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María ANGUIANO* and Francisco Javier SUÁREZ-GRAU†

Abstract

In this paper we study the asymptotic behavior of the stationary 3D magneto-micropolar fluid flow through a thin domain, whose thickness is given by a parameter $0 < \varepsilon \ll 1$. Assuming that the magnetic Reynolds number is written in terms of the thickness ε, we prove that there exists a critical magnetic Reynolds number, namely $Re_m^c = \varepsilon^{-2}$, such that for every magnetic Reynolds number Re_m with order smaller or equal than Re_m^c, the magneto-micropolar fluid flow in the thin domain can be modeled asymptotically when ε tends to zero by a 2D Reynolds-like model, whose expression is also given.

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1 Introduction

The magneto-micropolar fluid is the electric conductive micropolar fluid under the presence of a magnetic field. The equations that describe the motion of a viscous incompressible magneto-micropolar fluid in a domain $\Omega \subset \mathbb{R}^3$ in a time interval $[0,T]$, $0 < T < +\infty$, are given by (see [1] and [39] for instance):

$$\left\{\begin{array}{r} \partial_t u + (u\cdot\nabla)u - (\mu+\chi)\Delta u + \nabla\left(p + \frac{1}{2}|b|^2\right) = (b\cdot\nabla)b + 2\chi\mathrm{rot}(w) + f, \\ \partial_t w + (u\cdot\nabla)w - \kappa\Delta w - \lambda\nabla\,\mathrm{div}(w) + 4\chi w = 2\chi\mathrm{rot}(u) + g, \\ \partial_t b - \nu\Delta b + (u\cdot\nabla)b + \nabla q = (b\cdot\nabla)u, \\ \mathrm{div}(u) = 0, \quad \mathrm{div}(b) = 0, \end{array}\right. \tag{1.1}$$

supplemented with the following boundary conditions

$$u = 0, \quad w = 0, \quad b = 0 \quad \text{on } [0,T]\times\partial\Omega, \tag{1.2}$$

and initial conditions

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad b(0,x) = b_0(x), \quad x \in \Omega. \tag{1.3}$$

Here, $u(t,x), w(t,x), b(t,x) \in \mathbb{R}^3$ denote respectively the velocity field, the microrotational velocity and the magnetic field, and $p(t,x), q(t,x) \in \mathbb{R}$ denote respectively the hydrostatic pressure and the magnetic pressure at a point $x \in \Omega$ and in a time $t \in [0,T]$. The parameter μ denotes the kinematic viscosity, χ the microrotational viscosity, $1/\nu$ the magnetic Reynolds number, and κ and λ the angular viscosities. From physical reasons, we assume that these constants should satisfy the condition $\min\{\mu,\chi,\nu,\kappa+\lambda\} > 0$, see e.g. [1], [39]. The functions $f(t,x), g(t,x) \in \mathbb{R}^3$ are given external fields.

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We remark that if the microstructure of the fluid and the magnetic field are not taken into account, i.e. $w \equiv 0$ and $b \equiv 0$, the system (1.1) is the the classical Navier-Stokes equation. If the magnetic field is zero, i.e. $b \equiv 0$, then (1.1) coincides with the micropolar fluid system proposed by Eringen [20], which describes the motion of numerous real fluids better than the classical Navier-Stokes equations, see for example the book of Lukaszewicz [24]. Also, if $w \equiv 0$ and $\chi = 0$, then (1.1) reduces to the magneto-hydrodymanics (MHD) system, which describes the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes, see for example [19].

The magneto-micropolar model (1.1) was first introduced by Ahmadi and Shahinpoor [1]. Rojas-Medar and Boldrini [39] proved the existence of weak solutions in a bounded domain and also, the uniqueness for a two-dimensional domain. We refer to Ortega-Torres and Rojas-Medar [31], Perusato *et al.* [36], Cruz *et al.* [17] and Niche and Perusato [30] for different studies concerning the analysis of the solutions of system (1.1)-(1.3). With respect to the stationary magneto-micropolar fluids, we refer to Durán *et al.* in [18] and Kim and Ko [21].

On the other hand, engineering practice also stresses the interest of studying the effects of a domain with small thickness, i.e. a thin domain, on the behavior of the fluid flow. For that reason, it is important to deduce asymptotic models in thin domains. Throughout the mathematical literature, one can find many papers on the rigorous derivation of the asymptotic models describing the flow of Newtonian, non-Newtonian, micropolar and MHD fluid flows through thin domains, see e.g. [3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 25, 26, 27, 28, 34, 35, 40, 41, 42, 43] and references therein. However, it seems that there are no rigorous studies of the asymptotic behavior of magneto-micropolar fluid flows through thin domains, which represents the main objective of the present work.

In this sense, we consider in this paper a magneto-micropolar fluid flow through a thin domain $\Omega_\varepsilon \subset \mathbb{R}^3$, with thickness $0 < \varepsilon \ll 1$. The thin domain Ω_ε is the set of $\mathbb{R}^3$ with horizontal variables $x' = (x_1, x_2)$ belonging to a subset $\omega \subset \mathbb{R}^2$, and vertical variable x_3 such that $0 < x_3 < \varepsilon h(x')$, where h is a positive and smooth function (see Section 2). Thus, the bottom of the domain Ω_ε is described by the points $x' \in \omega, x_3 = 0$ and the upper wall of Ω_ε is described by the points satisfying

$$x' \in \omega, \quad x_3 = \varepsilon h(x').$$

This type of thin domain has been extensively studied for the isothermal flows, see e.g. [10, 16] for the classical Newtonian fluid flow, see e.g. [15, 29] for the flow of the generalized Newtonian fluid, and see e.g. [11, 33] for the micropolar fluid flow. In these papers, by using homogenization techniques, they derive different types of lower dimensional Reynolds equations. As far as the authors know, the flow of a magneto-micropolar fluids has not been yet considered in the above described setting. Thus, in this paper we consider a stationary magneto-micropolar fluid flow through the thin domain Ω_ε described above.

The behavior of the magneto-micropolar fluid flow can also be influenced by the order of the magnetic Reynolds number, as can be observed in several studies, see e.g. [2, 22, 23, 37, 44]. Then, to study its effects we assume that the magnetic Reynolds number $1/\nu$ is proportional to the thickness of the domain and denoted by $Re_m^\varepsilon = \varepsilon^{-\gamma}$, with $\gamma \in \mathbb{R}$. As a result, we consider the following system

$$\begin{cases} (u_\varepsilon \cdot \nabla)u_\varepsilon - (\mu + \chi)\Delta u_\varepsilon + \nabla\left(p_\varepsilon + \frac{1}{2}|b_\varepsilon|^2\right) = (b_\varepsilon \cdot \nabla)b_\varepsilon + 2\chi \mathrm{rot}(w_\varepsilon) + f_\varepsilon & \text{in } \Omega_\varepsilon, \\ (u_\varepsilon \cdot \nabla)w_\varepsilon - \kappa_c\varepsilon^2\Delta w_\varepsilon - \lambda_c\varepsilon^2\nabla \operatorname{div}(w_\varepsilon) + 4\chi w_\varepsilon = 2\chi \mathrm{rot}(u_\varepsilon) + g_\varepsilon & \text{in } \Omega_\varepsilon, \\ -\nu_c\varepsilon^\gamma\Delta b_\varepsilon + (u_\varepsilon \cdot \nabla)b_\varepsilon + \nabla q_\varepsilon = (b_\varepsilon \cdot \nabla)u_\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0, \quad \operatorname{div}(b_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0, \quad w_\varepsilon = 0, \quad b_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.4}$$

where $\nu_c, \kappa_c, \lambda_c = \mathcal{O}(1)$ and the external forces are of the form $f_\varepsilon = (f_1(x'), f_2(x'), 0)$ and $g_\varepsilon = \varepsilon(g_1(x'), g_2(x'), 0)$ (see Section 2 for more details). The goal of this paper is to study the asymptotic behavior of the weak solutions of the magneto-micropolar fluid in such domain when the thickness ε tends to zero. To do that, we first obtain the corresponding reduced linear limit system for a restricted range of values of γ, and finally, we derive a generalized 2D magneto-micropolar Reynolds equation for the hydrostatic and magnetic pressures.

More precisely, we first prove that there exists a critical value $\gamma_c = 2$, i.e. a critical magnetic Reynolds number $Re_m^c = \varepsilon^{-\gamma_c}$, such that for every $\gamma \leq \gamma_c$ (i.e. for every magnetic Reynolds number Re_m^ε of order smaller or equal

than $Re_m^c = \varepsilon^{-2}$) the nonlinear terms of system (1.4) vanish when ε tends to zero (they don't appear in the limit system) and so, the fluid will be able to be modeled by a linear 2D Reynolds-type equation, and above this quantity the nonlinear terms should be taken into account in the limit model. To prove this result, we apply a dilatation in the vertical variable x_3, i.e. we introduce the change of variables $z' = x'$ and $z_3 = x_3/\varepsilon$, which let us to introduce the unknowns $\widetilde{u}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{b}_\varepsilon, \widetilde{p}_\varepsilon$ and $\widetilde{q}_\varepsilon$ defined in a fixed domain, independent of ε, and denoted by $\Omega = \omega \times (0, h(z')) \subset \mathbb{R}^3$. By duality arguments and interpolation techniques, we are able to estimate both pressures and to derive the restriction $\gamma \leq 2$. Next, assuming this restriction on γ, we prove the homogenization process when ε tends to zero and identify the reduced linear limit system with unique solution denoted by $(\widetilde{u}, \widetilde{w}, \widetilde{b}, \widetilde{p}, \widetilde{q})$ (see Theorem 3.1).

We complete the paper by obtaining a generalized 2D magneto-micropolar Reynolds law for the hydrodynamic and magnetic pressures. Thus, we obtain that the averaged velocity $\widetilde{U}^\varepsilon = \int_0^{h(z')} \widetilde{u}_\varepsilon \, dz_3$, the averaged microrotation $\widetilde{W}^\varepsilon = \int_0^{h(z')} \widetilde{w}_\varepsilon \, dz_3$, the averaged magnetic field $\widetilde{B}^\varepsilon = \int_0^{h(z')} \widetilde{b}_\varepsilon \, dz_3$ and the hydrodynamic and magnetic pressures $\widetilde{p}_\varepsilon$ and $\widetilde{q}_\varepsilon$ satisfy the following approximations

$$\widetilde{U}_\varepsilon \approx \varepsilon^2 \widetilde{U}, \quad \widetilde{W}_\varepsilon \approx \varepsilon \widetilde{W}, \quad \widetilde{B}_\varepsilon \approx \varepsilon^{2-\frac{\gamma}{2}} \widetilde{B}, \quad \widetilde{p}_\varepsilon \approx p, \quad \widetilde{q}_\varepsilon \approx \varepsilon^{\frac{\gamma}{2}} q,$$

where $\widetilde{U} = \int_0^{h(z')} \widetilde{u} \, dz_3$, $\widetilde{W} = \int_0^{h(z')} \widetilde{w} \, dz_3$, $\widetilde{B} = \int_0^{h(z')} \widetilde{b} \, dz_3$ are given by

$$\begin{aligned}
&\widetilde{U}_1(z') = \tfrac{h^3(z')}{\mu} \Phi(h(z'), \kappa_c, \chi, \mu)(f_1(z') - \partial_{z_1} \widetilde{p}(z')),\\
&\widetilde{U}_2(z') = \tfrac{h^3(z')}{\mu} \Phi(h(z'), \kappa_c, \chi, \mu)(f_2(z') - \partial_{z_2} \widetilde{p}(z')),\\
&\widetilde{U}_3(z') = 0,\\
&\widetilde{W}_1(z') = \tfrac{h(z')}{4\chi} \Psi(h(z'), \kappa_c, \chi, \mu) g_1(z'),\\
&\widetilde{W}_2(z') = \tfrac{h(z')}{4\chi} \Psi(h(z'), \kappa_c, \chi, \mu) g_2(z'),\\
&\widetilde{W}_3(z') = 0,\\
&\widetilde{B}_1(z') = -\tfrac{h^3(z')}{12\nu_c} \partial_{z_1} \widetilde{q}(z'), \quad \widetilde{B}_2(z') = -\tfrac{h^3(z')}{12\nu_c} \partial_{z_2} \widetilde{q}(z'), \quad \widetilde{B}_3(z') = 0,
\end{aligned}$$

defined for a.e. $z' \in \omega$, where the functions Φ and Ψ are defined in (3.32) (see Theorem 3.4). Moreover, the limit hydrodynamic and magnetic pressures $\widetilde{p}$ and $\widetilde{q}$ are independent of z_3 and are the solution of the following generalized 2D magneto-micropolar Reynolds equation given by

$$-\mathrm{div}_{z'} \left(\frac{h^3(z')}{\mu} \Phi(h(z'), \kappa_c, \chi, \mu) \nabla_{z'} \widetilde{p}(z') \right) = -\mathrm{div}_{z'} \left(\frac{h^3(z')}{\mu} \Phi(h(z'), \kappa_c, \chi, \mu) f'(z') \right),$$

$$-\mathrm{div}_{z'} \left(\frac{h^3(z')}{12\nu_c} \nabla_{z'} \widetilde{q}(z') \right) = 0,$$

where $f'(z') = (f_1(z'), f_2(z'))$.

The structure of the paper is as follows. In Section 2 we introduce the domain and make the statement of the problem. The main results are given in Section 3. More precisely, in Subsection 3.1, we give the main result concerning the derivation of the reduced limit system (Theorem 3.1) and in Subsection 3.2, we give the expressions of the limit unknown and the generalized 2D magneto-micropolar Reynolds equation (Theorem 3.4). The proof of the main results are in Section 4. We finish the paper with a list of references.

2 Formulation of the problem and preliminaries

In this section, we first define the thin domain and some sets necessary to study the asymptotic behavior of the solutions. Next, we introduce the problem considered in the thin domain and also, the rescaled problem posed

in the domain of fixed height, together with the respective weak variational formulations.

2.1 The domain and some notation.

Along this paper, the points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

We consider ω as an open, smooth, bounded and connected set of $\mathbb{R}^2$, and a 3D thin domain defined by

$$\Omega_\varepsilon = \left\{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} \,:\, x' \in \omega,\ 0 < x_3 < h_\varepsilon(x')\right\}.$$

Here, the function $h_\varepsilon(x') = \varepsilon h(x')$ represents the real gap between the two surfaces. The small parameter ε is related to the film thickness. Function h is positive and smooth C^1 bounded function defined for x' such that $1/2 \leq h \leq 1$. We define the boundaries of Ω_ε as follows

$$\Gamma_0 = \omega \times \{0\}, \quad \Gamma_1^\varepsilon = \left\{(x', x_3) \in \mathbb{R}^3 \,:\, x' \in \omega,\ x_3 = h_\varepsilon(x')\right\}.$$

We denote by O_ε a generic real sequence which tends to zero with ε and can change from line to line. We denote by C a generic positive constant which can change from line to line.

