Sampling Random Cycle-Rooted Spanning Forests on Infinite Graphs

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Abstract. On a finite graph, there is a natural family of Boltzmann probability measures on cycle-rooted spanning forests, parametrized by weights on cycles. For a certain subclass of those weights, we construct Gibbs measures in infinite volume, as limits of probability measures on cycle-rooted spanning forests of increasing sequences of finite graphs. Those probability measures extend the family of already known random spanning forests and can be sampled by a random walks algorithm which generalizes Wilson's algorithm. We show that, unlike for uniform spanning forests, almost surely, all connected components are finite and two-points correlations decrease exponentially fast with the distance.

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Introduction

We call *cycle-rooted spanning forest*, on a finite connected graph, every subgraph which contains all vertices and all of whose connected components contain a unique cycle. Such a configuration of edges, endowed with a choice of orientation of cycles, can be seen as a discrete vector field on the graph: edges which are not in the cycle are oriented towards the cycle. Every vertex is associated with an edge starting from it.

Given a connected finite graph $G = (V, E)$, all of whose oriented cycles $(\gamma)_{\gamma \in \mathcal{C}(G)}$ are endowed with positive weights $(w(\gamma))$, we say that two oriented cycles are equivalent if they are equal after removal of their orientations. We define a probability measure on cycle-rooted spanning forests, induced by these cycle-weights as follows. Every spanning forest has a probability proportional to the product of weights of its cycles, counting both orientations:

$$\mu_w(F) = \frac{\prod_{[\gamma] \in \mathcal{C}(F)} (w(\gamma) + w(\gamma^{-1}))}{\mathcal{Z}_w}, \tag{1}$$

where γ, γ^{-1} are both oriented cycles of the equivalence class $[\gamma]$. The normalizing constant Z_w is called the *partition function* of the model and is defined as follows:

$$\mathcal{Z}_w = \sum_F \prod_{[\gamma] \in \mathcal{C}(F)} (w(\gamma) + w(\gamma^{-1})).$$

Recall that the usual model of uniform spanning tree is defined on finite graphs and its limit for infinite graphs is studied in [BP93, BLPS01]. In these papers, the existence of a measure for infinite graphs is shown. This measure is sampled by an extension in infinite volume of Wilson's algorithm [Wil96] which does not depend on the ordering of vertices; see [LP16] for a textbook treatment of this topic. These authors study the properties of the configurations under the measure in infinite volume, such as the number of connected components.

A model of random rooted spanning forests, all of whose connected components are rooted trees is studied in [BdTR17]. In this model, probability measures are associated with weights on vertices. If only one vertex has a weight, the probability measure associated with this weight has support in rooted trees, whose unique root is this vertex, and is equal to the uniform spanning tree measure after forgetting this root. In the model of rooted spanning forests, the random configurations are still sampled by an algorithm of loop-erased random walks.

The model of rooted spanning forests is a particular case of the model of cycle-rooted spanning forests, which is studied in this article. Indeed, weights on vertices can be interpreted as weights on small self-loops over vertices and the roots of the random configuration can be seen as the unique cycles of their connected component. From this point of view, the model of cycle-rooted spanning forests is a generalization of the rooted spanning forests and of the uniform spanning tree.

The measure on cycle-rooted spanning forests on finite graphs associated with a weight function on cycles for which every cycle has a weight smaller than 1 is studied in [KK17]. This measure is sampled by a loop-erased random walks algorithm inspired from the Propp-Wilson algorithm for the generation of a random spanning tree. This algorithm does not depend on the ordering of vertices. A similar algorithm is also introduced in [BBGJ07] to generate a random spanning web of square lattice annuli, using a "cycle-popping" inspired from the Propp-Wilson algorithm.

In this article, we study properties of probability measures on cycle-rooted spanning forests depending on a weight function on oriented cycles with values smaller than 1. We show that under an assumption on weights (Assumption 2.0.1), the weak limit of measures on spanning forests on growing finite graphs, with cycle-weights smaller than 1, is well-defined and sampled by an algorithm of loop-erased random walks (Theorem 3.3.1) which does not depend on the ordering of vertices (Theorem 3.3.2). We show furthermore that under this measure with a stronger assumption on weights (Assumption 4.0.1), all connected components are almost surely finite (Theorem 4.1.4) and the decay of edge-edge correlations is exponential (Theorem 4.2.2). Those properties show that under this assumption on weights, the measure which is constructed in infinite volume corresponds to a different "phase", in the sense of statistical mechanics, than the uniform spanning

forests measure studied in [BP93, BLPS01, LP16]. We also study probability measures on cycle-rooted spanning forests of finite graphs determined by a weight function w which can take values larger than 1. Those measures are no longer sampled by a loop-erased random walk algorithm. Nevertheless, we show that conditional on cycles of weights larger than 1, the measure is determined by a modified weight function w_- (Definition 5.1.1) which takes values smaller than 1 (Theorem 5.1.2). Assuming the existence of an infinite volume measure μ_w, we show that when Assumption 4.0.1 is satisfied by this modified weight function w_-, every connected component with a cycle is finite (Theorem 5.2.8). Combined with Proposition 3.1.1 which says that almost surely every finite connected component has a cycle, this result implies that, almost surely, every connected component is either a finite cycle-rooted tree or an infinite tree.

The paper is organized as follows. In Section 1, we define the probability measures on cycle-rooted spanning forests of finite graphs we are concerned with and give properties on these measures. In Section 2, we define the p-loop erased random walks which generalize the loop-erased random walks and which are used in the third section to define a measure in infinite volume sampled by a random walk algorithm. In Section 3, we study the weak limit of probability measures on spanning forests on growing finite graphs, which gives the existence of a probability measure in infinite volume. In Section 4, we study the long-range behavior of the configurations under this probability measure, such as the non-existence of an infinite connected component and the exponential rate of decay of edge-to-edge correlations. In Section 5, we study properties of infinite configurations sampled by infinite volume probability measures which are determined by unbounded weights on cycles, provided the limit exists.

1. Measures on Cycle-Rooted Spanning Forests on finite graphs

1.1. **Cycle-rooted spanning forests.** Let $G = (V, E)$ be a finite connected graph with vertex set V and edge set E. For every subgraph F of G, let $\mathcal{C}(F)$ be the set of unoriented simple cycles of F and $E(F)$ be the set of edges of F. If $[\gamma] \in \mathcal{C}(F)$ is a cycle of F, denote by γ and γ^{-1} the two oriented cycles obtained from γ. Let $\mathcal{C}_{\rightarrow}(F)$ be the set of oriented cycles of F.

We say that a subgraph of G is a *cycle-rooted spanning forest* (CRSF) if it contains all the vertices and if each of its connected components contains a unique cycle. Let $\mathcal{U}(G)$ be the set of CRSFs of G.

Let $w : \mathcal{C}_{\rightarrow}(G) \to \mathbb{R}_+$ be a non-zero function with non-negative values, defined on oriented cycles of G. There is a natural probability measure on $\mathcal{U}(G)$ associated with w, which is denoted by μ_w. It is defined for every CRSF $F \in \mathcal{U}(G)$ by:

$$\mu_w(F) = \frac{\prod_{[\gamma]\in\mathcal{C}(F)}(w(\gamma) + w(\gamma^{-1}))}{Z_w}, \tag{2}$$

where γ, γ^{-1} are both oriented cycles of the equivalence class $[\gamma]$, and Z_w is called the partition function of the model

$$Z_w = \sum_{F\in\mathcal{U}(G)} \prod_{[\gamma]\in\mathcal{C}(F)} (w(\gamma) + w(\gamma^{-1})).$$

We say that F is an *oriented cycle-rooted spanning forest* (OCRSF) if it is a CRSF and every cycle of F is given an orientation, that is to say if every connected component contains a unique oriented cycle. Let $\mathcal{U}_{\rightarrow}(G)$ be the set of OCRSFs of G. Every edge of an OCRSF is oriented towards the cycle of its connected component.

The partition function of the model can also be written as a sum of weights over OCRSF as follows:

$$Z_w = \sum_{F \text{ OCRSF}} \prod_{\gamma\in\mathcal{C}_{\rightarrow}(F)} w(\gamma),$$

and it gives a natural probability measure on OCRSF.

Let $W \subset V$ be a subset of vertices of G. We say that F is a wired cycle-rooted spanning forest or essential cycle-rooted spanning forest (ECRSF) with respect to W if every connected component of F is either a unicycle disjoint from W or an unrooted tree which contains a unique vertex of W, called a boundary-rooted tree. Let $\mathcal{U}_W(G)$ the set of ECRSF with respect to W.

Definition 1.1.1 (Wired boundary conditions)**.** *We define a measure on $\mathcal{U}_W(G)$ called the wired measure on ECRSF of G with boundary W whose configurations have weight proportional to the product of weights of cycles.*

$$\mu_w^W(F) = \frac{\prod_{[\gamma]\in\mathcal{C}(F)}(w(\gamma)+w(\gamma^{-1}))}{Z_w^W} \tag{3}$$

where γ, γ^{-1} are both oriented cycles of the equivalence class $[\gamma]$.

Notice that the measure defined in (2) corresponds to the case $W = \emptyset$.

1.2. **Wilson's algorithm.** When the weight function w is identically equal to 0, this measure μ_w^W has support on ECRSF all of whose connected component are boundary-rooted trees. In particular, when $W = \{r\}$ is a single vertex and the weight function w is identically equal to 0, this measure has support on spanning trees rooted at r. Since spanning trees on G are in 1-to-1 correspondence with spanning trees of G rooted at r, this measure is independent of the choice of vertex r and gives to every spanning tree the same weight. Therefore, this measure is the uniform spanning tree measure defined for every tree T by:

$$\mu(T) = \frac{1}{Z_{tree}} \tag{4}$$

where Z_{tree} is the partition function, that is the number of spanning trees of the graph G. This measure μ can be sampled by the Wilson algorithm ([Wil96]).

Assume that for every oriented cycle $\gamma \in \mathcal{C}_{\rightarrow}(G)$,

$$w(\gamma) \in [0,1]$$

We will write p instead of w in the following when this assumption is satisfied.

According to [KK17], the measure μ_p can be sampled by an algorithm of loop-erased random walk where we keep an oriented cycle γ, with probability $p(\gamma)$.

More precisely, let $x_1, \dots, x_n$ be an ordering of the vertex set V of G and let $\mathsf{F}_0 = \emptyset$. At each step i, let $(X_n^{(x_i)})_{n\geq 0}$ be a simple random walk on the graph G starting from x_i. Every time the random walk makes a loop, the oriented cycle γ is kept with probability $p(\gamma)$ or erased with probability $1-p(\gamma)$. The random walk $(X_n^{(x_i)})_{n\geq 0}$ is stopped when it reaches the set of already explored vertices denoted by $V(\mathsf{F}_{i-1})$ or when a cycle is kept. In the end of the i^{th} step, let $\mathsf{F}_i = \mathsf{F}_{i-1} \cup L(X_n^{(x_i)})$ where $L(X_n^{(x_i)})$ is obtained from $(X_n^{(x_i)})_{n\geq 0}$ after removing all the loops except the last one if a loop is kept at the end of the i^{th} step. At the end, $V(\mathsf{F}_n) = V(G_n)$. Notice that the algorithm always finishes if and only if there exists at least a loop γ in G such that $p(\gamma) > 0$.

