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Two-dimensional Jacobians \det and Det for bounded variation functions. Application.

Marc Briane^a & Juan Casado-Díaz^b

Abstract

The paper deals with the comparison in dimension two between the *strong* Jacobian determinant \det and the *weak* (or distributional) Jacobian determinant Det . Restricting ourselves to dimension two, we extend the classical results of Ball and Müller as well as more recent ones to bounded variation vector-valued functions, providing a sufficient condition on a vector-valued U in $BV(\Omega)^2$ such that the equality $\det(\nabla U) = \text{Det}(\nabla U)$ holds either in the distributional sense on Ω , or almost-everywhere in Ω when U is in $W^{1,1}(\Omega)^2$. The key-assumption of the result is the regularity of the Jacobian matrix-valued ∇U along the direction of a given non vanishing vector field $b \in C^1(\Omega)^2$, *i.e.* $\nabla U b$ is assumed either to belong to $C^0(\Omega)^2$ with one of its coordinates in $C^1(\Omega)$, or to belong to $C^1(\Omega)^2$. Two examples illustrate this new notion of two-dimensional distributional determinant. Finally, we prove the lower semicontinuity of a polyconvex energy defined for vector-valued functions U in $BV(\Omega)^2$, assuming that the vector field b and one of the coordinates of $\nabla U b$ lie in a compact set of regular vector-valued functions.

Keywords: Jacobian determinants, Det and \det , bounded variation function, ODE's flow, minimization under constraint, polyconvex energy

Mathematics Subject Classification: 34E10, 37C10, 37C40, 42B05

^aUniv Rennes, INSA Rennes, CNRS, IRMAR-UMR 6625 – mbriane@insa-rennes.fr

^bDpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla – jcasadod@us.es

1 Introduction

Following the seminal work of Morrey [22, 23] Ball showed in his famous article on nonlinearity elasticity [1, Sec. 6] (see also [10, Lemma 2.7]), that for any mapping $U = (u_1, \dots, u_N) : \Omega \rightarrow \mathbb{R}^N$ defined on a non-empty bounded open set Ω of \mathbb{R}^N , the two Jacobian determinants

$$\begin{cases} \det(DU) := \nabla u_1 \cdot (\nabla u_2 \times \cdots \times \nabla u_N) \\ \text{Det}(DU) := \text{div}(u_1 (\nabla u_2 \times \cdots \times \nabla u_N)), \end{cases} \quad (1.1)$$

which in dimension two with $U = (u, v)$, reads as

$$\det(DU) := \nabla u \cdot \nabla v^\perp \quad \text{and} \quad \text{Det}(DU) := \text{div}(u \nabla v^\perp), \quad (1.2)$$

agree in the distributions space $\mathcal{D}'(\Omega)$ if $U \in W^{1,N}(\Omega)^N$. But they may differ if $U \in W^{1,p}(\Omega)^2$ with $p < N$. He also conjectured that $\det(DU) = \text{Det}(DU)$ as soon as $\text{Det}(DU) \in L^1(\Omega)$. Müller [25] has answered to Ball's conjecture assuming the slightly stronger assumption that the vector-valued function U belongs to $W^{1,N^2/(N+1)}(\Omega)^N$. This extra condition is quite natural since it implies that $\text{Det}(DU) \in \mathcal{D}'(\Omega)$ as a distributional divergence of a vector-valued function in $L^1(\Omega)^N$.

Müller's result has been the object of several extensions: either the mapping U is assumed to be in $W^{1,N-1}(\Omega)^N$ and to satisfy some Lusin condition [20, 14], or the distributional determinant $\text{Det}(DU)$ is based on a current approach which provides some new properties as shown in [26, 9, 11, 12]. Moreover, Müller [24] has proved that the non-negativity of the Jacobian determinant implies some extra regularity of $\det(DU)$.

In the present note, we use a radically different approach restricted to dimension two. We construct a new distributional definition of the weak Jacobian determinant $\text{Det}(\nabla U)$, but also of the strong Jacobian determinant $\det(\nabla U)$ ^(c) (see Definition 2.5), which hold for any vector-valued function $U \in BV(\Omega)^2$ satisfying the following condition: the Jacobian matrix-valued ∇U is assumed to be regular along the direction of some non vanishing vector field $b \in C^1(\Omega)^2$, *i.e.* $\nabla U b$ is at least continuous in Ω .

Then, we prove (see Theorem 2.6) the distributional identity

$$\text{Det}(DU) = \det(DU) \quad \text{in } \mathcal{D}'(\Omega), \quad (1.3)$$

^cSince $\det(\nabla U)$ may be also distributional, $\det(\nabla U)$ will be called the strong Jacobian and $\text{Det}(\nabla U)$ will be called the weak Jacobian to avoid any confusion.

for any vector-valued function $U = (u, v)$ satisfying

$$\begin{aligned} & \nabla U b \in C^0(\Omega^2)^2, \\ & \begin{cases} U \in BV(\Omega)^2 & \text{with } (\nabla u \cdot b \in C^1(\Omega) \text{ or } \nabla v \cdot b \in C^1(\Omega)), \\ & \text{or} \\ U \in W^{1,1}(\Omega)^2 & \text{with } \nabla U b = (\nabla u \cdot b, \nabla v \cdot b) \in C^1(\Omega)^2. \end{cases} \end{aligned} \quad (1.4)$$

Moreover, under the second assumption of (1.4) equality (1.3) is shown to hold a.e. in Ω with $\det(DU)$ in $L^1_{\text{loc}}(\Omega)$. This provides a new approach of Ball's conjecture, and extends Müller's result [25] in dimension two. In particular, contrary to [25] the weak Jacobian determinant $\text{Det}(\nabla U)$ does not need to be *a priori* in $L^1(\Omega)$. The apparently unnatural condition (1.4) has been actually motivated (see [5, Corollary 2.8] and [5, Remark 2.9]) by the celebrated Franks-Misiurewicz ergodic theorem [17] on the Herman rotation set [21], when the weak Jacobian determinant $\text{Det}(DU)$ is null.

The proof of our main result (Theorem 2.6) is based on an original local decomposition (see Proposition 2.4 below) of any function $u \in BV(\Omega)$ such that $\nabla u \cdot b$ is regular, using to this end the ODE's flow induced by the vector field b .

