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# Two-dimensional Jacobians $\det$ and $\text{Det}$ for bounded variation functions. Application.

Marc Briane<sup>a</sup> & Juan Casado-Díaz<sup>b</sup>

## Abstract

The paper deals with the comparison in dimension two between the *strong* Jacobian determinant  $\det$  and the *weak* (or distributional) Jacobian determinant  $\text{Det}$ . Restricting ourselves to dimension two, we extend the classical results of Ball and Müller as well as more recent ones to bounded variation vector-valued functions, providing a sufficient condition on a vector-valued  $U$  in  $BV(\Omega)^2$  such that the equality  $\det(\nabla U) = \text{Det}(\nabla U)$  holds either in the distributional sense on  $\Omega$ , or almost-everywhere in  $\Omega$  when  $U$  is in  $W^{1,1}(\Omega)^2$ . The key-assumption of the result is the regularity of the Jacobian matrix-valued  $\nabla U$  along the direction of a given non vanishing vector field  $b \in C^1(\Omega)^2$ , *i.e.*  $\nabla U b$  is assumed either to belong to  $C^0(\Omega)^2$  with one of its coordinates in  $C^1(\Omega)$ , or to belong to  $C^1(\Omega)^2$ . Two examples illustrate this new notion of two-dimensional distributional determinant. Finally, we prove the lower semicontinuity of a polyconvex energy defined for vector-valued functions  $U$  in  $BV(\Omega)^2$ , assuming that the vector field  $b$  and one of the coordinates of  $\nabla U b$  lie in a compact set of regular vector-valued functions.

**Keywords:** Jacobian determinants,  $\text{Det}$  and  $\det$ , bounded variation function, ODE's flow, minimization under constraint, polyconvex energy

**Mathematics Subject Classification:** 34E10, 37C10, 37C40, 42B05

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# 1 Introduction

Following the seminal work of Morrey [22, 23] Ball showed in his famous article on nonlinearity elasticity [1, Sec. 6] (see also [10, Lemma 2.7]), that for any mapping  $U = (u_1, \dots, u_N) : \Omega \rightarrow \mathbb{R}^N$  defined on a non-empty bounded open set  $\Omega$  of  $\mathbb{R}^N$ , the two Jacobian determinants

$$\begin{cases} \det(DU) := \nabla u_1 \cdot (\nabla u_2 \times \cdots \times \nabla u_N) \\ \text{Det}(DU) := \text{div}(u_1 (\nabla u_2 \times \cdots \times \nabla u_N)), \end{cases} \quad (1.1)$$

which in dimension two with  $U = (u, v)$ , reads as

$$\det(DU) := \nabla u \cdot \nabla v^\perp \quad \text{and} \quad \text{Det}(DU) := \text{div}(u \nabla v^\perp), \quad (1.2)$$

agree in the distributions space  $\mathcal{D}'(\Omega)$  if  $U \in W^{1,N}(\Omega)^N$ . But they may differ if  $U \in W^{1,p}(\Omega)^2$  with  $p < N$ . He also conjectured that  $\det(DU) = \text{Det}(DU)$  as soon as  $\text{Det}(DU) \in L^1(\Omega)$ . Müller [25] has answered to Ball's conjecture assuming the slightly stronger assumption that the vector-valued function  $U$  belongs to  $W^{1,N^2/(N+1)}(\Omega)^N$ . This extra condition is quite natural since it implies that  $\text{Det}(DU) \in \mathcal{D}'(\Omega)$  as a distributional divergence of a vector-valued function in  $L^1(\Omega)^N$ .

Müller's result has been the object of several extensions: either the mapping  $U$  is assumed to be in  $W^{1,N-1}(\Omega)^N$  and to satisfy some Lusin condition [20, 14], or the distributional determinant  $\text{Det}(DU)$  is based on a current approach which provides some new properties as shown in [26, 9, 11, 12]. Moreover, Müller [24] has proved that the non-negativity of the Jacobian determinant implies some extra regularity of  $\det(DU)$ .

In the present note, we use a radically different approach restricted to dimension two. We construct a new distributional definition of the weak Jacobian determinant  $\text{Det}(\nabla U)$ , but also of the strong Jacobian determinant  $\det(\nabla U)$  <sup>(c)</sup> (see Definition 2.5), which hold for any vector-valued function  $U \in BV(\Omega)^2$  satisfying the following condition: the Jacobian matrix-valued  $\nabla U$  is assumed to be regular along the direction of some non vanishing vector field  $b \in C^1(\Omega)^2$ , *i.e.*  $\nabla U b$  is at least continuous in  $\Omega$ .

Then, we prove (see Theorem 2.6) the distributional identity

$$\text{Det}(DU) = \det(DU) \quad \text{in } \mathcal{D}'(\Omega), \quad (1.3)$$

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<sup>(c)</sup>Since  $\det(\nabla U)$  may be also distributional,  $\det(\nabla U)$  will be called the strong Jacobian and  $\text{Det}(\nabla U)$  will be called the weak Jacobian to avoid any confusion.

for any vector-valued function  $U = (u, v)$  satisfying

$$\begin{aligned} & \nabla U b \in C^0(\Omega^2)^2, \\ & \begin{cases} U \in BV(\Omega)^2 & \text{with } (\nabla u \cdot b \in C^1(\Omega) \text{ or } \nabla v \cdot b \in C^1(\Omega)), \\ \text{or} \\ U \in W^{1,1}(\Omega)^2 & \text{with } \nabla U b = (\nabla u \cdot b, \nabla v \cdot b) \in C^1(\Omega)^2. \end{cases} \end{aligned} \quad (1.4)$$

Moreover, under the second assumption of (1.4) equality (1.3) is shown to hold a.e. in  $\Omega$  with  $\det(DU)$  in  $L^1_{\text{loc}}(\Omega)$ . This provides a new approach of Ball's conjecture, and extends Müller's result [25] in dimension two. In particular, contrary to [25] the weak Jacobian determinant  $\text{Det}(\nabla U)$  does not need to be *a priori* in  $L^1(\Omega)$ . The apparently unnatural condition (1.4) has been actually motivated (see [5, Corollary 2.8] and [5, Remark 2.9]) by the celebrated Franks-Misiurewicz ergodic theorem [17] on the Herman rotation set [21], when the weak Jacobian determinant  $\text{Det}(DU)$  is null.

The proof of our main result (Theorem 2.6) is based on an original local decomposition (see Proposition 2.4 below) of any function  $u \in BV(\Omega)$  such that  $\nabla u \cdot b$  is regular, using to this end the ODE's flow induced by the vector field  $b$ .

Section 3 is devoted to examples. On the one hand, we get a counter-example of Ball's conjecture (see Example 3.1) for a vector-valued function  $U$  in  $BV(\Omega)^2$  rather than in  $W^{1,1}(\Omega)^2$ . On the other hand, Example 3.2 below illustrates the second condition of (1.4), and cannot be derived from Müller's result [25].