The measure μ_p^W defined in Definition 1.1.1 can also be sampled by an algorithm. We follow the same algorithm but every time the random walk meets W, the walk stops and a new random walk starts from the next vertex in the ordering. At the beginning of the algorithm, we set $\mathsf{F}_0 = W$ instead of $\mathsf{F}_0 = \emptyset$. The algorithm always finishes if and only if there exists at least a loop γ in $G\backslash W$ such that $p(\gamma) > 0$ or $W \neq \emptyset$.

Let us emphasize that when $W = \{r\}$ is a single vertex and the weight function w is equal to 0, then the sampling algorithm described just above is the classical Wilson algorithm which samples a uniform spanning tree on G rooted at r.

2. p-Loop erased random walks and rooting times

In the following, we will consider a countably infinite connected graph $G = (V, E)$, with finite degrees, exhausted by an increasing sequence $(G_n)_{n\geq 1}$ of connected induced subgraphs of G, with respective vertex set V_n. We denote by ∂G_n the subset of V_n of vertices which are connected by an edge to the complement of G_n in G.

For every $v \in V$, we denote by $\mathbb{P}_v$ the law of a simple random walk on G starting from v.

We consider a weight function $w = p \in [0, 1]$ and we make the following assumption on the exhaustion (G_n) of the graph G and the weight function p.

Assumption 2.0.1. *There exists $\alpha > 0$ and $\beta > 0$, such that for every $n \in \mathbb{N}^*$, for every random walk $(X_n)_{n\geq 0}$ on G, starting from a vertex v of ∂G_n, there exists a loop γ_v in $G_{n+1}\backslash(G_n \cup \partial G_{n+1})$ which satisfies $p(\gamma_v) \geq \alpha$ and $\mathbb{P}_v((X_1, \dots, X_{|\gamma|}) = \gamma) > \beta$.*

2.1. Hitting times, rooting time.

In the following we denote by v_0 a vertex of G_1.

Definition 2.1.1. *If C is a subset of the vertex set V, we define for a random walk (X_n) the hitting time of C, that is to say*

$$T_C := \min\{k \geq 0 | X_k \in C\}.$$

Notice that in this definition, T_C can be equal to 0 if the random walk starts from a vertex of C.

Definition 2.1.2. *Let (X_n) be a simple random walk starting from v_0. Let (T_n) be the sequence of random hitting times of ∂G_n for the random walk (X_n), that is to say*

$$T_n := T_{\partial G_n} = \min\{k \geq 0 | X_k \in \partial G_n\}.$$

Lemma 2.1.3. *The hitting-time T_n is finite almost-surely for every $n \in \mathbb{N}^*$. Furthermore,*

$$\lim_{k\to\infty} \mathbb{P}_{v_0}(T_n \geq k) = \mathbb{P}_{v_0}(T_n = \infty) = 0.$$

Proof. Let $n \in \mathbb{N}^*$. Almost surely, T_n is finite because almost surely if $k \geq 1$, there exists a time such that the random walk makes k consecutive steps in the same direction. Therefore, the random walk exits every finite ball in finite time almost surely. Since the events $(T_n \geq k)$ are decreasing in k (for a fixed n) with respect to inclusion, the monotone convergence theorem implies

$$\lim_{k\to\infty} \mathbb{P}_{v_0}(T_n \geq k) = \mathbb{P}_{v_0}(\bigcap_{k\geq 1} \{T_n \geq k\}) = \mathbb{P}_{v_0}(T_n = \infty),$$

which concludes the proof. □

Let (X_n) be a simple random walk on G starting from v_0 and let (Y_n) be a sequence of independent random variables of uniform law on $[0, 1]$, which are independent of the random variables X_n.

We want to define a p-loop erased random walk such that, if at time n, the random walk (X_n) closes a loop γ_n, the loop is kept if $Y_n \leq p(\gamma_n)$ and erased else.

Given (X_n, Y_n), we construct a sequence of random walks $((Z_n^k)_n)_k$ as follows. We define recursively $(Z_n^k)_{n\geq 1}$ for $k \in \mathbb{N}^*$. Let $(Z_n^1) = (X_n)$ and given $(Z_n^k)_n$, let us consider the first time n_k such that Z_n^k closes a loop that is to say

$$n_k = \min\{j > n_{k-1} \in \mathbb{N}^* | Z_j^k \in \{Z_0^k, \dots, Z_{j-1}^k\}\}.$$

Then, let n'_k be the time of the beginning of the loop, that is to say

$$n'_k = \min\{j \in \mathbb{N}, Z_j^k = Z_{n_k}^k\}.$$

Therefore, the loop which is closed at time n_k is the loop $\gamma_{n_k} := (Z^k_{n'_k}, \dots, Z^k_{n_k})$ Finally, if $Y_{n_k} \geq p(\gamma_{n_k})$, then define for every $n \in \mathbb{N}$,

$$Z^{k+1}_n = \begin{cases} Z^k_{n_k} & \text{if } n'_k \leq n \leq n_k \\ Z^k_n & \text{else.} \end{cases}$$

(Z^{k+1}_n) is obtained from (Z^k_n) erasing the loop γ_{n_k}.

Otherwise, if $Y_{n_k} \leq p(\gamma_{n_k})$, for every $m \geq k+1$ and $n \in \mathbb{N}$, let

$$Z^m_n = \begin{cases} Z^k_n & \text{if } n \leq n_k \\ Z^k_{n_k} & \text{else} \end{cases}$$

(Z^m_n) is obtained from (Z^k_n) stopping the random walk at time n_k.

Definition 2.1.4. *If (X_n) is a simple random walk on G starting from v_0 and (Y_n) is a sequence of independent random variables of uniform law on $[0,1]$, which are independent of the X_n, we say that $(n_k)_{n\geq 1}$ is the sequence of random times where the random walk (X_n) closes a loop γ_{n_k}. Let T_r be called the* random rooting time *for (X_n, Y_n) that is to say the first time where a loop is kept:*

$$T_r := \min\{n_k | Y_{n_k} \leq p(\gamma_{n_k})\},$$

where $\min \emptyset = +\infty$. If k is such that $T_r = n_k$, then let $(Z_n)_{n\leq T_r} = (Z^k_n)_{n\leq T_r}$ be called the p-loop erased random walk *obtained from (X_n, Y_n).*

Let us emphasize that if T_r is finite, then there exists a k such that $T_r = n_k$ and then the p-loop erased random walk $(Z_n)_{n\leq T_r}$ is well defined and is obtained from $(X_n)_{n\leq T_r}$, erasing every loop excepted the last one. Here, the loop-erased random walk is indexed on the same time set than the random walk (X_n). Nevertheless, Z_n does not depend only on $(X_k)_{k\leq n}$.

2.2. **The rooting time is almost surely finite.** We will show in this subsection that the rooting time T_r is a stopping time and that almost surely, it is finite.

Definition 2.2.1. *Let $(\mathcal{F}_n)_n$ be the filtration adapted to the process $((X_n, Y_n))_n$ that is defined by*

$$\mathcal{F}_n = \sigma(X_0, \dots, X_n, Y_0, \dots, Y_n),$$

which is the smallest sigma-field which makes the $(X_i)_{0\leq i\leq n}, (Y_i)_{1\leq i\leq n}$ measurable.

Lemma 2.2.2. *For every $m \in \mathbb{N}^*$, the hitting time T_m is a stopping time with respect to the filtration $(\mathcal{F}_n)_n$. The rooting time T_r is also a stopping time with respect to the filtration $(\mathcal{F}_n)_n$. Moreover, for every $n \in \mathbb{N}$, if we consider the σ-field adapted to the stopping time T_n, defined by*

$$\mathcal{F}_{T_n} = \{A \in \mathcal{F} : \forall k \geq 0, \{T_n \leq k\} \cap A \in \mathcal{F}_k\},$$

then, the event $\{T_n < T_r\}$ is in $\mathcal{F}_{T_n}$.

Proof. Let $n \in \mathbb{N}$. The events $\{T_n \geq k\} = \{X_1, \dots, X_k \in G_n \backslash \partial G_n\}$ are measurable with respect to $\mathcal{F}_k$ and therefore T_n is a stopping time.

From the construction of the rooting time, the event $\{T_r \leq k\}$ only depends on the steps of the random walk before time k, that is $((X_n, Y_n))_{n\leq k}$ and therefore the event $\{T_r \geq k\}$ is in $\mathcal{F}_k$.

The event $\{T_n < T_r\}$ is in $\mathcal{F}_{T_n}$ because if $k \geq 0$,

$$\{T_n \leq k\} \cap \{T_n < T_r\} = \bigcup_{1\leq i\leq k} (\{T_n = i\} \cap \{T_r > i\}) \in \mathcal{F}_k.$$

which concludes the proof. □

Let us emphasize that T_r is a stopping time for the filtration $(\mathcal{F}_n)$ even if (Z_n) is not adapted to the filtration $(\mathcal{F}_n)$. Indeed, Z_n depends on (X_k) for $k \geq n$. Lemma 2.2.2 is a useful tool to show that the rooting time is almost surely finite for a simple random walk starting from v_0.

Lemma 2.2.3. *Under Assumption 2.0.1, the rooting time T_r for a simple random walk (X_n) starting from v_0 and (Y_n) as defined in Definition 2.1.4 is finite almost surely and the sequence $(\mathbb{P}_{v_0}(T_r > T_n))_n$ decays exponentially fast to 0 with n. More precisely, there exists $\delta \in]0,1[$ such that*

$$\mathbb{P}_{v_0}(T_n < T_r) \leq \delta^n.$$

Proof. Let $n \in \mathbb{N}^*$ be fixed. The process $((X_k, Y_k))_{k\geq 0}$ satisfies the strong Markov property. Therefore, conditional on the event $\{T_n < \infty\}$ which is almost sure by Lemma 2.1.3, for every $k \geq 0$, the pair of random variables (X_{T_n+k}, Y_{T_n+k}) is independent of $\mathcal{F}_{T_n}$ given (X_{T_n}, Y_{T_n}).

From Assumption 2.0.1, there exists a loop $\gamma_{X_{T_n}}$ which lies inside $G_{n+1}\backslash(G_n \cup \partial G_{n+1})$ with weight larger than α and such that the probability that a random walk $(X_{T_n+k})_k$ makes this loop $\gamma_{X_{T_n}}$ is greater than β.