2.2 The problem

We consider the 3D stationary magneto-micropolar fluid equations by setting

$$u_\varepsilon(x) = (u'_\varepsilon(x), u_{3,\varepsilon}(x)), \quad b_\varepsilon(x) = (b'_\varepsilon(x), b_{3,\varepsilon}(x)), \quad w_\varepsilon(x) = (w'_\varepsilon(x), w_{3,\varepsilon}(x)),$$

at a point $x \in \Omega_\varepsilon$, and we set additional assumptions, which will be used along the paper:

- To derive coupled effects between velocity and microrotation, we assume (see for instance [11, 32]):

$$\kappa = \kappa_c \varepsilon^2, \quad \lambda = \lambda_c \varepsilon^2 \quad \kappa_c, \lambda_c = \mathcal{O}(1).$$

- To take into account the influence of the magnetic Reynolds number, we assume

$$\nu = \nu_c \varepsilon^\gamma, \quad \nu_c = \mathcal{O}(1), \quad \gamma \in \mathbb{R}.$$

- Due to the thickness of the domain, it is usual to assume that the vertical components of the external forces can be neglected and, moreover, the forces can be considered independent of the vertical variable. Thus, for sake of simplicity, given $f', g' \in L^2(\omega)^2$, along the paper we consider the following rescaling for f_ε, g_ε (see for instance [42, 43]):

$$f_\varepsilon = (f'(x'), 0), \quad g_\varepsilon = \varepsilon(g'(x'), 0). \tag{2.5}$$

Thus, as stated in the introduction, the 3D stationary magneto-micropolar system is expressed as follows

$$\begin{cases} (u_\varepsilon \cdot \nabla)u_\varepsilon - (\mu + \chi)\Delta u_\varepsilon + \nabla\left(p_\varepsilon + \frac{1}{2}|b_\varepsilon|^2\right) = (b_\varepsilon \cdot \nabla)b_\varepsilon + 2\chi \mathrm{rot}(w_\varepsilon) + f_\varepsilon & \text{in } \Omega_\varepsilon, \\ (u_\varepsilon \cdot \nabla)w_\varepsilon - \kappa_c \varepsilon^2 \Delta w_\varepsilon - \lambda_c \varepsilon^2 \nabla \,\mathrm{div}(w_\varepsilon) + 4\chi w_\varepsilon = 2\chi \mathrm{rot}(u_\varepsilon) + g_\varepsilon & \text{in } \Omega_\varepsilon, \\ -\nu_c \varepsilon^\gamma \Delta b_\varepsilon + (u_\varepsilon \cdot \nabla)b_\varepsilon + \nabla q_\varepsilon = (b_\varepsilon \cdot \nabla)u_\varepsilon & \text{in } \Omega_\varepsilon, \\ \mathrm{div}(u_\varepsilon) = 0, \quad \mathrm{div}(b_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{2.6}$$

with corresponding boundary conditions

$$u_\varepsilon = 0, \quad w_\varepsilon = 0, \quad b_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{2.7}$$

Definition 2.1. *For $\varepsilon > 0$, we say that $(u_\varepsilon, w_\varepsilon, b_\varepsilon, p_\varepsilon, q_\varepsilon)$ defined on Ω_ε is a weak solution of problem (2.6)-(2.7) if and only if the functions $u_\varepsilon, w_\varepsilon, b_\varepsilon, p_\varepsilon$ and q_ε satisfy*

$$u_\varepsilon, b_\varepsilon, w_\varepsilon \in H_0^1(\Omega_\varepsilon)^3, \quad p_\varepsilon, q_\varepsilon \in L_0^2(\Omega_\varepsilon),$$

where L_0^2 is the space of functions of L^2 with zero mean value, and also satisfy the following equations

$$\begin{aligned}(\mu+\chi)\int_{\Omega_\varepsilon} Du_\varepsilon : D\varphi\,dx + \int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)u_\varepsilon\cdot\varphi\,dx - \int_{\Omega_\varepsilon} p_\varepsilon\,\mathrm{div}(\varphi)\,dx - \frac{1}{2}\int_{\Omega_\varepsilon}|b_\varepsilon|^2\mathrm{div}(\varphi)\,dx \\ = \int_{\Omega_\varepsilon}(b_\varepsilon\cdot\nabla)b_\varepsilon\cdot\varphi\,dx + 2\chi\int_{\Omega_\varepsilon}\mathrm{rot}(w_\varepsilon)\cdot\varphi\,dx + \int_{\Omega_\varepsilon} f'\cdot\varphi'\,dx,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\kappa_c\varepsilon^2\int_{\Omega_\varepsilon} Dw_\varepsilon : D\phi\,dx + \lambda_c\varepsilon^2\int_{\Omega_\varepsilon}\mathrm{div}(w_\varepsilon)\,\mathrm{div}(\phi)\,dx + 4\chi\int_{\Omega_\varepsilon} w_\varepsilon\cdot\phi\,dx + \int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)w_\varepsilon\cdot\phi\,dx \\ = 2\chi\int_{\Omega_\varepsilon}\mathrm{rot}(u_\varepsilon)\cdot\phi\,dx + \varepsilon\int_{\Omega_\varepsilon} g'\cdot\phi'\,dx,\end{aligned} \tag{2.9}$$

$$\nu_c\varepsilon^\gamma\int_{\Omega_\varepsilon} Db_\varepsilon : D\psi\,dx + \int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)b_\varepsilon\cdot\psi\,dx - \int_{\Omega_\varepsilon} q_\varepsilon\,\mathrm{div}(\psi)\,dx = \int_{\Omega_\varepsilon}(b_\varepsilon\cdot\nabla)u_\varepsilon\cdot\psi\,dx, \tag{2.10}$$

$$\int_{\Omega_\varepsilon} u_\varepsilon\cdot\nabla\theta\,dx = 0, \quad \int_{\Omega_\varepsilon} b_\varepsilon\cdot\nabla\zeta\,dx = 0, \tag{2.11}$$

every $\varphi, \psi, \phi \in H_0^1(\Omega_\varepsilon)^3$ and $\theta, \zeta \in H_0^1(\Omega_\varepsilon)$.

Remark 2.2. *Under previous assumptions, [18, Proposition 3.1] gives the existence of at least one weak solution $(u_\varepsilon, w_\varepsilon, b_\varepsilon, p_\varepsilon, q_\varepsilon) \in H_0^1(\Omega_\varepsilon)^3 \times H_0^1(\Omega_\varepsilon)^3 \times H_0^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon) \times L_0^2(\Omega_\varepsilon)$.*

The objetive of this paper is to study the asymptotic behavior of the sequence of solutions $(u_\varepsilon, w_\varepsilon, b_\varepsilon, p_\varepsilon, q_\varepsilon)$ of the problem (2.6)-(2.7) when ε tends to zero. To do that, we introduce a classical change of variables in thin domains, the dilatation

$$z' = x', \quad z_3 = \varepsilon^{-1}x_3. \tag{2.12}$$

This change transforms Ω_ε into a fixed domain Ω of heigh of order h, defined as follows

$$\Omega = \left\{(z', z_3) \in \mathbb{R}^2\times\mathbb{R} \,:\, z'\in\omega,\ 0 < z_3 < h(z')\right\}. \tag{2.13}$$

The boundary of Ω is denoted by $\partial\Omega$, where the the top boundary of the rescaled domain Ω is defined by

$$\Gamma_1 = \left\{(z', z_3) \in \mathbb{R}^2\times\mathbb{R} \,:\, z'\in\omega,\ z_3 = h(z')\right\}.$$

Accordingly, we define the functions $\widetilde{u}_\varepsilon, \widetilde{w}_\varepsilon$ and $\widetilde{b}_\varepsilon$ by

$$\widetilde{u}_\varepsilon(z) = u_\varepsilon(z', \varepsilon z_3), \quad \widetilde{w}_\varepsilon(z) = w_\varepsilon(z', \varepsilon z_3), \quad \widetilde{b}_\varepsilon(z) = b_\varepsilon(z', \varepsilon z_3) \quad \text{a.e. } z\in\Omega, \tag{2.14}$$

and, similarly, the functions $\widetilde{p}_\varepsilon$ and $\widetilde{q}_\varepsilon$ by

$$\widetilde{p}_\varepsilon(z) = p_\varepsilon(z', \varepsilon z_3), \quad \widetilde{q}_\varepsilon(z) = q_\varepsilon(z', \varepsilon z_3) \quad \text{a.e. } z\in\Omega. \tag{2.15}$$

Let us now introduce some notation which will be useful in the following. For a vectorial function $\widetilde{\varphi} = (\widetilde{\varphi}', \widetilde{\varphi}_3)$ and a scalar function $\widetilde{\phi}$ obtained respectively from functions φ and ϕ by using the change of variables (2.12), we introduce the operators Δ_ε, div_ε, D_ε, ∇_ε and rot_ε by

$$\begin{aligned}&\Delta_\varepsilon\widetilde{\varphi} = \Delta_{z'}\widetilde{\varphi} + \varepsilon^{-2}\partial_{z_3}^2\widetilde{\varphi}, \quad \mathrm{div}_\varepsilon(\widetilde{\varphi}) = \mathrm{div}_{z'}\widetilde{\varphi}' + \varepsilon^{-1}\partial_{z_3}\widetilde{\varphi}_3,\\ &(D_\varepsilon\widetilde{\varphi})_{ij} = \partial_{z_j}\widetilde{\varphi}_i \text{ for } i=1,2,3,\ j=1,2, \quad (D_\varepsilon\widetilde{\varphi})_{i3} = \varepsilon^{-1}\partial_{z_3}\widetilde{\varphi}_i \text{ for } i=1,2,3,\\ &\nabla_\varepsilon\widetilde{\phi} = (\nabla_{z'}\widetilde{\phi}, \varepsilon^{-1}\partial_{z_3}\widetilde{\phi}), \quad \mathrm{rot}_\varepsilon(\widetilde{\varphi}) = \left(\mathrm{rot}_{z'}(\widetilde{\varphi}_3) + \varepsilon^{-1}\mathrm{rot}_{z_3}(\widetilde{\varphi}'), \mathrm{Rot}_{z'}(\widetilde{\varphi}')\right),\end{aligned} \tag{2.16}$$

where, denoting $(\widetilde{\varphi}')^{\perp}=(-\widetilde{\varphi}_2,\widetilde{\varphi}_1)$, we define

$$\mathrm{rot}_{z'}(\widetilde{\varphi}_3)=(\partial_{z_2}\widetilde{\varphi}_3,-\partial_{z_1}\widetilde{\varphi}_3),\quad \mathrm{rot}_{z_3}(\widetilde{\varphi}')=\partial_{z_3}(\widetilde{\varphi}')^{\perp},\quad \mathrm{Rot}_{z'}(\widetilde{\varphi}')=\partial_{z_1}\widetilde{\varphi}_2-\partial_{z_2}\widetilde{\varphi}_1. \tag{2.17}$$

Thus, using the change of variables (2.12) and taking into account the rescaling of the forces (2.5), the system (2.6)-(2.7) can be rewritten as

$$\begin{cases} (\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon-(\mu+\chi)\Delta_\varepsilon\widetilde{u}_\varepsilon+\nabla_\varepsilon\left(\widetilde{p}_\varepsilon+\dfrac{1}{2}|\widetilde{b}_\varepsilon|^2\right)=(\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon+2\chi\mathrm{rot}_\varepsilon(\widetilde{w}_\varepsilon)+f_\varepsilon & \text{in }\Omega,\\ (\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{w}_\varepsilon-\kappa_c\varepsilon^2\Delta_\varepsilon\widetilde{w}_\varepsilon-\lambda_c\varepsilon^2\nabla_\varepsilon\mathrm{div}_\varepsilon(\widetilde{w}_\varepsilon)+4\chi\widetilde{w}_\varepsilon=2\chi\mathrm{rot}_\varepsilon(\widetilde{u}_\varepsilon)+g_\varepsilon & \text{in }\Omega,\\ -\nu_c\varepsilon^\gamma\Delta_\varepsilon\widetilde{b}_\varepsilon+(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon+\nabla_\varepsilon\widetilde{q}_\varepsilon=(\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon & \text{in }\Omega,\\ \mathrm{div}_\varepsilon(\widetilde{u}_\varepsilon)=0,\quad \mathrm{div}_\varepsilon(\widetilde{b}_\varepsilon)=0 & \text{in }\Omega,\end{cases} \tag{2.18}$$

with corresponding boundary conditions

$$\widetilde{u}_\varepsilon=0,\quad \widetilde{w}_\varepsilon=0,\quad \widetilde{b}_\varepsilon=0\quad\text{on }\partial\Omega. \tag{2.19}$$

According to the change of variables (2.12) applied to the weak variational formulation (2.8)-(2.10), then, for $\varepsilon>0$, a rescaled weak solution $(\widetilde{u}_\varepsilon,\widetilde{w}_\varepsilon,\widetilde{b}_\varepsilon,\widetilde{p}_\varepsilon,\widetilde{q}_\varepsilon)$ of problem (2.18)-(2.19) satisfies

$$\begin{aligned}(\mu+\chi)\int_\Omega D_\varepsilon\widetilde{u}_\varepsilon:D_\varepsilon\widetilde{\varphi}\,dz+\int_\Omega(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon\cdot\widetilde{\varphi}\,dz-\int_\Omega\widetilde{p}_\varepsilon\,\mathrm{div}_\varepsilon(\widetilde{\varphi})\,dz-\frac{1}{2}\int_\Omega|\widetilde{b}_\varepsilon|^2\mathrm{div}_\varepsilon(\widetilde{\varphi})\,dz\\ =\int_\Omega(\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon\cdot\widetilde{\varphi}\,dz+2\chi\int_\Omega\mathrm{rot}_\varepsilon(\widetilde{w}_\varepsilon)\cdot\widetilde{\varphi}\,dz+\int_\Omega f'\cdot\widetilde{\varphi}'\,dz,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\kappa_c\varepsilon^2\int_\Omega D_\varepsilon\widetilde{w}_\varepsilon:D_\varepsilon\widetilde{\phi}\,dz+\lambda_c\varepsilon^2\int_\Omega\mathrm{div}_\varepsilon(\widetilde{w}_\varepsilon)\,\mathrm{div}_\varepsilon(\widetilde{\phi})\,dz+4\chi\int_\Omega\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz+\int_\Omega(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz\\ =2\chi\int_\Omega\mathrm{rot}_\varepsilon(\widetilde{u}_\varepsilon)\cdot\widetilde{\phi}\,dz+\varepsilon\int_\Omega g'\cdot\widetilde{\phi}'\,dz,\end{aligned} \tag{2.21}$$

$$\nu_c\varepsilon^\gamma\int_\Omega D_\varepsilon\widetilde{b}_\varepsilon:D_\varepsilon\widetilde{\psi}\,dz+\int_\Omega(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon\cdot\widetilde{\psi}\,dz-\int_\Omega\widetilde{q}_\varepsilon\,\mathrm{div}_\varepsilon(\widetilde{\psi})\,dz=\int_\Omega(\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon\cdot\widetilde{\psi}\,dz, \tag{2.22}$$

$$\int_\Omega\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon\widetilde{\theta}\,dz=0,\quad \int_\Omega\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon\widetilde{\zeta}\,dz=0, \tag{2.23}$$

for every $\widetilde{\varphi},\widetilde{\phi},\widetilde{\psi}$ and $\widetilde{\theta},\widetilde{\zeta}$ obtained from φ,ϕ,ψ and θ,ζ respectively by the change of variables (2.12).

3 Main results

In this section, we give the main results of this paper. In Subsection 3.1, we describe the asymptotic behavior of the rescaled $\widetilde{u}_\varepsilon,\widetilde{w}_\varepsilon,\widetilde{b}_\varepsilon,\widetilde{p}_\varepsilon$ and $\widetilde{q}_\varepsilon$, and we obtain the reduced limit system. In Subsection 3.2, we give the expressions of the solution of the reduced limit system and as consequence, we derive the generalized 2D magneto-micropolar Reynolds equation. The proof of the corresponding result is given in Section 4.

3.1 Effective system

Below we give the result concerning the asymptotic behavior of the rescaled functions $\widetilde{u}_\varepsilon$, $\widetilde{w}_\varepsilon$, $\widetilde{b}_\varepsilon$, $\widetilde{p}_\varepsilon$ and $\widetilde{q}_\varepsilon$ together with the restriction of the magnetic Reynolds number.