Section 3 is devoted to examples. On the one hand, we get a counter-example of Ball's conjecture (see Example 3.1) for a vector-valued function U in $BV(\Omega)^2$ rather than in $W^{1,1}(\Omega)^2$. On the other hand, Example 3.2 below illustrates the second condition of (1.4), and cannot be derived from Müller's result [25].

We conclude the paper with Section 4 by proving the lower semicontinuity (see Theorem 4.1) of the polyconvex functional defined by

$$\mathcal{F}(U) = |\nabla U|(\Omega) + \int_{\Omega} \psi(\det(\nabla U)) dx \quad \text{for } U \in BV(\Omega)^2, \quad (1.5)$$

involving the strong but distributional Jacobian determinant of Definition 2.5, and satisfying a compact set of constraints based on the regular conditions (1.4). The implication (2.18) of Theorem 2.6 shows actually that $\det(\nabla U) = \text{Det}(\nabla U)$ in (1.5). This result is in the spirit of the Müller-Spector minimization result [26, Theorem 4.2] (and the references therein), replacing the delicate stability of the positive Jacobian determinant [26, Theorem 4.2] (which is based on the global invertibility condition INV) for vector-valued functions in $W^{1,1}(\Omega)^N$, by the stability of the two-dimensional weak Jacobians of Definition 2.5 for vector-valued functions in $BV(\Omega)^2$. At this point, we have to mention the very fine weak continuity of the Jacobian determinant obtained by Brezis and

Nguyen [4] for *BMO* functions, and partly based on the famous div-curl approach of Coifman, Lions, Meyer and Semmes [8]. Restricting ourselves to dimension two, we have extended the weak continuity of [4] to *BV* functions [6, Theorem 3.8] using a different divergence-curl result [6, Theorem 2.1], and assuming in addition that one of the two gradients of the Jacobian matrix is equi-integrable in some Lorentz space. In the proof of Theorem 4.1 this condition is replaced by the regularity condition (1.4) satisfied by the Jacobian matrix along some vector field b .

Notation

- (e^1, e^2) denotes the canonical basis of \mathbb{R}^2 , and $0_{\mathbb{R}^2}$ denotes the null vector of \mathbb{R}^2 .
- I_2 is the unit matrix of $\mathbb{R}^{2 \times 2}$
- R_{\perp} denotes the (2×2) rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For any $\xi \in \mathbb{R}^2$, ξ^{\perp} denotes the perpendicular vector $R_{\perp}\xi$.
- “ \cdot ” denotes the scalar product and $|\cdot|$ the euclidean norm in \mathbb{R}^2 .
- $\mathbb{1}_A$ denotes the characteristic function of the set A .
- $B(x, R)$ denotes the euclidean open ball of \mathbb{R}^2 centered on $x \in \mathbb{R}^2$ and of radius $R > 0$.
- dx the Lebesgue measure on \mathbb{R}^2 , and $|A|$ denotes the Lebesgue measure of any measurable set in \mathbb{R}^2 .
- $\mathcal{D}'(\Omega)$ denotes the space of the distributions on an open set Ω of \mathbb{R}^2 .
- The Jacobian matrix of a mapping $\Phi \in L^1_{\text{loc}}(\Omega)^2$ is defined by the matrix-valued distribution $\nabla\Phi$ with entries

$$[\nabla\Phi]_{i,j} := \partial_{x_j}\Phi_i \in \mathcal{D}'(\Omega) \quad \text{for } i, j = 1, 2.$$

The divergence of Φ is defined by the distribution

$$\text{div } \Phi := \sum_{i=1}^2 \partial_{x_i}\Phi_i \in \mathcal{D}'(\Omega).$$

- The abbreviation “a.e.” for almost anywhere, will be used throughout the paper. The simple mention “a.e.” refers to the Lebesgue measure on \mathbb{R}^2 .

- $\mathcal{M}(\Omega)$ denotes the space of the bounded Borel measures μ on an open set Ω of \mathbb{R}^2 , and $|\mu|(\Omega)$ denotes the total variation of μ on Ω .
- $BV(\Omega)$ denotes the space of the bounded variation functions on an open set Ω of \mathbb{R}^2 , *i.e.* the set of the functions $u \in L^1(\Omega)$ such that the gradient distribution ∇u is in $\mathcal{M}(\Omega)^2$.
- $o(\delta) = \delta o_\delta(1)$ where $o_\delta(1)$ denotes a term satisfying $\lim_{\delta \rightarrow 0} o_\delta(1) = 0$.
 $O(\delta) = \delta O_\delta(1)$ where $O_\delta(1)$ denotes a bounded term with respect to δ .
- C denotes a positive constant which may vary from line to line.

2 Jacobians Det and det for bounded variation functions

In the sequel, Ω denotes a fixed non-empty bounded open set of \mathbb{R}^2 , and b denotes a fixed non vanishing vector field in $C^1(\Omega)^2$. For any vector-valued function $U = (u, v)$ satisfying the regularity conditions

$$U = (u, v) \in BV(\Omega)^2 \quad \text{and} \quad \nabla U b = (\nabla u \cdot b, \nabla v \cdot b) \in C^0(\Omega)^2, \quad (2.1)$$

we will be able to give a suitable definition of the weak Jacobian of U .

We start by the following result about the pointwise properties of any function in $BV(a, b)$ where (a, b) is a non-empty bounded open interval of \mathbb{R} .

Lemma 2.1. *Let $S \in BV(a, b)$ and let $(r_n)_{n \in \mathbb{N}}$ be the countable set of the discontinuity points of S . Then, for any sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $[0, 1]^{\mathbb{N}}$, there exists a sequence $(S_m)_{m \in \mathbb{N}}$ in $W^{1, \infty}(a, b)^{\mathbb{N}}$ such that for any $m \in \mathbb{N}$,*

$$\left\{ \begin{array}{l} \sup_{m \in \mathbb{N}} \|S_m\|_{L^\infty(a, b)} < \infty, \\ \lim_{m \rightarrow \infty} \|S'_m\|_{\mathcal{M}(a, b)} = \|S'\|_{\mathcal{M}(a, b)}, \\ \lim_{m \rightarrow \infty} S_m(t) = S(t), \quad \forall t \in (a, b) \setminus \{r_n\}, \\ \lim_{m \rightarrow \infty} S_m(r_n) = \alpha_n S(r_n^+) + (1 - \alpha_n) S(r_n^-), \quad \forall n \in \mathbb{N}, \end{array} \right. \quad (2.2)$$

where $S(r_n^+)$ and $S(r_n^-)$ denote respectively the limit from the right and the limit from the left of S at any point r_n .

