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1d backward stochastic differential equation: a unified theory, applications and open problems*

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Abstract

We develop a unified theory on the well-posedness of a one-dimensional backward stochastic differential equation (1d BSDE for short), where the generator g admits a one-sided linear/super-linear growth in the first unknown variable y and a no more than quadratic growth in the second unknown variable z . Several existence theorems and comparison theorems are established by virtue of the test function method and the a priori estimate technique, and then several existence and uniqueness results follow naturally. We also present an overview about relevant known results and introduce some practical applications of our theoretical results. Finally, some open problems on the well-posedness of 1d BSDEs are provided.

Keywords: Backward stochastic differential equation, A unified theory, Feynman-Kac formula
Existence and uniqueness, Comparison theorem, Open problems, g -expectation.

2010 MSC: 60H10

1. Introduction

Fix a real $T > 0$ and an integer $d \geq 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with augmented filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by a standard d -dimensional Brownian motion $(B_t)_{t \in [0, T]}$, and assume that $\mathcal{F}_T = \mathcal{F}$. The equality and inequality between random elements are usually stood in the sense of $\mathbb{P} - a.s.$ We are concerned with the following one-dimensional backward stochastic differential equation (1d BSDE in short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (1.1)$$

where ξ is called the terminal condition being an \mathcal{F}_T -measurable real random variable, the random field

$$g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

is called the generator of (1.1), which is (\mathcal{F}_t) -adapted for each (y, z) , and the pair of (\mathcal{F}_t) -adapted and $\mathbb{R} \times \mathbb{R}^d$ -valued processes $(Y_t, Z_t)_{t \in [0, T]}$ is called a solution of (1.1) if $\mathbb{P} - a.s.$, $t \mapsto Y_t$ is continuous,

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$t \mapsto |g(t, Y_t, Z_t)| + |Z_t|^2$ is integrable, and (1.1) is satisfied. Denote by BSDE(ξ, g) the BSDE with the terminal condition ξ and the generator g , which are the parameters of BSDEs.

For narrative convenience, in the whole paper let us always fix several constants $\alpha \in [1, 2]$, $\beta, \bar{\beta} \geq 0$, $\gamma > 0$, $\delta \in [0, 1]$, $\lambda \in \mathbb{R}$ and an (\mathcal{F}_t) -progressively measurable \mathbb{R}_+ -valued stochastic process $(f_t)_{t \in [0, T]}$. We are concerned with the 1d BSDE with the generator g satisfying $d\mathbb{P} \times dt - a.e.$,

$$\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{sgn}(y)g(\omega, t, y, z) \leq f_t(\omega) + \beta|y|(\ln(e + |y|))^\delta + \gamma|z|^\alpha(\ln(e + |z|))^\lambda. \quad (1.2)$$

We usually say that g has a one-sided linear growth in the state variable y when $\delta = 0$, and a one-sided super-linear growth in y when $\delta \in (0, 1]$. Furthermore, for the case of $\lambda = 0$, we say that g has a power sub-linear growth in the state variable z when $\alpha \in (0, 1)$, a linear growth in z when $\alpha = 1$, a sub-quadratic growth in z when $\alpha \in (1, 2)$, a quadratic growth in z when $\alpha = 2$, a super-quadratic growth in z when $\alpha > 2$, and for the case of $\alpha = 1$ and $\lambda \neq 0$, we say that g has a logarithmic sub-linear growth in z when $\lambda < 0$, and a logarithmic super-linear growth in z when $\lambda > 0$.

1.1. Overview of relevant known results

It is well known that the study on BSDEs can be dated back to [15, 16] for linear cases. Nonlinear BSDEs were initially introduced in [77], where an existence and uniqueness result was established for adapted solutions to multidimensional BSDEs with square-integrable parameters and uniformly Lipschitz continuity generators. Since then, BSDEs have been intensively investigated due to its theoretical beauty together with its surprising links with many topics such as partial differential equations (PDEs in short), mathematical finance, stochastic control, nonlinear mathematical expectation and so on. Readers are referred to for example [80, 81, 82, 83, 37, 79, 65, 62, 60, 61, 78] for more details.

Particularly, much attentions have been paid on the well-posedness of adapted solutions to BSDEs under the different growth and/or continuity assumptions of the generator g with respect to the two unknown variables (y, z) and different integrability assumptions on the parameters (ξ, f) . Generally speaking, these efforts can be classified into three different directions. The first one focuses on the $L^p(p \geq 1)$ solution of BSDEs. Relevant classical results are available in [66, 76, 17, 18, 58, 57, 52, 59, 53, 50, 51, 40, 41, 42, 43, 92] for the BSDEs with generator g having a linear/sub-linear growth in the unknown variable z . Readers are also referred to [2, 3, 9, 68, 6, 8, 11, 5, 93] for the BSDEs with generator g having a super-linear growth in the unknown variable z . The second one is devoted to the bounded solution of BSDEs with generator g having a quadratic/super-quadratic growth in the unknown variable z , see for example [67, 65, 41, 22, 91, 32, 19, 14, 41, 63, 54] for more details. The last one aims to studying the weakest possible integrability condition on (ξ, f) for existence and uniqueness of an adapted solution to BSDEs when the generator g satisfies some certain growth and/or continuity conditions in (y, z) . To the best of knowledge, these efforts in this direction were originated from [20, 21, 33, 89, 34] for the quadratic BSDEs, and were developed in [64, 23, 44, 75] for the linear growing BSDEs, and recently became mature in [45, 47, 48, 49] for the BSDEs with generator g having sub-quadratic, super-linear and logarithmic sub-linear growth in the unknown variable z , and the so-called localization procedure, θ -difference technique and test function method were initially put forward among them for the existence

and uniqueness of BSDEs and will be developed in the present paper. More specifically, let us present an overview about related results on the well-posedness of BSDEs with generator g satisfying (1.2).

Firstly, suppose that the generator g has a one-sided linear growth in y and a linear growth in z , i.e., (1.2) with $\delta = 0$, $\alpha = 1$ and $\lambda = 0$ holds for g . It is well known that if the terminal condition $|\xi| + \int_0^T f_s ds \in L^p$ for some $p > 1$, then BSDE(ξ, g) admits a solution in $\mathcal{S}^p \times \mathcal{M}^p$, and the solution is unique when g further satisfies the uniformly Lipschitz continuity assumption in (y, z) . Readers are referred to [77, 37, 66, 18, 50] for more details. Recently, [64, 23, 44, 75] generalized this result and obtained existence of an unbounded solution to a linearly growing BSDE(ξ, g) by supposing that the terminal condition $|\xi| + \int_0^T f_s ds \in L \exp(\mu\sqrt{2\ln L})$ for some $\mu \geq \gamma\sqrt{T}$, which is weaker than L^p ($p > 1$)-integrability and stronger than $L \ln L$ -integrability. They also established uniqueness of the unbounded solution provided that g satisfies a monotonicity condition in y and the uniformly Lipschitz continuity condition in z . Generally speaking, the generator g can allow a general growth in y when g satisfies the monotonicity condition in y . Relevant works are available in [76, 17, 18, 69, 22, 51, 40, 70, 41, 72].

Secondly, suppose that the generator g has a one-sided linear growth in y and a power sub-linear growth in z , i.e., (1.2) with $\delta = 0$, $\alpha \in (0, 1)$ and $\lambda = 0$ is satisfied for g . It follows from Briand et al. [18] that if the terminal condition $|\xi| + \int_0^T f_s ds \in L^1$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y_t)_{t \in [0, T]}$ is of class (D), and the solution is unique when g further satisfies an extended monotonicity condition in y and the uniformly Lipschitz continuity condition in z . See for example [20, 41, 43] for more details. Very recently, this existence and uniqueness result was further strengthened in [48] to the case that the generator g has a one-sided linear growth in y and a logarithmic sub-linear growth in z , i.e., (1.2) with $\delta = 0$, $\alpha = 1$ and $\lambda \in (-\infty, -1/2)$ is satisfied for g , see also [49] for more extensive arguments.

Thirdly, suppose that the generator g has a one-sided linear/super-linear growth in y and a logarithmic super-linear growth in z , i.e., (1.2) with $\delta \in [0, 1]$, $\alpha = 1$ and $\lambda \in [0, +\infty)$ is satisfied for g . Let $p := \delta \vee (\lambda + \frac{1}{2}) \vee (2\lambda) \in [\frac{1}{2}, +\infty)$. Very recently, it was shown in [47] that if the terminal condition $|\xi| + \int_0^T f_s ds \in L \exp(\mu(\ln L)^p)$ for some $\mu > \mu_0$ with a certain value μ_0 , then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(|Y_t| \exp(\mu(t)(\ln(e + |Y_t|))^p))_{t \in [0, T]}$ is of class (D) for some nonnegative and increasing function $\mu(t)$ defined on $[0, T]$ with $\mu(T) = \mu$, and the solution is unique when the generator g further satisfies an extended monotonicity condition in y and a uniform continuity condition in z , or a convexity/concavity condition in (y, z) , see assumptions (UN1)-(UN3) in [47] for more details. Furthermore, [6, 11] verified existence of a solution to BSDE(ξ, g) in the space of $\mathcal{S}^p \times \mathcal{M}^2$ for some sufficiently large $p > 2$, when the terminal condition $|\xi| + \int_0^T f_s ds \in L^p$ and the generator g satisfies (1.2) with $\delta = 1$, $\lambda = 1/2$ and $|g(\omega, t, y, z)|$ instead of the left side of (1.2). They also proved uniqueness of the solution when g further satisfies a kind of local monotonicity condition in (y, z) . Related works on super-linearly growing BSDEs can be found in [2, 3, 9, 67, 7, 10, 8, 71], where the solution of BSDE(ξ, g) in the space of $\mathcal{S}^p \times \mathcal{M}^p$ is considered under the assumption that the terminal condition $|\xi| + \int_0^T f_s ds \in L^p$ for some $p > 1$, and several kinds of locally Lipschitz continuity conditions or local monotonicity conditions of g in (y, z) are usually used in order to guarantee uniqueness of the solution of BSDE(ξ, g).

Fourthly, suppose that the generator g has a one-sided linear growth in y and a sub-quadratic growth

in z , i.e., (1.2) with $\delta = 0$, $\alpha \in (1, 2)$ and $\lambda = 0$ is satisfied for g . Let α^* represent the conjugate of α . It was proved in [45] that if the terminal condition $|\xi| + \int_0^T f_s ds \in \exp(\mu L^{\frac{2}{\alpha^*}})$ for some $\mu > \mu_0$ with a certain value μ_0 , which is weaker than $\exp(\mu L)$ -integrability and stronger than L^p ($p > 1$)-integrability, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(\exp(\mu(t)|Y_t|^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ is of class (D) for some nonnegative and increasing function $\mu(t)$ defined on $[0, T]$ with $\mu(T) = \mu$, and the solution is unique when $|\xi| + \int_0^T f_s ds \in \exp(\mu L^{\frac{2}{\alpha^*}})$ for each $\mu > 0$ and the generator g further satisfies an extended convexity/concavity condition in (y, z) , see assumption (H2') in [45] for more details.

Finally, suppose that the generator g has a one-sided linear growth in y and a quadratic growth in z , i.e., (1.2) with $\delta = 0$, $\alpha = 2$ and $\lambda = 0$ is satisfied for g . It is well known from [65] that if the terminal condition $|\xi| + \int_0^T |f_s| ds \in L^\infty$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y_t)_{t \in [0, T]} \in \mathcal{S}^\infty$, and the solution is unique if g further satisfies the uniformly Lipschitz continuity condition in y and a locally Lipschitz condition in z . Readers are referred to [19, 41, 54, 73] for more details on the bounded solution of quadratic BSDEs. After that, [20, 21, 33, 34] strengthened this result and proved existence and uniqueness of an unbounded solution to quadratic BSDE(ξ, g) by supposing that $|\xi| + \int_0^T f_s ds \in \exp(\mu L)$ for $\mu := \gamma e^{\beta T}$, where g is required to be uniformly Lipschitz continuous in y and convex/concave in z for the uniqueness of the solution, see also [14, 46] for more details. It should be mentioned that a class of quadratic BSDEs with L^p ($p > 1$)-integrable terminal conditions are studied in several recent works, see for example [5, 93, 4, 12]. In addition, it is also well known from [32] that general speaking, super-quadratic BSDEs, i.e., (1.2) with $\delta = 0$, $\alpha > 2$ and $\lambda = 0$ holds for the generator g , are not solvable and the solution is not unique even though the solution exists. Some relevant solvability results under the Markovian setting are available in [32, 74, 89, 27].

1.2. Organization of the present paper

The present paper will develop a unified theory on the well-posedness of 1d BSDEs to deal with most of cases mentioned above. The rest of this paper is organized as follows. In section 2, we formulate the test function method and establish a general existence result (see Definition 2.1 and Theorem 2.2) by the a priori estimate and localization technique, based on which several existence theorems on the adapted solutions of 1d BSDEs (see Theorems 2.3 and 2.7) are proved for the logarithmic quasi-linear growth case and the sub-quadratic/quadratic growth case, respectively, by appropriately choosing the test functions. Section 3 focuses on the comparison theorems of the adapted solutions of 1d BSDEs for the no more than linear growth case (see Theorem 3.3) and the super-linear but no more than quadratic growth case (see Theorem 3.9), respectively, and establishes some existence and uniqueness results (see Theorems 3.12 and 3.13). We first establish an important a priori estimate (see Proposition 3.1) associated with the test function, and prove Theorems 3.3 and 3.9 by picking the proper test function and using the θ -difference technique, respectively. This yields naturally the desired uniqueness results. Some examples and remarks are also provided in the last two sections to illustrate our theoretical results, see Remarks 2.4 and 2.9 and Examples 2.5 and 2.8 in section 2 as well as Remarks 3.2, 3.8, 3.10 and 3.11, Examples 3.4 to 3.6 and Proposition 3.7 in section 3. In section 4, several practical applications of our results are introduced including the conditional g -expectation (see Definition 4.1 and Propositions 4.2 and 4.3), the

dynamic utility process (see Proposition 4.5 and Theorem 4.6), risk measure (Example 4.8) and nonlinear Feynman-Kac formula (see Definition 4.9 and Theorem 4.10), and some commentaries on known related works are also made, see Remarks 4.4, 4.7 and 4.11. Finally, in section 5 we list several open problems on 1d BSDEs to be further studied, and in Appendix we prove a key inequality (see Proposition 2.6) used in section 2, which is interesting in its own right.

