



HAL
open science

An alternative to the theory of relativity based on conservation of acceleration

Jean-Paul Caltagirone

► **To cite this version:**

Jean-Paul Caltagirone. An alternative to the theory of relativity based on conservation of acceleration. 2023. hal-04182005

HAL Id: hal-04182005

<https://hal.science/hal-04182005v1>

Preprint submitted on 16 Aug 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

An alternative to the theory of relativity based on conservation of acceleration

Jean-Paul Caltagirone

calta@ipb.fr

Bordeaux INP, University of Bordeaux, CNRS UMR-5295,
Arts et Métiers Institute of Technology, INRAE, I2M Bordeaux,
351 Cours de la Libération - Bât. A11, 33405, Talence – France

Abstract

The proposal for an alternative to the theory of special and general relativity is based on the transformation of the Weak Equivalence Principle into a law of conservation of acceleration. This states that the intrinsic acceleration of a particle with or without mass is equal to the sum of the accelerations imposed on it by the exterior; the fundamental law of dynamics is therefore expressed exclusively in terms of accelerations. To establish it, a number of general concepts were revisited: (i) the principle of inertia, extended to uniform expansion and rotation motions; (ii) the compatibility of the Galilean and Lorentz transformations; (iii) the abandonment of the concept of space-time; (iv) the substitution of the notion of gravitational deformation of space-time by that of gravitational potentials. Maxwell's major idea of the unification of electrodynamics and magnetism, where direct and induced currents flow on the same support, is taken up to form intrinsic acceleration, an energy per unit of mass and length. The law of motion is a naturally relativistic wave equation obtained without recourse to the Lorentz transformation. Three emblematic examples taken up by the proposed formulation enable us to recover the results of observations, even though it is strictly disjoined from the theory of relativity.

Keywords

Conservation of Acceleration; Discrete Mechanics; Helmholtz-Hodge Decomposition; Special and General Relativity; Gravitational Lensing; Gravitational Redshift

1 Introduction

The theory of special and general relativity is not called into question, as it is perfectly coherent and compatible with observations made in the various fields of physics. It is therefore very difficult, if not impossible, to point out possible flaws, and this is not the aim of this presentation. Many excellent textbooks are devoted to this theory, for example [1, 2], and only those elements strictly necessary for the presentation of the formulation will be discussed.

The approach to the formalism described is strictly disjoint from the theory of special and general relativity. The aim is to show *in fine* that the results of observations are also rendered by this new formulation. It corresponds to the extension of discrete mechanics [3] developed around classical mechanics for fluid flows, solid behavior and wave propagation. It is presented as an alternative to the equations associated with these fields of physics, the Navier-Stokes equation, the Navier-Lamé equation or the wave equation. This starting point ensures that the equation of motion is representative of the most elementary motions, before extending its scope to relativistic motions.

The first part of this article is devoted to objections to the fundamental concepts on which relativity theory is based. The first of these concerns the physical meaning of the Weak Equivalence Principle (WEP); since Galileo, the interpretation of the free fall of bodies has focused on mass, i.e. the mass associated with inertia and equal to the mass associated with gravitation. Newton would later make this his second law, introducing the concept of momentum, the conservation of which is one of the pillars of mechanics and relativity. The WEP is revisited to extract a principle of conservation of acceleration, i.e. that the intrinsic acceleration of a particle with or without mass is equal to the sum of the accelerations applied to it. Other physics notions are

reconsidered, such as Galilean invariance, which is a form of the principle of inertia, or Newton's first law, according to which any body continues on a straight trajectory at constant velocity if no acceleration is applied to modify this state. In mechanics, rotational motion at constant angular velocity is considered accelerated motion. It is necessary to extend the concept of Galilean invariance to this rotational motion in order to establish an equation for accelerated motion; in particular, it is necessary to eliminate the singularity of rotational motion at infinity.

The properties of the discrete law of motion are directly derived from Maxwell's original model. It is naturally relativistic without the need to introduce the Lorentz transformation at any point. Its structure confers numerous conservation properties, including that of total energy per unit mass, the sum of translational and rotational energies, i.e. compressive and angular accelerations. By adding time invariance, we find the main symmetries of Noether's theorem, no longer in terms of momentum and total energy, but in terms of accelerations.

The aim of this article is to show that this formulation, presented as an alternative to the theory of special and general relativity, recovers the properties necessary to translate the fundamental notion of relativity. Three emblematic examples of SR and GR show how the exact result is obtained with a strictly disjoint formalism.

2 Objections to general concepts

2.1 Interpretation of the Weak Equivalence Principle

We have to go back to Galileo and his universal principle of the free fall of bodies to understand the origin of mechanics. All physics rests on this pillar, and all the advances made by Newton, Einstein and the countless scientists who followed have not called into question either the observation or the interpretation of it. The successive laws of physics have been built logically and coherently on the basis of their predecessors. The fact that some of them are based on incontestable experimental observations does not guarantee the validity of the assumptions and postulates used to arrive at them; moreover, there are parallel or alternative paths that may be of particular interest. The unification of the laws of physics has necessitated a return to the Galilean source to understand the path that led to attributing the notion of free fall to mass. All experiments since Galileo show that the Weak Equivalence Principle (WEP) is verified with great precision; the fall of two masses in space carried out recently *Microscope* [4] provides the same result with a precision of the order of 10^{-15} in terms of the Eötvös number. These experiments continue, despite the level of precision achieved today, in order to search for possible violations of this principle that could explain certain behaviors of quantum mechanics.

The formalization of mechanics by Newton's second law establishes, in a more modern version, that the force required to maintain acceleration is proportional to mass $\mathbf{F} = m \boldsymbol{\gamma}$. Applied to gravitational force, it becomes:

$$m_i \boldsymbol{\gamma} = m_g \mathbf{g}, \tag{1}$$

where m_i is the mass associated with inertia and m_g with gravitation. The WEP [5] assumes the equality of these two masses. If $m_i = m_g$, we can remove the mass from both members of this relationship:

$$\boldsymbol{\gamma} = \mathbf{g}. \tag{2}$$

This observation can be made in many areas of physics, including cosmology, where, for example, two bodies - a pulsar and a white dwarf - of very different masses, attracted by a larger object, move towards it with the same acceleration. This law has since been extended to all externally

imposed \mathbf{h} accelerations, as a postulate of discrete mechanics:

$$\boldsymbol{\gamma} = \mathbf{h}, \quad (3)$$

where this fundamental law reads as follows: "the intrinsic local acceleration of a particle, with or without mass, or material medium is equal to the sum of all accelerations applied to it".

2.2 Uniform rotational motion is not accelerated

Galilean or inertial invariance is defined by the satisfaction of the principle of inertia, according to which a particle retains its velocity on a straight trajectory if it is not subjected to any action. Its definition can be formulated by the condition:

$$\frac{d\mathbf{v}}{dt} = 0, \quad (4)$$

where \mathbf{v} is velocity and $\boldsymbol{\gamma} = d\mathbf{v}/dt$ is acceleration; inertial motion has zero acceleration.

Uniform rotational motion, i.e. motion in which the velocity of rotation about an axis is constant, is considered in physics to be accelerated motion. Consider the case of a particle whose angular velocity is equal to $\boldsymbol{\Omega}$ around the axis Oz , such that $\boldsymbol{\omega} = \boldsymbol{\Omega} \cdot \mathbf{e}_z$, so the local velocity is equal to $\mathbf{v}_{rot} = \boldsymbol{\Omega} \times \mathbf{r}$. In cylindrical coordinates, the only non-zero component of motion is that which follows the r coordinate, $\mathbf{v}_{rot} = \boldsymbol{\omega} r \mathbf{e}_\theta$. The equilibrium equation then becomes:

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega}^2 r \mathbf{e}_r - 2 \boldsymbol{\omega}^2 r \mathbf{e}_r = -\boldsymbol{\omega}^2 r \mathbf{e}_r, \quad (5)$$

where the two terms on the left-hand side of the relation (5) represent inertia in a Galilean frame of reference, and the term on the right-hand side corresponds to a fictitious centrifugal force, $-\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r}$. The conclusion of classical mechanics is that uniform rotational motion is accelerated motion.

For acceleration to be zero, motion must be stationary, $\partial\mathbf{v}/dt = 0$, and the sum of the inertia terms must also be zero. In continuum mechanics, the inertia term can be written in several equivalent forms:

$$\mathbf{v} \cdot \nabla \mathbf{v} = \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \mathbf{v} \times \nabla \times \mathbf{v} = \nabla \cdot (\mathbf{v} \mathbf{v}) - \mathbf{v} \nabla \cdot \mathbf{v}, \quad (6)$$

where $\boldsymbol{\mathcal{L}} = -\mathbf{v} \times \nabla \times \mathbf{v}$ is the Lamb vector. In the case of uniform motion with velocity \mathbf{v}_0 on a straight trajectory, it's easy to show that the inertia term is identically zero. The condition (4) is satisfied, but not for uniform rotational motion, whatever the form of the inertia term.

