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Ayoub Belhadji, Rémi Bardenet, Pierre Chainais. Signal reconstruction using determinantal sampling. 2023. hal-04181079

HAL Id: hal-04181079

<https://hal.science/hal-04181079>

Preprint submitted on 18 Aug 2023

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Signal reconstruction using determinantal sampling

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August 14, 2023

Abstract

We study the approximation of a square integrable function from a finite number of its evaluations at well-chosen nodes. The function is assumed to belong to a reproducing kernel Hilbert space (RKHS), the approximation to be one of a few natural finite-dimensional approximations, and the nodes to be drawn from one of two probability distributions. Both distributions are related to determinantal point processes, and use the kernel of the RKHS to favor RKHS-adapted regularity in the random design. While previous work on determinantal sampling relied on the RKHS norm, we prove mean-square guarantees in L^2 norm. We show that determinantal point processes and mixtures thereof can yield fast rates, and that they shed some light on how the rate changes as more smoothness is assumed, a phenomenon known as superconvergence. Besides, determinantal sampling generalizes i.i.d. sampling from the Christoffel function, a standard in the literature. In particular, determinantal sampling guarantees the so-called instance optimality property for a smaller number of function evaluations than i.i.d. sampling.

Keywords— Christoffel sampling; instance optimality property; finite-dimensional approximations; determinantal point processes; reproducing kernel Hilbert spaces.

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1 Introduction

The problem of reconstructing a continuous signal from a set of discrete samples is very common in signal processing and it has stimulated a considerable literature. This problem aims at approximating an unknown function f by a surrogate \hat{f} knowing a set of evaluations of f . The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem is arguably the most emblematic result in the field. It can be seen as an interpolation result for functions that are band-limited in the Fourier domain [Whittaker, 1928, Shannon, 1949, Kotel'nikov, 2006]. This theorem has been extended to functions that are band-limited with respect to other transforms than Fourier transform including for instance Sturm-Liouville transforms [Kramer, 1959, Campbell, 1964], Laguerre transforms [Jerri, 1976] and Jacobi transforms [Koornwinder and Walter, 1990].

Reproducing kernel Hilbert spaces (RKHSs) and sampling problems have a long common history. The definition of an RKHS, going back to [Aronszajn, 1943, 1950] can be interpreted by saying that any signal $f \in \mathcal{F}$ is a limit of weighted sums of kernel translates $k(x, \cdot)$. In a seminal paper, Yao [1967] derived sufficient conditions on a configuration $(x_i)_{i \in \mathcal{I}}$ of *nodes* to achieve exact reconstruction of any element f of the RKHS \mathcal{F} , i.e., to ensure that uniformly,

$$f = \sum_{i \in \mathcal{I}} f(x_i) k(x_i, \cdot). \quad (1)$$

This is all the more related to signal processing that Yao proved that the spaces of band-limited signals in the Fourier, Bessel, or cosine domains are all RKHSs. In particular, Yao's result generalizes the WSK theorem. Weaker sufficient conditions than Yao's for (1) to hold have been studied, see e.g. [Nashed and Walter, 1991].

However the WSK sampling theorem and its extensions to RKHSs are *asymptotic*: an infinite number of samples is required in order to guarantee the exact reconstruction of the function. In applications, only a finite number of evaluations $f(x_1), \dots, f(x_N)$, at nodes $x_1, \dots, x_N \in \mathcal{X}$, will be available. Therefore we seek *non-asymptotic* guarantees on reconstructions from N samples. Interestingly, the latter problem arose first in the order of events, as mentioned by Higgins [1985]. Indeed, an early form of sampling theorem may be traced back to an interpolation scheme due

to de La Vallée Poussin [1908]; see [Butzer and Stens, 1992] for a historical account. Non-asymptotic sampling have resurged in popularity recently [Cohen and Migliorati, 2017, Bach, 2017a, Avron et al., 2019]. In these works, the nodes x_1, \dots, x_N are taken to be independent draws from a particular probability distribution. The latter is closely related to extensions of the so-called *Christoffel function*, a classical tool in the theory of orthogonal polynomials [Nevai, 1986]. These configurations of independent particles were shown to yield approximations that satisfy the *instance optimality property*, a desirable multiplicative error bound that guarantees exact recovery on a finite-dimensional subspace, with an almost optimal number of particles. On the other hand, alternative designs have been proposed to achieve optimal reconstruction, using boosting [Haberstich et al., 2022] or sparsification techniques [Krieg and Ullrich, 2021, Dolbeault and Cohen, 2022b, Dolbeault et al., 2023, Chkifa and Dolbeault, 2023].

Our work presents a general approach for signal reconstruction, based on random sets of nodes drawn from two repulsive point processes. The key idea is to use random nodes sampled from mixtures of determinantal point processes (DPP). DPPs are distributions over configurations of points that encode repulsion in the form of a kernel: connecting the DPP’s kernel to the RKHS kernel allows the nodes to repel by an amount related to the smoothness of the target function. DPPs were introduced by Macchi [1975] as models for detection times in fermionic optics. Since then, they have been thoroughly studied in random matrix theory [Johansson, 2005], and have more recently been adopted in machine learning [Kulesza and Taskar, 2012], spatial statistics [Lavancier et al., 2015], and Monte Carlo methods [Bardenet and Hardy, 2020].

We have worked on non-asymptotic guarantees for function reconstruction with DPPs motivated by numerical integration [Belhadji et al., 2019, 2020]. In these works, we measured reconstruction performance using the RKHS norm, and we had to make a rather strong smoothness assumption that the function belongs to a particular strict subspace of the RKHS. Motivated this time by applications in signal reconstruction, and to overcome the smoothness assumption of our previous work, we conduct here an alternative analysis where we replace the RKHS norm by the L_2 norm. In particular, we study various finite-dimensional approximations of a function living in an RKHS, like the the least-squares approximation with respect to the L_2 norm. We show that its mean square error converges to 0, at a rate that depends on the eigenvalues of the RKHS kernel. Moreover, we show that the convergence is faster for functions living in a certain low-dimensional subspace of the RKHS. This sheds the light on the phenomenon of *superconvergence* observed in the literature of kernel-based approximations [Schaback, 2018], and gives new insight into the the question of *completeness* with DPPs [Lyons, 2014]. Yet, in spite of its remarkable theoretical properties, the least-squares approximation with respect to the L_2 norm cannot be evaluated using a finite number of evaluations of the target function. For this reason, we investigate more practical approximations based on two particular *transforms*, i.e. linear operators from the RKHS to finite-dimensional vector spaces, built through a compilation of quadrature rules. In particular, we prove that the instance optimality property holds for a particular transform-based approximation using DPPs, even with a minimal sampling budget. Numerical experiments in dimension one as well as on the hypersphere validate our results and show the empirical efficiency of the proposed transform-based approach. The performance is even better than the already interesting theoretical guarantees, which at the same time confirms the interest of the approach and leaves room for improvement of the bounds.

This article is organized as follows. Section 2 contains notations and definitions of the finite-dimensional approximations that we will study, as long as a brief review of previous work on the reconstruction of functions, based on discrete samples, living in RKHSs. In Section 3, we present the main results of this work. Finally, in Section 4 we illustrate our result and compare to related work using numerical simulations.

2 Sampling and reconstruction in RKHSs

In Section 2.1.1, for ease of reference, we introduce some basic notation and assumptions to be used throughout the paper. In Section 2.1.2, we define fractional subspaces, which correspond to increasing levels of smoothness within an RKHS. In Section 2.2, we recall the finite-dimensional approximations that we will work with in the rest of the article. To give historical context, Section 2.3 and Section 2.4 are devoted to related work on the topic. The contents of these two sections may be skipped in a first read except for Section 2.3.2 which is necessary for the statement of the main results presented in Section 3.

2.1 Kernels and RKHS

2.1.1 Notations and assumptions

Let \mathcal{X} be a metric set, equipped with a Borel measure ω . Let $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a symmetric, positive definite kernel, and consider the inner product defined on the space \mathcal{F}_0 of finite linear combinations of kernel translates by

$$\left\langle \sum_{i=1}^m a_i k(x_i, \cdot), \sum_{j=1}^n b_j k(y_j, \cdot) \right\rangle_{\mathcal{F}} = \sum_{i=1}^m \sum_{j=1}^n a_i b_j k(x_i, y_j), \quad (2)$$

where $a_1, b_1, \dots \in \mathbb{R}$ and $x_1, y_1, \dots \in \mathcal{X}$. In the sequel, $[N]$ will denote the set of indices ranging from 1 to N . The completion \mathcal{F} of \mathcal{F}_0 for $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ is the so-called reproducing kernel Hilbert space of kernel k . By the Moore-Aronszajn theorem, it is the unique Hilbert space of functions that satisfies the reproducing property, i.e., for all $f \in \mathcal{F}$ and $x \in \mathcal{X}$, $f(x) = \langle f, k(x, \cdot) \rangle_{\mathcal{F}}$; see e.g. [Berlinet and Thomas-Agnan, 2011] for a general reference.

The properties of a function in \mathcal{F} are often described in terms of the spectral characteristics of the integral operator of kernel k . More precisely, assume that

$$\Sigma f(x) = \int k(x, y) f(y) d\omega(y). \quad (3)$$

defines a compact operator Σ from the space $\mathbb{L}_2(\omega)$ of real-valued square-integrable functions to itself. A practical sufficient condition for this compactness to hold is given in [Steinwart and Scovel, 2012, Lemma 2.3].

Assumption 1. *The diagonal $x \mapsto k(x, x)$ is integrable.*

Under Assumption 1, the spectral theorem, see e.g. Chapter 6 in [Brezis, 2010], guarantees the existence of an orthonormal basis $(e_m)_{m \in \mathbb{N}^*}$ of $\mathbb{L}_2(\omega)$ and a family of non-increasing non-negative scalars $(\sigma_m)_{m \in \mathbb{N}^*}$ such that $\Sigma e_m = \sigma_m e_m$ for $m \in \mathbb{N}^*$. We now make extra assumptions to reach a Mercer decomposition.

Assumption 2. *The kernel k is continuous, ω has full support, and $\sigma_m > 0$ for all $m \in \mathbb{N}^*$.*

The continuity of the kernel guarantees that e_m may be taken to be a continuous function when $\sigma_m > 0$; see e.g. Theorem 4.7 in [Cucker and Zhou, 2007]. Moreover, the positivity of all σ_m s and the fact that ω has full support imply that \mathcal{F} is dense in $\mathbb{L}_2(\omega)$; see Section 4.2 in [Cucker and Zhou, 2007]. Letting $e_m^{\mathcal{F}} := \sqrt{\sigma_m} e_m$, the family $(e_m^{\mathcal{F}})_{m \in \mathbb{N}^*}$ is then an orthonormal basis of \mathcal{F} ; see Theorem 4.12 in [Cucker and Zhou, 2007]. In particular, for $f \in \mathcal{F}$ and $m \in \mathbb{N}^*$, we have $\langle f, e_m^{\mathcal{F}} \rangle_{\mathcal{F}} = \langle f, e_m \rangle_{\omega} / \sqrt{\sigma_m}$, so that

$$\|f\|_{\mathcal{F}}^2 = \sum_{m \in \mathbb{N}^*} \frac{\langle f, e_m \rangle_{\omega}^2}{\sigma_m} < +\infty. \quad (4)$$

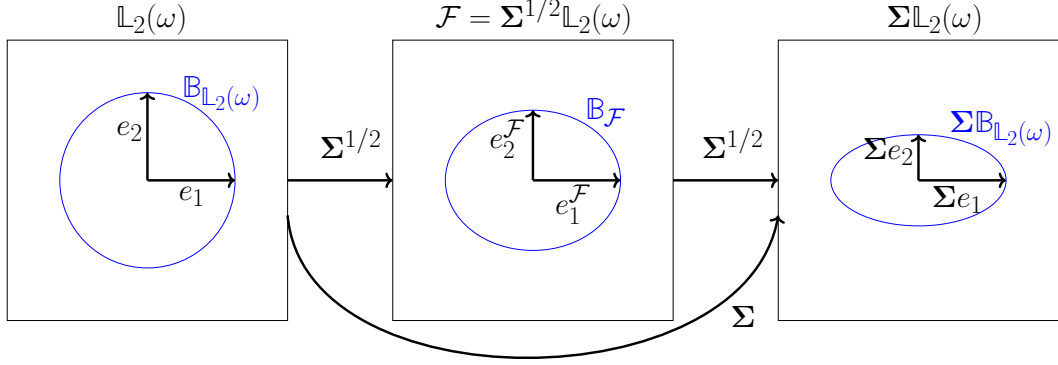


Figure 1: A schematic diagram illustrating the relationship between the unit ball $\mathbb{B}_{\mathbb{L}_2(\omega)}$ of $\mathbb{L}_2(\omega)$, the unit ball $\mathbb{B}_{\mathcal{F}}$ of \mathcal{F} , and the image of $\mathbb{B}_{\mathbb{L}_2(\omega)}$ by the integration operator Σ .

Moreover, by the reproducing property, for $x \in \mathcal{X}$ and $m \geq 1$, $\langle e_m^{\mathcal{F}}, k(x, \cdot) \rangle_{\mathcal{F}} = e_m^{\mathcal{F}}(x)$. Now, the decomposition of $k(x, \cdot)$ on the basis $(e_n^{\mathcal{F}})$ yields the *Mercer decomposition* of k ,

$$k(x, y) = \langle k(x, \cdot), k(y, \cdot) \rangle_{\mathcal{F}} = \sum_{m \in \mathbb{N}^*} e_m^{\mathcal{F}}(x) e_m^{\mathcal{F}}(y) = \sum_{m \in \mathbb{N}^*} \sigma_m e_m(x) e_m(y). \quad (5)$$

Actually, under Assumption 2, the convergence in (5) holds uniformly on any compact $A \times A \subset \mathcal{X} \times \mathcal{X}$; see [Steinwart and Scovel, 2012, Corollary 3.5]. Moreover, under Assumption 2 the pointwise convergence of the r.h.s. of (5) is equivalent to the injectivity of the embedding $I : \mathcal{F} \rightarrow \mathbb{L}_2(\omega)$; see [Steinwart and Scovel, 2012, Theorem 3.1] and [Steinwart and Christmann, 2008, Chapter 4]. We assume in this paper that Assumptions 1 and 2 hold.