We conclude the paper with Section 4 by proving the lower semicontinuity (see Theorem 4.1) of the polyconvex functional defined by

$$\mathcal{F}(U) = |\nabla U|(\Omega) + \int_{\Omega} \psi(\det(\nabla U)) \, dx \quad \text{for } U \in BV(\Omega)^2, \quad (1.5)$$

involving the strong but distributional Jacobian determinant of Definition 2.5, and satisfying a compact set of constraints based on the regular conditions (1.4). The implication (2.18) of Theorem 2.6 shows actually that  $\det(\nabla U) = \text{Det}(\nabla U)$  in (1.5). This result is in the spirit of the Müller-Spector minimization result [26, Theorem 4.2] (and the references therein), replacing the delicate stability of the positive Jacobian determinant [26, Theorem 4.2] (which is based on the global invertibility condition INV) for vector-valued functions in  $W^{1,1}(\Omega)^N$ , by the stability of the two-dimensional weak Jacobians of Definition 2.5 for vector-valued functions in  $BV(\Omega)^2$ . At this point, we have to mention the very fine weak continuity of the Jacobian determinant obtained by Brezis and

Nguyen [4] for *BMO* functions, and partly based on the famous div-curl approach of Coifman, Lions, Meyer and Semmes [8]. Restricting ourselves to dimension two, we have extended the weak continuity of [4] to *BV* functions [6, Theorem 3.8] using a different divergence-curl result [6, Theorem 2.1], and assuming in addition that one of the two gradients of the Jacobian matrix is equi-integrable in some Lorentz space. In the proof of Theorem 4.1 this condition is replaced by the regularity condition (1.4) satisfied by the Jacobian matrix along some vector field  $b$ .

## Notation

- $(e^1, e^2)$  denotes the canonical basis of  $\mathbb{R}^2$ , and  $0_{\mathbb{R}^2}$  denotes the null vector of  $\mathbb{R}^2$ .
- $I_2$  is the unit matrix of  $\mathbb{R}^{2 \times 2}$
- $R_{\perp}$  denotes the  $(2 \times 2)$  rotation matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . For any  $\xi \in \mathbb{R}^2$ ,  $\xi^{\perp}$  denotes the perpendicular vector  $R_{\perp}\xi$ .
- “ $\cdot$ ” denotes the scalar product and  $|\cdot|$  the euclidean norm in  $\mathbb{R}^2$ .
- $\mathbb{1}_A$  denotes the characteristic function of the set  $A$ .
- $B(x, R)$  denotes the euclidean open ball of  $\mathbb{R}^2$  centered on  $x \in \mathbb{R}^2$  and of radius  $R > 0$ .
- $dx$  the Lebesgue measure on  $\mathbb{R}^2$ , and  $|A|$  denotes the Lebesgue measure of any measurable set in  $\mathbb{R}^2$ .
- $\mathcal{D}'(\Omega)$  denotes the space of the distributions on an open set  $\Omega$  of  $\mathbb{R}^2$ .
- The Jacobian matrix of a mapping  $\Phi \in L^1_{\text{loc}}(\Omega)^2$  is defined by the matrix-valued distribution  $\nabla\Phi$  with entries

$$[\nabla\Phi]_{i,j} := \partial_{x_j}\Phi_i \in \mathcal{D}'(\Omega) \quad \text{for } i, j = 1, 2.$$

The divergence of  $\Phi$  is defined by the distribution

$$\text{div } \Phi := \sum_{i=1}^2 \partial_{x_i}\Phi_i \in \mathcal{D}'(\Omega).$$

- The abbreviation “a.e.” for almost anywhere, will be used throughout the paper. The simple mention “a.e.” refers to the Lebesgue measure on  $\mathbb{R}^2$ .

- $\mathcal{M}(\Omega)$  denotes the space of the bounded Borel measures  $\mu$  on an open set  $\Omega$  of  $\mathbb{R}^2$ , and  $|\mu|(\Omega)$  denotes the total variation of  $\mu$  on  $\Omega$ .
- $BV(\Omega)$  denotes the space of the bounded variation functions on an open set  $\Omega$  of  $\mathbb{R}^2$ , *i.e.* the set of the functions  $u \in L^1(\Omega)$  such that the gradient distribution  $\nabla u$  is in  $\mathcal{M}(\Omega)^2$ .
- $o(\delta) = \delta o_\delta(1)$  where  $o_\delta(1)$  denotes a term satisfying  $\lim_{\delta \rightarrow 0} o_\delta(1) = 0$ .  
 $O(\delta) = \delta O_\delta(1)$  where  $O_\delta(1)$  denotes a bounded term with respect to  $\delta$ .
- $C$  denotes a positive constant which may vary from line to line.

## 2 Jacobians Det and det for bounded variation functions

In the sequel,  $\Omega$  denotes a fixed non-empty bounded open set of  $\mathbb{R}^2$ , and  $b$  denotes a fixed non vanishing vector field in  $C^1(\Omega)^2$ . For any vector-valued function  $U = (u, v)$  satisfying the regularity conditions

$$U = (u, v) \in BV(\Omega)^2 \quad \text{and} \quad \nabla U b = (\nabla u \cdot b, \nabla v \cdot b) \in C^0(\Omega)^2, \quad (2.1)$$

we will be able to give a suitable definition of the weak Jacobian of  $U$ .

We start by the following result about the pointwise properties of any function in  $BV(a, b)$  where  $(a, b)$  is a non-empty bounded open interval of  $\mathbb{R}$ .

**Lemma 2.1.** *Let  $S \in BV(a, b)$  and let  $(r_n)_{n \in \mathbb{N}}$  be the countable set of the discontinuity points of  $S$ . Then, for any sequence  $(\alpha_n)_{n \in \mathbb{N}}$  in  $[0, 1]^{\mathbb{N}}$ , there exists a sequence  $(S_m)_{m \in \mathbb{N}}$  in  $W^{1, \infty}(a, b)^{\mathbb{N}}$  such that for any  $m \in \mathbb{N}$ ,*

$$\left\{ \begin{array}{l} \sup_{m \in \mathbb{N}} \|S_m\|_{L^\infty(a, b)} < \infty, \\ \lim_{m \rightarrow \infty} \|S'_m\|_{\mathcal{M}(a, b)} = \|S'\|_{\mathcal{M}(a, b)}, \\ \lim_{m \rightarrow \infty} S_m(t) = S(t), \quad \forall t \in (a, b) \setminus \{r_n\}, \\ \lim_{m \rightarrow \infty} S_m(r_n) = \alpha_n S(r_n^+) + (1 - \alpha_n) S(r_n^-), \quad \forall n \in \mathbb{N}, \end{array} \right. \quad (2.2)$$

where  $S(r_n^+)$  and  $S(r_n^-)$  denote respectively the limit from the right and the limit from the left of  $S$  at any point  $r_n$ .

**Remark 2.2.** Approximating the Lipschitz functions by smooth functions, it is clear that the functions  $S_m$  in Lemma 2.1 can be chosen in  $C^\infty([a, b])$ .

**Remark 2.3.** Lemma 2.1 shows that the representative of  $S$  which consists in defining

$$S(r_n) := \alpha_n S(r_n^+) + (1 - \alpha_n) S(r_n^-), \quad \forall n \in \mathbb{N},$$

is a bounded Borel function for any choice of  $(\alpha_n)_{n \in \mathbb{N}}$ . Therefore, the function  $S$  can be integrated with respect to any measure  $\mu$  in  $(a, b)$ , but this integral clearly depends on the choice of  $\alpha_n$  as soon as there exists some  $r_n$  with  $\mu(\{r_n\}) > 0$ .