From a physical point of view, there's no reason why a solid in uniform rotation around an axis not subject to friction shouldn't persist indefinitely in this motion. The only possible explanation calls into question the classical mechanical formulation of inertia, represented by one of three expressions (6).

The alternative definition of inertia in classical mechanics is symbolized by the form derived from discrete mechanics :

$$\boldsymbol{\kappa} = \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{v}|^2}{2} \mathbf{n} \right), \quad (7)$$

where $\boldsymbol{\kappa}$ is the inertial vector, \mathbf{n} , the unit vector orthogonal to the axis of rotation and the symbol $\nabla \otimes$, the dual curl. The quantity $\boldsymbol{\kappa}$ can be interpreted as the curvature of the inertial potential $|\mathbf{v}|^2/2$. For uniform translation and rotation about an axis, the inertia is identically zero and the condition (4) is well satisfied. This result has important consequences: (i) it removes the notion

of fictitious force or acceleration, leaving only real quantities in the laws of physics, and (ii) it opens up the possibility of annihilating all terms relating to these two motions in these same laws.

2.3 Space-time or space and time?

The notion of space-time, introduced in the 18th century by J. Lerond-d'Alembert, was taken up by the theory of special relativity, then developed into general relativity by A. Einstein. Until then, Newtonian mechanics had considered space and time as disjoint concepts. In special relativity, space-time is constructed from the classical coordinates of space, (x, y, z) , and a spatial coordinate defined by the product of celerity and time, $d = c_0 t$. Space-time is then characterized by the quadruplet $(c_0 t, x, y, z)$. In the theory of relativity, lengths are no longer fixed but depend on the time interval dt considered; using the distance $c_0 t$ as length, the metric is written:

$$ds^2 = dx^2 + dy^2 + dz^2 - c_0^2 dt^2, \quad (8)$$

where ds is the relativistic Minkowski length. The Minkowski spacetime metric is a relativistic invariant. Indeed, as $c_0 dt = \gamma(c_0 dt' + \mathbf{v}dx'/c)$, $dx = \gamma(dx' + \mathbf{v}dt')$, $dy = dy'$ and $dz = dz'$ then the relativistic distance:

$$ds^2 = dx^2 + dy^2 + dz^2 - c_0^2 dt^2 = dx'^2 + dy'^2 + dz'^2 - c_0^2 dt'^2, \quad (9)$$

remains unchanged during the transformation between the two inertial coordinate systems. Two different events can thus be separated by zero distance.

Length contraction and time dilation are *ad hoc* hypotheses in line with other relativity concepts, notably the Lorentz transformation. This conception of space-time is open to objections, even if it is no longer debated in the scientific community. Is it possible to arrive at the same conclusions and results by exploring potential avenues for a return to Newtonian notions of space and time?

A return to the concepts of disjoint space and time is not impossible. In this case, lengths are fixed and time flows smoothly, without depending on velocity. The concepts of relativity theory, originally based on thought experiments, the clock, proper time and relativistic time, can be abandoned; space and time no longer depend on the frame of reference. The undisputed results of special relativity and general relativity must then be found naturally, by the new laws of physics.

2.4 Compatibility of Galilean and Lorentz transformations

Sometimes presented as contradictory, they are in fact separate and complementary. Let's start with the Galilean transformation. A reference frame \mathbb{R}' moves at a constant velocity \mathbf{v}_0 relative to a reference frame \mathbb{R} . The spatial coordinate in the direction of the wave is represented by $\mathbf{r}(t)$ in the latter frame of reference and by $\mathbf{r}'(t)$ in the \mathbb{R}' frame of reference. To move from \mathbb{R} to \mathbb{R}' , simply apply the transformation:

$$\begin{cases} \mathbf{r}'(t) = \mathbf{r}(t) - \mathbf{v}_0 t. \\ t' = t \end{cases} \quad (10)$$

If a physical law is invariant under the effect of this transformation, it is said to respect Galileo's invariance. Applying Newton's law in the \mathbb{R}' reference frame, we can easily show that:

$$\mathbf{F}'(t) = m \boldsymbol{\gamma}'(t) = m \boldsymbol{\gamma}(t) = \mathbf{F}(t), \quad (11)$$

the law of the initial reference frame is valid in any reference frame. This interpretation of Galileo's transformation is relative to a change of reference frame, and is used to predict motion in the \mathbb{R}' reference frame, considering that the laws are the same in any Galilean or inertial reference frame. In fact, it should be noted that invariance can be implicitly realized by the corresponding law $\mathbf{F}(\mathbf{v}_0) = 0$, whatever its form.

The Lorentz transformation [6] is presented as an extension of Galileo's transformation to relativistic motion. One of its forms is written :

$$\begin{cases} \mathbf{r}' = \gamma (\mathbf{r} - \mathbf{v} t) \\ t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{r}}{c_0^2} \right), \end{cases} \quad (12)$$

where γ is the Lorentz factor $\gamma = 1/\sqrt{1 - \mathbf{v}^2/c_0^2}$. Note that the Lorentz transformation only affects the component corresponding to the direction of the wave, while the other components of space satisfy the Galilean transformation. Although this transformation (12) recovers the Galilean transformation (11) when $\mathbf{v} \rightarrow c_0$, they are not compatible, as the γ factor in front of the brackets leads to the appearance of phenomena such as the contraction of lengths and the dilation of durations. These constraints are in fact *ad hoc* abstractions that allow us to satisfy the behavior of a particle's velocity to tend towards the celerity of the medium on a rectilinear trajectory. The specific form of the Lorentz factor rules out any possibility of velocity above the medium's celerity. For the Galilean and Lorentz transformations to be compatible, without the abstractions of length contraction and time dilation being endorsed, they will have to be verified intrinsically by the laws of physics; this is the case of Maxwell's equations or the wave equation. The extension of Galileo's invariance to uniformly rotated motions is therefore essential.

3 Discrete equation of motion

3.1 Maxwell's local frame of reference

The local reference frame for discrete mechanics is named here "Maxwell's reference frame", in honor of J.C. Maxwell. Maxwell, who was the first to understand the entanglement of mutual interactions between the electrodynamic and magnetic effects of intertwined electrical circuits. This frame is illustrated in Figure 1.

The local reference frame shown in Figure 1 consists of two geometric structures, primal and dual. The primal structure is a rectilinear segment Γ oriented by the unit vector \mathbf{t} of length $dh = [a, b]$ named discrete horizon. The dual geometry is a closed contour Δ positively oriented by the unit vector \mathbf{n} . By construction, the two unit vectors are orthogonal, so $\mathbf{t} \cdot \mathbf{n} = 0$. A family of Γ^* segments can be constructed by joining several Γ segments by their common vertices to tessellate a complete physical Ω domain bounded by a $\partial\Omega$ surface. Scalar quantities are defined on the a or b vertices of the Γ segment as the celerity c_0 or the scalar potential ϕ ; the potential vector $\boldsymbol{\psi}$ is linked to the dual contour Δ . The vectors velocity \mathbf{v} and acceleration $\boldsymbol{\gamma}$ are fixed on the segment Γ . Velocity and acceleration are not vectors of three-dimensional space but their restrictions to the Γ segment, they are components of velocity and acceleration of space whose representations are not necessary. The formulation leaves aside the notions of tensor and the vectors are themselves scalars on oriented segments; the potential vector $\boldsymbol{\psi}$ is itself a scalar attached to a primal facet orthogonal to the unit vector \mathbf{n} .

The only dimension of space retained for the derivation of the fundamental law of discrete mechanics is that set by \mathbf{t} . Following Maxwell's idea of electromagnetism, the circulation of a current in the Δ circuit generates, in a time-varying regime, an induced current on Γ . Direct and

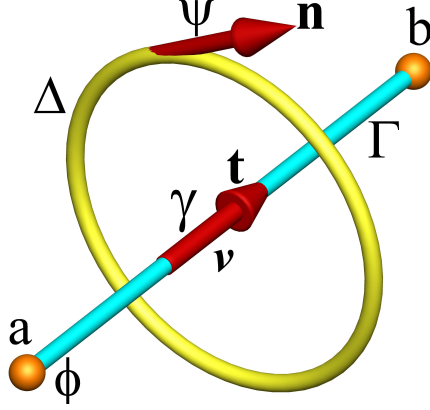


Figure 1. Native discrete mechanics model: a rectilinear segment Γ of length $dh = [a, b]$ oriented along the unit vector \mathbf{t} forms the primal structure. The dual contour Δ positively oriented by \mathbf{n} is such that $\mathbf{t} \cdot \mathbf{n} = 0$. Acceleration γ and velocity \mathbf{v} are vectors carried by the Γ oriented segment; scalar potential ϕ is assigned to its ends and vector potential ψ is fixed on the Δ contour.

induced currents are associated with the Γ segment alone. Interactions in the other dimensions of space are causal from one segment to the next, through their common vertices.