Theorem 3.1. *Assume $\gamma \leq 2$. Then, there exist $\widetilde{u}', \widetilde{w}', \widetilde{b}' \in H^1(0, h(z'); L^2(\omega)^2)$ and $\widetilde{p}, \widetilde{q} \in L^2_0(\Omega)$, independent of z_3, such that a solution $(\widetilde{u}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{b}_\varepsilon, \widetilde{p}_\varepsilon, \widetilde{q}_\varepsilon)$ of problem (2.18)-(2.19) satisfies the following convergences*

$$\varepsilon^{-2}\widetilde{u}_\varepsilon \rightharpoonup \widetilde{u} = (\widetilde{u}', 0) \quad \text{in } H^1(0, h(z'); L^2(\omega)^3), \quad \varepsilon^{-1}\widetilde{w}_\varepsilon \rightharpoonup \widetilde{w} = (\widetilde{w}', 0) \quad \text{in } H^1(0, h(z'); L^2(\omega)^3),$$
$$\widetilde{p}_\varepsilon \rightharpoonup \widetilde{p} \quad \text{in } L^2(\Omega),$$
$$\varepsilon^{\frac{\gamma}{2}-2}\widetilde{b}_\varepsilon \rightharpoonup \widetilde{b} = (\widetilde{b}', 0) \quad \text{in } H^1(0, h(z'); L^2(\omega)^3), \quad \varepsilon^{-\frac{\gamma}{2}}\widetilde{q}_\varepsilon \rightharpoonup \widetilde{q} \quad \text{in } L^2(\Omega).$$

Moreover, $(\widetilde{u}', \widetilde{w}', \widetilde{b}', \widetilde{p}, \widetilde{q})$ is the unique weak solution of the limit system

$$\begin{cases} -(\mu+\chi)\partial^2_{z_3}\widetilde{u}' + \nabla_{z'}\widetilde{p} = 2\chi \mathrm{rot}_{z_3}(\widetilde{w}') + f' & \text{in } \Omega, \\ -\kappa_c\partial^2_{z_3}\widetilde{w}' + 4\chi\widetilde{w}' = 2\chi \mathrm{rot}_{z_3}(\widetilde{u}') + g' & \text{in } \Omega, \\ -\nu_c\partial^2_{z_3}\widetilde{b}' + \nabla_{z'}\widetilde{q} = 0 & \text{in } \Omega, \\ \widetilde{u}' = \widetilde{w}' = \widetilde{b}' = 0 & \text{on } \Gamma_0 \cup \Gamma_1, \end{cases} \tag{3.24}$$

together with free divergence conditions

$$\mathrm{div}_{z'}\left(\int_0^{h(z')} \widetilde{u}'(z)\, dz_3\right) = 0 \quad \text{in } \omega, \quad \left(\int_0^{h(z')} \widetilde{u}'(z)\, dz_3\right)\cdot \mathrm{n} = 0 \quad \text{on } \partial\omega, \tag{3.25}$$

$$\mathrm{div}_{z'}\left(\int_0^{h(z')} \widetilde{b}'(z)\, dz_3\right) = 0 \quad \text{in } \omega, \quad \left(\int_0^{h(z')} \widetilde{b}'(z)\, dz_3\right)\cdot \mathrm{n} = 0 \quad \text{on } \partial\omega, \tag{3.26}$$

with n *the outside normal vector to $\partial\omega$.*

Remark 3.2. *We have proved that there exists a critical value $\gamma_c = 2$, i.e. a critical magnetic Reynolds number $Re^c_m = \varepsilon^{-\gamma_c} = \varepsilon^{-2}$, such that for every $\gamma \leq \gamma_c$, the nonlinear terms of the system (2.18)-(2.19) vanish in the limit. Observe that condition $\gamma \leq \gamma_c = 2$ means that the Re_m is of order smaller or equal that Re^c_m. Thus, for $\gamma \leq 2$, we have obtained the limit system (3.24)-(3.26), from which we will be able to derive a Reynolds-like equation for pressures (see Theorem 3.4).*

3.2 Generalized magneto-micropolar Reynolds equations

In this subsection, we obtain a generalized 2D magneto-micropolar Reynolds equation associated to the homogenized micropolar system given in Theorem 3.1. For sake of clarity, we rewrite system (3.24) by components, which will be useful to prove Theorem 3.4. Thus, $\widetilde{u}'$ and $\widetilde{w}'$ satisfy the following system

$$\begin{cases} -(\mu+\chi)\partial^2_{z_3}\widetilde{u}_1(z) + 2\chi\partial_{z_3}\widetilde{w}_2(z) &=& f_1(z') - \partial_{z_1}\widetilde{p}(z'), \\ -\kappa_c\partial^2_{z_3}\widetilde{w}_2(z) + 4\chi\widetilde{w}_2(z) - 2\chi\partial_{z_3}\widetilde{u}_1(z) &=& g_2(z'), \\ \widetilde{u}_1(z',0) = \widetilde{u}_1(z', h(z')) &=& 0, \\ \widetilde{w}_2(z',0) = \widetilde{w}_2(z', h(z')) &=& 0, \end{cases} \tag{3.27}$$

$$\begin{cases} -(\mu+\chi)\partial^2_{z_3}\widetilde{u}_2(z) - 2\chi\partial_{z_3}\widetilde{w}_1(z) &=& f_2(z') - \partial_{z_2}\widetilde{p}(z'), \\ -\kappa_c\partial^2_{z_3}\widetilde{w}_1(z) + 4\chi\widetilde{w}_1(z) + 2\chi\partial_{z_3}\widetilde{u}_2(z) &=& g_1(z'), \\ \widetilde{u}_2(z',0) = \widetilde{u}_2(z', h(z')) &=& 0, \\ \widetilde{w}_1(z',0) = \widetilde{w}_1(z', h(z')) &=& 0, \end{cases} \tag{3.28}$$

together with the free divergence condition

$$\partial_{z_1}\left(\int_0^{h(z')} \widetilde{u}_1(z)\, dz_3\right) + \partial_{z_2}\left(\int_0^{h(z')} \widetilde{u}_2(z)\, dz_3\right) = 0, \tag{3.29}$$

while $\widetilde{b}'$ satisfies the following one

$$\left\{\begin{array}{rcl} -\nu_c\partial_{z_3}^2\widetilde{b}_1(z)+\partial_{z_1}\widetilde{q}(z') &=& 0,\\ -\nu_c\partial_{z_3}^2\widetilde{b}_2(z)+\partial_{z_2}\widetilde{q}(z') &=& 0,\\ \widetilde{w}_1(z',0)=\widetilde{w}_1(z',h(z')) &=& 0,\end{array}\right. \tag{3.30}$$

together with the free divergence condition

$$\partial_{z_1}\left(\int_0^{h(z')}\widetilde{b}_1(z)\,dz_3\right)+\partial_{z_2}\left(\int_0^{h(z')}\widetilde{b}_2(z)\,dz_3\right)=0. \tag{3.31}$$

In the next lemma, we give the expression of $(\widetilde{u}',\widetilde{w}')$ in terms of f' and $\widetilde{p}$, and the expression of $\widetilde{b}'$ in terms of $\widetilde{q}$.

Lemma 3.3. *The solution $(\widetilde{u}_1,\widetilde{w}_2)$ of system (3.27) is given by the following expressions*

$$\begin{aligned}\widetilde{u}_1(z)=&\left[-\tfrac{z_3^2}{2\mu}-\tfrac{h(z')}{4\mu}\left(\tfrac{2\chi}{k(\mu+\chi)}\sinh(kz_3)-2z_3\right)\right.\\&\left.+\tfrac{\chi h(z')}{2k(\mu+\chi)\mu}(\cosh(kz_3)-1)\coth\left(\tfrac{kh(z')}{2}\right)\right](f_1(z')-\partial_{z_1}\widetilde{p}(z'))\\&+\left[-\tfrac{z_3}{2\chi}+\tfrac{h(z')}{2\chi}\left(\tfrac{2\chi}{k(\mu+\chi)}\sinh(kz_3)-2z_3\right)A_2(z')+\tfrac{h(z')}{k(\mu+\chi)}(\cosh(kz_3)-1)B_2(z')\right]g_2(z').\end{aligned}$$

$$\begin{aligned}\widetilde{w}_2(z)=&\left[-\tfrac{z_3}{2\mu}-(\cosh(kz_3)-1)\tfrac{h(z')}{4\mu}+\tfrac{h(z')}{4\mu}\sinh(kz_3)\coth\left(\tfrac{kh(z')}{2}\right)\right](f_1(z')-\partial_{z_1}\widetilde{p}(z'))\\&+\tfrac{h(z')}{2\chi}\left[(\cosh(kz_3)-1)A_2(z')+\sinh(kz_3)B_2(z')\right]g_2(z'),\end{aligned}$$

where $k=\sqrt{\frac{4\chi\mu}{\kappa_c(\mu+\chi)}}$ and the functions A_2,B_2 are given by

$$A_2(z')=-\frac{1}{2}\frac{1}{h(z')-\frac{2\chi}{k(\mu+\chi)}\tanh\left(\frac{kh(z')}{2}\right)},\quad B_2(z')=\frac{1}{2}\frac{\tanh\left(\frac{kh(z')}{2}\right)}{h(z')-\frac{2\chi}{k(\mu+\chi)}\tanh\left(\frac{kh(z')}{2}\right)}.$$

The solution $(\widetilde{u}_2,\widetilde{w}_1)$ of system (3.28) is given by the following expressions

$$\begin{aligned}\widetilde{u}_2(z)=&\left[-\tfrac{z_3^2}{2\mu}-\tfrac{h(z')}{4\mu}\left(\tfrac{2\chi}{k(\mu+\chi)}\sinh(kz_3)-2z_3\right)\right.\\&\left.+\tfrac{\chi h(z')}{2k(\mu+\chi)\mu}(\cosh(kz_3)-1)\coth\left(\tfrac{kh(z')}{2}\right)\right](f_2(z')-\partial_{z_2}\widetilde{p}(z'))\\&-\left[-\tfrac{z_3}{2\chi}+\tfrac{h(z')}{2\chi}\left(\tfrac{2\chi}{k(\mu+\chi)}\sinh(kz_3)-2z_3\right)A_2(z')+\tfrac{h(z')}{k(\mu+\chi)}(\cosh(kz_3)-1)B_2(z')\right]g_1(z').\end{aligned}$$

$$\begin{aligned}\widetilde{w}_1(z)=&-\left[-\tfrac{z_3}{2\mu}-(\cosh(kz_3)-1)\tfrac{h(z')}{4\mu}+\tfrac{h(z')}{4\mu}\sinh(kz_3)\coth\left(\tfrac{kh(z')}{2}\right)\right](f_2(z')-\partial_{z_2}\widetilde{p}(z'))\\&+\tfrac{h(z')}{2\chi}\left[(\cosh(kz_3)-1)A_2(z')+\sinh(kz_3)B_2(z')\right]g_1(z').\end{aligned}$$

Finally, the solution $(\widetilde{b}_1,\widetilde{b}_2)$ of system (3.30) is given by the following expressions

$$\widetilde{b}_1(z)=\frac{z_3^2-h(z')z_3}{2\nu_c}\partial_{z_1}\widetilde{q}(z'),\quad \widetilde{b}_2(z)=\frac{z_3^2-h(z')z_3}{2\nu_c}\partial_{z_2}\widetilde{q}(z').$$

As consecuence of the previous result, we give the main result concerning the generalized Reynolds equation.

Theorem 3.4 (Generalized 2D magneto-micropolar Reynolds equation)**.** *Defining the averaged limit velocity, microrotation and magnetic field respectively by*

$$\widetilde{U}(z')=\int_0^{h(z')}\widetilde{u}(z)\,dz_3,\quad \widetilde{W}(z')=\int_0^{h(z')}\widetilde{w}(z)\,dz_3,\quad \widetilde{B}(z')=\int_0^{h(z')}\widetilde{b}(z)\,dz_3,$$

we have the following expressions

$$\begin{aligned}
&\widetilde{U}'(z')=\tfrac{h^3(z')}{\mu}\Phi(h(z'),\kappa_c,\chi,\mu)(f'(z')-\nabla_{z'}\widetilde{p}(z')), && \widetilde{U}_3(z')=0,\\
&\widetilde{W}'(z')=\tfrac{h(z')}{4\chi}\Psi(h(z'),\kappa_c,\chi,\mu)g'(z'), && \widetilde{W}_3(z')=0,\\
&\widetilde{B}'(z')=-\tfrac{h^3(z')}{12\nu_c}\nabla_{z'}\widetilde{q}(z'), && \widetilde{B}_3(z')=0,
\end{aligned}$$

where the functions Φ *and* Ψ *are respectively defined by*

$$\begin{aligned}
\Phi(h(z'),\kappa_c,\chi,\mu) &= \frac{1}{12}+\frac{\kappa_c}{4h^2(z')}-\frac{1}{4h^2(z')}\sqrt{\frac{\kappa_c\chi}{\mu(\mu+\chi)}}\coth\left(h(z')\sqrt{\frac{\chi\mu}{\kappa_c(\mu+\chi)}}\right),\\
\Psi(h(z'),\kappa_c,\chi,\mu) &= \frac{1-\frac{1}{h(z')}\sqrt{\frac{\kappa_c(\mu+\chi)}{\chi\mu}}\tanh\left(h(z')\sqrt{\frac{\chi\mu}{\kappa_c(\mu+\chi)}}\right)}{1-\frac{\chi}{\mu+\chi}\frac{1}{h(z')}\sqrt{\frac{\kappa_c(\mu+\chi)}{\chi\mu}}\tanh\left(h(z')\sqrt{\frac{\chi\mu}{\kappa_c(\mu+\chi)}}\right)}.
\end{aligned}\tag{3.32}$$

The hydrostatic pressure $\widetilde{p}\in H^1(\Omega)\cap L^2_0(\Omega)$ *and the magnetic pressure* $\widetilde{q}\in H^1(\Omega)\cap L^2_0(\Omega)$ *are the unique solution of the Reynolds equations with Neumann boundary conditions*

$$\begin{cases}
-\mathrm{div}_{z'}\left(\dfrac{h^3(z')}{\mu}\Phi(h(z'),\kappa_c,\chi,\mu)\nabla_{z'}\widetilde{p}(z')\right)=-\mathrm{div}_{z'}\left(\dfrac{h^3(z')}{\mu}\Phi(h(z'),\kappa_c,\chi,\mu)f'(z')\right),\\
-\mathrm{div}_{z'}\left(\dfrac{h^3(z')}{12\nu_c}\nabla_{z'}\widetilde{q}(z')\right)=0,
\end{cases}\tag{3.33}$$

whose variational formulation is given as follows

$$\begin{cases}
\displaystyle\int_\omega\frac{h^3(z')}{\mu}\Phi(h(z'),\kappa_c,\chi,\mu)\,\nabla_{z'}\widetilde{p}(z')\cdot\nabla_{z'}\theta(z')\,dz'=\int_\omega\frac{h^3(z')}{\mu}\Phi(h(z'),\kappa_c,\chi,\mu)f'(z')\cdot\nabla_{z'}\theta(z')\,dz',\\
\displaystyle\int_\omega\frac{h^3(z')}{12\nu_c}\nabla_{z'}\widetilde{q}(z')\cdot\nabla_{z'}\zeta(z')\,dz'=0,
\end{cases}\tag{3.34}$$

for every $\theta,\zeta\in H^1(\omega)$.

4 Proof of the main results

In this section, we will prove the main results. In Subsection 4.1, we derive the estimates of the velocity, microrotation and magnetic field. Using these estimates, in Subsection 4.2, we derive the corresponding estimates of the hydrostatic and magnetic pressures together with a restriction of the value of γ. The compactness results and the proof of Theorem 3.1, corresponding to derivation of the limit system (3.24)-(3.26), will be developed in Subsection 4.3. Finally, the proof of Theorem 3.4 concerning the magneto-micropolar Reynolds equation is given in Subsection 4.4.

4.1 A priori velocity, microrotation and magnetic field estimates

In this subsection we give a priori estimates for the rescaled functions $\widetilde{u}_\varepsilon$, $\widetilde{w}_\varepsilon$ and $\widetilde{b}_\varepsilon$. First, we recall some important inequalities in a domain with thickness ε (see e.g. [24], [29]).

