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General integral identities for bianisotropic media and related equations, properties, and coupling expressions

J.M.L. Bernard

1) Introduction

We consider advanced mathematical equations and properties on electromagnetic fields derived from integral identities in bianisotropic media : generalized reciprocity principle for adjoint systems, equivalence of surface sources, uniqueness, links between impedance surface for adjoint systems, integral expressions of fields influenced by a perturbation of a system, and of fields due to the coupling between two arbitrary scatterers A and A' perturbing a complex bianisotropic system, which generalizes our previous works [1].

As a particular example of application of our results on couplings for bianisotropic systems, we consider the case of the evaluation of the radiation of a cavity (noted A) inserted in an infinite plate, when, in practice, the ends of the plate (noted A') perturb the result, and then present an efficient method which permits the numerical suppression of the first order couplings between A' and A by a simple and efficient postprocess. Thus, we can recover the scattering diagram of A when it is inserted in a plane (i.e. in the plate without ends), in a simple manner at any observation angle. Our method remains efficient for backscattering, from grazing to normal angles, in contrast of well-known difficulties at grazing incidence of common time domain filtering methods.

2) General integral identities for piecewise continuous media with finite losses

2.1) Bianisotropic media and adjoint characteristics

Let us consider the electromagnetic fields (E, H) and (E_0, H_0) in bianisotropic media, with characteristics (or constitutive parameters) $((\epsilon), (\xi), (\mu), (\varsigma))$ and $((\epsilon_0), (\xi_0), (\mu_0), (\varsigma_0))$ that are piecewise continuous in the domain Ω . For a time dependence $e^{i\omega t}$, the Maxwell equations are written [2]-[7],

$$\begin{aligned}
\text{curl}(H) &= i\omega((\epsilon)E + (\xi)H) + J, \\
\text{curl}(E) &= -i\omega((\mu)H + (\varsigma)E) - M, \\
\text{curl}(H_0) &= i\omega((\epsilon_0)E_0 + (\xi_0)H_0) + J_0, \\
\text{curl}(E_0) &= -i\omega((\mu_0)H_0 + (\varsigma_0)E_0) - M_0.
\end{aligned} \tag{1}$$

We can consider that they are valid, except at some isolated points (tip) or on some lines (edge) of discontinuity of parameters, where components of fields could be singular. In a similar manner, we define the fields (E_{0C}, H_{0C}) for media with characteristics $({}^t(\epsilon_0), -{}^t(\varsigma_0), {}^t(\mu_0), -{}^t(\xi_0))$, adjoint (or complementary) to $((\epsilon_0), (\xi_0), (\mu_0), (\varsigma_0))$, verifying,

$$\begin{aligned}
\text{curl}(H_{0C}) &= i\omega({}^t(\epsilon_0)E_{0C} - {}^t(\varsigma_0)H_{0C}) + J_{0C}, \\
\text{curl}(E_{0C}) &= -i\omega({}^t(\mu_0)H_{0C} - {}^t(\xi_0)E_{0C}) - M_{0C},
\end{aligned} \tag{2}$$

where ${}^t(\cdot)$ is the tranpose operator. The parameters $(\epsilon_{(0)})$ and $(\mu_{(0)})$ are the permittivity and the permeability tensors, while $(\xi_{(0)})$ and $(\varsigma_{(0)})$ are the bianisotropic coupling tensors [2]-[5]. The terms J_0 , M_0 , J_{0C} , M_{0C} , J , and M are the primary sources, and we can consider them in the sense of distributions (or generalized functions) so that we can take account of Dirac (dipolar) sources (see pp.35-37 of [6]). The fields in adjoint cases are strongly related, from some specific integral identities, in particular, from the generalized reciprocity theorem [2]-[4], that we will study and use in several configurations during our analysis.

Remark : The bianisotropic coupling tensors are related to the Tellegen (χ) and chirality (κ) tensors parameters, following $(\xi) = \frac{1}{c}((\chi) - i(\kappa))$ and $(\varsigma) = \frac{1}{c}((\chi) + i(\kappa))$, where c is the speed of light. In these expressions, (κ) is usually said to model reciprocal physical effects, while (χ) corresponds to nonreciprocal ones [2]-[5]. In the particular case where $(\chi) = 0$, ${}^t(\epsilon_0) = (\epsilon_0)$, ${}^t(\mu_0) = (\mu_0)$, and ${}^t(\kappa) = (\kappa)$, we note that the fields (E_{0C}, H_{0C}) and (E_0, H_0) satisfy identical Maxwell equations when $J_{0C} = J_0$ and $M_{0C} = M_0$.

2.2) Basic general properties of fields

We suppose afterwards the usual conditions on the fields [6] :

(a) The primary sources are bounded and the fields are composed of outward waves at infinity. Considering, from now on, arbitrary fixed non null small losses in unbounded part of media, the fields exponentially vanish if we let tend the observation point to infinity (as in limit absorption principle).

(b) We assume that the fields remain bounded in an arbitrarily thin transition plane layer, without any primary source in it. In this case, from (1) and (2), the curl of fields remain bounded, and, considering in the sense of the limit, the case of piecewise continuous media with finite parameters, we have the continuity of tangential components of fields at interfaces with smooth geometry.

(c) The fields can be singular at possible isolated points (tip of cone), or on some line of discontinuities (edge). Considering the primary sources apart, we suppose that the fields are integrable along any surface including these discontinuities, which leads us to consider any further surface integrals as improper. Besides, the fields component along any smooth line of discontinuities is continuous across it (which completes condition (b)).

Remark : In media with large losses, the fields generally decrease exponentially as we move away from an interface (skin effect), and skin depth vanishes as losses tend to ∞ .

Remark : from the continuity of tangential fields at smooth interfaces, the equations (1) and (2) can be directly considered, for piecewise continuous media with finite parameters, in the sense of distributions, without any additional surface sources distributions.

Remark : Let us note a point concerning mathematical definitions of parameters. In free space, we usually write $k = \omega\sqrt{\epsilon_0\mu_0}$ ($= \omega/c$) for the wave number, while, for modelisation of isotropic passive metamaterials, we will take the precaution to write $k = \omega\sqrt{\epsilon}\sqrt{\mu}$ ($= \omega\sqrt{\epsilon_r}\sqrt{\mu_r}/c$) to ensure passivity and $\text{Re}(ik) > 0$, so that e^{-ikR} vanishes at infinity. For that, elementary expressions commonly used for usual materials often need to be changed for metamaterials. In this case, we note for example that,

$$\frac{k}{\omega\epsilon}\tan(kd) = \frac{\sqrt{\mu}}{\sqrt{\epsilon}}\tan(\omega\sqrt{\epsilon}\sqrt{\mu}d) \text{ is not equivalent to } \sqrt{\frac{\mu}{\epsilon}}\tan(\omega\sqrt{\epsilon\mu}d). \quad (3)$$

2.3) Initial surface and volume integral identities for piecewise continuous media

Let us consider in first step that the media parameters are continuous in Ω , and the bounded primary sources are disjoint from $\partial\Omega$. Using the basic vectorial equalities,

$$\begin{aligned} \text{div}(U \wedge W) &= W.\text{curl}(U) - U.\text{curl}(W), \\ U.^t(P)W &= W.(P)U, \end{aligned} \quad (4)$$

with (1) and (2), and the divergence theorem [6], we can write,

$$\begin{aligned}
\int_{\Omega} \operatorname{div}(E \wedge H_{0C}) dV &= \int_{\partial\Omega} \hat{n}_s.(E \wedge H_{0C}) dS, \\
\int_{\Omega} \operatorname{div}(E \wedge H_{0C}) dV &= \int_{\Omega} (H_{0C}.\operatorname{curl} E - E.\operatorname{curl} H_{0C}) dV \\
&= \int_{\Omega} (-i\omega H_{0C}.(\mu)H - M.H_{0C} - i\omega E.^t(\epsilon_0)E_{0C} - J_{0C}.E) dV \\
&\quad + \int_{\Omega} (-i\omega H_{0C}.(\varsigma)E + i\omega E.^t(\varsigma_0)H_{0C}) dV \\
&= \int_{\Omega} (-i\omega H_{0C}.(\mu)H - M.H_{0C} - i\omega E_{0C}.(\epsilon_0)E - J_{0C}.E) dV \\
&\quad + \int_{\Omega} (-i\omega H_{0C}.((\varsigma) - (\varsigma_0))E) dV, \tag{5}
\end{aligned}$$