Let us denote by $A_{X_{T_n}}$ the event that the random walk $(X_{T_n+k})_k$ makes this loop $\gamma_{X_{T_n}}$, and let us denote by $B_{X_{T_n}}$ the event $\{Y_{T_n+|\gamma_{X_{T_n}}|} \leq p(\gamma_{X_{T_n}})\}$. Conditional on X_{T_n}, the events $A_{X_{T_n}}$ and $B_{X_{T_n}}$ are independent and have probabilities $\mathbb{P}_{X_{T_n}}(\gamma_{X_{T_n}}) \geq \beta$ and $p(\gamma_{X_{T_n}}) \geq \alpha$.

The event $A_{X_{T_n}} \cap \{Y_{T_n+|\gamma_{X_{T_n}}|} \leq p(\gamma_{X_{T_n}})\}$ for the random walk $(X_{T_n+k}, Y_{T_n+k})_{k\geq 0}$ starting from $(X_{T_n}, Y_{T_n}) \in \partial G_n$ has a probability greater than $\alpha\beta$. Conditional on (X_{T_n}, Y_{T_n}), it is independent of $\mathcal{F}_{T_n}$, therefore it is independent of the event $\{T_n < T_r\}$.

Let us show that conditional on $T_n < T_r$, if the event $A_{X_{T_n}} \cap \{Y_{T_n+|\gamma_{X_{T_n}}|} \leq p(\gamma_{X_{T_n}})\}$ is satisfied, then the event $\{T_{n+1} > T_r\}$ is satisfied. The idea is that on this event, the random walk keeps a loop before reaching ∂G_{n+1} and therefore $T_{n+1} > T_r$.

Let i be the largest integer such that $n_i \leq T_n$. Then, by construction of the p-loop-erased random walk, assuming $T_r > T_n \geq n_i$, (Z_k^{i+1}) coincides with (X_k) after time n_i and therefore after time T_n. For $k \leq T_n$, $Z_k^{i+1} \in \{X_0, \dots, X_{T_n}\}$ by construction and therefore $Z_k^{i+1} \in G_n$.

If the event $A_{X_{T_n}} \cap \{Y_{T_n+|\gamma_{X_{T_n}}|} \leq p(\gamma_{X_{T_n}})\}$ is satisfied, then, for k between $T_n + 1$ and $T_n + |\gamma_{X_{T_n}}|$, we have $Z_k^{i+1} \in G_{n+1}\backslash(G_n \cup \partial G_{n+1})$, and therefore, for such k,

$$Z_k^{i+1} \notin (Z_0^{i+1}, \dots, Z_{T_n}^{i+1}).$$

Since we have $n_{i+1} \geq T_n$ by assumption on i, we have necessarily $n_{i+1} = T_n + |\gamma_{X_{T_n}}|$. Since the event $\{Y_{T_n+|\gamma_{X_{T_n}}|} \leq p(\gamma_{X_{T_n}})\}$ is satisfied by assumption and $T_r > n_i$, we have $T_r = n_{i+1} = T_n + |\gamma_{X_{T_n}}|$, and since $A_{X_{T_n}}$ is satisfied, for $T_n + 1 \leq k \leq T_n + |\gamma_{X_{T_n}}|$, we have $X_k \in G_{n+1}\backslash(G_n \cup \partial G_{n+1})$ and therefore $T_{n+1} > T_n + |\gamma_{X_{T_n}}| = T_r$.

Therefore, denoting by $\delta := 1 - \alpha\beta < 1$,

$$\begin{aligned}
\mathbb{P}_{v_0}(T_{n+1} < T_r \mid T_n < T_r) &\leq 1 - \mathbb{P}_{(X_{T_n}, Y_{T_n})}(A_{X_{T_n}} \cap \{Y_{T_n+|\gamma_{X_{T_n}}|} \leq \alpha\} \mid T_n < T_r) \\
&= 1 - \mathbb{P}_{(X_{T_n}, Y_{T_n})}(A_{X_{T_n}} \cap \{Y_{T_n+|\gamma_{X_{T_n}}|} \leq \alpha\}) \\
&= 1 - \mathbb{P}_{X_{T_n}}(A_{X_{T_n}})\mathbb{P}(Y_{T_n+|\gamma_{X_{T_n}}|} \leq \alpha) \\
&\leq 1 - \alpha\beta = \delta.
\end{aligned}$$

This inequality holds for every $n \in \mathbb{N}^*$ and δ does not depend on n. Then, writing

$$\mathbb{P}_{v_0}(T_{n+1} < T_r) = \mathbb{P}_{v_0}(T_{n+1} < T_r \mid T_n < T_r)\mathbb{P}_{v_0}(T_n < T_r),$$

we obtain by induction on n the exponential decay of the following probability :

$$\mathbb{P}_{v_0}(T_n < T_r) \leq \delta^n.$$

Let $\varepsilon > 0$. For fixed n large enough, $\delta^n \leq \varepsilon$.
For k large enough, we have $\mathbb{P}_{v_0}(T_n \geq k) \leq \varepsilon$. Then for k large enough, we have

$$\begin{aligned}\mathbb{P}_{v_0}(T_r \geq k) &= \mathbb{P}_{v_0}(\{T_r \geq k\} \cap \{T_n \geq k\}) + \mathbb{P}_{v_0}(\{T_r \geq k\} \cap \{T_n \leq k-1\}) \\ &\leq \mathbb{P}_{v_0}(T_n \geq k) + \mathbb{P}_{v_0}(T_r > T_n) \leq 2\varepsilon\end{aligned}$$

Therefore we have shown that for every $\varepsilon > 0$, for k large enough, $\mathbb{P}_{v_0}(T_r \geq k) < 2\varepsilon$, which means that

$$\lim_{k\to\infty} \mathbb{P}_{v_0}(T_r \geq k) = 0.$$

Therefore, from the monotone convergence theorem,

$$\mathbb{P}_{v_0}(T_r = \infty) = \mathbb{P}_{v_0}(\bigcap_{k\geq 1} \{T_r \geq k\}) = \lim_{k\to\infty} \mathbb{P}_{v_0}(T_r \geq k) = 0$$

This concludes the proof. □

The proof of Lemma 2.2.3 can be adapted to show that the rooting time is almost surely finite for a random walk starting from another vertex of G, even if this vertex is not in G_1, as follows.

Lemma 2.2.4. *Let $(X_n^{(x)})_n$ be a random walk starting from $x \in G$ and and let $(Y_n)_n$ be the process defined in Definition 2.1.4. Under Assumption 2.0.1, the rooting time T_r for $(X_n^{(x)})_n$ is finite almost surely and $\mathbb{P}_x(T_r > T_n)$ decays exponentially fast to 0 with n.*

Proof. Notice that x is not anymore in G_1 and therefore the bound $\mathbb{P}_x(T_n \leq T_r) \leq \delta^n$ does not hold.

Nevertheless, if m_x is such that $x \in G_{m_x}$, then for $n \geq m_x$, the proof of Lemma 2.2.3 shows that for the loop-erased random walk starting from x,

$$\mathbb{P}_x(T_{n+1} < T_r | T_n < T_r) \leq \delta$$

and therefore for $n \geq m_x$,

$$\mathbb{P}_x(T_n < T_r) \leq \delta^{n-m_x}$$

Therefore $\mathbb{P}_x(T_n \leq T_r)$ tends to 0 exponentially fast with n and an argument similar to the one given in the proof of Lemma 2.2.3 gives that T_r is finite almost surely. □

Lemma 2.2.4 shows that if we start a simple random walk on G from a vertex v, almost surely T_r is finite. It implies that almost surely the sequence $((Z_n^k)_{n\geq 0})_{k\geq 0}$ is constant eventually and its limit $(Z_n)_{n\geq 0}$ is well defined with $(Z_n)_{n\geq 0}$ constant for $n \geq T_r$.

In the following, we define p-loop-erased random walks with wired boundary conditions in order to adapt the usual Wilson algorithm to sample cycle-rooted spanning forests of infinite graphs.

2.3. p-loop-erased random walk with a boundary condition.

Let us briefly recall our current notations. We still assume that (X_n) is a simple random walk on G starting from any vertex v, (Y_n) is a sequence of independent random variables of uniform law in $[0, 1]$, which are independent of the X_n and $W \subset V$ is a deterministic set of vertices.

We define in this subsection a p-loop erased random walk obtained from $(X_n, Y_n)_{n\geq 0}$ with the boundary condition W.

Definition 2.3.1. *Let T_W be the hitting time of W, and let T_r be the rooting time of $(X_n, Y_n)_{n\geq 0}$. Let $T_f = \min(T_r, T_W)$ be called the* ending time *of $(X_n, Y_n)_{n\geq 0}$ with boundary condition W.*

Given $(X_n, Y_n)_{n\leq T_W}$, we construct a p-loop erased random walk $(Z_n^W)_n$ with boundary conditions W as follows. We define recursively n_k^W and $(Z_n^{k,W})_{n\geq 0}$ for $k \in \mathbb{N}^*$. Let $(Z_n^{1,W})_{n\leq T_W} = (X_n)_{n\leq T_W}$ and $n_0^W = 0$.

Then, we define recursively a sequence $((Z_n^{i,W})_{n\leq T_W})_{i\geq 1}$ and a sequence $(n_i)_{i\leq k}$ as follows. Let $n_i^W : (X_0, \dots, X_k) \mapsto \min\{n_{i-1}^W < j \leq T_W | Z_j^{i,W} \in \{Z_0^{i,W}, \dots, Z_{j-1}^{i,W}\}\}$ be the i-th loop-closing time before reaching W, where $\min \emptyset = \infty$. If $n_i^W(X_0, \dots, X_k) = \infty$, let $Z^{i+1,W} = Z^{i,W}$.

Else, let $n_i'^W$ be the first time $j < n_i^W$ such that $Z_{n_i^W}^{i,W} = Z_{n_i'^W}^{i,W}$ and if $Y_{n_i^W} \leq p(\gamma_{n_i^W})$, let for every $m \geq i+1$,

$$Z_n^{m,W} = \begin{cases} Z_n^{i,W} & \text{if } n \leq n_i^W, \\ Z_{n_i^W}^{i,W} & \text{else,} \end{cases}$$

and otherwise, let for $n \leq T_W$

$$Z_n^{i+1,W} = \begin{cases} Z_{n_i'^W}^{i,W} & \text{if } n_i'^W \leq n \leq n_i^W, \\ Z_n^{i,W} & \text{else.} \end{cases}$$

Notice that $Z_n^{i+1,W}$ is obtained from $Z_n^{i,W}$ by erasing the first loop which ends before T_W. While i is small enough such that $n_i \leq T_W$, we have $n_i^W = n_i$ and $Z^{i+1,W} = Z^{i+1}$.