Lemma 4.1 (Poincaré and Ladyzhenskaya inequalities). *For all $\varphi \in H_0^1(\Omega^\varepsilon)^3$, there hold the following inequalities*

$$\|\varphi\|_{L^2(\Omega_\varepsilon)^3} \leq C\varepsilon \|D\varphi\|_{L^2(\Omega_\varepsilon)^{3\times 3}}, \quad \|\varphi\|_{L^4(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{1}{4}} \|D\varphi\|_{L^2(\Omega_\varepsilon)^{3\times 3}}. \tag{4.35}$$

Moreover, from the change of variables (2.12), there hold the following rescaled estimates

$$\|\tilde{\varphi}\|_{L^2(\Omega)^3} \leq C\varepsilon \|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}}, \quad \|\tilde{\varphi}\|_{L^4(\Omega)^3} \leq C\varepsilon^{\frac{1}{2}} \|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}}. \tag{4.36}$$

Lemma 4.2 (See Lemma 2 in [42]). *For $\varphi \in H_0^1(\Omega_\varepsilon)$ and $\tilde{\varphi}$ obtained from φ by the change of variables (2.12), the following inequalities hold*

$$\|\mathrm{rot}(\varphi)\|_{L^2(\Omega_\varepsilon)^3} \leq \|D\varphi\|_{L^2(\Omega_\varepsilon)^{3\times 3}}, \quad \|\mathrm{rot}_\varepsilon(\tilde{\varphi})\|_{L^2(\Omega)^3} \leq \|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}}. \tag{4.37}$$

Next, we give the *a priori* estimates of velocity, microrotation and magnetic fields.

Lemma 4.3 (Estimates for velocity, microrotation and magnetic field). *Let $(u_\varepsilon, w_\varepsilon, b_\varepsilon)$ be a weak solution of (2.6)-(2.7) and let $\gamma \in \mathbb{R}$. Then, there exists a positive constant C, independent of ε, such that we have the following estimates*

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}}, \quad \|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{3}{2}}, \tag{4.38}$$

$$\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}}, \quad \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{1}{2}}. \tag{4.39}$$

$$\|b_\varepsilon\|_{L^2(\Omega_\varepsilon)^3} \leq C\varepsilon^{\frac{5-\gamma}{2}}, \quad \|Db_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{3-\gamma}{2}}. \tag{4.40}$$

Moreover, after the change of variables (2.12), we have the following estimates:

$$\|\widetilde{u}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon^2, \quad \|D_\varepsilon \widetilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leq C\varepsilon, \tag{4.41}$$

$$\|\widetilde{w}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon, \quad \|D_\varepsilon \widetilde{w}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leq C. \tag{4.42}$$

$$\|\widetilde{b}_\varepsilon\|_{L^2(\Omega)^3} \leq C\varepsilon^{\frac{4-\gamma}{2}}, \quad \|D_\varepsilon \widetilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \leq C\varepsilon^{\frac{2-\gamma}{2}}. \tag{4.43}$$

Proof. We divide the proof in five steps.

Step 1. In this step, we derive some useful estimates for velocity that will be useful in the rest of the proof.

From (2.8) with $\varphi = u_\varepsilon$, we get

$$(\mu+\chi)\|Du_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} = \int_{\Omega_\varepsilon} (b_\varepsilon \cdot \nabla) b_\varepsilon \cdot u_\varepsilon \, dx + 2\chi \int_{\Omega_\varepsilon} \mathrm{rot}(w_\varepsilon) \cdot u_\varepsilon \, dx + \int_{\Omega_\varepsilon} f' \cdot u'_\varepsilon \, dx, \tag{4.44}$$

because $\int_{\Omega_\varepsilon} (u_\varepsilon \cdot \nabla) u_\varepsilon \cdot u_\varepsilon \, dx = 0$.

On the other hand, considering (2.10) with $\varphi = b_\varepsilon$, we get

$$\nu_c \varepsilon^\gamma \|Db_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} = \int_{\Omega_\varepsilon} (b_\varepsilon \cdot \nabla) u_\varepsilon \cdot b_\varepsilon \, dx, \tag{4.45}$$

because $\int_{\Omega_\varepsilon} (u_\varepsilon \cdot \nabla) b_\varepsilon \cdot b_\varepsilon \, dx = 0$.

Summing (4.44) and (4.45), we get

$$(\mu+\chi)\|Du_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} + \nu_c \varepsilon^\gamma \|Db_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}} = 2\chi \int_{\Omega_\varepsilon} \mathrm{rot}(w_\varepsilon) \cdot u_\varepsilon \, dx + \int_{\Omega_\varepsilon} f' \cdot u'_\varepsilon \, dx, \tag{4.46}$$

because $\int_{\Omega_\varepsilon}(b_\varepsilon\cdot\nabla)u_\varepsilon\cdot b_\varepsilon\,dx+\int_{\Omega_\varepsilon}(b_\varepsilon\cdot\nabla)b_\varepsilon\cdot u_\varepsilon\,dx=0$.

Observe that using Cauchy-Schwarz's inequality and taking into account that $1/2\leq h\leq 1$, we deduce

$$\begin{aligned}\int_{\Omega_\varepsilon} f'\cdot u_\varepsilon'\,dx &\leq \|f'\|_{L^2(\Omega_\varepsilon)^2}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &= \left(\int_\omega\int_0^{\varepsilon h(x')}|f'(x')|^2dx'dx_3\right)^{1/2}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &= \left(\varepsilon\int_\omega h(x')|f'(x')|^2dx'\right)^{1/2}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &\leq \varepsilon^{\frac12}\|f'\|_{L^2(\omega)^2}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}.\end{aligned}\tag{4.47}$$

Taking now into account that $\int_{\Omega_\varepsilon}\mathrm{rot}(w_\varepsilon)\cdot u_\varepsilon\,dx=\int_{\Omega_\varepsilon}\mathrm{rot}(u_\varepsilon)\cdot w_\varepsilon\,dx$ (see Exercise 1.3.3 in [24, Part II]) in (4.46), applying Cauchy-Schwarz's inequality, (4.47) and estimates (4.35) and $(4.37)_1$, we get

$$\begin{aligned}&(\mu+\chi)\|Du_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times3}}+\nu_c\varepsilon^\gamma\|Db_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times3}}\\ &\leq 2\chi\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}+\varepsilon^{\frac12}\|f'\|_{L^2(\omega)^2}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &\leq 2\chi\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}+\varepsilon^{\frac32}C\|f'\|_{L^2(\omega)^2}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\end{aligned}\tag{4.48}$$

which, in particular, implies

$$(\mu+\chi)\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}\leq 2\chi\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}+\varepsilon^{\frac32}C\|f'\|_{L^2(\omega)^2},\tag{4.49}$$

and

$$\nu_c\varepsilon^\gamma\|Db_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times3}}\leq 2\chi\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\|Du_\varepsilon\|_{L^2(\Omega)^{3\times3}}+\varepsilon^{\frac32}C\|f'\|_{L^2(\omega)^2}\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times3}}.\tag{4.50}$$

Thus, to derive estimates for velocity, we need to obtain estimates for w_ε, which will be derived in the following steps. The estimates for the magnetic field will be a direct consequence of the derived estimates for w_ε and u_ε together with (4.50).

Step 2. In this step, we derive some useful estimates for w_ε that will be useful in the rest of the proof. Similarly to the previous step, from (2.9) with $\phi=w_\varepsilon$, we get

$$\kappa_c\varepsilon^2\|Dw_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times3}}+\lambda_c\varepsilon^2\|\mathrm{div}(w_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)}+4\chi\|w_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^3}=2\chi\int_{\Omega_\varepsilon}\mathrm{rot}(u_\varepsilon)\cdot w_\varepsilon\,dx+\varepsilon\int_{\Omega_\varepsilon}g'\cdot w_\varepsilon'\,dx,\tag{4.51}$$

because $\int_{\Omega_\varepsilon}(u_\varepsilon\cdot\nabla)w_\varepsilon\cdot w_\varepsilon\,dx=0$.

Observe that using Cauchy-Schwarz's inequality and taking into account that $1/2\leq h\leq 1$, we deduce

$$\begin{aligned}\varepsilon\int_{\Omega_\varepsilon} g'\cdot w_\varepsilon'\,dx &\leq \varepsilon\|g'\|_{L^2(\Omega_\varepsilon)^2}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &= \varepsilon\left(\int_\omega\int_0^{\varepsilon h(x')}|g'(x')|^2dx'dx_3\right)^{1/2}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &= \varepsilon\left(\varepsilon\int_\omega h(x')|g'(x')|^2dx'\right)^{1/2}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\ &\leq \varepsilon^{\frac32}\|g'\|_{L^2(\omega)^2}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}.\end{aligned}\tag{4.52}$$

Applying Cauchy-Schwarz's inequality, (4.52) and estimate $(4.37)_1$, we get

$$\begin{aligned}
&\varepsilon^2\kappa_c\|Dw_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}}+\varepsilon^2\lambda_c\|\mathrm{div}(w_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)}+4\chi\|w_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^3}\\
&\leq 2\chi\|\mathrm{rot}(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^3}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}+\varepsilon^{\frac{3}{2}}\|g'\|_{L^2(\omega)^2}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\\
&\leq \left(2\chi\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}+\varepsilon^{\frac{3}{2}}\|g'\|_{L^2(\omega)^2}\right)\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3},
\end{aligned}\tag{4.53}$$

which, in particular, implies

$$2\chi\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\leq \chi\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}+\varepsilon^{\frac{3}{2}}\frac{1}{2}\|g'\|_{L^2(\omega^2)}.\tag{4.54}$$

Step 3. Let us obtain now the velocity estimates. By putting (4.54) into (4.49), we get

$$(\mu+\chi)\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\leq \varepsilon^{\frac{3}{2}}C\|f'\|_{L^2(\omega)^2}+\varepsilon^{\frac{3}{2}}\tfrac{1}{2}\|g'\|_{L^2(\omega)^2}+\chi\|Du_\varepsilon\|_{L^2(\Omega)^{3\times 3}},$$

i.e., we have

$$\mu\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\leq \varepsilon^{\frac{3}{2}}C\|f'\|_{L^2(\omega)^2}+\varepsilon^{\frac{3}{2}}\frac{1}{2}\|g'\|_{L^2(\omega)^2}.$$

This estimate implies $(4.38)_2$, and by Poincare's inequality (4.35), we get $(4.38)_1$. From the change of variables (2.12), we get (4.41).

Step 4. Let us obtain now the microrotation estimates. Using $\int_{\Omega_\varepsilon}\mathrm{rot}(u_\varepsilon)\cdot w_\varepsilon\,dx=\int_{\Omega_\varepsilon}\mathrm{rot}(w_\varepsilon)\cdot u_\varepsilon\,dx$ (see Exercise 1.3.3 in [24, Part II]) in (4.51), Cauchy-Schwarz's inequality, Poincaré's inequality, (4.52), and estimates (4.35), $(4.37)_1$ and $(4.38)_1$, we get

$$\begin{aligned}
&\kappa_c\varepsilon^2\|Dw_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}}+\lambda_c\varepsilon^2\|\mathrm{div}(w_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)}+4\chi\|w_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^3}\\
&\leq \varepsilon^{\frac{3}{2}}\|g'\|_{L^2(\omega)^2}\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}+2\chi\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\|\mathrm{rot}(w_\varepsilon)\|_{L^2(\Omega)^3}\\
&\leq C\left(\varepsilon^{\frac{5}{2}}\|g'\|_{L^2(\omega)^2}+2\chi\varepsilon^{\frac{5}{2}}\right)\|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}},
\end{aligned}\tag{4.55}$$

which, in particular, gives

$$\kappa_c\varepsilon^2\|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon)^{3\times 3}}\leq C\varepsilon^{\frac{5}{2}}.$$

This implies estimate $(4.39)_2$ and by the Poincaré inequality (4.35), we deduce $(4.39)_1$. From the change of variables (2.12), we get (4.42).

Step 5. We finish by deducing estimates for magnetic field (4.40). Putting estimates $(4.38)_2$ and $(4.39)_1$ into (4.50), we deduce

$$\nu_c\varepsilon^\gamma\|Db_\varepsilon\|^2_{L^2(\Omega_\varepsilon)^{3\times 3}}\leq C\varepsilon^3,$$

which implies estimate $(4.40)_2$, and from the Poincaré inequality, we get $(4.40)_1$. From the change of variables (2.12), we get (4.43) and so, the proof is finished.

□

4.2 A priori pressure estimates

Now, we will derive the estimate of the hydrostatic and magnetic pressures by using the *a priori* estimates for velocity, microrotation and magnetic field given in Lemma 4.3.

Lemma 4.4 (Estimates for hydrostatic pressure)**.** *Assume $\gamma\leq 2$. Then, there exists a positive constant C, independent of ε, such that we have the following estimates*

$$\|\widetilde{p}^\varepsilon\|_{L^2_0(\Omega)}\leq C,\quad \|\nabla_\varepsilon\tilde{p}_\varepsilon\|_{H^{-1}(\Omega)^3}\leq C.\tag{4.56}$$

Proof. Thank to the classical Nečkas estimate

$$\|\tilde{p}_\varepsilon\|_{L^2_0(\Omega)} \leq C\|\nabla_z \tilde{p}_\varepsilon\|_{H^{-1}(\Omega)^3}, \tag{4.57}$$

and taking into account that $\|\nabla_z \tilde{p}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C\|\nabla_\varepsilon \tilde{p}_\varepsilon\|_{H^{-1}(\Omega)^3}$, then we just need to obtain the estimate for $\nabla_\varepsilon \widetilde{p}_\varepsilon$ to derive the estimates for $\tilde{p}_\varepsilon$ given in (4.56) with the restriction imposed on the possible values of γ.

