$$

where

$$\sigma_\phi^2 = \frac{1}{(\omega - \mu_1)^4}\left((\omega - \mu_1)^2 \sigma_2^2 + \mu_2^2 \sigma_1^2 + \mu_2(\omega - \mu_1)\,\sigma_{1,2}\right) < \infty,$$

with σ_1^2 and σ_2^2 represent respectively the variances of X and Y and $\sigma_{1,2} = \int_0^1 \int_0^1 (\min(s,t) - st) dQ_1(1-s) dQ_2(1-t)$.

Note that in the case where the tail index γ is in the lower half of the unit interval, i.e. $0 < \gamma < \frac{1}{2}$, the second-order moments for the two random variables X and Y are finite. Consequently, the asymptotic normality of $\phi_n(u)$ in (2.10) holds. This result is not respected when the loss distribution is heavy-tailed with an index γ located in the upper half of the unit interval, i.e. $1/2 < \gamma < 1$, since the asymptotic variance σ_ϕ^2 is infinite, which is due in this case to the infitability of the second-order moments of the loss X. In this case, $\phi(u)$ must be estimated using another approach that would guarantee asymptotic normality.

To remedy this situation, [Rassoul (2014)] used extreme value theory taking into account Hill's estimator [Hill (1975)] estimator of the tail index γ and introduced a semi-parametric estimator for the probability of ruin $\phi(u)$ for heavy-tailed losses with infinite second-order moments.
The estimation of γ has been extensively studied in the literature, and γ is a positive-tail index. Hill's estimator is the most popular estimator of the positive-tail index γ in extreme value theory, and is defined as follows:

$$\widehat{\gamma}_{1,n,k} := k^{-1} \sum_{i=1}^{k} j \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right),$$

for an intermediate sequence $k = k(n)$, *i.e.*, are sequences such that $k \to \infty$ and $k/n \to 0$, as $n \to \infty$. On the other hand, the Hill estimator associated with the stop loss sample $(Y_1, \ldots, Y_n$ is given as follows:

$$\widehat{\gamma}_{2,n,\ell} := \ell^{-1} \sum_{i=1}^{\ell} j \left(\log Y_{n-j+1,n} - \log Y_{n-j,n}\right),$$

where $\ell = \ell(n)$ is another intermediate sequences satisfying $\ell \to \infty$ and $\ell/n \to 0$, as $n \to \infty$. In extreme value theory, Hill's estimator has been extensively studied, improved and even generalized to any real parameter γ (see e.g. [Dekkers *et al* (1989)], [Beirlant *et al.* (1999)], [Lo and Fall (2011)]). Its weak consistency was established under the condition of regular variation by [Mason (1982)] assuming only that the underlying distribution varies regularly at infinity. [Deheuvels *et al* (1988)] proved the strong consistency of Hill's estimator.
However, the asymptotic normality of Hill's estimator has been studied, under various conditions relating to the tail of the distribution, by many researchers, including [Csörgő *et al.* (1985)], [Beirlant and Teugels (1989)], [Dekkers *et al* (1989)].

Note that μ_1 and μ_2 can be rewritten respectively as follows :

$$\mu_1 = \int_0^{1-k/n} Q_1(s)\, ds + \int_{1-k/n}^{1} Q_1(s)\, ds \tag{2.11}$$

and

$$\mu_2 = \int_0^{1-\ell/n} Q_2(s)\, ds + \int_{1-\ell/n}^{1} Q_2(s)\, ds. \tag{2.12}$$

Let us define respectively the following estimators for $Q_1(s)$ and $Q_2(s)$, $s \in [0,1)$:

$$\widehat{Q}_{1,n,k}(s) = \begin{cases} Q_{1,n}(s) & \text{for } 0 \le s \le 1-k/n, \\ \\ Q^W_{1,n,k}(s) & \text{for } 1-k/n < s < 1, \end{cases}$$

and

$$\widehat{Q}_{2,n,\ell}(s) = \begin{cases} Q_{2,n}(s) & \text{for } 0 \le s \le 1-\ell/n, \\ \\ Q^W_{2,n,\ell}(s) & \text{for } 1-\ell/n < s < 1, \end{cases}$$

where $Q_{1,n}(s)$ and $Q_{2,n}(s)$ are the above empirical quantile estimators, $Q^W_{1,n,k}(s) = ((1-s)n/k)^{-\widehat{\gamma}^H_{1,n,k}} X_{n-k,n}$ and $Q^W_{2,n,\ell}(s) = ((1-s)n/\ell)^{-\widehat{\gamma}^H_{1,n,\ell}} Y_{n-\ell,n}$ are respectively the Weissman's estimators ([**?**]) of high quantiles $Q_1(s)$ and $Q_2(s)$ for $s \to 1$.

Replacing in (2.11), $Q_1(s)$ (resp. in (2.12), $Q_2(s)$) by $\widehat{Q}_{1,n,k}(s)$ (resp. by $\widehat{Q}_{2,n,\ell}(s)$), we arrive respectively at the following alternative estimators for the means m_1 and μ_2 when losses are heavy tailed with tail index γ in the upper haft of unit interval ($1/2 < \gamma < 1$):

$$\widehat{\mu}_{1,n,k} == n^{-1} \sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n} \frac{X_{n-k,n}}{(1-\widehat{\gamma}_{1,n,k})},$$

and

$$\widehat{\mu}_{2,n,\ell} = n^{-1} \sum_{i=1}^{n-\ell} Y_{i,n} + \frac{\ell}{n} \frac{Y_{n-\ell,n}}{(1-\widehat{\gamma}_{2,n,\ell})}.$$

These estimators of means were first studied by [Peng (2001)] and also generalized in [**?**], [Necir *et al.* (2010)], [**?**], [Deme *et al.* (2015)] and [Deme *et al.* (2021)] to assess financial and actuarial risk measures.

As in (2.9), substituting $\widehat{\mu}_{1,n}$ and $\widehat{\mu}_{2,n}$ with μ_1 and μ_2, respectively, on the right-hand side of the Equation (1.5), [Rassoul (2014)] introduced the following alternative estimator for the ruin probability $\phi(u)$:

$$\widetilde{\phi}_{n,k,\ell}(u) := \frac{\widehat{\mu}_{2,n,\ell}}{\omega - \widehat{\mu}_{1,n,k}} \quad \text{for } 1/2 < \gamma < 1. \tag{2.13}$$

[Rassoul (2014)] established the asymptotic normality of the estimator $\widetilde{\phi}_{n,k,\ell}(u)$ under certain restrictive assumptions. Finally, an asymptotic normal of $\phi(u)$, for $0 < \gamma < 1$ takes the following form:

$$\widehat{\phi}_n(u) := \begin{cases} \overline{\phi}_n(u), & \text{for } 0 < \gamma \le 1/2, \\ \widetilde{\phi}_{n,k,\ell}(u), & \text{for } 1/2 < \gamma < 1. \end{cases}$$

Note that the alternative ruin probability estimator $\widetilde{\phi}_{n,k,\ell}(u)$ is associated to the the Hill estimators $\widehat{\gamma}_{1,n,\ell}$ and $\widehat{\gamma}_{2,n,\ell}$.