Proposition 2.3.2. *Almost surely, T_f is finite and $((Z_n^{i,W})_{n\leq T_f})_{i\geq 0}$ is constant eventually. We define the p-loop erased random walk (p-LERW) with boundary conditions W as*

$$(Z_n^W)_{n\leq T_f} = \lim_{i\to\infty} (Z_n^{i,W})_{n\leq T_f}.$$

- *If $T_f = T_W$, $(Z_n^W)_{n\leq T_f} = (Z_n^{i_f})_{n\leq T_W}$ where $i_f = \min\{i | n_i > T_W\}$.*
- *If $T_f = T_r$, $(Z_n^W)_{n\leq T_f} = (Z_n)_{n\leq T_f}$ where (Z_n) is the p-loop erased random walk without any boundary condition.*

Proof. Recall from Lemma 2.2.4 that T_r is finite almost surely. Since $T_f \leq T_r$, the ending time T_f is almost surely finite. Assume that $T_r < \infty$. Recall that the sequence (n_i) is strictly increasing. If $T_W < T_r$, then T_W is finite and there exists i such that $n_i > T_W$ and then $n_i^W = \infty$ and $Z^{m,W} = Z^{i,W}$ for $m \geq i$. Let $i_f = \min\{i | n_i > T_W\}$. Then $n_{i_f-1} \leq T_W$. Then, $(Z_n^{i_f,W})_{n\leq T_W} = (Z_n^{i_f})_{n\leq T_W}$.

Then, for $m \geq i_f$, $n_m > T_W$ and then $n_m^W = \infty$. Then, for every $m \geq i_f$,

$$(Z_n^{m,W})_{n\leq T_f} = (Z_n^{i_f,W})_{n\leq T_f} = (Z_n^{i_f})_{n\leq T_f}.$$

Else, there exists i such that $T_r = n_i \leq T_W$. Then, $n_i^W = n_i$ and $Y_{n_i^W} \leq p(\gamma_{n_i^W})$ and for $m \geq i$, $(Z_n^{m,W})_{n\leq T_r} = (Z_n^{i,W})_{n\leq T_r} = (Z_n^i)_{n\leq T_r}$. Since $n_i = T_r$, then we have $(Z_n^i)_{n\leq T_r} = (Z_n)_{n\leq T_r}$ and therefore, $(Z_n^W)_{n\leq T_f} = (Z_n^W)_{n\leq T_f}$ where (Z_n) is the p-loop erased random walk without any boundary condition. □

Notice that a p-loop-erased random walk with boundary condition W is obtained from $(X_n, Y_n)_{n\leq \min(T_r, T_W)}$ erasing every loop except the last one if $T_r < T_W$.

Let us emphasize that when $T_r > T_W$, the p-loop erased random walk with boundary conditions $(Z_n^W)_{n\leq T_W}$ is not equal to the p-loop erased random walk $(Z_n)_{n\leq T_W}$ stopped at T_W.

We will define, in the next section, sequences of probability measures on CRSF on a growing exhaustion (G_n) of a countably infinite connected graph G, with boundary conditions. We will see that under some hypotheses, those sequences of probability measures on CRSF of finite graphs G_n converge to thermodynamic limits which are probability measures on CRSF of the infinite graph G. We will also define probability measures on

CRSF of infinite graphs from p-loop-erased random walks and compare those probability measures with limits of sequences of probability measures on finite graphs.

3. Measures on CRSF in infinite volume and thermodynamic limits

In the following, let $G = (V, E)$ be a countably infinite connected graph, with finite degrees, and let (G_n) be an exhaustion of G, in the sense of an increasing sequence of finite subgraphs of G whose union is G. Let v_0 be a vertex of G_1.

3.1. **Topological facts and boundary conditions.** Every subgraph of G can be seen as an element of $\{0,1\}^E$. Let us recall some topological facts about the space $\{0,1\}^E$.

Since $\{0,1\}$ is compact, $\Omega = \{0,1\}^E$ is compact for the product topology and this topology is compatible with the following metric

$$d(w, w') = \sum_{e \in E} 2^{-\|e_-\|} 1_{\{w_e \neq w'_e\}},$$

where $\|e_-\|$ is the length of the shortest path between v_0 and an extremity of the edge e. Therefore Ω is a compact metric space.

A function $f : \Omega \to \mathbb{R}$ is continuous for the product topology if for every $\varepsilon > 0$, there exists a finite subset $\Lambda \subset E$, such that

$$\sup_{w,w' \in \Omega : w_{|\Lambda} = w'_{|\Lambda}} |f(w) - f(w')| \leq \varepsilon.$$

A function $f : \Omega \to \mathbb{R}$ is called local if there exists a finite set $\Lambda \subset E$ such that $f(w)$ is entirely determined by $w_{|\Lambda}$. We say that an event $A \subset \Omega$ has finite support if the function 1_A is a local function. The set of local functions is dense in the set of continuous functions $(\mathcal{C}(\Omega), ||.||_\infty)$ which is a Banach-space.

We consider $\mathcal{C}$ the smallest σ-field which makes the cylinders $\mathcal{C}_{\Lambda,\eta} = \{w \in \Omega, w_\Lambda = \eta\}$ measurable for every finite subset $\Lambda \subset E$ and every finite configuration $\eta \in \{0,1\}^\Lambda$. We say that a sequence of probability measures (μ_n) converges to the measure μ on $(\Omega, \mathcal{C})$ if and only if

$$\lim_{n \to \infty} \int_\Omega f d\mu_n = \int_\Omega f d\mu,$$

for every local function f. Since the set of local functions is dense in the set of continuous functions, this topology on the set of measures on $(\Omega, \mathcal{C})$ is the weak convergence.

In this section, we are interested in the sequences of measures $(\mu_n)_{n \geq 1}$ on CRSF on growing subgraphs G_n. If such a sequence of measures converges weakly towards an infinite volume measure, we have the following result on the limit measure.

Proposition 3.1.1. *Assume that a sequence $(\mu_n)_{n \geq 1}$ of measures on CRSF on growing subgraphs G_n of G converges weakly towards a measure μ, and let F be distributed according to μ. Then μ-almost surely, every finite connected component of F has exactly one cycle and its cycle has non-trivial weight.*

Proof. Let $x \in G$ and let T be a finite connected subgraph of G which contains x and which satisfies one of the following properties:

- T has strictly more than one cycle;
- T has a cycle of trivial weight;
- T has no cycle.

Let $cc(x)$ be the connected component of x in F. Notice that the event $\{cc(x) = T\}$ is an event with finite support since its support is included in the set of edges which have at least one extremity in T.

Let m be large enough such that $T \subset G_{m-1}$. Then, the event

$$\{cc(x) = T\} = \{cc(x) \cap G_m = T\}$$

has support in G_m. For every $n \geq m$, μ_n is supported on CRSF whose cycles have non trivial weight on G_n. Therefore,

$$\mu_n(cc(x) \cap G_m = T) = \mu_n(cc(x)_{F_n} = T) = 0.$$

We have the convergence $\mu_n \to \mu$ on configurations with finite support. Then

$$\mu_n(cc(x) \cap G_m = T) \to \mu(cc(x) \cap G_m = T).$$

Finally, we obtain $\mu(cc(x) = T) = 0$. Since G is countable, almost surely, every finite connected component of F has exactly one cycle and its cycle has non-trivial weight. □

We can consider sequences of measures $(\mu_n)_{n\geq 1}$ on CRSF on growing subgraphs G_n of G with boundary conditions, such as free and wired boundary conditions.

Definition 3.1.2 (Free boundary conditions)**.** *We define the free measure on CRSF of G_n as the measure on CRSF of G_n whose configurations have weight proportional to the product of weights of cycles. This measure is denoted by μ_n^F.*

Definition 3.1.3 (Wired boundary conditions)**.** *We define the wired measure on CRSF on G_n as the measure on CRSF of graph G_n whose configurations are either trees connected to ∂G_n or unicycles and have weight proportional to the product of weights of cycles. This measure is denoted by μ_n^W.*

3.2. **Sampling algorithm for a fixed ordering on an infinite graph.** In the following, we still consider a countably infinite connected graph G, an exhaustion (G_n) and a weight function $w = p \in [0, 1]$ on cycles which satisfies Assumption 2.0.1.

We now construct a probability measure on CRSF of G for weights p which is sampled by an algorithm determined by a fixed ordering of the vertex set V.

Let φ be an ordering of the vertex set V of G, in the sense of a bijection $\varphi : \mathbb{N} \to V$. Let $(v_i)_{i\geq 1}$ be the sequence of vertices of G with ordering φ, that is $(v_i)_i = (\varphi(i))_i$.

We will construct a measure on CRSF of G by means of a family of p-loop-erased random walks with boundary conditions which are defined recursively. This family will be obtained deterministically from a family of independent simple random walks following results of Section 2.

We still denote by $\mathcal{C}$ the smallest σ-field which makes the cylinders $C_{\Lambda,\eta}$ measurable.

Definition 3.2.1. *Let $((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}$ be independent random variables such that for all $x \in G$, $(X_n^{(x)})_n$ is a simple random walk on G starting from x and $(Y_n^{(x)})_n$ is a sequence of independent random variables with uniform law on $[0, 1]$. For a fixed x, consider the sequence $\left((X_n^{(x)}, Y_n^{(x)})\right)_{n\geq 1}$ and denote by T_r^x the rooting time of the p-LERW that is to say the first time n such that $(X_n^{(x)})_n$ closes a loop γ_n such that the inequality $p(\gamma_n) \geq Y_n^{(x)}$ holds.*

Definition 3.2.2. *For a fixed random data $((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}$ as above, we construct the subgraphs (F_i) recursively. Let $\mathsf{F}_0 = \emptyset$. Let $i \in \mathbb{N}^*$ and assume that F_{i-1} is constructed. Denote by $T_f^{\overline{v}_i}$ the ending time of $((X_n^{(v_i)})_{n\geq 1}, (Y_n^{(v_i)})_{n\geq 1})$ with boundary condition $V(\mathsf{F}_{i-1})$, that is*

$$T_f^{\overline{v}_i} = \min(T_r^{v_i}, T_{V(\mathsf{F}_{i-1})}).$$

Let $\mathsf{F}_i = \mathsf{F}_{i-1} \cup L(v_i)$ where $L(v_i)$ is the p-LERW with boundary condition $V(\mathsf{F}_{i-1})$ obtained from $\left((X_n^{(v_i)}, Y_n^{(v_i)})\right)_{n\geq 1}$ until $T_f^{\overline{v}_i}$.

Each step i of the algorithm finishes either if the random walk reaches a connected component created during a previous step or if the random walk is rooted to a loop. Notice that $T_f^{\overline{v}_i}$ is the time where the i^{th}-step of the algorithm with ordering φ finishes.

Recall that under Assumption 2.0.1 on p, Proposition 2.3.2 implies that $T_f^{v_i}$ is finite almost surely.