2.1.2 Increasing levels of smoothness

Equation (4) yields more intuition on why RKHSs are considered spaces of smooth functions. Indeed, by definition, f lying in such an RKHS means that f has generalized Fourier coefficients $\langle f, e_m \rangle_{\omega}$ that decay fast enough in modulus for the sum in (4) to converge, thus ensuring smoothness. Note also that, for $f \in \mathcal{F}$,

$$\tilde{f} := \sum_{m \in \mathbb{N}^*} \langle f, e_m^{\mathcal{F}} \rangle_{\mathcal{F}} e_m \in \mathbb{L}_2(\omega)$$

satisfies $f = \Sigma^{1/2} \tilde{f}$ and $\|f\|_{\mathcal{F}} = \|\tilde{f}\|_{\omega}$. In other words, $\mathcal{F} = \Sigma^{1/2} \mathbb{L}_2(\omega)$, which illustrates the smoothing nature of the integral operator Σ as well. Similarly, for $r \geq 0$, an element $g \in \mathbb{L}_2(\omega)$ belongs to $\Sigma^{r+1/2} \mathbb{L}_2(\omega)$ if and only if $\sum_{m \in \mathbb{N}^*} \langle g, e_m \rangle_{\omega}^2 / \sigma_m^{2r+1} < +\infty$. Moreover, since (σ_m) is a (non-increasing) sequence of positive real numbers, we have

$$r \geq r' \geq 0 \implies \Sigma^{r+1/2} \mathbb{L}_2(\omega) \subset \Sigma^{r'+1/2} \mathbb{L}_2(\omega). \quad (6)$$

In particular, since $\mathcal{F} = \Sigma^{1/2} \mathbb{L}_2(\omega)$, we know that for $r \geq 0$, $\Sigma^{r+1/2} \mathbb{L}_2(\omega) \subset \mathcal{F}$: these subspaces define increasing levels of regularity in the RKHS \mathcal{F} . Figure 1 illustrates this hierarchy of functional spaces and gives some more intuition.

2.2 Finite-dimensional approximations in RKHSs

The reconstruction of a function f that belongs to an RKHS \mathcal{F} based on its evaluations over a finite set of nodes $x_1, \dots, x_N \in \mathcal{X}$ can be achieved by different families of finite-dimensional approximations. The following approximations will be of interest for the rest of the article.

2.2.1 Approximations based on mixtures of kernel translates

By definition of an RKHS, a natural choice of approximation is a weighted sum of kernel functions. Formally, the objective is thus to build a set of nodes (x_i) , a.k.a. a *design*, and weights (w_i) , such that a suitable norm of the residual

$$f - \sum_{i \in [N]} w_i k(x_i, \cdot) \quad (7)$$

is small. Assuming the design $\mathbf{x} = (x_i)_{i \in [N]}$ is fixed, we now consider two possible choices for this norm that induce different sets of weights.

Minimizing the $\mathbb{L}_2(\omega)$ norm of the residual (7) yields the classical least-squares (LS) approximation

$$\hat{f}_{\text{LS}, \mathbf{x}} = \arg \min_{\mathbf{w} \in \mathbb{R}^N} \|f - \sum_{i \in [N]} w_i k(x_i, \cdot)\|_{\omega}^2. \quad (8)$$

The least-squares approximation typically enjoys strong theoretical properties. However, we shall see in Section 3.1 that computing the optimal weights in (8) requires the evaluation of Σf at the nodes x_1, \dots, x_N . This makes the method impractical, and calls for more tractable approximations.

The RKHS norm of the residual is a natural alternative objective to minimize, and yields the so-called *optimal kernel approximation* (OKA)

$$\hat{f}_{\text{OKA}, \mathbf{x}} = \arg \min_{\mathbf{w} \in \mathbb{R}^N} \|f - \sum_{i \in [N]} w_i k(x_i, \cdot)\|_{\mathcal{F}}. \quad (9)$$

We shall sometimes write \hat{f}_{OKA} for $\hat{f}_{\text{OKA}, \mathbf{x}}$ when the design \mathbf{x} is clear from the context. Let us observe that the approximation (9) is uniquely defined if the matrix $\mathbf{K}(\mathbf{x}) = (k(x_i, x_{i'}))_{i, i' \in [N]}$ is non-singular. The optimal vector of weights is then simply $\hat{\mathbf{w}}_{\text{OKA}, \mathbf{x}} := \mathbf{K}(\mathbf{x})^{-1} f(\mathbf{x})$; unlike the least-squared approximation, computing the weights solely depends on being able to evaluate f at the nodes. Moreover, OKA comes with a remarkable interpolation property: for $i \in [N]$, $\hat{f}_{\text{OKA}, \mathbf{x}}(x_i) = f(x_i)$.

2.2.2 Approximations living in eigenspaces

Another way to recover a continuous signal $f \in \mathcal{F}$ from its evaluations $f(x_1), \dots, f(x_N)$ at fixed nodes, is to seek an approximation that belongs to the eigenspace $\mathcal{E}_M = \text{Span}(e_1, \dots, e_M)$ for some $M \in \mathbb{N}^*$. We consider two such approaches.

Transform-based approximations. Consider the projection of f on \mathcal{E}_M ,

$$f_M := \sum_{m \in [M]} \langle f, e_m \rangle_{\omega} e_m. \quad (10)$$

Note that, for $m \in [M]$, $\langle f, e_m \rangle_\omega$ is the integral $I_m(f) := \int_{\mathcal{X}} f(x) e_m(x) d\omega(x)$. Since we want to assume only the availability of the evaluation of f at the nodes, we consider approximating each integral $I_m(f)$ by a weighted sum

$$\hat{I}_m(f) := \sum_{i \in [N]} \alpha_{m,i} f(x_i), \quad (11)$$

where $\alpha_{m,i}$ are quadrature weights to be discussed shortly. The collection of operators \hat{I}_m form a *transform* Φ , defined as $\Phi(f) = (\hat{I}_m(f))_{m \in [M]} \in \mathbb{R}^M$. The resulting approximation \hat{f}_Φ will be called a *transform-based approximation* of f , and we observe that

$$\hat{f}_\Phi := \sum_{m \in [M]} \hat{I}_m(f) e_m = \sum_{i \in [N]} \left(\sum_{m \in [M]} \alpha_{m,i} \right) f(x_i), \quad (12)$$

is a weighted sum of evaluations of f at the nodes (x_i) .

For the construction of the quadrature weights $(\alpha_{m,i})$ in (11), this work focuses on *interpolative quadrature rules* [Larkin, 1972]: given a kernel $\kappa : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, the weights $\alpha_{m,i}$ of an interpolative quadrature are chosen so that

$$\forall i \in [N], m \in [N], \hat{I}_m(\kappa(x_i, \cdot)) = I_m(\kappa(x_i, \cdot)). \quad (13)$$

Under the assumption that the functions $\kappa(x_1, \cdot), \dots, \kappa(x_N, \cdot)$ are linearly independent, the sequence $(\alpha_{m,i})_{(m,i) \in [N] \times [N]}$ is uniquely defined by (13).

In general, (13) only holds for $\hat{I}_1, \dots, \hat{I}_N$. Yet, when $\kappa = k$, the property (13) extends to \hat{I}_m for $m > N$ as well. Indeed, when $\kappa = k$, we have $I_m(\kappa(x, \cdot)) = \sigma_m e_m(x)$ for $x \in \mathcal{X}$, so that (13) implies that the vector $\boldsymbol{\alpha}_m := (\alpha_{m,i})_{i \in [N]}$ satisfies

$$\boldsymbol{\alpha}_m = \sigma_m \mathbf{K}(\mathbf{x})^{-1} e_m(\mathbf{x}), \quad (14)$$

under the assumption that the matrix $\mathbf{K}(\mathbf{x})$ is non-singular. In other words, the vector $\boldsymbol{\alpha}_m$ is nothing but the one that minimizes (9) when f is $\boldsymbol{\Sigma} e_m = \sigma_m e_m$. We name the resulting quadrature rule the *optimal kernel quadrature* (OKQ). In the rest of the article, the corresponding transform will be called the *optimal kernel quadrature transform*, the corresponding quadrature rules denoted by $\hat{I}_1^{\text{OKQ}}, \dots, \hat{I}_M^{\text{OKQ}}$, and the resulting approximation by $\hat{f}_{\text{OKQ}, M, \mathbf{x}}$.

Alternatively, by taking κ equal to $k_N(x, y) := \sum_{n \in [N]} e_n(x) e_n(y)$, (13) boils down to

$$\forall m, m' \in [N], \hat{I}_m(e_{m'}) = \delta_{m, m'}, \quad (15)$$

which is reminiscent of a property satisfied by Gaussian quadrature [Gautschi, 2004]. Moreover, for $m \in [N]$, we have

$$\boldsymbol{\alpha}_m = \mathbf{K}_N(\mathbf{x})^{-1} e_m(\mathbf{x}), \quad (16)$$

where $\mathbf{K}_N(\mathbf{x}) := (k_N(x_i, x_{i'}))_{i, i' \in [N]}$. Note the similarity with (14). Except for some exceptional families of functions such as orthogonal polynomials, the quadrature weights (16) do not guarantee that (13) can be extended to $m > N$. Finally, note that the resulting approximation, denoted by $\hat{f}_{\text{QI}, N, \mathbf{x}}$ has been called *quasi-interpolant* or *hyperinterpolant* in the literature [Sloan, 1995], since

$$\forall f \in \mathcal{E}_M = \text{Span}(e_1, \dots, e_M), \hat{f}_{\text{QI}, M, \mathbf{x}} = f = f_M. \quad (17)$$

In the rest of the article, we denote the corresponding quadrature rules by $\hat{I}_1^{\text{QI}}, \dots, \hat{I}_N^{\text{QI}}$.¹

¹Note that for $m \in [N]$, \hat{I}_m^{QI} depends on N and \mathbf{x} . Yet, we drop N and \mathbf{x} from the notation for simplicity.

The empirical least-squares approximation. Let $\mathbf{x} \in \mathcal{X}^N$ and $q : \mathcal{X} \rightarrow \mathbb{R}_+^*$. Consider the so-called *empirical* semi-norm $\|\cdot\|_{q,\mathbf{x}}$ defined on $\mathbb{L}_2(\omega)$ by

$$\|h\|_{q,\mathbf{x}}^2 := \frac{1}{N} \sum_{i=1}^N q(x_i) h(x_i)^2. \quad (18)$$

The empirical least-squares estimator yields yet another approximation

$$\hat{f}_{\text{ELS},M,\mathbf{x}} := \arg \min_{\hat{f} \in \mathcal{E}_M} \|f - \hat{f}\|_{q,\mathbf{x}}^2, \quad (19)$$

We discuss the choice of q and the design \mathbf{x} in Section 2.3.1. For the moment, the intuition is that, for a well-chosen weight q and design \mathbf{x} , the semi-norm $\|\cdot\|_{q,\mathbf{x}}$ is supposed to “mimic” the $\|\cdot\|_\omega$ norm as N goes to infinity. Similarly, $\hat{f}_{\text{ELS},M,\mathbf{x}}$ is supposed to inherit some of the properties as f_M , the projection of f on \mathcal{E}_M . Compared to $\hat{f}_{\text{LS},\mathbf{x}}$, the approximation $\hat{f}_{\text{ELS},M,\mathbf{x}}$ has the advantage of being computable given the evaluations of f at the nodes \mathbf{x} . Indeed, Cohen et al. [2013] show that writing $\hat{f}_{\text{ELS},M,\mathbf{x}} = \sum_{m=1}^M w_m e_m$ yields

$$\mathbf{G}_{q,\mathbf{x}} \mathbf{w} = \mathbf{d}_{q,\mathbf{x}}, \quad (20)$$

where $\mathbf{G}_{q,\mathbf{x}} = (\langle e_m, e_{m'} \rangle_{q,\mathbf{x}})_{m,m' \in [M]} \in \mathbb{R}^{M \times M}$ is the Gramian matrix of the family $(e_m)_{m \in [M]}$ with respect to the inner product defined by (18), and $\mathbf{d}_{q,\mathbf{x}} = (d_m)_{m \in [M]} \in \mathbb{R}^M$ is defined by

$$d_m := \sum_{i=1}^N q(x_i) f(x_i) e_m(x_i) / N, \quad m \in [M].$$

In other words, the numerical evaluation of $\hat{f}_{\text{ELS},M,\mathbf{x}}$ requires to solve the linear system (20). This in turn requires the evaluation of $\mathbf{G}_{q,\mathbf{x}}$ and $\mathbf{d}_{q,\mathbf{x}}$, which boils down to the evaluation of f , the functions e_m and q on the nodes x_1, \dots, x_N . Moreover, $\hat{f}_{\text{ELS},M,\mathbf{x}}$ is uniquely defined if and only if $\mathbf{G}_{q,\mathbf{x}}$ is non-singular.