and,

$$\begin{aligned}
\int_{\Omega} \operatorname{div}(H \wedge E_{0C}) dV &= \int_{\partial\Omega} \hat{n}_s.(H \wedge E_{0C}) dS, \\
\int_{\Omega} \operatorname{div}(H \wedge E_{0C}) dV &= \int_{\Omega} (E_{0C}.\operatorname{curl}(H) - H.\operatorname{curl}(E_{0C})) dV \\
&= \int_{\Omega} (i\omega E_{0C}.(\epsilon)E + J.E_{0C} + i\omega H.^t(\mu_0)H_{0C} + M_{0C}.H) dV \\
&\quad + \int_{\Omega} (i\omega E_{0C}.(\xi)H - i\omega H.^t(\xi_0)E_{0C}) dV \\
&= \int_{\Omega} (i\omega E_{0C}.(\epsilon)E + J.E_{0C} + i\omega H_{0C}.(\mu_0)H + M_{0C}.H) dV \\
&\quad + \int_{\Omega} (i\omega E_{0C}.((\xi) - (\xi_0))H) dV, \tag{6}
\end{aligned}$$

where \hat{n}_s is the unit normal to $\partial\Omega$ outward Ω .

After addition of these equations, we then obtain integral identities in the following form,

$$\begin{aligned}
&\int_{\partial\Omega} (E_{0C}.(\hat{n}_s \wedge H) + H_{0C}.(\hat{n}_s \wedge E)) dS \\
&= \int_{\Omega} i\omega E_{0C}.((\epsilon) - (\epsilon_0))E dV - \int_{\Omega} i\omega H_{0C}.((\mu) - (\mu_0))H dV \\
&\quad - \int_{\Omega} i\omega H_{0C}.((\varsigma) - (\varsigma_0))E dV + \int_{\Omega} i\omega E_{0C}.((\xi) - (\xi_0))H dV \\
&\quad + \int_{\Omega} (M_{0C}.H - J_{0C}.E) dV - \int_{\Omega} (H_{0C}.M - E_{0C}.J) dV, \tag{7}
\end{aligned}$$

and thus, with $\hat{n} = -\hat{n}_s$ the unit normal to $\partial\Omega$ inward Ω ,

$$\begin{aligned}
& \int_{\partial\Omega} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) dS \\
&= - \int_{\Omega} i\omega E_{0C} \cdot ((\epsilon) - (\epsilon_0)) E dV + \int_{\Omega} i\omega H_{0C} \cdot ((\mu) - (\mu_0)) H dV \\
&+ \int_{\Omega} i\omega H_{0C} \cdot ((\varsigma) - (\varsigma_0)) E dV - \int_{\Omega} i\omega E_{0C} \cdot ((\xi) - (\xi_0)) H dV \\
&+ \int_{\Omega} (J_{0C} \cdot E - M_{0C} \cdot H) dV - \int_{\Omega} (E_{0C} \cdot J - H_{0C} \cdot M) dV.
\end{aligned} \tag{8}$$

From continuity of tangential components of fields at boundaries when losses are finite, i.e. at interface of media with finite parameters, we can consider, after concatenation of domains, that,

$$\begin{aligned}
& \text{the equation (8) continues to be valid in general when } \Omega \text{ is filled} \\
& \text{by piecewise continuous media with finite parameters,}
\end{aligned} \tag{9}$$

(note : infinite losses have to be taken in the sense of the limit).

3) Generalized reciprocity theorem and associated fields properties

Considering the fields (E, H) and (E_{0C}, H_{0C}) satisfying Maxwell equations (1) and (2) in media respectively adjoint (or complementary) each other in Ω , with characteristics,

$$\begin{aligned}
& ((\epsilon) = (\epsilon_0), (\xi) = (\xi_0), (\mu) = (\mu_0), (\varsigma) = (\varsigma_0)) \\
& \text{and } ({}^t(\epsilon_0), -{}^t(\varsigma_0), {}^t(\mu_0), -{}^t(\xi_0)) \text{ in } \Omega,
\end{aligned} \tag{10}$$

piecewise continuous in Ω , we can apply (8) with,

$$(\epsilon) - (\epsilon_0) = (\xi) - (\xi_0) = (\mu) - (\mu_0) = (\varsigma) - (\varsigma_0) = 0,$$

and obtain the generalized reciprocity theorem in bianisotropic media for adjoint systems, following,

$$\begin{aligned}
& \int_{\partial\Omega} E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E) dS \\
&= \int_{\Omega} (J_{0C} \cdot E - M_{0C} \cdot H) dV - \int_{\Omega} (E_{0C} \cdot J - H_{0C} \cdot M) dV,
\end{aligned} \tag{11}$$

where \hat{n} is the unit normal to $\partial\Omega$, inward Ω . The surface integral is considered as improper, i.e. with integration apart from the discontinuities on lines or at isolated points.

Remark : The case of media with infinite losses, or of sources in contact with discontinuities, can be considered in (11) in the sense of the limit, but also, of distributions theory.

Remark : We can also take the conjugate of fields and obtain some other generalized equalities, as those for well-known Poynting vector (see appendix).

3.1) Property 1 : Principle of equivalence of surface sources enclosing the domain of a volume source in open space

Let the domain Ω be simply connected (i.e. no holes in it), and Ω_1 a subdomain of it with $\partial\Omega \cap \partial\Omega_1 \equiv \emptyset$ so that $\partial(\Omega \setminus \Omega_1) \equiv \partial\Omega \cup \partial\Omega_1$ (i.e. Ω_1 is a subset of Ω strictly disjoint from the frontier of Ω). We then consider that Ω_1 holds J and M , while $\Omega \setminus \Omega_1$ contains J_{0C} and M_{0C} . We have from (11),

$$\begin{aligned} \int_{\Omega_1} E_{0C} \cdot J - H_{0C} \cdot M \, dV &= - \int_{\partial\Omega_1} E_{0C} \cdot (\hat{n}_1 \wedge H) + H_{0C} \cdot (\hat{n}_1 \wedge E) \, dS, \\ \int_{\Omega \setminus \Omega_1} J_{0C} \cdot E - M_{0C} \cdot H \, dV &= - \int_{\partial\Omega_1} E_{0C} \cdot (\hat{n}_1 \wedge H) + H_{0C} \cdot (\hat{n}_1 \wedge E) \, dS \\ &+ \int_{\partial\Omega} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) \, dS, \end{aligned} \quad (12)$$

where \hat{n} is the inward unit normal to $\partial\Omega$, and $-\hat{n}_1 = \hat{n}_s$ is the unit normal to $\partial\Omega_1$ outward Ω_1 (note : $\Omega \setminus \Omega_1$ is the set of points of Ω not belonging to Ω_1). We then let $\Omega \rightarrow R^3$. For that, we need to consider the behaviour of fields at infinity assumed in condition (a), with small losses at infinity. The fields exponentially vanish at infinity, and we can write in (12),

$$\lim_{\Omega \rightarrow R^3} \int_{\partial\Omega} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) \, dS = 0. \quad (13)$$

Therefore, considering that J_{0C} and M_{0C} in $\Omega \setminus \Omega_1$ can be chosen arbitrarily, we conclude from (12) when $\Omega \rightarrow R^3$, that,

$$\begin{aligned} &\text{for } (E, H) \text{ in } R^3 \setminus \overline{\Omega_1}, \text{ vanishing at infinity for small losses, the primary} \\ &\text{sources } J \text{ and } M \text{ enclosed by } \partial\Omega_1 \text{ can be seen as strictly equivalent to} \\ &\text{the surface distributions : } J_e = (\hat{n}_s \wedge H) \delta_{S_1} \text{ and } M_e = -(\hat{n}_s \wedge E) \delta_{S_1}. \end{aligned} \quad (14)$$

(note : this equivalence of sources can also be derived from the theory of distributions applied to maxwell equations).