*Proof of Lemma 2.1.* Let  $H := \mathbb{1}_{(0, \infty)}$  be the Heaviside function (which is not well defined at the point 0). The function  $S$  can be written

$$S(r) = g_c(r) + \sum_{n \in \mathbb{N}} (S(r_n^+) - S(r_n^-)) H(r - r_n), \quad \forall r \in (a, b) \setminus \{r_n, n \in \mathbb{N}\},$$

where  $g_c$  is a continuous function in  $(a, b)$ . Then, define

$$H_{m,n}(s) = \begin{cases} 0 & \text{if } s < -\frac{\alpha_n}{m} \\ ms + \alpha_n & \text{if } -\frac{\alpha_n}{m} \leq s \leq \frac{1 - \alpha_n}{m} \\ 1 & \text{if } \frac{1 - \alpha_n}{m} < s, \end{cases}$$

and let  $(\rho_m)_{m \in \mathbb{N}}$  be a sequence of mollifiers in  $C^\infty(\mathbb{R})$ . Finally, extending the function  $g_c$  on  $\mathbb{R}$  by the constants  $g_c(a)$  and  $g_c(b)$  respectively in  $(-\infty, a)$  and in  $(b, \infty)$ , we get that the functions  $S_m$  for  $m \in \mathbb{N}$ , defined by

$$S_m(t) := (\rho_m * g_c)(t) + \sum_{n \in \mathbb{N}} (S(x_n^+) - S(x_n^-)) H_{m,n}(t - x_n), \quad t \in \mathbb{R},$$

satisfy the desired conditions (2.2). □

The next result provides a decomposition for any function  $u$  in  $BV(\Omega)$  such that  $\nabla u \cdot b$  belongs to  $C^0(\Omega)$ .

**Proposition 2.4.** *For any point  $x_0 \in \Omega$ , there exists  $\delta > 0$  and an open set  $O \Subset \Omega$  containing  $x_0$ , such that the mapping  $\Phi : (-\delta, \delta)^2 \rightarrow O$  defined by the dynamical system*

$$\partial_t \Phi(t, r) = b(\Phi(t, r)), \quad \Phi(0, r) = x_0 + r b^\perp(x_0), \quad (2.3)$$

is a  $C^1$ -diffeomorphism from  $(-\delta, \delta)^2$  onto  $O$ .

Moreover, for any function  $u \in BV(\Omega)$  and any function  $f \in C^0(O)$  satisfying

$$\nabla u \cdot b = f \quad \text{in } O, \quad (2.4)$$

there exists a function  $S \in BV(-\delta, \delta)$  such that the following decomposition holds:

$$u = \hat{u} + \tilde{u} \quad \text{in } O, \quad (2.5)$$

where  $\hat{u} \in BV(O) \cap C^0(\bar{O})$  defined by

$$\hat{u}(\Phi(t, r)) := \int_0^t f(\Phi(s, r)) ds - \frac{1}{2\delta} \int_{-\delta}^{\delta} \left( \int_0^q f(\Phi(s, r)) ds \right) dq, \quad (2.6)$$

and where  $\tilde{u} \in BV(O)$  defined by

$$\tilde{u}(x) := S(\Psi_2(x)), \quad \text{with } \Psi := \Phi^{-1}, \quad (2.7)$$

are solutions to

$$\nabla \hat{u} \cdot b = f \quad \text{and} \quad \nabla \tilde{u} \cdot b = \nabla \Psi_2 \cdot b = 0 \quad \text{in } O. \quad (2.8)$$

*Proof of Proposition 2.4.* Since  $\nabla \Phi(x_0) = (b(x_0), b(x_0)^\perp)$  is invertible, the existence of  $\delta$  and  $O$  is a simple consequence of the inverse function theorem.

Next, let  $u \in BV(\Omega)$  be a function satisfying the assumptions of Proposition 2.4, and define the functions  $w$ ,  $\hat{w}$  and  $\tilde{w}$  by

$$\begin{cases} w(t, r) := u(\Phi(t, r)), \\ \hat{w}(t, r) := \int_0^t f(\Phi(s, r)) ds - \frac{1}{2\delta} \int_{-\delta}^{\delta} \left( \int_0^q f(\Phi(s, r)) ds \right) dq, \\ \tilde{w} := w - \hat{w}. \end{cases}$$

The function  $\hat{w}$  clearly belongs to  $C^0([-\delta, \delta]^2)$ , and taking into account that  $\partial_t w = \partial_t \hat{w}$ , we get that  $\tilde{w}$  does not depend on  $t$ . Thus,  $\tilde{w} = S(r)$  for some function  $S \in L^\infty(-\delta, \delta)$ . Hence, it follows that

$$\begin{aligned} w(t, r) - \frac{1}{2\delta} \int_{-\delta}^{\delta} w(q, r) dq &= \hat{w}(t, r) + S(r) - \frac{1}{2\delta} \int_{-\delta}^{\delta} (\hat{w}(q, r) + S(r)) dq \\ &= \hat{w}(t, r), \end{aligned}$$

which combined with  $w$  in  $BV((-\delta, \delta)^2)$ , implies that  $\hat{w}$  and  $\tilde{w} = w - \hat{w}$  belong to  $BV((-\delta, \delta)^2)$ . Therefore,  $S$  also belongs to  $BV(-\delta, \delta)$ .



Finally, we easily deduce from the definition (2.6) of  $\hat{u}$  that  $\nabla\hat{u}\cdot b = f$  in  $O$ , which combined with (2.4) yields  $\nabla\tilde{u}\cdot b = 0$  in  $O$ . Moreover, taking the derivative with respect to the variable  $t$  of the equality  $r = \Psi_2(\Phi(t, r))$  and using (2.3), we get that

$$0 = \nabla\Psi_2(\Phi(t, r)) \cdot b(\Phi(t, r)), \quad \forall (t, r) \in (-\delta, \delta)^2,$$

which implies that  $\nabla\Psi_2\cdot b = 0$  in  $O$ . We have just obtained the equalities (2.8), which concludes the proof of Proposition 2.4.  $\square$

Now, consider a vector-valued  $U = (u, v)$  satisfying the regularity condition (2.1). The classical way to define the weak Jacobian of  $U$  is to use the distributional formula

$$\text{Det}(\nabla U) := \text{div}(u \nabla v^\perp) \quad \text{on } \Omega, \quad (2.9)$$

for which it is enough to define  $u \nabla v^\perp$  as a suitable distribution on  $\Omega$ . By the well-known properties of the distributions, we have only to define  $u \nabla v^\perp$  locally in  $\mathcal{D}'(\Omega)$ . For this purpose, we may use the decomposition given by (2.5) both for  $u$  and  $v$ . Take for the function  $S$  one of the representatives given by Lemma 2.1. Then, using that  $\hat{u}$  is continuous, that  $S$  is  $BV$  and that  $\tilde{u}$  is a Borel function, we deduce that  $u$  is integrable with respect to the vector-valued measure  $\nabla v$ . Therefore, we could define  $u \nabla v^\perp$  as

$$\langle u \nabla v^\perp, \zeta \rangle = \int_{\Omega} u \zeta \cdot d(\nabla v^\perp) \quad \text{for } \zeta \in C_c^\infty(\Omega)^2. \quad (2.10)$$

But, as above mentioned the previous integral depends *a priori* on the choice of the function  $S$ , which could make the definition (2.10) inappropriate.