It is well known that these Hill estimators are both pseudo-maximum likelihood estimators based on the exponential approximation of normalized log-spacings, i.e., $V_{1,j} := j(\log X_{n-j+1,n} - \log X_{n-j,n})$, for $j = 1, ..., k$ and $V_{2,j} := j(\log Y_{n-j+1,n} - \log Y_{n-j,n})$, for $j = 1, ..., \ell$, see, eg. [Beirlant *et al.* (2004)], [Beirlant *et al.* (1999)]. Clearly, these Hill estimators depend respectively on the choice of sample fractions k, ℓ and their influence functions are slowly increasing but not bounded. Consequently, these estimators are not very robust to large values of $V_{bullet,j}$, which makes the estimator of the probability of ruin $widetilde\phi_{n,k,\ell}\,(u)$ sensitive. This constitutes a serious problem in terms of bias and root mean square error (RMSE). To overcome this problem, we introduce in the next section a robust estimator of the probability of ruin $\phi(u)$ for heavy-tailed distributions whose index lies in the upper half of the unit interval, and establish its asymptotic properties.

3 Robust Estimator and main results

3.1 Robust Estimator for the ruin probability $\phi(u)$

To solve the aforementioned problem of the classical Hill estimator, [Fabián and Stehlík (2009)] proposed a sensible distribution known as the t-Hill estimator (t-score or score moment estimate) for the tail index of any distribution that varies the tail regularly. In addition, [Jordanova and Pancheva (2012)] discovered the limiting distribution of the t-Hill estimator in the case where the rank $S = k,\ \ell$ of the higher-order statistic is $o(n)$ and proved its asymptotic normality.

This estimator of the score moment has been studied in [Stehlík *et al*(2010)] and [Stehlík *et al*(2012)]. According to these authors, this estimator is more robust than the classic Hill estimator. Recently, several studies on t-Hill have been published, see [Beran *et al* (2014)] and [Jordanova *et al* (2016)]. In order to improve the quality of the averages μ_1 and μ_2 given respectively in (2.11) and (2.12), which allows us to improve the quality of the infinite-time ruin probability for a heavy-tailed distribution, instead of implementing the Hill estimator, we propose to estimate the tail index γ by the so-called t-score moment procedure, in order to obtain a robust result.

The formula of the t-Hill estimators of γ are given by:

$$\widehat{\gamma}^{\,tH}_{1,n,k} =: \left(\frac{1}{k}\sum_{i=1}^{k}\frac{X_{n-k,n}}{X_{n-i+1,n}}\right)^{-1} - 1 \tag{3.14}$$

and

$$\widehat{\gamma}^{\,tH}_{2,n,\ell} =: \left(\frac{1}{\ell}\sum_{i=1}^{\ell}\frac{Y_{n-\ell,n}}{Y_{n-i+1,n}}\right)^{-1} - 1. \tag{3.15}$$

For other robust estimators of the tail index γ, we may refer the reader to [[Juárez SF and Schucany WR (2004)], [Kim and Lee (2008)], [Peng and Welsh (2001)], [Vandewalle *et al*(2017)]].

As already mentioned, the $\widehat{\phi}_{n,k,\ell}\,(u)$ estimator given in (1.5) is not robust. To this end, we provide a solution using the t-Hill estimator of γ to derive a robust estimator of the infinite-time probability of ruin for heavy-tailed $\phi\,(u)$ distributions. We follow the same method and steps as [Rassoul (2014)] to write this new estimator, but instead of the simple tail index estimators $\widehat{\gamma}^{\,tH}_{1,n,k}$ and $\widehat{\gamma}^{\,tH}_{2,n,\ell}$ defining respectively in

(3.14) and (3.15).and we introduce the following robust estimators of the ruin probability $\phi(u)$:

$$\widehat{\phi}_{n,k,\ell}^{tH}(u) = \frac{\widehat{\mu}_{2,n,\ell}^{tH}}{\omega - \widehat{\mu}_{1,n,k}^{tH}}, \text{ for } 1/2 < \gamma < 1, \tag{3.16}$$

where

$$\widehat{\mu}_{1,n,k}^{tH} := n^{-1} \sum_{i=1}^{n-k} X_{i,n} + \frac{k}{n} \frac{X_{n-k,n}}{(1 - \widehat{\gamma}_{1,n,k}^{tH})}$$

and

$$\widehat{\mu}_{2,n,\ell}^{tH} := n^{-1} \sum_{i=1}^{n-\ell} Y_{i,n} + \frac{\ell}{n} \frac{Y_{n-\ell,n}}{(1 - \widehat{\gamma}_{2,n,\ell}^{tH})}.$$

In the next subsection, we establish the asymptotic properties of our proposed estimator of the probability of ruin.

3.2 Asymptotic Results of the estimator $\widehat{\phi}_{n,k,\ell}^{tH}(u)$

As usual in the extreme value framework, to prove asymptotic normality results, we need a second-order condition on the function $\mathbb{U}_i(x) = Q_i(1 - 1/x)$, $x > 1$, $i = 1, 2$, such as the following:

Condition $(R_{\mathbb{U}_i})$. There exist a function $A_i(x) \longrightarrow 0$ as $x \longrightarrow \infty$ of constant sign for large values of x and a second-order parameter $\rho_i < 0$ such that, for every $x > 0$

$$\lim_{t \to \infty} \frac{\log \mathbb{U}_i(tx) - \log \mathbb{U}_i(t) - \gamma \log(x)}{A_i(t)} = \frac{x^{\rho_i} - 1}{\rho_i}, \qquad i = 1, 2.$$

Note that condition $(\mathcal{R}_{\mathbb{U}_i})$ implies that $|A_i|$ is regularly varying with index ρ_i (see, e.g., [Geluk and de Hann (1987)], [de Haan and Ferreira (2006)]). It is satisfied for most of the classical distribution functions such as the Pareto, Burr, and Fréchet ones.