2.3 Designs for finite-dimensional approximations

An abundant literature provides theoretical guarantees for the finite-dimensional approximations presented in Section 2.2; this includes, but is not limited to, [Erdős and Turán, 1937, Sloan, 1995, Wendland, 2004, Schaback and Wendland, 2006]. These results deal with specific RKHSs, and cannot be easily generalized to an arbitrary RKHS. Recently, a new tendency has emerged that looks for a universal sampling approach that is valid for a large class of RKHSs. In this section, we first provide an overview of this universal sampling literature. In the same spirit, we then show that determinantal point processes (DPPs) offer an adequate framework to design configurations of nodes with strong theoretical guarantees for finite-dimensional approximations, in a wide family of RKHSs.

2.3.1 Independent samples from the Christoffel function

The last decade has seen significant progress in the study of function reconstruction based on randomized configurations [Cohen et al., 2013, Hampton and Doostan, 2015, Cohen and Migliorati, 2017, Adcock and Cardenas, 2020, Adcock et al., 2022, Dolbeault and Cohen, 2022a]. These works focused on the study of the empirical least-squares approximation; see Section 2.2.2. In

particular, the so-called *instance optimality property* (IOP) has been a matter of extensive investigation. The idea is to find assumptions on the configuration \mathbf{x} , the function $q : \mathcal{X} \rightarrow \mathbb{R}_+^*$, and the order $M \leq N$, under which we can certify that²

$$\forall f \in \mathcal{F}, \quad \|f - \hat{f}_{\text{ELS},M,\mathbf{x}}\|_\omega \leq C \|f - f_M\|_\omega, \quad (21)$$

where $C > 0$ is a constant. Instance optimality implies that $\hat{f}_{\text{ELS},M,\mathbf{x}} = f$ for $f \in \mathcal{E}_M$, so that the reconstruction is exact in the eigenspace \mathcal{E}_M . Moreover, it provides an upper bound of the approximation error $\|f - \hat{f}_{\text{ELS},M,\mathbf{x}}\|_\omega$ for generic functions. Investigating necessary conditions for the IOP to hold unraveled the importance of the study of the eigenvalues of the Gramian matrix $\mathbf{G}_{q,\mathbf{x}}$ [Cohen and Migliorati, 2017]. In particular, it was proved that

$$\|\mathbf{G}_{q,\mathbf{x}} - \mathbb{1}_M\|_{\text{op}} \leq \delta \iff \forall f \in \mathcal{E}_M, \quad (1 - \delta)\|f\|_\omega^2 \leq \|f\|_{q,\mathbf{x}}^2 \leq (1 + \delta)\|f\|_\omega^2, \quad (22)$$

where $\mathbb{1}_M \in \mathbb{R}^{M \times M}$ is the identity matrix of order M , and $\|\cdot\|_{\text{op}}$ is the operator norm. In other words, the $\|\cdot\|_{q,\mathbf{x}}$ norm is equivalent to $\|\cdot\|_\omega$ on \mathcal{E}_M if and only if the matrix $\mathbf{G}_{q,\mathbf{x}}$ is close to $\mathbb{1}_M$. Interestingly, Gröchenig [2020] showed that the condition (22) was connected to Marcinkiewicz-Zygmund inequalities, which are common tools in the literature of non-uniform sampling [Gröchenig, 1993, Ortega-Cerdà and Saludes, 2007, Filbir and Mhaskar, 2011]; see [Gröchenig, 2020] for more references. Cohen and Migliorati [2017] proved necessary conditions for $\|\mathbf{G}_{q,\mathbf{x}} - \mathbb{1}_M\|_{\text{op}} \leq 1/2$ to hold with high probability, when x_1, \dots, x_N are i.i.d. draws from the measure $\tilde{\omega}$ defined by $d\tilde{\omega} = 1/q d\omega$. In particular, they proved that the sampling budget N should scale as $\|q/c_M\|_\infty \log(2M)$, where c_M is the inverse of the so-called *Christoffel function*

$$c_M(x) := \sum_{m=1}^M e_m(x)^2; \quad x \in \mathcal{X}, \quad (23)$$

and $\|\cdot\|_\infty$ is the infinity norm on \mathcal{X} . Since the required sampling budget grows with $\|q/c_M\|_\infty$, it is preferable to choose the function q in such a way that $\|q/c_M\|_\infty$ is minimized, under the constraint that $\int_{\mathcal{X}} 1/q(x) d\omega(x) = 1$. In particular, if we take $q = c_M/M$, the constant $\|q/c_M\|_\infty$ grows linearly with M , and the required sampling budget scales as $M \log(M)$. This is to be compared with the situation when q is taken to be a constant, where the required sampling budget would scale as $\mathcal{O}(M \|1/c_M\|_\infty \log(M))$. The constant $\|1/c_M\|_\infty$, also called the *Nikolskii constant* in approximation theory, is known to be at best linear in M for many families of orthogonal polynomials [Nevai, 1986, Bos, 1994, Xu, 1996, Totik, 2000], hinting at a suboptimal sampling budget in $\mathcal{O}(M^2 \log(M))$. Achieving the IOP using a sampling budget that scales linearly in M was proved to be possible using sparsification techniques [Dolbeault and Cohen, 2022b, Chkifa and Dolbeault, 2023]. These results rely on the extension of techniques developed to solve the Kadison-Singer problem [Marcus et al., 2015]. However, as was shown in [Dolbeault and Cohen, 2022b], the algorithmic complexity of these techniques is exponential in N .

Alternately, a similar approach was adopted by Bach [2017a], yet with a kernel-based approximation that solves a regularized variant of the optimization problem (9). In particular, his analysis relied on the so-called *regularized leverage score function*

$$\tilde{c}_\lambda(x) := \sum_{m=1}^{+\infty} \frac{\sigma_m}{\sigma_m + \lambda} e_m(x)^2, \quad (24)$$

where $\lambda > 0$ is a regularization constant. In particular, Bach [2017a] showed that when the nodes are i.i.d. draws from $\tilde{c}_\lambda d\omega$, the resulting approximation converges to 0 at an almost optimal rate, as λ is made to go to zero with N at a suitable rate. One practical downside, compared to (23), is the need for an infinite summation.

²This is a specific formulation of the IOP. See Cohen and Migliorati [2017] for a more generic formulation.

2.3.2 Determinantal sampling

In [Belhadji et al., 2019] and [Belhadji et al., 2020], we have investigated two random designs defined using determinants and Gram matrices.

Definition 1 (A projection DPP). *Let $N \geq 1$, and*

$$k_N(x, y) = \sum_{n \in [N]} e_n(x)e_n(y), \quad (25)$$

where (e_n) are the eigenfunctions in the Mercer decomposition (5) of the RKHS kernel k . The design $\mathbf{x} = (x_1, \dots, x_N)$ is said to have for distribution the DPP of kernel k_N and reference measure ω if

$$\mathbf{x} = (x_1, \dots, x_N) \sim \frac{1}{N!} \text{Det} \left(k_N(x_i, x_j) \right)_{(i,j) \in [N] \times [N]} d\omega(x_1) \dots d\omega(x_N). \quad (26)$$

Since k_N is a projection kernel, the resulting point process is a projection DPP and (26) integrates to 1. In the remainder of the paper, \mathbb{E}_{DPP} is to be understood as an expectation under (26). DPPs were introduced by Macchi [1975], and possess many interesting properties [Hough et al., 2006]. For instance, any point of \mathbf{x} in (26) has marginal distribution

$$x_i \sim \frac{1}{N} \sum_{n=1}^N e_n(x_i)^2 d\omega(x_i), \quad (27)$$

which is related to the inverse of the Christoffel function (23). In that sense, (26) generalizes previous work on i.i.d. designs sampled from the inverse of the Christoffel function, but adding a kernel-dependent correlation among the nodes. Another useful property of DPPs with projection kernels like (26) is that the chain rule for \mathbf{x} amounts to a product of explicit “base-times-height” terms. This yields a polynomial-time, *exact* sampling algorithm colloquially known as HKPV, after the authors of [Hough et al., 2006].

Continuous volume sampling, introduced in [Belhadji et al., 2020], is a closely related distribution for nodes.

Definition 2 (continuous volume sampling). *Assume that $\int_{\mathcal{X}} k(x, x) d\omega(x) < \infty$. The design $\mathbf{x} = (x_1, \dots, x_N)$ is said to have a distribution according to continuous volume sampling if*

$$\mathbf{x} = (x_1, \dots, x_N) \sim \frac{1}{Z} \text{Det} \left(k(x_i, x_j) \right)_{(i,j) \in [N] \times [N]} d\omega(x_1) \dots d\omega(x_N), \quad (28)$$

where $Z < \infty$ is a normalization constant.

In the remainder of the paper, \mathbb{E}_{CVS} is to be understood as an expectation under (28). Note that unlike (26), the normalization constant is not explicit. Yet Hadamard’s inequality yields

$$\begin{aligned} Z_{\kappa, \omega} &= \int_{\mathcal{X}^N} \text{Det} \left(\kappa(x_i, x_j) \right) \prod_{n \in [N]} d\omega(x_n) \leq \int_{\mathcal{X}^N} \prod_{n \in [N]} \kappa(x_n, x_n) d\omega(x_n) \\ &\leq \left(\int_{\mathcal{X}} \kappa(x, x) d\omega(x) \right)^N < +\infty. \end{aligned}$$

While the chain rule for continuous volume sampling is not as simple as for the DPP in (26), continuous volume sampling can actually be shown to be a statistical mixture of projection DPPs. In words, \mathbf{x} in (28) can be drawn by first sampling a N -uplet $I \in \mathbb{N}^N$ proportionally to $\prod_{i \in I} \sigma_i$, and then drawing \mathbf{x} from (26) with k_N replaced by $\sum_{i \in I} e_i(x)e_i(y)$. In that sense, continuous volume sampling is a “soft” modification of the DPP (26), which is the component with largest weight in the mixture; see [Belhadji et al., 2020] for more details.

2.4 Existing results on RKHS sampling using DPPs and CVS

This section gathers existing works on function reconstruction using determinantal distributions. We provide some details because we shall use some results in Section 3.

2.4.1 The Ermakov-Zolotukhin quadrature rule

Non-asymptotic reconstruction guarantees for functions living in Hilbert spaces can be traced back to [Ermakov and Zolotukhin \[1960\]](#). Indeed, in this work, the authors studied quadrature rules $\hat{I}_1, \dots, \hat{I}_N$ obtained by taking the configuration of the nodes \mathbf{x} to follow the distribution of the projection DPP as in Definition 1 with $\kappa = k_N$ in Section 2.3.2, and the corresponding vectors of weights $\alpha_1, \dots, \alpha_N$ to be equal to (16). In particular, they proved that for a continuous function $f : \mathcal{X} \rightarrow \mathbb{R}$ living in $\mathbb{L}_2(\omega)$, we have

$$\forall n \in [N], \begin{cases} \mathbb{E}_{\text{DPP}} \hat{I}_n(f) &= I_n(f) \\ \mathbb{V}_{\text{DPP}} \hat{I}_n(f) &= \sum_{m \geq N+1} \langle f, e_m \rangle_\omega^2 \end{cases}. \quad (29)$$

[Gautier et al. \[2019\]](#) revisited this result and proved that

$$\forall n_1, n_2 \in [N], n_1 \neq n_2 \implies \text{Cov}_{\text{DPP}}(\hat{I}_{n_1}(f), \hat{I}_{n_2}(f)) = 0. \quad (30)$$

As it was mentioned in [[Kassel and Lévy, 2022](#)], (29) and (30) imply that the resulting transform³ yields an approximation $\hat{h}_{\text{EZ}, \mathbf{x}}$ that satisfies

$$\forall f \in \mathbb{L}_2(\omega), \mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{QI}, N, \mathbf{x}}\|_\omega^2 = N \|f - f_N\|_\omega^2. \quad (31)$$

In other words, the quasi-interpolant satisfies an instance optimality. Yet, the corresponding constant grows to infinity with N . Observe that (31) holds in $\mathbb{L}_2(\omega)$ and f is not assumed to live in a particular RKHS.

2.4.2 The optimal kernel approximation using determinantal sampling

The study of quadrature rules instigated the inquiry of non-asymptotic guarantees for finite-dimensional approximations based on determinantal nodes for functions living in RKHSs. Indeed, in the context of numerical integration, we have for any function h living in the RKHS \mathcal{F} and for any $g \in \mathbb{L}_2(\omega)$ [[Muandet et al., 2017](#)]

$$\left| \int_{\mathcal{X}} h(x)g(x)d\omega(x) - \sum_{i=1}^N w_i h(x_i) \right|^2 \leq \|h\|_{\mathcal{F}}^2 \|f - \sum_{i=1}^N w_i k(x_i, \cdot)\|_{\mathcal{F}}^2, \quad (32)$$

where $f = \Sigma g$. Moreover, $\|f - \sum_{i=1}^N w_i k(x_i, \cdot)\|_{\mathcal{F}}^2$ corresponds to the squared worst case integration error (WCE) on the unit ball of the RKHS \mathcal{F} . In other words, the squared residual provides an upper bound of the squared error of the approximation of the integral $\int_{\mathcal{X}} h(x)g(x)d\omega(x)$ by the quadrature rule $\sum_{i=1}^N w_i h(x_i)$. This is especially applicable to the optimal kernel quadrature mentioned at Section 2.2.2: by taking the vector of weights α_g is taken to be equal to $\mathbf{K}(\mathbf{x})^{-1}f(\mathbf{x})$, we extend (14) to g and the squared WCE of the corresponding quadrature rule is equal to $\|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2$. Now, it was proven in [[Belhadji et al., 2019](#)] that

$$\mathbb{E}_{\text{DPP}} \sup_{f \in \Sigma \mathbb{B}_{\mathbb{L}_2(\omega)}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 = \mathcal{O}(N^2 r_{N+1}), \quad (33)$$

³See the end of Section 2.2.

where $r_{N+1} := \sum_{m \geq N+1} \sigma_m$. Moreover, it was proven in Theorem 3 of [Belhadji, 2021] that

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 \leq 4\|g\|_{\omega}^2 r_{N+1}. \quad (34)$$

The first upper bound (33) deals with the worst interpolation error on the set $\Sigma\mathbb{B}_{\mathbb{L}_2(\omega)}$. In contrast, the second upper bound (34) is punctual since the expected squared error of interpolation depends on the function f . These upper bounds highlight the importance of the eigenvalues σ_m for the study of the convergence of $\hat{f}_{\text{OKA}, \mathbf{x}}$ under the projection DPP of Definition 1: these quantities converge to 0 if the convergence of σ_m , and moreover of r_N , to 0 is fast enough. These results give convergence rates for the interpolation under the distribution of the projection DPP, that scale, at the best, as $\mathcal{O}(r_N)$, which is slower than the empirical convergence rate $\mathcal{O}(\sigma_{N+1})$ observed in [Belhadji et al., 2019, Belhadji, 2020]. Indeed, when $\sigma_N = N^{-2s}$ for $s > 1/2$, we have $\mathcal{O}(r_N) = \mathcal{O}(N^{1-2s})$, which is slower, by a factor of N , than $\mathcal{O}(\sigma_{N+1})$, which corresponds to the optimal rate of convergence in the following sense: for $N \in \mathbb{N}^*$, there exists $g \in \mathbb{L}_2(\omega)$ such that $\|g\|_{\omega} \leq 1$ and $\|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 \geq \sigma_{N+1}$ for $\mathbf{x} \in \mathcal{X}^N$ such that $\text{Det } \kappa(\mathbf{x}) > 0$; see Section 2.5. in Belhadji et al. [2020] for a proof.