1.3. Notations and spaces

In this subsection, we give some necessary notations and spaces used in this paper. Let $\mathbb{R}_+ := [0, +\infty)$. For $a, b \in \mathbb{R}$, we denote $a \wedge b := \min\{a, b\}$, $a^+ := \max\{a, 0\}$ and $a^- := -\min\{a, 0\}$, and $\text{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x\leq 0}$, where $\mathbf{1}_A$ is the indicator function of set A . Let \mathbf{S} denote the set of \mathbb{R}_+ -valued continuously differentiable functions $\phi(s, x)$ defined on $[0, T] \times \mathbb{R}_+$ such that $\phi_s(\cdot, \cdot) \geq 0$, $\phi_x(s, \cdot) > 0$ and $\phi_{xx}(s, \cdot) > 0$, where $\phi_s(\cdot, \cdot)$ denotes the first-order partial derivative of $\phi(\cdot, \cdot)$ with respect to the first variable, and by $\phi_x(\cdot, \cdot)$ and $\phi_{xx}(\cdot, \cdot)$ respectively the first-order and second order partial derivative of $\phi(\cdot, \cdot)$ with respect to the second variable. Let $\bar{\mathbf{S}}$ denote the set of \mathbb{R}_+ -valued functions $h(t, x, \bar{x})$ defined on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$ such that $h(t, \cdot, \bar{x})$ is nondecreasing for each $(t, \bar{x}) \in [0, T] \times \mathbb{R}_+$. Let $\mathcal{S}^\infty([0, T]; \mathbb{R})$ (or \mathcal{S}^∞) denote the set of (\mathcal{F}_t) -adapted and continuous bounded real processes $(Y_t)_{t \in [0, T]}$. For each $p > 0$, let $\mathcal{S}^p([0, T]; \mathbb{R})$ (or \mathcal{S}^p) be the set of (\mathcal{F}_t) -adapted and continuous real processes $(Y_t)_{t \in [0, T]}$ satisfying

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] \right)^{\frac{1}{p} \wedge 1} < +\infty,$$

and $\mathcal{M}^p([0, T]; \mathbb{R}^d)$ (or \mathcal{M}^p) the set of all (\mathcal{F}_t) -adapted \mathbb{R}^d -valued processes $(Z_t)_{t \in [0, T]}$ satisfying

$$\|Z\|_{\mathcal{M}^p} := \left\{ \mathbb{E} \left[\left(\int_0^T |Z_t|^2 dt \right)^{p/2} \right] \right\}^{\frac{1}{p} \wedge 1} < +\infty.$$

Denote by Σ_T the set of all (\mathcal{F}_t) -stopping times τ valued in $[0, T]$. For an (\mathcal{F}_t) -adapted real process $(X_t)_{t \in [0, T]}$, if the family $\{X_\tau : \tau \in \Sigma_T\}$ is uniformly integrable, then we call that it is of class (D).

Now, fix $t \in [0, T]$. For $p, \mu > 0$, we denote by $L^p(\mathcal{F}_t)$ and $L^\infty(\mathcal{F}_t)$ the set of \mathcal{F}_t -measurable real random variables ξ such that $\mathbb{E}[|\xi|^p] < +\infty$ and $|\xi| \leq M$ for some real $M > 0$, respectively, and denote the following three spaces of \mathcal{F}_t -measurable real random variables:

$$L(\ln L)^p(\mathcal{F}_t) := \{ \xi \in \mathcal{F}_t \mid \mathbb{E} [|\xi| (\ln(e + |\xi|))^p] < +\infty \},$$

$$L \exp[\mu(\ln L)^p](\mathcal{F}_t) := \{ \xi \in \mathcal{F}_t \mid \mathbb{E} [|\xi| \exp(\mu(\ln(e + |\xi|))^p)] < +\infty \}$$

and

$$\exp(\mu L^p)(\mathcal{F}_t) := \{ \xi \in \mathcal{F}_t \mid \mathbb{E} [\exp(\mu|\xi|^p)] < +\infty \}.$$

It is clear that for each $0 < p < q$ and $0 < \bar{\mu}, \tilde{\mu} < \mu$, we have

$$L^\infty(\mathcal{F}_t) \subset L^q(\mathcal{F}_t) \subset L^p(\mathcal{F}_t), \quad L(\ln L)^q(\mathcal{F}_t) \subset L(\ln L)^p(\mathcal{F}_t),$$

$$L \exp[\mu(\ln L)^q](\mathcal{F}_t) \subset L \exp[\bar{\mu}(\ln L)^q](\mathcal{F}_t) \subset L \exp[\tilde{\mu}(\ln L)^p](\mathcal{F}_t),$$

and

$$\exp(\mu L^q)(\mathcal{F}_t) \subset \exp(\bar{\mu} L^q)(\mathcal{F}_t) \subset \exp(\tilde{\mu} L^p)(\mathcal{F}_t).$$

It can also be verified that for each $\mu, \bar{\mu}, r > 0$ and $0 < p < 1 < q$, we have

$$L^\infty(\mathcal{F}_t) \subset \exp(\mu L^r)(\mathcal{F}_t) \subset L \exp[\bar{\mu}(\ln L)^q](\mathcal{F}_t) \subset L \exp[\mu \ln L](\mathcal{F}_t) = L^{1+\mu}(\mathcal{F}_t)$$

and

$$L^q(\mathcal{F}_t) \subset L \exp[\mu(\ln L)^p](\mathcal{F}_t) \subset L(\ln L)^r(\mathcal{F}_t) \subset L^1(\mathcal{F}_t).$$

Furthermore, we would like to mention that for each $p, \mu > 0$ and $0 < \bar{p} \leq 1 < \tilde{p}$, the spaces

$$L(\ln L)^p(\mathcal{F}_t), \quad L \exp[\mu(\ln L)^{\bar{p}}](\mathcal{F}_t), \quad \bigcup_{\bar{\mu} > \mu} L \exp[\bar{\mu}(\ln L)^{\tilde{p}}](\mathcal{F}_t) \quad \text{and} \quad \bigcap_{\bar{\mu} > 0} \exp(\bar{\mu} L^p)(\mathcal{F}_t)$$

are all the Orlicz hearts corresponding, respectively, to the following Young functions

$$x(\ln(e+x))^p, \quad x \exp[\mu(\ln(e+x))^{\bar{p}}], \quad x \exp[\mu(\ln(e+x))^{\tilde{p}}] \quad \text{and} \quad \exp(x^p) - 1.$$

More details on the Orlicz space, the Orlicz class and the Orlicz heart can be referred to [36, 26]. Finally, it should be noted that in all notations of the spaces on random variables, the expression of (\mathcal{F}_T) at the end of the above notations is usually omitted without causing confusion.

2. Existence results

2.1. The test function method and a general existence result

Let us first introduce the following assumptions on the generator g .

(EX1) $d\mathbb{P} \times dt - a.e.$, $g(\omega, t, \cdot, \cdot)$ is continuous.

(EX2) There exist two \mathbb{R}_+ -valued functions $H(\omega, t, x)$ and $\Gamma(\omega, t, x)$ defined on $\Omega \times [0, T] \times \mathbb{R}_+$, which are (\mathcal{F}_t) -progressively measurable for each $x \in \mathbb{R}_+$ and nondecreasing with respect to the variable x , such that $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|g(\omega, t, y, z)| \leq H(\omega, t, |y|) + \Gamma(\omega, t, |y|)|z|^2,$$

and $\mathbb{P} - a.s.$, for each $x \in \mathbb{R}_+$, $\Gamma(\omega, \cdot, x)$ is left-continuous on $[0, T]$ and

$$\int_0^T H(\omega, t, x) dt + \sup_{t \in [0, T]} \Gamma(\omega, t, x) < +\infty.$$

(EX3) There exists a function $h(\cdot, \cdot, \cdot) \in \bar{\mathbf{S}}$ such that $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\text{sgn}(y)g(\omega, t, y, z) \leq f_t(\omega) + h(t, |y|, |z|).$$

Definition 2.1. Assume that the generator g satisfies assumption (EX3). A function $\varphi(\cdot, \cdot) \in \mathbf{S}$ is called a test function for g or h if it satisfies that for each $(s, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$,

$$-\varphi_x(s, x)h(s, x, \bar{x}) + \frac{1}{2}\varphi_{xx}(s, x)|\bar{x}|^2 + \varphi_s(s, x) \geq 0. \quad (2.1)$$

Theorem 2.2. Assume that ξ is an \mathcal{F}_T -measurable random variable and the generator g satisfies the above assumptions (EX1)-(EX3). If there exists a test function $\varphi(\cdot, \cdot) \in \mathbf{S}$ for g such that

$$\mathbb{E} \left[\varphi \left(T, |\xi| + \int_0^T f_s ds \right) \right] < +\infty, \quad (2.2)$$

then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(\varphi(t, |Y_t| + \int_0^t f_s ds))_{t \in [0, T]}$ is of class (D). Furthermore, the process $(\varphi(t, |Y_t| + \int_0^t f_s ds))_{t \in [0, T]}$ is a sub-martingale on $[0, T]$. In particular, we have

$$\varphi \left(t, |Y_t| + \int_0^t f_s ds \right) \leq \mathbb{E} \left[\varphi \left(T, |\xi| + \int_0^T f_s ds \right) \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (2.3)$$

Proof. The whole proof will be divided into two steps.

Step 1. We prove that under the assumptions of Theorem 2.2, if $|\xi| + \int_0^T f_s ds \in L^\infty$ and $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^\infty([0, T]; \mathbb{R}) \times \mathcal{M}^2([0, T]; \mathbb{R}^d)$ is a solution of BSDE(ξ, g), then (2.3) holds. Define

$$\bar{Y}_t := |Y_t| + \int_0^t f_s ds \quad \text{and} \quad \bar{Z}_t := \text{sgn}(Y_t)Z_t, \quad t \in [0, T].$$

It follows from Itô-Tanaka's formula that

$$\bar{Y}_t = \bar{Y}_T + \int_t^T (\text{sgn}(Y_s)g(s, Y_s, Z_s) - f_s) ds - \int_t^T \bar{Z}_s \cdot dB_s - \int_t^T dL_s, \quad t \in [0, T],$$

where L is the local time of Y at 0. Applying Itô-Tanaka's formula to the process $\varphi(s, \bar{Y}_s)$ and using assumption (EX3) yield that for each $s \in [0, T]$,

$$\begin{aligned} d\varphi(s, \bar{Y}_s) &= \varphi_x(s, \bar{Y}_s) (-\text{sgn}(Y_s)g(s, Y_s, Z_s) + f_s) ds + \varphi_x(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s \\ &\quad + \varphi_x(s, \bar{Y}_s) dL_s + \frac{1}{2} \varphi_{xx}(s, \bar{Y}_s) |\bar{Z}_s|^2 ds + \varphi_s(s, \bar{Y}_s) ds \\ &\geq \left[-\varphi_x(s, \bar{Y}_s) h(s, |Y_s|, |Z_s|) + \frac{1}{2} \varphi_{xx}(s, \bar{Y}_s) |Z_s|^2 + \varphi_s(s, \bar{Y}_s) \right] ds + \varphi_x(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s. \end{aligned}$$

Then, in view of $|Y_s| \leq \bar{Y}_s$ and the fact that $h(t, \cdot, \bar{x})$ is nondecreasing, by (2.1) we know that

$$d\varphi(s, \bar{Y}_s) \geq \varphi_x(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s, \quad s \in [0, T],$$

which means that

$$\varphi(T, \bar{Y}_T) - \varphi(t, \bar{Y}_t) \geq \int_t^T \varphi_x(s, \bar{Y}_s) \bar{Z}_s \cdot dB_s, \quad t \in [0, T].$$

Thus, since $|\xi| + \int_0^T f_s ds \in L^\infty$ and $(Y_t, Z_t)_{t \in [0, T]} \in \mathcal{S}^\infty([0, T]; \mathbb{R}) \times \mathcal{M}^2([0, T]; \mathbb{R}^d)$, by taking the conditional expectation with respect to \mathcal{F}_t on both sides of the above inequalities we get (2.3).

Step 2. Based on step 1, we use the localization procedure used initially in Briand and Hu [20] to construct the desired solution. For each $n, p \geq 1$ and $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$, denote

$$\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p \quad \text{and} \quad g^{n,p}(\omega, t, y, z) := g^+(\omega, t, y, z) \wedge n - g^-(\omega, t, y, z) \wedge p. \quad (2.4)$$

It is clear that $|\xi^{n,p}| \leq |\xi| \wedge (n \vee p)$ and $|g^{n,p}| \leq |g| \wedge (n \vee p)$ for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. It can also be verified that the generator $g^{n,p}$ satisfies assumptions (EX1)-(EX3) with $f \wedge (n \vee p)$ instead of f . Then,

according to [65], the following BSDE $(\xi^{n,p}, g^{n,p})$ has a maximal bounded solution $(Y_t^{n,p}, Z_t^{n,p})_{t \in [0, T]}$ in the space of processes $\mathcal{S}^\infty([0, T]; \mathbb{R}) \times \mathcal{M}^2([0, T]; \mathbb{R}^d)$:

$$Y_t^{n,p} = \xi^{n,p} + \int_t^T g^{n,p}(s, Y_s^{n,p}, Z_s^{n,p}) ds - \int_t^T Z_s^{n,p} \cdot dB_s, \quad t \in [0, T]. \quad (2.5)$$

It follows from the comparison theorem that $(Y_t^{n,p})_{t \in [0, T]}$ is nondecreasing in n and non-increasing in p . Furthermore, by step 1 we know that for each $n, p \geq 1$,

$$\begin{aligned} \varphi \left(t, |Y_t^{n,p}| + \int_0^t [f_s \wedge (n \vee p)] ds \right) &\leq \mathbb{E} \left[\varphi \left(T, |\xi^{n,p}| + \int_0^T [f_s \wedge (n \vee p)] ds \right) \middle| \mathcal{F}_t \right] \\ &\leq \mathbb{E} \left[\varphi \left(T, |\xi| + \int_0^T f_s ds \right) \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \end{aligned} \quad (2.6)$$

Now, for each pair of integers $m, l \geq 1$, we define the following stopping times:

$$\tau_m := \inf \left\{ t \in [0, T] : \mathbb{E} \left[\varphi \left(T, |\xi| + \int_0^T f_s ds \right) \middle| \mathcal{F}_t \right] \geq \varphi(t, m) \right\} \wedge T$$

and

$$\sigma_{m,l} := \inf \left\{ t \in [0, T] : \int_0^t H(s, m) ds + \sup_{s \in [0, t]} \Gamma(s, m) \geq l \right\} \wedge \tau_m$$

with the convention that $\inf \emptyset = +\infty$. Then $(Y_{m,l}^{n,p}(t), Z_{m,l}^{n,p}(t))_{t \in [0, T]} := (Y_{t \wedge \sigma_{m,l}}^{n,p}, Z_{t \wedge \sigma_{m,l}}^{n,p} \mathbf{1}_{t \leq \sigma_{m,l}})_{t \in [0, T]}$ is a solution in the space of processes $\mathcal{S}^\infty([0, T]; \mathbb{R}) \times \mathcal{M}^2([0, T]; \mathbb{R}^d)$ to the following BSDE:

$$Y_{m,l}^{n,p}(t) = Y_{\sigma_{m,l}}^{n,p} + \int_t^{\sigma_{m,l}} \mathbf{1}_{s \leq \sigma_{m,l}} g^{n,p}(s, Y_{m,l}^{n,p}(s), Z_{m,l}^{n,p}(s)) ds - \int_t^{\sigma_{m,l}} Z_{m,l}^{n,p}(s) \cdot dB_s, \quad t \in [0, T].$$