Let us prove that  $\text{div}(u \nabla v^\perp)$  does not actually depend on the choice of  $S$ . To this end, consider the function  $S_m \in C^\infty(\mathbb{R})$  (recall Remark 2.2) for  $m \in \mathbb{N}$ , defined in Lemma 2.1. We then have

$$\text{div}((\hat{u} + S_m(\Psi_2)) \nabla v^\perp) = \text{div}(\hat{u} \nabla v^\perp) + S'_m(\Psi_2) \nabla\Psi_2 \cdot (\nabla\hat{v}^\perp + \nabla\tilde{v}^\perp),$$

However, since by (2.8)  $\nabla\Psi_2$  is parallel to  $b^\perp$  and  $\nabla\tilde{v}^\perp \cdot b^\perp = \nabla\tilde{v} \cdot b = 0$ , we deduce that for any  $m \in \mathbb{N}$ ,

$$\begin{aligned} & \text{div}((\hat{u} + S_m(\Psi_2)) \nabla v^\perp) \\ &= \text{div}(\hat{u} \nabla v^\perp) + S'_m(\Psi_2) \nabla\Psi_2 \cdot \nabla\hat{v}^\perp \\ &= \text{div}(\hat{u} \nabla v^\perp) - \text{div}(\hat{v} \nabla(S_m(\Psi_2))^\perp) \\ &= \text{div}(\hat{u} \nabla\hat{v}^\perp) + \text{div}(\hat{u} \nabla\tilde{v}^\perp) - \text{div}(\hat{v} \nabla(S_m(\Psi_2))^\perp). \end{aligned} \quad (2.11)$$

Hence, passing to the limit as  $m \rightarrow \infty$  in the distributional sense owing to the properties (2.2) of the sequence  $(S_m)_{m \in \mathbb{N}}$  together with decomposition (2.5), we obtain that

$$\operatorname{div}(u \nabla v^\perp) = \operatorname{div}(\hat{u} \nabla \hat{v}^\perp) + \operatorname{div}(\hat{u} \nabla \tilde{v}^\perp) - \operatorname{div}(\hat{v} \nabla \tilde{u}^\perp). \quad (2.12)$$

Due to the continuity of  $\hat{u}$  and  $\hat{v}$ , the right-hand side of (2.12) is well defined in  $\mathcal{D}'(O)$ , and indeed does not depend on  $S$ .

Then, taking into account the existence of two measures  $\mu$  and  $\nu$  on  $O$  such that

$$\nabla \tilde{u} = b^\perp \mu \quad \text{and} \quad \nabla \tilde{v} = b^\perp \nu \quad \text{in } \mathcal{M}(O), \quad (2.13)$$

and using the approximation of  $\hat{u} = S(\Psi_2)$  by  $S_m(\Psi_2)$  as in (2.11), respectively for  $\hat{v}$ , we also get the equalities

$$\begin{cases} \operatorname{div}(\hat{v} \nabla \tilde{u}^\perp) = (\nabla \hat{v} \cdot b) \mu = (\nabla v \cdot b) \mu, \\ \operatorname{div}(\hat{u} \nabla \tilde{v}^\perp) = (\nabla \hat{u} \cdot b) \nu = (\nabla u \cdot b) \nu. \end{cases}$$

Putting these equalities in (2.12) we thus get the alternative expression of the weak Jacobian in  $\mathcal{D}'(O)$ ,

$$\operatorname{Det}(\nabla U) = \operatorname{div}(u \nabla v^\perp) = \operatorname{div}(\hat{u} \nabla \hat{v}^\perp) + (\nabla u \cdot b) \nu - (\nabla v \cdot b) \mu \quad (2.14)$$

The other issue that is related naturally to the weak determinant, is to give a sense to the strong determinant for any vector field  $U = (u, v)$  in  $BV(\Omega)^2$ , which is formally defined by

$$\ll \operatorname{det}(\nabla U) := \nabla u \cdot \nabla v^\perp \gg.$$

Using the equalities  $\nabla \tilde{u} = b^\perp \mu$  and  $\nabla \tilde{v} = b^\perp \nu$  as above, we are led to the formal definition

$$\ll \nabla u \cdot \nabla v^\perp := \nabla \hat{u} \cdot \nabla \hat{v}^\perp + (\nabla \hat{u} \cdot b) \nu - (\nabla \hat{v} \cdot b) \mu \gg. \quad (2.15)$$

The problem is now to give a sense to  $\nabla \hat{u} \cdot \nabla \hat{v}^\perp$ , which can be formally defined by

$$\ll \nabla \hat{u} \cdot \nabla \hat{v}^\perp = \operatorname{div}(\hat{u} \nabla \hat{v}^\perp) \gg.$$

For example, a sufficient condition to justify the previous definition is to assume that  $\nabla u \cdot b$  belongs to  $C^1(\Omega)$ , which by (2.6) (see the proof of Theorem 2.6 below) implies that  $\hat{u} \in C^1(\Omega)$ .

To conclude, we have just justified the following definitions of  $\operatorname{Det}$  and  $\operatorname{det}$  for  $BV$  functions.

**Definition 2.5.** Let  $U = (u, v)$  be a vector-valued function satisfying condition (2.1).

Then, the weak Jacobian Det may be defined in the distributional sense on  $\Omega$  by

$$\text{Det}(\nabla U) = \text{div}(u \nabla v^\perp) := \text{div}(\hat{u} \nabla \hat{v}^\perp) + \text{div}(\hat{u} \nabla \tilde{v}^\perp) - \text{div}(\hat{v} \nabla \tilde{u}^\perp) \quad (2.16)$$

where the pair  $(\hat{u}, \tilde{u})$ , respectively  $(\hat{v}, \tilde{v})$ , satisfies the local decomposition  $u = \hat{u} + \tilde{u}$ , respectively  $v = \hat{v} + \tilde{v}$ , provided by Proposition 2.4. This definition can be also written (2.14) in terms of the measures  $\mu$  and  $\nu$  given by (2.13).

Moreover, if one of the two functions  $\nabla u \cdot b$  or  $\nabla v \cdot b$  belongs to  $C^1(\Omega)$ , the strong Jacobian det may be also defined in the distributional sense on  $\Omega$  by

$$\det(\nabla U) := \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \text{div}(\hat{u} \nabla \tilde{v}^\perp) - \text{div}(\hat{v} \nabla \tilde{u}^\perp). \quad (2.17)$$

This definition can be also written (2.15) in terms of the measures  $\mu$  and  $\nu$  given by (2.13).

Now, we can state the main result of the paper.