Theorem 3.1 *Assume that F_i satisfies $(\mathcal{R}_{\mathbb{U}_i})$ with $\gamma \in (1/2, 1)$. Then for any sequence of integer $k = k(n)$ and $\ell = \ell(n)$ satisfying $k \to \infty$, $k/n \to 0$, $\sqrt{k} A_1(n/k) \to 0$, $\ell \to \infty$, $\ell/n \to 0$, $\sqrt{\ell} A_2(n/\ell) \to 0$ and $\ell/k \to \theta < \infty$ as $n \to \infty$, we have:*

$$\frac{\sqrt{n}\left(\widehat{\phi}_{n,k,\ell}^{tH}(u) - \phi(u)\right)}{(k/n)^{1/2} \mathbb{U}_1(n/k)} \overset{\mathcal{D}}{=} \kappa_1 \sum_{i=1}^{3} \mathbb{W}_{n,i} + \theta^{(1/2-\gamma)} \kappa_2 \sum_{i=1}^{3} \overline{\mathbb{W}}_{n,i} + o_{\mathbb{P}}(1)$$

where

$$\begin{cases} \mathbb{W}_{n,1} = -\sqrt{\frac{n}{k}} \int_0^{1-\frac{k}{n}} \frac{\mathbb{B}_n(s)}{Q_1\left(1 - \frac{k}{n}\right)} dQ_1(s), \\ \mathbb{W}_{n,2} = -\frac{\gamma}{(1-\gamma)} \sqrt{\frac{n}{k}} \mathbb{B}_n\left(1 - \frac{k}{n}\right), \\ \mathbb{W}_{n,3} = \frac{\gamma(\gamma+1)^2}{(1-\gamma)^2} \int_0^1 s^{\gamma-1} \mathbb{B}_n(s)\, ds, \end{cases} \quad \begin{cases} \overline{\mathbb{W}}_{n,1} = -\sqrt{\frac{n}{\ell}} \int_0^{1-\frac{\ell}{n}} \frac{\mathbb{B}_n(s)}{Q_2\left(1 - \frac{\ell}{n}\right)} dQ_2(s), \\ \overline{\mathbb{W}}_{n,2} = -\frac{\gamma}{(1-\gamma)} \sqrt{\frac{n}{\ell}} \mathbb{B}_n\left(1 - \frac{\ell}{n}\right), \\ \overline{\mathbb{W}}_{n,3} = \frac{\gamma(\gamma+1)^2}{(1-\gamma)^2} \int_0^1 s^{\gamma-1} \mathbb{B}_n(s)\, ds, \end{cases}$$

with $\kappa_1 = \mu_2/(\omega-\mu_1)^2$, $\kappa_2 = (\omega-\mu_1)^{-1}$ *and* $\mathbb{B}_n(s)$, $0 \le s \le 1$ *is a sequence of Brownian, Bridges.*

Now, by computing the asymptotic variances of the different processes appearing in Theorem 3.1, we deduce the following corollary:

Corollary 3.1 *Under the assumptions of Theorem 3.1, we have*

$$\frac{\sqrt{n}\left(\widehat{\phi}^{tH}_{n,k,\ell}(u) - \phi(u)\right)}{(k/n)^{1/2}\,\mathbb{U}_1(n/k)} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2(\gamma,\theta)\right)$$

where

$$\begin{aligned}\sigma^2(\gamma,\theta)) &= \frac{\gamma^2}{(1-\gamma)^2(2\gamma-1)}\left(\kappa_1^2 + \kappa_2^2\,\theta^{(1-2\gamma)} + 2\,\kappa_1\kappa_2\right) \\ &\quad + \frac{\gamma^2(\gamma+1)^2}{(2\gamma+1)(1-\gamma)^4}\left(\kappa_1 + \kappa_2\,\theta^{(1/2-\gamma)}\right)^2.\end{aligned}$$

Remark 3.1 *In the case where* $k \sim \ell$, *as* $n \to \infty$, *we have* $\theta = 1$ *and the asymptotic variance* $\sigma^2(\gamma,\theta)$ *is equal to* $4\gamma^5(\kappa_1+\kappa_2)^2/(1-\gamma)^4(4\gamma^2-1)$.

4 Simulation Study

4.1 Performance and comparative study

In this simulation study, we examine the performance of the new estimator $\widehat{\phi}^{tH}_{n,k,\ell}(u)$ given in (3.16) with the classical estimator $\widehat{\phi}_{n,k,\ell}(u)$ proposed by [Rassoul (2014)] and defined in (1.5). Thus, we generate $N = 1000$ samples $(X_1, ..., X_n)$ with the sample size $n = 1000,\ 1500,\ 2000$ from a Pareto distribution function defined as: $F_1(x) = 1 - x^{-1/\gamma}$, $x \ge 1$ with extreme value index $\gamma \in \{2/3,\ 3/4\}$. For a given initial large reserve $u = 1.5$, we derive for each sample its corresponding excess of loss $(Y_1, ..., Y_n)$, where $Y_j = \max(X_j - u, 0)$.

The ruin probability estimators $\widehat{\phi}^{tH}_{n,k,\ell}(u)$ and $\widehat{\phi}_{n,k,\ell}(u)$ are computed with the parameter $\omega = c/\lambda = 18$ (in order to ensure that $\mu_2 < \omega - \mu_1$) and with respectively the tail index estimators $\gamma^{H}_{i,n,S}$ and $\gamma^{tH}_{i,n,S}$, $(i,S) \in \{(1,k),\ (2,\ell)\}$, for different sample fractional numbers of top order statistics $k = 1, \ldots, n-1$ and $\ell = 1, \ldots, m_n - 1$, where m_n is the number of positive values of $Y_j \neq 0$, $j = 1, \ldots, n$. Employing the algorithm of [Reiss and Thomas (2007)], Page 137, the optimal values k^* and ℓ^* of the number of top extremes of k and ℓ to compute the ruin probability estimators are respectively values k^* and ℓ^* defined as:

$$k^* = \arg\min_k \frac{1}{k}\sum_{j=1}^{k} j^{\delta}\left|\widehat{\gamma}^{\bullet}_{1,n,j} - \text{median}\left(\widehat{\gamma}^{\bullet}_{1,n,1}, ..., \widehat{\gamma}^{\bullet}_{1,n,k}\right)\right|,\ 1 \le k \le n-1, \tag{4.17}$$

and

$$\ell^* = \arg\min_\ell \frac{1}{\ell}\sum_{j=1}^{\ell} j^{\delta}\left|\widehat{\gamma}^{\bullet}_{2,n,j} - \text{median}\left(\widehat{\gamma}^{\bullet}_{2,n,1}, ..., \widehat{\gamma}^{\bullet}_{2,n,\ell}\right)\right|,\ 1 \le \ell \le m_n - 1, \tag{4.18}$$

where $0 \le \delta < 1/2$ and $\widehat{\gamma}^{\bullet}_{1,n,k}$ (respectively $\widehat{\gamma}^{\bullet}_{2,n,\ell}$, is either the Hill's or the t-Hill's estimator of the tail index γ computed with the sample $(X_1, \ldots X_n)$, respectively with the excess sample $(Y_1, \ldots, Y_n)$. By the way, choosing $\delta = 1/4$, we compute the optimal values k^* and ℓ^* as in (4.17) and (4.18) for each tail index estimator used in the computation of their associated ruin probability estimators.