Remark 2.2. Approximating the Lipschitz functions by smooth functions, it is clear that the functions S_m in Lemma 2.1 can be chosen in $C^\infty([a, b])$.

Remark 2.3. Lemma 2.1 shows that the representative of S which consists in defining

$$S(r_n) := \alpha_n S(r_n^+) + (1 - \alpha_n) S(r_n^-), \quad \forall n \in \mathbb{N},$$

is a bounded Borel function for any choice of $(\alpha_n)_{n \in \mathbb{N}}$. Therefore, the function S can be integrated with respect to any measure μ in (a, b) , but this integral clearly depends on the choice of α_n as soon as there exists some r_n with $\mu(\{r_n\}) > 0$.

Proof of Lemma 2.1. Let $H := \mathbb{1}_{(0, \infty)}$ be the Heaviside function (which is not well defined at the point 0). The function S can be written

$$S(r) = g_c(r) + \sum_{n \in \mathbb{N}} (S(r_n^+) - S(r_n^-)) H(r - r_n), \quad \forall r \in (a, b) \setminus \{r_n, n \in \mathbb{N}\},$$

where g_c is a continuous function in (a, b) . Then, define

$$H_{m,n}(s) = \begin{cases} 0 & \text{if } s < -\frac{\alpha_n}{m} \\ ms + \alpha_n & \text{if } -\frac{\alpha_n}{m} \leq s \leq \frac{1 - \alpha_n}{m} \\ 1 & \text{if } \frac{1 - \alpha_n}{m} < s, \end{cases}$$

and let $(\rho_m)_{m \in \mathbb{N}}$ be a sequence of mollifiers in $C^\infty(\mathbb{R})$. Finally, extending the function g_c on \mathbb{R} by the constants $g_c(a)$ and $g_c(b)$ respectively in $(-\infty, a)$ and in (b, ∞) , we get that the functions S_m for $m \in \mathbb{N}$, defined by

$$S_m(t) := (\rho_m * g_c)(t) + \sum_{n \in \mathbb{N}} (S(x_n^+) - S(x_n^-)) H_{m,n}(t - x_n), \quad t \in \mathbb{R},$$

satisfy the desired conditions (2.2). □

The next result provides a decomposition for any function u in $BV(\Omega)$ such that $\nabla u \cdot b$ belongs to $C^0(\Omega)$.

Proposition 2.4. *For any point $x_0 \in \Omega$, there exists $\delta > 0$ and an open set $O \Subset \Omega$ containing x_0 , such that the mapping $\Phi : (-\delta, \delta)^2 \rightarrow O$ defined by the dynamical system*

$$\partial_t \Phi(t, r) = b(\Phi(t, r)), \quad \Phi(0, r) = x_0 + r b^\perp(x_0), \quad (2.3)$$

is a C^1 -diffeomorphism from $(-\delta, \delta)^2$ onto O .

Moreover, for any function $u \in BV(\Omega)$ and any function $f \in C^0(O)$ satisfying

$$\nabla u \cdot b = f \quad \text{in } O, \quad (2.4)$$

there exists a function $S \in BV(-\delta, \delta)$ such that the following decomposition holds:

$$u = \hat{u} + \tilde{u} \quad \text{in } O, \quad (2.5)$$

where $\hat{u} \in BV(O) \cap C^0(\bar{O})$ defined by

$$\hat{u}(\Phi(t, r)) := \int_0^t f(\Phi(s, r)) ds - \frac{1}{2\delta} \int_{-\delta}^{\delta} \left(\int_0^q f(\Phi(s, r)) ds \right) dq, \quad (2.6)$$

and where $\tilde{u} \in BV(O)$ defined by

$$\tilde{u}(x) := S(\Psi_2(x)), \quad \text{with } \Psi := \Phi^{-1}, \quad (2.7)$$

are solutions to

$$\nabla \hat{u} \cdot b = f \quad \text{and} \quad \nabla \tilde{u} \cdot b = \nabla \Psi_2 \cdot b = 0 \quad \text{in } O. \quad (2.8)$$

Proof of Proposition 2.4. Since $\nabla \Phi(x_0) = (b(x_0), b(x_0)^\perp)$ is invertible, the existence of δ and O is a simple consequence of the inverse function theorem.

Next, let $u \in BV(\Omega)$ be a function satisfying the assumptions of Proposition 2.4, and define the functions w , \hat{w} and \tilde{w} by

$$\begin{cases} w(t, r) := u(\Phi(t, r)), \\ \hat{w}(t, r) := \int_0^t f(\Phi(s, r)) ds - \frac{1}{2\delta} \int_{-\delta}^{\delta} \left(\int_0^q f(\Phi(s, r)) ds \right) dq, \\ \tilde{w} := w - \hat{w}. \end{cases}$$

The function \hat{w} clearly belongs to $C^0([-\delta, \delta]^2)$, and taking into account that $\partial_t w = \partial_t \hat{w}$, we get that \tilde{w} does not depend on t . Thus, $\tilde{w} = S(r)$ for some function $S \in L^\infty(-\delta, \delta)$. Hence, it follows that

$$\begin{aligned} w(t, r) - \frac{1}{2\delta} \int_{-\delta}^{\delta} w(q, r) dq &= \hat{w}(t, r) + S(r) - \frac{1}{2\delta} \int_{-\delta}^{\delta} (\hat{w}(q, r) + S(r)) dq \\ &= \hat{w}(t, r), \end{aligned}$$

which combined with w in $BV((-\delta, \delta)^2)$, implies that \hat{w} and $\tilde{w} = w - \hat{w}$ belong to $BV((-\delta, \delta)^2)$. Therefore, S also belongs to $BV(-\delta, \delta)$.