**Theorem 2.6.** *Let  $\Omega$  be a non-empty bounded open set of  $\mathbb{R}^2$ , and let  $b$  be a non vanishing vector field in  $C^1(\Omega)^2$ .*

*Then, according to Definition 2.16 any vector-valued function  $U = (u, v)$  with regularity (2.1) satisfies the implication*

$$\begin{array}{c} \underbrace{\nabla u \cdot b \in C^1(\Omega) \text{ or } \nabla v \cdot b \in C^1(\Omega)} \\ \downarrow \\ \overbrace{\text{Det}(\nabla U) = \det(\nabla U) \text{ in } \mathcal{D}'(\Omega)} \end{array} \quad (2.18)$$

*Moreover, any vector-valued function  $U = (u, v)$  in  $W^{1,1}(\Omega)^2$  satisfies the implication*

$$\begin{array}{c} \underbrace{(\nabla u \cdot b, \nabla v \cdot b) \in C^1(\Omega)^2} \\ \downarrow \\ \overbrace{\text{Det}(\nabla U) = \det(\nabla U) = \nabla u \cdot \nabla v^\perp \in L^1_{\text{loc}}(\Omega) \text{ a.e. in } \Omega} \end{array} \quad (2.19)$$

*Proof of Theorem 2.6.* We use the notations of Proposition 2.4 taking a neighborhood  $O \Subset \Omega$  of some point  $x_0$ . Assume for example that  $f := \nabla u \cdot b \in C^1(\Omega)$ . Then, the definition (2.7) of  $\hat{u}$  yields for any  $(t, r) \in (-\delta, \delta)^2$ ,

$$\begin{cases} \partial_t (\hat{u}(\Phi(t, r))) = f(\Phi(t, r)) \\ \partial_r (\hat{u}(\Phi(t, r))) = \int_0^t \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) ds, \end{cases} \quad (2.20)$$

which implies that  $\hat{u} \in C^1(O)$  (recall that  $O \Subset \Omega$ ). Hence, we deduce the equality

$$\operatorname{div}(\hat{u} \nabla \hat{v}^\perp) = \nabla \hat{u} \cdot \nabla \hat{v}^\perp \quad \text{in } O, \quad (2.21)$$

where  $\nabla \hat{v}$  is a vector-valued measure in  $\mathcal{M}(O)^2$ . Therefore, implication (2.18) is an immediate consequence of the definition (2.16) of the weak Jacobian  $\operatorname{Det}(\nabla U)$  and of the definition (2.17) of the strong Jacobian  $\det(\nabla U)$  under the continuity condition (2.1) satisfied by  $\nabla U$ .

On the other hand, when the vector-valued  $U = (u, v)$  is in  $W^{1,1}(\Omega)^2$  and satisfies the left-hand side of implication (2.19), the formulas (2.20) applied to  $\hat{v}$  with  $\nabla v \cdot b \in C^1(\Omega)$ , imply that  $\hat{v}$  belongs to  $C^1(O)$ . Hence, equality (2.21) holds in  $C^0(O) \subset L^1_{\text{loc}}(O)$ . Finally, comparing (2.16) to (2.17) and using that  $(\hat{u}, \hat{v}) \in C^1(O)^2$  with  $(\tilde{u}, \tilde{v}) \in W^{1,1}(O)^2$ , we obtain that

$$\begin{aligned} \operatorname{Det}(\nabla U) &= \det(\nabla U) \\ &= \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \operatorname{div}(\hat{u} \nabla \tilde{v}^\perp) - \operatorname{div}(\hat{v} \nabla \tilde{u}^\perp) \\ &= \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \nabla \hat{u} \cdot \nabla \tilde{v}^\perp + \nabla \tilde{u} \cdot \nabla \hat{v}^\perp \in L^1_{\text{loc}}(O). \end{aligned} \quad (2.22)$$

Moreover, due to the second equality of (2.8) and to the fact that the vector field  $b$  is non vanishing, we get that

$$\nabla \tilde{u} \cdot \nabla \tilde{v}^\perp = 0 \quad \text{a.e. in } \Omega,$$

which implies that a.e. in  $\Omega$ ,

$$\operatorname{Det}(\nabla U) = \det(\nabla U) = \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \nabla \hat{u} \cdot \nabla \tilde{v}^\perp + \nabla \tilde{u} \cdot \nabla \hat{v}^\perp = \nabla u \cdot \nabla v^\perp.$$

This combined (2.19) and the arbitrariness of the open sets  $O \Subset \Omega$ , yields the right-hand side of (2.19). The proof of Theorem 2.6 is thus complete.  $\square$

**Remark 2.7.** We can replace in condition (2.18) and (2.19) the  $C^1$ -regularity of the function  $f := \nabla u \cdot b$  (or/and  $\nabla v \cdot b$ ) by the  $C^1$ -regularity

outside a discrete set  $A$  on which  $\nabla f$  may blow-up. Without loss of generality we can assume that  $A$  is a unit set of  $\Omega$ .

More precisely, assume that there exist  $x_0 := \Phi(s_0, r_0)$  in  $O$ ,  $\sigma_0 > 0$  (which will be chosen later with respect to  $\varepsilon_0$ ) and a non-negative non-increasing function  $h \in L^1(0, \sigma_0)$  satisfying

$$\begin{cases} f \in C^0(\Omega) \cap C^1(\Omega \setminus \{x_0\}) \\ |\nabla f(x)| \leq C h(|x - x_0|), \quad \forall x \in B(x_0, \varepsilon_0). \end{cases} \quad (2.23)$$

Let us prove that the function  $\hat{u}$  defined by (2.6) is in  $C^1(B(x_0, \varepsilon_0))$  under condition (2.23). Recalling (2.20) we have

$$\begin{cases} \partial_t (\hat{u}(\Phi(t, r))) = f(\Phi(t, r)) \\ \partial_r (\hat{u}(\Phi(t, r))) = \int_0^t \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) ds, \end{cases} \quad (2.24)$$

for any  $(t, r) \in (-\delta, \delta) \times (-\delta, \delta) \setminus \{r_0\}$ . Since  $|\det(\nabla \Phi)| = |\det(\partial_s \Phi, \partial_r \Phi)|$  is bounded from below by a positive constant in any compact set of  $(-\delta, \delta)^2$ , by virtue of the one-order Taylor-Young expansion at the point  $(s_0, r_0)$ , there exists  $\alpha_0 > 0$  such that for  $\varepsilon_0$  small enough and for any  $(s, r) \in \Phi^{-1}(B(x_0, \varepsilon_0))$ , we get that

$$\begin{aligned} & |\Phi(s, r) - x_0| \\ &= \left| \partial_s \Phi(s_0, r_0) (s - s_0) + \partial_r \Phi(s_0, r_0) (r - r_0) + o(|s - s_0| + |r - r_0|) \right| \\ &\geq \alpha_0 (|s - s_0| + |r - r_0|). \end{aligned}$$