Time flows continuously, invariant to translation. Duration is noted here as dt , and is strictly independent of velocity, contrary to the theory of special relativity. Physically, it represents the passage of time between two mechanical equilibria, themselves defined by the solution of the equation of motion at two instants, t^o and $t = t^o + dt$. The length $dh = c_0 dt$ corresponds to the discrete horizon, the distance $[a, b]$ where a signal emitted at a begins to be perceived by an observer located at b . The length dh is not contracted and time dt does not undergo dilation as a function of the ratio v/c_0 , the definitions and concepts *ad hoc* of the theory of relativity are abandoned; the Lorentz transformation is not used for the derivation of the law of motion.

Four operators can be used to transform the information established on the primary structure and apply it to the dual structure, and vice versa. The discrete gradient of a scalar $\nabla\phi = (\phi_b - \phi_a)/dh$ is the restriction of the classical gradient to the segment Γ . The primal curl $\nabla \times \mathbf{v}$ corresponds to the circulation of the velocity vector along the segments of the Γ^* family. The $\nabla \cdot \mathbf{v}$ operator is the divergence of the velocity assigned to the a vertex of the flows on all segments having this vertex in common. Finally, the dual curl $\nabla \otimes \psi$ is calculated as the circulation of ψ along the dual contour Δ . The symbol $\nabla \otimes$ is sometimes used to define the tensor product, but the absence of tensors in the presented formulation avoids any possible confusion. These differential geometry operators, used in certain approaches notably linked to exterior calculus, have remarkable properties, notably to mimic certain established continuous media, $\nabla \times (\nabla\phi) = 0$ and $\nabla \cdot (\nabla \otimes \psi) = 0$ whatever the polygonal or polyhedral geometric structures used. Direct currents are represented by the potential gradient $\nabla\phi$ and induced currents by the operator $\nabla \otimes \psi$, which projects the result onto Γ . In fact, the acceleration γ fixed on Γ is the only quantity that defines the problem at hand, and the equation of motion is built around it.

3.2 Fundamental law of dynamics

The interpretation of the WEP leads to a conservation of accelerations on the segment Γ , $\gamma = \mathbf{h}$, of the proper or intrinsic acceleration γ of the particle or material medium and of those applied to it, \mathbf{h} . The law of dynamics is as follows:

$$\gamma = -\nabla\phi + \nabla \otimes \psi, \quad (13)$$

where $-\nabla\phi$ is the applied direct acceleration and $\nabla \otimes \boldsymbol{\psi}$ is the induced acceleration. These are the only two ways in which $\boldsymbol{\gamma}$ can be modified by external actions. The expression (13) corresponds to the Helmholtz-Hodge decomposition of acceleration, and ϕ and $\boldsymbol{\psi}$ are its scalar and vector potentials. In this case, the decomposition is locally orthogonal [7] and unique [8]. Indeed, all uniform translational and rotational motions have been eliminated from the outset, and the decomposition has no harmonic terms with both divergence-free and curl-free.

The physical modeling of the potentials translates their evolutions $\phi = \phi^o + d\phi$, $\boldsymbol{\psi} = \boldsymbol{\psi}^o + d\boldsymbol{\psi}$ between times t^o and $t^o + dt$, the quantities ϕ^o and $\boldsymbol{\psi}^o$ are the retarded potentials in the sense of electromagnetism [9] :

$$\phi^o = - \int_0^{t^o} c_0^2 \nabla \cdot \mathbf{v} d\tau, \quad \boldsymbol{\psi}^o = - \int_0^{t^o} c_0^2 \nabla \times \mathbf{v} d\tau. \quad (14)$$

The increments $d\phi$ and $d\boldsymbol{\psi}$ are energies per unit mass which are functions of the celerity of light c_0 and the velocity \mathbf{v} . Their modeling leads to $d\phi = c_0^2 dt \nabla \cdot \mathbf{v}$ and $d\boldsymbol{\psi} = c_0^2 dt \nabla \times \mathbf{v}$ [3], these quantities represent compression or expansion energy and rotational energy respectively.

The equation of motion can only be defined over limited space and time intervals, (i) the discrete horizon $dh = [a, b]$ in Figure 1 and $dt = t - t^o$ the time lapse between two mechanical equilibria, each defined by the solution of the equation of motion. Progression in space is by cause and effect from one segment to the next, and in parallel, progression in time is ensured by the time derivative.

In classical mechanics, acceleration is the material derivative of velocity, which can take various forms: $\mathbf{v} \cdot \nabla \mathbf{v}$, or $\nabla(|\mathbf{v}|^2/2) - \mathbf{v} \times \nabla \mathbf{v}$ or $\nabla \cdot (\mathbf{v} \mathbf{v}) - \mathbf{v} \nabla \cdot \mathbf{v}$. The term $\boldsymbol{\mathcal{L}} = -\mathbf{v} \times \nabla \times \mathbf{v}$ is Lamb's vector [10]. In discrete mechanics, the material derivative of velocity is expressed in three terms: the time derivative, the gradient of the inertial potential $\nabla(|\mathbf{v}|^2/2)$ and a third non-classical term $\nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n})$ [11]:

$$\boldsymbol{\gamma} \equiv \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{v}|^2}{2} \mathbf{n} \right), \quad (15)$$

where the last two terms are those of a Helmholtz-Hodge decomposition of the inertial potential, and their sum represents the curvature $\boldsymbol{\kappa}$ of this potential. In general, the two components of a Helmholtz-Hodge decomposition can be interpreted as the curvature of a certain potential, e.g. that of gravitation is expressed not only as a gradient but also as a dual curl. The Lamb vector $\boldsymbol{\mathcal{L}}$ is not divergence-free, even if the velocity itself is divergence-free. The discrete form (15) is essential to exclude all harmonic solutions of an equation, such as that corresponding to uniform rotational motion. The average acceleration over the Γ segment is then presented as the integral:

$$\int_{\Gamma} \frac{d\mathbf{v}}{dt} dl = - \int_{\Gamma} \nabla \phi dl + \int_{\Gamma} \nabla \otimes \boldsymbol{\psi} dl, \quad (16)$$

which is a form of total energy per unit mass. The first term of the second member is the conservation of compression energy on Γ and the last term is the conservation of angular acceleration.

By replacing the increments of the potentials and the inertia terms, the law of dynamics reads:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(\phi^o + \frac{|\mathbf{v}|^2}{2} - c_0^2 dt \nabla \cdot \mathbf{v} \right) + \nabla \otimes \left(\boldsymbol{\psi}^o + \frac{|\mathbf{v}|^2}{2} \mathbf{n} - c_0^2 dt \nabla \times \mathbf{v} \right). \quad (17)$$

The law of motion (17) is an integro-differential equation when retarded potentials (14) are inserted. Its integration is complex, but can be replaced by explicit updates deduced from their

definitions.

$$\text{updates} \begin{cases} \alpha_l \phi^o - c_0^2 dt \nabla \cdot (\mathbf{v} - \mathbf{v}^o) \mapsto \phi^o \\ \alpha_t \psi^o - c_0^2 dt \nabla \times (\mathbf{v} - \mathbf{v}^o) \mapsto \psi^o \\ \mathbf{v}^o + \gamma dt \mapsto \mathbf{v}^o, \end{cases} \quad (18)$$

where the symbol \mapsto indicates the replacement of the quantity known at time t^o by the current quantity. Updating the velocity \mathbf{v} can be done directly when solving the equation of motion. Although γ is the main variable considered as an absolute quantity, it is the velocity \mathbf{v} that is the unknown in the equation of motion.

The quantities α_l and α_t are attenuation factors for longitudinal and transverse waves respectively. In the case of light propagation in a vacuum, the celerity is equal to c_0 and the restitution factors are equal to unity.

The potentials are updated from the difference in velocities $(\mathbf{v} - \mathbf{v}^o)$ at times t^o and $t^o + dt$. The extension of Galilean invariance is thus guaranteed *de facto* by the law of motion and its improvements by the elimination of harmonic solutions. Lorentz invariance becomes a constraint that must be intrinsically guaranteed by the equation (17) itself. This law has many properties, including the preservation of energy, mass, angular momentum and so on.

Real media are not considered here, but there is no difficulty in modeling wave propagation in such media. In this case, the energy restitution factors α_l or α_t are different from unity if longitudinal or transverse waves are attenuated. In addition, the celerity itself may be affected by a change of medium, so that the values of the longitudinal c_l and transverse c_t celerities may be different. For example, monochromatic light is diffracted by an air-water interface with different refractive indices. Many physical phenomena concerning waves in general and light in particular can be simulated by the equation of motion (17) without modifying their structure, interference, absorption, reflection, etc. This law of motion can be perceived as a law of motion.

This law of motion can be perceived on all time scales dt and space scales dh , from very small time lapses $dt \approx 10^{-20} s$ for light propagation to very large time constants for stationary phenomena. From a practical point of view, the solutions of this equation are always consistent when solved with increments dt and dh compatible with the physics to be captured.