On the other hand, it is possible to derive better convergence guarantees using continuous volume sampling. Indeed, Belhadji et al. [2020] showed that

$$\mathbb{E}_{\text{CVS}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 = \sum_{m \in \mathbb{N}^*} \langle g, e_m \rangle_{\omega}^2 \epsilon_m(N), \quad (35)$$

where

$$\epsilon_m(N) = \sigma_m \frac{\sum_{U \in \mathcal{U}_N; m \notin U} \prod_{u \in U} \sigma_u}{\sum_{U \in \mathcal{U}_N} \prod_{u \in U} \sigma_u}; \quad \mathcal{U}_N := \{U \subset \mathbb{N}^*; |U| = N\}. \quad (36)$$

The identity (35) gives an explicit expression of $\mathbb{E}_{\text{CVS}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2$ in terms of the coefficients of g on the o.n.b. $(e_m)_{m \in \mathbb{N}^*}$ and the $\epsilon_m(N)$ defined by (36). Moreover, by observing that the sequence $(\epsilon_m(N))_{m \in \mathbb{N}^*}$ is non-increasing, the formula (35) implies that

$$\sup_{f \in \Sigma\mathbb{B}_{\mathbb{L}_2(\omega)}} \mathbb{E}_{\text{CVS}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 \leq \sup_{m \in \mathbb{N}^*} \epsilon_m(N) = \epsilon_1(N). \quad (37)$$

Moreover, as it was proven in Theorem 4 of [Belhadji et al., 2020], we have

$$\epsilon_1(N) \leq \sigma_{N+1} (1 + \beta_N), \quad (38)$$

where $\beta_N := \min_{M \in [2; N+1]} [(N - M + 2)\sigma_{N+1}]^{-1} \sum_{m \geq M} \sigma_m$. In particular, under the assumption that the sequence $(\beta_N)_{N \in \mathbb{N}^*}$ is bounded, which is the case as soon as the sequence (σ_m) decreases polynomially or exponentially, (38) yields

$$\sup_{f \in \Sigma\mathbb{B}_{\mathbb{L}_2(\omega)}} \mathbb{E}_{\text{CVS}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 = \mathcal{O}(\sigma_{N+1}), \quad (39)$$

which corresponds to the optimal rate of convergence.

The results reviewed until now are restricted to functions that belong to $\Sigma\mathbb{L}_2(\omega)$, are mostly relevant in the study of kernel-based quadrature using determinantal sampling. Yet, as it was mentioned, $\Sigma\mathbb{L}_2(\omega)$ is strictly included in the RKHS \mathcal{F} . Still, it is possible to extend (39) to functions belonging to $\Sigma^{r+1/2}\mathbb{L}_2(\omega)$, where $r \in [0, 1/2]$ is a parameter that interpolates between the set of the embeddings $\Sigma\mathbb{L}_2(\omega)$ and the RKHS $\mathcal{F} = \Sigma^{1/2}\mathbb{L}_2(\omega)$. Indeed, as it was shown in

Belhadji et al. [2020], $\mathbb{E}_{\text{CVS}} \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}^2 = \mathcal{O}(\sigma_{N+1}^{2r})$ under the assumption that the sequence $(\beta_N)_{N \in \mathbb{N}^*}$ is bounded. This result is an extension of (39) to $r \leq 1/2$: the rate of convergence is $\mathcal{O}(\sigma_{N+1}^{2r})$ which is slower than $\mathcal{O}(\sigma_{N+1})$ and it gets worse as r goes to 0. In other words, an additional level of smoothness controlled by $r > 0$ is needed to achieve the convergence with respect to the RKHS norm $\|\cdot\|_{\mathcal{F}}$. Nevertheless, we can expect that the convergence hold with respect to a weaker norm that is $\|\cdot\|_{\omega}$ as we will see in Section 3.

3 Theoretical guarantees

While previous work on determinantal sampling measured performance in RKHS norm, we investigate here the mean squared reconstruction error in $\mathbb{L}_2(\omega)$ norm. We emphasize that a few key results play a fundamental role, like the two properties (29) and (30) of the Ermakov-Zolotukhin quadrature, and the ‘Pythagorean’ formula (35) of continuous volume sampling.

3.1 The least-squares approximation

The following result gives a bound for the mean-square error of $\hat{f}_{\text{LS}, \mathbf{x}}$ under the projection DPP of Definition 1.

Theorem 1. *Consider $N \in \mathbb{N}^*$. Let $f \in \mathcal{F}$, and let f_N be its projection onto the eigenspace \mathcal{E}_N defined by (10). Then*

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{LS}, \mathbf{x}}\|_{\omega}^2 \leq 2 \left(\|f - f_N\|_{\omega}^2 + \|f_N\|_{\omega}^2 \sum_{m=N+1}^{+\infty} \sigma_m^2 \right). \quad (40)$$

The relative importance of the two terms of the r.h.s. of (40) depends on the smoothness of f , as characterized by the nested spaces in Section 2.1.2. To be more precise, if $f \in \Sigma^{1/2+r} \mathbb{L}_2(\omega)$ with $r \geq 0$, there exists $g \in \mathbb{L}_2(\omega)$ such that

$$\forall m \in \mathbb{N}^*, \langle f, e_m \rangle_{\omega} = \sigma_m^{r+1/2} \langle g, e_m \rangle_{\omega}, \quad (41)$$

so that the first term of the r.h.s. of (40) satisfies

$$\|f - f_N\|_{\omega}^2 = \sum_{m=N+1}^{+\infty} \langle f, e_m \rangle_{\omega}^2 = \sum_{m=N+1}^{+\infty} \frac{\langle f, e_m \rangle_{\omega}^2}{\sigma_m^{2r+1}} \sigma_m^{2r+1} \quad (42)$$

$$= \sigma_{N+1}^{2r+1} \sum_{m=N+1}^{+\infty} \langle g, e_m \rangle_{\omega}^2 \leq \sigma_{N+1}^{2r+1} \|g\|_{\omega}^2. \quad (43)$$

Meanwhile, the second term in the r.h.s. of (40) is $o(\sigma_{N+1})$, upon noting that $\|f_N\|_{\omega} \leq \|f\|_{\omega}$ and

$$\sum_{m=N+1}^{+\infty} \sigma_m^2 \leq \sigma_{N+1} \sum_{m=N+1}^{+\infty} \sigma_m.$$

So whatever $r \geq 0$, the r.h.s. of (40) is $\mathcal{O}(\sigma_{N+1})$, and as soon as $r > 0$, it is even $o(\sigma_{N+1})$. As r grows, and more smoothness is assumed, the second term in the r.h.s. gets to dominate. In particular, if $r \geq 1/2$, the first term is $\mathcal{O}(\sigma_{N+1}^2)$: this is the fastest rate we can expect from the second term, however large r is. This potential change of convergence rate when r increases is called *superconvergence* in the literature; see [Schaback, 2018] and references therein. In particular, we shall observe a convergence in $\mathcal{O}(\sigma_{N+1}^2)$ in the numerical experiments of Section 4.

The following lemma, which we use to prove Theorem 1, underlines a practical limitation of the least-squares approximation.

Lemma 1. *Let $f \in \mathcal{F}$, $\mathbf{w} \in \mathbb{R}^N$ and $\mathbf{x} \in \mathcal{X}^N$. Then*

$$\|f - \sum_{i \in [N]} w_i k(x_i, \cdot)\|_{\omega}^2 = \|f\|_{\omega}^2 - 2 \sum_{i \in [N]} w_i \mu_f(x_i) + \mathbf{w}^T \mathbf{K}_2(\mathbf{x}) \mathbf{w}, \quad (44)$$

where $\mu_f := \Sigma f$ and $\mathbf{K}_2(\mathbf{x})$ is the Gram matrix associated to the kernel

$$k_2(x, y) := \sum_{m=1}^{+\infty} \sigma_m^2 e_m(x) e_m(y), \quad (45)$$

where the convergence holds uniformly on compact subsets of $\mathcal{X} \times \mathcal{X}$. In particular, for a given $\mathbf{x} \in \mathcal{X}^N$ such that the matrix $\mathbf{K}_2(\mathbf{x})$ is non-singular, the associated least-squares approximation $\hat{f}_{\text{LS}, \mathbf{x}}$ is equal to $\sum_{i \in [N]} \hat{w}_i k(x_i, \cdot)$, where

$$\hat{\mathbf{w}} = \mathbf{K}_2(\mathbf{x})^{-1} \mu_f(\mathbf{x}). \quad (46)$$

As a consequence, the most direct way to evaluate $\hat{f}_{\text{LS}, \mathbf{x}}$ requires evaluating μ_f rather than f , and evaluating the kernel k_2 instead of k . Both may not have tractable expressions, which can be an important practical limitation. The proof of Lemma 1 is based on the Mercer decomposition (5) and it is given in Section 3.4.1, while the crux of the proof of Theorem 1, given in Section 3.4.2, is a property of the Ermakov-Zolotukhin quadrature rule proved in [Belhadji, 2021].

3.2 The transform based on the optimal kernel quadrature

To bypass the practical limitations of the least-squares approximation, we study the transform $\hat{f}_{\text{OKQ}, M, \mathbf{x}}$ based on the optimal kernel quadrature, defined in Section 2.2.2. The latter transform can be seen as a surrogate of $\hat{f}_{\text{LS}, \mathbf{x}}$, and has the advantage to only require evaluating f at the design. The following result makes use of the recent progress in the field of numerical integration using determinantal sampling to prove bounds on the mean square error of this approximation.

Proposition 1. *Let $f \in \mathcal{F}$ and $N \in \mathbb{N}^*$. For a given $\mathbf{x} \in \mathcal{X}^N$, denote by $\hat{f}_{\text{OKQ}, M, \mathbf{x}}$ the approximation obtained through the optimal kernel quadrature transform of order $M \in \mathbb{N}^*$ associated to the configuration \mathbf{x} , which is defined by (12) where the vectors $\boldsymbol{\alpha}_m := (\alpha_{m,i})$ are defined by (14) for $m \in [M]$. Then*

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{OKQ}, M, \mathbf{x}}\|_{\omega}^2 \leq M r_{N+1} \|f\|_{\mathcal{F}}^2 + \|f - f_M\|_{\omega}^2 \quad (47)$$

and

$$\mathbb{E}_{\text{CVS}} \|f - \hat{f}_{\text{OKQ}, M, \mathbf{x}}\|_{\omega}^2 \leq M \epsilon_1(N) \|f\|_{\mathcal{F}}^2 + \|f - f_M\|_{\omega}^2, \quad (48)$$

where $f_M := \sum_{m=1}^M \langle f, e_m \rangle_{\omega} e_m$, $r_{N+1} := \sum_{m \geq N+1} \sigma_m$ and $\epsilon_1(M)$ is given by (36).

Two comments are in order. First, the two upper bounds (47) and (48) both involve the squared residual $\|f - f_M\|_{\omega}^2$ and a term that depends on $\|f\|_{\mathcal{F}}$. The residual can be seen as a trace of approximating f_M instead of f , while the RKHS norm is a trace of performing numerical quadrature. Note that the two upper bounds are not multiplicative in the squared residual $\|f - f_M\|_{\omega}^2$: for $f = e_1^{\mathcal{F}}$, the residual is zero as long as $M \geq 1$, while the RKHS term

is positive. In other words, a priori the IOP⁴ does not hold. We will see in Section 3.3 that the quasi-interpolant satisfies the IOP under the distribution of the projection DPP. Second, the upper bounds (47) and (48) yield convergence rates. Indeed, observe that

$$\|f - f_M\|_\omega^2 = \sum_{m \geq M+1} \langle f, e_m \rangle_\omega^2 = \sum_{m \geq M+1} \sigma_m \frac{\langle f, e_m \rangle_\omega^2}{\sigma_m} \leq \sigma_{M+1} \|f - f_M\|_{\mathcal{F}}^2. \quad (49)$$

In particular, when $M = N$, (47) yields a convergence rate under the DPP in $\mathcal{O}((N+1)r_N)$, while (48) yields a convergence rate under CVS that scales as $\mathcal{O}((N+1)\sigma_{N+1})$ when $\epsilon_1(N) = \mathcal{O}(\sigma_{N+1})$. Both are slower than the rate of convergence of $\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{LS}, \mathbf{x}}\|_\omega^2 = \mathcal{O}(\sigma_{N+1})$ proved in Theorem 1. It would be interesting to investigate whether (47) and (48) can be sharpened. Indeed, the RHS of (47) and (48) goes to infinity as $M \rightarrow +\infty$. The following result shows that for a given configuration $\mathbf{x} \in \mathcal{X}^N$ such that $\mathbf{K}(\mathbf{x})$ is non-singular, $\|f - \hat{f}_{\text{OKQ}, M, \mathbf{x}}\|_\omega$ seen as a function of M is bounded.