To do this, we consider $\widetilde{\varphi} \in H^1_0(\Omega)^3$, and taking into account the variational formulation (2.20), we get

$$\begin{aligned}\langle \nabla_\varepsilon \widetilde{p}_\varepsilon, \widetilde{\varphi}\rangle = \; & -(\mu+\chi)\int_\Omega D_\varepsilon \widetilde{u}_\varepsilon : D_\varepsilon \widetilde{\varphi}\, dz - \int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\varphi}\, dz - \frac{1}{2}\int_\Omega \nabla_\varepsilon(|\widetilde{b}_\varepsilon|^2)\widetilde{\varphi}\, dz \\ & + \int_\Omega (\widetilde{b}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\varphi}\, dz + 2\chi \int_\Omega \mathrm{rot}_\varepsilon(\widetilde{w}_\varepsilon)\cdot \widetilde{\varphi}\, dz + \int_\Omega f' \cdot \widetilde{\varphi}'\, dz,\end{aligned} \tag{4.58}$$

where brackets are for the duality products between H^{-1} and H^1_0. We estimate the terms on the right-hand side of (4.58) taking into account that $\|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}} \leq \varepsilon^{-1}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}$:

- First term. Applying Cauchy-Schwarz's inequality and estimate (4.41), we get

$$\begin{aligned}\left|(\mu+\chi)\int_\Omega D_\varepsilon \widetilde{u}_\varepsilon : D_\varepsilon \widetilde{\varphi}\, dz\right| &\leq C\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^{-1}\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3} \\ &\leq C\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}.\end{aligned}$$

- Second term. Applying Hölder's inequality and using estimate $(4.36)_2$ and (4.41), we have

$$\begin{aligned}\left|\int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\varphi}\, dz\right| &\leq \|\tilde{u}_\varepsilon\|_{L^4(\Omega)^3}\|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\tilde{\varphi}\|_{L^4(\Omega)^3} \\ &\leq C\varepsilon\|D_\varepsilon \tilde{u}_\varepsilon\|^2_{L^2(\Omega)^{3\times 3}}\|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^2\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}.\end{aligned}$$

- Third term. Applying Hölder's inequality and using estimate $(4.36)_2$ and (4.43), we have

$$\begin{aligned}\left|\frac{1}{2}\int_\Omega \nabla_\varepsilon(|\widetilde{b}_\varepsilon|^2)\widetilde{\varphi}\, dz\right| = \left|\int_\Omega \widetilde{b}_\varepsilon \nabla_\varepsilon \widetilde{b}_\varepsilon\, \widetilde{\varphi}\, dz\right| &\leq \|\tilde{b}_\varepsilon\|_{L^4(\Omega)^3}\|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\tilde{\varphi}\|_{L^4(\Omega)^3} \\ &\leq C\varepsilon\|D_\varepsilon \tilde{b}_\varepsilon\|^2_{L^2(\Omega)^{3\times 3}}\|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^{2-\gamma}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}.\end{aligned}$$

- Fourth term. Applying Hölder's inequality and using estimate $(4.36)_2$ and (4.43), we have

$$\begin{aligned}\left|\int_\Omega (\widetilde{b}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\varphi}\, dz\right| &\leq \|\tilde{b}_\varepsilon\|_{L^4(\Omega)^3}\|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\tilde{\varphi}\|_{L^4(\Omega)^3} \\ &\leq C\varepsilon\|D_\varepsilon \tilde{b}_\varepsilon\|^2_{L^2(\Omega)^{3\times 3}}\|D_\varepsilon \tilde{\varphi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^{2-\gamma}\|\tilde{\varphi}\|_{H^1_0(\Omega)^3}.\end{aligned}$$

- Fifth term. Applying Cauchy-Schwarz's inequality and estimates $(4.37)_2$ and (4.42), we get

$$\begin{aligned}\left|2\chi\int_\Omega \mathrm{rot}_\varepsilon(\widetilde{w}_\varepsilon)\cdot \widetilde{\varphi}\, dz\right| &\leq C\|\mathrm{rot}_\varepsilon(\widetilde{w}_\varepsilon)\|_{L^2(\Omega)^3}\|\widetilde{\varphi}\|_{L^2(\Omega)^3} \\ &\leq C\|D_\varepsilon \widetilde{w}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\widetilde{\varphi}\|_{H^1_0(\Omega)^3} \\ &\leq C\|\widetilde{\varphi}\|_{H^1_0(\Omega)^3}.\end{aligned}$$

- Sixth term. Applying Cauchy-Schwarz's inequality, we get

$$\left|\int_\Omega f'\,\widetilde{\varphi}'\,dz\right| \leq C\|\widetilde{\varphi}\|_{L^2(\Omega)^3} \leq C\|\widetilde{\varphi}\|_{H_0^1(\Omega)^3}.$$

Finally, taking into account all the previous estimates, we get

$$\|\nabla_\varepsilon \tilde{p}\|_{H^{-1}(\Omega)^3} \leq \begin{cases} C & \text{if } \gamma \leq 2, \\ C\varepsilon^{2-\gamma} & \text{if } \gamma > 2. \end{cases}$$

To prevent that the nonlinear terms will be those of the highest order, then we only consider that $\gamma \leq 2$, which finishes the proof.

□

Lemma 4.5 (Estimates for magnetic pressure)**.** *Assume $\gamma \leq 2$. Then, there exists a positive constant C, independent of ε, such that we have the following estimates*

$$\|\widetilde{q}^\varepsilon\|_{L^2_0(\Omega)} \leq C\varepsilon^{\frac{\gamma}{2}}, \quad \|\nabla_\varepsilon \widetilde{q}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C\varepsilon^{\frac{\gamma}{2}}. \tag{4.59}$$

Proof. Analogously to the previous Lemma, we just need to obtain the estimate for $\nabla_\varepsilon \widetilde{q}_\varepsilon$ to derive the estimates for $\widetilde{q}_\varepsilon$ given in (4.59) and the restriction on the value of γ.

To do this, we consider $\widetilde{\psi} \in H_0^1(\Omega)^3$, and taking into account the variational formulation (2.22), we get

$$\left\langle \nabla_\varepsilon \widetilde{q}^\varepsilon, \widetilde{\psi} \right\rangle = -\nu_c \varepsilon^\gamma \int_\Omega D_\varepsilon \widetilde{b}_\varepsilon : D_\varepsilon \widetilde{\psi}\,dz - \int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\psi}\,dz + \int_\Omega (\widetilde{b}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\psi}\,dz. \tag{4.60}$$

We estimate the terms on the right-hand side of (4.60) taking into account that it holds the estimate $\|D_\varepsilon \tilde{\psi}\|_{L^2(\Omega)^{3\times 3}} \leq \varepsilon^{-1}\|\tilde{\psi}\|_{H_0^1(\Omega)^3}$:

- First term. Applying the Cauchy-Schwarz inequality and estimate (4.43), we get

$$\begin{aligned} \left|\nu_c \varepsilon^\gamma \int_\Omega D_\varepsilon \widetilde{b}_\varepsilon : D_\varepsilon \widetilde{\psi}\,dz\right| &\leq C\varepsilon^\gamma \|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|D_\varepsilon \tilde{\psi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^{\gamma-1} \|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|\tilde{\psi}\|_{H_0^1(\Omega)^3} \\ &\leq C\varepsilon^{\frac{\gamma}{2}} \|\tilde{\psi}\|_{H_0^1(\Omega)^3}. \end{aligned}$$

- Second term. Applying Hölder's inequality and using estimates $(4.36)_2$ and (4.41) and (4.43), we have

$$\begin{aligned} \left|\int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\psi}\,dz\right| &\leq \|\tilde{u}_\varepsilon\|_{L^4(\Omega)^3} \|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|\tilde{\psi}\|_{L^4(\Omega)^3} \\ &\leq C\varepsilon \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|D_\varepsilon \tilde{\psi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^{2-\frac{\gamma}{2}} \|\tilde{\psi}\|_{H_0^1(\Omega)^3}. \end{aligned}$$

- Third term. Applying Hölder's inequality and using estimates $(4.36)_2$ and (4.41) and (4.43), we have

$$\begin{aligned} \left|\int_\Omega (\widetilde{b}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\psi}\,dz\right| &\leq \|\tilde{b}_\varepsilon\|_{L^4(\Omega)^3} \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|\tilde{\psi}\|_{L^4(\Omega)^3} \\ &\leq C\varepsilon \|D_\varepsilon \tilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \|D_\varepsilon \tilde{\psi}\|_{L^2(\Omega)^{3\times 3}} \\ &\leq C\varepsilon^{2-\frac{\gamma}{2}} \|\tilde{\psi}\|_{H_0^1(\Omega)^3}. \end{aligned}$$

Finally, taking into account all the previous estimates, then we get for any value of γ that

$$\|\nabla_\varepsilon \tilde{q}\|_{H^{-1}(\Omega)^2} \leq \begin{cases} C\varepsilon^{\frac{\gamma}{2}} & \text{if } \gamma \leq 2, \\ C\varepsilon^{2-\frac{\gamma}{2}} & \text{if } \gamma > 2. \end{cases}$$

Similarly to what happens with the estimate of the hydrostatic pressure, to prevent that the non linear terms will be those of the highest order, then we only consider that $\gamma \leq 2$, which finishes the proof.

□

4.3 Compactness results and proof of Theorem 3.1

First, we give some compactness results about the behavior of the rescaled sequences $\widetilde{u}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{b}_\varepsilon$ and $\widetilde{p}_\varepsilon$, $\widetilde{q}_\varepsilon$ satisfying respectively the *a priori* estimates given in Lemmas 4.3, 4.4 and 4.5.

Lemma 4.6. *Assume $\gamma \leq 2$. Then, there exist:*

- *A subsequence, still denoted by $(\widetilde{u}_\varepsilon, \widetilde{w}_\varepsilon, \widetilde{b}_\varepsilon, \widetilde{p}_\varepsilon, \widetilde{q}_\varepsilon)$, chosen from a sequence of solutions of system (2.18)-(2.19).*
- *$\widetilde{u} = (\widetilde{u}', 0)$ with $\widetilde{u}' \in H^1(0, h(z'); L^2(\omega)^2)$ and $\widetilde{u} = 0$ on $\Gamma_0 \cup \Gamma_1$ such that*

$$\varepsilon^{-2}\widetilde{u}_\varepsilon \rightharpoonup \widetilde{u} \quad in \quad H^1(0, h(z'); L^2(\omega)^3), \tag{4.61}$$

$$\mathrm{div}_{z'} \left(\int_0^{h(z')} \widetilde{u}'(z)\, dz_3 \right) = 0 \quad in \quad \omega, \quad \left(\int_0^{h(z')} \widetilde{u}'(z)\, dz_3 \right) \cdot \mathrm{n} = 0 \quad on \quad \partial\omega, \tag{4.62}$$

- *$\widetilde{w} = (\widetilde{w}', 0)$ with $\widetilde{w}' \in H^1(0, h(z'); L^2(\omega)^2)$ and $\widetilde{w} = 0$ on $\Gamma_0 \cup \Gamma_1$ such that*

$$\varepsilon^{-1}\widetilde{w}_\varepsilon \rightharpoonup \widetilde{w} \quad in \quad H^1(0, h(z'); L^2(\omega)^3), \tag{4.63}$$

- *$\widetilde{b} = (\widetilde{b}', 0)$ with $\widetilde{b}' \in H^1(0, h(z'); L^2(\omega)^2)$ and $\widetilde{b} = 0$ on $\Gamma_0 \cup \Gamma_1$ such that*

$$\varepsilon^{\frac{\gamma}{2}-2}\widetilde{b}_\varepsilon \rightharpoonup \widetilde{b} \quad in \quad H^1(0, h(z'); L^2(\omega)^3), \tag{4.64}$$

$$\mathrm{div}_{z'} \left(\int_0^{h(z')} \widetilde{b}'(z)\, dz_3 \right) = 0 \quad in \quad \omega, \quad \left(\int_0^{h(z')} \widetilde{b}'(z)\, dz_3 \right) \cdot \mathrm{n} = 0 \quad on \quad \partial\omega, \tag{4.65}$$

- *$\tilde{p}, \tilde{q} \in L^2(\Omega)$, independent of z_3, such that*

$$\widetilde{p}_\varepsilon \rightharpoonup \widetilde{p} \quad in \quad L^2(\Omega), \tag{4.66}$$

$$\varepsilon^{-\frac{\gamma}{2}}\widetilde{q}_\varepsilon \rightharpoonup \widetilde{q} \quad in \quad L^2(\Omega), \tag{4.67}$$

with

$$\int_\Omega \widetilde{p}(z')\, dz = \int_\omega h(z')\widetilde{p}(z')\, dz' = 0, \quad \int_\Omega \widetilde{q}(z')\, dz = \int_\omega h(z')\widetilde{q}(z')\, dz' = 0. \tag{4.68}$$

Proof. The convergences given in the lemma are a direct consequence of the estimates given in Lemmas 4.3, 4.4 and 4.5. We will only give some remarks, and for more details we refer the reader to [8, 29, 42]:

– *Velocity.* Estimates (4.41) imply the existence of a function $\widetilde{u} \in H^1(0, h(z'); L^2(\omega)^3)$ such that convergence (4.61) holds. The continuity of the trace applications from the space of $\widetilde{u}$ such that $\|\widetilde{u}\|_{L^2}$ and $\|\partial_{z_3}\widetilde{u}\|_{L^2}$ are bounded to $L^2(\Gamma_0)$ and to $L^2(\Gamma_1)$ implies that $\widetilde{u} = 0$ on $\Gamma_0 \cup \Gamma_1$.

From the condition $\mathrm{div}_\varepsilon \widetilde{u}_\varepsilon = 0$ in Ω, it can be deduced that $\widetilde{u}_3$ is independent of z_3, which combined with the boundary conditions satisfied by $\widetilde{u}_3$ on $\Gamma_0 \cup \Gamma_1$ implies $\widetilde{u}_3 = 0$.

Finally, from the free divergence condition and the convergence (4.61) of $\varepsilon^{-2}\widetilde{u}_\varepsilon$, it is straightforward the corresponding free divergence condition in a thin domain given in (4.62).

– *Microrotation.* From estimates (4.42), then convergence (4.63) and $\widetilde{w} = 0$ on $\Gamma_0 \cup \Gamma_1$ are obtained directly. It remains to prove $\widetilde{w}_3 = 0$. To do this, taking into account that from estimates (4.41) and (4.42) it holds

$$\left|\varepsilon^{-1}\int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{w}_\varepsilon \cdot \widetilde{\phi}\, dz\right| \leq \varepsilon^{-1}\|\widetilde{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)^3}\|D_\varepsilon \widetilde{w}_\varepsilon\|_{L^2(\Omega)^3}\|\widetilde{\phi}\|_{L^\infty(\Omega)^3} \leq C\varepsilon \to 0,$$

then we consider a test function $\widetilde{\phi} = (0, 0, \varepsilon^{-1}\widetilde{\phi}_3)$ in the variational formulation (2.21), which gives

$$\begin{aligned}\kappa_c\varepsilon\int_\Omega \nabla_\varepsilon \widetilde{w}_{3,\varepsilon}\cdot\nabla_\varepsilon\widetilde{\phi}_3\, dz + \lambda_c\int_\Omega \varepsilon^{-1}\partial_{z_3}\widetilde{w}_{3,\varepsilon}\,\partial_{z_3}\widetilde{\phi}_3\, dz + 4\chi\int_\Omega \varepsilon^{-1}\widetilde{w}_{3,\varepsilon}\,\widetilde{\phi}_3\, dz \\ = 2\chi\int_\Omega \varepsilon^{-1}\mathrm{Rot}_{z'}(\widetilde{u}'_\varepsilon)\,\widetilde{\phi}_3\, dz + O_\varepsilon,\end{aligned} \tag{4.69}$$

with O_ε devoted to tend to zero when $\varepsilon \to 0$. Then, passing to the limit in (4.69) when $\varepsilon \to 0$ by using convergences (4.61) and (4.63), we get

$$(\kappa_c + \lambda_c)\int_\Omega \partial_{z_3}\widetilde{w}_3\,\partial_{z_3}\widetilde{\phi}_3\, dz + 4\chi\int_\Omega \widetilde{w}_3\,\widetilde{\phi}_3\, dz = 0,$$

which is equivalent to problem $-(\kappa_c + \lambda_c)\partial^2_{z_3}\widetilde{w}_3 + 4\chi\widetilde{w}_3 = 0$. This equation together with the boundary conditions $\widetilde{w}_3 = 0$ on $\Gamma_0 \cup \Gamma_1$ implies $\widetilde{w}_3 = 0$.

– *Magnetic field.* This follows similarly to the case of velocity just taking into account estimates (4.43).

– *Pressures.* Estimates $(4.56)_1$ and $(4.59)_1$ imply, up to a subsequence, the existence of $\widetilde{p}, \widetilde{q} \in L^2(\Omega)$ such that convergences (4.66) and (4.67) hold, respectively. Also, from $(4.56)_2$ and $(4.59)_2$, by noting that $\varepsilon^{-1}\partial_{z_3}\widetilde{p}_\varepsilon$ and $\varepsilon^{-1-\frac{\gamma}{2}}\partial_{z_3}\widetilde{q}_\varepsilon$ also converge weakly in $H^{-1}(\Omega)$, we deduce $\partial_{z_3}\widetilde{p} = \partial_{z_3}\widetilde{q} = 0$. Then, $\widetilde{p}$ and $\widetilde{q}$ are independent of z_3. To finish, it remains to prove that $p, q \in L^2_0(\Omega)$, that is properties (4.68). Passing to the limit when ε tends to zero in

$$\int_\Omega \widetilde{p}_\varepsilon\, dz = 0, \quad \text{and} \quad \int_\Omega \varepsilon^{-\frac{\gamma}{2}}\widetilde{q}_\varepsilon\, dz = 0,$$

we respectively deduce

$$\int_\Omega \widetilde{p}(z')\, dz = \int_\omega h(z')\widetilde{p}(z')\, dz' = 0, \quad \text{and} \quad \int_\Omega \widetilde{q}(z')\, dz = \int_\omega h(z')\widetilde{q}(z')\, dz' = 0,$$

and so that $\widetilde{p}$ and $\widetilde{q}$ has null mean value in Ω. This ends the proof.