4 Properties of Discrete equation

4.1 A naturally relativistic law

The law of motion (17) is a naturally relativistic wave equation obtained without the use *a priori* of the Lorentz transformation, which it nevertheless satisfies identically [12]. To show that this equation is relativistic, it is essential to eliminate all uniform translational and rotational motion from the outset. To eliminate any dependence on the translational velocity \mathbf{v}_0 , it is possible to change the reference frame from an absolute reference frame \mathcal{R} to a relative reference frame \mathcal{R}' :

$$\begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t \\ t' = t, \end{cases} \quad (19)$$

where \mathbf{r} and \mathbf{r}' are the coordinates along the chosen rectilinear axis corresponding to the velocity vector \mathbf{v}_0 and t and t' are the times associated with the reference frames \mathcal{R} and \mathcal{R}' .

This uniform translational motion is insufficient to describe all non-accelerated motions. Indeed, an isolated observer has no way of observing a uniform expansion of the Universe such

as $\nabla \cdot \mathbf{v} = cte$ and similarly, a uniform rotation at constant angular velocity such as $\nabla \times \mathbf{v} = cte$ cannot be perceived by this observer. In classical mechanics, the latter motion is considered accelerated, but this is no longer the case in discrete mechanics; this formalism establishes an equation free of such motions in order to represent only those that are accelerated $\gamma \neq 0$. The aim of this section is to show that the discrete mechanics equation (17) can be used to extend the concept of Galilean relativity to all uniform motions. Uniform motions of expansion and rotation play a very special role in mechanics. The laws of physics must eliminate the terms associated with these motions from the outset. Uniform expansion motion is defined by the velocity $\mathbf{v}_{exp} = \mathbf{v}_0 + a\mathbf{r}$ where \mathbf{v}_0 and a are constants; Galilean invariance is a special case of expanding motion where $a = 0$. Uniform rotational motion is fixed by the velocity $\mathbf{v}_{rot} = \mathbf{\Omega} \times \mathbf{r}$ where $\mathbf{\Omega}$ is a constant rotational velocity. Both motions are stationary and unaccelerated. Applying the divergence and curl operators leads to $\nabla \cdot \mathbf{v}_{exp} = a$ and $\nabla \times \mathbf{v}_{rot} = 2\mathbf{\Omega}$, both constant. Let's look at the divergence of the expansion velocity in different coordinate systems:

$$\frac{\partial \mathbf{v}_{exp}}{\partial r} = \frac{1}{r} \frac{\partial(r \mathbf{v}_{exp})}{\partial r} = \frac{1}{r^2} \frac{\partial(r^2 \mathbf{v}_{exp})}{\partial r} = a. \quad (20)$$

A constant divergence of the axial velocity vector in Cartesian, polar or spherical coordinates always gives the same result for $\mathbf{v}_{exp} = \mathbf{v}_0 + a\mathbf{r}$; it does not depend on the dimension of space. A particle or material medium whose motion we follow in such a flow undergoes expansion if $\nabla \cdot \mathbf{v}_{exp} > 0$ or compression if $\nabla \cdot \mathbf{v}_{exp} < 0$. In Lagrange coordinates, velocity divergence is expressed as the ratio of acceleration to velocity:

$$\nabla \cdot \mathbf{v}_{exp} = \frac{d\mathbf{v}_{exp}}{dr} = \frac{1}{\mathbf{v}_{exp}} \frac{d\mathbf{v}_{exp}}{dt} = \frac{\gamma_{exp}}{\mathbf{v}_{exp}}. \quad (21)$$

This time-independent divergence (21) reflects the local expansion or contraction that follows a material medium as it moves. In practice, the expansion of a gas cannot continue indefinitely without the velocity of the fluid reaching an unsurpassable limit celerity; in this case, the velocity becomes equal to this limit velocity and acceleration becomes zero, expansion is complete. In cosmology, the Universe underwent a phase of rapid expansion known as cosmic inflation, which would explain certain phenomena such as the horizon problem. There is a link between the evolution over time of velocity divergence, which is the measure of expansion, and the flatness of the Universe, i.e. its curvature becomes zero. The notion of cosmological fluid is based on a reworking of the equations of mechanics, taking into account the expansion of the Universe and positing $\mathbf{v} = \mathcal{H} \mathbf{r}$ where \mathcal{H} is the Hubble parameter [13]; this reformulation is based on the use of comoving coordinates. As with expansion/compression, the primal curl $\nabla \times \mathbf{v}_{rot}$ of the rotational motion will be positive or negative depending on the direction of rotation on the facet oriented by \mathbf{n} . The superposition of these motions will have the same properties.

Uniform dilation and rotation motions have remarkable properties that are very useful for their immediate elimination, called filtering, in the equation of these motions:

$$\left\{ \begin{array}{ll} \mathbf{v}_{rot} = \mathbf{\Omega} \times \mathbf{r}, & \mathbf{v}_{exp} = \mathbf{v}_0 + a \mathbf{r} \\ \nabla \cdot \mathbf{v}_{exp} = a, & \nabla \times \mathbf{v}_{rot} = 2\mathbf{\Omega} \\ \nabla \cdot \mathbf{v}_{rot} = 0, & \nabla \times \mathbf{v}_{exp} = 0 \\ \nabla(\nabla \cdot \mathbf{v}_{exp}) = 0, & \nabla \otimes (\nabla \times \mathbf{v}_{rot}) = 0. \end{array} \right. \quad (22)$$

In practice, continuous expansion cannot be pursued physically, as the velocity of a gas in one dimension of space cannot exceed the celerity of sound, $\mathbf{v} < c_s$. The analogy with the velocity of

photons leads to a limitation of the celerity of light, $\mathbf{v} < c_0$. Just as uniform rotational motion presents a singularity when $\mathbf{r} \rightarrow \infty$, circumferential velocity also tends towards infinity. However, unlike expansion, rotational velocities at the periphery of some systems are much higher than the celerity of sound. Transposing this to the case of light would be risky, as light can only undergo deflection and rotation in the presence of considerable gravity, such as that observed in the vicinity of black holes. In the absence of any curvature, the velocity of photons is equal to the celerity of light in vacuum, an invariant where $c_0 = 3 \cdot 10^8 \text{ m/s}$, and this postulate of special relativity is not disputed. In the presence of spatial curvature, the wave's celerity remains equal to c_0 , but the massless photon possesses inertia.

Galileo introduced the concept of relativity, which later became the principle of relativity; it is closely linked to the principle of inertia, Newton's first law. It assumes that the sum of the forces applied to a material point is equal to zero. The only motion considered is translation at constant velocity, such that $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$. The principle of inertia expresses, according to the interpretation given here, that a particle not subjected to acceleration continues its rectilinear trajectory at the same velocity \mathbf{v}_0 . In this context, it makes no reference to mass and applies whether the particle has mass or not. The point is to extend the Galilean principle of relativity to the two uniform motions of expansion and rotation, and no longer to consider the latter as an accelerated motion as in classical mechanics or relativity theory. Translational motion at uniform velocity is replaced by expansion motion at constant divergence, and rotational motion is eliminated from the fundamental law of dynamics. Restrictions on the existence of these motions when \mathbf{r} increases indefinitely will no longer be necessary.

The extension of Galilean invariance (22) to uniform motions of expansion and rotation allows us to dispense with satisfying this invariance and concentrate on the second requirement, the limitation of velocity to the celerity c_0 of light in vacuum, a constraint satisfied by the Lorentz transformation. Indeed, the two invariances, Galileo's and Lorentz's, would be incompatible and non-comparable if they were of the same nature, which is not the case. Indeed, the former aims to eliminate an absolute reference frame, which is an undeniable advance in mechanics, and the latter reflects a local phenomenon whereby a particle or material medium cannot exceed the celerity of the medium in which it evolves. Note that this limitation applies only to one-dimensional motion, i.e. the trajectory of a photon or other particle along a rectilinear axis.

Let's consider the two classic reference frames, an absolute \mathcal{R} and a relative \mathcal{R}' . The change of reference frame traditionally brings out the three spatial and temporal coordinates, even for a four-vector formulation. The discrete mechanics point of view is based on a one-dimensional spatial model defined by the \mathbf{r} coordinate, the direction of the γ segment of the Figure 1. This concept is not incompatible with that of special relativity. Indeed, the other two directions y and z , orthogonal in a Cartesian reference frame to the direction of propagation, are not necessary, as relativistic transformations do not modify these coordinates in the relative reference frame $\mathcal{R}'(x', y', z')$ with respect to the reference frame \mathcal{R} . The theory of special relativity, essentially due to Lorentz, Poincaré and Einstein, introduces the eponymous Lorentz transformation, whose most common form is:

$$\begin{cases} \mathbf{r}' = \gamma (\mathbf{r} - \mathbf{v} t) \\ t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{r}}{c_0^2} \right), \end{cases} \quad (23)$$

where γ is the Lorentz factor $\gamma = 1/\sqrt{1 - \mathbf{v}^2/c_0^2}$.

It's easy to see that this Lorentz transformation is incompatible with Galileo's transformation (19), ($\mathbf{r}' = \mathbf{r} - \mathbf{v} t$, $t' = t$), if length and time in a frame of reference in relative motion had the same meanings as in the \mathcal{R} frame of reference. Einstein's idea for resolving this incompatibility

was to define a distance and duration that depend on the Lorentz factor γ ; thus, in the course of motion, time undergoes a dilation and length a contraction.