Proposition 2. *Let $f \in \mathcal{F}$ and $N \in \mathbb{N}^*$. For a given $\mathbf{x} \in \mathcal{X}^N$ such that the matrix $\mathbf{K}(\mathbf{x})$ is non-singular, denote by $\hat{f}_{\text{OKQ}, M, \mathbf{x}}$ the approximation defined through the optimal kernel quadrature transform of order $M \in \mathbb{N}^*$ associated to the configuration \mathbf{x} , which is defined by (12) where the vectors $\boldsymbol{\alpha}_m := (\alpha_{m,i})$ are defined by (14) for $m \in [M]$. Then*

$$\forall M \in \mathbb{N}^*, \|f - \hat{f}_{\text{OKQ}, M, \mathbf{x}}\|_\omega^2 \leq \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_\omega^2 + \|f - f_M\|_\omega^2, \quad (50)$$

where $f_M := \sum_{m=1}^M \langle f, e_m \rangle_\omega e_m$.

Compared to (47) and (48), the first term of the RHS of (50) is independent of M . In particular, $\hat{f}_{\text{OKA}, \mathbf{x}}$ can be seen as limit of $\hat{f}_{\text{OKQ}, M, \mathbf{x}}$ as $M \rightarrow +\infty$. This result would yield an improved convergence rate of the squared mean error of $\hat{f}_{\text{OKQ}, M, \mathbf{x}}$ if we manage to prove sharp upper bounds of the squared mean error of $\hat{f}_{\text{OKA}, \mathbf{x}}$ under one of the two distributions defined in Section 2.3.2. Such a result is not yet available. However, the study of $\hat{f}_{\text{LS}, \mathbf{x}}$ in Section 3.1, which is close to $\hat{f}_{\text{OKA}, \mathbf{x}}$ in the construction, suggests that the squared mean error of $\hat{f}_{\text{OKA}, \mathbf{x}}$, under determinantal sampling, scales as $\mathcal{O}(\sigma_{N+1})$. This intuition is corroborated by numerical simulations presented in Section 4.1. The proof of Proposition 1, given in Section 3.4.3, is based on (34) and (37). The proof of Proposition 2 is given in Section 3.4.4.

3.3 The instance optimality property under projection DPPs

The empirical least-squares approximation (19) was extensively studied in the literature as mentioned in Section 2.3.1. A large proportion of these works focus on the study of the instance optimality property of $\hat{f}_{\text{ELS}, M, \mathbf{x}}$ as defined by (21) when \mathbf{x} is a random configuration of N nodes of \mathcal{X} . It was shown that the IOP holds when the nodes are i.i.d. draws according to the inverse of the Christoffel function defined by (23) and when the sampling budget N scales as $\mathcal{O}(M \log(M))$, which is optimal up to the logarithmic factor. Seeking alternative sampling algorithms that require a minimal budget of sampling that scale linearly with M is an active topic of research as it was mentioned in Section 2.3.1. To this aim, the projection DPP defined in Section 2.3.2 is a good candidate as a non-i.i.d. extension of the Christoffel sampling with a negative correlation property. This section investigates the IOP of the empirical least-squares approximation under this determinantal distribution. More precisely, this section studies a variant of this approximation which is amenable to theoretical analysis under the distribution of the projection

⁴Defined in Section 2.3.1 for the empirical least square approximation $\hat{f}_{\text{ELS}, \mathbf{x}}$, but it can be extended to any finite-dimensional approximation.

DPP of Definition 1. Given a positive function $q : \mathcal{X} \rightarrow \mathbb{R}_+^*$, and given $N, M \in \mathbb{N}^*$ such that $M \leq N$, we define the *truncated empirical least-squares approximation of order M* associated to the configuration $\mathbf{x} \in \mathcal{X}^N$ as

$$\hat{f}_{\text{tELS},M,\mathbf{x}} := \sum_{m=1}^M \langle \hat{f}_{\text{ELS},N,\mathbf{x}}, e_m \rangle_{\omega} e_m, \quad (51)$$

where $\hat{f}_{\text{ELS},N,\mathbf{x}}$ is the empirical least-squares approximation of f of dimension N associated to the function q and the configuration \mathbf{x} of size N . A subtle but important remark is that, in general, $\hat{f}_{\text{tELS},M,\mathbf{x}} \neq \hat{f}_{\text{ELS},M,\mathbf{x}}$: $\hat{f}_{\text{tELS},M,\mathbf{x}}$ is the projection on \mathcal{E}_M of the ELS approximation in \mathcal{E}_N while $\hat{f}_{\text{ELS},M,\mathbf{x}}$ is the ELS approximation in \mathcal{E}_M . As we shall now see, $\hat{f}_{\text{tELS},M,\mathbf{x}}$ is even invariant to the choice of the function q , which is not the case of $\hat{f}_{\text{ELS},M,\mathbf{x}}$.

Proposition 3. *Consider $N \in \mathbb{N}^*$ and let $q : \mathcal{X} \rightarrow \mathbb{R}_+^*$ a positive function. Let $\mathbf{x} \in \mathcal{X}^N$ such that the matrix $E_N(\mathbf{x}) := (e_m(x_i))_{(i,m) \in [N] \times [N]}$ is non-singular, then*

$$\hat{f}_{\text{ELS},N,\mathbf{x}} = \hat{f}_{\text{QI},N,\mathbf{x}}, \quad (52)$$

where $\hat{f}_{\text{QI},N,\mathbf{x}}$ is the quasi-interpolant defined in Section 2.2.2. As a result,

$$\forall M \in [N], \hat{f}_{\text{tELS},M,\mathbf{x}} = \sum_{m=1}^M \langle \hat{f}_{\text{QI},N,\mathbf{x}}, e_m \rangle_{\omega} e_m. \quad (53)$$

In other words, the empirical least-squares approximation of dimension N is invariant to the choice of the function q and coincides with the quasi-interpolant defined in Section 2.2.2. As a result, $\hat{f}_{\text{tELS},M,\mathbf{x}}$ ⁵ is nothing but the projection of $\hat{f}_{\text{QI},N,\mathbf{x}}$ on the eigenspace \mathcal{E}_M . Therefore, the numerical evaluation of $\hat{f}_{\text{tELS},M,\mathbf{x}}$ boils down to the evaluation of the quadrature rules $\hat{I}_1(f), \dots, \hat{I}_M(f)$ defined by (15) that implies the evaluation of $\alpha_1, \dots, \alpha_N$ via (16). Compared to the evaluation of (20) to get $\hat{f}_{\text{ELS},M,\mathbf{x}}$, this involves an $N \times N$ rather than $M \times M$ matrix. Therefore the evaluation of the coefficients of $\hat{f}_{\text{tELS},M,\mathbf{x}}$ in the o.n.b. $(e_m)_{m \in \mathbb{N}^*}$ is numerically more expensive than $\hat{f}_{\text{ELS},M,\mathbf{x}}$. However, this newly introduced approximation can be studied theoretically when the configuration \mathbf{x} is a projection DPP, which yields the following result.

Proposition 4. *Consider $N, M \in \mathbb{N}^*$ such that $M \leq N$. Then, for $f \in \mathcal{F}$*

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{tELS},M,\mathbf{x}}\|_{\omega}^2 = \|f - f_M\|_{\omega}^2 + M \|f - f_N\|_{\omega}^2, \quad (54)$$

As a direct consequence,

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{tELS},M,\mathbf{x}}\|_{\omega}^2 \leq (1 + M) \|f - f_M\|_{\omega}^2. \quad (55)$$

In other words, under the projection DPP, $\hat{f}_{\text{tELS},M,\mathbf{x}}$ satisfies the IOP with constant $(1 + M)$, for any sampling budget that satisfies $N \geq M$. For comparison, the required sampling budget for i.i.d. Christoffel sampling scales as $O(M \log(M))$. We emphasize that this approximation bound in $O(M)$ is a concrete benefit of the negative correlation property of projection DPPs for sampling.

The proof of Proposition 3 is given in Section 3.4.5, and the proof of Proposition 4, based on the identities (29) and (30), is given in Section 3.4.6.

⁵We drop q from the notation thanks to the invariance with respect to q .

3.4 Proofs

3.4.1 Proof of Lemma 1

Let $f \in \mathcal{F}$, $\mathbf{w} \in \mathbb{R}^N$ and $\mathbf{x} \in \mathcal{X}^N$. The squared residual $\|f - \sum_{i \in [N]} w_i k(x_i, \cdot)\|_\omega^2$ writes

$$\|f\|_\omega^2 - 2\langle f, \sum_{i \in [N]} w_i k(x_i, \cdot) \rangle_\omega + \left\| \sum_{i \in [N]} w_i k(x_i, \cdot) \right\|_\omega^2. \quad (56)$$

The identity (44) follows from evaluating the latter two terms. First note that

$$\mu_f = \Sigma f = \sum_{m \in \mathbb{N}^*} \sigma_m \langle f, e_m \rangle_\omega e_m.$$

Moreover, by (5), we have $k(x_i, \cdot) = \sum_{m=1}^{+\infty} \sigma_m e_m(x_i) e_m(\cdot)$ for $i \in [N]$, so that

$$\begin{aligned} \langle f, k(x_i, \cdot) \rangle_\omega &= \sum_{m \in \mathbb{N}^*} \langle f, e_m \rangle_\omega \sigma_m e_m(x_i) \\ &= \sum_{m \in \mathbb{N}^*} \langle \mu_f, e_m \rangle_\omega e_m(x_i) = \mu_f(x_i), \end{aligned} \quad (57)$$

and

$$\forall i, i' \in [N]^2, \langle k(x_i, \cdot), k(x_{i'}, \cdot) \rangle_\omega = k_2(x_i, x_{i'}). \quad (58)$$

Plugging (57) and (58) into (56) yields the desired formula.

3.4.2 Proof of Theorem 1

First, observe that when \mathbf{x} follows the distribution of the projection DPP of Definition 1, the Gram matrix $\mathbf{K}_N(\mathbf{x})$ associated to the kernel k_N defined by (25) is almost surely non-singular. Similarly, the Gram matrix $\mathbf{K}_{2,N}(\mathbf{x})$ associated to the kernel $k_{2,N}(x, y) = \sum_{m=1}^N \sigma_m^2 e_m(x) e_m(y)$ is almost surely non-singular. Now, observe that

$$\mathbf{K}_{2,N}(\mathbf{x}) = \sum_{m=1}^N \sigma_m^2 e_m(\mathbf{x}) e_m(\mathbf{x})^\top \prec \mathbf{K}_2(\mathbf{x}) = \sum_{m=1}^{+\infty} \sigma_m^2 e_m(\mathbf{x}) e_m(\mathbf{x})^\top, \quad (59)$$

so that the matrix $\mathbf{K}_2(\mathbf{x})$ is also almost surely non-singular.

Now, let $f \in \mathcal{F}$, $\mathbf{x} \in \mathcal{X}^N$ such that the matrices $\mathbf{K}_N(\mathbf{x})$, $\mathbf{K}_{2,N}(\mathbf{x})$ are non-singular. In short, we use the Ermakov-Zolotukhin quadrature of Section 2.4.1 as a pivot. Define

$$\hat{f}_{\text{EZ}, \mathbf{x}} := \sum_{i=1}^N \hat{w}_i^{\text{EZ}} k(x_i, \cdot),$$

where $\hat{\mathbf{w}}^{\text{EZ}} := \mathbf{K}_N(\mathbf{x})^{-1} f_N(\mathbf{x})$ and f_N is defined by (10). By definition of $\hat{f}_{\text{LS}, \mathbf{x}}$, $\|f - \hat{f}_{\text{LS}, \mathbf{x}}\|_\omega^2 \leq \|f - \hat{f}_{\text{EZ}, \mathbf{x}}\|_\omega^2$, so that

$$\|f - \hat{f}_{\text{LS}, \mathbf{x}}\|_\omega^2 \leq 2 \left(\|f - f_N\|_\omega^2 + \|f_N - \hat{f}_{\text{EZ}, \mathbf{x}}\|_\omega^2 \right).$$

Therefore

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{LS}, \mathbf{x}}\|_\omega^2 \leq 2 \left(\|f - f_N\|_\omega^2 + \mathbb{E}_{\text{DPP}} \|f_N - \hat{f}_{\text{EZ}, \mathbf{x}}\|_\omega^2 \right).$$