Lemma 3.2.3. *There exists a measure μ_φ on $(\mathcal{U}(G),\mathcal{C})$ which is sampled by the previous algorithm with ordering φ. The measure on finite cylinders corresponds to finite random configurations which are sampled in a finite time.*

Proof. Sample a sequence $((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}$ such as in Definition 3.2.1.

From Definition 3.2.2, we obtain a CRSF of G by setting $\mathsf{F} = \cup_{i\geq 1}\mathsf{F}_i$. The configuration F is well defined since it is a deterministic function of

$$((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}.$$

Let μ_φ be the law of F associated with a random choice of $((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}$, that is to say the push-forward by the algorithm of the measure which gives

$$((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}.$$

Then, μ_φ is a probability measure on $(\mathcal{U}(G),\mathcal{C})$. Let B be a finite subset of size n of E, with edges $e_1,\dots,e_n$ and let $\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}^n$. Let K be the finite set of vertices containing all the extremities of edges of B and vertices which are preceding those vertices for the order φ. Let us consider the previous algorithm for vertices $v_1,\dots,v_{|K|}$ for the ordering φ. Almost surely, the algorithm to construct $\mathsf{F}_{|K|}$ finishes in a finite time. The constructed graph $\mathsf{F}_{|K|}$ is a random subgraph of G which is spanning for K and therefore it is spanning for B. Then $\mu_\varphi(C_{\varepsilon_1,\dots,\varepsilon_n})$ is the probability that the random configuration $F_{|B}$ obtained from the previous construction satisfies

$$\mathsf{F}_{|B} \in C_{\varepsilon_1,\dots,\varepsilon_n}.$$

The measure μ_φ restricted to $(2^B,\mathcal{C})$ is the law of the random configuration $\mathsf{F}_{|B}$, which is sampled in a finite time. □

In the following, we will show that the measure in infinite volume constructed from an enumeration of the vertex set V does not depend on the choice of the enumeration. The proof of this statement will rely on a comparison between the measure μ_φ for an ordering φ and a measure on CRSF on a large finite subgraph of G. We will see that, under some hypotheses, the thermodynamic limit coincides with the measure sampled by the previous algorithm and does not depend on the ordering of the infinite vertex set.

3.3. **Thermodynamic limits of the Wilson measures.** Assume that Assumption 2.0.1 on the existence of a lower bound $\alpha > 0$ on the weight of a family of loops still holds.

Let $n \in \mathbb{N}$ and φ_n be an ordering of the vertex set V_n of the graph G_n. The measures μ_n^F and μ_n^W are sampled by the algorithm described in the first section.

Let us consider the sequence of measures (μ_n) on CRSF of $(G_n)_{n\geq 1}$ which are defined by the previous algorithm but if the walk meets ∂G_n, the walk is stopped. According to [KK17], the measure does not depend on the ordering of the vertices of $G_n\backslash\partial G_n$.

Theorem 3.3.1. *Let φ be an ordering of G in the sense of a bijection $\varphi : \mathbb{N} \to G$. Let (G_n) be an increasing exhaustion of G, and let $(\mu_n^F), (\mu_n^W)$ be the corresponding sequences of probability measures on CRSF of G_n with free and wired boundary conditions, respectively. The sequences of probability measures (μ_n^F) and (μ_n^W) converges weakly to the measure μ_φ.*

Proof. We consider an event $B \in 2^E$ which depends only on finitely many edges, and we consider K_0 the set of vertices incident to the edges on which B depends. Let K be the union of K_0 and the set of vertices that precede some vertex in K_0 in the ordering φ of the vertices. Let n be large enough such that $K \subset G_n$.

Let us construct a coupling $(\mathsf{F}, \tilde{\mathsf{F}}_n^F, \tilde{\mathsf{F}}_n^W)$ of random configurations, obtained from the same random data $((X_n^{(x)})_{n\geq 1})_{x\in G}, ((Y_n^{(x)})_{n\geq 1})_{x\in G}$, such that the law of F is μ_φ, the law

of $\tilde{\mathsf{F}}_n^F$ is μ_n^F and the law of $\tilde{\mathsf{F}}_n^W$ is μ_n^W and such that the three configurations coincide with high probability on B.

We denote $\tilde{\varphi}_n$ the ordering induced by the ordering φ on G_n.

We follow the algorithm for every vertex of K following the ordering $\varphi_{|K}$. If at one step of the algorithm, the random walk $(X_n^{(x)})$ starting from a vertex $x \in G_n$ reaches ∂G_n, the configuration F is obtained following the random walk in the infinite graph G until the end of the step and $\tilde{\mathsf{F}}_n^F$, $\tilde{\mathsf{F}}_n^W$ are obtained following the random walk with boundary conditions. More precisely, $\tilde{\mathsf{F}}_n^W$ is obtained from p-loop erased random walks with boundary conditions ∂G_n and $\tilde{\mathsf{F}}_n^F$ is obtained following the random walk on the graph G_n, until the end of the step, that is the ending time of the process $(X_n^{(x)}, Y_n^{(x)})_n$.

Once every vertex of K has been explored, we complete the configuration F following the ordering φ and we complete the configurations $\tilde{\mathsf{F}}_n^F$, $\tilde{\mathsf{F}}_n^W$ on G_n following the ordering $\tilde{\varphi}_n$, with boundary conditions.

Let us denote by $E(K)$ the set of edges whose vertices are in K. The configurations F, $\tilde{\mathsf{F}}_n^F$ and $\tilde{\mathsf{F}}_n^W$ obtained from the algorithm are respectively subgraphs of G and G_n. The three configurations are spanning subgraphs of G_n and in particular spanning subgraphs of $(K, E(K))$.

$\mathsf{F}_{E(K)}$ is the restriction to $(K, E(K))$ of a random configuration following the law μ_φ. Since G_n is finite, $(\tilde{\mathsf{F}}_n)_{E(K)}^{W,F}$ is the restriction to $(K, E(K))$ of a random configuration following the law $\mu_n^{W,F}$ and this law does not depend on $\tilde{\varphi}_n$ (see [KK17]).

Since B depends only on edges whose endpoints are in K_0, we know that

$$|\mu_\varphi(B) - \mu_n^{F,W}(B)| \leq \mathbb{P}(\mathsf{F}_{E(K)} \neq (\tilde{\mathsf{F}}_n)_{E(K)})$$

Following the previous algorithm, while each step starting from a vertex of K finishes before the random walk reaches ∂G_n, both configurations $\mathsf{F}_{E(K)}, (\tilde{\mathsf{F}}_n)_{E(K)}$ which are obtained are equal. Therefore, from the union bound,

$$\begin{aligned}\mathbb{P}(\mathsf{F}_{E(K)} \neq (\tilde{\mathsf{F}}_n)_{E(K)}) &\leq \mathbb{P}\left(\bigcup_{i\in[|K|]} \{T_n^{v_i} \leq T_f^{\underline{v}_i}\}\right) \leq \sum_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_f^{\underline{v}_i}) \\ &\leq \sum_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_r^{v_i}) \leq |K| \max_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_r^{v_i})\end{aligned}$$

From Lemma 2.2.4, we obtain when $n \to \infty$,

$$|\mu_\varphi(B) - \mu_n^{W,F}(B)| \leq |K| \max_{i\in[|K|]} \mathbb{P}(T_n^{v_i} \leq T_r^{v_i}) \to 0$$

which implies the weak convergence of $(\mu_n^{W,F})$ towards μ_φ. □

Recall that for every n, the measure μ_n^W is sampled by an algorithm and does not depend on the ordering of vertices of G_n chosen in the algorithm. Combined with Theorem 3.3.1, this independence implies the following result.

Theorem 3.3.2. *Let φ be an ordering of the vertices, that is a bijection $\varphi : \mathbb{N} \to V$. Let p be a weight function satisfying Assumption 2.0.1. Let μ_φ be the measure on the cycle-rooted spanning forests of G associated with the algorithm of loop-erased random walk with weights $p(\gamma)$. The measure μ_φ does not depend on φ.*

Proof. Let φ, τ be two orderings, with $(v_i) = (\varphi(i))$, $(w_i) = (\tau(i))$ let K_1 and K_2 be respectively the sets of vertices that precede some vertex in K_0 in the ordering φ (resp. τ) and let n large enough such that $K_1 \cup K_2 \subset G_n$. Then, from Theorem 3.3.1,

$$|\mu_\varphi(B) - \mu_\tau(B)| \leq |K_1| \max_{i\in[|K_1|]} \mathbb{P}_{v_i}(T_n \leq T_r) + |K_2| \max_{i\in[|K_2|]} \mathbb{P}_{w_i}(T_n \leq T_r) \to 0$$

This shows that both distributions in infinite volume coincide on cylinders and therefore the measure in infinite volume does not depend on the ordering of the vertices. □

For a weight function p satisfying Assumption 2.0.1, we will denote by μ_p the corresponding probability measure on CRSFs of G, which does not depend on the ordering of the vertices.

4. Study of the configurations sampled under the Wilson measure

In this section, we will study the asymptotic behavior of configurations and the rate of decay of correlations with the distance for the measure μ_p which is sampled by the previous algorithm of p-loop erased random walks in infinite volume and which will be called the Wilson measure, under the following assumption.

Assumption 4.0.1. *There exists $\alpha > 0, \beta > 0, M, M' > 0, C > 0, d \in \mathbb{N}$, a family of loops $\Gamma \subset \mathcal{C}(G)$ and for every $x \in G$, an increasing sequence (B_n^x) of subgraphs of G, exhausting G and containing x such that :*

- *For every $\gamma \in \Gamma$, $\alpha \leq w(\gamma) \leq 1$.*
- *For every $v \in \partial B_n^x$, there exists a loop $\gamma_v \in \Gamma \cap (B_{n+1}^x \backslash (B_n \cup \partial B_{n+1}^x))$ such that the probability for a random walk starting from v of making this loop γ_v is greater than β.*
- *For every x, for every $n \in \mathbb{N}$, $M'n \leq d(x, \partial B_n^x) \leq Mn$, and $|\partial B_n^x| \leq Cn^d$.*

Assumption 4.0.1 implies Assumption 2.0.1 and is satisfied in particular if the graph and the weight function w on cycles are invariant under translations and if Assumption 2.0.1 is satisfied for an exhaustion $(G_n)_n$ such that $d(0, \partial G_n) \sim Mn$ when n tends to infinity, where $M > 0$.

4.1. Every connected component is finite for the Wilson measure.

Definition 4.1.1. *For a vertex x and a subset $A \subset G$, we denote by $\{x \leftrightarrow A\}$ the event $\{A \cap C_x \neq \emptyset\}$ where C_x is the connected component of x in the random configuration sampled under μ. In particular, $\{x \leftrightarrow y\}$ means that x and y are in the same connected component.*

We will denote by $T_{m,x}^x$ the hitting-time of ∂B_m^x for the random walk $(X_n^{(x)})$. Recall that T_m^x is the hitting-time of ∂G_m for the random walk $(X_n^{(x)})$.