• Next, we compare the performance of the above-mentioned ruin probability estimators by computing the absolute value of the mean as well as the mean square errors (MSE) based on the $N = 500$ simulated samples, and defined as follows:

$$\text{ABias}\left(\phi^{\bullet}_{n,k^*,\ell^*}(u)\right) := \left|\frac{1}{N}\sum_{j=1}^{N}\frac{\phi^{\bullet,j}_{n,k^*,\ell^*}(u)}{\phi(u)} - 1\right|$$

and

$$\text{MSE}\left(\phi^{\bullet}_{n,k^*,\ell^*}(u)\right) := \frac{1}{N}\sum_{j=1}^{N}\left(\frac{\phi^{\bullet,j}_{n,k^*,\ell^*}(u)}{\phi(u)} - 1\right)^2,$$

where $\phi(u)$ is the true value of the ruin probability and $\phi^{\bullet,j}_{n,k^*,\ell^*}(u)$ is the j-th value ($j = 1, ..., N$) of any ruin probability estimator $\phi^{\bullet}_{n,k^*,\ell^*}(u)$ of $\phi(u)$, evaluated at their optimal numbers of higher-order statistics.

The point estimates of the probability of ruin at their optimal k^* values as well as their Abias and MSE are summarized in the following Tables 4.1.

$n = 1000$							
$\gamma = 2/3, \quad \phi(u) = 0.1$				$\gamma = 3/4, \quad \phi(u) = 0.1576$			
$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.0893	$\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$	0.0943	$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.1504	$\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$	0.1638
ABais	0.1069	ABais	0.0648	ABais	0.0615	ABais	0.0585
MSE	0.0114	MSE	0.0042	MSE	0.0037	MSE	0.0034
$n = 1500$							
$\gamma = 2/3, \quad \phi(u) = 0.1$				$\gamma = 3/4, \quad \phi(u) = 0.1576$			
$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.0905	$\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$	0.0953	$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.1638	$\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$	0.1523
ABais	0.0946	ABais	0.0545	ABais	0.0526	ABais	0.0520
MSE	0.0089	MSE	0.0029	MSE	0.0027	MSE	0.0023
$n = 2000$							
$\gamma = 2/3, \quad \phi(u) = 0.1$				$\gamma = 3/4, \quad \phi(u) = 0.1576$			
$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.0925	$\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$	0.0995	$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.1649	$\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$	0.1546
ABais	0.0751	ABais	0.0354	ABais	0.0502	ABais	0.0358
MSE	0.0056	MSE	0.0013	MSE	0.0025	MSE	0.0012

Table 4.1: Estimation results of $\widehat{\phi}_{k^*\ell^*,n}(u)$ and $\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$ estimators of the ruin probability $\phi(u) = 0.1$ for $u = 1.5$ and$\omega = c/\lambda = 18$, computed with optimal numbers of top statistics k^* and ℓ^*, based on $N = 1000$ samples of size $n = 1000$; 1500; 2000, from the distribution $F_1(x) = 1 - x^{-1/\gamma}$, $\gamma = 2/3;\ 3/4$.

Examination of the table leads to two conclusions, whatever the situation. Firstly, we note that the absolute bias of both probability of ruin estimators decrease to zero when the sample size n becomes large. Secondly, we find that the MSEs of the $\widehat{\phi}^{tH}_{n,k,\ell}(u)$ estimator converge faster to zero as n increases, compared with the MSEs of $\widehat{\phi}_{n,k,\ell}(u)$. In this context, these numerical results show that the estimator $\widehat{\phi}^{tH}_{n,k,\ell}(u)$ is the best.

4.2 Comparative robustness study

One way of increasing robustness is to create a contamination model, which is considered to replace some of the variables of the data X with outliers. Thus, to assess the robustness of our estimator, a simulation was performed

with contaminated data for each estimator. The main points here are to consider a $\varepsilon-$ contamination model, which consists in considering a Pareto distribution $F_1(x) = 1 - x^{-1/\gamma}$ polluted by variables extracted from the other Pareto distribution $F_{1,a}(x) = 1 - \left(\frac{x}{a}\right)^{-1/\gamma}$ and to use the following mixing distribution:

$$F_1^c(x) = (1-\varepsilon) \times F_1(x) + \varepsilon \times F_{1,a}(x) = 1 - (1-\varepsilon)x^{-1/\gamma} + \varepsilon \left(\frac{x}{a}\right)^{-1/\gamma},$$

where $\varepsilon \in (0,1)$ is the contamination rate. Now, for a given $\varepsilon = 10\%$ and $a = 3$, we generate also $N = 1000$ samples of size $n = 1000,\ 1500,\ 2000$ from the contaminated Pareto distribution $F_1^c(x)$. This kind of ε-contaminated model is used in [Fabián and Stehlík (2009)], [Brahim and Kenioua (2016)] and [Bouali *et al* (2021)] to evaluate insured.

Next, as in subsection 4.1, we compare the new estimator $\widehat{\phi}^{tH}_{n,k,\ell}(u)$ with the classical estimator $\widehat{\phi}_{n,k,\ell}(u)$ of the ruin probability $\phi(u)$, by calculating the Absolute bias and the MSE. The results are shown in Table 4.2. It turned out that the effect of contamination gets immediately apparent.

$n = 1000$							
$\gamma = 2/3, \quad \phi(u) = 0.1$				$\gamma = 3/4, \quad \phi(u) = 0.1576008$			
$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.0886	$\widehat{\phi}^{t.H}_{k^*,\ell^*,n}(u)$	0.0995	$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.1424	$\widehat{\phi}^{t.H}_{k^*,\ell^*,n}(u)$	0.1597
ABais	0.1145	ABais	0.0492	ABais	0.0985	ABais	0.0542
MSE	0.0131	MSE	0.0024	MSE	0.0097	MSE	0.0029
$n = 1500$							
$\gamma = 2/3, \quad \phi(u) = 0.1$				$\gamma = 3/4, \quad \phi(u) = 0.1576008$			
$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.0905	$\widehat{\phi}^{t.H}_{k^*,\ell^*,n}(u)$	0.1012	$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.1724	$\widehat{\phi}^{t.H}_{k^*,\ell^*,n}(u)$	0.1568
ABais	0.0981	ABais	0.0418	ABais	0.0974	ABais	0.0476
MSE	0.0096	MSE	0.0017	MSE	0.00948	MSE	0.0022
$n = 2000$							
$\gamma = 2/3, \quad \phi(u) = 0.1$				$\gamma = 3/4, \quad \phi(u) = 0.1576008$			
$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.0901	$\widehat{\phi}^{t.H}_{k^*,\ell^*,n}(u)$	0.1008	$\widehat{\phi}_{k^*,\ell^*,n}(u)$	0.1422	$\widehat{\phi}^{t.H}_{k^*,\ell^*,n}(u)$	0.1567
ABais	0.0947	ABais	0.0305	ABais	0.0939	ABais	0.0348
MSE	0.0089	MSE	0.0012	MSE	0.0088	MSE	0.0013

Table 4.2: Estimation results of $\widehat{\phi}_{k^*\ell^*,n}(u)$ and $\widehat{\phi}^{tH}_{k^*,\ell^*,n}(u)$ estimators of the ruin probability $\phi(u) = ..$ for $u = ..$, computed with optimal numbers of top statistics k^* and ℓ^*, based on $N = 1000$ samples of size $n = 1000;\ 1500;\ 2000$, from the distribution $F_1^c(x) = 1 - (1-\varepsilon)x^{-1/\gamma} + \varepsilon \left(\frac{x}{a}\right)^{-1/\gamma}$, $\gamma = 2 - 3,\ 3/4$.