□

Proof of Theorem 3.1. From Lemma 4.6, it remains to prove the equations $(3.24)_1$, $(3.24)_2$ and $(3.24)_3$. To do this, we divide the proof in four steps.

Step 1. We pass to the limit in the variational formulation (2.20) for every $\widetilde{\varphi} \in \mathcal{D}(\Omega)^3$ with $\widetilde{\varphi}_3 = 0$, which is given by

$$\begin{aligned}&(\mu+\chi)\int_\Omega D_\varepsilon\widetilde{u}_\varepsilon : D_\varepsilon\widetilde{\varphi}\,dz + \int_\Omega (\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon\cdot\widetilde{\varphi}\,dz - \int_\Omega \widetilde{p}_\varepsilon \operatorname{div}_\varepsilon(\widetilde{\varphi})\,dz + \frac{1}{2}\int_\Omega \nabla_\varepsilon(|\widetilde{b}_\varepsilon|^2)\widetilde{\varphi}\,dz\\ &= 2\chi\int_\Omega \operatorname{rot}_\varepsilon(\widetilde{w}_\varepsilon)\cdot\widetilde{\varphi}\,dz + \int_\Omega (\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon\cdot\widetilde{\varphi}\,dz + \int_\Omega f'\cdot\widetilde{\varphi}'\,dz.\end{aligned} \tag{4.70}$$

To do this, let us pass to the limit when ε tends to zero in every terms:

- First term on the left-hand side of (4.70). Taking into account convergence (4.61), we get

$$(\mu+\chi)\int_\Omega D_\varepsilon\widetilde{u}_\varepsilon : D_\varepsilon\widetilde{\varphi}\,dz = (\mu+\chi)\int_\Omega \varepsilon^{-2}\partial_{z_3}\widetilde{u}'_\varepsilon\cdot\partial_{z_3}\widetilde{\varphi}'\,dz + O_\varepsilon = (\mu+\chi)\int_\Omega \partial_{z_3}\widetilde{u}'\cdot\partial_{z_3}\widetilde{\varphi}'\,dz + O_\varepsilon.$$

- Second term on the left-hand side of (4.70). Taking into account that $\widetilde{\varphi}' \in \mathcal{D}(\Omega)^2$, applying Cauchy-Schwarz's inequality and estimates (4.41), we get that

$$\left|\int_\Omega (\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon\cdot\widetilde{\varphi}\,dz\right| \le \|\widetilde{u}_\varepsilon\|_{L^2(\Omega)^3}\|D_\varepsilon\widetilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\widetilde{\varphi}'\|_{L^\infty(\Omega)^2} \le C\varepsilon^3,$$

  which implies

$$\int_\Omega (\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{u}_\varepsilon\cdot\widetilde{\varphi}\,dz \to 0.$$

- Third term on the left-hand side of (4.70). Taking into account that $\widetilde{\varphi}_3 = 0$ and convergence (4.66), we get

$$\int_\Omega \widetilde{p}_\varepsilon \operatorname{div}_\varepsilon(\widetilde{\varphi})\,dz = \int_\Omega \widetilde{p}_\varepsilon \operatorname{div}_{z'}(\widetilde{\varphi}')\,dz \to \int_\Omega \widetilde{p}\operatorname{div}_{z'}(\widetilde{\varphi}')\,dz.$$

- Fourth term on the left-hand side of (4.70). Applying the Cauchy-Schwarz inequality and estimates (4.43), we get

$$\left|\frac{1}{2}\int_\Omega \nabla_\varepsilon(|\widetilde{b}_\varepsilon|^2)\widetilde{\varphi}\,dz\right| = \left|\int_\Omega \widetilde{b}_\varepsilon\nabla_\varepsilon\widetilde{b}_\varepsilon\,\widetilde{\varphi}\,dz\right| \le \|\widetilde{\varphi}'\|_{L^\infty(\Omega)^2}\|\widetilde{b}_\varepsilon\|_{L^2(\Omega)^3}\|D_\varepsilon\widetilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \le C\varepsilon^{3-\gamma},$$

  and since $\gamma \le 2$, then we get

$$\frac{1}{2}\int_\Omega \nabla(|\widetilde{b}_\varepsilon|^2)\widetilde{\varphi}\,dz \to 0.$$

- First term on the right-hand side of (4.70). Taking into account the definition of $\operatorname{rot}_\varepsilon(\widetilde{w}_\varepsilon)$ and convergence (4.63), we get

$$2\chi\int_\Omega \operatorname{rot}_\varepsilon(\widetilde{w}_\varepsilon)\cdot\widetilde{\varphi}\,dz = 2\chi\int_\Omega \varepsilon^{-1}\operatorname{rot}_{z_3}(\widetilde{w}'_\varepsilon)\cdot\widetilde{\varphi}'\,dz + O_\varepsilon = 2\chi\int_\Omega \operatorname{rot}_{z_3}(\widetilde{w}')\cdot\widetilde{\varphi}'\,dz + O_\varepsilon.$$

- Second term on the right-hand side of (4.70). Applying the Cauchy-Schwarz inequality and estimates (4.43), we get

$$\left|\int_\Omega (\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon\cdot\widetilde{\varphi}\,dz\right| \le \|\widetilde{b}_\varepsilon\|_{L^2(\Omega)^3}\|D_\varepsilon\widetilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}}\|\widetilde{\varphi}'\|_{L^\infty(\Omega)^2} \le C\varepsilon^{3-\gamma},$$

  and since $\gamma \le 2$, then we get

$$\int_\Omega (\widetilde{b}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{b}_\varepsilon\cdot\widetilde{\varphi}\,dz \to 0.$$

Finally, from the above convergences when $\varepsilon \to 0$, we derive the following limit system

$$(\mu+\chi)\int_\Omega \partial_{z_3}\widetilde{u}'\cdot\partial_{z_3}\widetilde{\varphi}'\,dz - \int_\Omega \widetilde{p}\,\mathrm{div}_{z'}(\widetilde{\varphi}')\,dz = 2\chi\int_\Omega \mathrm{rot}_{z_3}(\widetilde{w}')\cdot\widetilde{\varphi}'\,dz + \int_\Omega f'\cdot\widetilde{\varphi}'\,dz,$$

for every $\widetilde{\varphi}' \in \mathcal{D}(\Omega)^2$.

Step 2. We pass to the limit in the variational formulation (2.21) for every $\widetilde{\phi} \in \mathcal{D}(\Omega)^3$ with $\widetilde{\phi}_3 = 0$, which after dividing by ε, is given by

$$\begin{aligned}\kappa_c\varepsilon\int_\Omega D_\varepsilon\widetilde{w}_\varepsilon : D_\varepsilon\widetilde{\phi}\,dz + \lambda_c\varepsilon\int_\Omega \mathrm{div}_\varepsilon(\widetilde{w}_\varepsilon)\,\mathrm{div}_\varepsilon(\widetilde{\phi})\,dz + 4\chi\int_\Omega \varepsilon^{-1}\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz + \int_\Omega \varepsilon^{-1}(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz \\ = 2\chi\int_\Omega \varepsilon^{-1}\mathrm{rot}_\varepsilon(\widetilde{u}_\varepsilon)\cdot\widetilde{\phi}\,dz + \int_\Omega g'\cdot\widetilde{\phi}'\,dz.\end{aligned} \tag{4.71}$$

To do this, let us pass to the limit when ε tends to zero in every terms:

- First term on the left-hand side of (4.71). Taking into account convergence (4.63), we get

$$\kappa_c\varepsilon\int_\Omega D_\varepsilon\widetilde{w}_\varepsilon : D_\varepsilon\widetilde{\phi}\,dz = \kappa_c\int_\Omega \varepsilon^{-1}\partial_{z_3}\widetilde{w}'_\varepsilon\cdot\partial_{z_3}\widetilde{\phi}'\,dz + O_\varepsilon = \kappa_c\int_\Omega \partial_{z_3}\widetilde{w}'\cdot\partial_{z_3}\widetilde{\phi}'\,dz + O_\varepsilon.$$

- Second term on the left-hand side of (4.71). Taking into account that $\widetilde{\phi}_3 = 0$ and convergence (4.63), we get

$$\lambda_c\varepsilon\int_\Omega \mathrm{div}_\varepsilon(\widetilde{w}_\varepsilon)\,\mathrm{div}_\varepsilon(\widetilde{\phi})\,dz = \lambda_c\varepsilon\int_\Omega \mathrm{div}_\varepsilon(\widetilde{w}_\varepsilon)\,\mathrm{div}_{z'}(\widetilde{\phi}')\,dz = \lambda_c\int_\Omega \partial_{z_3}\widetilde{w}_{3,\varepsilon}\,\mathrm{div}_{z'}(\widetilde{\phi}')\,dz + O_\varepsilon \to 0.$$

- Third term on the left-hand side of (4.71). Taking into account convergence (4.63), we get

$$4\chi\int_\Omega \varepsilon^{-1}\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz = 4\chi\int_\Omega \varepsilon^{-1}\widetilde{w}'_\varepsilon\cdot\widetilde{\phi}'\,dz = 4\chi\int_\Omega \widetilde{w}\cdot\widetilde{\phi}\,dz + O_\varepsilon.$$

- Fourth term on the left-hand side of (4.71). By using Cauchy-Schwarz's inequality and estimates (4.41) and (4.42), we get

$$\left|\int_\Omega \varepsilon^{-1}(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz\right| \le \varepsilon^{-1}\|\widetilde{\phi}'\|_{L^\infty(\Omega)^2}\|\widetilde{u}_\varepsilon\|_{L^2(\Omega)^3}\|D_\varepsilon\widetilde{w}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \le C\varepsilon,$$

  which implies

$$\int_\Omega \varepsilon^{-1}(\widetilde{u}_\varepsilon\cdot\nabla_\varepsilon)\widetilde{w}_\varepsilon\cdot\widetilde{\phi}\,dz \to 0.$$

- First term on the right-hand side of (4.71). By using the definition of $\mathrm{rot}_\varepsilon(\widetilde{u}_\varepsilon)$ and convergence (4.61), we get

$$2\chi\int_\Omega \varepsilon^{-1}\mathrm{rot}_\varepsilon(\widetilde{u}_\varepsilon)\cdot\widetilde{\phi}\,dz = 2\chi\int_\Omega \varepsilon^{-2}\mathrm{rot}_{z_3}(\widetilde{u}'_\varepsilon)\cdot\widetilde{\phi}'\,dz + O_\varepsilon = 2\chi\int_\Omega \mathrm{rot}_{z_3}(\widetilde{u}')\cdot\widetilde{\phi}'\,dz + O_\varepsilon.$$

Finally, from the above convergences, we get the limit variational formulation

$$\kappa_c\int_\Omega \partial_{z_3}\widetilde{w}'\cdot\partial_{z_3}\widetilde{\phi}'\,dz + 4\chi\int_\Omega \widetilde{w}'\cdot\widetilde{\phi}'\,dz = 2\chi\int_\Omega \mathrm{rot}_{z_3}(\widetilde{u}')\cdot\widetilde{\phi}'\,dz + \int_\Omega g'\cdot\widetilde{\phi}'\,dz,$$

for every $\widetilde{\phi}' \in \mathcal{D}(\Omega)^2$.

Step 3. We pass to the limit in the variational formulation (2.22) for every $\widetilde{\psi} \in \mathcal{D}(\Omega)^3$ with $\widetilde{\psi}_3 = 0$, which after dividing by $\varepsilon^{\frac{\gamma}{2}}$ is given by

$$\nu_c \varepsilon^{\frac{\gamma}{2}} \int_\Omega D_\varepsilon \widetilde{b}_\varepsilon : D_\varepsilon \widetilde{\psi}\, dz + \varepsilon^{-\frac{\gamma}{2}} \int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\psi}\, dz - \varepsilon^{-\frac{\gamma}{2}} \int_\Omega \widetilde{q}_\varepsilon \operatorname{div}_\varepsilon(\widetilde{\psi})\, dz = \varepsilon^{-\frac{\gamma}{2}} \int_\Omega (\widetilde{b}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\psi}\, dz. \tag{4.72}$$

To do this, let us pass to the limit when ε tends to zero in every terms:

- First term on the left-hand side of (4.72). Taking into account convergence (4.64), we get

$$\nu_c \varepsilon^{\frac{\gamma}{2}} \int_\Omega D_\varepsilon \widetilde{b}_\varepsilon : D_\varepsilon \widetilde{\psi}\, dz = \nu_c \int_\Omega \varepsilon^{\frac{\gamma}{2}-2} \partial_{z_3} \widetilde{b}'_\varepsilon \cdot \partial_{z_3} \widetilde{\psi}'\, dz + O_\varepsilon = \nu_c \int_\Omega \partial_{z_3} \widetilde{b}' \cdot \partial_{z_3} \widetilde{\psi}'\, dz + O_\varepsilon.$$

- Second term on the left-hand side of (4.72). Applying the Cauchy-Schwarz inequality and taking into account estimates (4.41) and (4.43), we get

$$\varepsilon^{-\frac{\gamma}{2}} \int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\psi}\, dz \le \varepsilon^{-\frac{\gamma}{2}} \|\widetilde{\psi}'\|_{L^\infty(\Omega)^2} \|\widetilde{u}_\varepsilon\|_{L^2(\Omega)^3} \|D_\varepsilon \widetilde{b}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \le C\varepsilon^{3-\gamma},$$

  and since $\gamma \le 2$, then we get

$$\varepsilon^{-\frac{\gamma}{2}} \int_\Omega (\widetilde{u}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{b}_\varepsilon \cdot \widetilde{\psi}\, dz \to 0.$$

- Third term on the left-hand side of (4.72). Taking into account that $\widetilde{\psi}_3 = 0$ and convergence (4.67), we get

$$\varepsilon^{-\frac{\gamma}{2}} \int_\Omega \widetilde{q}_\varepsilon \operatorname{div}_\varepsilon(\widetilde{\psi})\, dz = \int_\Omega \varepsilon^{-\frac{\gamma}{2}} \widetilde{q}_\varepsilon \operatorname{div}_{z'}(\widetilde{\psi}')\, dz + O_\varepsilon = \int_\Omega \widetilde{q} \operatorname{div}_{z'}(\widetilde{\psi}')\, dz + O_\varepsilon.$$

- First term on the right-hand side of (4.72). Applying the Cauchy-Schwarz inequality and taking into account estimates (4.41) and (4.43), we get

$$\left| \varepsilon^{-\frac{\gamma}{2}} \int_\Omega (\widetilde{b}_\varepsilon \cdot \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\psi}\, dz \right| \le \varepsilon^{-\frac{\gamma}{2}} \|\widetilde{\psi}\|_{L^\infty(\Omega)^2} \|\widetilde{b}_\varepsilon\|_{L^2(\Omega)^3} \|D_\varepsilon \widetilde{u}_\varepsilon\|_{L^2(\Omega)^{3\times 3}} \le C\varepsilon^{3-\gamma},$$

  and since $\gamma \le 2$, then we get

$$\varepsilon^{-\frac{\gamma}{2}} \int_\Omega (\widetilde{b}_\varepsilon\, \nabla_\varepsilon)\widetilde{u}_\varepsilon \cdot \widetilde{\psi}\, dz \to 0.$$

Finally, from the above convergences, we get the limit variational formulation

$$\nu_c \int_\Omega \partial_{z_3} \widetilde{b}' \cdot \partial_{z_3} \widetilde{\psi}'\, dz - \int_\Omega \widetilde{q} \operatorname{div}_{z'}(\widetilde{\psi}')\, dz = 0,$$

for every $\psi' \in \mathcal{D}(\Omega)^2$.