Finally, we easily deduce from the definition (2.6) of \hat{u} that $\nabla\hat{u}\cdot b = f$ in O , which combined with (2.4) yields $\nabla\tilde{u}\cdot b = 0$ in O . Moreover, taking the derivative with respect to the variable t of the equality $r = \Psi_2(\Phi(t, r))$ and using (2.3), we get that

$$0 = \nabla\Psi_2(\Phi(t, r)) \cdot b(\Phi(t, r)), \quad \forall (t, r) \in (-\delta, \delta)^2,$$

which implies that $\nabla\Psi_2\cdot b = 0$ in O . We have just obtained the equalities (2.8), which concludes the proof of Proposition 2.4. \square

Now, consider a vector-valued $U = (u, v)$ satisfying the regularity condition (2.1). The classical way to define the weak Jacobian of U is to use the distributional formula

$$\text{Det}(\nabla U) := \text{div}(u \nabla v^\perp) \quad \text{on } \Omega, \quad (2.9)$$

for which it is enough to define $u \nabla v^\perp$ as a suitable distribution on Ω . By the well-known properties of the distributions, we have only to define $u \nabla v^\perp$ locally in $\mathcal{D}'(\Omega)$. For this purpose, we may use the decomposition given by (2.5) both for u and v . Take for the function S one of the representatives given by Lemma 2.1. Then, using that \hat{u} is continuous, that S is BV and that \tilde{u} is a Borel function, we deduce that u is integrable with respect to the vector-valued measure ∇v . Therefore, we could define $u \nabla v^\perp$ as

$$\langle u \nabla v^\perp, \zeta \rangle = \int_{\Omega} u \zeta \cdot d(\nabla v^\perp) \quad \text{for } \zeta \in C_c^\infty(\Omega)^2. \quad (2.10)$$

But, as above mentioned the previous integral depends *a priori* on the choice of the function S , which could make the definition (2.10) inappropriate.

Let us prove that $\text{div}(u \nabla v^\perp)$ does not actually depend on the choice of S . To this end, consider the function $S_m \in C^\infty(\mathbb{R})$ (recall Remark 2.2) for $m \in \mathbb{N}$, defined in Lemma 2.1. We then have

$$\text{div}((\hat{u} + S_m(\Psi_2)) \nabla v^\perp) = \text{div}(\hat{u} \nabla v^\perp) + S'_m(\Psi_2) \nabla\Psi_2 \cdot (\nabla\hat{v}^\perp + \nabla\tilde{v}^\perp),$$

However, since by (2.8) $\nabla\Psi_2$ is parallel to b^\perp and $\nabla\tilde{v}^\perp \cdot b^\perp = \nabla\tilde{v} \cdot b = 0$, we deduce that for any $m \in \mathbb{N}$,

$$\begin{aligned} & \text{div}((\hat{u} + S_m(\Psi_2)) \nabla v^\perp) \\ &= \text{div}(\hat{u} \nabla v^\perp) + S'_m(\Psi_2) \nabla\Psi_2 \cdot \nabla\hat{v}^\perp \\ &= \text{div}(\hat{u} \nabla v^\perp) - \text{div}(\hat{v} \nabla(S_m(\Psi_2))^\perp) \\ &= \text{div}(\hat{u} \nabla\hat{v}^\perp) + \text{div}(\hat{u} \nabla\tilde{v}^\perp) - \text{div}(\hat{v} \nabla(S_m(\Psi_2))^\perp). \end{aligned} \quad (2.11)$$

Hence, passing to the limit as $m \rightarrow \infty$ in the distributional sense owing to the properties (2.2) of the sequence $(S_m)_{m \in \mathbb{N}}$ together with decomposition (2.5), we obtain that

$$\operatorname{div}(u \nabla v^\perp) = \operatorname{div}(\hat{u} \nabla \hat{v}^\perp) + \operatorname{div}(\hat{u} \nabla \tilde{v}^\perp) - \operatorname{div}(\hat{v} \nabla \tilde{u}^\perp). \quad (2.12)$$

Due to the continuity of \hat{u} and \hat{v} , the right-hand side of (2.12) is well defined in $\mathcal{D}'(O)$, and indeed does not depend on S .

Then, taking into account the existence of two measures μ and ν on O such that

$$\nabla \tilde{u} = b^\perp \mu \quad \text{and} \quad \nabla \tilde{v} = b^\perp \nu \quad \text{in } \mathcal{M}(O), \quad (2.13)$$

and using the approximation of $\hat{u} = S(\Psi_2)$ by $S_m(\Psi_2)$ as in (2.11), respectively for \hat{v} , we also get the equalities

$$\begin{cases} \operatorname{div}(\hat{v} \nabla \tilde{u}^\perp) = (\nabla \hat{v} \cdot b) \mu = (\nabla v \cdot b) \mu, \\ \operatorname{div}(\hat{u} \nabla \tilde{v}^\perp) = (\nabla \hat{u} \cdot b) \nu = (\nabla u \cdot b) \nu. \end{cases}$$

Putting these equalities in (2.12) we thus get the alternative expression of the weak Jacobian in $\mathcal{D}'(O)$,

$$\operatorname{Det}(\nabla U) = \operatorname{div}(u \nabla v^\perp) = \operatorname{div}(\hat{u} \nabla \hat{v}^\perp) + (\nabla u \cdot b) \nu - (\nabla v \cdot b) \mu \quad (2.14)$$

The other issue that is related naturally to the weak determinant, is to give a sense to the strong determinant for any vector field $U = (u, v)$ in $BV(\Omega)^2$, which is formally defined by

$$\ll \operatorname{det}(\nabla U) := \nabla u \cdot \nabla v^\perp \gg.$$

Using the equalities $\nabla \tilde{u} = b^\perp \mu$ and $\nabla \tilde{v} = b^\perp \nu$ as above, we are led to the formal definition

$$\ll \nabla u \cdot \nabla v^\perp := \nabla \hat{u} \cdot \nabla \hat{v}^\perp + (\nabla \hat{u} \cdot b) \nu - (\nabla \hat{v} \cdot b) \mu \gg. \quad (2.15)$$

The problem is now to give a sense to $\nabla \hat{u} \cdot \nabla \hat{v}^\perp$, which can be formally defined by

$$\ll \nabla \hat{u} \cdot \nabla \hat{v}^\perp = \operatorname{div}(\hat{u} \nabla \hat{v}^\perp) \gg.$$

For example, a sufficient condition to justify the previous definition is to assume that $\nabla u \cdot b$ belongs to $C^1(\Omega)$, which by (2.6) (see the proof of Theorem 2.6 below) implies that $\hat{u} \in C^1(\Omega)$.