5 Proofs

Let $E_1, ..., E_n$ be independent and identically distributed random variables from the unit Pareto distribution G, defined as $G(t) = 1 - t^{-1}$, $t \geq 1$. For each $n \geq 1$, let $E_{1,n} \leq ... \leq E_{n,n}$ be the order statistics pertaining to $E_1, \ldots, E_n$. Clearly $X_{j,n} \stackrel{d}{=} \mathbb{U}_1(E_{j,n})$ and $Y_{j,n} \stackrel{d}{=} \mathbb{U}_2(E_{j,n})$ $j = 1, \ldots, n$. In order to use the results from [Csörgő *et al.* (1986)], a probability space $(\Omega, \mathbb{A}, \mathbb{P})$ is constructed carrying a sequence $\xi_1, \xi_2, \ldots$ of independent random variables uniformly distributed on $(0, 1)$ and a sequence of Brownian bridges $\mathbb{B}_n(s)$, $0 \leq s \leq 1, n = 1, 2 \ldots$ such that for all $0 \leq \nu < 1/2$ and $\lambda > 0$

$$\sup_{\lambda/n \leq s \leq 1-\lambda/n} \frac{|\beta_n(s) - \mathbb{B}_n(s)|}{(s(1-s))^{1/2-\nu}} = O_{\mathbb{P}}(n^{-\nu}), \tag{5.19}$$

where β_n is following the uniform quantile process

$$\beta_n(t) = \sqrt{n}\,(t - V_n(t))$$

with V_n denoting the empirical uniform quantile function defined to be $V_n(t) = \xi_{j,n}$, $\frac{j-1}{n} < t \leq \frac{j}{n}, j = 1, \ldots, n$ and $V_n(0) = 0$.

Proof of Theorem 3.1

Recall that:

$$\widehat{\phi}^{\,tH}_{n,k,\ell}(u) - \phi(u) = \frac{\widehat{\mu}^{\,tH}_{2,n,\ell}}{\omega - \widehat{\mu}^{\,tH}_{1,n,k}} - \frac{\mu_2}{\omega - \mu_1},$$

where

$$\widehat{\mu}^{\,tH}_{1,n,k} := \int_0^{1-k/n} Q_{1,n}(s)ds + \frac{k}{n}\frac{X_{n-k,n}}{1 - \widehat{\gamma}^{\,tH}_{1,n,k}},$$

and

$$\widehat{\mu}^{\,tH}_{2,n,\ell} := \int_0^{1-\ell/n} Q_{2,n}(s)ds + \frac{\ell}{n}\frac{Y_{n-\ell,n}}{1 - \widehat{\gamma}^{\,tH}_{2,n,\ell}}.$$

Then, one may be rewrite the random variable $\widehat{\phi}^{\,tH}_{n,k,\ell}(u) - \phi(u)$ as follows:

$$\widehat{\phi}^{\,tH}_{n,k,\ell}(u) - \phi(u) = \frac{\mu_2}{\left(\omega - \widehat{\mu}^{\,tH}_{1,n,k}\right)(\omega - \mu_1)}\left(\widehat{\mu}^{\,tH}_{1,n,k} - \mu_1\right) + \frac{1}{\omega - \widehat{\mu}^{\,tH}_{1,n,k}}\left(\widehat{\mu}^{\,tH}_{2,n,\ell} - \mu_2\right).$$

Under the assumption (1.6), we have from from Theorem 2.4.1 in [de Haan and Ferreira (2006)], $X_{n-k,n} = Q_1(1 - k/n)\{1 + o_{\mathbb{P}}(1)\}$, as $n \to \infty$. Since the relation (1.6) is equivalent to $Q_1(1-s) = s^{-\gamma}\ell_{Q_1}(s)$, $s \in (0, 1)$, where ℓ_{Q_1} is a slowly varying function, more precisely $\ell_{Q_1}(sx)/\ell_{Q_1}(s) \to 1$, as $s \to 0$, then $(k/n)Q_1(1-k/n) = (k/n)^{1-\gamma}\ell_{Q_1}(k/n)$. Thus, for a given $\gamma \in (1/2, 1)$, we have from Proposition 1.3.6 in [Bingham *et al.* (1987)], $(k/n)^{1-\gamma}\ell_{Q_1}(k/n) \to 0$, as $n \to \infty$. Therefore, using the weak consistency of the estimator $\widehat{\gamma}^{\,tH}_{1,n,k}$ to γ (see, [Jordanova *et al* (2016)]), we obtain $(k/n)X_{n-k,n}/(1 - \widehat{\gamma}^{\,tH}_{1,n,k}) \stackrel{\mathbb{P}}{\to} 0$, as $n \to \infty$.

Also, under assumption we have from [Peng (2001)],

$$\frac{\sqrt{n}\int_0^{1-k/n}(Q_{1,n}(s) - Q_1(s))ds}{(k/n)^{1/2}Q_1(1-k/n)} \stackrel{d}{=} -\frac{\int_0^{1-k/n}\mathbb{B}_n(s)dQ_1(s)}{(k/n)^{1/2}Q_1(1-k/n)} + o_{\mathbb{P}}(1). \tag{5.20}$$

Since the right term in (5.20) is bounded in probability, it comes for all large values of n,

$$\int_0^{1-k/n} Q_{1,n}(s)ds = \int_0^{1-k/n} Q_1(s)ds + o_{\mathbb{P}}(1).$$

Remarking that $k/n \to 0$, as $n \to \infty$, we have $\int_0^{1-k/n} Q_1(s)ds = \int_0^1 Q_1(s)ds\{1 + o_{\mathbb{P}}(1)\}$, as $n \to \infty$. Which leads to the convergence in probability of $\widehat{\mu}^{\,tH}_{1,n,k}$ to the mean μ_1. Next, let's denote by $\kappa_1 = \mu_2/(\omega - \mu_1)^2$ and $\kappa_2 = 1/(\omega - \mu_1)$. Then, for all n large enough, the random variable $\widehat{\phi}^{\,tH}_{n,k,\ell}(u) - \phi(u)$ can be also represented as follows

$$\widehat{\phi}^{\,tH}_{n,k,\ell}(u) - \phi(u) \stackrel{d}{=} \kappa_1\left(\widehat{\mu}^{\,tH}_{1,n,k} - \mu_1\right) + \kappa_2\left(\widehat{\mu}^{\,tH}_{2,n,\ell} - \mu_2\right).$$