Lemma 4.1.2. *Under Assumption 4.0.1, there exists $\delta > 0$ such that the following inequality holds for every m, for every $x \in G$*

$$\mathbb{P}_x(\{T_r^x \geq T_{m,x}^x\}) \leq \delta^m.$$

Proof. Let $x \in G$. Under Assumption 4.0.1, Assumption 2.0.1 is satisfied for the vertex x, and therefore, if we denote by $T_{m,x}^x$ the hitting time of ∂B_m^x for a p-loop erased random walk starting from x, and T_r^x its rooting time, Lemma 2.2.3 gives the existence of a $0 < \delta < 1$ such that the following inequality holds for every m,

$$\mathbb{P}_x(\{T_r^x \geq T_{m,x}^x\}) \leq \delta^m,$$

where $\delta = 1 - \alpha\beta$ for parameters α, β of 4.0.1. In particular, δ does not depend on x, which concludes the proof. □

Lemma 4.1.3. *Let $\delta > 0$ as in Lemma 4.1.2 and $M > 0$ as in Assumption 4.0.1. Let x, y be two vertices of G and denote by $d(x, y)$ the distance between x and y, that is to say the length of the shortest path from x to y. Then, if $n \leq \frac{d(x,y)}{2M}$,*

$$\mu_p(x \leftrightarrow y) \leq 2\delta^n.$$

Proof. According to Theorem 3.3.2, the measure μ_p does not depend on the ordering of the vertices. We may choose an ordering φ in which x and y are the first two vertices. Since $n \leq \frac{d(x,y)}{2M}$, we have $d(x, \partial B_n^x) \leq Mn \leq \frac{d(x,y)}{2}$ and $d(y, \partial B_n^y) \leq Mn \leq \frac{d(x,y)}{2}$.

If x and y are in the same connected component in a configuration obtained from this algorithm, we know that either for the p-loop erased random walk starting from x or for the one starting from y, we have $\{T_r^x \geq T_{f\varphi}^x \geq T_{n,x}^x\}$ or $\{T_r^y \geq T_{f\varphi}^y \geq T_{n,x}^y\}$. Indeed, if $T_{f\varphi}^x \leq T_{n,x}^x$ and $T_{f\varphi}^y \leq T_{n,y}^x$, then the p-loop erased random walk starting from x and from y cannot intersect, and form two disjoint connected component in the configuration. Therefore, from the union bound,

$$\mu_p(x \leftrightarrow y) \leq \mathbb{P}_x(\{T_r^x \geq T_{n,x}^x\}) + \mathbb{P}_y(\{T_r^y \geq T_{n,y}^y\}) \leq 2\delta^n$$

which concludes the proof. □

Theorem 4.1.4. *μ_p-almost surely, for every vertex $x \in V$ of G, the connected component of x is finite.*

Proof. Let $x \in G$. For every $n \in \mathbb{N}$, for every $y \in \partial B_n^x$, $d(x,y) \geq M'n$. Let $n' = \lfloor \frac{M'n}{2M} \rfloor$. Then $n' \leq \frac{d(x,y)}{2M}$ and therefore, from Lemma 4.1.3,

$$\mu_p(x \leftrightarrow y) \leq 2\delta^{n'}.$$

Then, from the union bound, the following upper bound on the probability that the connected component of x contains vertices of the boundary of B_n^x holds for every $n \in \mathbb{N}$, with $\delta' = \delta^{\frac{M'}{2M}}$:

$$\mathbb{P}(x \leftrightarrow \partial B_n^x) \leq \sum_{y \in \partial B_n^x} \mathbb{P}(x \sim y) \leq 2|\partial B_n^x| \delta^{\lfloor \frac{M'n}{2M} \rfloor} \leq 2Cn^d \delta'^n.$$

Then, from the monotone convergence theorem, we have

$$\mathbb{P}(x \leftrightarrow \infty) = \mathbb{P}(\cap_n \{x \leftrightarrow \partial B_n^x\}) = \lim_{n \to \infty} \mathbb{P}(x \leftrightarrow \partial B_n^x) = 0.$$

Since G is countable, we know that μ-almost surely, for every $x \in G$, the connected component of x is finite. □

4.2. **Exponential decay of correlations for the Wilson measure.** We still assume that weights are in $[0,1]$ and satisfy Assumption 4.0.1.

Let $m \in \mathbb{N}$ and let $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ be such that $d(\{x_1, y_1\}, \{x_2, y_2\}) \geq m$.

If F is a CRSF following the law μ_p it can be sampled from the algorithm described in Section 3.2 and from Theorem 3.3.2, it does not depend on the chosen ordering of vertices, therefore we may assume that the first four vertices of the ordering φ are x_1, y_1, x_2, y_2.

Let us consider in the following, four independent couples of sequences of random variables $(X_n^{x_1}, Y_n^{x_1})$, $(X_n^{y_1}, Y_n^{y_1})$, $(X_n^{x_2}, Y_n^{x_2})$, $(X_n^{y_2}, Y_n^{y_2})$, as defined in Section 4.1.

For $i \in \{1,2\}$, let us denote by A_i the event that both p-loop erased random walks obtained from $(X_n^{x_i}, Y_n^{x_i})_n$, $(X_n^{y_i}, Y_n^{y_i})_n$ starting from x_i, y_i are rooted before leaving the subgraphs $B_{m/2}^{x_i}, B_{m/2}^{y_i}$, that is to say that

$$A_i = \{T_r^{x_i} < T_{x_i, m/2}^{x_i}\} \cap \{T_r^{y_i} < T_{y_i, m/2}^{y_i}\}$$

Lemma 4.2.1. *Conditional on $A_1 \cap A_2$, the events $\{e_1 \in \mathsf{F}\}$ and $\{e_2 \in \mathsf{F}\}$ are independent.*

Proof. Let F_4 be the subgraph obtained after the first four runs of the algorithm.

Notice that, once F_4 has been sampled, during every subsequent run of the algorithm, the p-loop erased random walk stops if it reaches x_1, y_1, x_2, y_2 because F_4 contains x_1, y_1, x_2, y_2. Therefore, for F the configuration obtained following the algorithm in infinite volume, we have the following equality of events for $i \in [1,2]$,

$$\{e_i \in \mathsf{F}\} = \{e_i \in \mathsf{F}_4\}.$$

Let $(Z_n^{x_1})_{n\leq T_r^{x_1}}$ be the p-loop erased random walk obtained from $(X_n^{x_1}, Y_n^{x_1})$ and let

$$W_1 = V((Z_n^{x_1})_{n\leq T_r^{x_1}})$$

be the set of vertices explored by this p-loop erased random walk. Let $(Z_n^{y_1})_{n\leq \min(T_r^{y_1}, T_{W_1}^{y_1})}$ be the p-loop erased random walk starting from y_1 with boundary condition W_1. Let $\mathfrak{F}_1$ be the subgraph given by $(Z_n^{x_1})_{n\leq T_r^{x_1}}, (Z_n^{y_1})_{n\leq \min(T_r^{y_1}, T_{W_1}^{y_1})}$.

Let $(Z_n^{x_2})_{n\leq T_r^{x_2}}$ be the p-loop erased random walk obtained from $(X_n^{x_2}, Y_n^{x_2})$ and let

$$W_2 = V((Z_n^{x_2})_{n\leq T_r^{x_2}})$$

be the set of vertices explored by this p-loop erased random walk. Let $(Z_n^{y_2})_{n\leq \min(T_r^{y_2}, T_{W_2}^{y_2})}$ be the p-loop erased random walk starting from y_1 with boundary condition W_2. Let $\mathfrak{F}_2$ be the subgraph given by $(Z_n^{x_2})_{n\leq T_r^{x_2}}, (Z_n^{y_2})_{n\leq \min(T_r^{y_2}, T_{W_2}^{y_2})}$.

Let us emphasize that the p-loop erased random walk corresponding to the third and the fourth runs of the algorithm has boundary conditions $V(\mathfrak{F}_1)$, corresponding to the configuration created during the first two runs of the algorithm. Therefore, in general, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are not disjoint and their union is not the component created after four runs of the algorithm.

If A_1 is satisfied, $\mathfrak{F}_1$ is contained in $B_{m/2}^{x_1} \cup B_{m/2}^{y_1}$ and if A_2 is satisfied,

$$\begin{cases} T_r^{x_2} < T_{x_2,m/2}^{x_2} < T_{V(\mathfrak{F}_1)} \\ T_r^{y_2} < T_{y_2,m/2}^{y_2} < T_{V(\mathfrak{F}_1)} \end{cases}$$

and therefore, the third and the fourth runs finish before the p-loop erased random walks reach $V(\mathfrak{F}_1)$, that is to say that the p-loop erased random walks with boundary condition $V(\mathfrak{F}_1)$ coincides with the p-loop erased random walks without this boundary condition (see Proposition 2.3.2). Therefore, if A_1 and A_2 are satisfied, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are disjoint connected components and their union is exactly the component created after four runs of the algorithm.

In particular, if $A_1 \cap A_2$ is satisfied, for $i \in [1,2]$, $\{e_i \in \mathsf{F}\}$ is satisfied if and only if $\{e_i \in \mathfrak{F}_i\}$ is satisfied.

We show that conditional on $A_1 \cap A_2$, the random configurations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are independent. Recall that $(Z_n^{x_1}), (Z_n^{y_1})$ and $(Z_n^{x_2}), (Z_n^{y_2})$ are independent and $\mathfrak{F}_1, A_1$ only depends on $(Z_n^{x_1}), (Z_n^{y_1})$ and $\mathfrak{F}_2, A_2$ only depends on $(Z_n^{x_2}), (Z_n^{y_2})$. Therefore, if F_1, F_2 are some fixed configurations,

$$\mathbb{P}(\mathfrak{F}_1 = F_1, A_1, \mathfrak{F}_2 = F_2, A_2) = \mathbb{P}(\mathfrak{F}_1 = F_1, A_1)\mathbb{P}(\mathfrak{F}_2 = F_2, A_2).$$

Therefore, using independence of $\{\mathfrak{F}_i = F_i\} \cap A_i$ and A_j for $i \neq j$, we obtain

$$\begin{aligned} \mathbb{P}(\mathfrak{F}_1 = F_1, \mathfrak{F}_2 = F_2 | A_1, A_2) &= \frac{\mathbb{P}(\mathfrak{F}_1 = F_1, A_1)\mathbb{P}(\mathfrak{F}_2 = F_2, A_2)}{\mathbb{P}(A_1 \cap A_2)} \\ &= \mathbb{P}(\mathfrak{F}_1 = F_1 | A_1)\mathbb{P}(\mathfrak{F}_2 = F_2 | A_2) \\ &= \mathbb{P}(\mathfrak{F}_1 = F_1 | A_1 \cap A_2)\mathbb{P}(\mathfrak{F}_2 = F_2 | A_1 \cap A_2). \end{aligned}$$

Therefore, conditional on $A_1 \cap A_2$, the random variables $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are still independent.