3.2) Property 2 : generalized reciprocity theorem in open space and generalized reciprocity principle

For systems with media adjoint each other in $\Omega = R^3 \setminus \Omega'$, with Ω' a bounded subdomain of R^3 , we can apply the generalized reciprocity theorem (11). We then rewrite it with (13), and obtain for the (infinite) open space $\Omega = R^3 \setminus \Omega'$,

$$\begin{aligned} & \int_{\partial\Omega'} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) dS \\ &= \int_{R^3 \setminus \Omega'} (J_{0C} \cdot E - M_{0C} \cdot H) dV - \int_{R^3 \setminus \Omega'} (E_{0C} \cdot J - H_{0C} \cdot M) dV, \end{aligned} \quad (15)$$

where \hat{n} is the unit normal to $\partial\Omega'$ outward Ω' . In the case where $\Omega \equiv R^3$, this leads us to the generalized reciprocity principle in bianisotropic media,

$$\int_{R^3} (J_{0C} \cdot E - M_{0C} \cdot H) dV = \int_{R^3} (E_{0C} \cdot J - H_{0C} \cdot M) dV. \quad (16)$$

3.3) Uniqueness properties

3.3.1) Property 3 : uniqueness property when primary sources J and M vanish

Considering the fields (E, H) with $J = M = 0$, satisfying (1) and the conditions (a), (b) and (c), we can apply (16) and write,

$$\int_{R^3} (J_{0C} \cdot E - M_{0C} \cdot H) dV = 0. \quad (17)$$

Since the choice of J_{0C} and M_{0C} is free, we then derive that $E = H = 0$. In these circumstances, two fields (E, H) satisfying (1) and the conditions (a), (b) and (c) in whole R^3 with the same primary sources, are necessarily identical, while any field, verifying (1) and (a), (b) and (c) with no primary sources, is null (note : a surface wave that propagates on a plane with no primary sources is in contradiction with condition (a))

3.3.2) Property 4 : uniqueness when $\hat{n} \wedge E$ and $\hat{n} \wedge H$ both vanish on an open surface

Let us consider Ω' an open connected subdomain of Ω , with the characteristics that are analytic with respect to coordinates in Ω . Taking $J = M = M_{0C} = 0$ in Ω' and

$J_{0C} = J'_{0C}\delta(r - r_0)$ with $r_0 \in \Omega$, we have, from the application of (11),

$$\begin{aligned} J'_{0C}.E(r_0) &= \int_{\partial\Omega'} (E_{0C}.(\hat{n} \wedge H) + H_{0C}.(\hat{n} \wedge E))dS \text{ for } r_0 \in \Omega', \\ \text{and } \int_{\partial\Omega'} (E_{0C}.(\hat{n} \wedge H) + H_{0C}.(\hat{n} \wedge E))dS &= 0 \text{ for } r_0 \in \Omega \setminus \overline{\Omega'}, \end{aligned} \quad (18)$$

We now take the case where,

$$(\hat{n} \wedge H) = (\hat{n} \wedge E) = 0 \text{ on a regular part } S_h \text{ of } \partial\Omega', \quad (19)$$

making like a hole in $\partial\Omega'$, so that we can replace $\partial\Omega'$ by the open surface $\partial\Omega' \setminus S_h$ in the surface integral function of r_0 given in first and second line of (18). Following analyticity of parameters, we can assume that (E_{0C}, H_{0C}) , solution of (2) in Ω , is an analytic function with respect to coordinates of r_0 in Ω , and thus, that the surface integral function is continuous through the hole S_h . This function then continues to vanish from outside to inside Ω' . Therefore, we have $J'_{0C}.E(r_0) = 0$ in Ω for arbitrary vector J'_{0C} , and thus,

$$E(r_0) = 0 \text{ in } \Omega. \quad (20)$$

In the same manner, we can also consider $J = M = J_{0C} = 0$ in Ω' and $M_{0C} = M'_{0C}\delta(r - r_0)$ in (11), and then derive, when (19) applies, that $M'_{0C}.H(r_0) = 0$ in Ω for arbitrary vector M'_{0C} , and thus $H(r_0) = 0$ in Ω .

This implies the following uniqueness property,

$$\begin{aligned} &\text{if the characteristics of media are analytic in } \Omega \\ &\text{and } (\hat{n} \wedge H) = (\hat{n} \wedge E) = 0 \text{ on an open surface } S_h \subset \Omega, \\ &\text{then } E = H = 0 \text{ in } \Omega. \end{aligned} \quad (21)$$

This property also applies by continuity if S_h is on the frontier of Ω . Considering now the continuity of tangential fields at interface of media with finite losses in a connected domain Ω with piecewise analytic characteristics, we can apply (21) in whole generality if the open surface S_h belongs to $\overline{\Omega}$ (note : we avoid domains having some parts with strictly infinite losses, i.e. strictly impenetrable).

3.4) property 5 : vanishing surface integral equation

For a domain Ω' free of sources for (E, H) and (E_{0C}, H_{0C}) (i.e. J, M, J_{0C} and M_{0C} vanish in Ω'), with respective media adjoint each other in Ω' , we have, following (11),

$$\int_{\partial\Omega'} (E_{0C}.(\hat{n} \wedge H) + H_{0C}.(\hat{n} \wedge E))dS = 0. \quad (22)$$

We now consider elementary applications of this vanishing surface integral equation.

3.4.1) Property 6 : équivallence of two surface sources

For first case of application of (22), we use linearity of Maxwell equations. We consider the fields (E_{0C}, H_{0C}) associated to sources J_{0C} and M_{0C} vanishing in Ω' , and two fields (E', H') and (E'', H'') associated to sources identical in Ω' . We then note that, when media considered for them are adjoint to media considered for (E_{0C}, H_{0C}) in Ω' , we have, from (22),

$$\int_{\partial\Omega'} (E_{0C} \cdot (\hat{n} \wedge (H' - H'')) + H_{0C} \cdot (\hat{n} \wedge (E' - E''))) dS = 0, \quad (23)$$

and thus the equivalence of contributions in this integral of $(\hat{n} \wedge E', \hat{n} \wedge H')|_{\partial\Omega'}$ and $(\hat{n} \wedge E'', \hat{n} \wedge H'')|_{\partial\Omega'}$, which leads us to,

$$\begin{aligned} & \int_{\partial\Omega'} (E_{0C} \cdot (\hat{n} \wedge H' + H_{0C} \cdot (\hat{n} \wedge E')) dS \\ &= \int_{\partial\Omega'} (E_{0C} \cdot (\hat{n} \wedge H'') + H_{0C} \cdot (\hat{n} \wedge E'')) dS. \end{aligned} \quad (24)$$

3.4.2) Property 7 : Links between surface impedance parameters for adjoint cases

For second case of application of (22), we can consider the case where Ω' encloses a part of the interface S of two distinct media 1 and 2 so that $\partial\Omega' \equiv S'_1 \cup S'_2$, where S'_1 is an open surface above S and S'_2 is an open one below S . We can write, from (22),

$$\begin{aligned} & \int_{S'_1} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) dS \\ &+ \int_{S'_2} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) dS = 0. \end{aligned} \quad (25)$$

If we let losses tending to infinity in Ω' below, we have, at the limit, the fields that vanish below the interface,

$$E = E_{0C} = 0 \text{ and } H = H_{0C} = 0 \text{ on } S'_2, \quad (26)$$

which implies at the limit, from (25),

$$\int_{S'_1} (E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)) dS = 0. \quad (27)$$

Considering the case where S'_1 tends to a smooth part of the interface S just above it, named S^+ , we can let vary it, and then deduce that,

$$\begin{aligned} E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E) &= 0 \text{ on } S^+, \\ \text{i.e. } (\hat{n} \wedge H_{0C}) \cdot (\hat{n} \wedge (\hat{n} \wedge E)) &= (\hat{n} \wedge H) \cdot (\hat{n} \wedge (\hat{n} \wedge E_{0C})) \text{ on } S^+. \end{aligned} \quad (28)$$

Independently, in these circumstances, at interface of media with losses tending to infinity in the medium below the interface, we can assume a surface impedance boundary conditions on tangential components of fields [6], following,

$$\begin{aligned} (\hat{n}_+ \wedge (\hat{n}_+ \wedge E)) &= - (Z_0) (\hat{n}_+ \wedge H) \text{ on } S^+, \\ (\hat{n}_+ \wedge (\hat{n}_+ \wedge E_{0C})) &= - (Z_{0C}) (\hat{n}_+ \wedge H_{0C}) \text{ on } S^+, \end{aligned} \quad (29)$$

where \hat{n}_+ is the unit normal to S^+ , outward it. This implies, from (28),

$$(\hat{n} \wedge H) \cdot ((Z_{0C}) - {}^t(Z_0)) (\hat{n} \wedge H_{0C}) = 0 \text{ on } S^+, \quad (30)$$

where (Z_0) and (Z_{0C}) are tensors, assumed invertible and independent of fields (note : the cases with (Z_0) and (Z_{0C}) null or infinite can be considered in the limit sense).