Step 4. From the previous steps and by density, we deduce that the limit variational formulation are the following

$$(\mu + \chi) \int_\Omega \partial_{z_3} \widetilde{u}' \cdot \partial_{z_3} \widetilde{\varphi}'\, dz - \int_\Omega \widetilde{p} \operatorname{div}_{z'}(\widetilde{\varphi}')\, dz = 2\chi \int_\Omega \operatorname{rot}_{z_3}(\widetilde{w}') \cdot \widetilde{\varphi}'\, dz + \int_\Omega f' \cdot \widetilde{\varphi}'\, dz,$$

$$\kappa_c \int_\Omega \partial_{z_3} \widetilde{w}' \cdot \partial_{z_3} \widetilde{\phi}'\, dz + 4\chi \int_\Omega \widetilde{w}' \cdot \widetilde{\phi}'\, dz = 2\chi \int_\Omega \operatorname{rot}_{z_3}(\widetilde{u}') \cdot \widetilde{\phi}'\, dz + \int_\Omega g' \cdot \widetilde{\phi}'\, dz,$$

$$\nu_c \int_\Omega \partial_{z_3} \widetilde{b}' \cdot \partial_{z_3} \widetilde{\psi}'\, dz - \int_\Omega \widetilde{q} \operatorname{div}_{z'}(\widetilde{\psi}')\, dz = 0,$$

for every $\widetilde{\varphi}', \widetilde{\phi}', \widetilde{\psi}' \in H_0^1(\Omega)^2$, which are respectively equivalent to the equations $(3.24)_1$, $(3.24)_2$ and $(3.24)_3$. From the uniqueness of solution of (3.24) (see Lemma 3.3 and Theorem 3.4), it follows that the whole sequences $\widetilde{u}_\varepsilon$, $\widetilde{w}_\varepsilon$, $\widetilde{b}_\varepsilon$, $\widetilde{p}_\varepsilon$ and $\widetilde{q}_\varepsilon$ converge weakly. This ends the proof.

□

4.4 Proofs of Lemma 3.3 and Theorem 3.4

Proof of Lemma 3.3. We divide the proof in three steps.

Step 1. We derive the expressions of $\widetilde{u}_1$ and $\widetilde{w}_2$, as solution of system (3.27). First, from the first equation in (3.27), we have

$$\partial_{z_3}\widetilde{u}_1(z) = -\frac{1}{\mu+\chi}\left(f_1(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3 + \frac{2\chi}{\mu+\chi}\widetilde{w}_2(z) + K_1(z'), \tag{4.73}$$

with K_1 an unknown function. Putting this into the second equation in (3.27), we have

$$\partial^2_{z_2}\widetilde{w}_2(z) - \frac{4\chi\mu}{\kappa_c(\mu+\chi)}\widetilde{w}_2(z) = \frac{2\chi}{\kappa_c(\mu+\chi)}\left(f(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3 - \frac{2\chi}{\kappa_c}K_1(z') - \frac{1}{\kappa_c}g_2(z'), \tag{4.74}$$

whose solution is

$$\widetilde{w}_2(z) = C_1(z')e^{kz_3} + C_2(z')e^{-kz_3} - \frac{1}{2\mu}\left(f(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3 + \frac{\mu+\chi}{2\mu}K_1(z') + \frac{\mu+\chi}{4\chi\mu}g_2(z'), \tag{4.75}$$

with $k = \sqrt{\frac{4\chi\mu}{\kappa_c(\mu+\chi)}}$ and C_1 and C_2 are unknowns functions. Using this expression in (4.73), we rewrite $\partial_{z_3}\widetilde{u}_1$ as follows

$$\partial_{z_3}\widetilde{u}_1(z) = \frac{2\chi}{\mu+\chi}\left(C_1(z')e^{kz_3} + C_2(z')e^{-kz_3}\right) - \frac{1}{\mu}\left(f(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3 + \frac{\chi+\mu}{\mu}K_1(z') + \frac{1}{2\mu}g_2(z'), \tag{4.76}$$

and integrating with respect to z_3, we obtain

$$\begin{aligned}\widetilde{u}_1(z) &= \frac{2\chi}{k(\mu+\chi)}\left(C_1(z')e^{kz_3} - C_2(z')e^{-kz_3}\right)\\ &\quad -\frac{1}{2\mu}\left(f(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3^2 + \frac{\chi+\mu}{\mu}K_1(z')z_3 + \frac{1}{2\mu}g_2(z')z_3 + K_2(z'),\end{aligned} \tag{4.77}$$

with K_2 an unknown function.

Now, denoting $A(z') = C_1(z') + C_2(z')$ and $B(z') = C_1(z') - C_2(z')$, we rewrite the expressions of $\widetilde{u}_1$ and $\widetilde{w}_2$ as follows

$$\begin{aligned}\widetilde{u}_1(z) &= \frac{2\chi}{k(\mu+\chi)}\left(A(z')\sinh(kz_3) + B\cosh(kz_3)\right)\\ &\quad -\frac{1}{2\mu}\left(f(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3^2 + \frac{\chi+\mu}{\mu}K_1(z')z_3 + \frac{1}{2\mu}g_2(z')z_3 + K_2(z'),\\ \widetilde{w}_2(z) &= A(z')\cosh(kz_3) + B(z')\sinh(kz_3)\\ &\quad -\frac{1}{2\mu}\left(f(z') - \partial_{z_1}\widetilde{p}(z')\right) z_3 + \frac{\mu+\chi}{2\mu}K_1(z') + \frac{\mu+\chi}{4\chi\mu}g_2(z'),\end{aligned} \tag{4.78}$$

where A, B and K_1, K_2 are unknown functions to be determined using the boundary conditions.

From the boundary condition $\widetilde{u}_1(z', 0) = 0$ and $\widetilde{w}_2(z', 0) = 0$, we get

$$K_1(z') = -\frac{2\mu}{\mu+\chi}A(z') - \frac{1}{2\chi}g_2(z'), \quad K_2(z') = -\frac{2\chi}{k(\mu+\chi)}B(z'). \tag{4.79}$$

Then, (4.78) are rewritten as follows

$$\begin{aligned}\widetilde{u}_1(z) &= \left(\frac{2\chi}{k(\mu+\chi)}\sinh(kz_3)-2z_3\right)A(z')+\frac{2\chi}{k(\mu+\chi)}(\cosh(kz_3)-1)B(z')\\ &\quad -\frac{1}{2\mu}\left(f(z')-\partial_{z_1}\widetilde{p}(z')\right)z_3^2-\frac{1}{2\chi}g_2(z')z_3,\\ \widetilde{w}_2(z) &= (\cosh(kz_3)-1)\,A(z')+\sinh(kz_3)B(z')-\frac{1}{2\mu}\left(f(z')-\partial_{z_1}\widetilde{p}(z')\right)z_3.\end{aligned} \tag{4.80}$$

Using the boundary conditions $\widetilde{u}_1(z',h(z'))=0$ and $\widetilde{w}_2(z',h(z'))=0$ in (4.80), we get the following system

$$Q\begin{pmatrix}A\\B\end{pmatrix}=\frac{1}{2\mu}h(z')(f(z')-\partial_{z_1}\widetilde{p}(z'))\begin{pmatrix}h(z')\\1\end{pmatrix}+\frac{1}{2\chi}h(z')g_2(z')\begin{pmatrix}1\\0\end{pmatrix}, \tag{4.81}$$

where Q is the matrix defined by

$$Q=\begin{pmatrix}\frac{2\chi}{k(\mu+\chi)}\sinh(kh(z'))-2h(z') & \frac{2\chi}{k(\mu+\chi)}(\cosh(kh(z'))-1)\\ \cosh(kh(z'))-1 & \sinh(kh(z'))\end{pmatrix}.$$

The solution of this system is given by

$$\begin{aligned}A(z')&=\frac{1}{2\mu}h(z')(f(z')-\partial_{z_1}\widetilde{p}(z'))A_1(z')+\frac{1}{2\chi}h(z')g_2(z')A_2(z'),\\ B(z')&=\frac{1}{2\mu}h(z')(f(z')-\partial_{z_1}\widetilde{p}(z'))B_1(z')+\frac{1}{2\chi}h(z')g_2(z')B_2(z'),\end{aligned} \tag{4.82}$$

where A_1, B_1 and A_2, B_2 are respectively solutions of

$$Q\begin{pmatrix}A_1\\B_1\end{pmatrix}=\begin{pmatrix}h(z')\\1\end{pmatrix}\quad\text{and}\quad Q\begin{pmatrix}A_2\\B_2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}.$$

To compute A_i, B_i, $i=1,2$, we take into account that

$$\begin{aligned}|Q| &= -2h(z')\sinh(kh(z'))-\frac{4\chi}{k(\mu+\chi)}(1-\cosh(kh(z')))\\ &= -2\sinh(kh(z'))\left(h(z')-\frac{2\chi}{k(\mu+\chi)}\frac{\cosh(kh(z'))-1}{\sin(kh(z'))}\right),\end{aligned}$$

that Q^{-1} is given by

$$Q^{-1}=\frac{1}{|Q|}\begin{pmatrix}\sinh(kh(z')) & \frac{2\chi}{k(\mu+\chi)}(1-\cosh(kh(z')))\\ 1-\cosh(kh(z')) & \frac{2\chi}{k(\mu+\chi)}\sinh(kh(z'))-2h(z')\end{pmatrix},$$

and that

$$\tanh\left(\frac{kh(z')}{2}\right)=\frac{\sinh(kh(z'))}{1+\cosh(kh(z'))}=\frac{\cosh(kh(z'))-1}{\sinh(kh(z'))}.$$

Thus, we get that A_i, B_i, $i = 1, 2$, are given by

$$\begin{aligned}
A_1(z') &= -\frac{1}{2},\\
A_2(z') &= \frac{\sinh(kh(z'))}{-2h(z')\sinh(kh(z')) - \frac{4\chi}{k(\mu+\chi)}(1-\cosh(kh(z')))} = -\frac{1}{2}\frac{1}{h(z') - \frac{2\chi}{k(\mu+\chi)}\tanh\left(\frac{kh(z')}{2}\right)},\\
B_1(z') &= \frac{1}{2}\coth\left(\frac{kh(z')}{2}\right),\\
B_2(z') &= \frac{1-\cosh(kh(z'))}{-2h(z')\sinh(kh(z')) - \frac{4\chi}{k(\mu+\chi)}(1-\cosh(kh(z')))} = \frac{1}{2}\frac{\tanh\left(\frac{kh(z')}{2}\right)}{h(z') - \frac{2\chi}{k(\mu+\chi)}\tanh\left(\frac{kh(z')}{2}\right)},
\end{aligned} \tag{4.83}$$

and then $\widetilde{u}_1$ and $\widetilde{w}_2$ are obtained by (4.80) as functions of $\widetilde{p}$, f_1 and g_2, by the desired expressions given in Lemma 3.3.

Step 2. We derive the expressions of $\widetilde{u}_2$ and $\widetilde{w}_1$, as solution of system (3.28). To do this, we note that $(\widetilde{u}_1, \widetilde{w}_2)$ with external forces (f_1, g_2), and $(\widetilde{u}_2, -\widetilde{w}_1)$ with external forces $(f_2, -g_1)$ satisfy the same equations and boundary conditions. Then, by using this fact and the expressions of $(\widetilde{u}_1, \widetilde{w}_2)$ given in Lemma 3.3, then we easily deduce expressions for $(\widetilde{u}_2, \widetilde{w}_1)$ given in Lemma 3.3.

Step 3. We derive the expressions of $\widetilde{b}_1$ and $\widetilde{b}_2$, as solution of system (3.30). To do this, we rewrite (3.30) as follows,

$$\partial_{z_3}^2 \widetilde{b}'(z) = \frac{1}{\nu_c}\nabla_{z'}\widetilde{q}(z').$$

Integrating twice with respect to z_3, we deduce

$$\widetilde{b}'(z) = \frac{z_3^2}{2\nu_c}\nabla_{z'}\widetilde{q}(z') + C_1(z')z_3 + C_2(z'),$$

with C_1 and C_2 unknown functions. By using the boundary conditions $\widetilde{b}'(z', 0) = \widetilde{b}'(z', h(z')) = 0$, we deduce

$$\widetilde{b}'(z) = \frac{z_3^2 - h(z')z_3}{2\nu_c}\nabla_{z'}\widetilde{q}(z'),$$

which is the desired expression and concludes the proof.

□

Proof of Theorem 3.4. We divide the proof in five steps.

Step 1. We derive the expression for the averaged $\widetilde{U}$. First, since $\widetilde{u}_3 = 0$, then $\widetilde{U}_3 = 0$, so it only remains to derive the expression for $\widetilde{U}'$. Integrating with respect to z_3 between 0 and $h(z')$ the expressions of $\widetilde{u}_i$, $i = 1, 2$,

given in Lemma 3.3, we get

$$\begin{aligned}\widetilde{U}_1(z') = \int_0^{h(z')} \widetilde{u}_1(z)dz_3 &= \Big[-\tfrac{h^3(z')}{6\mu} - \tfrac{h(z')}{4\mu}\Big(\tfrac{2\chi}{k^2(\mu+\chi)}(\cosh(kh(z'))-1) - h^2(z')\Big)\\ &\quad + \tfrac{\chi h(z')}{2k(\mu+\chi)\mu}\big(\tfrac{1}{k}\sinh(kh(z')) - h(z')\big)\coth\Big(\tfrac{kh(z')}{2}\Big)\Big](f_1(z') - \partial_{z_1}\widetilde{p}(z'))\\ &\quad + \Big[-\tfrac{h^2(z')}{4\chi} + \tfrac{h(z')}{2\chi}\Big(\tfrac{2\chi}{k^2(\mu+\chi)}(\cosh(kh(z'))-1) - h(z')\Big)A_2\\ &\quad + \tfrac{h(z')}{k(\mu+\chi)}\big(\tfrac{1}{k}\sinh(kh(z')) - h(z')\big)B_2\Big]g_2(z')\\ &= \Theta_1(f_1(z') - \partial_{z_1}\widetilde{p}(z')) + \Theta_2 g_2(z'),\end{aligned}$$

$$\begin{aligned}\widetilde{U}_2(z') = \int_0^{h(z')} \widetilde{u}_2(z)dz_3 &= \Big[-\tfrac{h^3(z')}{6\mu} - \tfrac{h(z')}{4\mu}\Big(\tfrac{2\chi}{k^2(\mu+\chi)}(\cosh(kh(z'))-1) - h^2(z')\Big)\\ &\quad + \tfrac{\chi h(z')}{2k(\mu+\chi)\mu}\big(\tfrac{1}{k}\sinh(kh(z')) - h(z')\big)\coth\Big(\tfrac{kh(z')}{2}\Big)\Big](f_2(z') - \partial_{z_2}\widetilde{p}(z'))\\ &\quad - \Big[-\tfrac{h^2(z')}{4\chi} + \tfrac{h(z')}{2\chi}\Big(\tfrac{2\chi}{k^2(\mu+\chi)}(\cosh(kh(z'))-1) - h(z')\Big)A_2\\ &\quad + \tfrac{h(z')}{k(\mu+\chi)}\big(\tfrac{1}{k}\sinh(kh(z')) - h(z')\big)B_2\Big]g_1(z')\\ &= \Theta_1(f_2(z') - \partial_{z_2}\widetilde{p}(z')) - \Theta_2 g_1(z').\end{aligned}$$

Now, we develop Θ_1 as follows

$$\begin{aligned}\Theta_1 &= -\tfrac{h^3(z')}{6\mu} - \tfrac{h(z')}{4\mu}\Big(\tfrac{2\chi}{k^2(\mu+\chi)}(\cosh(kh(z'))-1) - h^2(z')\Big)\\ &\quad + \tfrac{\chi h(z')}{2k(\mu+\chi)\mu}\big(\tfrac{1}{k}\sinh(kh(z')) - h(z')\big)\coth\Big(\tfrac{kh(z')}{2}\Big)\\ &= \tfrac{h^3(z')}{12\mu} + \tfrac{h(z')\chi}{\mu k^2(\mu+\chi)} - \tfrac{\chi h^2(z')}{2\mu k(\mu+\chi)}\coth\Big(\tfrac{kh(z')}{2}\Big)\\ &= \tfrac{h^3(z')}{\mu}\Big(\tfrac{1}{12} + \tfrac{\chi}{h^2(z')k^2(\mu+\chi)} - \tfrac{\chi}{2h(z')k(\mu+\chi)}\coth\Big(\tfrac{kh(z')}{2}\Big)\Big),\end{aligned}$$

and Θ_2 as follows

$$\begin{aligned}\Theta_2 &= \Big[-\tfrac{h^2(z')}{4\chi} + \tfrac{h(z')}{2\chi}\Big(\tfrac{2\chi}{k^2(\mu+\chi)}(\cosh(kh(z'-1)) - h^2(z')\Big)A_2\\ &\quad + \tfrac{h(z')}{k(\mu+\chi)}\big(\tfrac{1}{k}\sinh(kh(z')) - h(z')\big)B_2\Big] = 0.\end{aligned}$$

Then, we get

$$\widetilde{U}_1(z') = \frac{h^3(z')}{\mu}\Phi(h(z'),\mu,\chi,\nu_c)(f_1(z') - \partial_{z_1}\widetilde{p}(z')),$$

$$\widetilde{U}_2(z') = \frac{h^3(z')}{\mu}\Phi(h(z'),\mu,\chi,\nu_c)(f_2(z') - \partial_{z_2}\widetilde{p}(z')),$$

where Φ is defined in (3.32).