It follows from (2.23) and from the decrease of  $h$ , the existence of a constant  $C > 0$  such that for any  $(s, r) \in \Phi^{-1}(B(x_0, \varepsilon_0))$ ,  $r \neq r_0$ ,

$$\left| \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) \right| \leq C h(\alpha_0 (|s - s_0| + |r - r_0|)) \leq C h(\alpha_0 |s - s_0|),$$

which (choosing  $\sigma_0$  be such that  $\alpha_0 |s - s_0| < \sigma_0$ ) implies the domination condition

$$\left| \mathbb{1}_{[0, t]}(s) \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) \right| \leq C h(|s - s_0|) \in L^1(-\sigma_0, \sigma_0),$$

for any  $(s, r) \in \Phi^{-1}(B(x_0, \varepsilon_0))$ ,  $r \neq r_0$ . By the Lebesgue theorem applied to the second integral of (2.24) parametrized by the pair  $(t, r)$ , it follows that

$$\begin{aligned} & \lim_{(t, r) \rightarrow (s_0, r_0)} \partial_r (\hat{u}(\Phi(t, r))) \\ &= \lim_{(t, r) \rightarrow (s_0, r_0)} \left( \text{sgn}(s_0) \int_{-\delta}^{\delta} \mathbb{1}_{[0, t]}(s) \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) ds \right) \\ &= \int_0^{s_0} \nabla f(\Phi(s, r_0)) \cdot \partial_r \Phi(s, r_0) ds. \end{aligned}$$

Moreover, the continuity of  $f$  in  $\Omega$  yields immediately

$$\lim_{(t,r) \rightarrow (s_0, r_0)} \partial_t (\hat{u}(\Phi(t, r))) = f(\Phi(s_0, r_0)).$$

Therefore, the function  $\hat{u}$  belongs to  $C^1(B(x_0, \varepsilon_0))^2$ .

### 3 Examples

The following example shows that Ball's conjecture cannot be extended from mappings in  $W^{1,1}(\Omega)^2$  to mappings in  $BV(\Omega)^2$ .

**Example 3.1.** Let  $\Omega := (-\frac{1}{2}, \frac{1}{2})^2$  and let  $b$  the vector field defined by

$$b(x) := \nabla w(x) \text{ where } w(x) := x_1 - \cos(2\pi x_2) \text{ for } x \in \Omega, \quad (3.1)$$

which has been used in [7, Example 2.12].

The Borel measure defined by

$$\mu(dx) := dx_1 \otimes \delta_0(dx_2)$$

is invariant for the flow  $X$  associated with  $b$  (3.1). Indeed, we have for any vector-valued function  $\varphi \in C_c^1(\Omega)$ ,

$$\begin{aligned} & \int_{\Omega} \nabla \varphi(x) \cdot b(x) \mu(dx) \\ &= \int_{\Omega} (\partial_{x_1} \varphi(x) + 2\pi \sin(2\pi x_2) \partial_{x_2} \varphi(x)) (dx_1 \times \delta_0(dx_2)) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_{x_1} \varphi(x_1, 0) dx_1 = 0, \end{aligned}$$

or equivalently,  $\mu b$  is divergence free in  $\Omega$ . Then, any function  $u$  which agrees a.e. in  $\Omega$  with the characteristic function ( $x \mapsto \mathbb{1}_{\mathbb{R}_+}(x_2)$ ) whose support is  $(-\frac{1}{2}, \frac{1}{2}) \cap [0, \frac{1}{2})$  in  $\Omega$ , satisfies

$$\mu b = \mu e^1 = \nabla u^1 \text{ on } \Omega. \quad (3.2)$$

Indeed, on the one hand, we have for any vector-valued function  $\zeta$  in  $C_c^1(\Omega)^2$ ,

$$\begin{aligned} & \int_{\Omega} \zeta(x) \cdot b(x) \mu(dx) = \int_{\Omega} (\zeta_1(x) + 2\pi \sin(2\pi x_2) \zeta_2(x)) (dx_1 \times \delta_0(dx_2)) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta_1(x_1, 0) dx_1 = \int_{\Omega} \zeta(x) \cdot e^1 \mu(dx), \end{aligned}$$

and on the other hand, integrating by parts we get that

$$\begin{aligned}
\int_{\Omega} \zeta(x) \cdot d(\nabla u^\perp)(dx) &= \int_{\Omega} \operatorname{div}(\zeta^\perp)(x) u(x) dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\frac{1}{2}} (\partial_{x_1} \zeta_2(x) - \partial_{x_2} \zeta_1(x)) dx_1 dx_2 \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta_1(x_1, 0) dx_1 = \int_{\Omega} \zeta(x) \cdot e^1 \mu(dx).
\end{aligned}$$

Since by (3.2) we have  $\nabla u \cdot b = 0$ , the vector-valued function  $U$  satisfies the assumptions of Theorem 2.6. Hence, from formula (2.16) we deduce that

$$\operatorname{Det}(DU) = \operatorname{div}(u \nabla u^\perp) = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (3.3)$$

However, the formal identity

$$\ll \det(\nabla U) := \nabla u \cdot \nabla u^\perp = 0 \gg$$

has no sense since the product of measures  $\mu \times \mu$  is not defined.

Note that a more regular function  $\tilde{u}$  which is solution, similarly to  $u$ , to equation  $b \cdot \nabla \tilde{u} = 0$  in  $\Omega$  (whose existence is proved generally by Proposition 2.4) is given explicitly by

$$\tilde{u}(x) := \arctan(e^{4\pi^2 x_1} \tan(\pi x_2)) \quad \text{for } x \in \Omega.$$

Finally, formula (3.3) can be directly deduced from (3.2) by

$$u \nabla u^\perp = u \mu e^1 = \mu e^1 \quad \text{and} \quad \operatorname{div}(\mu e^1) = \partial_{x_1} \mu = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

The next example illustrates Theorem 2.6 with functions in  $W^{1,1}(\Omega)$ .

**Example 3.2.** Let  $\Omega := B(0_{\mathbb{R}^2}, \frac{1}{2})$ , let  $\alpha \in (0, 1)$ , let  $b$  the vector field and let  $g, u, v$  be the functions defined for  $x \in \Omega$ , by

$$\begin{aligned}
b(x) &:= x_1 e^1 - \alpha(x_2 + 2) e^2, \\
g(x) &:= |x_1|^{\alpha-2} x_1, \\
\begin{cases} u(x) &:= \int_{\frac{1}{2}}^{|x|} \frac{dr}{\ln r} \\ v(x) &:= |x_1|^\alpha (x_2 + 2) \end{cases} & \quad (3.4)
\end{aligned}$$

Hence, we have for any  $x \in \Omega$ ,

$$\begin{aligned}
U &:= (u, v) \in C^1(\Omega) \times W^{1,1}(\Omega) \text{ and } g \in L^1(\Omega), \\
g(x)b(x) &= |x_1|^\alpha e^1 - \alpha |x_1|^{\alpha-2} x_1 (x_2 + 2) e^2 = \nabla v(x)^\perp \text{ for } x_1 \neq 0, \\
(\nabla u \cdot b)(x) &= \frac{x_1^2 - \alpha x_2(x_2 + 2)}{|x| (\ln |x|)^2}, \\
|(\nabla(\nabla u \cdot b))(x)| &\leq \frac{C}{|x| (\ln |x|)^2}, \\
h(r) &:= \frac{C}{r (\ln r)^2} \text{ is in } L^1(0_{\mathbb{R}^2}, \frac{1}{2}).
\end{aligned}$$

It follows that the function  $\nabla u \cdot b$  satisfies condition (2.23) rather the left-hand side of (2.19). Moreover, we have  $\nabla v \cdot b = 0$ . Therefore, by virtue of the implication (2.19) combined with Remark 2.7 we obtain that a.e. in  $\Omega$ ,

$$\text{Det}(DU) = \det(DU) = \frac{x_1 (|x_1|^\alpha - \alpha |x_1|^{\alpha-2} x_2(x_2 + 2))}{|x| (\ln |x|)^2} \in L^{\frac{1}{1-\alpha}}(\Omega).$$

However, for any  $p \in (1, 2)$ , the function  $v$  does not belong to  $W^{1,p}(\Omega)$  when  $\alpha := 1 - 1/p$ . Finally, the present example cannot be derived from the result of Müller [25].