From uniqueness, the tangential electric and magnetic field cannot be both null on a segment of S^+ , except if the fields vanish everywhere above the interface. We then deduce, from (28), (29), (30) and the free choice of primary sources exciting (E, H) and (E_{0C}, H_{0C}) above the interface, that we have the following link between respective surface impedances tensors attached to adjoint cases,

$$(Z_{0C}) = {}^t(Z_0). \quad (31)$$

3.4.3) Property 8 : impedance boundary conditions and holes in surface integrals

We note that, for fields in adjoint cases, on any smooth part of a surface S^c holding impedance boundary conditions, with the link (31) between impedances tensors, we can write,

$$\begin{aligned} (\hat{n}_1 \wedge (\hat{n}_1 \wedge E))|_{S^c} &= - (Z) (\hat{n}_1 \wedge H)|_{S^c}, \\ (\hat{n}_1 \wedge (\hat{n}_1 \wedge E_{0C}))|_{S^c} &= - {}^t(Z) (\hat{n}_1 \wedge H_{0C})|_{S^c}, \end{aligned} \quad (32)$$

where \hat{n}_1 is the unit normal to S^c , outward the domain with infinite losses. We then have,

$$\begin{aligned} [E_{0C} \cdot (\hat{n} \wedge H) + H_{0C} \cdot (\hat{n} \wedge E)]_{S^c} \\ = [{}^t(Z) (\hat{n}_1 \wedge H_{0C}) \cdot (\hat{n} \wedge H) - (\hat{n}_1 \wedge H_{0C}) \cdot (Z) (\hat{n} \wedge H)]_{S^c} = 0, \end{aligned} \quad (33)$$

and thus a hole in $\partial\Omega$, i.e. $\partial\Omega$ replaced by $\partial\Omega \setminus S^c$, in all surface integrals previously considered for fields in adjoint cases (note : this result is intimately related to the limit case where we have a part of the surface included in a medium for which we let losses tend to infinity).

4) Generalized reciprocity principle and dyadic tensors fields

4.1) Dirac sources in open space and reciprocity principle

Let the primary sources $J_0 = J'_0 \delta_{r_0}$ and $M_0 = M'_0 \delta_{r_0}$ attached to (E_0, H_0) , and the primary sources $J_{0C} = J'_{0C} \delta_{r_{0C}}$, $M_{0C} = M'_{0C} \delta_{r_{0C}}$ attached to (E_{0C}, H_{0C}) , satisfying (1) and (2), with respective adjoint characteristics $((\epsilon_0), (\xi_0), (\mu_0), (\varsigma_0))$ and $({}^t(\epsilon_0), -{}^t(\varsigma_0), {}^t(\mu_0), -{}^t(\xi_0))$ in $\Omega \equiv R^3$ (note : by definition $\delta_{r_v} \equiv \delta(r - r_v)$). We then consider the system as composed of a bianisotropic object holded by the domain Ω_0 , and of the open space $R^3 \setminus \Omega_0$ which holds the primary sources.

The generalized reciprocity principle (16) then applies, and the fields in adjoint cases are equivalent, in the sense that they simply derive each from other when the source and observation positions exchange, following,

$$\int_{R^3} (J_{0C} \cdot E_0 - M_{0C} \cdot H_0) dV = \int_{R^3} (E_{0C} \cdot J_0 - H_{0C} \cdot M_0) dV. \quad (34)$$

From now on, we will specify case by case the source terms in the arguments of fields.

4.2) Generalized reciprocity for co-dipolar dyadic tensors

We can define co-dipolar dyadic tensors, $[E_{0(C)}(r', r)]$ and $[H_{0(C)}(r', r)]$, in writing,

$$\begin{aligned} \int_{R^3} E_{0C}(J'_{0C} \delta_{r_{0C}}, M_{0C} = 0; r) \cdot J_0 dV &= J'_{0C} \cdot [E_{0C}(r_{0C}, r_0)] \cdot J'_0, \\ \int_{R^3} E_0(J'_0 \delta_{r_0}, M_0 = 0; r) \cdot J_{0C} dV &= J'_0 \cdot [E_0(r_0, r_{0C})] \cdot J'_{0C}, \\ \int_{R^3} H_{0C}(J_{0C} = 0, M'_{0C} \delta_{r_{0C}}; r) \cdot M_0 dV &= M'_{0C} \cdot [H_{0C}(r_{0C}, r_0)] \cdot M'_0, \\ \int_{R^3} H_0(J_0 = 0, M'_0 \delta_{r_0}; r) \cdot M_{0C} dV &= M'_0 \cdot [H_0(r_0, r_{0C})] \cdot M'_{0C}, \end{aligned} \quad (35)$$

and letting successively in (34), $M'_0 = M'_{0C} = 0$, then $J'_0 = J'_{0C} = 0$, we deduce that,

$$\begin{aligned} [E_{0C}(r', r)] &= {}^t[E_0(r, r')], \\ [H_{0C}(r', r)] &= {}^t[H_0(r, r')], \end{aligned} \quad (36)$$

where ${}^t[\mathcal{H}]$ is the transpose of $[\mathcal{H}]$, such that $U.[\mathcal{H}].W = W.{}^t[\mathcal{H}].U$ (note : for details on dyadic tensors, the reader can refer to the sections on cartesian (dyadic) tensors in [6]).

4.3) Generalized reciprocity for cross-dipolar dyadic tensors

In the same manner, we can define cross-dipolar dyadic tensors, $[E'_{0(C)}(r', r)]$ and $[H'_{0(C)}(r', r)]$, in writing,

$$\begin{aligned} \int_{R^3} E_{0C}(J_{0C} = 0, M'_{0C}\delta_{r_{0C}}; r).J_0 dV &= M'_{0C}.[E'_{0C}(r_{0C}, r_0)].J'_0, \\ \int_{R^3} H_0(J'_0\delta_{r_0}, M_0 = 0; r).M_{0C} dV &= J'_0.[H'_0(r_0, r_{0C})].M'_{0C}, \\ \int_{R^3} E_0(J_0 = 0, M'_0\delta_{r_0}; r).J_{0C} dV &= M'_0.[E'_0(r_0, r_{0C})].J'_{0C}, \\ \int_{R^3} H_{0C}(J'_{0C}\delta_{r_{0C}}, M_{0C} = 0; r).M_0 dV &= J'_{0C}.[H'_{0C}(r_{0C}, r_0)].M'_0, \end{aligned} \quad (37)$$

and, letting successively $J'_{0C} = M'_0 = 0$, then $J'_0 = M'_{0C} = 0$, directly derive from (34),

$$\begin{aligned} [E'_{0C}(r', r)] &= -{}^t[H'_0(r, r')], \\ [E'_0(r', r)] &= -{}^t[H'_{0C}(r, r')]. \end{aligned} \quad (38)$$

4.4) Dyadic tensorial integral expressions of fields and reciprocity

We can apply (35)-(36) and (37)-(38) to the dyadic tensorial expressions of fields. These formal expressions represent the fields influenced by arbitrary bounded primary sources J and M [6], following,

$$\begin{aligned} E_{0(C)}(J, M; r) &= \int_{\Omega_1} Z_0 J(r').[\underline{\mathcal{F}}^0_{he,t(C)}(r', r)] - M(r').[\underline{\mathcal{D}}^0_{eh,t(C)}(r', r)] dV_{r'}, \\ Z_0 H_{0(C)}(J, M; r) &= \int_{\Omega_1} Z_0 J(r').[\underline{\mathcal{D}}^0_{he,t(C)}(r', r)] + M(r').[\underline{\mathcal{F}}^0_{eh,t(C)}(r', r)] dV_{r'}, \end{aligned} \quad (39)$$

where Ω_1 is a domain holding J and M . We then directly deduce from reciprocity relations (35)-(36) and (37)-(38) that the dyadic tensors $[\underline{\mathcal{F}}^0_{he,t(C)}], [\underline{\mathcal{D}}^0_{eh,t(C)}], [\underline{\mathcal{D}}^0_{he,t(C)}]$ and $[\underline{\mathcal{F}}^0_{eh,t(C)}]$ verify,

$$\begin{aligned} [\underline{\mathcal{F}}_{he(,eh),t}^0(r_1, r'_1)] &= {}^t[\underline{\mathcal{F}}_{he(,eh),tC}^0(r'_1, r_1)], \\ [\underline{\mathcal{D}}_{he(,eh),t}^0(r_1, r'_1)] &= {}^t[\underline{\mathcal{D}}_{eh(,he),tC}^0(r'_1, r_1)], \end{aligned} \quad (40)$$

where the index C corresponds to the choice of media with adjoint characteristics.