Observe that for each fixed $m, l \geq 1$, $(Y_{m,l}^{n,p}(t))_{t \in [0, T]}$ is nondecreasing in n and non-increasing in p , and that $d\mathbb{P} \times dt - a.e.$, $(g^{n,p})_{n,p}$ converges locally uniformly in (y, z) to g as $n, p \rightarrow \infty$. Since $\varphi(t, \cdot)$ is nondecreasing for each $t \in [0, T]$, by (2.6) together with the definitions of τ_m and $\sigma_{m,l}$ we can obtain that $d\mathbb{P} \times dt - a.e.$,

$$\sup_{n,p \geq 1} |Y_{m,l}^{n,p}(t)| \leq m.$$

Furthermore, since $|g^{n,p}| \leq |g|$ and g satisfies assumption (EX2), we know that $d\mathbb{P} \times ds - a.e.$,

$$\forall (y, z) \in [-m, m] \times \mathbb{R}^d, \quad \sup_{n,p \geq 1} (\mathbf{1}_{s \leq \sigma_{m,l}} |g^{n,p}(s, y, z)|) \leq \mathbf{1}_{s \leq \sigma_{m,l}} H(s, m) + l|z|^2$$

with $\int_0^T \mathbf{1}_{s \leq \sigma_{m,l}} H(s, m) ds \leq l$. Thus, for each fixed $m, l \geq 1$, we can apply the stability result for the bounded solutions of BSDEs (see for example Proposition 3.1 in [73]). Setting $Y_{m,l}(t) := \inf_p \sup_n Y_{t \wedge \sigma_{m,l}}^{n,p}$, then $(Y_{m,l}(t))_{t \in [0, T]}$ is continuous and there exists a $(Z_{m,l}(t))_{t \in [0, T]}$ such that $\lim_{n,p \rightarrow \infty} Z_{t \wedge \sigma_{m,l}}^{n,p} \mathbf{1}_{t \leq \sigma_{m,l}} = Z_{m,l}(t)$ in $\mathcal{M}^2([0, T]; \mathbb{R}^d)$ and

$$Y_{m,l}(t) = \inf_{p \geq 1} \sup_{n \geq 1} Y_{\sigma_{m,l}}^{n,p} + \int_t^{\sigma_{m,l}} \mathbf{1}_{s \leq \sigma_{m,l}} g(s, Y_{m,l}(s), Z_{m,l}(s)) ds - \int_t^{\sigma_{m,l}} Z_{m,l}(s) \cdot dB_s, \quad t \in [0, T].$$

Finally, in view of the above equation and by virtue of the stability of stopping times τ_m and $\sigma_{m,l}$ and the facts that $d\mathbb{P} \times dt - a.e.$, for each $m, l \geq 1$,

$$Y_{m+1, l+1}(t \wedge \sigma_{m,l}) = Y_{m, l+1}(t \wedge \sigma_{m,l}) = Y_{m,l}(t \wedge \sigma_{m,l}) = \inf_{p \geq 1} \sup_{n \geq 1} Y_{t \wedge \sigma_{m,l}}^{n,p}$$

and

$$Z_{m+1,l+1}\mathbf{1}_{t \leq \sigma_{m,l}} = Z_{m,l+1}\mathbf{1}_{t \leq \sigma_{m,l}} = Z_{m,l}\mathbf{1}_{t \leq \sigma_{m,l}} = \lim_{n,p \rightarrow \infty} Z_t^{n,p}\mathbf{1}_{t \leq \sigma_{m,l}},$$

we can check that $(Y_t, Z_t)_{t \in [0, T]}$ is an adapted solution to BSDE (ξ, g) , where for each $t \in [0, T]$,

$$Y_t := \inf_p \sup_n Y_t^{n,p} \quad \text{and} \quad Z_t := \sum_{m=1}^{+\infty} \left(\sum_{l=1}^{+\infty} Z_{m,l}(t) \mathbf{1}_{t \in [\sigma_{m,l-1}, \sigma_{m,l})} \right) \mathbf{1}_{t \in [\tau_{m-1}, \tau_m)}$$

with $\tau_0 := 0$ and $\sigma_{m,0} := 0$ for each $m \geq 1$. And, (2.3) follows from (2.6) by sending n and p to infinity. In addition, according to (2.5), similar to step 1 we can also verify that for each $n, p \geq 1$ and $0 \leq t \leq r \leq T$,

$$\varphi \left(t, |Y_t^{n,p}| + \int_0^t [f_s \wedge (n \vee p)] ds \right) \leq \mathbb{E} \left[\varphi \left(r, |Y_r^{n,p}| + \int_0^r [f_s \wedge (n \vee p)] ds \right) \middle| \mathcal{F}_t \right].$$

Thus, in view of (2.6), by sending n and p to infinity and using Lebesgue's dominated convergence theorem in the above inequality we can conclude that the process $(\varphi(t, |Y_t| + \int_0^t f_s ds))_{t \in [0, T]}$ is indeed a sub-martingale. The proof is then complete. \square

As applications of Theorem 2.2, we will prove the following Theorem 2.3 and Theorem 2.7, where the function $h(\cdot, \cdot, \cdot)$ in (EX3) are picked as the following specific form:

$$h(t, x, \bar{x}) := \beta x (\ln(e + x))^\delta + \gamma \bar{x}^\alpha (\ln(e + \bar{x}))^\lambda, \quad (t, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+. \quad (2.7)$$

These two theorems can be compared with some known existence results on adapted solutions to one-dimensional BSDEs obtained for example in [65, 20, 64, 44, 45, 48, 47].

2.2. The logarithmic quasi-linear growth case

Let us first consider the case that the generator g has a logarithmic quasi-linear growth in the unknown variables (y, z) , i.e., the case of $\alpha = 1$ in (2.7).

Theorem 2.3. *Assume that ξ is an \mathcal{F}_T -measurable random variable and the generator g satisfies (EX1)-(EX3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\alpha = 1$. Then, the following assertions hold.*

(i) *Let $\delta = 0$ and $\lambda \in (-\infty, -\frac{1}{2})$. If $\xi + \int_0^T f_s ds \in L^1$, then BSDE (ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(Y_t)_{t \in [0, T]}$ is of class (D);*

(ii) *Let $\delta = 0$ and $\lambda = -\frac{1}{2}$. If $\xi + \int_0^T f_s ds \in L(\ln L)^p$ for some $p > 0$, then BSDE (ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(|Y_t|(\ln(e + |Y_t|))^p)_{t \in [0, T]}$ is of class (D);*

(iii) *If $p := \delta \vee (\lambda + \frac{1}{2}) \vee (2\lambda) \in (0, +\infty)$ and $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} L \exp[\mu(\ln L)^p]$, then BSDE (ξ, g) has a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(|Y_t| \exp(\mu(\ln(e + |Y_t|))^p))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$;*

(iv) *Let $\delta = 0$ and $\lambda = 0$. If $\xi + \int_0^T f_s ds \in L^p$ for some $p > 1$, then BSDE (ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(|Y_t|^p)_{t \in [0, T]}$ is of class (D).*

Remark 2.4. *It is worth noting that when the growth conditions of the generator g in the unknown variables (y, z) becomes weaker, some stronger integrability conditions on $\xi + \int_0^T f_s ds$ are required to ensure the existence of a solution of BSDE (ξ, g) . In addition, to the best of our knowledge, the case of $\lambda \in [-\frac{1}{2}, 0)$ is at the first time studied in (ii) and (iii) of Theorem 2.3.*

Example 2.5. Consider the following simple BSDE:

$$Y_t = \xi + \int_t^T (f_s + \beta|Y_s| + \gamma|Z_s|)ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (2.8)$$

where $\xi \geq 0$ and $f \in L^1(0, T)$. It can be regarded as a special case of $\delta = 0$, $\lambda = 0$ and $\alpha = 1$ in (2.7). It follows from Theorem 2.1 in [64] that BSDE (2.8) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $Y \geq 0$ if and only if there exists a locally bounded process \bar{Y} such that

$$\operatorname{ess\,sup}_{q \in \mathcal{A}} \left\{ \mathbb{E}_q \left[e^{\beta(T-t)} |\xi| \middle| \mathcal{F}_t \right] \right\} + \int_t^T e^{\beta(s-t)} f_s ds \leq \bar{Y}_t,$$

where \mathcal{A} is the set of (\mathcal{F}_t) -progressively measurable \mathbb{R}^d -valued process $(q_t)_{t \in [0, T]}$ such that $|q| \leq \gamma$, and

$$\frac{d\mathbb{Q}^q}{d\mathbb{P}} := M_T^q$$

with

$$M_t^q := \exp \left\{ \int_0^t q_s \cdot dB_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right\}, \quad t \in [0, T]$$

and \mathbb{E}_q is the expectation with respect to \mathbb{Q}^q . In particular, if BSDE (2.8) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $Y \geq 0$, then $\xi e^{\gamma|B_T|} \in L^1$.

The following specific example comes from Example 2.3 in [64]. Set $d = 1$, $T = 1$, $\gamma = 1$ and the terminal variable $\xi := e^{\frac{1}{2}(|B_1| - 1)^2} - 1$. Then, BSDE (2.8) does not admit a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $Y \geq 0$, as $\xi e^{|B_T|}$ does not belong to L^1 by the following calculate:

$$\mathbb{E} \left[\xi e^{|B_T|} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{1}{2}(|x| - 1)^2} e^{|x|} e^{-\frac{1}{2}x^2} dx = +\infty.$$

Whereas it can be directly checked that this ξ belongs to the space of $\cap_{0 < \mu < \sqrt{2}} L \exp[\mu(\ln L)^p]$ and then $\cap_{q > 0} L(\ln L)^q$, but does not belong to $L \exp[\sqrt{2}(\ln L)^p]$, where $p = \frac{1}{2}$ is defined in (iii) of Theorem 2.3.

Furthermore, it has been shown in [64, 44] that in order to get a solution to a linearly growing BSDE (ξ, g) , an $L \exp(\mu\sqrt{\ln L})$ -integrability condition of $\xi + \int_0^T f_s ds$ for $\mu = \gamma\sqrt{2T}$ is sufficient, but it is not enough for any $0 < \mu < \gamma\sqrt{2T}$, which can not follow from (iii) of Theorem 2.3.

To prove Theorem 2.3, we introduce the following Proposition 2.6, whose proof is given in Appendix.

Proposition 2.6. For each $p > 1$ and $\lambda \in \mathbb{R}$, there exists a sufficiently large positive constant $k_{\lambda, p} \geq e$ depending only on (λ, p) such that for each $k \geq k_{\lambda, p}$,

$$2xy (\ln(k + y))^\lambda \leq px^2 (\ln(k + x))^{2\lambda} + y^2, \quad \forall x, y > 0. \quad (2.9)$$

In particular, when $p = 1$, the constant k such that (2.9) holds does not exist except $\lambda = 0$.

Proof of Theorem 2.3. Since the generator g satisfies (EX1)-(EX3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\alpha = 1$, a function $\varphi(\cdot, \cdot) \in \mathbf{S}$ is a test function for g if for each $(s, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$, it holds that

$$-\varphi_x(s, x) \left(\beta x (\ln(e + x))^\delta + \gamma \bar{x} (\ln(e + \bar{x}))^\lambda \right) + \frac{1}{2} \varphi_{xx}(s, x) |\bar{x}|^2 + \varphi_s(s, x) \geq 0.$$

It follows from Proposition 2.6 with $p = 2$ that there exists a sufficient positive constant $k_\lambda \geq e$ depending only on λ such that for each $k \geq k_\lambda$ and $(s, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{aligned} & -\gamma\varphi_x(s, x)\bar{x}(\ln(k + \bar{x}))^\lambda + \frac{1}{2}\varphi_{xx}(s, x)|\bar{x}|^2 \\ &= \frac{\varphi_{xx}(s, x)}{2} \left(-2\frac{\gamma\varphi_x(s, x)}{\varphi_{xx}(s, x)}\bar{x}(\ln(k + \bar{x}))^\lambda + |\bar{x}|^2 \right) \\ &\geq -\frac{\gamma^2(\varphi_x(s, x))^2}{\varphi_{xx}(s, x)} \left(\ln \left(k + \frac{\gamma\varphi_x(s, x)}{\varphi_{xx}(s, x)} \right) \right)^{2\lambda}. \end{aligned}$$

Thus, if a function $\varphi(\cdot, \cdot) \in \mathbf{S}$ satisfies that for some $k \geq k_\lambda \geq e$ and each $(s, x) \in [0, T] \times \mathbb{R}_+$,

$$-\beta\varphi_x(s, x)(k + x)(\ln(k + x))^\delta - \frac{\gamma^2(\varphi_x(s, x))^2}{\varphi_{xx}(s, x)} \left(\ln \left(k + \frac{\gamma\varphi_x(s, x)}{\varphi_{xx}(s, x)} \right) \right)^{2\lambda} + \varphi_s(s, x) \geq 0, \quad (2.10)$$

then it is a test function for the generator g .

(i) Let $\delta = 0$ and $\lambda \in (-\infty, -\frac{1}{2})$. By a similar computation as in [48], one can verify that for each $k \geq k_\lambda$ sufficient large and each $c \geq 2\beta - \frac{8\gamma^2}{1+2\lambda}$, the following function

$$\varphi(s, x) = (k + x) \left(1 - (\ln(k + x))^{1+2\lambda} \right) \exp(cs), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

satisfies the inequality (2.10) with $\delta = 0$ and $\lambda < -1/2$, and hence is a test function for the generator g .

Note that for each $s \in [0, T]$,

$$\lim_{x \rightarrow +\infty} \frac{\varphi(s, x)}{x} = \exp(cs) \in [1, \exp(cT)].$$

It follows from Theorem 2.2 that if $\xi + \int_0^T f_s ds \in L^1$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(Y_t)_{t \in [0, T]}$ is of class (D).

(ii) Let $\delta = 0$ and $\lambda = -\frac{1}{2}$. It is not very hard to verify that for each $p > 0$, $k \geq k_\lambda$ sufficient large and $c \geq 2\beta + \frac{4\gamma^2}{p}$, the following function

$$\varphi(s, x) = (k + x) (\ln(k + x))^p \exp(cs), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

satisfies the inequality (2.10) with $\delta = 0$ and $\lambda = -1/2$, and hence is a test function for the generator g .

Note that for each $s \in [0, T]$,

$$\lim_{x \rightarrow +\infty} \frac{\varphi(s, x)}{x (\ln(e + x))^p} = \exp(cs) \in [1, \exp(cT)].$$

It follows from Theorem 2.2 that if $\xi + \int_0^T f_s ds \in L(\ln L)^p$ for some $p > 0$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(|Y_t|(\ln(e + |Y_t|))^p)_{t \in [0, T]}$ is of class (D).