To conclude, we have just justified the following definitions of Det and det for BV functions.

Definition 2.5. Let $U = (u, v)$ be a vector-valued function satisfying condition (2.1).

Then, the weak Jacobian Det may be defined in the distributional sense on Ω by

$$\text{Det}(\nabla U) = \text{div}(u \nabla v^\perp) := \text{div}(\hat{u} \nabla \hat{v}^\perp) + \text{div}(\hat{u} \nabla \tilde{v}^\perp) - \text{div}(\hat{v} \nabla \tilde{u}^\perp) \quad (2.16)$$

where the pair (\hat{u}, \tilde{u}) , respectively (\hat{v}, \tilde{v}) , satisfies the local decomposition $u = \hat{u} + \tilde{u}$, respectively $v = \hat{v} + \tilde{v}$, provided by Proposition 2.4. This definition can be also written (2.14) in terms of the measures μ and ν given by (2.13).

Moreover, if one of the two functions $\nabla u \cdot b$ or $\nabla v \cdot b$ belongs to $C^1(\Omega)$, the strong Jacobian det may be also defined in the distributional sense on Ω by

$$\det(\nabla U) := \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \text{div}(\hat{u} \nabla \tilde{v}^\perp) - \text{div}(\hat{v} \nabla \tilde{u}^\perp). \quad (2.17)$$

This definition can be also written (2.15) in terms of the measures μ and ν given by (2.13).

Now, we can state the main result of the paper.

Theorem 2.6. *Let Ω be a non-empty bounded open set of \mathbb{R}^2 , and let b be a non vanishing vector field in $C^1(\Omega)^2$.*

Then, according to Definition 2.16 any vector-valued function $U = (u, v)$ with regularity (2.1) satisfies the implication

$$\begin{array}{c} \underbrace{\nabla u \cdot b \in C^1(\Omega) \text{ or } \nabla v \cdot b \in C^1(\Omega)} \\ \downarrow \\ \overbrace{\text{Det}(\nabla U) = \det(\nabla U) \text{ in } \mathcal{D}'(\Omega)} \end{array} \quad (2.18)$$

Moreover, any vector-valued function $U = (u, v)$ in $W^{1,1}(\Omega)^2$ satisfies the implication

$$\begin{array}{c} \underbrace{(\nabla u \cdot b, \nabla v \cdot b) \in C^1(\Omega)^2} \\ \downarrow \\ \overbrace{\text{Det}(\nabla U) = \det(\nabla U) = \nabla u \cdot \nabla v^\perp \in L^1_{\text{loc}}(\Omega) \text{ a.e. in } \Omega} \end{array} \quad (2.19)$$

Proof of Theorem 2.6. We use the notations of Proposition 2.4 taking a neighborhood $O \Subset \Omega$ of some point x_0 . Assume for example that $f := \nabla u \cdot b \in C^1(\Omega)$. Then, the definition (2.7) of \hat{u} yields for any $(t, r) \in (-\delta, \delta)^2$,

$$\begin{cases} \partial_t (\hat{u}(\Phi(t, r))) = f(\Phi(t, r)) \\ \partial_r (\hat{u}(\Phi(t, r))) = \int_0^t \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) ds, \end{cases} \quad (2.20)$$

which implies that $\hat{u} \in C^1(O)$ (recall that $O \Subset \Omega$). Hence, we deduce the equality

$$\operatorname{div}(\hat{u} \nabla \hat{v}^\perp) = \nabla \hat{u} \cdot \nabla \hat{v}^\perp \quad \text{in } O, \quad (2.21)$$

where $\nabla \hat{v}$ is a vector-valued measure in $\mathcal{M}(O)^2$. Therefore, implication (2.18) is an immediate consequence of the definition (2.16) of the weak Jacobian $\operatorname{Det}(\nabla U)$ and of the definition (2.17) of the strong Jacobian $\det(\nabla U)$ under the continuity condition (2.1) satisfied by ∇U .

On the other hand, when the vector-valued $U = (u, v)$ is in $W^{1,1}(\Omega)^2$ and satisfies the left-hand side of implication (2.19), the formulas (2.20) applied to \hat{v} with $\nabla v \cdot b \in C^1(\Omega)$, imply that \hat{v} belongs to $C^1(O)$. Hence, equality (2.21) holds in $C^0(O) \subset L^1_{\text{loc}}(O)$. Finally, comparing (2.16) to (2.17) and using that $(\hat{u}, \hat{v}) \in C^1(O)^2$ with $(\tilde{u}, \tilde{v}) \in W^{1,1}(O)^2$, we obtain that

$$\begin{aligned} \operatorname{Det}(\nabla U) &= \det(\nabla U) \\ &= \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \operatorname{div}(\hat{u} \nabla \tilde{v}^\perp) - \operatorname{div}(\hat{v} \nabla \tilde{u}^\perp) \\ &= \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \nabla \hat{u} \cdot \nabla \tilde{v}^\perp + \nabla \tilde{u} \cdot \nabla \hat{v}^\perp \in L^1_{\text{loc}}(O). \end{aligned} \quad (2.22)$$

Moreover, due to the second equality of (2.8) and to the fact that the vector field b is non vanishing, we get that

$$\nabla \tilde{u} \cdot \nabla \tilde{v}^\perp = 0 \quad \text{a.e. in } \Omega,$$

which implies that a.e. in Ω ,

$$\operatorname{Det}(\nabla U) = \det(\nabla U) = \nabla \hat{u} \cdot \nabla \hat{v}^\perp + \nabla \hat{u} \cdot \nabla \tilde{v}^\perp + \nabla \tilde{u} \cdot \nabla \hat{v}^\perp = \nabla u \cdot \nabla v^\perp.$$

This combined (2.19) and the arbitrariness of the open sets $O \Subset \Omega$, yields the right-hand side of (2.19). The proof of Theorem 2.6 is thus complete. \square

Remark 2.7. We can replace in condition (2.18) and (2.19) the C^1 -regularity of the function $f := \nabla u \cdot b$ (or/and $\nabla v \cdot b$) by the C^1 -regularity

outside a discrete set A on which ∇f may blow-up. Without loss of generality we can assume that A is a unit set of Ω .