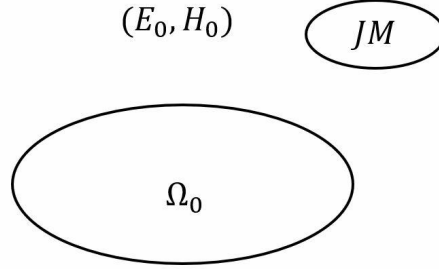


figure 1 : radiation in presence of unperturbed object

5) Fields influenced by a perturbation A of a bianisotropic object

5.1) Definitions of fields

As previously, (E_0, H_0) is the field radiated by primary sources in presence of a bianisotropic object held by Ω_0 , that we now call the unperturbed system. We then consider (E_{1S}^0, H_{1S}^0) the field in presence of a scatterer A inserted in this initial system, and define S as a closed surface enclosing A and limiting the simply connected domain Ω_S (no hole in it). We will note (E_{1S}, H_{1S}) the field identical to (E_{1S}^0, H_{1S}^0) except inside S , radiated by primary sources outside S , and by equivalent sources, enclosing A but inside S , in presence of the unperturbed system attached to (E_0, H_0) .

5.2) Definition of primary sources

The primary sources, for the fields (E_0, H_0) , (E_{1S}^0, H_{1S}^0) and (E_{1S}, H_{1S}) , are assumed to be the electric and magnetic dipolar sources positioned at $r_v \in R^3 \setminus \overline{\Omega_S}$,

$$J_v \delta_{r_v} \text{ and } M_v \delta_{r_v} \text{ with } \delta_{r_v}(r) \equiv \delta(r - r_v), \quad (41)$$

where δ is the Dirac function [6]. For media with adjoint characteristics, the primary sources, for the fields (E_{0C}, H_{0C}) , (E_{1SC}^0, H_{1SC}^0) and (E_{1SC}, H_{1SC}) , are assumed to be

the electric and magnetic dipolar sources positioned at $r' \in R^3 \setminus \Omega_S$,

$$J' \delta_{r'} \text{ and } M' \delta_{r'} \text{ with } \delta_{r'} = \delta(r - r'). \quad (42)$$

5.3) Integral equalities for (E_{1S}, H_{1S}) and (E_{1S}^0, H_{1S}^0) on S

Let us consider that the field (E_{1S}^0, H_{1S}^0) on S is known. We can write from the generalized integral reciprocity theorem in open space (15),

$$\begin{aligned} & \int_{R^3 \setminus \Omega_S} J' \delta_{r'} \cdot E_{1S} - M' \delta_{r'} \cdot H_{1S} dV_r - \int_{R^3 \setminus \Omega_S} J_v \delta_{r_v} \cdot E_{0C} - M_v \delta_{r_v} \cdot H_{0C} dV_r \\ &= \int_S E_{0C}(r) \cdot (\hat{n} \wedge H_{1S}^0(r)) + H_{0C}(r) \cdot (\hat{n} \wedge E_{1S}^0(r)) dS_r, \\ & \int_{R^3 \setminus \Omega_S} J' \delta_{r'} \cdot E_0 - M' \delta_{r'} \cdot H_0 dV_r - \int_{R^3 \setminus \Omega_S} J_v \delta_{r_v} \cdot E_{0C} - M_v \delta_{r_v} \cdot H_{0C} dV_r = 0, \end{aligned} \quad (43)$$

where \hat{n} is the outward unit normal to $\partial\Omega_S \equiv S$. By definition, we have,

$$(E_{1S}^0, H_{1S}^0) \equiv (E_{1S}, H_{1S}) \text{ on and outside } S, \quad (44)$$

with Ω_S containing A , while (E_0, H_0) and (E_{0C}, H_{0C}) are the unperturbated fields, in adjoint cases, in whole space including $\overline{\Omega}_0$. The domain $\Omega = R^3 \setminus \Omega_S$ holds the primary sources and a part of the unperturbated system.

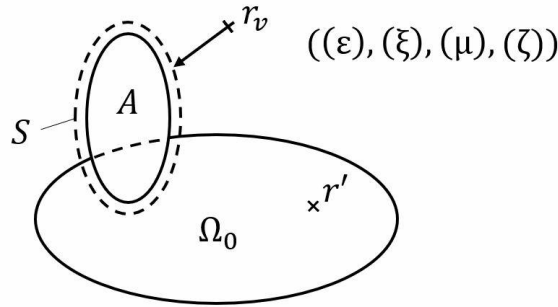


figure 2 : Radiation in presence of the system perturbed by A

5.4) Integral expressions for (E_{1S}, H_{1S}) from fields known on S

We now use (43) and present integral expressions for (E_{1S}, H_{1S}) , outside and on S (up to S) in bianisotropic media, from (E_{1S}, H_{1S}) known on S .

5.4.1) Definition of indicator function

Let Ω_a be an open simply connected domain, Ω_S the open domain limited by S , and $\Omega \equiv \Omega_a \setminus \Omega_S$. We can write, for $J'.E_{1S}$ continuous at point $r' \in \Omega$,

$$\int_{\Omega} J' \delta_{r'} . E_{1S} dV_r = 1_{\Omega}(r') J' . E_{1S} \text{ with } 1_{\Omega}(r') = \int_{\Omega} \delta(r' - r) dV_r. \quad (45)$$

The function $1_{\Omega}(r')$ is the indicator (also said characteristic) function of the domain $\Omega \equiv \Omega_a \setminus \Omega_S$, equal to 1 when $r' \in \Omega_a \setminus \overline{\Omega_S}$, equal to 0 when $r' \notin \overline{\Omega_a} \setminus \Omega_S$, and generally fractional when $r' \in S$. Letting Ω'_ϵ the ball of radius ϵ centered at $r' \in \Omega$, this function verifies (see (131)-(133) in appendix),

$$\begin{aligned} 1_{\Omega}(r') &= \lim_{\epsilon \rightarrow 0} \int_{\Omega} \text{div} \left(\frac{-1_{R^3 \setminus \Omega_\epsilon}}{4\pi|r' - r|} \right) dV_r = \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi\epsilon^2} \int_{\partial\Omega'_\epsilon \cap \Omega} dS, \\ 1_{\Omega}(r') &= \frac{1}{4\pi} pv \int_{\partial\Omega} \hat{n} . \text{grad}_r \left(\frac{1}{|r - r'|} \right) dS_r = \frac{1}{4\pi} pv \int_{\partial\Omega} \frac{\hat{n} . (r' - r)}{|r' - r|^3} dS_r, \end{aligned} \quad (46)$$

where \hat{n} is the normal to $\partial\Omega$ directed inside Ω . We then note that $4\pi 1_{R^3 \setminus \Omega_S}(r')$ is the solid angle, locally view from r' , of the free space domain limited by S (note : on regular part of S , $1_{\Omega}(r') = \frac{1}{2}$).

5.4.2) Integral expressions for E_{1S} and H_{1S} outside and on S , for finite losses

From the definitions of fields (E_{1S}, H_{1S}) , we can assume that $J' . (E_{1S} - E_0)$ is continuous in the domain $R^3 \setminus \Omega_S$ (up to S) for $J' = J_s$, where J_s is defined as an arbitrary vector, except on possible lines of discontinuities of E_{1S} on S , along which we define J_s as a vector locally tangent to them (note : we then take account of the property that, on any smooth part of lines of discontinuities, the fields component along it is continuous as we cross it).