The proof of the relativistic nature of the equation (17) lies in its natural transformation into a wave equation. Like Maxwell's equations, the wave propagation equation is relativistic, i.e. the Lorentz transformation allows us to conclude that it is true whatever the inertial reference frame considered. Using the formula for calculating the vector $\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$ we obtain:

$$\frac{1}{c_0^2} \frac{d^2 \mathbf{u}}{dt^2} - \nabla^2 \mathbf{u} = -\nabla \phi^o + \nabla \times \boldsymbol{\psi}^o, \quad (24)$$

where $\mathbf{u} = \mathbf{v} dt$ is the displacement and the second member corresponds to the Helmholtz-Hodge decomposition of the retarded acceleration. The first member of this equation is an d'Alembertian in a Lagrangian reference frame; in an Eulerian formulation the wave equation is written:

$$\square \mathbf{u} \equiv \frac{1}{c_0^2} \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla^2 \mathbf{u}. \quad (25)$$

Given the definition of the retarded potentials ϕ^o and $\boldsymbol{\psi}^o$ their sum also forms a Laplacian of the retarded displacement \mathbf{u}^o independent of time and the equation (24) becomes:

$$\frac{1}{c_0^2} \frac{d^2 \mathbf{u}}{dt^2} - \nabla^2 (\mathbf{u} - \mathbf{u}^o) = 0. \quad (26)$$

The difference between (24) and the classical form of a wave equation (25) is the inertia. The former expression involves inertia terms of the form $\nabla (|\mathbf{v}|^2/2) - \nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n})$ introduced in [11].

The final step is to show that the wave equation is relativistic; this demonstration is classical, but is repeated here in the one-dimensional spatial framework. As the equation of motion is derived on a single rectilinear segment, interactions in other directions are taken into account from cause to effect, via the common vertices of neighboring segments. What is shown here for the \mathbf{r} direction alone will be valid for all the others. Applying the inverse Lorentz transformation (23) and differentiating each expression twice with respect to \mathbf{r} and t , we obtain:

$$\frac{\partial^2}{\partial \mathbf{r}^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} = \gamma^2 \left(1 - \frac{\mathbf{v}^2}{c_0^2}\right) \frac{\partial^2}{\partial \mathbf{r}'^2} - \frac{1}{c_0^2} \gamma^2 \left(1 - \frac{\mathbf{v}^2}{c_0^2}\right) \frac{\partial^2}{\partial t'^2}. \quad (27)$$

Given the definition of the Lorentz parameter, we find that $\square = \square'$, the wave equation is indeed invariant with respect to the Lorentz transformation. The model represented by the discrete equation (17) satisfies both the extension of the Galilean transformation to uniform motions of expansion and rotation, and Lorentz invariance for rectilinear wave propagation limited by the celerity of the material medium. This approach leaves aside the need to consider the contraction of distances and the dilation of time.

4.2 Laws of conservation of compressive and rotational energy

Given the intrinsically orthogonal nature of the gradient and dual curl operators, the corresponding compression or expansion and rotation energies cannot be spontaneously exchanged. The submission of one to the other must be ensured by the intrinsic acceleration that entangles the two physical effects. Before combining them in the same formulation, it is useful to analyze what the separate balances of each represent. The notion of equivalence between mass and energy must be strictly demonstrated, as must an analogous law for rotation.

Let's consider the compressive energy ϕ contained in the dual volume Ω centered around a vertex a of the primary structure that we follow in its motion. The Ω volume remains fixed,

while a flow of energy is injected through its Σ surface. The compression energy per unit mass initially contained in Ω is denoted ϕ^o and its variation over time is a function of the injected flow rate represented by the divergence of the velocity normal to the boundary, $\nabla \cdot \mathbf{v}$. This variation is therefore equal to $d\phi$ for a time lapse of dt . Similarly, the rotational energy per unit mass ψ about an axis \mathbf{n} orthogonal to each facet of the primary structure is associated with the rate of rotation characterized by the vortex vector $\boldsymbol{\omega} = \nabla \times \mathbf{v}$; the potential vector $\boldsymbol{\psi}$ and the vortex vector $\boldsymbol{\omega}$ are, of course, collinear. Thus, the conservation of compression and rotation energies is given by the laws:

$$\begin{cases} \gamma_c = \frac{d\phi}{dt} = -\phi \nabla \cdot \mathbf{v} \\ \gamma_r = \frac{d\psi}{dt} = -\boldsymbol{\psi} \nabla \times \mathbf{v}, \end{cases} \quad (28)$$

where $\boldsymbol{\psi} \nabla \times \mathbf{v}$ is the symbolic notation for $(\boldsymbol{\psi} \cdot \nabla \times \mathbf{v}) \mathbf{n}$, both vectors being collinear with the unit vector \mathbf{n} . If the time interval $dt = t - t^o$ is small enough for $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ to be constant between two observations of mechanical equilibrium, then:

$$\begin{cases} \phi = -\phi^o e^{-dt \nabla \cdot \mathbf{v}} \\ \boldsymbol{\psi} = -\boldsymbol{\psi}^o e^{-dt \nabla \times \mathbf{v}}. \end{cases} \quad (29)$$

In all cases, $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ are local and instantaneous quantities determined by solving the equation of motion. The groupings $dt \nabla \cdot \mathbf{v}$ and $dt \nabla \times \mathbf{v}$ must be compared with unity for a coherent temporal representation of the phenomena; the time lapse or the value of the velocity operators have no quantitative significance in themselves.

Let's return to continuum mechanics and consider a mass of fluid m contained in the volume Ω that we follow in its motion. The volume Ω remains fixed while a flow of matter is injected through its surface Σ at velocity \mathbf{v} . The local mass conservation equation relates directly to m or mass per unit volume ρ , the density. Similarly, let's consider the evolution over time of the moment of inertia \mathbf{J}_Δ subjected to a constraint designed to modify its angular momentum. The conservation of mass and moment of inertia can be written as follows:

$$\begin{cases} \frac{dm}{dt} = -m \nabla \cdot \mathbf{v} & \iff & \frac{d\phi}{dt} = -\phi \nabla \cdot \mathbf{v} \\ \frac{dJ_\Gamma}{dt} = -J_\Gamma \nabla \times \mathbf{v} & \iff & \frac{d\boldsymbol{\psi}}{dt} = -\boldsymbol{\psi} \nabla \times \mathbf{v}. \end{cases} \quad (30)$$

A comparison of the classical equilibrium laws of mass and moment of inertia (30) with those of discrete mechanics (28) clearly shows the equivalence between mass and compression energy, on the one hand, and moment of inertia and rotational energy, on the other; these are equivalences, not simple proportionalities. The units of mass and compression energy may not be the same, but their balance equations are, and this is what expresses the notion of equivalence.

In the theory of special relativity, the equivalence of mass and energy is summed up by A. Einstein's famous formula $E = m c_0^2$ where m is the moving mass. However, this law shows a simple proportionality between E and c_0^2 , the factor of which is mass. If we now consider energy per unit mass $e = E/m$, we see that energy can be identified with the scalar potential $\phi = c_0^2$. These laws (28) are very general and apply not only to light, but also to the celerities of the medium, for example the celerity of sound c_l in the case of fluid flows. Note that these balances are expressed in local Lagrangian coordinates, so the advection of the quantities considered must be taken into account.

For example, the equivalence of mass and energy in special relativity only concerns compression energy; rotation energy is not mentioned. Total energy is considered to be the sum of potential energy and kinetic energy. In discrete mechanics, the kinetic energy carried by the potential $|\mathbf{v}|^2/2$ is a component of the intrinsic acceleration integrated into the material derivative in the form of a Helmholtz-Hodge decomposition.

Intrinsic energy is directly defined as the discrete integral over the Γ segment, and is equal to the sum of compression and rotation energy:

$$\int_{\Gamma} \boldsymbol{\gamma} \cdot \mathbf{t} \, dl = - \int_{\Gamma} \nabla \phi \cdot \mathbf{t} \, dl + \int_{\Gamma} \nabla \otimes \boldsymbol{\psi} \cdot \mathbf{t} \, dl. \quad (31)$$

This equation (31), derived from the fundamental law of discrete physics, expresses the conservation of total energy per unit mass on the segment Γ . The two terms in the second member represent, respectively, the direct energy of compression and the induced energy of rotation. Remember that these last two energies cannot be exchanged without an intrinsic acceleration; this mutual dependence is called mechanical entanglement. It plays a fundamental role in highly non-linear phenomena, such as fluid turbulence.

4.3 Invariances and Noether's theorem

Noether's theorem [14, 15, 16] establishes the link between certain symmetries in the action of systems and the conservation of certain quantities in physics, (i) directional invariance to the conservation of linear momentum, (ii) rotational invariance to the conservation of angular momentum, (iii) translational invariance in time to the conservation of energy, for the three main ones. This theorem removes the notion of absolute reference for each of these motions in the establishment of the laws of physics. It was developed within the framework of Lagrangian or Hamiltonian mechanics.