Consequently,

$$\begin{aligned}\frac{\sqrt{n}\left(\widehat{\phi}_{n,k,\ell}^{\,tH}(u)-\phi(u)\right)}{(k/n)^{1/2}Q_1(1-k/n)} &\overset{d}{=} \kappa_1\frac{n^{1/2}\left(\widehat{\mu}_{1,n,k}^{\,tH}-\mu_1\right)}{(k/n)^{1/2}Q_1(1-k/n)}\\ &\quad+\kappa_2\sqrt{\ell/k}\left(\frac{Q_2(1-\ell/n)}{Q_1(1-k/n))}\right)\frac{n^{1/2}\left(\widehat{\mu}_{2,n,\ell}^{\,tH}-\mu_2\right)}{(\ell/n)^{1/2}Q_2(1-\ell/n)},\\ &:= A_{1,n}+A_{2,n}.\end{aligned}$$

Now, let We first compute the $A_{1,n}$ term. By substituting $\widehat{\mu}_{1,n,k}^{\,tH}$ and μ_1 with their expressions, we have

$$\begin{aligned}A_{1,n} &= \kappa_1\frac{\sqrt{n}\int_0^{1-k/n}(Q_{1,n}(s)-Q_1(s))ds}{(k/n)^{1/2}Q_1(1-k/n)}\\ &\quad+\kappa_1\frac{n^{1/2}\left(\frac{(k/n)X_{n-k,n}}{1-\widehat{\gamma}_{1,n,k}^{\,tH}}-\int_{1-k/n}^1 Q_1(s)ds\right)}{(k/n)^{1/2}Q_1(1-k/n)}\\ &:= A_{1,n}^{(1)}+A_{1,n}^{(2)}.\end{aligned}$$

From the statement in (5.20), we have

$$A_{1,n}^{(1)}\overset{d}{=}-\kappa_1\mathbb{W}_{n,1}+o_{\mathbb{P}}(1), \tag{5.21}$$

where

$$\mathbb{W}_{n,1}:=\frac{\int_0^{1-k/n}\mathbb{B}_n(s)dQ_1(s)}{(k/n)^{1/2}Q_1(1-k/n)}.$$

Next, remarking that $\mathbb{U}_1(n/k)=Q_1(1-k/n)$ and $X_{n-k,n}\overset{d}{=}\mathbb{U}_1(E_{n-k,n})$, we have

$$A_{1,n}^{(2)}\overset{d}{=}\sum_{i=1}^4 T_{n,i},$$

where

$$\begin{aligned}T_{n,1}&=\frac{\kappa_1}{1-\widehat{\gamma}_{1,n,k}^{\,tH}}\sqrt{k}\left[\frac{\mathbb{U}_1(E_{n-k,n})}{\mathbb{U}_1(n/k)}-\left(\frac{k}{n}E_{n-k,n}\right)^{\gamma}\right],\\ T_{n,2}&=\frac{\kappa_1}{1-\widehat{\gamma}_{1,n,k}^{\,tH}}\sqrt{k}\left(\left(\frac{k}{n}E_{n-k,n}\right)^{\gamma}-1\right),\\ T_{n,3}&=\frac{\kappa_1}{(1-\gamma)\left(1-\widehat{\gamma}_{1,n,k}^{\,tH}\right)}\sqrt{k}\left(\widehat{\gamma}_{1,n,k}^{\,tH}-\gamma\right),\\ T_{n,4}&=\kappa_1\sqrt{k}\left[\frac{1}{1-\gamma}-\frac{\int_1^{+\infty}s^{-2}\mathbb{U}_1(ns/k)\,ds}{\mathbb{U}_1(n/k)}\right].\end{aligned}$$

We study each term separately.

Term $T_{n,1}$. According to [de Haan and Ferreira (2006)], Theorem 2.3.9) , for any $\delta>0$, we have

$$\sqrt{k}\left(\frac{\mathbb{U}_1(E_{n-k,n})}{\mathbb{U}_1(n/k)}-\left(\frac{k}{n}E_{n-k,n}\right)^{\gamma}\right)=\sqrt{k}\;A_1\left(\frac{n}{k}\right)\left\{\left(\frac{k}{n}E_{n-k,n}\right)^{\gamma}\frac{\left(\frac{k}{n}E_{n-k,n}\right)^{\rho}-1}{\rho}+o_{\mathbb{P}}(1)\left(\frac{k}{n}E_{n-k,n}\right)^{\gamma+\rho\pm\delta}\right\},$$

Thus, since $kE_{n-k,n}/n\to 1$, $\sqrt{k}\;A_1(n/k)\to 0$ and $\widehat{\gamma}_{1,n,k}^{\,tH}\overset{\mathbb{P}}{\to}\gamma$, as $n\to\infty$ (see, [Jordanova *et al* (2016)]), it readily follows that

$$T_{n,1}=o_{\mathbb{P}}(1). \tag{5.22}$$

Term $T_{n,2}$. The equality $E_{n-k,n}\overset{d}{=}(1-\xi_{n-k,n})^{-1}$ yields:

$$\begin{aligned}\sqrt{k}\left[\left(\frac{k}{n}E_{n-k,n}\right)^{\gamma}-1\right] &\overset{d}{=} \sqrt{k}\left(\left(\frac{n}{k}(1-\xi_{n-k,n})\right)^{-\gamma}-1\right)\\ &= -\gamma\sqrt{k}\left(\frac{n}{k}(1-\xi_{n-k,n})-1\right)(1+o_{\mathbb{P}}(1))\quad\text{by a Taylor expansion}\\ &= -\gamma\sqrt{\frac{n}{k}}\beta_n\left(1-\frac{k}{n}\right)(1+o_{\mathbb{P}}(1))\\ &= -\gamma\sqrt{\frac{n}{k}}\left(\mathbb{B}_n\left(1-\frac{k}{n}\right)+O_{\mathbb{P}}(n^{-\nu})\left(\frac{k}{n}\right)^{1/2-\nu}\right)(1+o_{\mathbb{P}}(1)),\end{aligned}$$

for $0 \leq \nu < 1/2$, by [Csörgő *et al.* (1986)]. Thus, using again the weak consistency of $\widehat{\gamma}_{1,n,k}^{tH}$ to γ, it follows that:

$$T_{n,2} \quad \stackrel{d}{=} \quad \kappa_1 \mathbb{W}_{n,2}(1+o_{\mathbb{P}}(1)), \tag{5.23}$$

where

$$\mathbb{W}_{n,2} = -\frac{\gamma}{(1-\gamma)}\sqrt{\frac{n}{k}}\mathbb{B}_n\left(1-\frac{k}{n}\right).$$