Letting $\Omega \equiv R^3 \setminus \Omega_S$, we can then rewrite (43) with the use of (45) and (34), when $J' = J_s$ and $M' = M_v = 0$, and obtain, as $r' \in \Omega$ (outside and on S) and $r_v \in R^3 \setminus \overline{\Omega_S}$,

$$\begin{aligned} 1_{\Omega}^c(r') J' . E_{1S}(J_v \delta_{r_v}, M = 0; r') - J' . E_0(J_v \delta_{r_v}, M = 0; r') &= \\ &= \text{FP} \int_S E_{0C}(r) . (\hat{n} \wedge H_{1S}(r)) + H_{0C}(r) . (\hat{n} \wedge E_{1S}(r)) dS_r \\ &= \text{FP} \int_S (H_{0C}(r) \wedge \hat{n}) . E_{1S}(r) dS_r + \text{FP} \int_S (E_{0C}(r) \wedge \hat{n}) . H_{1S}(r) dS_r, \end{aligned} \quad (47)$$

where

$$1_{\Omega}^c(r') \equiv 1_{\Omega}(r') \text{ when the losses in the unperturbed system remains finite,} \quad (48)$$

the vector \hat{n} is the outward unit normal to $\partial\Omega_S \equiv S$, and FP is the finite part of integrals [7]. The FP is an operator regularizing integrals with singular kernels (see details in appendix). In our case, these kernels are $E_{0C} \wedge \hat{n}$ and $H_{0C} \wedge \hat{n}$ depending of r and r' , which can be singular when $r' \rightarrow S$.

In a similar manner, we can consider an integral expression for H_{1S} outside and on S . We then assume $M' \cdot (H_{1S} - H_0)$ continuous in $R^3 \setminus \Omega_S$ for $M' = M_s$, where M_s is defined as an arbitrary vector, except on possible lines of discontinuities of H_{1S} on S , on which we define M_s as locally tangent to them. As previously for E_{1S} , we then rewrite (43) with the use of (45) and (34), and derive for H_{1S} , when $M' = M_s$ and $J' = J_v = 0$,

$$\begin{aligned} & 1_{\Omega}^c(r') M' \cdot Z_0 H_{1S}(J = 0, M_v \delta_{r_v}; r') - M' \cdot Z_0 H_0(J = 0, M_v \delta_{r_v}; r') = \\ & = -\text{FP} \int_S E_{0C}(r) \cdot (\hat{n} \wedge H_{1S}(r)) dS_r + H_{0C}(r) \cdot (\hat{n} \wedge E_{1S}(r)) dS_r \\ & = -\text{FP} \int_S (E_{0C}(r) \wedge \hat{n}) \cdot H_{1S}(r) dS_r - \text{FP} \int_S (H_{0C}(r) \wedge \hat{n}) \cdot E_{1S}(r) dS_r. \end{aligned} \quad (49)$$

5.4.3) Integral expressions outside and on S , taking account of infinite losses

We modify (47) to extend its application to the case where S contains a subdomain of the unperturbed system with infinite losses whose surface partly coincides with S . This case is encountered, for example, if we want to consider an infinite plane with impedance boundary conditions, pertubated by a cavity inserted in it. For that, we begin by analysing some particular properties of kernels.

We note that, taking $J_s - (J_s \cdot \hat{n}_d(r')) \hat{n}_d(r')$ (resp. $(J_s \cdot \hat{n}_d(r')) \hat{n}_d(r')$) instead of J' with $\hat{n}_d(r')$ a function coinciding with \hat{n} for $r' \in S$, we can generally assume in (47) that finite part of the integral with $(E_{0C}(r) \wedge \hat{n})$ (resp. $(H_{0C}(r) \wedge \hat{n})$) kernel is continuous at r' , up to S included (note : FP generally implies boundedness but not necessarily the continuity up to S included).

Therefore, we can extend and slightly modify (47) to take account of a part S^c of S , piecewise smooth, coinciding with the boundary of a subdomain of the unperturbed system with infinite losses. We then assume impedance boundary conditions, following,

$$(\hat{n} \wedge (\hat{n} \wedge E_{0C})) = -(Z_{0C})(\hat{n} \wedge H_{0C}) \text{ on } S^c \subset S, \quad (50)$$

as in (29). In these circumstances, the finite part of integral with kernel $(\hat{n} \wedge H_{0C})$ (resp. $(\hat{n} \wedge E_{0C})$) is continuous as the one with $(\hat{n} \wedge E_{0C})$ (resp. $(\hat{n} \wedge H_{0C})$) is, when we

consider $J' = J_s - (J_s \cdot \hat{n}_d(r')) \hat{n}_d(r')$ (resp. $J' = (J_s \cdot \hat{n}_d(r')) \hat{n}_d(r')$) on S^c , so that we can extend the domain of continuity up to S^c (note : this behaviour is also related to the presence of an image of $J' \delta_{r'}$ relatively to S^c). Therefore, for the extension of (47) to this case, we then have to complete the definition of $1_\Omega^c(r')$, following,

$$\begin{aligned} 1_\Omega^c(r') &\equiv 1_\Omega(r') \text{ as } r' \notin \overline{S^c}, \\ 1_\Omega^c(r') &\equiv 1 \text{ as } r' \in S^c \setminus \partial S^c \text{ with } J' = J_s, \end{aligned} \quad (51)$$

and in (49),

$$\begin{aligned} 1_\Omega^c(r') &\equiv 1_\Omega(r') \text{ as } r' \notin \overline{S^c}, \\ 1_\Omega^c(r') &\equiv 1 \text{ as } r' \in S^c \setminus \partial S^c \text{ with } M' = M_s. \end{aligned} \quad (52)$$

where S^c is a piecewise smooth subdomain of S , where the fields (E_{0C}, H_{0C}) (and thus also (E_0, H_0)) for the unperturbed system, satisfy impedance boundary conditions (note : $1_\Omega^c(r')$ on ∂S^c depends on the media configuration of the unperturbed system on this frontier, but in practice, we often consider problems with $\partial S^c \equiv 0$ or $\partial S^c \rightarrow \infty$).

5.4.4) General integral tensorial expressions of (E_{1S}, H_{1S})

We can use the dyadic tensors expressions (39), with (40), to express (47) and (49) in a tensorial form. Considering (47), we have from (39),

$$E_{0C}(r) = Z_0 J' \cdot [\underline{\mathcal{F}}_{he,tC}^0(r', r)], H_{0C}(r) = J' \cdot [\underline{\mathcal{D}}_{he,t(C)}^0(r', r)], \quad (53)$$

and (47) can be rewritten,

$$\begin{aligned} 1_\Omega^c(r') J' \cdot E_{1S}(J_v \delta_{r_v}, M = 0; r') - J' \cdot E_0(J_v \delta_{r_v}, M = 0; r') &= \\ = \text{FP}(J' \cdot \int_S [\underline{\mathcal{F}}_{he,tC}^0(r', r)] \cdot Z_0 J_{es}(r) - [\underline{\mathcal{D}}_{he,tC}^0(r', r)] \cdot M_{es}(r) dS_r) &= \\ = \text{FP}(J' \cdot \int_S Z_0 J_{es}(r) \cdot ([\underline{\mathcal{F}}_{he,t}^0(r, r')]) - M_{es}(r') \cdot ([\underline{\mathcal{D}}_{eh,t}^0(r, r')])) dS_r, \end{aligned} \quad (54)$$

with,

$$\begin{aligned} J_{es}(r) &= \hat{n} \wedge H_{1S}(J_v \delta_{r_v}, M = 0; r), \\ M_{es}(r) &= -\hat{n} \wedge E_{1S}(J_v \delta_{r_v}, M = 0; r), \end{aligned} \quad (55)$$

while, for (49), we can write from (39),

$$E_{0C}(r) = -M' \cdot [\underline{\mathcal{D}}_{eh,t(C)}^0(r', r)], H_{0C}(r) = M' \cdot [\underline{\mathcal{F}}_{eh,t(C)}^0(r', r)] / Z_0, \quad (56)$$