Therefore, conditional on $A_1 \cap A_2$, $\{e_1 \in \mathsf{F}\} = \{e_1 \in \mathfrak{F}_1\}$ and $\{e_2 \in \mathsf{F}\} = \{e_2 \in \mathfrak{F}_2\}$ are independent. □

As a consequence, we obtain the following decay of correlations.

Theorem 4.2.2. *There exists a parameter $0 < \iota < 1$ such that for every m large enough,*

$$\mu_p(e_2 \in \mathsf{F})\mu_p(e_1 \in \mathsf{F}) - \iota^m \leq \mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\}) \leq \mu_p(e_2 \in \mathsf{F})\mu_p(e_1 \in \mathsf{F}) + \iota^m$$

Proof. Let us compute $\mu_p(\{e_1, e_2 \in \mathsf{F}\})$ using the following decomposition :

$$\{e_1, e_2 \in \mathsf{F}\} = (\{e_1, e_2 \in \mathsf{F}\} \cap A_1 \cap A_2) \cup \left(\{e_1, e_2 \in \mathsf{F}\} \cap (A_1^{\complement} \cup A_2^{\complement})\right).$$

Since $\{e_1, e_2 \in \mathsf{F}\} \cap (A_1^{\complement} \cup A_2^{\complement})$ is included in $A_1^{\complement} \cup A_2^{\complement}$, it has probability less than the quantity $\mu_p(A_1^{\complement} \cup A_2^{\complement})$.

From Lemma 2.2.3, there exists some $\delta < 1$ such that from the union bound, we get

$$\mu_p(A_1^{\complement} \cup A_2^{\complement}) \leq 4\delta^{m/2}.$$

For the other term, we use the independence of $\{e_1 \in \mathsf{F}\}$ and $\{e_2 \in \mathsf{F}\}$ conditional on $A_1 \cap A_2$ proved in Lemma 4.2.1, which implies

$$\begin{aligned}\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) &= \mu_p(\{e_1 \in \mathsf{F}\} \cap \{e_2 \in \mathsf{F}\} | A_1 \cap A_2)\mu_p(A_1 \cap A_2) \\ &= \frac{\mu_p(\{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2)\mu_p(\{e_2 \in \mathsf{F}\} \cap A_1 \cap A_2)}{\mu_p(A_1 \cap A_2)} \\ &\leq \frac{\mu_p(\{e_1 \in \mathsf{F}\})\mu_p(\{e_2 \in \mathsf{F}\})}{\mu_p(A_1 \cap A_2)}.\end{aligned}$$

Using again the lower bound on $\mu_p(A_1 \cap A_2)$ which comes from Lemma 2.2.3, we have

$$\mu_p(A_1 \cap A_2) \geq 1 - 4\delta^{m/2}.$$

Let $\eta < 1$ be such that for m large enough,

$$4\delta^{m/2} < \eta^m$$

Therefore, we have the following upper bound on $\frac{1}{\mu_p(A_1 \cap A_2)}$,

$$\frac{1}{\mu_p(A_1 \cap A_2)} \leq \frac{1}{1-\eta^m} = \sum_{k \geq 0} \eta^{mk} = 1 + \sum_{k \geq 1} \eta^{mk} \leq 1 + \sum_{k \geq m} \eta^k \leq 1 + \frac{\eta^m}{1-\eta}.$$

Therefore we get

$$\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) \leq \mu_p(e_1 \in \mathsf{F})\mu_p(e_2 \in \mathsf{F})(1 + \frac{\eta^m}{1-\eta}).$$

For the other inequality, notice that

$$\begin{aligned}&\frac{\mu_p(\{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2)\mu_p(\{e_2 \in \mathsf{F}\} \cap A_1 \cap A_2)}{\mu_p(A_1 \cap A_2)} \\ &\geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\mu_p(A_1^{\complement} \cup A_2^{\complement}) \\ &\geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\eta^m.\end{aligned}$$

Therefore,

$$\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\} \cap A_1 \cap A_2) \geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\eta^m,$$

and

$$\mu_p(\{e_2 \in \mathsf{F}\} \cap \{e_1 \in \mathsf{F}\}) \geq (\mu_p(\{e_1 \in \mathsf{F}\}\mu_p(\{e_2 \in \mathsf{F}\}) - 2\eta^m.$$

Considering $\iota < 1$ such that for m large enough, $2\eta^m < \iota^m$ and $\eta^m \frac{1}{1-\eta} + 4\delta^{m/2} < \iota^m$ concludes the proof. □

5. Study of the configurations sampled under an infinite volume measure μ_w.

In this section, we consider a non-negative weight function w on oriented cycles of G, which can take values larger than 1. We assume that the sequence of measures (μ_n) on cycle-rooted spanning forests of G_n associated with the weight function w converges weakly towards an infinite volume measure μ and that this measure does not depend on the free or wired boundary conditions.

5.1. **Algorithm conditional on cycles with weights larger than 1.** In this subsection, we assume that G is a finite connected graph. Let $W \subset G$ be a set of vertices of G (which can be empty). Let $\mathcal{C}_+(G\backslash W)$ be the set of cycles of G of weight strictly larger than 1, so-called positive cycles and $\mathcal{C}_-(G\backslash W)$ the set of cycles of G of weight less than 1, so-called negative cycles.

Definition 5.1.1. *The weight function w_- on cycles of the graph G associated with the weight function w is defined by the following restrictions*

$$\begin{cases} w_{-\mathcal{C}_-(G\backslash W)} = w_{\mathcal{C}_-(G\backslash W)}, \\ w_{-\mathcal{C}_+(G\backslash W)} = 0. \end{cases}$$

The following result gives a way to sample a cycle-rooted spanning forest conditional on its positive cycles.

Theorem 5.1.2. *Let C be a subset of $\mathcal{C}_+(G\backslash W)$ and let A be the set of vertices which are extremities of edges in C. Let F a ECSRF with respect to W sampled according μ^W. Conditional on $\mathcal{C}_+(\mathsf{F}) = C$, $\mathsf{F}\backslash C$ has the same law than a ECRSF with respect to $A \cup W$ with weight function w_-.*

Proof. Let $F_0 \in \mathcal{U}_W(G)$.

$$\mu^W(\mathsf{F} = F_0 | \mathcal{C}_+(F) = C) = \frac{\mu(\mathsf{F} = F_0 \cap \mathcal{C}_+(\mathsf{F}) = C)}{\mu(C_+(\mathsf{F}) = C)}.$$

Notice that this quantity is zero if $C_+(F_0) \neq C$. Then, if $C_+(F_0) = C$ and $F_0 \in \mathcal{U}_W(G)$, every connected component of F_0 either contains a unique cycle in $C_-(G)$ or a unique cycle in C or is connected to a unique point in W. Therefore, every connected component of $F_0\backslash C$ either contains a unique cycle in $C_-(G)$ or or is connected to a unique point in $W \cup A$ which means that $F_0\backslash C \in \mathcal{U}_{W\cup A}(G)$.

Then, the measure $\mu^W(.|C_+(F) = C)$ has support in

$$\mathcal{U}_W^C(G) = \{F \in \mathcal{U}_W(G) | C_+(F) = C\} = \{F \in \mathcal{U}_W(G) | F = C \cup F_-, F_- \in \mathcal{U}_{W\cup A}(G)\},$$

and if $F_0 \in \mathcal{U}_W^C(G)$,

$$\begin{aligned} \mu^W(\mathsf{F} = F_0 | \mathcal{C}_+(F) = C) &= \frac{\prod_{\gamma\in C} w(\gamma) \prod_{\gamma\in\mathcal{C}_-(F_0)} w(\gamma)}{\sum_{F\in\mathcal{U}_W(G)|\mathcal{C}_+(F)=C} \prod_{\gamma\in C} w(\gamma) \prod_{\gamma\in\mathcal{C}_-(F)} w(\gamma)} \\ &= \frac{\prod_{\gamma\in\mathcal{C}_-(F_0)} w(\gamma)}{\sum_{F\in\mathcal{U}_W^C(G)} \prod_{\gamma\in\mathcal{C}_-(F)} w(\gamma)}. \end{aligned}$$

Writing every $F \in \mathcal{U}_W^C(G)$ on a unique way as $C \cup F_-$ with $F_- \in \mathcal{U}_{W\cup A}(G)$,

$$\begin{aligned} \mu^W(\mathsf{F} = F_0 | \mathcal{C}_+(\mathsf{F}) = C) &= \frac{\prod_{\gamma\in\mathcal{C}_-(F_{0-})} w(\gamma)}{\sum_{F_-\in\mathcal{U}_{W\cup A}(G)} \prod_{\gamma\in\mathcal{C}_-(F_-)} w(\gamma)} \\ &= \mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}(F_{0-}) = \mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}(F_0\backslash C). \end{aligned}$$

Finally,

$$\mu^W(\mathsf{F}\backslash C = .|\mathcal{C}_+(\mathsf{F}) = C) = \mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}(.),$$

which concludes the proof. □

Since $w_{\mathcal{C}_-(G\backslash W)}$ takes values in $[0, 1]$ by definition of $C_-(G\backslash W)$, the measure $\mu^{W\cup A}_{w_{\mathcal{C}_-(G\backslash W)}}$ can be sampled by the wired Wilson algorithm with boundary conditions $A \cup W$.

Therefore, under the measure μ^W, conditional on $\mathcal{C}_+(F)$, a ECRSF with respect to W has the same law as a a ECRSF with respect to W and extremities of edges in $\mathcal{C}_+(F)$ and can be sampled from a loop-erased random walk algorithm.

5.2. **All connected components with a cycle are finite.** We assume that Assumption 4.0.1 holds for the weight function w_- as defined in Definition 5.1.1 and for a family of cycles $\Gamma \subset \mathcal{C}_-(G)$. In particular, Assumption 2.0.1 also holds for w_-, but the weight function w can take values larger than 1.

We will show in this subsection that under this assumption, every connected component with a cycle is finite.

We will use the following lemma on the exponential decay of the tail distribution of ending times, which is a corollary of Section 2.

Lemma 5.2.1. *Let $m \in \mathbb{N}$, $n \geq m$. Let $C \subset G_n$ be a subgraph of G_n. Let $\mathbb{P}_x$ be the law of a w_--loop erased random walk (X_n) starting from x and T_r be the rooting time of (X_n). Let T_C and $T_{m,x}$ be the hitting times of C and ∂B_m^x. Under Assumption 2.0.1 on w_-, the following inequality holds*

$$\mathbb{P}_x(\min(T_C, T_r) \geq T_{m,x}) \leq \delta^m.$$

Proof. Applying Lemma 2.2.3 to the random walk (X_n) with weight function w_- which satisfies Assumption 2.0.1, we get

$$\mathbb{P}_x(T_r \geq T_{m,x}) \leq \delta^m.$$

But since $\min(T_C, T_r) \leq T_r$, we have

$$\mathbb{P}_x(\min(T_C, T_r) \geq T_m^x) \leq \mathbb{P}_x(T_r \geq T_{m,x}) \leq \delta^m,$$

which concludes the proof. □

Let $x \in V$ be a fixed vertex of G. We introduce some events with compact support which depend on x and prove an upper bound on the probability of those events.