More precisely, assume that there exist $x_0 := \Phi(s_0, r_0)$ in O , $\sigma_0 > 0$ (which will be chosen later with respect to ε_0) and a non-negative non-increasing function $h \in L^1(0, \sigma_0)$ satisfying

$$\begin{cases} f \in C^0(\Omega) \cap C^1(\Omega \setminus \{x_0\}) \\ |\nabla f(x)| \leq C h(|x - x_0|), \quad \forall x \in B(x_0, \varepsilon_0). \end{cases} \quad (2.23)$$

Let us prove that the function \hat{u} defined by (2.6) is in $C^1(B(x_0, \varepsilon_0))$ under condition (2.23). Recalling (2.20) we have

$$\begin{cases} \partial_t (\hat{u}(\Phi(t, r))) = f(\Phi(t, r)) \\ \partial_r (\hat{u}(\Phi(t, r))) = \int_0^t \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) ds, \end{cases} \quad (2.24)$$

for any $(t, r) \in (-\delta, \delta) \times (-\delta, \delta) \setminus \{r_0\}$. Since $|\det(\nabla \Phi)| = |\det(\partial_s \Phi, \partial_r \Phi)|$ is bounded from below by a positive constant in any compact set of $(-\delta, \delta)^2$, by virtue of the one-order Taylor-Young expansion at the point (s_0, r_0) , there exists $\alpha_0 > 0$ such that for ε_0 small enough and for any $(s, r) \in \Phi^{-1}(B(x_0, \varepsilon_0))$, we get that

$$\begin{aligned} & |\Phi(s, r) - x_0| \\ &= \left| \partial_s \Phi(s_0, r_0) (s - s_0) + \partial_r \Phi(s_0, r_0) (r - r_0) + o(|s - s_0| + |r - r_0|) \right| \\ &\geq \alpha_0 (|s - s_0| + |r - r_0|). \end{aligned}$$

It follows from (2.23) and from the decrease of h , the existence of a constant $C > 0$ such that for any $(s, r) \in \Phi^{-1}(B(x_0, \varepsilon_0))$, $r \neq r_0$,

$$\left| \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) \right| \leq C h(\alpha_0 (|s - s_0| + |r - r_0|)) \leq C h(\alpha_0 |s - s_0|),$$

which (choosing σ_0 be such that $\alpha_0 |s - s_0| < \sigma_0$) implies the domination condition

$$\left| \mathbb{1}_{[0, t]}(s) \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) \right| \leq C h(|s - s_0|) \in L^1(-\sigma_0, \sigma_0),$$

for any $(s, r) \in \Phi^{-1}(B(x_0, \varepsilon_0))$, $r \neq r_0$. By the Lebesgue theorem applied to the second integral of (2.24) parametrized by the pair (t, r) , it follows that

$$\begin{aligned} & \lim_{(t, r) \rightarrow (s_0, r_0)} \partial_r (\hat{u}(\Phi(t, r))) \\ &= \lim_{(t, r) \rightarrow (s_0, r_0)} \left(\text{sgn}(s_0) \int_{-\delta}^{\delta} \mathbb{1}_{[0, t]}(s) \nabla f(\Phi(s, r)) \cdot \partial_r \Phi(s, r) ds \right) \\ &= \int_0^{s_0} \nabla f(\Phi(s, r_0)) \cdot \partial_r \Phi(s, r_0) ds. \end{aligned}$$

Moreover, the continuity of f in Ω yields immediately

$$\lim_{(t,r) \rightarrow (s_0,r_0)} \partial_t (\hat{u}(\Phi(t,r))) = f(\Phi(s_0,r_0)).$$

Therefore, the function \hat{u} belongs to $C^1(B(x_0, \varepsilon_0))^2$.

3 Examples

The following example shows that Ball's conjecture cannot be extended from mappings in $W^{1,1}(\Omega)^2$ to mappings in $BV(\Omega)^2$.

Example 3.1. Let $\Omega := (-\frac{1}{2}, \frac{1}{2})^2$ and let b the vector field defined by

$$b(x) := \nabla w(x) \text{ where } w(x) := x_1 - \cos(2\pi x_2) \text{ for } x \in \Omega, \quad (3.1)$$

which has been used in [7, Example 2.12].

The Borel measure defined by

$$\mu(dx) := dx_1 \otimes \delta_0(dx_2)$$

is invariant for the flow X associated with b (3.1). Indeed, we have for any vector-valued function $\varphi \in C_c^1(\Omega)$,

$$\begin{aligned} & \int_{\Omega} \nabla \varphi(x) \cdot b(x) \mu(dx) \\ &= \int_{\Omega} (\partial_{x_1} \varphi(x) + 2\pi \sin(2\pi x_2) \partial_{x_2} \varphi(x)) (dx_1 \times \delta_0(dx_2)) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_{x_1} \varphi(x_1, 0) dx_1 = 0, \end{aligned}$$

or equivalently, μb is divergence free in Ω . Then, any function u which agrees a.e. in Ω with the characteristic function $(x \mapsto \mathbb{1}_{\mathbb{R}_+}(x_2))$ whose support is $(-\frac{1}{2}, \frac{1}{2}) \cap [0, \frac{1}{2})$ in Ω , satisfies

$$\mu b = \mu e^1 = \nabla u^1 \text{ on } \Omega. \quad (3.2)$$

Indeed, on the one hand, we have for any vector-valued function ζ in $C_c^1(\Omega)^2$,

$$\begin{aligned} & \int_{\Omega} \zeta(x) \cdot b(x) \mu(dx) = \int_{\Omega} (\zeta_1(x) + 2\pi \sin(2\pi x_2) \zeta_2(x)) (dx_1 \times \delta_0(dx_2)) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta_1(x_1, 0) dx_1 = \int_{\Omega} \zeta(x) \cdot e^1 \mu(dx), \end{aligned}$$

and on the other hand, integrating by parts we get that

$$\begin{aligned}
\int_{\Omega} \zeta(x) \cdot d(\nabla u^\perp)(dx) &= \int_{\Omega} \operatorname{div}(\zeta^\perp)(x) u(x) dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\frac{1}{2}} (\partial_{x_1} \zeta_2(x) - \partial_{x_2} \zeta_1(x)) dx_1 dx_2 \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \zeta_1(x_1, 0) dx_1 = \int_{\Omega} \zeta(x) \cdot e^1 \mu(dx).
\end{aligned}$$