Term $T_{n,3}$. From the Proposition 1 in [Brahim and Kenioua (2016)], page 877, we have

$$\sqrt{k}\left(\widehat{\gamma}_{1,n,k}^{tH}-\gamma\right) \stackrel{d}{=} \gamma\left(\gamma+1\right)^2\int_0^1 s^{\gamma-1}\mathbb{B}_n\left(s\right)ds + o_{\mathbb{P}}(1).$$

And by using again the consistency in probability of $\widehat{\gamma}_{1,n,k}^{tH}$ to γ, we get for all n large enough:

$$T_{n,3} = \kappa_1\frac{\gamma\left(\gamma+1\right)^2}{\left(1-\gamma\right)^2}\int_0^1 s^{\gamma-1}\mathbb{B}_n\left(s\right)ds + o_{\mathbb{P}}(1) = \kappa_1\mathbb{W}_{n,3} + o_{\mathbb{P}}(1). \tag{5.24}$$

Term $T_{n,4}$. A change of variables and an integration by parts yield

$$\begin{aligned} T_{n,4} &= \kappa_1\sqrt{k}\left\{\frac{1}{1-\gamma} - \int_1^\infty x^{-2}\frac{\mathbb{U}_1(nx/k)}{\mathbb{U}_1(n/k)}dx\right\} \\ &= -\kappa_1\sqrt{k}\int_1^\infty x^{-2}\left(\frac{\mathbb{U}_1(nx/k)}{\mathbb{U}_1(n/k)} - x^{\gamma}\right)dx. \end{aligned}$$

Theorem 2.3.9 in [de Haan and Ferreira (2006)] entails that, for $\gamma \in (1/2, 1)$,

$$\begin{aligned} T_{n,4} &= -\kappa_1\sqrt{k}A_1\left(\frac{n}{k}\right)\int_1^\infty x^{\gamma-2}\,\frac{x^\rho-1}{\rho}dx\,(1+o_{\mathbb{P}}(1)) \\ &= \kappa_1\sqrt{k}A_1\left(\frac{n}{k}\right)\frac{1}{(1-\gamma)(\gamma+\rho-1)}(1+o_{\mathbb{P}}(1)). \\ &= o_{\mathbb{P}}(1), \qquad \text{by} \;\; \sqrt{k}\;A_1(n/k)\to 0. \end{aligned} \tag{5.25}$$

Combining (5.21), (5.22), (5.23), (5.24) and (5.25), we get

$$A_{n,1} \stackrel{d}{=} \kappa_1\Big(\mathbb{W}_{n,1}+\mathbb{W}_{n,2}+\mathbb{W}_{n,3}\Big) + o_{\mathbb{P}}(1). \tag{5.26}$$

Now let's compute the $A_{2,n}$ term. We first have: $Q_2(1-\frac{\ell}{n}) \sim Q_1(1-\frac{\ell}{n}) = Q_1(1-\frac{\ell}{k}\frac{k}{n}) \sim (\frac{\ell}{k})^{-\gamma}Q_1(1-\frac{k}{n})$, as $n\to\infty$. Since by assumption $\ell/k\to\theta>0$, then $Q_2(1-\frac{\ell}{n}) \sim \theta^{-\gamma}Q_1(1-\frac{k}{n})$, as $n\to\infty$. This leads for large values of n to

$$A_{2,n} \stackrel{d}{=} \theta^{(1/2-\gamma)}\,\kappa_2\,\frac{n^{1/2}\left(\widehat{\mu}_{2,n,\ell}^{tH}-\mu_2\right)}{\left(\ell/n\right)^{1/2}Q_2(1-\ell/n)}.$$

Further, Substituting $\widehat{\mu}_{2,n,\ell}^{tH}$ and μ_2 with their expressions and using similar arguments as those developed to show the expression $A_{1,n}$ in (5.26) together with $Y_{n-\ell,n} \stackrel{d}{=} \mathbb{U}_2(E_{n-\ell,n})$, it comes:

$$A_{n,2} \stackrel{d}{=} \theta^{(1/2-\gamma)}\,\kappa_2\left(\overline{\mathbb{W}}_{n,1}+\overline{\mathbb{W}}_{n,2}+\overline{\mathbb{W}}_{n,3}\right) + o_{\mathbb{P}}(1), \tag{5.27}$$

where

$$\begin{aligned} \overline{\mathbb{W}}_{n,1} &:= -\frac{\int_0^{1-\ell/n}\mathbb{B}_n(s)dQ_2(s)}{(\ell/n)^{1/2}Q_2(1-\ell/n)}, \\ \overline{\mathbb{W}}_{n,2} &:= -\frac{\gamma}{1-\gamma}\sqrt{\frac{n}{\ell}}\mathbb{B}_n\left(1-\frac{\ell}{n}\right), \\ \overline{\mathbb{W}}_{n,3} &:= \frac{\gamma\;(\gamma+1)^2}{(1-\gamma)^2}\int_0^1 s^{\gamma-1}\mathbb{B}_n\left(s\right)ds. \end{aligned}$$

Finally, for all n large enough, the Theorem 3.1 holds with

$$\frac{\sqrt{n}\left(\widehat{\phi}_{n,k,\ell}^{tH}\left(u\right)-\phi\left(u\right)\right)}{\left(k/n\right)^{1/2}Q_1(1-k/n)} \stackrel{d}{=} \kappa_1\sum_{i=1}^3\mathbb{W}_{n,i} + \theta^{(1/2-\gamma)}\,\kappa_2\sum_{i=1}^3\overline{\mathbb{W}}_{n,i} + o_{\mathbb{P}}(1)$$