(iii) Let $p := \delta \vee (\lambda + \frac{1}{2}) \vee (2\lambda) \in (0, +\infty)$. By a similar computation as in [47], one can verify that for each $k \geq k_\lambda$ sufficient large, $c_1 \geq 1$ and $c_2 \geq (p + 1)\beta - 4^{\lambda^+} \gamma^2$, the following function

$$\varphi(s, x) = (k + x) \exp(c_1 \exp(c_2 s) (\ln(k + x))^p), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

satisfies the inequality (2.10) with $p > 0$, and hence is a test function for the generator g . Note that for each $s \in [0, T]$,

$$\lim_{x \rightarrow +\infty} \frac{\varphi(s, x)}{x \exp(c_1 \exp(c_2 s) (\ln(e + x))^p)} = 1.$$

It follows from Theorem 2.2 that if $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} L \exp[\mu(\ln L)^p]$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that for each $\mu > 0$, $(|Y_t| \exp(\mu(\ln(e + |Y_t|))^p))_{t \in [0, T]}$ is of class (D).

(iv) Let $\delta = 0$ and $\lambda = 0$. It is easy to verify that for each $p > 1$, $k \geq k_\lambda$ and $c \geq p\beta + \frac{p}{p-1}\gamma^2$, the following function

$$\varphi(s, x) = (k + x)^p \exp(cs), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

satisfies the inequality (2.10) with $\delta = 0$ and $\lambda = 0$, and hence is a test function for the generator g . Note that for each $s \in [0, T]$,

$$\lim_{x \rightarrow +\infty} \frac{\varphi(s, x)}{x^p} = \exp(cs) \in [1, \exp(cT)].$$

It follows from Theorem 2.2 that if $\xi + \int_0^T f_s ds \in L^p$ for some $p > 1$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(|Y_t|^p)_{t \in [0, T]}$ is of class (D). \square

2.3. The sub-quadratic/quadratic growth case

Let us further consider the case that the generator g has a one-sided linear growth in the unknown variable y and has a super-linear and no more than quadratic growth in the unknown variable z .

Theorem 2.7. *Assume that ξ is an \mathcal{F}_T -measurable random variable and the generator g satisfies assumptions (EX1)-(EX3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\delta = 0$, $\lambda = 0$ and $\alpha \in (1, 2]$. Let α^* denote the conjugate of α , i.e.,*

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1.$$

If $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^}})$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(\exp(\mu|Y_t|^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$. In particular, If $\xi + \int_0^T f_s ds \in L^\infty$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y_t)_{t \in [0, T]} \in \mathcal{S}^\infty([0, T]; \mathbb{R})$.*

Example 2.8. *Let us consider the following well known BSDE:*

$$Y_t = \xi + \int_t^T \frac{1}{2} |Z_s|^2 ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T]. \quad (2.11)$$

The change of variables leads to the equation

$$e^{Y_t} = e^\xi - \int_t^T e^{Y_s} Z_s \cdot dB_s, \quad t \in [0, T],$$

which has a solution as soon as $e^\xi \in L^1$. On the other hand, since $\{e^{Y_t}\}_{t \in [0, T]}$ is a positive supermartingale, it has been proved in Theorem 3.1 of [22] that the condition of $e^\xi \in L^1$ is also the necessary condition to obtain a solution for this BSDE.

Furthermore, we consider the case that the generator g satisfies assumptions (EX1)-(EX3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\delta = 0$, $\lambda = 0$ and $\alpha = 2$. It has been shown in [20] that in this case, in order to get a solution to BSDE(ξ, g), an $\exp(\mu L)$ -integrability condition of $\xi + \int_0^T f_s ds$ for $\mu = 2\gamma e^{\beta T}$ is sufficient, but it is not enough for any $0 < \mu < 2\gamma e^{\beta T}$, which can not follow from Theorem 2.7.

Proof of Theorem 2.7. We distinguish two cases of $\alpha = 2$ and $\alpha \in (1, 2)$ to prove this theorem.

(i) The case of $\alpha = 2$. In this case, a function $\varphi(\cdot, \cdot) \in \mathbf{S}$ is a test function for g if for each $(s, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$, it holds that

$$-\varphi_x(s, x) (\beta x + \gamma \bar{x}^2) + \frac{1}{2} \varphi_{xx}(s, x) |\bar{x}|^2 + \varphi_s(s, x) \geq 0.$$

It is easy to verify that for each $c_1 \geq 2\gamma$ and $c_2 \geq \beta$, the following function

$$\varphi(s, x) = \exp(c_1 \exp(c_2 s) x), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

satisfies the last inequality, and hence is a test function for the generator g . It follows from Theorem 2.2 that if $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} \exp(\mu L)$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(\exp(\mu |Y_t|))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$, which is the desired assertion since $\alpha^* = 2$ in this case.

(ii) The case of $\alpha \in (1, 2)$. In this case, a function $\varphi(\cdot, \cdot) \in \mathbf{S}$ is a test function for g if for each $(s, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$, it holds that

$$-\varphi_x(s, x) (\beta x + \gamma \bar{x}^\alpha) + \frac{1}{2} \varphi_{xx}(s, x) |\bar{x}|^2 + \varphi_s(s, x) \geq 0.$$

It follows from Young's inequality that for each $(s, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{aligned} & -\gamma \varphi_x(s, x) \bar{x}^\alpha + \frac{1}{2} \varphi_{xx}(s, x) |\bar{x}|^2 \\ &= \varphi_{xx}(s, x) \left(-\frac{\gamma \varphi_x(s, x)}{\varphi_{xx}(s, x)} \bar{x}^\alpha + \frac{1}{2} |\bar{x}|^2 \right) \\ &\geq -\frac{2-\alpha}{2\alpha} \cdot \frac{(\alpha \gamma \varphi_x(s, x))^{\frac{2}{2-\alpha}}}{(\varphi_{xx}(s, x))^{\frac{\alpha}{2-\alpha}}} \geq -\frac{(2\gamma \varphi_x(s, x))^{\frac{2}{2-\alpha}}}{(\varphi_{xx}(s, x))^{\frac{2}{2-\alpha}}}. \end{aligned}$$

Thus, if a function $\varphi(\cdot, \cdot) \in \mathbf{S}$ satisfies that for each $(s, x) \in [0, T] \times \mathbb{R}_+$,

$$-\beta \varphi_x(s, x) x - \frac{(2\gamma \varphi_x(s, x))^{\frac{2}{2-\alpha}}}{(\varphi_{xx}(s, x))^{\frac{2}{2-\alpha}}} + \varphi_s(s, x) \geq 0, \quad (2.12)$$

then it is a test function for the generator g . Furthermore, by a similar computation as in [45], it can be verified that for each $c_1 \geq 1$, $k \geq k_{\alpha, c_1}$ with k_{α, c_1} being a positive constant depending only on α and c_1 , and $c_2 \geq \beta + (1 + c_1) 2^{\frac{6}{2-\alpha}} (2\alpha - 2)^{\frac{2-2\alpha}{2-\alpha}} \gamma^{\frac{2}{2-\alpha}}$, the following function

$$\varphi(s, x) = \exp\left(c_1 \exp(c_2 s) (x + k)^{\frac{2}{\alpha^*}}\right), \quad (s, x) \in [0, T] \times \mathbb{R}_+$$

satisfies the inequality (2.12), and hence is a test function for the generator g . Note that

$$\exp\left(x^{\frac{2}{\alpha^*}}\right) \leq \varphi(s, x) \leq \exp\left(c_1 \exp(c_2 T) k^{\frac{2}{\alpha^*}}\right) \exp\left(c_1 \exp(c_2 T) x^{\frac{2}{\alpha^*}}\right), \quad \forall (s, x) \in [0, T] \times \mathbb{R}_+.$$

It follows from Theorem 2.2 that if $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}})$, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(\exp(\mu |Y_t|^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$.

Finally, by (2.3) it can be concluded that if $\xi + \int_0^T f_s ds \in L^\infty$, then $(\varphi(t, |Y_t| + \int_0^t f_s ds))_{t \in [0, T]}$ is a bounded process, and then $(Y_t)_{t \in [0, T]} \in \mathcal{S}^\infty([0, T]; \mathbb{R})$. The proof is complete. \square

Remark 2.9. Some more accurate integrability conditions on the terminal variable $\xi + \int_0^T f_t dt$ can be explored to ensure the existence of an adapted solution to BSDE(ξ, g) by picking more appropriate test functions different from those in the proof of Theorem 2.3 and Theorem 2.7. The readers are referred to [20, 44, 45, 47] for more details.

3. Comparison theorems and existence and uniqueness results

In this section, we will establish two comparison theorems under different growth conditions of the generator g in the unknown variables (y, z) , which directly yields the desired uniqueness results and can be compared with some known comparison results established in for example [37, 65, 62, 21, 41, 45, 47, 48]. Before that, we need to establish the following a priori estimate.

Proposition 3.1. *Assume that there exists a function $h(\cdot, \cdot, \cdot) \in \bar{\mathbf{S}}$ such that $d\mathbb{P} \times dt - a.e.$,*

$$\mathbf{1}_{Y_t > 0} g(t, Y_t, Z_t) \leq f_t + h(t, Y_t^+, |Z_t|),$$

and that $\varphi(\cdot, \cdot) \in \mathbf{S}$ is a test function for h . If $(\varphi(t, Y_t^+ + \int_0^t f_s ds))_{t \in [0, T]}$ is of class (D), then we have

$$\varphi\left(t, Y_t^+ + \int_0^t f_s ds\right) \leq \mathbb{E} \left[\varphi\left(T, \xi^+ + \int_0^T f_s ds\right) \middle| \mathcal{F}_t \right], \quad t \in [0, T].$$

Proof. Note that $(\varphi(t, Y_t^+ + \int_0^t f_s ds))_{t \in [0, T]}$ belongs to class (D). The desired conclusion can be easily obtained by a similar computation to step 1 in the proof of Theorem 2.2 with $Y_t^+, Y_s^+, \mathbf{1}_{Y_t > 0}$ and $\mathbf{1}_{Y_s > 0}$ instead of $|Y_t|, |Y_s|, \text{sgn}(Y_t)$ and $\text{sgn}(Y_s)$, respectively. \square

3.1. Comparison results for the no more than linear growth case

Now, let us introduce the following assumptions on the generator g , where g has a one-sided linear growth in y and a no more than linear growth in z .

(UN1) g satisfies an extended monotonicity condition in y , i.e., there exists a continuous, increasing and concave function $\rho(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and

$$\int_{0^+} \frac{du}{\rho(u)} := \lim_{\varepsilon \rightarrow 0^+} \int_0^\varepsilon \frac{du}{\rho(u)} = +\infty$$

such that $d\mathbb{P} \times dt - a.e.$, for each $(y_1, y_2, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$,

$$\text{sgn}(y_1 - y_2) (g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \leq \rho(|y_1 - y_2|).$$

(UN2) g satisfies a logarithmic uniform continuity condition in z , i.e., there exist a non-positive constant $\lambda \in (-\infty, 0]$ and a nondecreasing continuous function $\kappa(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with linear growth and $\kappa(0) = 0$ such that $d\mathbb{P} \times dt - a.e.$, for each $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \kappa(|z_1 - z_2| (\ln(e + |z_1 - z_2|))^\lambda) \leq \kappa(|z_1 - z_2|).$$

Remark 3.2. *Since the function $\rho(\cdot)$ appearing in (UN1) is nondecreasing and concave with $\rho(0) = 0$, one can verify that $\rho(\cdot)$ is of linear growth. We always assume that there exists a $A > 0$ such that*

$$\forall u \in \mathbb{R}_+, \quad \rho(u) \leq A(u + 1) \quad \text{and} \quad \kappa(u) \leq A(u + 1). \quad (3.1)$$

Thus, if the generator g satisfies (UN1) and (UN2), then we have $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} \text{sgn}(y)g(\omega, t, y, z) &\leq |g(t, 0, 0)| + \rho(|y|) + \kappa(|z| (\ln(e + |z|))^\lambda) \\ &\leq |g(t, 0, 0)| + 2A + A|y| + A|z| (\ln(e + |z|))^\lambda, \end{aligned}$$

which means that the generator g has a one-sided linear growth in y and a logarithmic sub-linear/linear growth in the unknown variable z , and then satisfies assumption (EX3) with $f_t := |g(t, 0, 0)| + 2A$ and $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\beta = \gamma = A$, $\delta = 0$, $\alpha = 1$ and $\lambda \leq 0$. In addition, we would like to mention that in the case of $\lambda = 0$, assumption (UN2) is equivalent to say that the function $g(t, \omega, y, z)$ is uniformly continuous in the variable z uniformly with respect to the variables (t, ω, y) , and that the smaller the λ , the stronger the assumption (UN2).

Theorem 3.3. *Suppose that ξ and ξ' are two terminal conditions such that $\xi \leq \xi'$, the generator g (resp. g') satisfies assumptions (UN1) and (UN2), $(Y_t, Z_t)_{t \in [0, T]}$ and $(Y'_t, Z'_t)_{t \in [0, T]}$ are, respectively, adapted solutions to BSDE(ξ, g) and BSDE(ξ', g'), and that*

$$\mathbf{1}_{Y_t > Y'_t} (g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t)) \leq 0 \quad (\text{resp. } \mathbf{1}_{Y_t > Y'_t} (g(t, Y_t, Z_t) - g'(t, Y_t, Z_t)) \leq 0). \quad (3.2)$$

If one of the following four conditions holds, then for each $t \in [0, T]$, we have $Y_t \leq Y'_t$:

(i) $\lambda < -1/2$ and both processes $(Y_t)_{t \in [0, T]}$ and $(Y'_t)_{t \in [0, T]}$ are of class (D).

(ii) $\lambda = -1/2$ and both processes $(|Y_t|(\ln(e + |Y_t|))^p)_{t \in [0, T]}$ and $(|Y'_t|(\ln(e + |Y'_t|))^p)_{t \in [0, T]}$ are of class (D) for some constant $p > 0$.

(iii) $\lambda \in (-1/2, 0]$ and both processes $(|Y_t| \exp(\mu(\ln(e + |Y_t|))^{\lambda + \frac{1}{2}}))_{t \in [0, T]}$ and $(|Y'_t| \exp(\mu(\ln(e + |Y'_t|))^{\lambda + \frac{1}{2}}))_{t \in [0, T]}$ are of class (D) for each $\mu > 0$.

(iv) $\lambda = 0$ and both processes $(|Y_t|^p)_{t \in [0, T]}$ and $(|Y'_t|^p)_{t \in [0, T]}$ are of class (D) for some $p > 1$.