## 4 Minimization of a polyconvex energy under constraint

Let  $\Omega$  be a  $C^1$ -regular bounded open set of  $\mathbb{R}^2$ .

Let  $B$  be a non-empty compact set of  $(C^1(\bar{\Omega})^2, \|\cdot\|_{W^{1,\infty}(\Omega)^2})$  such that

$$\exists \beta \in (0, \infty), \forall b \in B, \quad |b| \geq \beta \text{ in } \Omega, \quad (4.1)$$

and which contains the constant vectors in  $\mathbb{R}^2$  satisfying (4.1). Moreover, let  $\mathbf{B}$  be a non-empty compact set of  $(C^0(\bar{\Omega})^2, \|\cdot\|_{L^\infty(\Omega)^2})$ , and let  $\mathbf{b}$  be a compact set of  $(C^1(\bar{\Omega}), \|\cdot\|_{W^{1,\infty}(\Omega)})$ .

Now, define the set  $\mathcal{B}_{\mathbf{b}, \mathbf{B}, B}$  of the admissible vector-field displacements, which only depends on the sets  $\mathbf{b}$ ,  $\mathbf{B}$ ,  $B$ , by

$$\mathcal{B}_{\mathbf{b}, \mathbf{B}, B} := \left\{ U = (u, v) \in BV(\Omega)^2 : \begin{array}{l} b \in B \\ \nabla U b \in \mathbf{B} \\ \nabla u \cdot b \in \mathbf{b} \text{ or } \nabla v \cdot b \in \mathbf{b} \end{array} \right\}. \quad (4.2)$$



Let  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function which satisfies the coerciveness and the boundedness properties

$$\exists \alpha \in (0, 1), \exists p \in (1, \infty), \forall t \in \mathbb{R}, \quad \alpha |t|^p \leq \psi(t) \leq \frac{|t|^p + 1}{\alpha}. \quad (4.3)$$

Then, define the functional

$$\mathcal{F}(U) := \int_{\Omega} |\nabla U| dx + \int_{\Omega} \psi(\det(\nabla U)) dx \quad \text{for } U \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B}, \quad (4.4)$$

where  $\det(\nabla U)$  is the strong (distributional) Jacobian defined by (2.17).

**Theorem 4.1.** *There exist  $\bar{U} \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B}$  and  $\bar{b} \in B$  such that*

$$\mathcal{F}(\bar{U}) = \inf \left\{ \mathcal{F}(U), U \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B} \right\}$$

*with* (4.5)

$$\bar{U} \bar{b} \in \mathbf{B} \quad \text{and} \quad (\nabla u \cdot b \in \mathbf{b} \text{ or } \nabla v \cdot b \in \mathbf{b}),$$

where the strong (distributional) Jacobian  $\det(\nabla \bar{U})$  belongs to  $L^p(\Omega)$ .

**Remark 4.2.** Note that by definition (4.1) and implication (2.18) we have

$$\det(\nabla U) = \text{Det}(\nabla U) \quad \text{for any } U \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B},$$

which is generally a distribution on  $\Omega$ . Theorem 4.1 shows that the functional  $\mathcal{F}$  defined by (4.4) is lower-semi continuous with respect to the  $BV(\Omega)^2$ -norm under the compact constraints (4.1) satisfied by the admissible set  $\mathcal{B}_{\mathbf{b}, \mathbf{B}, B}$  (4.1). However, the  $L^p$  *a priori* estimate on  $\det(\nabla U)$  in  $\mathcal{F}(U)$  implies that the distribution  $\text{Det}(\nabla \bar{U})$  belongs actually to  $L^p(\Omega)$ .

**Remark 4.3.** By virtue of Ascoli's theorem, if any function in  $\mathbf{b}$  and each coordinate of any vector-valued  $B$  belong to  $C^2(\bar{\Omega})$ , and satisfy for some constant  $C \in (0, \infty)$  the boundedness condition

$$\|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)^2} + \|\nabla^2 f\|_{L^\infty(\Omega)^{2 \times 2}} \leq C,$$

then  $\mathbf{b}$  and  $B$  are compact sets respectively of  $C^1(\bar{\Omega})$  and  $C^1(\bar{\Omega})^2$ . Similarly, if each coordinate of any vector-valued in  $\mathbf{B}$  belongs to  $C^1(\bar{\Omega})$ , and satisfies for some constant  $C \in (0, \infty)$  the boundedness condition

$$\|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)^2} \leq C,$$

then  $\mathbf{B}$  is a compact set of  $C^0(\bar{\Omega})^2$ .

*Proof of Theorem 4.1.* First of all, note that the infimum non negative energy  $\mathcal{F}(\bar{U})$  is finite, since  $U := \varepsilon \text{Id}$  for  $\varepsilon \in (0, 1)$  small enough, is an admissible vector-field displacement in  $\mathcal{B}_{\mathfrak{b}, \mathfrak{B}, B}$  (4.2), with for example the admissible constant vector field  $b := \beta e^1$ .

Let  $U_n = (u_n, v_n) \in BV(\Omega)^2$  be a minimizing sequence of  $\mathcal{F}$  satisfying  $\nabla U_n b_n \in \mathfrak{B}$  for some vector field  $b_n \in B$ , and for example (up to extract a subsequence)  $\nabla u_n \cdot b_n \in \mathfrak{b}$ . It is clear that  $U_n$  is bounded in  $BV(\Omega)^2$  and  $\det(DU_n)$  is bounded in  $L^p(\Omega)$ . Then, using (4.3), there exist two subsequences of  $U_n, b_n$ , still denoted by  $U_n, b_n$ , a vector-valued function  $\bar{U} = (\bar{u}, \bar{v}) \in BV(\Omega)^2$ , a vector field  $\bar{b} \in B$  with  $\nabla \bar{U} \bar{b} \in \mathfrak{B}$  and  $\nabla \bar{u} \cdot \bar{b} \in \mathfrak{b}$ , and  $\theta \in L^p(\Omega)$  such that the following convergences hold

$$\nabla U_n \rightharpoonup \nabla \bar{U} \quad \text{in } \mathcal{M}(\Omega)^{2 \times 2} *, \quad \begin{cases} b_n \rightarrow \bar{b} & \text{in } C^1(\bar{\Omega})^2 \\ \nabla U_n b_n \rightarrow \nabla \bar{U} \bar{b} & \text{in } C^0(\bar{\Omega})^2 \\ \nabla u_n \cdot b_n \rightarrow \nabla \bar{u} \cdot \bar{b} & \text{in } C^1(\bar{\Omega}), \\ \det(\nabla U_n) \rightharpoonup \theta & \text{in } L^p(\Omega). \end{cases} \quad (4.6)$$