and obtain,

$$\begin{aligned}
& 1_{\Omega}^c(r')M'.Z_0H_{1S}(J=0, M_v\delta_{r_v}; r') - M'.Z_0H_0(J=0, M_v\delta_{r_v}; r') = \\
& = \text{FP}(M' \cdot \int_S [\underline{\mathcal{D}}_{eh,tC}^0(r', r)] \cdot Z_0J_{ms}(r) + [\underline{\mathcal{F}}_{eh,tC}^0(r', r)] \cdot M_{ms}(r) dS_r) \\
& = \text{FP}(M' \cdot \int_S Z_0J_{ms}(r) \cdot [\underline{\mathcal{D}}_{he,t}^0(r, r')] + M_{ms}(r) \cdot [\underline{\mathcal{F}}_{eh,t}^0(r, r')] dS_r), \tag{57}
\end{aligned}$$

with,

$$\begin{aligned}
J_{ms}(r) &= \hat{n} \wedge H_{1S}(J=0, M_v\delta_{r_v}; r), \\
M_{ms}(r) &= -\hat{n} \wedge E_{1S}(J=0, M_v\delta_{r_v}; r), \tag{58}
\end{aligned}$$

where J' and M' are defined as previously for (47) and (49), and r_v is strictly exterior to S . As a direct consequence, we note that we have,

$$\begin{aligned}
& 1_{\Omega}^c(r')J'.Z_0H_{1S}(J_v\delta_{r_v}, M=0; r') - J'.Z_0H_0(J_v\delta_{r_v}, M=0; r') \\
& = \text{FP}(J' \cdot \int_S [\underline{\mathcal{D}}_{eh,tC}^0(r', r)] \cdot Z_0J_{es}(r) + [\underline{\mathcal{F}}_{eh,tC}^0(r', r)] \cdot M_{es}(r) dS_r) \\
& = \text{FP}(J' \cdot \int_S Z_0J_{es}(r) \cdot [\underline{\mathcal{D}}_{he,t}^0(r, r')] + M_{es}(r) \cdot [\underline{\mathcal{F}}_{eh,t}^0(r, r')] dS_r), \tag{59}
\end{aligned}$$

and,

$$\begin{aligned}
& 1_{\Omega}^c(r')M'.E_{1S}(J=0, M_v\delta_{r_v}; r') - M'.E_0(J=0, M_v\delta_{r_v}; r') \\
& = \text{FP}(M' \cdot \int_S [\underline{\mathcal{F}}_{he,tC}^0(r', r)] \cdot Z_0J_{ms}(r) - [\underline{\mathcal{D}}_{he,tC}^0(r', r)] \cdot M_{ms}(r) dS_r) \\
& = \text{FP}(M' \cdot \int_S Z_0J_{ms}(r) \cdot ([\underline{\mathcal{F}}_{he,t}^0(r, r')]) - M_{ms}(r) \cdot ([\underline{\mathcal{D}}_{eh,t}^0(r, r')])) dS_r). \tag{60}
\end{aligned}$$

Remark : It is worth noticing that, we can take, for the definition of $J_{es(ms)}$ and $M_{es(ms)}$, (E_{1S}^0, H_{1S}^0) in place of (E_{1S}, H_{1S}) on S , i.e.

$$J_{es(ms)} = \hat{n} \wedge H_{1S}^0, \quad M_{es(ms)} = -\hat{n} \wedge E_{1S}^0, \tag{61}$$

but also, from (21), if r_v and $r' \notin S$ (we then exclude S),

$$J_{es(ms)} = \hat{n} \wedge (H_{1S} - H_0), \quad M_{es(ms)} = -\hat{n} \wedge (E_{1S} - E_0). \tag{62}$$

Remark : In our expressions of fields, we can let S be composed by piecewise continuous open parts S_i , $i = 1, \dots, n$, where E and H are continuous on S_i except possibly on its contour \mathcal{L}_i where only $\hat{t}.E$ and $\hat{t}.H$ are continuous across \mathcal{L}_i , with \hat{t} the unit tangential vector along \mathcal{L}_i .

Remark : Let us consider an impedance surface S^c of the unperturbed system outside the surface S enclosing A . If we deform S to be just in contact with S^c , we can then reduce S to an open surface $S \setminus S^c$ in the surface integral expressions.

5.5) Definition of influence tensors from $E_{1S} - E_0$ and $H_{1S} - H_0$

Let r_v and $r' \notin S$, so that we can take $1_Q^c(r') = 1$ and $FP \equiv \text{Identity}$ in previous equations. We now define, for the perturbation of the field given by the equivalent sources on S , the influence tensors $[\underline{\mathcal{F}}_{c,S}^1]$, $[\underline{\mathcal{F}}_{m,S}^1]$, $[\underline{\mathcal{D}}_{m,S}^1]$, and $[\underline{\mathcal{D}}_{c,S}^1]$, as follows,

$$\begin{aligned} Z_0 J_v \cdot [\underline{\mathcal{F}}_{c,S}^1(r_v, r)] &= (E_{1S} - E_0)(J = J_v \delta_{r_v}, M = 0; r), \\ -M_v \cdot [\underline{\mathcal{D}}_{m,S}^1(r_v, r)] &= (E_{1S} - E_0)(J = 0, M = M_v \delta_{r_v}; r), \\ M_v \cdot [\underline{\mathcal{F}}_{m,S}^1(r_v, r)] &= (Z_0 H_{1S} - Z_0 H_0)(J = 0, M = M_v \delta_{r_v}; r), \\ Z_0 J_v \cdot [\underline{\mathcal{D}}_{c,S}^1(r_v, r)] &= (Z_0 H_{1S} - Z_0 H_0)(J = J_v \delta_{r_v}, M = 0; r), \end{aligned} \quad (63)$$

where J and M are electrical and magnetic sources. In the same manner, we define (E_{1SC}, H_{1SC}) , $[\underline{\mathcal{F}}_{c(m),SC}^1]$ and $[\underline{\mathcal{D}}_{c(m),SC}^1]$ for the adjoint system. The application of the generalized principle of reciprocity (16) then gives us,

$$\begin{aligned} J' \cdot [\underline{\mathcal{F}}_{c,SC}^1(r', r_v)] \cdot J_v &= J_v \cdot [\underline{\mathcal{F}}_{c,S}^1(r_v, r')] \cdot J', [\underline{\mathcal{F}}_{c,SC}^1(r', r)] = {}^t[\underline{\mathcal{F}}_{c,S}^1(r, r')], \\ M' \cdot [\underline{\mathcal{F}}_{m,SC}^1(r', r_v)] \cdot M_v &= M_v \cdot [\underline{\mathcal{F}}_{m,S}^1(r_v, r')] \cdot M', [\underline{\mathcal{F}}_{m,SC}^1(r', r)] = {}^t[\underline{\mathcal{F}}_{m,S}^1(r, r')], \\ M' \cdot [\underline{\mathcal{D}}_{m,SC}^1(r', r_v)] \cdot J_v &= J_v \cdot [\underline{\mathcal{D}}_{c,S}^1(r_v, r')] \cdot M', [\underline{\mathcal{D}}_{m,SC}^1(r', r)] = {}^t[\underline{\mathcal{D}}_{c,S}^1(r, r')]. \end{aligned} \quad (64)$$

In other respects, concerning the system perturbed by A' bounded by S' in place of A bounded by S , we have equivalent properties and definitions with S replaced by S' .

6) Fields influenced by coupling between two scatterers A and A'

We consider the coupling of two scatterers A and A' limited by disjoint surfaces S and S' inserted in a complex system, composed of bianisotropic media and metamaterials. The scatterers A and A' can be considered as complex local perturbations of characteristics of the initial system. The field $(E_{1S(C)}, H_{1S(C)})$ is defined as previously for the system without A' , and $(E_{1S'(C)}, H_{1S'(C)})$ in a similar manner for the system without A .