It is well known from [37] that the strict comparison theorem for solutions of two BSDEs is also true when one of two generators is uniformly Lipschitz continuous in (y, z) . However, the following two examples indicate that the strict comparison theorem does not hold in general when the generator g satisfies only assumptions (UN1) and (UN2), which are provided in Section 5.3 of [78] and Example 3.2 of [59], respectively. In economics, this means that there exist infinitely many opportunities of arbitrage.

Example 3.4. *Let $d = 1$ and consider the following BSDE:*

$$Y_t = \xi + \int_t^T (-2\sqrt{Y_s^+}) ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T]. \quad (3.3)$$

Clearly, the generator $g(\omega, t, y, z) := -2\sqrt{y^+}$ satisfies assumptions (UN1) and (UN2) with $\rho(x) = x$ and $\kappa(x) \equiv 0$. It is not hard to verify that $(Y_t, Z_t)_{t \in [0, T]} := (0, 0)_{t \in [0, T]}$ and $(Y'_t, Z'_t)_{t \in [0, T]} := (t^2, 0)_{t \in [0, T]}$ are respectively the unique solution to (3.3) with $\xi = 0$ and $\xi = T^2$ such that $(|Y_t|^p)_{t \in [0, T]}$ and $(|Y'_t|^p)_{t \in [0, T]}$ belong to class (D) for each $p > 0$. Note that $Y'_T = T^2 > 0 = Y_T$, but $Y_0 = Y'_0 = 0$.

Example 3.5. *Let $d = 1$ and consider the following BSDE:*

$$Y_t = \xi + \int_t^T (-3|Z_s|^{\frac{2}{3}}) ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T]. \quad (3.4)$$

Clearly, the generator $g(\omega, t, y, z) := -3|z|^{\frac{2}{3}}$ satisfies (UN1) and (UN2) with $\rho(x) = x$ and $\kappa(x) = 3x^{\frac{2}{3}}$. It is not hard to verify that $(Y_t, Z_t)_{t \in [0, T]} := (0, 0)_{t \in [0, T]}$ and $(Y'_t, Z'_t)_{t \in [0, T]} := (\frac{1}{4}B_t^4, B_t^3)_{t \in [0, T]}$ are respectively the unique solution to (3.4) with $\xi = 0$ and $\xi = \frac{1}{4}B_T^4$ such that $(|Y_t|^p)_{t \in [0, T]}$ and $(|Y'_t|^p)_{t \in [0, T]}$ belong to class (D) for each $p > 0$. Note that $\mathbb{P}(Y'_T > Y_T) = 1 > 0$, but $Y_0 = Y'_0 = 0$.

Proof of Theorem 3.3. We only prove the case that the generator g satisfies assumptions (UN1) and (UN2), and $d\mathbb{P} \times dt - a.e.$,

$$\mathbf{1}_{Y_t > Y'_t} (g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t)) \leq 0. \quad (3.5)$$

The other case can be proved in the same way. According to Theorem 2.1 in [41] and the above assumptions we know that it suffices to prove that the process $((Y_t - Y'_t)^+)_{t \in [0, T]}$ is bounded whenever either of four conditions (i)-(iv) holds.

Define $\hat{Y} := Y - Y'$ and $\hat{Z} := Z - Z'$. Then, the pair of processes $(\hat{Y}_t, \hat{Z}_t)_{t \in [0, T]}$ verifies

$$\hat{Y}_t = \hat{\xi} + \int_t^T \hat{g}(s, \hat{Y}_s, \hat{Z}_s) ds - \int_t^T \hat{Z}_s \cdot dB_s, \quad t \in [0, T], \quad (3.6)$$

where $\hat{\xi} := \xi - \xi'$ and

$$\hat{g}(s, \hat{Y}_s, \hat{Z}_s) := g(s, Y_s, Z_s) - g'(s, Y'_s, Z'_s).$$

From assumptions (UN1) and (UN2) together with inequalities (3.5) and (3.1) it follows that

$$\begin{aligned} \mathbf{1}_{\hat{Y}_t > 0} \hat{g}(t, \hat{Y}_t, \hat{Z}_t) &= \mathbf{1}_{\hat{Y}_t > 0} (g(s, Y_s, Z_s) - g'(s, Y'_s, Z'_s)) \\ &= \mathbf{1}_{\hat{Y}_t > 0} (g(s, Y_s, Z_s) - g(s, Y'_s, Z_s) + g(s, Y'_s, Z_s) - g(s, Y'_s, Z'_s) + g(s, Y'_s, Z'_s) - g'(s, Y'_s, Z'_s)) \\ &\leq \rho(\hat{Y}_t^+) + \kappa \left(|\hat{Z}_t| (\ln(e + |\hat{Z}_t|))^\lambda \right) \\ &\leq 2A + A\hat{Y}_t^+ + A|\hat{Z}_t| \left(\ln(e + |\hat{Z}_t|) \right)^\lambda = f_t + h(t, \hat{Y}_t^+, |\hat{Z}_t|), \end{aligned} \quad (3.7)$$

where $f_t := 2A$ and for each $(t, x, \bar{x}) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$,

$$h(t, x, \bar{x}) := Ax + A\bar{x} (\ln(e + \bar{x}))^\lambda. \quad (3.8)$$

In the sequel, according to Proposition 3.1 together with (3.7) and (3.8) we will verify that \hat{Y}_t^+ is a bounded process whenever anyone of conditions (i)-(iv) holds.

(i) Let $\lambda < -1/2$ and both processes $(Y_t)_{t \in [0, T]}$ and $(Y'_t)_{t \in [0, T]}$ be of class (D). For each $k \geq e$ sufficient large and each $c \geq 2A - \frac{8A^2}{1+2\lambda}$, define the following function

$$\varphi(s, x) = (k + x) \left(1 - (\ln(k + x))^{1+2\lambda} \right) \exp(cs), \quad (s, x) \in [0, T] \times \mathbb{R}_+.$$

Since $0 \leq \varphi(s, x) \leq (k + x) \exp(cT)$ for each $(s, x) \in [0, T] \times \mathbb{R}_+$, it is clear that $\{\varphi(t, \hat{Y}_t^+ + 2At)\}_{t \in [0, T]}$ is of class (D). On the other hand, a similar analysis to that in (i) of the proof of Theorem 2.3 yields that the last function $\varphi(\cdot, \cdot)$ satisfies (2.1), and hence is a test function for h defined in (3.8). It then follows from Proposition 3.1 that

$$\varphi(t, Y_t^+ + 2At) \leq \mathbb{E} [\varphi(T, \xi^+ + 2AT) | \mathcal{F}_t] = \varphi(T, 2AT), \quad t \in [0, T],$$

which means that $(\hat{Y}_t^+)_{t \in [0, T]}$ is a bounded process.

(ii) Let $\lambda = -1/2$ and both processes $(|Y_t|(\ln(e + |Y_t|))^p)_{t \in [0, T]}$ and $(|Y'_t|(\ln(e + |Y'_t|))^p)_{t \in [0, T]}$ be of class (D) for some constant $p > 0$. For each $k \geq e$ sufficient large and each $c \geq 2A + \frac{4A^2}{p}$, define the following function

$$\varphi(s, x) = (k + x) (\ln(k + x))^p \exp(cs), \quad (s, x) \in [0, T] \times \mathbb{R}_+.$$

Since $0 \leq \varphi(s, x) \leq Kx (\ln(e+x))^p$ for each $(s, x) \in [0, T] \times \mathbb{R}_+$ and some positive constant $K > 0$ depending only on (k, T) , we can deduce that $\{\varphi(t, \hat{Y}_t^+ + 2At)\}_{t \in [0, T]}$ is of class (D). On the other hand, a similar analysis to that in (ii) of the proof of Theorem 2.3 yields that the last function $\varphi(\cdot, \cdot)$ satisfies (2.1), and hence is a test function for h defined in (3.8). Thus, the boundedness of the process $(\hat{Y}_t^+)_{t \in [0, T]}$ follows immediately as in (i).

(iii) Let $\lambda \in (-\frac{1}{2}, 0]$ and both processes $(|Y_t| \exp(\mu(\ln(e + |Y_t|))^{\lambda + \frac{1}{2}}))_{t \in [0, T]}$ and $(|Y'_t| \exp(\mu(\ln(e + |Y'_t|))^{\lambda + \frac{1}{2}}))_{t \in [0, T]}$ be of class (D) for any $\mu > 0$. For each $k \geq e$ sufficient large, $c_1 \geq 1$ and $c_2 \geq (\lambda + 3/2)\beta - 4\lambda^+ \gamma^2$, define the following function

$$\varphi(s, x) = (k+x) \exp\left(c_1 \exp(c_2 s) (\ln(k+x))^{\lambda + \frac{1}{2}}\right), \quad (s, x) \in [0, T] \times \mathbb{R}_+.$$

Since $0 \leq \varphi(s, x) \leq Kx \exp\left(K (\ln(e+x))^{\lambda + \frac{1}{2}}\right)$ for each $(s, x) \in [0, T] \times \mathbb{R}_+$ and some positive constant $K > 0$ depending only on (k, T) , we can deduce that $\{\varphi(t, \hat{Y}_t^+ + 2At)\}_{t \in [0, T]}$ is of class (D). On the other hand, a similar analysis to that in (iii) of the proof of Theorem 2.3 yields that the last function $\varphi(\cdot, \cdot)$ satisfies (2.1), and is a test function for h defined in (3.8). Thus, the boundedness of the process $(\hat{Y}_t^+)_{t \in [0, T]}$ follows immediately as in (i).

(iv) Let $\lambda = 0$ and both processes $(|Y_t|^p)_{t \in [0, T]}$ and $(|Y'_t|^p)_{t \in [0, T]}$ be of class (D) for some $p > 1$. Note that for each $\mu > 0$, there exists a positive constant $K > 0$ depending only on (μ, p) such that

$$0 \leq x \exp\left(\mu(\ln(e+x))^{\frac{1}{2}}\right) \leq Kx^p, \quad x \geq 1.$$

The desired assertion is a direct consequence of (iii). \square

The following example comes from Remark 6 in [58], which indicates that the uniform continuity of the generator g in the unknown variable y does not guarantee uniqueness of the solution to a BSDE(ξ, g).

Example 3.6. *Let us consider the following BSDE:*

$$Y_t = \int_t^T \sqrt{|Y_s|} ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T]. \quad (3.9)$$

Clearly, $g(\omega, t, y, z) := \sqrt{|y|}$ is uniformly continuous. It is not hard to check that for each $c \in [0, T]$,

$$(Y_t, Z_t)_{t \in [0, T]} := \left(\frac{[(c-t)^+]^2}{4}, 0 \right)_{t \in [0, T]}$$

is a solution to (3.9) such that $(|Y_t|^p)_{t \in [0, T]}$ belongs to class (D) for each $p > 0$.

3.2. Comparison results for the super-linear but no more than quadratic growth case

In the following comparison theorem, we will use the following assumption on the generator g , where the generator g admits a super-linear but no more than quadratic growth in (y, z) in general.

(UN3) It holds that $d\mathbb{P} \times dt - a.e.$,

$$\mathbf{1}_{\delta_\theta y > 0} \frac{g(\omega, t, y_1, z) - \theta g(\omega, t, y_2, z)}{1 - \theta} \leq f_t(\omega) + \bar{\beta}(|y_1| + |y_2|) + h(t, (\delta_\theta y)^+, |\delta_\theta z|), \quad (3.10)$$

$$\forall (y_1, y_2, z_1, z_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \text{ and } \theta \in (0, 1),$$

or

$$-\mathbf{1}_{\delta_\theta y < 0} \frac{g(\omega, t, y_1, z) - \theta g(\omega, t, y_2, z)}{1 - \theta} \leq f_t(\omega) + \bar{\beta}(|y_1| + |y_2|) + h(t, (\delta_\theta y)^-, |\delta_\theta z|), \quad (3.11)$$

$$\forall (y_1, y_2, z_1, z_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \text{ and } \theta \in (0, 1),$$

where $h(\cdot, \cdot, \cdot) \in \bar{\mathbf{S}}$ is defined in (2.7) for $\lambda \geq 0$,

$$\delta_\theta y := \frac{y_1 - \theta y_2}{1 - \theta} \text{ and } \delta_\theta z := \frac{z_1 - \theta z_2}{1 - \theta}.$$

With respect to the above assumption, we would like to give some explanations. First, the following Proposition 3.7 can be proved by a similar way as Proposition 3.5 in [45]. We omit its proof here.

Proposition 3.7. *Assume that the generator g satisfies (EX3) with $h(\cdot, \cdot, \cdot) \in \bar{\mathbf{S}}$ being defined in (2.7) for $\lambda \geq 0$. Then, assumption (UN3) holds for g if it satisfies any one of the following three conditions:*

- (i) $d\mathbb{P} \times dt - a.e.$, $g(\omega, t, \cdot, \cdot)$ is convex or concave;
- (ii) $d\mathbb{P} \times dt - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\omega, t, \cdot, z)$ is Lipschitz and $g(\omega, t, y, \cdot)$ is convex or concave;
- (iii) $g(t, y, z) \equiv l(y)q(z)$, where both $l : \mathbb{R} \rightarrow \mathbb{R}$ and $q : \mathbb{R}^d \rightarrow \mathbb{R}$ are bounded Lipschitz functions, and the function $q(z)$ has a bounded support.

Remark 3.8. *One typical example of (UN3) is $g(\omega, t, y, z) := g_1(y) + g_2(y)$, where $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ is convex or concave with a one-sided logarithmic sup-linear growth, i.e., there exists a nonnegative constant $A \geq 0$ such that for each $y \in \mathbb{R}$,*

$$\text{sgn}(y)g_1(y) \leq A + \beta|y|(\ln(e + |y|))^\delta,$$

and $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function. In other words, g is a Lipschitz perturbation of some convex (concave) function. Another typical example of (UN3) is $\bar{g}(\omega, t, y, z) := g_3(z) + g_4(z)$, where $g_3 : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex or concave with a logarithmic sub-quadratic growth, i.e., there exists a nonnegative constant $A \geq 0$ such that for each $z \in \mathbb{R}^d$,

$$|g_3(z)| \leq A + \gamma|z|^\alpha(\ln(e + |z|))^\lambda,$$

and $g_4 : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz function with bounded support. In other words, \bar{g} is a local Lipschitz perturbation of some convex (concave) function. Furthermore, it is easy to verify that if both generators g_1 and g_2 satisfies (3.10) (resp. (3.11)), then $g_1 + g_2$ also satisfies (3.10) (resp. (3.11)). Consequently, the generator g satisfying (UN3) may be not necessarily convex (concave) or Lipschitz in (y, z) , and it can have a general growth in y . Finally, a similar argument to that in Section 4 of [55] gives that if the generator g satisfies (UN3), then it has to be local Lipschitz in (y, z) , and then satisfies (EX1).