Adapting *e.g.* the proof of [26, Theorem 5.4] (see also [2, Theorem 5.4]) thanks to the lower semicontinuity of the variation measure (see, *e.g.*, [15, Section 5.2.1]) and to the convexity of the function  $\psi$ , and using the three strong convergences in (4.6), we get that

$$|\nabla \bar{U}|(\Omega) + \int_{\Omega} \psi(\theta) dx \leq \liminf_{n \rightarrow \infty} \mathcal{F}(U_n) = \mathcal{F}(\bar{U}) \quad (4.7)$$

with

$$\nabla \bar{U} \bar{b} \in \mathfrak{B} \quad \text{and} \quad \nabla \bar{u} \cdot \bar{b} \in \mathfrak{b}.$$

Moreover, by virtue of Proposition 2.4 combined with the three strong convergences in (4.6), the sequences  $\hat{u}_n, \tilde{u}_n$  and  $\hat{v}_n, \tilde{v}_n$  respectively associated with the sequences  $u_n = \hat{u}_n + \tilde{u}_n$  and  $v_n = \hat{v}_n + \tilde{v}_n$ , satisfy the following convergences (up to extract new subsequences)

$$(\hat{u}_n, \hat{v}_n) \rightarrow (\hat{u}, \hat{v}) \quad \text{in } C^1(\bar{\Omega})^2, \quad (\nabla \tilde{u}_n, \nabla \tilde{v}_n) \rightharpoonup (\nabla \tilde{u}, \nabla \tilde{v}) \quad \text{in } \mathcal{M}(\Omega)^{2 \times 2} *,$$

which, taking into account the definition (2.17) of  $\det$  and the weak convergence of  $\det(\nabla U_n)$  in (4.6), imply that

$$\det(\nabla U_n) \rightharpoonup \det(\nabla \bar{U}) = \theta \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore, putting this in (4.7) we obtain the desired equality (4.5), which concludes the proof of Theorem 4.1.  $\square$

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## References

- [1] J.M. BALL: “Convexity conditions and existence theorems in non-linear elasticity”, *Arch. Rat. Mech. Anal.*, **63** (1976/77), 337-403.
- [2] J.M. BALL, J.C. CURRIE; P. J. OLVER: “Null Lagrangians, weak continuity, and variational problems of arbitrary order”, *J. Functional Analysis*, **41** (2) (1981), 135-174.
- [3] J.M. BALL & F. MURAT: “Remarks on rank-one convexity and quasiconvexity”, *Ordinary and partial differential equations, Vol. III*, Dundee, 1990, 25-37. Pitman Res. Notes Math. Ser. 254, Longman Sci. Tech., Harlow, 1991.
- [4] H. BREZIS & H. NGUYEN: “The Jacobian determinant revisited”, *Invent. Math.*, **185** (1) (2011), 17-54.
- [5] M. BRIANE & J. CASADO DÍAZ: “A divergence-curl result for measures. Application to the two-dimensional ODE’s flow”, HAL: <https://hal.science/hal-04088840> (2023), pp. 25.
- [6] M. BRIANE & J. CASADO DÍAZ: “A new div-curl result. Applications to the homogenization of elliptic systems and to the weak continuity of the Jacobian”, *J. Diff. Equa.*, **260** (2016), 5678-5725.
- [7] M. BRIANE, G.W. MILTON & A. TREIBERGS : “Which electric fields are realizable in conducting materials?”, *ESAIM: Math. Model. Numer. Anal.*, **48** (2) (2014), 307-323.
- [8] R. COIFMAN, P.L. LIONS, Y. MEYER, S. SEMMES: “Compensated compactness and Hardy spaces”, *J. Math. Pures Appl.*, **72** (3) (1993), 247-286.
- [9] S. CONTI & C. DE LELLIS: “Some remarks on the theory of elasticity for compressible Neohookean materials”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **5** Vol. II (3) (2003), 521-549.
- [10] B. DACOROGNA: *Direct Methods in the Calculus of Variations*, Applied Mathematical Sciences **78**, Springer-Verlag, Berlin, 1989, 308 pp.

- [11] C. DE LELLIS: “Some fine properties of currents and applications to distributional Jacobians”, *Proc. Roy. Soc. Edinburgh Sect. A*, **132** (4) (2002), 815-842.
- [12] C. DE LELLIS & F. GHIRALDIN: “An extension of the identity  $\text{Det}=\det$ ”, *C. R. Math. Acad. Sci. Paris*, **348** (17-18) (2010), 973-976.
- [13] G. DE PHILIPPIS: “Weak notions of Jacobian determinant and relaxation”, *ESAIM Control Optim. Calc. Var.*, **18** (1) (2012), 181-207.
- [14] L. D’ONOFRIO, S. HENCL, J. MALÝ & R. SCHIATTARELLA: “Note on Lusin (N) condition and the distributional determinant”, *J. Math. Anal. Appl.*, **439** (1) (2016), 171-182.
- [15] L.C. EVANS, & R.F. GARIEPY: *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992, 268 pp.
- [16] I. FONSECA, G. LEONI & J. MALÝ: “Weak continuity and lower semicontinuity results for determinants”, *Arch. Ration. Mech. Anal.* **178** (3) (2005), 411-448.
- [17] J. FRANKS & M. MISIUREWICZ: “Rotation sets of toral flows”, *Proc. Amer. Math. Soc.*, **109** (1) (1990), 243-249.
- [18] F. GIANNETTI, T. IWANIEC, J. ONNINEN & A. VERDE: “Estimates of Jacobians by subdeterminants”, *J. Geom. Anal.*, **12** (2) (2002), 223-254.
- [19] L. GRECO, T. IWANIEC & U. SUBRAMANIAN: “Another approach to biting convergence of Jacobians”, *Illinois J. Math.*, **47** (3) (2003), 815-830.
- [20] D. HENAO & C. MORA-CORRAL: “Lusin’s condition and the distributional determinant for deformations with finite energy”, *Adv. Calc. Var.*, **5** (2012) 355-409.
- [21] M.R. HERMAN: “Existence et non existence de tores invariants par des difféomorphismes symplectiques” (French), [Existence and nonexistence of tori invariant under symplectic diffeomorphisms], *Séminaire sur les Équations aux Dérivées Partielles 1987-1988*, **XIV**, École Polytech. Palaiseau, 1988, 24 pp.

- [22] C.B. MORREY: "Quasi-convexity and the lower semicontinuity of multiple integrals", *Pacific J. Math.*, **2** (1952), 25-53.
- [23] C.B. MORREY: *Multiple Integrals in the Calculus of Variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130 Springer-Verlag New York, Inc., New York 1966, 506 pp.
- [24] S. MÜLLER: "A surprising higher integrability property of mappings with positive determinant", *Bull. Amer. Math. Soc. (N.S.)*, **21** (2) (1989), 245-248.
- [25] S. MÜLLER: "Det= $\det$ . A remark on the distributional determinant", *C.R. Acad. Sci. Paris Sér. I Math.*, **311** (1) (1990), 13-17.
- [26] S. MÜLLER & S. SPECTOR: "An existence theory for nonlinear elasticity that allows for cavitation", *Arch. Rational Mech. Anal.*, **131** (1) (1995), 1-66.