■

Proof of Corollary 3.1

From Theorem 3.1, we need to compute the asymptotic variance $\sigma^2(\gamma,\theta)$ of the limiting process. More precisely, have

$$\begin{aligned}
\sigma^2(\gamma,\theta) \quad = \quad & \lim_{n\to\infty}\Bigg((\kappa_1)^2 \left\{ \mathbb{E}\left(\mathbb{W}_{n,1}^2\right) + \mathbb{E}\left(\mathbb{W}_{n,2}^2\right) + \mathbb{E}\left(\mathbb{W}_{n,3}^2\right) \right\} \\
& + (\kappa_2)^2\, \theta^{(1-2\gamma)} \left\{ \mathbb{E}(\overline{\mathbb{W}}_{n,1}^2) + \mathbb{E}(\overline{\mathbb{W}}_{n,2}^2) + \mathbb{E}(\overline{\mathbb{W}}_{n,3}^2) \right\} \\
& + 2\,(\kappa_1)^2 \left\{ \mathbb{E}(\mathbb{W}_{n,1}\mathbb{W}_{n,2}) + \mathbb{E}(\mathbb{W}_{n,1}\mathbb{W}_{n,3}) + \mathbb{E}(\mathbb{W}_{n,2}\mathbb{W}_{n,3}) \right\} \\
& + 2\,(\kappa_2)^2\, \theta^{(1-2\gamma)} \left\{ \mathbb{E}(\overline{\mathbb{W}}_{n,1}\overline{\mathbb{W}}_{n,2}) + \mathbb{E}(\overline{\mathbb{W}}_{n,1}\overline{\mathbb{W}}_{n,3}) + \mathbb{E}(\overline{\mathbb{W}}_{n,2}\overline{\mathbb{W}}_{n,3}) \right\} \\
& + 2\,\kappa_1\kappa_2\, \theta^{(1/2-\gamma)} \left\{ \mathbb{E}(\mathbb{W}_{n,1}\overline{\mathbb{W}}_{n,1}) + \mathbb{E}(\mathbb{W}_{n,1}\overline{\mathbb{W}}_{n,2}) + \mathbb{E}(\mathbb{W}_{n,1}\overline{\mathbb{W}}_{n,3}) \right\} \\
& + 2\,\kappa_1\kappa_2\, \theta^{(1/2-\gamma)} \left\{ \mathbb{E}(\mathbb{W}_{n,2}\overline{\mathbb{W}}_{n,1}) + \mathbb{E}(\mathbb{W}_{n,2}\overline{\mathbb{W}}_{n,2}) + \mathbb{E}(\mathbb{W}_{n,2}\overline{\mathbb{W}}_{n,3}) \right\} \\
& + 2\,\kappa_1\kappa_2\, \theta^{(1/2-\gamma)} \left\{ \mathbb{E}(\mathbb{W}_{n,3}\overline{\mathbb{W}}_{n,1}) + \mathbb{E}(\mathbb{W}_{n,3}\overline{\mathbb{W}}_{n,2}) + \mathbb{E}(\mathbb{W}_{n,3}\overline{\mathbb{W}}_{n,3}) \right\} \Bigg).
\end{aligned}$$

Recall that $Q_1(1-su)/Q_1(1-u) \to s^{-\gamma}$, $Q_2(1-su)/Q_2(1-u) \to s^{-\gamma}$ and $Q_1(1-u) \sim Q_2(1-u)$ as $u \to 0$, $u \to 0u \to 0$. Following [Deme *et al.* (2015)] with some algebraic operations, we get, as $n \to \infty$:

$$\mathbb{E}\left(\mathbb{W}_{n,1}^2\right) \longrightarrow \frac{2\gamma}{2\gamma-1}, \qquad \mathbb{E}\left(\mathbb{W}_{n,2}^2\right) \longrightarrow \frac{\gamma^2}{(1-\gamma)^2},$$

$$\mathbb{E}\left(\mathbb{W}_{n,3}^2\right) = \frac{\gamma^2\,(\gamma+1)^2}{(2\gamma+1)\,(\gamma-1)^4}, \qquad \mathbb{E}\left(\mathbb{W}_{n,1}\mathbb{W}_{n,2}\right) \longrightarrow \frac{\gamma}{1-\gamma},$$

$$\mathbb{E}\left(\mathbb{W}_{n,1}\mathbb{W}_{n,3}\right) = 0 + o(1), \qquad \mathbb{E}\left(\mathbb{W}_{n,2}\mathbb{W}_{n,3}\right) = 0 + o(1),$$

$$\mathbb{E}\left(\overline{\mathbb{W}}_{n,1}^2\right) \longrightarrow \frac{2\gamma}{2\gamma-1}, \qquad \mathbb{E}\left(\overline{\mathbb{W}}_{n,2}^2\right) \longrightarrow \frac{\gamma^2}{(1-\gamma)^2},$$

$$\mathbb{E}\left(\overline{\mathbb{W}}_{n,3}^2\right) = \frac{\gamma^2\,(\gamma+1)^2}{(2\gamma+1)\,(\gamma-1)^4}, \qquad \mathbb{E}\left(\overline{\mathbb{W}}_{n,1}\overline{\mathbb{W}}_{n,2}\right) \longrightarrow \frac{\gamma}{1-\gamma},$$

$$\mathbb{E}\left(\overline{\mathbb{W}}_{n,1}\overline{\mathbb{W}}_{n,3}\right) = 0 + o(1), \qquad \mathbb{E}\left(\overline{\mathbb{W}}_{n,2}\overline{\mathbb{W}}_{n,3}\right) = 0 + o(1),$$

$$\mathbb{E}\left(\mathbb{W}_{n,1}\overline{\mathbb{W}}_{n,1}\right) \longrightarrow \frac{\gamma}{1-\gamma}\left(\frac{1}{2\gamma-1} - \theta^{(1-\gamma)}\right)\theta^{(\gamma-1/2)}, \qquad \mathbb{E}\left(\mathbb{W}_{n,1}\overline{\mathbb{W}}_{n,2}\right) \longrightarrow \frac{\gamma}{1-\gamma}\theta^{1/2},$$

$$\mathbb{E}\left(\mathbb{W}_{n,1}\overline{\mathbb{W}}_{n,3}\right) = 0 + o(1), \qquad \mathbb{E}\left(\mathbb{W}_{n,2}\overline{\mathbb{W}}_{n,1}\right) \longrightarrow \frac{\gamma(1-\gamma\,\theta^{(1-\gamma)})}{(1-\gamma)^2}\theta^{(\gamma-1/2)},$$

$$\mathbb{E}\left(\mathbb{W}_{n,2}\overline{\mathbb{W}}_{n,2}\right) \longrightarrow \frac{\gamma^2}{(1-\gamma)^2}\theta^{1/2}, \qquad \mathbb{E}\left(\mathbb{W}_{n,2}\overline{\mathbb{W}}_{n,3}\right) = 0 + o(1),$$

$$\mathbb{E}\left(\mathbb{W}_{n,3}\overline{\mathbb{W}}_{n,1}\right) = 0 + o(1), \qquad \mathbb{E}\left(\mathbb{W}_{n,3}\overline{\mathbb{W}}_{n,2}\right) = 0 + o(1), \qquad \mathbb{E}\left(\mathbb{W}_{n,3}\overline{\mathbb{W}}_{n,3}\right) = \frac{\gamma^2(1+\gamma)^2}{(1+2\gamma)(1-\gamma)^4}.$$

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