The discrete equation does not derive from a variational formulation, and the link with Noether's theorem cannot be formally established. Furthermore, the quantities used in this theorem to define actions - mass, momentum, force, etc. - are no longer quantities used in mechanics.

The law of motion (17) is not directly derived from Lagrange's formalism for a number of reasons, (i) the discrete formulation is derived from Maxwell's founding idea, (ii) the actions associated with Lagrangians are kinetic energy and potential energy are not the only ones, (iii) mass is always present in Lagrangian mechanics is an abandoned notion and (iv) invariants are formulated in terms of accelerations.

To clarify the nature of the phenomena present in the equation (17) of accelerated motion, let's write it in the form:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) = -\nabla (\phi^o - c_t^2 dt \nabla \cdot \mathbf{v}) + \nabla \otimes (\boldsymbol{\psi}^o - c_t^2 dt \nabla \times \mathbf{v}), \quad (32)$$

where the time derivative is the sum of a curl-free term and a divergence-free term. Like inertia and the material derivative itself, this is a two-term Helmholtz-Hodge decomposition.

The interpretation of the law (32) turns out to be in agreement with Noether's theorem, although the approaches are very different. This law expresses the conservation of total energy linked to the acceleration $\boldsymbol{\gamma} \equiv d\mathbf{v}/dt$ of the first member, where the acceleration itself is energy per unit mass and length. The first term of the second member is an acceleration related to the conservation of translational energy (compression and expansion). The second term, associated with rotational energy, is an angular acceleration. Remember that these last two terms are free from the contributions of uniform-velocity translational and rotational motion. These two terms are orthogonal, which from a physical point of view means that they are disjoint, that an angular

acceleration cannot be described by the gradient of a scalar potential and that a translational acceleration cannot be represented by a dual curl of a vector potential. The table 1 summarizes the three main invariances of Noether's theorem, but in terms of accelerations:

Property of the system	Symmetry	Invariant
translation in time	time invariance	conservation of total energy compression and rotation
homogeneous space	invariance by translation	conservation of compression energy and of translation acceleration mass conservation
isotropic space	invariance by rotation	conservation of rotational energy and of angular acceleration conservation of angular momentum

Table 1. Invariance and conservation of the law of motion (32)

The deeper meaning of the law (32) refers above all to the principle of inertia, i.e. if the second member is zero, then the material derivative is zero and uniform translational motion and uniform rotational motion persist indefinitely. This equation of motion is therefore specifically a law of accelerated motion. When the inertia terms are transferred to each of the two right-hand member terms for stationary motion, the time derivative $\partial \mathbf{v} / \partial t = 0$ separates the two orthogonal contributions into two equivalent equations where the second members are harmonic functions. In other words, the scalar potential ϕ^o does not depend on the vector potential ψ^o . In the general case, all the terms in equation (32) are intertwined, i.e. a change in the scalar potential generates a change in the intrinsic acceleration of the particle or material medium, which in turn modifies the rotational energy described by the vector potential. This interweaving of contributions is strictly linked to accelerated motions, which are physically identified with oscillators. Each oscillator of the second member is itself composed of a potential and an operator, $\nabla \cdot \mathbf{u}$ or $\nabla \times \mathbf{u}$, weighted by the celerity, which translates the exchanges between potential energy and kinetic energy of the same nature, compression or rotation. For the record, the Table 1 reproduces the invariants of classical mechanics concerning mass and angular momentum. It is therefore directly the law of discrete motion that establishes the links, in terms of accelerations, between the invariants and conservations of Noether's theorem.

4.4 A Helmholtz-Hodge decomposition of acceleration

The aim is to show that the intrinsic acceleration of a particle or material medium γ is an absolute quantity associated with the two potentials of the relation (13) excluding the addition of divergence-free and curl-free term of a three-term Helmholtz-Hodge decomposition.

The Helmholtz or Helmholtz-Hodge decomposition is widely used in mathematics to project a field onto a zero-divergence space, as well as in image processing, fingerprint recognition and many other applications [17, 18, 19, 11, 20]. The problem posed here is of a different nature, we want to show that absolute acceleration can be formulated as a Helmholtz-Hodge decomposition. This suggestion is based on Maxwell's idea of associating [21] direct and induced currents with temporal dynamics. Let's consider any \mathcal{V} vector field, which can be generally decomposed into three terms:

$$\mathcal{V} = -\nabla \mathcal{A} + \nabla \times \mathcal{B} + \mathcal{H}, \quad (33)$$

where \mathcal{H} is a vector that can be written as a harmonic term, with zero or constant divergence and zero or constant curl. Indeed, the application *a posteriori* of the divergence and curl operators

will make the constant disappear. The two conditions for $\nabla \cdot \mathcal{H} = cte$ are as follows:

$$\nabla \cdot \mathcal{H} = cte \Rightarrow \begin{cases} \mathcal{H} = \nabla \times \boldsymbol{\beta} \\ \mathcal{H} = a \mathbf{r}, \end{cases} \quad (34)$$

where $\boldsymbol{\beta}$ is any vector with zero divergence and \mathbf{r} is any radial vector with constant divergence. Similarly, consider the conditions under which the rotation vector \mathcal{H} satisfies the equality $\nabla \times \mathcal{H} = cte$:

$$\nabla \times \mathcal{H} = cte \Rightarrow \begin{cases} \mathcal{H} = \nabla \alpha \\ \mathcal{H} = \boldsymbol{\Omega} \times \mathbf{r}, \end{cases} \quad (35)$$

where it is then necessary for \mathcal{H} to be the gradient of a scalar $\mathcal{H} = \nabla \alpha$ or for it to be constant $\boldsymbol{\Omega} \times \mathbf{r}$. But as it is necessary that $\nabla \boldsymbol{\beta} = 0$ as well as $\nabla \cdot \nabla \alpha = 0$ it is necessary to fix $\alpha = \beta = 0$ and the harmonic field becomes:

$$\mathcal{H} = a \mathbf{r} + \boldsymbol{\Omega} \times \mathbf{r}, \quad (36)$$

the stationary field \mathcal{H} has constant divergence $\nabla \cdot \mathcal{H} = a$ and constant rotation $\nabla \times \mathcal{H} = 2 \boldsymbol{\Omega}$ for continuous functions $a \mathbf{r}$ and $\boldsymbol{\Omega} \times \mathbf{r}$.

We need to add a condition to these two stationary motions: they must also have zero inertia. This is the case in discrete mechanics, where the material derivative cancels out:

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \left(\frac{1}{2} |\mathcal{H}|^2 \right) - \nabla \otimes \left(\frac{1}{2} |\mathcal{H}|^2 \mathbf{n} \right) = 0. \quad (37)$$

Since velocity is reduced to its harmonic component $\boldsymbol{\mathcal{V}} = \mathcal{H}$, its acceleration is zero, since \mathcal{H} is a stationary field and its material derivative is equal to $d\mathcal{H}/dt = 0$.

In conclusion, the notion of galilean or inertial reference frame associated with uniform translational motion alone in continuum mechanics is extended, in discrete mechanics, to uniform dilatational motion and uniform rotational motion. From the point of view of the equation of motion, the two uniform motions are filtered out, as they concern acceleration and not velocity. Thus, the Helmholtz-Hodge decomposition of acceleration $\boldsymbol{\gamma} = -\nabla \phi + \nabla \otimes \boldsymbol{\psi}$ is unique.

5 Three emblematic examples

5.1 Particle under constant acceleration

Equation (17) applies equally to fluid flows under conditions where the velocity is of the order of magnitude of the celerity of sound $c = c_l$ and to phenomena related to special or general relativity $c = c_0$. The case of a single particle with or without mass subjected to an external acceleration $\mathbf{h}_s = g \mathbf{e}_x = \nabla \phi_s$ from a state of rest is an emblematic case of special relativity which has an exact solution. Consider a particle at rest at the initial time, set in motion by means of an acceleration imposed from the outside, a gravitational field or a magnetic field, to the charged particles. In a Lagrangian description movement is described by the time-dependent position of the particle $x(t)$ and velocity $v(t) = dx/dt$ with $\mathbf{v} = v \mathbf{e}_x$. The equation of the discrete motion (17) to the rectilinear movement of the particle from its origin located at $x = 0$ to $t = 0$ is rewritten:

$$\frac{dv}{dt} = -\frac{d}{dx} \left(\phi^o - dt v^2 \frac{dv}{dx} \right) + g. \quad (38)$$

In order to quantify the movement of the particle in terms of velocity and position in a more general vision, let us pose the following quantities to resize distance, time, velocity and energy per unit of mass:

$$x^* = \frac{x g}{c^2}; \quad t^* = \frac{t g}{c}; \quad v^* = \frac{v}{c}; \quad \phi^{*o} = \frac{\phi^o}{c^2}, \quad (39)$$

where the starred quantities are the dimensionless variables. To lighten the writing, the same symbols are used to express the dimensionless quantities. The equation thus becomes:

$$\frac{dv}{dt} = -\frac{d}{dx} \left(\phi^o - dt v^2 \frac{dv}{dx} \right) + 1. \quad (40)$$