Since by (3.2) we have $\nabla u \cdot b = 0$, the vector-valued function U satisfies the assumptions of Theorem 2.6. Hence, from formula (2.16) we deduce that

$$\operatorname{Det}(DU) = \operatorname{div}(u \nabla u^\perp) = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (3.3)$$

However, the formal identity

$$\ll \det(\nabla U) := \nabla u \cdot \nabla u^\perp = 0 \gg$$

has no sense since the product of measures $\mu \times \mu$ is not defined.

Note that a more regular function \tilde{u} which is solution, similarly to u , to equation $b \cdot \nabla \tilde{u} = 0$ in Ω (whose existence is proved generally by Proposition 2.4) is given explicitly by

$$\tilde{u}(x) := \arctan(e^{4\pi^2 x_1} \tan(\pi x_2)) \quad \text{for } x \in \Omega.$$

Finally, formula (3.3) can be directly deduced from (3.2) by

$$u \nabla u^\perp = u \mu e^1 = \mu e^1 \quad \text{and} \quad \operatorname{div}(\mu e^1) = \partial_{x_1} \mu = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

The next example illustrates Theorem 2.6 with functions in $W^{1,1}(\Omega)$.

Example 3.2. Let $\Omega := B(0_{\mathbb{R}^2}, \frac{1}{2})$, let $\alpha \in (0, 1)$, let b the vector field and let g, u, v be the functions defined for $x \in \Omega$, by

$$\begin{aligned}
b(x) &:= x_1 e^1 - \alpha(x_2 + 2) e^2, \\
g(x) &:= |x_1|^{\alpha-2} x_1, \\
\begin{cases} u(x) := \int_{\frac{1}{2}}^{|x|} \frac{dr}{\ln r} \\ v(x) := |x_1|^\alpha (x_2 + 2) \end{cases} & \quad (3.4)
\end{aligned}$$

Hence, we have for any $x \in \Omega$,

$$\begin{aligned}
U &:= (u, v) \in C^1(\Omega) \times W^{1,1}(\Omega) \text{ and } g \in L^1(\Omega), \\
g(x)b(x) &= |x_1|^\alpha e^1 - \alpha |x_1|^{\alpha-2} x_1 (x_2 + 2) e^2 = \nabla v(x)^\perp \text{ for } x_1 \neq 0, \\
(\nabla u \cdot b)(x) &= \frac{x_1^2 - \alpha x_2(x_2 + 2)}{|x| (\ln |x|)^2}, \\
|(\nabla(\nabla u \cdot b))(x)| &\leq \frac{C}{|x| (\ln |x|)^2}, \\
h(r) &:= \frac{C}{r (\ln r)^2} \text{ is in } L^1(0_{\mathbb{R}^2}, \frac{1}{2}).
\end{aligned}$$

It follows that the function $\nabla u \cdot b$ satisfies condition (2.23) rather the left-hand side of (2.19). Moreover, we have $\nabla v \cdot b = 0$. Therefore, by virtue of the implication (2.19) combined with Remark 2.7 we obtain that a.e. in Ω ,

$$\text{Det}(DU) = \det(DU) = \frac{x_1 (|x_1|^\alpha - \alpha |x_1|^{\alpha-2} x_2(x_2 + 2))}{|x| (\ln |x|)^2} \in L^{\frac{1}{1-\alpha}}(\Omega).$$

However, for any $p \in (1, 2)$, the function v does not belong to $W^{1,p}(\Omega)$ when $\alpha := 1 - 1/p$. Finally, the present example cannot be derived from the result of Müller [25].

4 Minimization of a polyconvex energy under constraint

Let Ω be a C^1 -regular bounded open set of \mathbb{R}^2 .

Let B be a non-empty compact set of $(C^1(\bar{\Omega})^2, \|\cdot\|_{W^{1,\infty}(\Omega)^2})$ such that

$$\exists \beta \in (0, \infty), \forall b \in B, \quad |b| \geq \beta \text{ in } \Omega, \quad (4.1)$$

and which contains the constant vectors in \mathbb{R}^2 satisfying (4.1). Moreover, let \mathbf{B} be a non-empty compact set of $(C^0(\bar{\Omega})^2, \|\cdot\|_{L^\infty(\Omega)^2})$, and let \mathbf{b} be a compact set of $(C^1(\bar{\Omega}), \|\cdot\|_{W^{1,\infty}(\Omega)})$.

Now, define the set $\mathcal{B}_{\mathbf{b}, \mathbf{B}, B}$ of the admissible vector-field displacements, which only depends on the sets \mathbf{b} , \mathbf{B} , B , by

$$\mathcal{B}_{\mathbf{b}, \mathbf{B}, B} := \left\{ U = (u, v) \in BV(\Omega)^2 : \begin{array}{l} b \in B \\ \nabla U b \in \mathbf{B} \\ \nabla u \cdot b \in \mathbf{b} \text{ or } \nabla v \cdot b \in \mathbf{b} \end{array} \right\}. \quad (4.2)$$

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function which satisfies the coerciveness and the boundedness properties

$$\exists \alpha \in (0, 1), \exists p \in (1, \infty), \forall t \in \mathbb{R}, \quad \alpha |t|^p \leq \psi(t) \leq \frac{|t|^p + 1}{\alpha}. \quad (4.3)$$

Then, define the functional

$$\mathcal{F}(U) := \int_{\Omega} |\nabla U| dx + \int_{\Omega} \psi(\det(\nabla U)) dx \quad \text{for } U \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B}, \quad (4.4)$$

where $\det(\nabla U)$ is the strong (distributional) Jacobian defined by (2.17).