Step 2. We derive the expression for the averaged $\widetilde{W}$. First, since $\widetilde{w}_3 = 0$, then $\widetilde{W}_3 = 0$, so it only remains to derive the expression for $\widetilde{W}'$. Integrating with respect to z_3 between 0 and $h(z')$ the expressions of $\widetilde{w}_i$, $i = 1, 2$, given in Lemma 3.3, we get

$$\widetilde{W}_1(z') = \int_0^{h(z')} \widetilde{w}_1(z)dz_3 = -\overline{\Theta}_1\,(f_2(z') - \partial_{z_2}\widetilde{p}(z')) + \overline{\Theta}_2\,g_1(z'),$$

$$\widetilde{W}_2(z') = \int_0^{h(z')} \widetilde{w}_2(z)dz_3 = \overline{\Theta}_1\,(f_1(z') - \partial_{z_1}\widetilde{p}(z')) + \overline{\Theta}_2\,g_2(z'),$$

where

$$\begin{aligned}\overline{\Theta}_1 &= -\tfrac{h(z')}{4\mu} - \tfrac{h(z')}{4\mu}\left(\tfrac{1}{k}\sinh(kh(z')) - h(z')\right) + \tfrac{h(z')}{4\mu k}(\cosh(kh(z')) - 1)\coth\left(\tfrac{kh(z')}{2}\right)\\ &= -\tfrac{h(z')}{4\mu k}\sinh(kh(z')) + \tfrac{h(z')}{4\mu k}(\cosh(kh(z')) - 1)\tfrac{\sinh(kh(z'))}{\cosh(kh(z'))-1} = 0,\end{aligned}$$

$$\begin{aligned}\overline{\Theta}_2 &= \tfrac{h(z')}{2\chi}\left(\tfrac{1}{k}\sinh(kh(z')) - h(z')\right)A_2 + \tfrac{h(z')}{2\chi k}(\cosh(kh(z')) - 1)B_2\\ &= \frac{h(z')}{4\chi}\frac{1 - \frac{2}{kh(z')}\tanh\left(\frac{kh(z')}{2}\right)}{1 - \frac{\chi}{\mu+\chi}\frac{2}{kh(z')}\tanh\left(\frac{kh(z')}{2}\right)}.\end{aligned}$$

Then, we get

$$\widetilde{W}_1(z') = \frac{h(z')}{4\chi}\Psi(h(z'),\mu,\chi,\nu_c)g_1(z'), \quad \widetilde{W}_2(z') = \frac{h(z')}{4\chi}\Psi(h(z'),\mu,\chi,\nu_c)g_2(z'),$$

where Ψ is defined in (3.32).

Step 3. We derive the expression for the averaged $\widetilde{B}$. First, since $\widetilde{b}_3 = 0$, then $\widetilde{B}_3 = 0$, so it only remains to derive the expression for $\widetilde{B}'$. Integrating with respect to z_3 between 0 and $h(z')$ the expressions of $\widetilde{b}_i$, $i = 1, 2$, given in Lemma 3.3, we get

$$\widetilde{B}'(z') = \int_0^{h(z')} \widetilde{b}'(z)\,dz_3 = -\frac{h^3(z')}{12\nu_c}\nabla_{z'}\widetilde{q}(z').$$

Step 4. Finally, taking $\theta \in H^1(\omega)$ as test function in (3.29), and integrating by parts, we deduce

$$\begin{aligned}&\int_\omega \frac{h^3(z')}{\mu}\Phi(h(z'),\mu,\chi,\nu_c)\partial_{z_1}\widetilde{p}(z')\,\partial_{z_1}\theta(z')\,dz' + \int_\omega \frac{h^3(z')}{\mu}\Phi(h(z'),\mu,\chi,\nu_c)\partial_{z_2}\widetilde{p}(z')\,\partial_{z_2}\theta(z')\,dz'\\ &- \int_\omega \frac{h^3(z')}{\mu}\Phi(z')f_1(z')\,\partial_{z_1}\theta(z')\,dz' - \int_\omega \frac{h^3(z')}{\mu}\Phi(z')f_2(z')\,\partial_{z_2}\theta\,dz'(z') = 0,\end{aligned}$$

which is equivalent to the equation for the hydrostatic pressure given in $(3.33)_1$. It has a unique solution because $\widetilde{p} \in L^2_0(\Omega)$. Observe that from the expression of $\tilde{U}'$ and its regularity, it holds $\widetilde{p} \in H^1(\Omega)$.

Step 5. We derive the equation for the magnetic pressure. Putting this expression in (3.31) and integrating by parts, we deduce for every $\zeta \in H^1(\omega)$ that

$$\int_\omega \frac{h^3(z')}{12\nu_c}\partial_{z_1}\widetilde{q}(z')\partial_{z_1}\zeta(z')\,dz' + \int_\omega \frac{h^3(z')}{12\nu_c}\partial_{z_2}\widetilde{q}(z')\partial_{z_2}\zeta(z')\,dz' = 0,$$

which is equivalent to the equation for the magnetic pressure given in $(3.33)_2$. It has a unique solution because $\widetilde{q} \in L^2_0(\Omega)$. Observe that from the expression of $\tilde{B}'$ and its regularity, it holds $\widetilde{q} \in H^1(\Omega)$.

□

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References

[1] G. Ahmadi and M. Shahinpoor, *Universal stability of magneto-micropolar fluid motions.* Int. J. Enging. Sci. **12** (1974), 657–663.

[2] M.K. Alam, K. Bibi, A. Khan, U. Fernandez-Gamiz and S. Noeiaghdam, *The Effect of Variable Magnetic Field on Viscous Fluid between 3-D Rotatory Vertical Squeezing Plates: A Computational Investigation.* Energies **15** (2022), 2473.

[3] M. Anguiano, *On the non-stationary non-Newtonian flow through a thin porous medium.* ZAMM Z. Angew. Math. Mech. **97** (2017), 895–915.

[4] M. Anguiano, *Existence, uniqueness and homogenization of nonlinear parabolic problems with dynamical boundary conditions in perforated media.* Mediterr. J. Math. **17** (2020), 18.

[5] M. Anguiano, *On p-Laplacian Reaction–Diffusion Problems with Dynamical Boundary Conditions in Perforated Media.* Mediterr. J. Math. **20** (2023), 124.

[6] M. Anguiano and R. Bunoiu, *Homogenization of Bingham flow in thin porous media.* Netw. Heterog. Media **15** (2020), 87–110.

[7] M. Anguiano and F.J. Suárez-Grau, *Derivation of a coupled Darcy-Reynolds equation for a fluid flow in a thin porous medium including a fissure.* Z. Angew. Math. Phys. **68** (2017), 52.

[8] M. Anguiano and F.J. Suárez-Grau, *Nonlinear Reynolds equations for non-Newtonian thin-film fluid flows over a rough boundary.* IMA J. Appl. Math. **84** (2019), 63–95.

[9] M. Anguiano and F.J. Suárez-Grau, *Sharp Pressure Estimates for the Navier–Stokes System in Thin Porous Media.* Bull. Malays. Math. Sci. Soc. **46** (2023), 117.

[10] G. Bayada and M. Chambat, *The transition between the Stokes equations and the Reynolds equation: A mathematical proof.* Appl. Math. Optim. **14** (1986), 73–93.

[11] G. Bayada and G. Lukaszewicz, *On micropolar fluids in the theory of lubrication. Rigorous derivation of an analogue of the Reynolds equation.* Internat. J. Engrg. Sci. **34** (1996), 1477–1490.

[12] G. Bayada, N. Benhaboucha and M. Chambat, *New models in micropolar fluid and their application to lubrication.* Math. Models Methods Appl. Sci. **15** (2005), 343–374.

[13] M. Bonnivard, I. Pažanin and F.J. Suárez-Grau, *Effects of rough boundary and nonzero boundary conditions on the lubrication process with micropolar fluid.* Eur. J. Mech. B Fluids **72** (2018), 501–518.

[14] M. Bonnivard, I. Pažanin and F.J. Suárez-Grau, *A generalized Reynolds equation for micropolar flows past a ribbed surface with nonzero boundary conditions.* ESAIM: Math. Model. Numer. Anal **56** (2022), 1255 – 1305.

[15] F. Boughanim and R. Tapiéro, *Derivation of The Two-dimensional Carreau Law for a Quasi-Newtonian Fluid Flow Through a Thin Slab.* Appl. Anal. **57** (1995), 243–269.

[16] G. Cimatti, *A rigorous justification of the Reynolds equation.* Q. Appl. Math. **45** (1987), 627–644.

[17] F.W. Cruz, C.F. Perusato, M.A. Rojas-Medar and P.R. Zingano, *Large time behavior for MHD micropolar fluids in* $\mathbb{R}^n$. J. Differ. Equ. **312** (2022), 1–44.

[18] M. Durán, E. Ortega-Torres and M.A. Rojas-Medar, *Stationary solutions of magneto-micropolar fluid equations in exterior domains.* Proyecciones **22** (2003), 63-79.

[19] G. Duvaut and J.L. Lions, *Inéquations en thermoélasticité et magnéto-hydrodynamique.* Arch. Rational Mech. Anal. **46** (1972), 241–279.

[20] A.C. Eringen, *Theory of micropolar fluids.* J. Math. Mech. 16 (1966), 1–18.

[21] J.-M. Kim and S. Ko, *Some Liouville-type theorems for the stationary 3D magneto-micropolar fluids.* Authorea (2022).

[22] B. Knaepen, S. Kassinos and D. Carati, *MHD turbulence at moderate magnetic Reynolds number.* J. Fluid Mech. **513** (2004), 199–220.

[23] B. Knaepen and R. Moreau, *Magnetohydrodynamic Turbulence at Low Magnetic Reynolds Number.* Annu. Rev. Fluid Mech. **40** (2008), 25-45.

[24] G. Lukaszewicz, *Micropolar Fluids: Theory and Applications.* Birkhauser, Boston (1999).

[25] U.S. Mahabaleshwar, I. Pažanin, M. Radulović and F.J. Suárez-Grau, *Effects of small boundary perturbation on the MHD duct flow.* Theor. App. Mech. **44** (2017), 83–101.

[26] E. Marušić-Paloka, I. Pažanin and S. Marušić, *An effective model for the lubrication with micropolar fluid.* Mech. Res. Commun. **52** (2013), 69–73.

[27] E. Marušić-Paloka and I. Pažanin, *A note on the MHD flow in a porous channel.* Theor. App. Mech. **49** (2022), 49–60.

[28] E. Marušić-Paloka, I. Pažanin and M. Radulović, *MHD flow through a perturbed channel filled with a porous medium.* Bull. Malaysian Math. Sci. Soc. **45** (2022), 2441–2471.

[29] A. Mikelić and R. Tapiero, *Mathematical derivation of the power law describing polymer flow through a thin slab.* RAIRO Modél. Math. Anal. Numér. **29** (1995), 3–21.

[30] C.J. Niche and C.F. Perusato, *Sharp decay estimates and asymptotic behaviour for 3D magneto-micropolar fluids.* Z. Angew. Math. Phys. **73**, 48 (2022).

[31] E.E. Ortega-Torres and M.A. Rojas-Medar, *Magneto-micropolar fluid motion: global existence of strong solutions.* Abstr. Appl. Anal. **4** (1999), 109–125.

[32] I. Pažanin, *On the micropolar flow in a circular pipe: the effects of the viscosity coefficients.* Theor. Appl. Mech. Lett. **1** (2011), 062004.

[33] I. Pažanin, *Asymptotic Analysis of the Lubrication Problem with Nonstandard Boundary Conditions for Microrotation.* Filomat **30** (2016), 2233–2247.

[34] I. Pažanin and F.J. Suárez-Grau, *Analysis of the thin film flow in a rough domain filled with micropolar fluid.* **68** (2014), 1915–1932.

[35] I. Pažanin and F.J. Suárez-Grau, *Homogenization of the Darcy-Lapwood-Brinkman Flow in a Thin Domain with Highly Oscillating Boundaries.* Bull. Malays. Math. Sci. Soc. **42** (2019), 3073–310915.

[36] C.F. Perusato, W.G. Melo, R.H. Guterres and J.R. Nunes, *Time asymptotic profiles to the magneto-micropolar system.* Appl. Anal. **99** (2020), 2678–2691.

[37] S. Riasat, M. Ramzan, S. Kadry and Y.-M. Chu, *Significance of magnetic Reynolds number in a three-dimensional squeezing Darcy-Forchheimer hydromagnetic nanofuid thin-film flow between two rotating disks.* Sci Rep **10** (2020), 17208.

[38] M.A. Rojas-Medar, *Magneto-micropolar fluid motion: existence and uniqueness of strong solution.* Math. Nachr. **188** (1997), 301–319.

[39] M.A. Rojas-Medar and J. L. Boldrini, *Magneto-micropolar fluid motion: existence of weak solutions.* Rev. Mat. Complut. **11** (1998), 443-460.

[40] C. Singh and P. Sinha, *The three-dimensional Reynolds' equation for micropolar fluid lubricated bearings.* Wear 76 (1982), 199–209.

[41] F.J. Suárez-Grau, *Asymptotic behavior of a non-Newtonian flow in a thin domain with Navier law on a rough boundary.* Nonlinear Anal-Theor. **117** (2015), 99–123.

[42] F.J. Suárez-Grau, *Analysis of the Roughness Regimes for Micropolar Fluids via Homogenization.* Bull. Malays. Math. Sci. Soc. **44** (2021), 1613–1652.

[43] F.J. Suárez-Grau, *Mathematical modeling of micropolar fluid flows through a thin porous medium.* J. Eng. Math. **126** (2021), 7.

[44] L. Zhang, M.B. Arain, M.M. Bhatti, A. Zeeshan and H. Hal-Sulami, *Effects of magnetic Reynolds number on swimming of gyrotactic microorganisms between rotating circular plates filled with nanofluids.* Appl. Math. Mech. **41** (2020), 637–654.