Theorem 3.9. *Suppose that ξ and ξ' are two terminal conditions such that $\xi \leq \xi'$, the generator g (resp. g') satisfies assumption (UN3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\lambda \geq 0$, $(Y_t, Z_t)_{t \in [0, T]}$ and $(Y'_t, Z'_t)_{t \in [0, T]}$ are, respectively, adapted solutions to BSDE(ξ, g) and BSDE(ξ', g'), and that*

$$g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t) \text{ (resp. } g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t) \text{)}.$$

Then the following two assertions holds.

(i) Let $\alpha = 1$ and $p := \delta \vee (\lambda + \frac{1}{2}) \vee (2\lambda)$. If $\int_0^T f_s ds \in \cap_{\mu>0} L \exp[\mu(\ln L)^p]$ and both processes $(|Y_t| \exp(\mu(\ln(e + |Y_t|))^p))_{t \in [0, T]}$ and $(|Y'_t| \exp(\mu(\ln(e + |Y'_t|))^p))_{t \in [0, T]}$ are of class (D) for each $\mu > 0$, then for each $t \in [0, T]$, we have $Y_t \leq Y'_t$.

(ii) Let $\delta = 0$, $\lambda = 0$, $\alpha \in (1, 2]$ and α^* be the conjugate of α . If $\int_0^T f_s ds \in \exp(\mu L^{\frac{2}{\alpha^*}})$ and both processes $(\exp(\mu(|Y_t|)^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ and $(\exp(\mu(|Y'_t|)^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ are of class (D) for each $\mu > 0$, then for each $t \in [0, T]$, we have $Y_t \leq Y'_t$. In particular, if the random variable $\int_0^T f_s ds$ and both processes Y and Y' are all bounded, then for each $t \in [0, T]$, $Y_t \leq Y'_t$.

Remark 3.10. Let us suppose that the generator g satisfies assumption (EX3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\delta = 0$, $\lambda = 0$, $\alpha = 2$. For the case of the bounded terminal condition, it has been shown in [41] and [65] that in order to ensure that the (strictly) comparison result in Theorem 3.9 holds, the assumption (UN3) can be weakened such that the generator g further satisfies assumption (UN1) and the following locally Lipschitz condition in z : $d\mathbb{P} \times dt - a.e.$, for each $(y, z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{1 \times d} \times \mathbb{R}^{1 \times d}$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \gamma(1 + |z_1|^\delta + |z_2|^\delta)|z_1 - z_2| \quad (3.12)$$

with $\delta \in [0, 1]$ and $\gamma > 0$. For the case of the unbounded terminal condition, it has been shown in [46] that in order to ensure that the comparison result in Theorem 3.9 holds, the assumption (UN3) can be weakened such that the generator g is uniformly Lipschitz continuous in y and further satisfies (3.12) with $\delta \in [0, 1)$ and an additional strictly positive/negative quadratic condition of the generator g in z , see assumptions (H3) and (H3') in [46] for more details.

Proof of Theorem 3.9. (i) We first consider the case when the generator g satisfies (3.10) in assumption (UN3), and $d\mathbb{P} \times dt - a.e.$,

$$g(t, Y'_t, Z'_t) \leq g'(t, Y'_t, Z'_t). \quad (3.13)$$

The θ -technique put forward initially in Briand and Hu [21] will be used in the following argument. For each fixed $\theta \in (0, 1)$, define

$$\delta_\theta U := \frac{Y - \theta Y'}{1 - \theta} \quad \text{and} \quad \delta_\theta V := \frac{Z - \theta Z'}{1 - \theta}. \quad (3.14)$$

Then the pair $(\delta_\theta U_t, \delta_\theta V_t)_{t \in [0, T]}$ verifies the following BSDE:

$$\delta_\theta U_t = \delta_\theta U_T + \int_t^T \delta_\theta g(s, \delta_\theta U_s, \delta_\theta V_s) ds - \int_t^T \delta_\theta V_s \cdot dB_s, \quad t \in [0, T], \quad (3.15)$$

where $d\mathbb{P} \times ds - a.e.$,

$$\delta_\theta g(s, \delta_\theta U_s, \delta_\theta V_s) := \frac{(g(s, Y_s, Z_s) - \theta g(s, Y'_s, Z'_s)) + \theta(g(s, Y'_s, Z'_s) - g'(s, Y'_s, Z'_s))}{1 - \theta}. \quad (3.16)$$

It follows from the assumptions that $d\mathbb{P} \times ds - a.e.$, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathbf{1}_{\delta_\theta U_s > 0} \delta_\theta g(s, \delta_\theta U_s, \delta_\theta V_s) \leq \bar{f}_s + h(s, (\delta_\theta U_s)^+, |\delta_\theta V_s|) \quad (3.17)$$

with

$$\bar{f}_s := f_s + \bar{\beta}(|Y_s| + |Y'_s|).$$

On the other hand, for each $k \geq e$ sufficient large, $c_1 \geq 1$ and $c_2 \geq (p+1)\beta - 4^{\lambda^+}\gamma^2$, define the following function

$$\varphi(s, x) = (k+x) \exp(c_1 \exp(c_2 s) (\ln(k+x))^p), \quad (s, x) \in [0, T] \times \mathbb{R}_+.$$

Since $0 \leq \varphi(s, x) \leq Kx \exp(K \ln(e+x)^p)$ for each $(s, x) \in [0, T] \times \mathbb{R}_+$ and some positive constant $K > 0$ depending only on (k, T) , according to the assumptions it is not hard to verify that the process $\{\varphi(t, (\delta_\theta U_t)^+ + \int_0^t \bar{f}_s ds)\}_{t \in [0, T]}$ is of class (D). On the other hand, by a similar analysis to that in (iii) of the proof of Theorem 2.3 we can conclude that the last function $\varphi(\cdot, \cdot)$ satisfies (2.1), and hence is a test function for h defined in (2.7) with $\alpha = 1$ and $\lambda \geq 0$. It then follows from Proposition 3.1 that

$$(\delta_\theta U_t)^+ \leq \varphi\left(t, (\delta_\theta U_t)^+ + \int_0^t \bar{f}_s ds\right) \leq \mathbb{E}\left[\varphi\left(T, (\delta_\theta U_T)^+ + \int_0^T \bar{f}_s ds\right) \middle| \mathcal{F}_t\right], \quad t \in [0, T]. \quad (3.18)$$

Moreover, since

$$\delta_\theta U_T^+ = \frac{(\xi - \theta\xi')^+}{1 - \theta} = \frac{[\xi - \theta\xi + \theta(\xi - \xi')]^+}{1 - \theta} \leq \xi^+, \quad (3.19)$$

it follows that

$$(Y_t - \theta Y_t')^+ \leq (1 - \theta) \mathbb{E}\left[\varphi\left(T, \xi^+ + \int_0^T \bar{f}_s ds\right) \middle| \mathcal{F}_t\right], \quad t \in [0, T].$$

Thus, the desired assertion follows by sending θ to 1 in the last inequality.

For the case that (3.13) holds and the generator g satisfies (3.11), we need to use $\theta Y - Y'$ and $\theta Z - Z'$, respectively, instead of $Y - \theta Y'$ and $Z - \theta Z'$ in (3.14). In this case, the generator $\delta_\theta g$ in (3.15) and (3.16) should be

$$\delta_\theta g(s, \delta_\theta U_s, \delta_\theta V_s) := \frac{(\theta g(s, Y_s, Z_s) - g(s, Y_s', Z_s')) + (g(s, Y_s', Z_s') - g'(s, Y_s', Z_s'))}{1 - \theta}.$$

It follows from (3.11) that the generator $\delta_\theta g$ still satisfies (3.17). Consequently, (3.18) still holds. Moreover, by using

$$\delta_\theta U_T^+ = \frac{(\theta\xi - \xi')^+}{1 - \theta} = \frac{[\theta\xi - \xi + (\xi - \xi')]^+}{1 - \theta} \leq (-\xi)^+ = \xi^-$$

instead of (3.19), by virtue of (3.18) we deduce that

$$(\theta Y_t - Y_t')^+ \leq (1 - \theta) \mathbb{E}\left[\varphi\left(T, \xi^- + \int_0^T \bar{f}_s ds\right) \middle| \mathcal{F}_t\right], \quad t \in [0, T].$$

Thus, the desired assertion follows by sending θ to 1 in the above inequality.

Finally, in the same way we can prove the desired assertion under the conditions that the generator g' satisfies assumption (UN3) and $d\mathbb{P} \times dt - a.e.$, $g(t, Y_t, Z_t) \leq g'(t, Y_t, Z_t)$.

(ii) The desired assertion can be proved in the same way as in (i). The only difference lies in that the test function used in (i) needs to be replaced with those used, respectively, in (i) and (ii) of the proof of Theorem 2.7 for two different cases of $\alpha = 2$ and $\alpha \in (1, 2)$. \square

Remark 3.11. *It is worth noting that both (ii) and (iii) in Theorem 3.3 are explored at the first time in this paper, and that the key idea employed to prove Theorem 3.3 and Theorem 3.9 can be dated back to [20, 41, 45, 48].*

3.3. Existence and uniqueness

By virtue of Theorem 2.3, Theorem 2.7, Theorem 3.3 and Theorem 3.9, one can easily verify the following two existence and uniqueness results, whose proofs are omitted here.

Theorem 3.12. *Assume that ξ is an \mathcal{F}_T -measurable random variable and the generator g satisfies (EX1)-(EX3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\alpha = 1$. Then, the following assertions hold.*

(i) *Let $\delta = 0$ and $\lambda \in (-\infty, -\frac{1}{2})$. If $\xi + \int_0^T f_s ds \in L^1$ and the generator g further satisfies (UN1) and (UN2), then BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y_t)_{t \in [0, T]}$ is of class (D);*

(ii) *Let $\delta = 0$ and $\lambda = -\frac{1}{2}$. If $\xi + \int_0^T f_s ds \in L(\ln L)^p$ for some $p > 0$ and the generator g further satisfies (UN1) and (UN2), then BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(|Y_t|(\ln(e + |Y_t|))^p)_{t \in [0, T]}$ is of class (D);*

(iii) *Let $p := \delta \vee (\lambda + \frac{1}{2}) \vee (2\lambda) \in (0, +\infty)$. If $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} L \exp[\mu(\ln L)^p]$ and the generator g further satisfies (UN1) and (UN2) for the case of $\lambda \in (-\frac{1}{2}, 0]$ and (UN3) for the case of $\lambda \in [0, +\infty)$, then BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(|Y_t| \exp(\mu(\ln(e + |Y_t|))^p))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$;*

(iv) *Let $\delta = 0$ and $\lambda = 0$. If $\xi + \int_0^T f_s ds \in L^p$ for some $p > 1$ and g further satisfies (UN1) and (UN2), then BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(|Y_t|^p)_{t \in [0, T]}$ is of class (D).*

Theorem 3.13. *Assume that ξ is an \mathcal{F}_T -measurable random variable and the generator g satisfies assumptions (EX2), (EX3) and (UN3) with $h(\cdot, \cdot, \cdot)$ being defined in (2.7) for $\delta = 0$, $\lambda = 0$ and $\alpha \in (1, 2]$. If $\xi + \int_0^T f_s ds \in \cap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}})$ with α^* being the conjugate of α , then BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that the process $(\exp(\mu |Y_t|^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$. In particular, If $\xi + \int_0^T f_s ds \in L^\infty(\mathcal{F}_T)$, then BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $(Y_t)_{t \in [0, T]} \in \mathcal{S}^\infty([0, T]; \mathbb{R})$.*

4. Applications

In this section, we will introduce some applications of our theoretical results obtained in the last two sections, which are enlightened by for example [80, 81, 84, 21, 57, 45, 48, 47].

4.1. The (conditional) g -expectation defined on $L^1(\mathcal{F}_T)$

First of all, we extend the notion of (conditional) g -expectation of [84] defined on the space $L^2(\mathcal{F}_T)$ of squarely integrable random variables to the larger one $L^1(\mathcal{F}_T)$ of integrable random variables.

Definition 4.1. *Let the generator g satisfy assumptions (EX1)-(EX2) and (UN1)-(UN2) with $\lambda \in (-\infty, -\frac{1}{2})$ and $\int_0^T f_s ds \in L^1$. Assume further that g satisfies the following assumption:*

$$d\mathbb{P} \times dt - a.e., \quad g(\omega, t, y, 0) \equiv 0, \quad \forall y \in \mathbb{R}. \quad (4.1)$$

By virtue of (i) in Theorem 3.12, for each $\xi \in L^1(\mathcal{F}_T)$ and $t \in [0, T]$, we can denote the conditional g -expectation $\mathcal{E}_g[\xi | \mathcal{F}_t]$ of ξ with respect to \mathcal{F}_t by the following formula:

$$\mathcal{E}_g[\xi | \mathcal{F}_t] := Y_t^\xi, \quad (4.2)$$

where $(Y_t^\xi, Z_t^\xi)_{t \in [0, T]}$ is the unique solution of BSDE (ξ, g) such that Y_t^ξ belongs to class (D). In particular, we call $\mathcal{E}_g[\xi] := \mathcal{E}_g[\xi | \mathcal{F}_0]$ the g -expectation of ξ .

It is clear that the conditional g -expectation operator $\mathcal{E}_g[\cdot | \mathcal{F}_t]$ defined by (4.2) maps $L^1(\mathcal{F}_T)$ to $L^1(\mathcal{F}_t)$ for each $t \in [0, T]$, which shares the same domain with the classical mathematical expectation operator. Furthermore, proceeding identically as in [85] and [61], from (i) of Theorem 3.3 and (i) of Theorem 3.12 together with (4.1) we easily (thus omitting the proof) have the following two propositions.

Proposition 4.2. $\mathcal{E}_g[\cdot]$ possesses the following properties:

- (i) **Preserving of constants:** For each constant $c \in \mathbb{R}$, $\mathcal{E}_g[c] = c$;
- (ii) **Monotonicity:** For each $\xi_1, \xi_2 \in L^1(\mathcal{F}_T)$, if $\xi_1 \geq \xi_2$, then $\mathcal{E}_g[\xi_1] \geq \mathcal{E}_g[\xi_2]$.

Proposition 4.3. For each $t \in [0, T]$, $\mathcal{E}_g[\cdot | \mathcal{F}_t]$ possesses the following properties:

- (i) If $\xi \in L^1(\mathcal{F}_t)$, then $\mathcal{E}_g[\xi | \mathcal{F}_t] = \xi$;
- (ii) For each $\xi_1, \xi_2 \in L^1(\mathcal{F}_T)$, if $\xi_1 \geq \xi_2$, then $\mathcal{E}_g[\xi_1 | \mathcal{F}_t] \geq \mathcal{E}_g[\xi_2 | \mathcal{F}_t]$;
- (iii) For each $\xi \in L^1(\mathcal{F}_T)$ and $r \in [0, T]$, we have $\mathcal{E}_g[\mathcal{E}_g[\xi | \mathcal{F}_t] | \mathcal{F}_r] = \mathcal{E}_g[\xi | \mathcal{F}_{t \wedge r}]$;
- (iv) For each $A \in \mathcal{F}_t$ and $\xi \in L^1(\mathcal{F}_T)$, $\mathcal{E}_g[\mathbf{1}_A \xi | \mathcal{F}_t] = \mathbf{1}_A \mathcal{E}_g[\xi | \mathcal{F}_t]$ and $\mathcal{E}_g[\mathbf{1}_A \xi] = \mathcal{E}_g[\mathbf{1}_A \mathcal{E}_g[\xi | \mathcal{F}_t]]$.