Compression energy ϕ^o is that accumulated at time t from the initial time:

$$\phi^o(t) = -\int_0^t v^2 \frac{dv}{dx} d\tau = -\int_0^t \frac{dt}{dx} v^2 dv = -\frac{v^2}{2}. \quad (41)$$

Thus, at each instant, the compression energy of the wave is exactly equal to the kinetic energy acquired by the particle. As $g \mathbf{e}_x = \nabla(g x)$ in real variables, the scalar potential of the acceleration becomes equal to:

$$\phi(t) = x - \phi^o(t) = x - \frac{v^2}{2}. \quad (42)$$

Equation (40) becomes:

$$\frac{dv}{dt} = \frac{d}{dx} \left(\frac{v^2}{2} \right). \quad (43)$$

Position of the particle x and time t are not independent and the relation (43) is not an equation but an identity; each member expresses the material derivative in a Lagrangian description, because dx/dt is none other than the velocity v . In fact the equation of discrete motion is not a simple differential equation. It involves a temporal process called "accumulation" where the description of the mechanical equilibrium at the instant t is deduced from that at the previous equilibrium instant t^o by a Lagrangian formulation composed of an equation on acceleration and its updates on velocity, energy and spatial coordinates:

$$\left\{ \begin{array}{l} \gamma = 1 - \frac{1}{v} \frac{d}{dt} \left(\phi^o - v \frac{dv}{dt} \right) \\ \phi^o - v \frac{dv}{dt} \mapsto \phi^o \\ v^o + dt \gamma \mapsto v^o \\ x^o + dt v \mapsto x^o. \end{array} \right. \quad (44)$$

It is possible to find a robust solution by considering the energy necessary for the particle to overcome inertia and a velocity which tends towards celerity. This energy, by definition equal to $c^2/2$, is expressed as a function of the integral of the acceleration from zero to infinity, i.e. in dimensionless form:

$$\phi_c = \int_0^\infty \gamma(x) dx = \frac{1}{2}. \quad (45)$$

Acceleration is thus a decreasing function of x in $\mathcal{O}(x^{-n})$, whereby $n = 3$ in order to satisfy the condition (45). The initial condition corresponding to an acceleration equal to unity for $x = 0$, satisfied by the system equation (44), leads to the following form:

$$\gamma = \frac{1}{(1+x)^3}. \quad (46)$$

Imposed acceleration $\mathbf{h}_s = 1$ is transformed partly into compression energy equal to ϕ^o . This compression energy is transferred to the particle to increase its own kinetic energy ϕ_c and therefore its own acceleration γ . At each instant we have $\phi_c = \phi^o$ and the vector potential of the acceleration is equal to $\phi = x - \phi^o$; equilibrium is reached when $-\nabla\phi + 1 = 0$. The imposed acceleration \mathbf{h}_s then no longer acts on the particle, the movement is incompressible, $\nabla \cdot v = 0$, and the particle moves at a velocity equal to the celerity $v = c$.

Solution to the problem in dimensionless form obtained by the incremental system (44) corresponds very precisely to the solution obtained in the context of special relativity:

$$\begin{cases} \gamma(t) = \frac{1}{(1+x)^3} = \frac{1}{(1+t^2)^{3/2}} \\ v(t) = \frac{\sqrt{x(2+x)}}{(1+x)} = \frac{t}{\sqrt{1+t^2}} \\ \phi(t) = x - \frac{v^2}{2}. \end{cases} \quad (47)$$

Figure (2) shows the evolutions obtained numerically by the system (44), the acceleration $\gamma(t)$, the velocity $v(t)$ and the scalar potential $\phi^o(t)$ which tends towards the value $1/2$. The divergence of velocity $\nabla \cdot v = \gamma/v = 1/(1+t^2)t$ shows that the movement becomes incompressible, explaining that the velocity of the particle tends towards celerity.

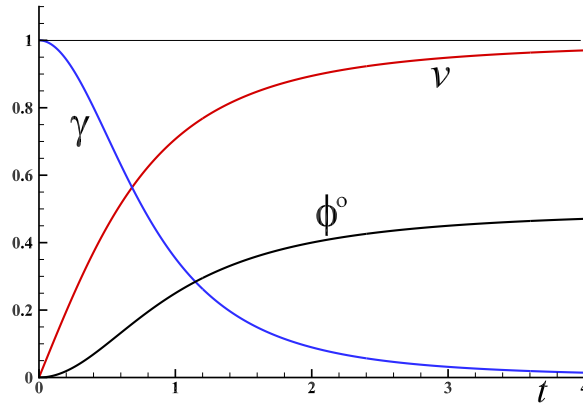


Figure 2. Acceleration of an isolated particle with or without mass over time: $v(t)$ its velocity, $\gamma(t)$, its acceleration and $\phi^o(t)$ the scalar potential.

With short times, the acceleration is equal to unity, the velocity increases in the form $v(t) \approx t^2/2$ and the divergence of the velocity is then in $\mathcal{O}(t^{-1})$; at this point the accumulated energy is very close to zero. When the time increases beyond $t \approx 0.5$ the potential $\phi = x - \phi^o$ increases, the intrinsic acceleration γ decreases and the growth in velocity slows down. The acceleration then tends towards zero and the velocity towards the celerity of the medium. The root cause

of this limitation is the divergence of the motion of the particle, which goes from $\mathcal{O}(t^{-1})$ for Newtonian mechanics to $\mathcal{O}(t^{-3})$ in relativity and in discrete mechanics when time increases.

5.2 Gravitational lensing

This problem of light being deflected by the sun is also known as gravitational lensing, a deflection of light by the presence of bodies, stars or clusters, present in the vicinity of its trajectory. The deviation is measured by an angle Φ defined as the angle formed by the two trajectories upstream and downstream of the body producing the deviation. The first scientific studies on this subject were carried out by Soldner [22], but the problem was the subject of experimental and theoretical studies by Dyson and Eddington [23, 24, 25, 26, 27, 5], who showed that the theory of general relativity developed a few years earlier by Einstein was able to predict the observed result. This theoretical value is very close to Eddington's observed value of $\Phi = 4 \mathcal{G} M_{\odot} / (c_0^2 R_{\odot})$ where \mathcal{G} is the gravitational constant, M_{\odot} the mass of the sun and R_{\odot} its mean radius.

In discrete mechanics, proper mass is a concept that has been abandoned for the description of physical phenomena in general and gravitation in particular. The gravitational potential becomes:

$$\phi_g^o = \mathcal{G} M_{\odot} / r, \quad (48)$$

where ϕ_g^o is the energy per unit mass, independent of its own mass.

From a physical point of view, two actions are in competition: (i) the inertia of particles, which tends to keep them on a straight trajectory, and (ii) gravitation, which attracts photons towards the massive body. Here, it is no longer space-time that is curved, but the acceleration of gravity that applies to any particle, with or without mass. The attraction exerted on this particle does not depend at all on its mass, as suggested by the WEP revisited by discrete mechanics. Generally speaking, inertia is fixed by the material derivative:

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{v}|^2}{2} \mathbf{n} \right), \quad (49)$$

where the last two terms of (49) represent the curvature κ_i of the inertial potential. Photons have no mass, but do possess kinetic energy. Gravitation acts on photons travelling close to the Sun, attracting them as soon as they enter its sphere of influence and accelerating them. As with geometric curvature, the spherical character of inertial potential has twice the curvature of a cylinder. The gravitational potential (48) induces two accelerations, the first as a direct action and the second corresponding to an action induced to form the curvature of the gravitational potential :

$$\kappa_g = -\nabla \phi_g + \nabla \otimes \psi_g. \quad (50)$$

In discrete mechanics, the acceleration vector is always represented by two contributions, the gradient of a scalar potential and the dual curl of the inertial potential, here $\psi_g = \phi_g \mathbf{n}$. Assuming the phenomenon is stationary, the equation of motion is written :

$$\nabla \left(\frac{|\mathbf{v}|^2}{2} \right) = -\nabla \left(\frac{\mathcal{G} M_{\odot}}{R_{\odot}} \right) + \nabla \otimes \left(\frac{\mathcal{G} M_{\odot}}{R_{\odot}} \mathbf{n} \right) = -2 \nabla \left(\frac{\mathcal{G} M_{\odot}}{R_{\odot}} \right). \quad (51)$$

Given the small deviation of light $\cos \Phi \approx 1$, the approximation $\mathbf{v} \approx c_0$ is adopted and the value of the deviation is equal to:

$$\Phi \approx \frac{4 \mathcal{G} M_{\odot}}{c_0^2 R_{\odot}}, \quad (52)$$

is the correct value obtained by general relativity and corroborated by observation.

The reasoning followed is close to that of Newtonian mechanics, where inertia and gravitation are confronted. But here we're talking about equality between the potential curves, that of inertia and that of gravitation, i.e. $\kappa_i = \kappa_g$. In summary, Newtonian mechanics leads to an angle $\Phi = 4.250 \cdot 10^{-6} rd$ or $\Phi = 0.875$ arc second, while the theory of general relativity predicts an angle twice as large $\Phi = 8.498 \cdot 10^{-6} rd$ or $\Phi = 1.750$ arc second, and experiments confirm this prediction [23].