Now, using the identity (44) of Lemma 1, $\|f_N - \hat{f}_{\text{EZ},\mathbf{x}}\|_\omega^2$ writes

$$\|f_N\|_\omega^2 - 2 \sum_{i \in [N]} \hat{w}_i^{\text{EZ}} \mu_{f_N}(x_i) + \hat{\mathbf{w}}^{\text{EZ}\top} \mathbf{K}_2(\mathbf{x}) \hat{\mathbf{w}}^{\text{EZ}}, \quad (60)$$

which is equal to $\|\mu_{f_N} - \sum_{i \in [N]} \hat{w}_i^{\text{EZ}} k_2(x_i, \cdot)\|_{\mathcal{F}_2}^2$, where $\mu_{f_N} = \boldsymbol{\Sigma} f_N$, and \mathcal{F}_2 is defined to be the RKHS associated to the kernel k_2 and $\|\cdot\|_{\mathcal{F}_2}$ its norm. Indeed, for $i, i' \in [N]$, we have $k_2(x_i, x_{i'}) = \langle k_2(x_i, \cdot), k_2(x_{i'}, \cdot) \rangle_{\mathcal{F}_2}$. Moreover, denote by $\boldsymbol{\Sigma}_2$ the integration operator associated to the kernel k_2 and the measure ω , and observe that $\boldsymbol{\Sigma}_2^{1/2} = \boldsymbol{\Sigma}$, so that

$$\|f_N\|_\omega^2 = \|\boldsymbol{\Sigma}_2^{1/2} f_N\|_{\mathcal{F}_2}^2 = \|\boldsymbol{\Sigma} f_N\|_{\mathcal{F}_2}^2 = \|\mu_{f_N}\|_{\mathcal{F}_2}^2.$$

Now, since $f_N \in \mathcal{E}_N = \text{Span}(e_1, \dots, e_N)$, Theorem 3 in [Belhadji, 2021] yields

$$\mathbb{E}_{\text{DPP}} \|\mu_{f_N} - \sum_{i \in [N]} \hat{w}_i^{\text{EZ}} k_2(x_i, \cdot)\|_{\mathcal{F}_2}^2 = \|f_N\|_\omega^2 \sum_{m=N+1}^{+\infty} \sigma_m^2.$$

To sum up, we obtain

$$\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{LS},\mathbf{x}}\|_\omega^2 \leq 2 \left(\|f - f_N\|_\omega^2 + \|f_N\|_\omega^2 \sum_{m=N+1}^{+\infty} \sigma_m^2 \right).$$

3.4.3 Proof of Proposition 1

Let $f \in \mathcal{F}$. We have $f = \sum_{m \in \mathbb{N}^*} I_m(f) e_m$, where $I_m(f) := \langle f, e_m \rangle_\omega$. Moreover, by definition (12), any transform-based approximation writes

$$\hat{f}_\Phi = \sum_{m \in [M]} \hat{I}_m(f) e_m,$$

where \hat{I}_m are quadrature rules. In particular,

$$f - \hat{f}_\Phi = \sum_{m \in [M]} (I_m(f) - \hat{I}_m(f)) e_m + \sum_{m \geq M+1} \langle f, e_m \rangle_\omega e_m \quad (61)$$

and

$$\|f - \hat{f}_\Phi\|_\omega^2 = \sum_{m \in [M]} (I_m(f) - \hat{I}_m(f))^2 + \|f - f_M\|_\omega^2. \quad (62)$$

Now, when Φ corresponds to the transform based on the optimal kernel quadrature, we have by (12) and (14)

$$\forall m \in [M], \hat{I}_m(f) = \sum_{i \in [N]} \alpha_{m,i} f(x_i) = \mu_m(\mathbf{x})^\top \mathbf{K}(\mathbf{x})^{-1} f(\mathbf{x}), \quad (63)$$

where $\mu_m := \sigma_m e_m = \boldsymbol{\Sigma} e_m$.

Now, let $m \in [M]$ and define $\mu := \mu_m$. By definition (9), we have $\hat{\mu}_{\text{OKA},\mathbf{x}} = \sum_{i \in [N]} w_i k(x_i, \cdot)$, where $\mathbf{w} = \mathbf{K}(\mathbf{x})^{-1} \mu(\mathbf{x})$. Thus, by (63), we have

$$\hat{I}_m(f) = f(\mathbf{x})^\top \mathbf{w} = \langle f, \sum_{i \in [N]} w_i k(x_i, \cdot) \rangle_{\mathcal{F}} = \langle f, \hat{\mu}_{\text{OKA},\mathbf{x}} \rangle_{\mathcal{F}}. \quad (64)$$

Moreover, we have $\langle e_m^{\mathcal{F}}, \varphi \rangle_{\mathcal{F}} = \langle e_m, \varphi \rangle_{\omega} / \sqrt{\sigma_m}$ for any $\varphi \in \mathcal{F}$; see Section 2.1.1. Thus

$$I_m(f) = \langle f, e_m \rangle_{\omega} = \sqrt{\sigma_m} \langle f, e_m^{\mathcal{F}} \rangle_{\mathcal{F}} = \langle f, \mu \rangle_{\mathcal{F}}. \quad (65)$$

Combining (64) and (65), we get

$$|I_m(f) - \hat{I}_m(f)| = |\langle f, \mu - \hat{\mu}_{\text{OKA}, \mathbf{x}} \rangle_{\mathcal{F}}| \leq \|f\|_{\mathcal{F}} \|\mu - \hat{\mu}_{\text{OKA}, \mathbf{x}}\|_{\mathcal{F}}. \quad (66)$$

Since $\|e_m\|_{\omega} = 1$, $\mu \in \Sigma \mathbb{B}_{\mathbb{L}_2(\omega)}$. Thus by (34) and (37), we get

$$\mathbb{E}_{\text{DPP}} |I_m(f) - \hat{I}_m(f)|^2 \leq 4r_N \|f\|_{\mathcal{F}}^2, \quad (67)$$

and

$$\mathbb{E}_{\text{CVS}} |I_m(f) - \hat{I}_m(f)|^2 \leq \epsilon_m(N) \|f\|_{\mathcal{F}}^2. \quad (68)$$

By taking the sum over $m \in [M]$ in (67) and (68), and by using (62), we prove (47) and (48).

3.4.4 Proof of Proposition 2

Let $f \in \mathcal{F}$. We proceed as in Section 3.4.3, and we prove that

$$\forall M \in \mathbb{N}^*, \|f - \hat{f}_{\text{OKQ}, M, \mathbf{x}}\|_{\omega}^2 = \sum_{m \in [M]} (I_m(f) - \hat{I}_m(f))^2 + \|f - f_M\|_{\omega}^2, \quad (69)$$

where $I_m(f) = \langle f, e_m \rangle_{\omega}$ for $m \in \mathbb{N}^*$, and $\hat{I}_1, \dots, \hat{I}_m$ are given by (63). Thus, we only need to prove that $\sum_{m \in [M]} (I_m(f) - \hat{I}_m(f))^2 \leq \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\omega}^2$.

Now, let $m \in [M]$. By definition (9), we have $\hat{f}_{\text{OKA}, \mathbf{x}} = \sum_{i \in [N]} w_i k(x_i, \cdot)$, where $\mathbf{w} = \mathbf{K}(\mathbf{x})^{-1} f(\mathbf{x})$. Thus, by (63), we have

$$\hat{I}_m(f) = \mu_m(\mathbf{x})^{\text{T}} \mathbf{w} = \langle \mu_m, \sum_{i \in [N]} w_i k(x_i, \cdot) \rangle_{\mathcal{F}} = \langle \mu_m, \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\mathcal{F}}. \quad (70)$$

Moreover, we have $\langle e_m^{\mathcal{F}}, \varphi \rangle_{\mathcal{F}} = \langle e_m, \varphi \rangle_{\omega} / \sqrt{\sigma_m}$ for any $\varphi \in \mathcal{F}$; see Section 2.1.1. Thus, since $\mu_{e_m} = \Sigma e_m = \sigma_m e_m = \sqrt{\sigma_m} e_m^{\mathcal{F}}$, we have

$$\langle \mu_m, \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\mathcal{F}} = \sqrt{\sigma_m} \langle e_m, \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\omega} / \sqrt{\sigma_m} = \langle e_m, \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\omega}. \quad (71)$$

On the other hand, we have $I_m(f) = \langle e_m, f \rangle_{\omega}$. Combining the latter with (71), we get

$$(I_m(f) - \hat{I}_m(f))^2 = \langle e_m, f - \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\omega}^2, \quad (72)$$

so that

$$\begin{aligned} \sum_{m=1}^M (I_m(f) - \hat{I}_m(f))^2 &= \sum_{m=1}^M \langle e_m, f - \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\omega}^2 \leq \sum_{m=1}^{+\infty} \langle e_m, f - \hat{f}_{\text{OKA}, \mathbf{x}} \rangle_{\omega}^2 \\ &= \|f - \hat{f}_{\text{OKA}, \mathbf{x}}\|_{\omega}^2. \end{aligned}$$

3.4.5 Proof of Proposition 3

Let $f \in \mathcal{F}$, $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{X}^N$, and $\mathbf{w} = (w_1, \dots, w_N) \in \mathbb{R}^N$. Define

$$\begin{cases} \beta(\mathbf{x}) & := \frac{1}{N} \sum_{i \in [N]} q(x_i) f(x_i)^2 \\ \delta(\mathbf{x}) & := \left(\frac{1}{N} \sum_{i \in [N]} q(x_i) f(x_i) e_n(x_i) \right)_{n \in [N]} \in \mathbb{R}^N \\ \gamma(\mathbf{x}) & := \left(\frac{1}{N} \sum_{i \in [N]} q(x_i) e_{n_1}(x_i) e_{n_2}(x_i) \right)_{(n_1, n_2) \in [N] \times [N]} \in \mathbb{R}^{N \times N}. \end{cases} \quad (73)$$

We have

$$\|f - \sum_{m \in [N]} w_m e_m\|_{q, \mathbf{x}}^2 = \frac{1}{N} \sum_{i \in [N]} q(x_i) \left(f(x_i) - \sum_{m \in [N]} w_m e_m(x_i) \right)^2 \quad (74)$$

$$= \beta(\mathbf{x}) - 2\mathbf{w}^T \delta(\mathbf{x}) + \mathbf{w}^T \gamma(\mathbf{x}) \mathbf{w}. \quad (75)$$

Under the assumption that $\gamma(\mathbf{x})$ is non-singular, let $\hat{\mathbf{w}} = \gamma(\mathbf{x})^{-1} \delta(\mathbf{x})$. The function $\hat{f} := \sum_{m \in [N]} \hat{w}_m e_m$ is then the unique function belonging to \mathcal{E}_N that satisfies

$$\|f - \hat{f}\|_{q, \mathbf{x}} = \min_{\tilde{f} \in \mathcal{E}_N} \|f - \tilde{f}\|_{q, \mathbf{x}}. \quad (76)$$

Thus $\hat{f}_{\text{ELS}, N, q, \mathbf{x}} = \sum_{m \in [N]} \hat{w}_m e_m$, and we only need to prove that $\hat{w}_m = \hat{I}_m^{\text{QI}}(f)$ for $m \in [N]$, where

$$\hat{I}_m^{\text{QI}}(f) = f(\mathbf{x})^T \mathbf{K}_N(\mathbf{x})^{-1} e_m(\mathbf{x})$$

are the interpolative quadrature rules that define the quasi-interpolant in Section 2.2.2. For this purpose, observe that $\gamma(\mathbf{x})$ and $\delta(\mathbf{x})$ write

$$\begin{cases} \gamma(\mathbf{x}) & := \frac{1}{N} E_N(\mathbf{x})^T Q(\mathbf{x}) E_N(\mathbf{x}) \\ \delta(\mathbf{x}) & := \frac{1}{N} E_N(\mathbf{x})^T Q(\mathbf{x}) f(\mathbf{x}) \end{cases}, \quad (77)$$

where $Q(\mathbf{x}) := \text{Diag}(q(x_1), \dots, q(x_N))$, $E_N(\mathbf{x}) := (e_n(x_i))_{(i, n) \in [N] \times [N]}$, and $f(\mathbf{x}) = (f(x_i))_{i \in [N]}$. Thus, still under the assumption that $\gamma(\mathbf{x})$ is non-singular, we have

$$\hat{\mathbf{w}} = \gamma(\mathbf{x})^{-1} \delta(\mathbf{x}) = E_N(\mathbf{x})^{-1} Q(\mathbf{x})^{-1} E_N(\mathbf{x})^T E_N(\mathbf{x}) f(\mathbf{x}) = E_N(\mathbf{x})^{-1} f(\mathbf{x}). \quad (78)$$

In particular, for $m \in [N]$,

$$\hat{w}_m = v_m^T E_N(\mathbf{x})^{-1} f(\mathbf{x}), \quad (79)$$

where v_m is the m -th element of the Euclidean basis of \mathbb{R}^N . By observing that $e_m(\mathbf{x}) = E_N(\mathbf{x}) v_m$, we deduce from (79) that

$$\hat{w}_m = e_m(\mathbf{x})^T E_N(\mathbf{x})^T E_N(\mathbf{x})^{-1} f(\mathbf{x}) = e_m(\mathbf{x})^T (E_N(\mathbf{x}) E_N(\mathbf{x})^T)^{-1} f(\mathbf{x}) \quad (80)$$

$$= e_m(\mathbf{x})^T \mathbf{K}_N(\mathbf{x})^{-1} f(\mathbf{x}) \quad (81)$$

$$= \hat{I}_m^{\text{QI}}(f). \quad (82)$$

Finally, we need to check that $\gamma(\mathbf{x})$ is non-singular if and only if $E_N(\mathbf{x})$ is non-singular. This claim can be proved by observing that $\gamma(\mathbf{x})$ is non-singular if and only if the matrices $Q(\mathbf{x})$ and $E_N(\mathbf{x})$ are non-singular. We conclude by observing that $\text{Det } Q(\mathbf{x}) = \prod_{i \in [N]} q(x_i) > 0$, since q is positive by assumption.