It can be indicated from both propositions that the (conditional) g -expectation preserves essential properties (but except linearity) of the classical expectations. Some extensive issues on the (conditional) g -expectation still remain to be further studied along the lines of [84, 25, 85, 28, 88, 86, 87, 61, 31].

Remark 4.4. In the same way as above, by Theorems 3.3, 3.9, 3.12 and 3.13 one can define the (conditional) g -expectation via the solutions of BSDEs on the spaces

$$L(\ln L)^p \ (p > 0), \quad \bigcap_{\mu > 0} L \exp[\mu(\ln L)^p] \ (p > 0), \quad L^p \ (p > 1), \quad \bigcap_{\mu > 0} \exp(\mu L^{\frac{2}{\alpha^*}}) \ (\alpha^* \geq 2) \quad \text{and} \quad L^\infty$$

respectively. It is clear that the generator g of BSDEs needs to satisfy some stronger conditions as the space becomes larger. In particular, when $g(t, y, z) := \gamma|z|$, the corresponding conditional g -expectation $\mathcal{E}_g[\cdot | \mathcal{F}_t]$ for $t \in [0, T]$ can be defined on the space $\bigcap_{\mu > 0} L \exp[\mu\sqrt{\ln L}]$, which is bigger than L^p ($p > 1$) used for example in [84, 25, 28, 88, 86, 87, 90, 57, 59]. Furthermore, according to (iii) of Theorem 3.3 and (iii) of Theorem 3.12, we can verify that for each $t \in [0, T]$ and $\xi \in \bigcap_{\mu > 0} L \exp[\mu\sqrt{\ln L}]$,

$$\mathcal{E}_g[\xi | \mathcal{F}_t] := \operatorname{ess\,sup}_{q \in \mathcal{A}} \mathbb{E}_q[\xi | \mathcal{F}_t]$$

with \mathcal{A} being defined in Example 2.5. This is just the maximal conditional expectation on \mathcal{A} .

4.2. Dynamic utility process and risk measure

In the sequel, we introduce an application of our theoretical results to mathematical finance. For notational simplicity, for each $t \in [0, T]$ and $\alpha \in (1, 2]$ we denote

$$E^\alpha(\mathcal{F}_t) := \bigcap_{\mu > 0} \exp[\mu L^{\frac{2}{\alpha^*}}](\mathcal{F}_t)$$

with $\alpha^* := \frac{\alpha}{\alpha-1} \geq 2$ being the conjugate of α . Clearly, $E^\alpha(\mathcal{F}_t)$ is a linear space containing $L^\infty(\mathcal{F}_t)$ of bounded random variables. The following proposition is a direct consequence of (iii) of Theorem 3.12, and the proof is omitted.

Proposition 4.5. *Suppose that the generator $g(z) : \mathbb{R}^d \rightarrow \mathbb{R}$ is a concave function satisfying $g(0) = 0$ and*

$$|g(z)| \leq a + \gamma|z|^\alpha \quad (4.3)$$

with $a \geq 0$ and $\alpha \in (1, 2]$ being two given constants. Then, for each $\xi \in E^\alpha(\mathcal{F}_T)$, BSDE(ξ, g) admits a unique solution $(Y_t, Z_t)_{t \in [0, T]}$ such that $Y_t \in E^\alpha(\mathcal{F}_t)$ for each $t \in [0, T]$.

Now, by virtue of Proposition 4.5, for each $\xi \in E^\alpha(\mathcal{F}_T)$ we can define

$$U_t^g(\xi) := Y_t^\xi, \quad t \in [0, T], \quad (4.4)$$

where $(Y_t^\xi, Z_t^\xi)_{t \in [0, T]}$ is the unique solution of BSDE(ξ, g) such that $Y_t^\xi \in E^\alpha(\mathcal{F}_t)$ for each $t \in [0, T]$.

The following theorem indicates that the family of operators $\{U_t^g(\cdot)\}_{t \in [0, T]}$ defined via (4.4) constitutes a dynamic utility process defined on $E^\alpha(\mathcal{F}_T)$.

Theorem 4.6. *For each $t \in [0, T]$, the mapping $U_t^g(\cdot) : E^\alpha(\mathcal{F}_T) \rightarrow E^\alpha(\mathcal{F}_t)$ defined via (4.4) satisfies the following properties:*

- (i) **Positivity:** $U_t^g(0) = 0$ and $U_t^g(\xi) \geq 0$ for each nonnegative random variable $\xi \in E^\alpha(\mathcal{F}_T)$;
- (ii) **Monotonicity:** for each $\xi, \eta \in E^\alpha(\mathcal{F}_T)$, if $\xi \geq \eta$, then $U_t^g(\xi) \geq U_t^g(\eta)$;
- (iii) **Monetary:** $U_t^g(\xi + \eta) = U_t^g(\xi) + \eta$ for each $\xi \in E^\alpha(\mathcal{F}_T)$ and $\eta \in E^\alpha(\mathcal{F}_t)$;
- (iv) **Concavity:** $U_t^g(\theta\xi + (1 - \theta)\eta) \geq \theta U_t^g(\xi) + (1 - \theta)U_t^g(\eta)$ for all $\xi, \eta \in E^\alpha(\mathcal{F}_T)$ and $\theta \in (0, 1)$.

Proof. In view of $g(0) = 0$ and the fact that g is independent of y , (i)-(iii) are the direct consequences of (iii) of Theorem 3.12 and (i) of Theorem 3.9. Furthermore, proceeding identically as Proposition 3.5 in [37], by virtue of (i) of Theorem 3.9 and the concavity of g we can get (iv). \square

Now, we let the function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be convex, satisfy $f(0) = 0$, and $\liminf_{|x| \rightarrow \infty} f(x)/|x|^2 > 0$. For each $\xi \in L^\infty(\mathcal{F}_T)$, define

$$\bar{U}_t(\xi) := \text{essinf} \left\{ \mathbb{E}_q \left[\xi + \int_t^T f(q_u) du \middle| \mathcal{F}_t \right] \middle| \mathbb{Q}^q \sim \mathbb{P} \right\}, \quad t \in [0, T], \quad (4.5)$$

where $\mathbb{E}_q[\cdot | \mathcal{F}_t]$ is the conditional expectation operator with respect to \mathcal{F}_t under the probability measure \mathbb{Q}^q , which is equivalent to \mathbb{P} and

$$\mathbb{E} \left[\frac{d\mathbb{Q}^q}{d\mathbb{P}} \middle| \mathcal{F}_t \right] = \exp \left\{ \int_0^t q_u \cdot dB_u - \frac{1}{2} \int_0^t |q_u|^2 du \right\}, \quad t \in [0, T].$$

It is not difficult to check that $\{\bar{U}_t(\cdot)\}_{t \in [0, T]}$ defined via (4.5) constitutes a dynamic utility process defined on $L^\infty(\mathcal{F}_T)$. And, it follows from Theorems 2.1-2.2 in [32] that there exists a $(Z_t)_{t \in [0, T]} \in \mathcal{M}^2$ such that $(\bar{U}_t(\xi), Z_t)_{t \in [0, T]}$ is the unique bounded solution of the following BSDE

$$Y_t = \xi + \int_t^T g(Z_s) ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (4.6)$$

where

$$g(z) := - \sup_{x \in \mathbb{R}^d} (-z \cdot x - f(x)) = \inf_{x \in \mathbb{R}^d} (z \cdot x + f(x)) \leq 0, \quad \forall z \in \mathbb{R}^d \quad (4.7)$$

is a concave function with $g(0) = 0$, and $\limsup_{|x| \rightarrow \infty} |g(x)|/|x|^2 < \infty$. For example, if $c > 0$ and

$$f(x) = c|x|^{\alpha^*} \geq 0, \quad x \in \mathbb{R}^d,$$

then

$$0 \geq g(z) = - \frac{1}{c^{\alpha^*-1}} \frac{\alpha^* - 1}{(\alpha^*)^\alpha} |z|^\alpha, \quad z \in \mathbb{R}^d,$$

which means that g satisfies the conditions in Proposition 4.5.

Remark 4.7. *It is well known that (4.5) is usually used to define the utility of bounded endowments in mathematical finance, see for example [32] for details. However, it is only defined on the space $L^\infty(\mathcal{F}_T)$. This motivates the definition (4.4) via a BSDE, which can be defined on a larger space $E^\alpha(\mathcal{F}_T)$ than $L^\infty(\mathcal{F}_T)$. Some relevant results are available in [37, 35, 47]. In a symmetric way to the above, we can also define a convex risk measure on $E^\alpha(\mathcal{F}_T)$, see [56, 61] among others for more details.*

Example 4.8. *Let the generator*

$$g(z) := c(\mathbf{1}_{\lambda \geq 0} - \mathbf{1}_{\lambda < 0})|z|(\ln(e + |z|))^\lambda, \quad z \in \mathbb{R}^d,$$

where $c > 0$ and $\lambda \in \mathbb{R}$ are two given constants. According to Theorem 3.12, we can easily verify that the following asserts hold:

- (i) *If $\lambda \in (-\infty, -\frac{1}{2})$, then for each $\xi \in L^1$, BSDE(ξ, g) admits a unique solution $(Y_t^\xi, Z_t^\xi)_{t \in [0, T]}$ such that $(Y_t^\xi)_{t \in [0, T]}$ is of class (D);*
- (ii) *If $\lambda = -\frac{1}{2}$, then for each $\xi \in L(\ln L)^p$ with $p > 0$, BSDE(ξ, g) admits a unique solution $(Y_t^\xi, Z_t^\xi)_{t \in [0, T]}$ such that the process $(|Y_t^\xi|(\ln(e + |Y_t^\xi|))^p)_{t \in [0, T]}$ is of class (D);*
- (iii) *If $\lambda \in (-\frac{1}{2}, +\infty)$ and $p := (\lambda + \frac{1}{2}) \vee (2\lambda)$, then for each $\xi \in \cap_{\mu > 0} L \exp[\mu(\ln L)^p]$, BSDE(ξ, g) admits a unique solution $(Y_t^\xi, Z_t^\xi)_{t \in [0, T]}$ such that the process $(|Y_t^\xi| \exp(\mu(\ln(e + |Y_t^\xi|))^p))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$.*

Thus, for the last three different cases of range of λ , we can define the following operator

$$\varrho(\xi) := Y_0^{-\xi}$$

on three different spaces of contingent claims: L^1 , $L(\ln L)^p$ and $\cap_{\mu > 0} L \exp[\mu(\ln L)^p]$. Moreover, according to the properties of the generator g together with Theorems 3.3, 3.9 and 3.12, in the same spirit as in for example [1, 56, 61, 26, 31, 24], we can verify that $\varrho(\cdot)$ is a convex risk measure, a star-shaped risk measure and a coherent risk measure on the corresponding spaces of contingent claims for the cases of $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$, respectively. In addition, in the same way as above we can also define the corresponding dynamic risk measure by virtue of the solution $Y_t^{-\xi}$ of BSDE(ξ, g) for $t \in [0, T]$.

4.3. Nonlinear Feynman-Kac formula

As another application of our theoretical results, in this subsection we will derive a nonlinear Feynman-Kac formula for PDEs which are no more than quadratic with respect to the gradient of the solution. Let us consider the following semi-linear PDE:

$$\partial_t u(t, x) + \mathcal{L}u(t, x) + g(t, x, u(t, x), \sigma^* \nabla_x u(t, x)) = 0, \quad u(T, \cdot) = h(\cdot), \quad (4.8)$$

where \mathcal{L} is the infinitesimal generator of the solution $X^{t,x}$ to the following SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r, \quad t \leq s \leq T, \quad \text{and} \quad X_s^{t,x} = x, \quad 0 \leq s < t. \quad (4.9)$$

The nonlinear Feynman-Kac formula consists in proving that the function defined by

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, \quad u(t, x) := Y_t^{t,x}, \quad (4.10)$$

where, for each $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, $(Y_t^{t_0, x_0}, Z_t^{t_0, x_0})_{t \in [0, T]}$ represents the solution to the BSDE

$$Y_t = h(X_T^{t_0, x_0}) + \int_t^T g(s, X_s^{t_0, x_0}, Y_s, Z_s) ds + \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (4.11)$$

is a viscosity solution to PDE (4.8).

Definition 4.9. A continuous function $u(t, x) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $u(T, \cdot) = h(\cdot)$ is said to be a viscosity super-solution (resp. sub-solution) to PDE (4.8) if for each smooth function $\varphi(\cdot, \cdot)$,

$$\partial_t u(t_0, x_0) + \mathcal{L}u(t_0, x_0) + g(t_0, x_0, u(t_0, x_0), \sigma^* \nabla_x \varphi(t_0, x_0)) \leq 0 \quad (\text{resp.} \quad \geq 0)$$

as soon as the function $u - \varphi$ has a local minimum (resp. maximum) at the point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. Moreover, a viscosity solution is both a viscosity super-solution and a viscosity sub-solution.

Let us now introduce the following assumptions on the coefficients of SDE (4.9).

(A1) $b(t, x) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma(t, x) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ are jointly continuous and there exists a positive constant $K > 0$ such that for each $(t, x, x') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|b(t, 0)| + |\sigma(t, x)| \leq K \quad \text{and} \quad |b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq K|x - x'|.$$

Classical results on SDEs show that under the assumption (A1), SDE (4.9) has a unique solution $X^{t,x} \in \cap_{q \geq 1} \mathcal{S}^q$ for each $(t, x) \in [0, T] \times \mathbb{R}^n$. And, since σ is bounded, the argument in page 563 of Briand and Hu [21] indicates that for each $q \in [1, 2)$ and $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$\forall \mu > 0, \quad \mathbb{E} \left[\sup_{s \in [0, T]} \exp(\mu |X_s^{t,x}|^q) \right] \leq C \exp(\mu C |x|^q), \quad (4.12)$$

where $C > 0$ is a constant depending only on (q, μ, T, K) .

Let us further give our assumptions on the generator g and the terminal condition of BSDE (4.11).