5.3 Gravitational redshift

The redshift of radiation corresponds to a loss of energy as it passes through an inhomogeneous gravitational field, whose intensity decreases along its path. The phenomenon of gravitational blue-shift is observed when the observer is in a region where the gravitational field is greater than that in which the radiation is emitted.

$$\nabla \left(\frac{|\mathbf{v}|^2}{2} \right) = \nabla \left(\frac{\mathcal{G} M}{r} \right). \quad (53)$$

The Galilean approximation assumes that the velocity of photons is equal to the celerity of light; under these conditions, the ratio of frequencies is equal to

$$\frac{\Delta f}{f} = \frac{2 \mathcal{G} M}{c_0^2 r}, \quad (54)$$

where the second member is none other than the Schwarzschild radius R_s . In special relativity, the calculation is completed by an additional term; the argument developed by A. Einstein is based on the existence of two times, the one observed in the absence of the gravitational field dt and the proper time $d\tau < dt$ such that $d\tau = \sqrt{1 - R_s/r} dt$. This phenomenon is called gravitational time dilation, but the result is the same when the concept of length contraction is invoked.

The discrete formulation is more complex, as the solution is derived from the law of motion, where the photon's velocity and radius vary along a trajectory:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(\frac{|\mathbf{v}|^2}{2} - c_0^2 dt \nabla \cdot \mathbf{v} \right) + \nabla \left(\frac{\mathcal{G} M}{r} \right). \quad (55)$$

In the general case, the photon's velocity is no longer equal to c_0 , but is reduced by the gravity field. The velocity divergence is no longer zero, since it varies along the rectilinear trajectory and the gravity field depends on the radius. As the velocity \mathbf{v} decreases during expansion, the divergence is positive, so energy is borrowed from the kinetic energy $|\mathbf{v}|^2/2$. The frequency ratio is therefore modified in this relativistic framework. The differences between the simplified analysis (54) and the complete approach (55) are very small, even in the case of stars like the sun. Introducing the displacement $d\mathbf{u} = \mathbf{v} dt$ and considering stationary motion, equation (55) reads as follows:

$$2 \nabla \cdot \mathbf{u} = \left(\frac{|\mathbf{v}|^2}{c_0^2} - \frac{R_s}{r} \right) \approx \left(1 - \frac{R_s}{r} \right), \quad (56)$$

or, by identification with the time ratio of general relativity, $2 \nabla \cdot \mathbf{u} \approx d\tau^2/dt^2$. Using the Schwarzschild metric centered on a spherically symmetrical solid body, we find the coefficient of the time coordinate $g_{00} = 1 - R_s/r$ [27, 5].

6 Conclusions

The concepts of special relativity and general relativity, whose legitimacy is not called into question, are simply discarded to establish a disjoint formulation. The abandonment of the notion of mass, the omnipresent quantity in the equations of physics, as soon as the Weak Equivalence Principle is stated eliminates any similarity with Newtonian mechanics and the theory of relativity. This approach also implies abandoning the classical quantities of mechanics: momentum, force, energy... All that remains is acceleration, a quantity considered absolute in discrete mechanics. Acceleration gives rise to relative quantities such as velocity or energy per unit mass. Maxwell's model unifies electrodynamics and magnetism in a dynamic representation, electromagnetism, emphasizing the interconnection of longitudinal compression and transverse propagation.

The law of discrete motion (17), already demonstrated in classical mechanics by numerous examples, proves perfectly compatible with the results of relativity theory. The dissimilarities and divergences concern fundamental concepts, such as the existence of the notion of space-time, replaced by the independent notions of space and time. The formalism used corresponds to a parsimonious vision of the physical model and considerably simplifies the understanding of observed phenomena, replacing, for example, the use of high-order tensors with simple differential operators.

References

- [1] L. Landau, E. Lifchitz, *The Classical Theory of Fields*, Third Revised English Edition, Pergamon Press Ltd, Oxford, 1971.
- [2] B. Dolan, *Einstein's General Theory of Relativity: A Concise Introduction*, Cambridge University Press, 2023. doi:10.1017/9781009263689.
- [3] J.-P. Caltagirone, *Discrete Mechanics, concepts and applications*, ISTE, John Wiley & Sons, London, 2019. doi:10.1002/9781119482826.
- [4] P. Touboul, al., *Microscope mission: Final results of the test of the equivalence principle*, *Phys. Rev. Lett.* 129 (2022) 121102. doi:10.1103/PhysRevLett.129.121102.
- [5] C. Will, *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge, United Kingdom, 2018.
- [6] H. Poincaré, *Sur la dynamique de l'électron*, *C.R. Acad. Scien.* 140 (1905) 1504–1508.
- [7] J.-P. Caltagirone, S. Vincent, *On primitive formulation in fluid mechanics and fluid-structure interaction with constant piecewise properties in velocity-potentials of acceleration*, *Acta Mechanica* 231 (6) (2020) 2155–2171. doi:10.1007/s00707-020-02630-w.
- [8] J.-P. Caltagirone, *Extension of galilean invariance to uniform motions for a relativistic equation of fluid flows*, *Phys. Fluids* 35 (2023). doi:10.1063/5.0128422.
- [9] A. Liénard, *Champ électrique et magnétique produit par une charge électrique concentrée en un point et animée d'un mouvement quelconque*, *L'Éclairage électrique* XVI (27) (1898) 5–112.

- [10] C. Hamman, J. Klewick, R. Kirby, On the Lamb vector divergence in Navier-Stokes flows, *Journal Fluid Mechanics* 610 (2008) 261–284. doi:10.1017/S0022112008002760.
- [11] J.-P. Caltagirone, On Helmholtz-Hodge decomposition of inertia on a discrete local frame of reference, *Physics of Fluids* 32 (2020) 083604. doi:10.1063/5.0015837.
- [12] J.-P. Caltagirone, Physique discrète et relativité, *Annales de la Fondation Louis de Broglie* 44 (2019) 1–13.
- [13] A. Friedmann, On the Curvature of Space, *General Relativity and Gravitation* 31 (1999) 1991. doi:10.1023/A:1026751225741.
- [14] Y. Kosmann-Schwarzbach, *Noether Theorems. Invariance and Conservations Laws*, Springer-Verlag, New York, 2011. doi:10.1007/978-0-387-87868-3.
- [15] M. Tavel, English translation of "invariante variationsprobleme," *nachr. d. konig. gesellsch.d. wiss. zu gottingen, math-phys. klasse*, (1918), *Transport Theory and Statistical Physics* 1 (3) (1971) 235–257.
- [16] N. Byers, E. Noether’s discovery of the deep connection between symmetries and conservation laws, arXiv preprint physics/9807044 (1998).
- [17] F. Denaro, On the application of the Helmholtz-Hodge decomposition in projection methods for incompressible flows with general boundary conditions, *International Journal for Numerical Methods in Fluids* 43 (2003) 43–49.
- [18] H. Bhatia, G. Norgard, V. Pascucci, P. Bremer, The Helmholtz-Hodge Decomposition - A Survey, *IEEE Transactions on Visualization and Computer Graphics* 19 (8) (2013) 1386–1404. doi:10.1109/TVCG.2012.316.
- [19] H. Ranocha, K. Ostaszewski, H. P., Discrete Vector Calculus and Helmholtz Hodge Decomposition for Classical Finite Difference Summation by Parts Operators, arXiv:1908.08732v1 [math.NA] 23 Aug 2019 (2019) 1–33.
- [20] J.-P. Caltagirone, On a reformulation of Navier-Stokes equations based on Helmholtz-Hodge decomposition, *Physics of Fluids* 33 (2021) 063605. doi:10.1063/5.0053412.
- [21] J. Maxwell, A dynamical theory of the electromagnetic field, *Philosophical Transactions of the Royal Society of London* 155 (1865) 459–512.
- [22] J. Soldner, Ueber die Ablenkung eines Lichtstrals von seiner geradlinigen Bewegung, durch die Attraktion eines Weltkörpers, an welchem er nahe vorbeigeht, *Berliner Astronomisches Jahrbuch* (1804) 161–172.
- [23] F. Dyson, A. Eddington, C. Davidson, A determination of the deflection of light by the sun’s gravitational field, from observations made at the total eclipse of may 29,1919, *Philosophical Transactions of the Royal Society of London, Series A* 220 (1920) 291–333. URL <http://www.jstor.org/stable/91137>
- [24] A. Eddington, A generalisation of Weyl’s theory of the electromagnetic and gravitational field, *Philosophical Transactions of the Royal Society of London, Series A* 99 (1921) 104–122.
- [25] A. Eddington, *The Mathematical Theory of Relativity*, Cambridge University Press, Cambridge, 1923.

- [26] C. Will, Henry Cavendish, Johann von Soldner, and the deflection of light, *American Journal of Physics* 56 (1988) 413–415. doi:10.1119/1.15622.
- [27] C. Will, The confrontation between general relativity and experiment, *Space Sci. Rev.* 148 (2009) 3–13. doi:10.1007/s11214-009-9541-6.