Theorem 4.1. *There exist $\bar{U} \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B}$ and $\bar{b} \in B$ such that*

$$\mathcal{F}(\bar{U}) = \inf \left\{ \mathcal{F}(U), U \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B} \right\}$$

with (4.5)

$$\bar{U} \bar{b} \in \mathbf{B} \quad \text{and} \quad (\nabla u \cdot b \in \mathbf{b} \text{ or } \nabla v \cdot b \in \mathbf{b}),$$

where the strong (distributional) Jacobian $\det(\nabla \bar{U})$ belongs to $L^p(\Omega)$.

Remark 4.2. Note that by definition (4.1) and implication (2.18) we have

$$\det(\nabla U) = \text{Det}(\nabla U) \quad \text{for any } U \in \mathcal{B}_{\mathbf{b}, \mathbf{B}, B},$$

which is generally a distribution on Ω . Theorem 4.1 shows that the functional \mathcal{F} defined by (4.4) is lower-semi continuous with respect to the $BV(\Omega)^2$ -norm under the compact constraints (4.1) satisfied by the admissible set $\mathcal{B}_{\mathbf{b}, \mathbf{B}, B}$ (4.1). However, the L^p *a priori* estimate on $\det(\nabla U)$ in $\mathcal{F}(U)$ implies that the distribution $\text{Det}(\nabla \bar{U})$ belongs actually to $L^p(\Omega)$.

Remark 4.3. By virtue of Ascoli's theorem, if any function in \mathbf{b} and each coordinate of any vector-valued B belong to $C^2(\bar{\Omega})$, and satisfy for some constant $C \in (0, \infty)$ the boundedness condition

$$\|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)^2} + \|\nabla^2 f\|_{L^\infty(\Omega)^{2 \times 2}} \leq C,$$

then \mathbf{b} and B are compact sets respectively of $C^1(\bar{\Omega})$ and $C^1(\bar{\Omega})^2$. Similarly, if each coordinate of any vector-valued in \mathbf{B} belongs to $C^1(\bar{\Omega})$, and satisfies for some constant $C \in (0, \infty)$ the boundedness condition

$$\|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)^2} \leq C,$$

then \mathbf{B} is a compact set of $C^0(\bar{\Omega})^2$.

Proof of Theorem 4.1. First of all, note that the infimum non negative energy $\mathcal{F}(\bar{U})$ is finite, since $U := \varepsilon \text{Id}$ for $\varepsilon \in (0, 1)$ small enough, is an admissible vector-field displacement in $\mathcal{B}_{\mathfrak{b}, \mathfrak{B}, B}$ (4.2), with for example the admissible constant vector field $b := \beta e^1$.

Let $U_n = (u_n, v_n) \in BV(\Omega)^2$ be a minimizing sequence of \mathcal{F} satisfying $\nabla U_n b_n \in \mathfrak{B}$ for some vector field $b_n \in B$, and for example (up to extract a subsequence) $\nabla u_n \cdot b_n \in \mathfrak{b}$. It is clear that U_n is bounded in $BV(\Omega)^2$ and $\det(DU_n)$ is bounded in $L^p(\Omega)$. Then, using (4.3), there exist two subsequences of U_n, b_n , still denoted by U_n, b_n , a vector-valued function $\bar{U} = (\bar{u}, \bar{v}) \in BV(\Omega)^2$, a vector field $\bar{b} \in B$ with $\nabla \bar{U} \bar{b} \in \mathfrak{B}$ and $\nabla \bar{u} \cdot \bar{b} \in \mathfrak{b}$, and $\theta \in L^p(\Omega)$ such that the following convergences hold

$$\nabla U_n \rightharpoonup \nabla \bar{U} \quad \text{in } \mathcal{M}(\Omega)^{2 \times 2} *, \quad \begin{cases} b_n \rightarrow \bar{b} & \text{in } C^1(\bar{\Omega})^2 \\ \nabla U_n b_n \rightarrow \nabla \bar{U} \bar{b} & \text{in } C^0(\bar{\Omega})^2 \\ \nabla u_n \cdot b_n \rightarrow \nabla \bar{u} \cdot \bar{b} & \text{in } C^1(\bar{\Omega}), \\ \det(\nabla U_n) \rightharpoonup \theta & \text{in } L^p(\Omega). \end{cases} \quad (4.6)$$

Adapting *e.g.* the proof of [26, Theorem 5.4] (see also [2, Theorem 5.4]) thanks to the lower semicontinuity of the variation measure (see, *e.g.*, [15, Section 5.2.1]) and to the convexity of the function ψ , and using the three strong convergences in (4.6), we get that

$$|\nabla \bar{U}|(\Omega) + \int_{\Omega} \psi(\theta) dx \leq \liminf_{n \rightarrow \infty} \mathcal{F}(U_n) = \mathcal{F}(\bar{U}) \quad (4.7)$$

with

$$\nabla \bar{U} \bar{b} \in \mathfrak{B} \quad \text{and} \quad \nabla \bar{u} \cdot \bar{b} \in \mathfrak{b}.$$

Moreover, by virtue of Proposition 2.4 combined with the three strong convergences in (4.6), the sequences \hat{u}_n, \tilde{u}_n and \hat{v}_n, \tilde{v}_n respectively associated with the sequences $u_n = \hat{u}_n + \tilde{u}_n$ and $v_n = \hat{v}_n + \tilde{v}_n$, satisfy the following convergences (up to extract new subsequences)

$$(\hat{u}_n, \hat{v}_n) \rightarrow (\hat{u}, \hat{v}) \quad \text{in } C^1(\bar{\Omega})^2, \quad (\nabla \tilde{u}_n, \nabla \tilde{v}_n) \rightharpoonup (\nabla \tilde{u}, \nabla \tilde{v}) \quad \text{in } \mathcal{M}(\Omega)^{2 \times 2} *,$$

which, taking into account the definition (2.17) of \det and the weak convergence of $\det(\nabla U_n)$ in (4.6), imply that

$$\det(\nabla U_n) \rightharpoonup \det(\nabla \bar{U}) = \theta \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore, putting this in (4.7) we obtain the desired equality (4.5), which concludes the proof of Theorem 4.1. \square

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