(A2) $g(t, x, y, z) : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $h(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ are jointly continuous and there exist three real constants $k \geq 0$, $\alpha \in (1, 2]$ and $p \in [1, \alpha^*)$ with α^* being the conjugate of α such that for each $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$,

$$\operatorname{sgn}(y)g(t, x, y, z) \leq k(1 + |x|^p + |y| + |z|^\alpha), \quad (4.13)$$

$$|g(t, x, y, z)| + |h(x)| \leq k\left(1 + |x|^p + \exp(k|y|^{\frac{2}{\alpha^*}}) + |z|^2\right) \quad (4.14)$$

and it holds that

$$\mathbf{1}_{y-\theta y' > 0} (g(t, x, y, z) - \theta g(t, x, y', z')) \leq (1 - \theta)k \left(1 + |x|^p + |y'| + \left(\frac{y - \theta y'}{1 - \theta}\right)^+ + \left|\frac{z - \theta z'}{1 - \theta}\right|^\alpha\right),$$

$$\forall (t, x, y, y', z, z') \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \text{ and } \theta \in (0, 1),$$

or

$$-\mathbf{1}_{y-\theta y' < 0} (g(t, x, y, z) - \theta g(t, x, y', z')) \leq (1 - \theta)k \left(1 + |x|^p + |y'| + \left(\frac{y - \theta y'}{1 - \theta}\right)^- + \left|\frac{z - \theta z'}{1 - \theta}\right|^\alpha\right),$$

$$\forall (t, x, y, y', z, z') \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \text{ and } \theta \in (0, 1).$$

Since $p \in [1, \alpha^*)$, it follows from (4.12) that for each $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ and each $\mu > 0$, we have

$$\mathbb{E} \left[\exp \left(\mu (|X_T^{t_0, x_0}|^p)^{\frac{2}{\alpha^*}} \right) \right] \leq \mathbb{E} \left[\sup_{s \in [0, T]} \exp \left(\mu |X_s^{t_0, x_0}|^{\frac{2p}{\alpha^*}} \right) \right] \leq \bar{C} \exp(\mu \bar{C} |x_0|^{\frac{2p}{\alpha^*}}) < +\infty$$

and

$$\mathbb{E} \left[\exp \left(\mu \left(\int_0^T |X_s^{t_0, x_0}|^p ds \right)^{\frac{2}{\alpha^*}} \right) \right] \leq \mathbb{E} \left[\sup_{s \in [0, T]} \exp \left(\mu T^{\frac{2}{\alpha^*}} |X_s^{t_0, x_0}|^{\frac{2p}{\alpha^*}} \right) \right] < +\infty,$$

where $\bar{C} > 0$ is a constant depending only on (p, α, μ, T, K) . Then, in view of the assumption (A2) and the last two inequalities, we can apply Theorem 3.13 to construct a unique solution $(Y_t^{t_0, x_0}, Z_t^{t_0, x_0})_{t \in [0, T]}$ to BSDE (4.11) such that $(\exp(\mu |Y_t^{t_0, x_0}|^{\frac{2}{\alpha^*}}))_{t \in [0, T]}$ is of class (D) for each $\mu > 0$. Furthermore, a classical argument yields that u defined via (4.10) is a deterministic function.

The following theorem is the main result of this subsection, whose proof is available in [45] and [21]. Similar results can also be found in [29] and [30].

Theorem 4.10. *Let assumptions (A1) and (A2) hold. Then, the function $u(t, x)$ defined in (4.10) is continuous on $[0, T] \times \mathbb{R}^n$ and there exists a constant $C > 0$ such that*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n, \quad |u(t, x)| \leq C(1 + |x|^p).$$

Moreover, $u(t, x)$ is a viscosity solution to PDE (4.8).

Remark 4.11. *To the best of our knowledge, the nonlinear Feynman-Kac formula for PDEs was at first time studied in [81], along the spirit of which several related works were successively presented in for example [82, 83, 76, 79, 65, 57, 21, 78, 45]. In fact, it should be noted that according to Theorems 3.3, 3.9, 3.12 and 3.13, one can establish a one-to-one correspondence between PDEs and BSDEs via Feynman-Kac formulas, as mentioned in section 4 of [57]. Generally speaking, the generator g and the terminal condition h appearing in BSDE (4.11) can admit a more general growth condition in the unknown variable x when g satisfies a stronger growth condition in the unknown variable z .*

5. Open problems

In this section, we will list several open problems to be further studied. The first two problems concern the existence of a BSDE and a PDE, the next three problems focus on the uniqueness.

(i) Consider the following BSDE:

$$Y_t = \xi + \int_t^T \frac{|Z_s|}{\sqrt{\ln(e + |Z_s|)}} ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T]. \quad (5.1)$$

For each $\xi \in L^1$, does BSDE (5.1) admit a solution $(Y_t, Z_t)_{t \in [0, T]}$ such that Y is of class (D)?

(ii) Suppose that the functions g and h only satisfy (4.13) and (4.14) of assumption (A2) in the last section. Does the semi-linear PDE (4.8) admit a viscosity solution?

(iii) Let $p > 1$ and consider the following BSDE:

$$Y_t = \xi + \int_t^T |Z_s| \sin |Z_s| ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T]. \quad (5.2)$$

For each $\xi \in L^p$, is the solution $(Y_t, Z_t)_{t \in [0, T]}$ to BSDE (5.2) with $(|Y_t|^p)_{t \in [0, T]}$ being of class (D) unique? We note that the generator g of (5.2) is not uniformly continuous in z .

(iv) Consider the following BSDE:

$$Y_t = \xi + \int_t^T \frac{1}{2} |Z_s^1|^2 ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (5.3)$$

where $Z := (Z^1, Z^2)$. For each $\xi \in \exp(L)$, is the solution $(Y_t, Z_t)_{t \in [0, T]}$ to BSDE (5.3) with $(\exp(|Y_t|))_{t \in [0, T]}$ being of class (D) unique? We note that the generator g of (5.3) is convex in z .

(v) Let $d = 2$ and consider the following BSDE:

$$Y_t = \xi + \int_t^T (|Z_s^1|^2 - |Z_s^2|^2) ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T], \quad (5.4)$$

where $Z := (Z^1, Z^2)$. For each $\xi \in \cap_{\mu > 0} \exp(\mu L)$, is the solution $(Y_t, Z_t)_{t \in [0, T]}$ to BSDE (5.4) with $(\exp(\mu |Y_t|))_{t \in [0, T]}$ being of class (D) for each $\mu > 0$ unique? We note that the generator g of (5.4) is neither convex nor concave in z , but satisfies (3.12) with $\delta = 1$ and $\gamma = 2$.

Appendix A. The Proof of Proposition 2.6

Proof of Proposition 2.6. The case of $\lambda = 0$ is clear. Let us consider the case of $\lambda \neq 0$. Given $k \geq k_{\lambda, p}$ with $\bar{p} := p^{\frac{1}{2+|\lambda|}} > 1$, $\bar{\lambda} := 2|\lambda|(2 + |\lambda - 1|)$, $k_{\lambda, p} \geq e^{|\lambda-1|+1}$,

$$\bar{\lambda} (\ln k_{\lambda, p})^{\lambda-1} < k_{\lambda, p}^{1-\frac{1}{\bar{p}}}, \quad k_{\lambda, p}^{\bar{p}} - k_{\lambda, p} - 2\sqrt{p}k_{\lambda, p} (\ln k_{\lambda, p})^\lambda > 0 \quad \text{and} \quad k_{\lambda, p}^{\frac{1}{\bar{p}}} < \sqrt{p}k_{\lambda, p} (\ln k_{\lambda, p})^\lambda. \quad (A.1)$$

For each $(x, y) \in (0, +\infty) \times (0, +\infty)$, define the function

$$\begin{aligned} f(x, y) &:= y^2 - 2xy (\ln(k + y))^\lambda + px^2 (\ln(k + x))^{2\lambda} \\ &= \left(y - x (\ln(k + y))^\lambda \right)^2 + px^2 (\ln(k + x))^{2\lambda} - x^2 (\ln(k + y))^{2\lambda}. \end{aligned} \quad (A.2)$$

Clearly, in order to prove (2.9), it is enough to prove that $f(x, y) \geq 0$ for each $x, y > 0$. Fix arbitrarily $x \in (0, +\infty)$ and let $\bar{f}(y) := f(x, y)$, $y \in (0, +\infty)$. A simple calculate gives that for each $y \in (0, +\infty)$,

$$\bar{f}'(y) = 2y - 2x \left(y (\ln(k+y))^\lambda \right)' = 2y - 2x (\ln(k+y))^\lambda \left[1 + \frac{\lambda y}{(k+y) \ln(k+y)} \right] \quad (\text{A.3})$$

and

$$\bar{f}''(y) = 2 - 2x \left(y (\ln(k+y))^\lambda \right)'' = 2 - \frac{2\lambda x [(2k+y) \ln(k+y) - (1-\lambda)y]}{(k+y)^2 (\ln(k+y))^{2-\lambda}}. \quad (\text{A.4})$$

Furthermore, let $y_0 > 0$ be the unique constant depending only on (p, k, λ, x) and satisfying

$$p (\ln(k+x))^{2\lambda} = (\ln(k+y_0))^{2\lambda} \quad \text{or equivalently, } x = (k+y_0)^{\frac{1}{p}} - k \quad \text{or } y_0 = (k+x)^{p\frac{1}{2\lambda}} - k. \quad (\text{A.5})$$

It then follows from (A.2) that $\bar{f}(y_0) = f(x, y_0) \geq 0$.

In the sequel, we will distinguish two different cases to prove the desired inequality (2.9).

Case 1: $\lambda > 0$. In this case, by (A.2) and (A.5) we know that

$$\begin{aligned} \bar{f}(y) &\geq px^2 (\ln(k+x))^{2\lambda} - x^2 (\ln(k+y))^{2\lambda} \\ &\geq x^2 \left[p (\ln(k+x))^{2\lambda} - (\ln(k+y_0))^{2\lambda} \right] = 0, \quad y \in (0, y_0]. \end{aligned}$$

Hence, it suffices to verify that $\bar{f}(y) \geq 0$ for $y \in [y_0, +\infty)$. In fact, by (A.4), (A.5) and (A.1) we have

$$\begin{aligned} \bar{f}''(y) &\geq 2 - \frac{2|\lambda|x[2(k+y) \ln(k+y) + |\lambda-1|(k+y) \ln(k+y)]}{(k+y)^2 (\ln(k+y))^{2-\lambda}} \\ &= 2 - \frac{\bar{\lambda} (\ln(k+y))^{\lambda-1}}{k+y} \left((k+y_0)^{\frac{1}{p}} - k \right) \\ &\geq 2 - \frac{\bar{\lambda} (\ln(k+y_0))^{\lambda-1}}{(k+y_0)^{1-\frac{1}{p}}} > 2 - \frac{\bar{\lambda} (\ln k)^{\lambda-1}}{k^{1-\frac{1}{p}}} > 1, \quad y \in [y_0, +\infty). \end{aligned}$$

And, by (A.3), (A.5) and (A.1) we can deduce that, in view of $\bar{p} > 1$,

$$\begin{aligned} \bar{f}'(y_0) &\geq 2y_0 - 2x (\ln(k+y_0))^\lambda \left[1 + \frac{\lambda}{\ln(k+y_0)} \right] \\ &\geq 2y_0 - 4x (\ln(k+y_0))^\lambda \\ &= 2(k+x)^{\bar{p}} - 2k - 4\sqrt{p}x (\ln(k+x))^\lambda \\ &\geq 2(k+x)^{\bar{p}} - 2(k+x) - 4\sqrt{p}(k+x) (\ln(k+x))^\lambda \\ &\geq 2k^{\bar{p}} - 2k - 4\sqrt{p}k (\ln k)^\lambda > 0. \end{aligned} \quad (\text{A.6})$$

Consequently, for each $y \in [y_0, +\infty)$, we have $\bar{f}'(y) \geq \bar{f}'(y_0) > 0$ and then $\bar{f}(y) \geq \bar{f}(y_0) \geq 0$.

Case 2: $\lambda < 0$. In this case, by (A.2) and (A.5) we know that

$$\begin{aligned} \bar{f}(y) &\geq px^2 (\ln(k+x))^{2\lambda} - x^2 (\ln(k+y))^{2\lambda} \\ &\geq x^2 \left[p (\ln(k+x))^{2\lambda} - (\ln(k+y_0))^{2\lambda} \right] = 0, \quad y \in [y_0, +\infty). \end{aligned}$$

Hence, it suffices to verify that $\bar{f}(y) \geq 0$ for $y \in (0, y_0]$. In fact, by (A.4) and (A.1) we have

$$\begin{aligned} \bar{f}''(y) &\geq 2 + \frac{2|\lambda|x[(k+y) \ln(k+y) - |1-\lambda|(k+y)]}{(k+y)^2 (\ln(k+y))^{2-\lambda}} \\ &= 2 + \frac{2|\lambda|x[\ln(k+y) - |1-\lambda|]}{(k+y) (\ln(k+y))^{2-\lambda}} > 2, \quad y \in (0, y_0]. \end{aligned}$$

And, by (A.3), (A.5) and (A.1) we can deduce that, in view of $\bar{p} > 1$,

$$\begin{aligned}
\bar{f}'(y_0) &\leq 2y_0 - 2x(\ln(k + y_0))^\lambda \\
&\leq 2(k + x)^{\frac{1}{\bar{p}}} - 2k - 2\sqrt{\bar{p}}x(\ln(k + x))^\lambda \\
&\leq 2(k + x)^{\frac{1}{\bar{p}}} - 2\sqrt{\bar{p}}(k + x)(\ln(k + x))^\lambda \\
&\leq 2k^{\frac{1}{\bar{p}}} - 2\sqrt{\bar{p}}k(\ln k)^\lambda < 0.
\end{aligned} \tag{A.7}$$

Consequently, for each $y \in (0, y_0]$, we have $\bar{f}'(y) \leq \bar{f}'(y_0) < 0$ and then $\bar{f}(y) \geq \bar{f}(y_0) \geq 0$.

In conclusion, (2.9) holds. Finally, we verify that when $p = 1$ and $\lambda \neq 0$, the constant k such that (2.9) holds does not exist. In fact, assume that (2.9) holds for some $k \geq e$. Let $x, y > 0$ satisfy

$$y = x(\ln(k + y))^\lambda.$$

It is clear that $y > x$ for $\lambda > 0$, and $y < x$ for $\lambda < 0$. Then, in view of (A.2),

$$y^2 - 2xy(\ln(k + y))^\lambda + x^2(\ln(k + x))^{2\lambda} = x^2 \left[(\ln(k + x))^{2\lambda} - (\ln(k + y))^{2\lambda} \right] < 0,$$

from which the desired assertion follows immediately. The proof is then complete. \square

Remark A.1. *The case of $\lambda < 0$ in Proposition 2.6 has been established in Proposition 3.2 of [48]. However, our proof method is more direct and simple than that in [48]. The case of $\lambda > 0$ in Proposition 2.6 can be comparable with Proposition 3.2 in [47], where the constant p appearing in (2.9) is required to be strictly bigger than $4^{(\lambda-1)^+}$. From this point of view, Proposition 2.6 strengthens Proposition 3.2 in [47] for the case of $\lambda > 1$.*

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