3.4.6 Proof of Proposition 4

Let $\mathbf{w} = (w_m)_{m \in [N]} \in \mathbb{R}^N$. We have

$$\|f - \sum_{m=1}^M w_m e_m\|_\omega^2 = \|f\|_\omega^2 - 2 \sum_{m=1}^M w_m \langle f, e_m \rangle_\omega + \sum_{m=1}^M w_m^2. \quad (83)$$

Therefore

$$\|f - \hat{f}_{\text{ELS}, M, \mathbf{x}}\|_\omega^2 = \|f\|_\omega^2 - 2 \sum_{m=1}^M \hat{w}_m \langle f, e_m \rangle_\omega + \sum_{m=1}^M \hat{w}_m^2, \quad (84)$$

where $\hat{\mathbf{w}} = \gamma(\mathbf{x})^{-1} \delta(\mathbf{x})$, with $\gamma(\mathbf{x})$ and $\delta(\mathbf{x})$ given by (77). By Proposition 3, we get

$$\forall n \in [N], \hat{w}_n = \hat{I}_n^{\text{QI}}(f). \quad (85)$$

Based on (85), (29) implies that

$$\forall n \in [N], \mathbb{E}_{\text{DPP}} \hat{w}_n = \langle f, e_n \rangle_\omega, \quad (86)$$

and (30) implies that

$$\begin{aligned} \forall n \in [N], \mathbb{E}_{\text{DPP}} \hat{w}_n^2 &= \mathbb{E}_{\text{DPP}} \hat{I}_n^{\text{QI}}(f)^2 \\ &= I_n(f)^2 + 2I_n(f) \mathbb{E}_{\text{DPP}} (\hat{I}_n^{\text{QI}}(f) - I_n(f)) + \mathbb{E}_{\text{DPP}} (\hat{I}_n^{\text{QI}}(f) - I_n(f))^2 \\ &= \langle f, e_n \rangle_\omega^2 + \sum_{m=N+1}^{+\infty} \langle f, e_m \rangle_\omega^2. \end{aligned} \quad (87)$$

Finally, combining (84), (86) and (87), we get

$$\begin{aligned} \mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{ELS}, M, \mathbf{x}}\|_\omega^2 &= \|f\|_\omega^2 - 2 \sum_{n=1}^M \langle f, e_n \rangle_\omega^2 + \sum_{n=1}^M (\langle f, e_n \rangle_\omega^2 + \sum_{m=N+1}^{+\infty} \langle f, e_m \rangle_\omega^2) \\ &= \sum_{n=M+1}^{+\infty} \langle f, e_n \rangle_\omega^2 + \sum_{n=1}^M \sum_{m=N+1}^{+\infty} \langle f, e_m \rangle_\omega^2 \\ &= \|f - f_M\|_\omega^2 + M \|f - f_N\|_\omega^2. \end{aligned}$$

4 Numerical illustrations

In this section we illustrate the results of Section 3 on three families of RKHSs.

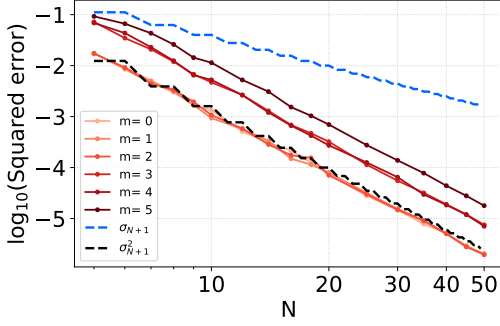
4.1 Periodic Sobolev spaces

Let $\mathcal{X} = [0, 1]$ equipped with the uniform measure ω , and define for $s \in \mathbb{N}^*$ the kernel

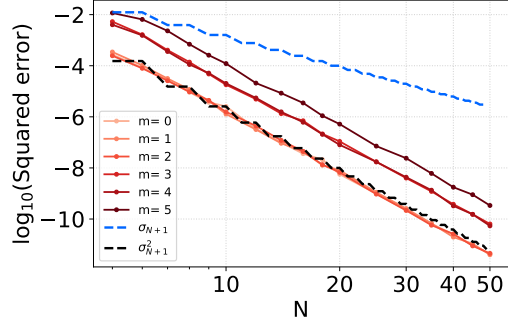
$$k_s(x, y) = 1 + 2 \sum_{m \in \mathbb{N}^*} \frac{1}{m^{2s}} \cos(2\pi m(x - y)), \quad (88)$$

where the convergence holds uniformly on $\mathcal{X} \times \mathcal{X}$. The kernel k_s can be expressed in closed form using Bernoulli polynomials [Wahba, 1990],

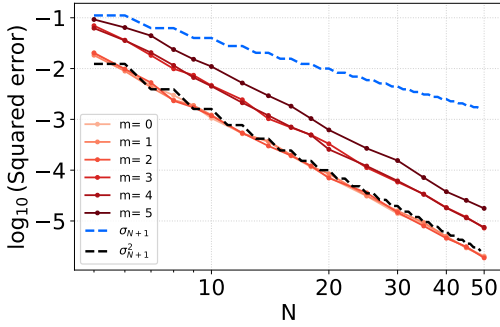
$$k_s(x, y) = 1 + \frac{(-1)^{s-1} (2\pi)^{2s}}{(2s)!} B_{2s}(\{x - y\}). \quad (89)$$



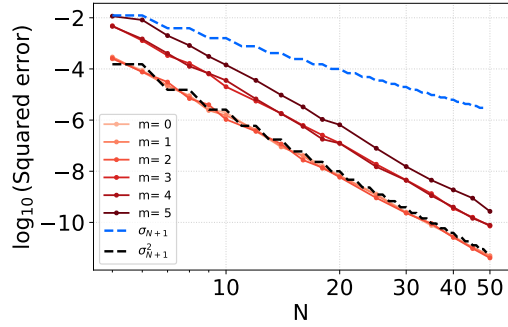
(a) The residual $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ vs. N ($s = 1$).



(b) The residual $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ vs. N ($s = 2$).



(c) The residual $\|f - \hat{f}_{\text{OKA},\mathbf{x}}\|_{\omega}^2$ vs. N ($s = 1$).



(d) The residual $\|f - \hat{f}_{\text{OKA},\mathbf{x}}\|_{\omega}^2$ vs. N ($s = 2$).

Figure 2: The reconstruction error for $f = e_m^{\mathcal{F}}$, when \mathcal{F} is the periodic Sobolev space of order $s \in \{1, 2\}$.

The corresponding RKHS $\mathcal{F} = \mathcal{S}_s$ is the periodic Sobolev space of order s ; an element of \mathcal{S}_s is a function f defined on $[0, 1]$, that has a derivative of order s in the sense of distributions such that $f^{(s)} \in \mathbb{L}_2(\omega)$, and

$$\forall i \in \{0, \dots, s-1\}, f^{(i)}(0) = f^{(i)}(1); \quad (90)$$

see Chapter 7 of [Berlinet and Thomas-Agnan, 2011]. This class of RKHSs is ideal to validate the theoretical guarantees obtained in Section 3, since the eigenvalues σ_m and the eigenfunctions e_m are known explicitly.

Figure 2a and Figure 2b show log-log plots of $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ when $f = e_m^{\mathcal{F}}$ with $m \in \{1, 2, 3, 4, 5\}$ w.r.t. N , averaged over 50 independent draws from the the projection DPP, when \mathcal{F} is the periodic Sobolev spaces of order $s = 1$ and $s = 2$, respectively. We observe that the expected squared residual converges to 0 at the same rate as $\sigma_{N+1}^2 = \mathcal{O}(N^{-4s})$, which is slightly faster than the rate of convergence of $\sum_{m \geq N+1} \sigma_m^2 = \mathcal{O}(N^{1-4s})$ predicted by Theorem 1. Interestingly, this fast rate of convergence is also observed for the kernel-based interpolant $\hat{f}_{\text{OKA},\mathbf{x}}$, as shown in Figure 2c and Figure 2d. Indeed, these figures show log-log plots of $\|f - \hat{f}_{\text{OKA},\mathbf{x}}\|_{\omega}^2$ when $f = e_m^{\mathcal{F}}$ with $m \in \{1, 2, 3, 4, 5\}$ w.r.t. N , averaged over 50 independent DPP samples. The squared residuals are evaluated using the formula (44).

Now, we move to another experiment, and consider f to be a random function

$$f = \sum_{m=1}^M \xi_m e_m^{\mathcal{F}}, \quad (91)$$

where $M \in \mathbb{N}^*$ and the ξ_m are i.i.d. standard Gaussians. In other words, f is a random element of the eigenspace $\mathcal{E}_M^{\mathcal{F}}$. Figure 3a and Figure 3b show log-log plots of $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ w.r.t. N , averaged over 50 independent DPP samples, and for a set of 50 functions f sampled according to (91), when \mathcal{F} is the periodic Sobolev spaces of order $s = 1$. To be clear, for every function f we use different DPP samples. We observe that the empirical convergence rate scales as $\mathcal{O}(\sigma_{N+1}^2)$, which is slightly faster than $\mathcal{O}(\sum_{m \geq N+1} \sigma_m^2)$. Again, this fast convergence is also observed for the kernel-based interpolant $\hat{f}_{\text{OKA},\mathbf{x}}$, as shown in Figure 3c and Figure 3d.

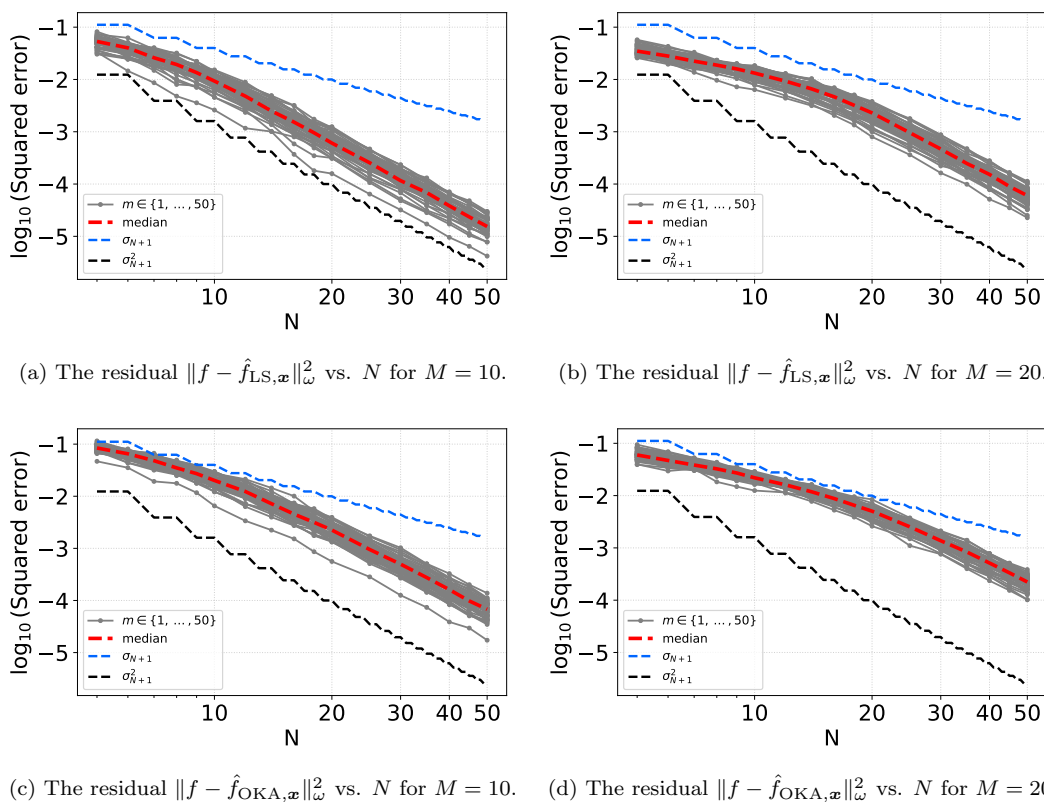
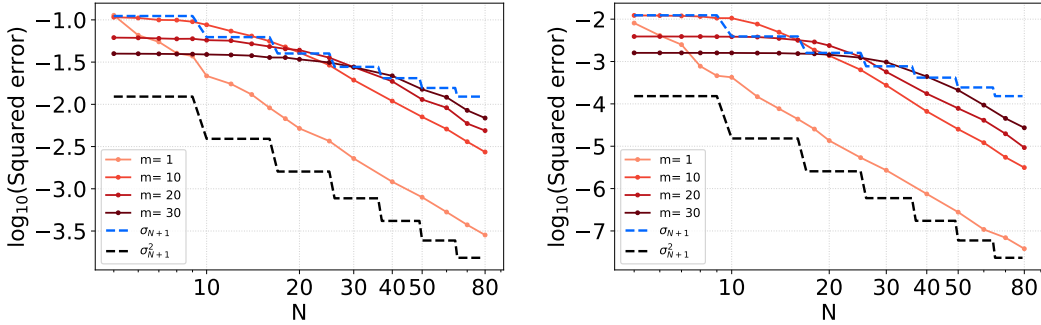


Figure 3: The reconstruction error for 10 samples of f from the distribution defined by (91) when \mathcal{F} is the periodic Sobolev space of order $s = 1$.

4.2 An RKHS with a rotation-invariant kernel on the hypersphere

The RKHS framework allow treating high-dimensional and/or non-Euclidean domains, such as hyperspheres. The definition of a positive definite kernel on a hypersphere dates back to [Schoenberg, 1942]. According to Schoenberg's seminal work, a kernel $k : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ is said to be



(a) The residual $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ vs. N ($s = 1$).

(b) The residual $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ vs. N ($s = 2$).