Definition 5.2.2. *For $m \in \mathbb{N}$, we denote by $A_m = \{x \leftrightarrow \partial B_{2m}^x\}$ the event that x and ∂B_{2m}^x are connected in F, that is to say that there exists a path between x and ∂B_{2m}^x in B_{2m}^x. If $n \in \mathbb{N}$, we denote by $\Gamma_n^- = \{x \leftrightarrow C_-(F_{G_n})\}$ the event that x is connected to a closed cycle in G_n of weight less than 1.*

Lemma 5.2.3. *Let $m \in \mathbb{N}$. For $n \geq m$ large enough, if F_n is distributed according to the free measure μ_n on G_n,*

$$\mu_n(A_m \cap \Gamma_n^-) \leq (|\partial B_m^x| + 1)\delta^m.$$

Proof. Let n be a large enough integer such that for every $y \in \partial B_{2m}^x$, we have $B_m^y \subset G_n$. Let $C \subset \mathcal{C}_+(G_n)$. From Theorem 5.1.2, conditional on $\mathcal{C}_+(F_n) = C$, F_n is given by an algorithm of w_--loop erased random walks with boundary conditions on C. The proof relies on the same ideas as that in the proof of Lemma 4.1.3 and Theorem 4.1.4.

The event $A_m \cap \Gamma_n^-$ is satisfied if there exists $y \in \{x\} \cup \partial B_{2m}^x$ such that the w_--loop erased random walk starting from y has left B_m^y before being rooted to a cycle in $\mathcal{C}_-(G_n)$ and before touching C.

From Lemma 5.2.1, for every $y \in \{x\} \cup \partial B_{2m}^x$,

$$\mathbb{P}_y(\min(T_C, T_r) \geq T_{m,y}) \leq \delta^m.$$

Then, the union bound concludes the proof. □

Lemma 5.2.4. *The previous lemma implies that*

$$\mu(\{x \leftrightarrow C_-(F)\} \cap \{|cc(x)| = \infty\}) = 0.$$

Proof. Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ fixed, large enough such that $(|\partial B_m^x| + 1)\delta^m < \varepsilon$. We consider the notations from Definition 5.2.2.

Since (Γ_n^-) is increasing, if we let $\Gamma^- := \{x \leftrightarrow \mathcal{C}_-(F)\} = \cup\Gamma_n^-$, then

$$\mu(A_m \cap \Gamma^-) = \mu(A_m \cap \cup\Gamma_n^-) = \mu(\cup_n (A_m \cap \Gamma_n^-)) = \lim_n \mu(A_m \cap \Gamma_n^-).$$

Let us consider n_0 large enough such that the inequality from Lemma 5.2.3 holds. Since for $n \geq n_0$, $\mu_n(A_m \cap \Gamma_{n_0}^-) \leq \mu_n(A_m \cap \Gamma_n^-) \leq \varepsilon$,

We obtain

$$\mu(A_m \cap \Gamma_{n_0}^-) = \lim_n \mu_n(A_m \cap \Gamma_{n_0}^-) \leq \varepsilon.$$

It holds for every n_0 large enough and therefore, $\mu(A_m \cap \Gamma^-) \leq \varepsilon$.

Since this inequality holds for m large enough, we have when $m \to \infty$, $\mu(A_m \cap \Gamma^-) \to 0$.

Therefore, since (A_m) is decreasing and $\cap_m A_m = \{|cc(x)| = \infty\}$, the monotone convergence theorem concludes the proof. □

Definition 5.2.5. *For $l \in \mathbb{N}$, let $\Gamma_l^+ = \{x \leftrightarrow_{B_l^x} C_+(F_{B_l^x})\}$ be the event that x is connected inside B_l^x to a cycle with weight larger than 1 which is inside B_l^x, that is to say the event that in $F_{B_l^x}$, the connected component of x contains a cycle with weight larger than 1. For $m \in \mathbb{N}$, let $A_m := \{x \leftrightarrow \partial B_m^x\}$ be the event that x is connected to the boundary of B_m^x.*

Lemma 5.2.6. *Let $m_0 \in \mathbb{N}$. For m large enough, there exists n_0 such that if $n \geq n_0$ and if F_n is distributed according to μ_n, then,*

$$\mu_n(A_m \cap \Gamma_l^+) \leq |\partial B_{m_0}^x| \delta^{m_0}.$$

Proof. Let $m_0, m \in \mathbb{N}$. Assume that m is large enough such that for every $y \in \partial B_m^x$, the equality $B_{m_0}^y \cap B_l^x = \emptyset$ holds. Let n be large enough such that for every $y \in \partial B_m^x$, the inclusion $B_{m_0}^y \subset G_n$ holds.

Let $C \in \mathcal{C}_+(G_n)$. Conditional on $\mathcal{C}_+(F_n) = C$, F_n is given by an algorithm of w_--loop erased random walks with wired conditions on C.

The event $\Gamma_l^+ \cap A_m$ is satisfied if the w_--loop erased random walk starting from x hits a cycle in $C_+(F_{B_l^x})$ before leaving B_l^x and before being rooted to another cycle and if one of the w_--loop erased random walks starting from points of ∂B_m^x reaches B_l^x before being rooted to a cycle or hitting C. Therefore, from Lemma 5.2.1 and from the union bound,

$$\mu_n(A_m \cap \Gamma_l^+) \leq |\partial B_{m_0}^x| \delta^{m_0}$$

which concludes the proof. □

Lemma 5.2.7. *For every vertex $x \in V$ of G, we have*

$$\mu(\{x \leftrightarrow C_+(F)\} \cap \{|cc(x)| = \infty\}) = 0.$$

Proof. Let $x \in V$ be a fixed vertex of G and let

$$A := \{|cc(x)| = \infty\} = \bigcap_m A_m$$

be the event that the connected component of 0 is infinite, where A_m was defined in Definition 5.2.5.

Let $l \in \mathbb{N}$. Let $\varepsilon > 0$. Let m_0 large enough such that $|\partial B_{m_0}^x| \delta^{m_0} \leq \varepsilon$. Let m be large enough such that for every $y \in \partial B_m^x$, $B_{m_0}^y \cap B_l^x = \emptyset$. Let n be large enough such that for every $y \in \partial B_m^x$, $B_{m_0}^y \subset G_n$.

Let F_n distributed according μ_n. Then, from Lemma 5.2,

$$\mu_n(A_m \cap \Gamma_l^+) \leq |\partial B_{m_0}^x| \delta^{m_0} \leq \varepsilon.$$

Since this inequality holds for every n large enough and $A_m \cap \Gamma_l^+$ depends on finitely many edges,

$$\mu(A_m \cap \Gamma_l^+) = \lim_n \mu_n(A_m \cap \Gamma_l^+) \leq \varepsilon.$$

Since this inequality holds for every ε for m large enough, we obtain when $m \to \infty$,

$$\mu(A_m \cap \Gamma_l^+) \to 0.$$

Since the sequence of events $(A_m)_m$ is decreasing,

$$\mu(A \cap \Gamma_l^+) = \mu\left(\bigcap_m A_m \cap \Gamma_l^+\right) = \lim_m \mu(A_m \cap \Gamma_l^+) = 0.$$

Since the sequence of events $(\Gamma_l^+)_l$ is increasing and

$$\Gamma^+ = \{x \leftrightarrow C_+(F)\} = \bigcup_l \Gamma_l^+,$$

we have

$$\mu(A \cap \Gamma^+) = \mu\left(A \cap \left(\bigcup_l \Gamma_l^+\right)\right) = \mu\left(\bigcup_l (A \cap \Gamma_l^+)\right) = \lim_l \mu(A \cap \Gamma_l^+) = 0.$$

which is precisely what we wanted to prove. □

Let us emphasize that Lemma 5.2.7 and Lemma 5.2.4 show that for every vertex $x \in V$ of G, almost surely, if x is connected to a cycle in the random configuration F, the connected component of x is finite. Therefore, since G is countable, we immediately deduce the following theorem.

Theorem 5.2.8. *Under a measure μ_w such that w_- satisfies Assumption 4.0.1, every connected component with a cycle is finite.*

From Proposition 3.1.1, we know that every finite connected component has a cycle then, if a connected component does not have a cycle, it is necessarily an infinite tree. Therefore, almost surely every connected component is either a finite cycle-rooted tree or an infinite tree.

Conclusion and open questions

When a positive weight function on oriented cycles takes values in $[0, 1]$ and satisfies an assumption of minoration of weights, it gives rise to a unique infinite volume measure on cycle-rooted spanning forests, which is sampled by an algorithm of loop-erased random walks and which is the thermodynamic limit of finite volume measures, with respect to free or wired boundary conditions. Under this measure, almost surely, all connected components are finite and the edge-to-edge correlations decay is exponential.

By contrast, when the weight function is constant equal to 0, the model is the uniform spanning tree and the thermodynamic limit in infinite volume of finite volume measures is the free or wired uniform spanning forests measure, depending on boundary conditions. On a large class of graphs (amenable graphs for instance), the infinite volume measure does not depend on the boundary conditions and is sampled by the Wilson algorithm of loop-erased random walks (see [BP93, BLPS01, LP16]). Under this measure, almost surely, every connected component is an infinite tree and the edge-to-edge correlations have long range (for instance, they decay polynomially for $\mathbb{Z}^d$).

Considering these two cases as instances of a same model, we thus observe two qualitatively distinct phases, depending on the weight function on cycles.

For determinantal measures on cycle-rooted spanning forests (see [Ken11]) associated to a unitary connection, sequences of measures on finite growing subgraphs also converge towards infinite volume measures (see [Ken19, Sun16, KK17, KL22]). When the connection satisfies some assumptions, the infinite volume measure does not depend on the boundary conditions (see [KL23, Con23]).

These determinantal measures are associated to a weight function on cycles which can take values larger than 1, like in Section 5. Under some assumptions on the connection, the assumption of minoration of cycle weights (Assumption 4.0.1) is satisfied and therefore, by Theorem 5.2.8, almost surely all connected components are either finite cycle-rooted

trees or infinite trees. We also observe two phases (polynomial versus exponential decay of edge-to-edge correlations) depending on the unitary connection (see [Con23]).

A relevant question is to know if under the assumption of minoration on the weight function, there are infinite trees with a positive probability under the infinite volume measure, in particular in the case where the measure is determinantal and associated to a weight function which is provided by a unitary connection.

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