Figure 4: The reconstruction error for $f = e_m^{\mathcal{F}}$, when \mathcal{F} is the periodic Sobolev space of order $s \in \{1, 2\}$ in the hypersphere \mathbb{S}^{d-1} where $d = 3$.

a *dot-product kernel* if there exists a function $\varphi : [-1, 1] \rightarrow \mathbb{R}$ such that $k(x, y) = \varphi(\langle x, y \rangle)$ for $x, y \in \mathbb{S}^{d-1}$. This defines a large class of rotation-invariant kernels on \mathbb{S}^{d-1} . Moreover, the integration operator Σ associated to a dot-product kernel, when \mathbb{S}^{d-1} is equipped with the uniform measure, decomposes in the basis of spherical harmonics, and Mercer's decomposition holds in the form

$$k(x, y) = \sum_{\ell=0}^{+\infty} \sigma_{\ell} \sum_{k=1}^{N(d, \ell)} Y_{\ell, k}(x) Y_{\ell, k}(y), \quad (92)$$

where for $\ell \in \mathbb{N}$, $\{Y_{\ell, k} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}, k = 1, \dots, N(d, \ell)\}$ is the basis of spherical harmonics of exact degree ℓ and $N(d, \ell) := (2\ell + d - 1)\Gamma(\ell + d - 1)/(\Gamma(d)\Gamma(\ell + 1))$; see [Groemer, 1996]. The exact expression of σ_m or the eigenvalue decay may be found in [Cui and Freeden, 1997, Smola et al., 2000, Bach, 2017b, Azevedo and Menegatto, 2014, Scetbon and Harchaoui, 2021].

In order to illustrate the superconvergence phenomenon of Theorem 1, we consider the kernel obtained by taking $d = 3$ and $\sigma_{\ell} = (1 + \ell)^{-2s}$ in (92). The corresponding RKHS is akin to a Sobolev space of order s ; an element of \mathcal{F} is a function f defined on \mathbb{S}^{d-1} , which has a derivative of order s in the sense of distributions such that $f^{(s)} \in \mathbb{L}_2(\omega)$ [Hesse, 2006].

Figure 4a and Figure 4b show log-log plots of $\|f - \hat{f}_{\text{LS},\mathbf{x}}\|_{\omega}^2$ when $f = e_m^{\mathcal{F}}$ with $m \in \{1, 10, 20, 30\}$ w.r.t. N , averaged over 50 independent DPP samples, for $s = 1$ and $s = 2$ respectively. Again, we observe that the expected squared residual converges to 0 at the same rate as $\sigma_{N+1}^2 = \mathcal{O}(N^{-4s})$, which is slightly faster than the rate of convergence predicted by Theorem 1 of $\sum_{m \geq N+1} \sigma_m^2 = \mathcal{O}(N^{1-4s})$. Moreover, the superconvergence regime corresponds to $N \geq m$, as predicted by Theorem 1.

4.3 The RKHS spanned by the uni-dimensional PSWFs

Let now $\mathcal{X} = [-T/2, T/2]$ equipped with ω the uniform measure, and let $F > 0$. Consider the Sinc kernel

$$k_F(x, y) := \text{Sinc}(F(x - y)) = \frac{\sin(F(x - y))}{F(x - y)}. \quad (93)$$

This kernel defines an RKHS that corresponds to the space of band-limited functions restricted to the interval \mathcal{X} . Slepian, Landau, and Pollak proved that the eigenfunctions of the integration

operator Σ associated to (93) and the measure ω satisfy a differential equation known in physics as the prolate spheroidal wave equation (PSW; Slepian and Pollak, 1961). Since this seminal work, the eponymous functions were subject to extensive research. In particular, a detailed description of the eigenfunctions was carried out in [Osipov, 2013, Bonami and Karoui, 2014, Osipov and Rokhlin, 2012]. In particular, these functions were shown to be well represented in the orthonormal basis defined by the Legendre polynomials [Boyd, 2005, Osipov et al., 2013].

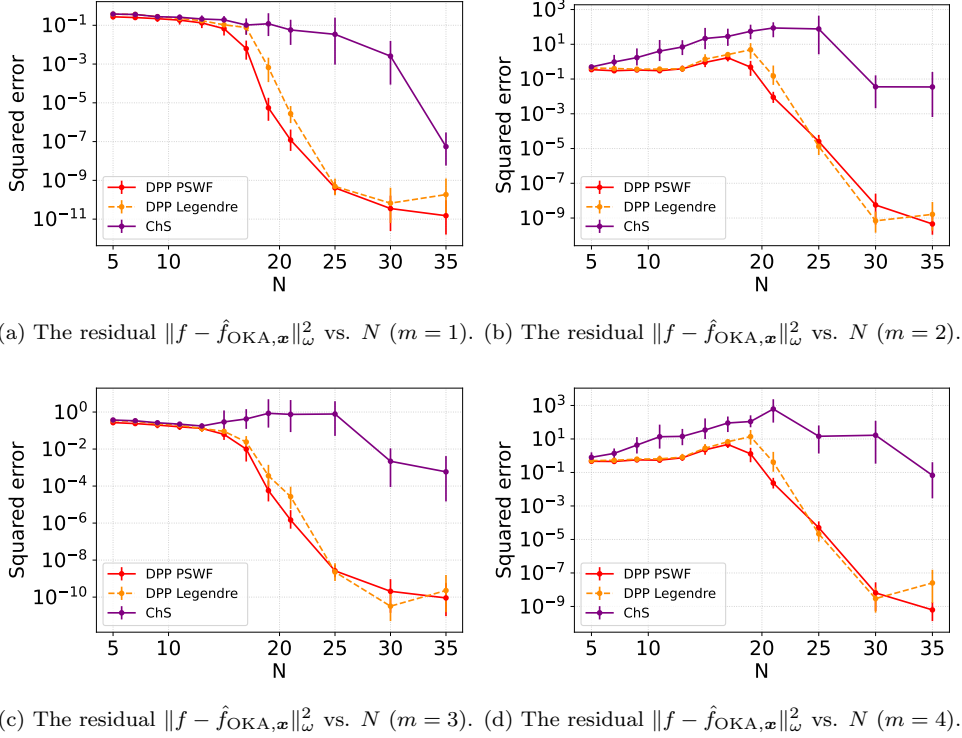


Figure 5: The reconstruction error for $f = e_m^{\mathcal{F}}$ in the RKHS associated to the Sinc kernel.

The asymptotics of the eigenvalues (σ_m) of Σ in the limit $c := TF \rightarrow +\infty$ were investigated too [Landau and Widom, 1980]: for $\epsilon > 0$, in this asymptotic limit, Σ has approximately TF eigenvalues in the interval $[1 - \epsilon, 1]$, $\mathcal{O}(\log(TF) \log(1/\epsilon))$ eigenvalues in the interval $[\epsilon, 1 - \epsilon]$, and the remaining eigenvalues decrease to 0 at an exponential rate.

In this set of experiments, we study the influence of the design \mathbf{x} on the convergence of $\hat{f}_{\text{OKA},\mathbf{x}}$ to f with respect to the norm $\|\cdot\|_{\omega}$. We compare the following random designs: (i) the projection DPP defined in Section 2.3.2 associated to the first PSW functions (DPP-PSWF), (ii) the projection DPP associated to the first normalized Legendre polynomials (DPP-Legendre), and (iii) Christoffel i.i.d. sampling (ChS). Figure 5 shows log-log plots of $\|f - \hat{f}_{\text{OKA},\mathbf{x}}\|_{\omega}^2$ when $f = e_m^{\mathcal{F}}$ with $m \in \{1, 2, 3\}$ w.r.t. N , averaged over 50 independent samples of each of the three distributions. We observe that both DPP-PSWF and DPP-Legendre significantly improve over Christoffel sampling. Moreover, the expected squared residual $\|f - \hat{f}_{\text{OKA},\mathbf{x}}\|_{\omega}^2$ under the two DPPs converges to 0 at an exponential rate when $N \geq TW$, which corresponds to the asymptotics described in [Landau and Widom, 1980].

5 Discussion

In this article, we established guarantees in L^2 norm for the reconstruction of functions living in an RKHS, based on repulsive nodes defined by determinantal distributions. In the following, we give further high-level comments and discuss possible extensions.

First, we insist that generating different random designs requires access to different quantities. Christoffel sampling in general will require rejection sampling. One should thus be able to evaluate the density $x \mapsto \sum_{m=1}^N e_m(x)^2/N$, as well as find a suitable proposal. The study of the ‘shape’ of the Christoffel function is thus crucial for sampling, and is an active topic of research [Pauwels et al., 2018, Avron et al., 2019, Dolbeault and Cohen, 2022b]. A good sampler for the Christoffel function is also relevant to simulate the DPP of Equation (26), as the p.d.f. of Christoffel sampling can be used as a proposal distribution when sampling the sequential conditionals of the HKPV algorithm [Hough et al., 2006]; see [Gautier et al., 2019]. When the p.d.f. of Christoffel sampling cannot be evaluated, one can still resort to continuous volume sampling. Indeed, while the only known *exact* sampling algorithm for CVS still relies on possibly hard-to-evaluate projection kernels, there are approximate samplers that leverage the fact that evaluating the p.d.f. of CVS in Equation (28) only requires evaluating the RKHS kernel k . In particular, Rezaei and Gharan [2019] have studied a natural Markov chain Monte Carlo sampler, whose mixing time scales as $\mathcal{O}(N^5 \log(N))$. Moreover, while each iteration of their Markov chain requires a rejection sampling step, the expected number of rejections is shown to be $\mathcal{O}(1/\sum_{m \geq N+1} \sigma_m)$. In other words, the smoother the kernel, the harder it is to run the MCMC algorithm. It would be interesting to investigate whether alternative MCMC algorithms can circumvent this ‘smoothness curse’.

Second, we comment on the instance optimal property (IOP), which motivated the introduction of Christoffel sampling [Cohen and Migliorati, 2017]. In particular, it was proven that for some variants of the empirical least-squares approximation, the IOP holds when the sampling budget is as low as $\mathcal{O}(M \log(M))$. These variants essentially exclude configurations of nodes \mathbf{x} for which the Gramian matrix $\mathbf{G}_{q,\mathbf{x}}$ is ill-conditioned. Similarly, determinantal sampling implicitly favors configurations of nodes \mathbf{x} so that $\text{Det } \mathbf{G}_{q,\mathbf{x}}$ is large. Moreover, Proposition 4 shows that the IOP holds under a suitable projection DPP, with a minimum sampling budget $N \geq M$. The price to pay is the constant $M + 1$ in the IOP. Now, when $N = M$, this constant is $N + 1$, which actually improves upon the constant $N^2 + 1$ proven in [Chkifa and Dolbeault, 2023] for an algorithm based on the so-called *effective resistances*. The latter algorithm generates randomized configurations \mathbf{x} in a greedy fashion so that the ‘redundancy’ of sampling is reduced.

Third, we might seek approximation schemes that are optimal in some worst-case sense, rather than looking for ones that satisfy the IOP. This is the approach adopted in [Krieg and Ullrich, 2021, Dolbeault et al., 2023], where the authors investigate the *sampling numbers*

$$g_N(F) := \inf_{\substack{x_1, \dots, x_N \in \mathcal{X} \\ \varphi_1, \dots, \varphi_N \in \mathbb{L}_2(\omega)}} \sup_{f \in \mathcal{F}} \|f - \sum_{i=1}^N f(x_i) \varphi_i\|_\omega. \quad (94)$$

The sampling numbers measure the complexity of the recovery of a function $f \in \mathcal{F}$ based on its discrete samples $f(x_1), \dots, f(x_N)$. It was shown by Dolbeault et al. [2023] that there is a universal constant $c \in \mathbb{N}^*$ such that

$$\forall N \in \mathbb{N}^*, g_{cN}(\mathbb{B}_{\mathcal{F}}) \leq \sqrt{\frac{1}{N} \sum_{m \geq N+1} d_m^2(\mathbb{B}_{\mathcal{F}})}, \quad (95)$$

where the d_m are the *Kolmogorov widths* defined by

$$d_N(\mathbb{B}_{\mathcal{F}}) := \inf_{\substack{\ell_1, \dots, \ell_N: \mathbb{B}_{\mathcal{F}} \rightarrow \mathbb{R} \\ \varphi_1, \dots, \varphi_N \in \mathbb{L}_2(\omega)}} \sup_{f \in \mathbb{B}_{\mathcal{F}}} \|f - \sum_{i=1}^N \ell_i(f) \varphi_i\|_{\omega}. \quad (96)$$

The Kolmogorov N -width corresponds to the lowest worst-case approximation error by a linear subspace of dimension N . Because \mathcal{F} is an RKHS, the Kolmogorov N -width (96) corresponds to the so-called *linear* N -width, defined by restricting the functions ℓ_1, \dots, ℓ_N to be linear; see Remark 5 in [Dolbeault et al., 2023]. In particular, in this case, $d_N(\mathbb{B}_{\mathcal{F}}) = \sigma_{N+1}$, so that the RHS of (95) scales as $\mathcal{O}(\sigma_{N+1})$ for many sequences $(\sigma_m)_{m \in \mathbb{N}^*}$, although this is not always the case [Krieg and Ullrich, 2021]. Moreover, the bound (95) is optimal [Hinrichs et al., 2022]. Our results in Section 3 can match these rates, but only for a single f so far; Theorem 1, for instance, implies that $\mathbb{E}_{\text{DPP}} \|f - \hat{f}_{\text{LS}, \mathbf{x}}\|_{\omega}^2 = \mathcal{O}(\sigma_{N+1})$ for every function f in the RKHS \mathcal{F} . Such a worst-case optimality would further connect to a string of results on the *completeness of DPPs* [Lyons, 2003, Ghosh, 2015, Bufetov et al., 2021], which look for conditions under which a sample \mathbf{x} from a DPP is a *uniqueness set* in the sense that two elements of \mathcal{F} that coincide on \mathbf{x} are equal.

Acknowledgments

AB acknowledges support from the AllegroAssai ANR project ANR-19-CHIA-0009. RB acknowledges support from the ERC grant BLACKJACK ERC-2019-STG-851866 and the Baccarat ANR project ANR-20-CHIA-0002. PC acknowledges support from the Sherlock ANR project ANR-20-CHIA-0031-01, the programme d'investissements d'avenir ANR-16-IDEX-0004 ULNE and Région Hauts-de-France.

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