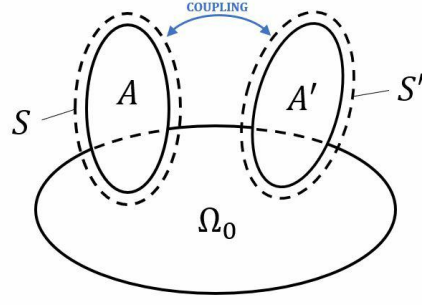


figure 3 : presence of perturbations A and A'

6.1) Definitions of first order coupling terms $E_{S,S'}$ and $H_{S,S'}$

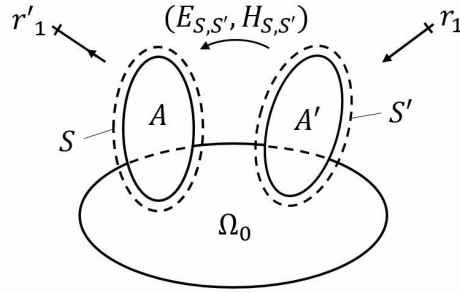


figure 4 : coupling influence of A' on A

We now consider the system that consists of two scatterers A and A' that can be partly inserted in an imperfectly conducting object in Ω_0 , illuminated by primary electric and magnetic dipoles at r_1 ,

$$J = J_1 \delta_{r_1} \text{ and } M = M_1 \delta_{r_1} \text{ with } r_1 \text{ strictly outside } S \text{ and } S', \quad (65)$$

while, for adjoint characteristics of media, we consider illumination by primary electric and magnetic dipoles at r'_1 ,

$$J = J'_1 \delta_{r'_1} \text{ and } M = M'_1 \delta_{r'_1} \text{ with } r'_1 \text{ strictly outside } S \text{ and } S'. \quad (66)$$

The sources $J_{1S'} = \hat{n} \wedge H_{1S'}^0(r'_{S'}) \delta_{S'}$ and $M_{1S'} = -\hat{n} \wedge E_{1S'}^0(r'_{S'}) \delta_{S'}$ influence A , and we define (E_2, H_2) as the field in presence of $J_{1S'}$ and $M_{1S'}$, of J and M , and of A partly inserted in Ω_0 , but without A' .

We can say that the field (E_2, H_2) is partial since it contains a term of coupling of A' on A , that we name $(E_{S,S'}, H_{S,S'})$, but no terms of coupling of A on A' . Thus, we can write,

$$\begin{aligned} (E_{S,S'}, H_{S,S'}) &= ((E_2, H_2) - (E_{1S}, H_{1S})) \\ &\quad - ((E_{1S'}, H_{1S'}) - (E_0, H_0)), \end{aligned} \quad (67)$$

where the notation $(U_0, V_0) - (U_1, V_1) = (U_0 - U_1, V_0 - V_1)$ is used. The total term of coupling will be,

$$(E_{ctot}, H_{ctot}) = (E_{S,S'}, H_{S,S'}) + (E_{S',S}, H_{S',S}). \quad (68)$$

From now on, to simplify, a field U at r'_1 for primary sources $J = J_1\delta_{r_1}$ and $M = M_1\delta_{r_1}$ at r_1 , will be denoted $U(r_1, r'_1)$, or $U(J_1\delta_{r_1}, M_1\delta_{r_1}; r'_1)$, or $U(J_1\delta_{r_1}; r'_1)$ when $M = 0$, or $U(M_1\delta_{r_1}; r'_1)$ when $J = 0$.

6.2) Expressions of $(E_{S,S'}, H_{S,S'})$ for $M'_1 = M_1 = 0$ and for $J'_1 = J_1 = 0$

To express $E_{S,S'}$ and $H_{S,S'}$, we use two times the generalized integral reciprocity equation (11) for dipoles sources at points r_1 and r'_1 , strictly outside of S and S' . In what follows, we note Ω' the domain outside of S' . Coupling expressions and coupling tensors, for the fields due to couplings, will be respectively determined for $M'_1 = M_1 = 0$ and for $J'_1 = J_1 = 0$.

6.2.1) General combinations of integrals

The systems for (E_2, H_2) and (E_{1S}^0, H_{1S}^0) are identical in the domain Ω' , outside and on S' , while the field (E_{1S}, H_{1S}) is identical to (E_{1S}^0, H_{1S}^0) outside and on S . We can then use (11), and, for r_1 and r'_1 strictly outside S , we can write,

$$\begin{aligned} &\int_{\Omega'} (J'_1\delta(r' - r'_1)E_2(r_1, r') - M'_1\delta(r' - r'_1)H_2(r_1, r') \\ &\quad - J_1\delta(r' - r_1)E_{1SC}(r'_1, r') + M_1\delta(r' - r_1)H_{1SC}(r'_1, r'))dV_{r'} \\ &= \int_{S'} (E_{1SC})(r'_1, r').(\hat{n} \wedge H_2(r_1, r')) \\ &\quad + (H_{1SC})(r'_1, r').(\hat{n} \wedge E_2(r_1, r'))dS_{r'}, \end{aligned} \quad (69)$$

In other respects, the systems for $(E_{1S'}, H_{1S'})$ and (E_0, H_0) are identical in the domain Ω' , outside and on S' , so that we have,

$$\begin{aligned}
& \int_{\Omega'} (J'_1 \delta(r' - r'_1) E_{1S'}(r_1, r') - M'_1 \delta(r' - r'_1) H_{1S'}(r_1, r')) \\
& - J_1 \delta(r' - r_1) E_{0C}(r'_1, r') + M_1 \delta(r' - r_1) H_{0C}(r'_1, r')) dV_{r'} \\
& = \int_{S'} ((E_{0C})(r'_1, r') \cdot (\hat{n} \wedge H_{1S'}(r_1, r')) \\
& + (H_{0C})(r'_1, r') \cdot (\hat{n} \wedge E_{1S'}(r_1, r'))) dS_{r'}, \tag{70}
\end{aligned}$$

where the lower index C means that the parameters of the systems are adjoint, and \hat{n} is the outward unit normal to S' . Since $(E_2 - E_{1S'}, H_2 - H_{1S'})$ and (E_{1SC}, H_{1SC}) correspond to systems adjoint each other, and without sources, inside and on S' , we can consider (23), and use that,

$$\int_{S'} E_{1SC} \cdot (\hat{n} \wedge (H_2 - H_{1S'})) + H_{1SC} \cdot (\hat{n} \wedge (E_2 - E_{1S'})) dS_{r'} = 0, \tag{71}$$

and thus replace $(\hat{n} \wedge H_2, -\hat{n} \wedge E_2)$ by $(\hat{n} \wedge H_{1S'}, -\hat{n} \wedge E_{1S'})$ in (69) on S' . We then obtain after subtraction,

$$\begin{aligned}
& \int_{\Omega'} (J'_1 \delta_{r'_1} (E_2 - E_{1S'})(r_1, r') - M'_1 \delta_{r'_1} (H_2 - H_{1S'})(r_1, r')) \\
& - J_1 \delta_{r_1} (E_{1SC} - E_{0C})(r'_1, r') + M_1 \delta_{r_1} (H_{1SC} - H_{0C})(r'_1, r')) dV_{r'} \\
& = \int_{S'} ((E_{1SC} - E_{0C})(r'_1, r') \cdot (\hat{n} \wedge H_{1S'}(r_1, r')) \\
& + (H_{1SC} - H_{0C})(r'_1, r') \cdot (\hat{n} \wedge E_{1S'}(r_1, r'))) dS_{r'}, \tag{72}
\end{aligned}$$

where we suppose known $(\hat{n} \wedge (E_{1SC} - E_{0C}))$, $(\hat{n} \wedge (H_{1SC} - H_{0C}))$, $(\hat{n} \wedge H_{1S'})$, and $(\hat{n} \wedge E_{1S'})$ on S' (note : $U \cdot (\hat{n} \wedge W) = -(\hat{n} \wedge (\hat{n} \wedge U)) \cdot (\hat{n} \wedge W)$ for any U, W).

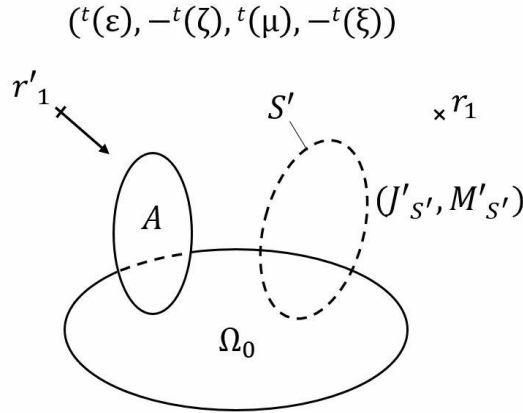


figure 5 : $J'_{S'} = (\hat{n} \wedge (H_{1SC} - H_{0C}))$ and $M'_{S'} = -(\hat{n} \wedge (E_{1SC} - E_{0C}))$ on S' for primary source at r'_1

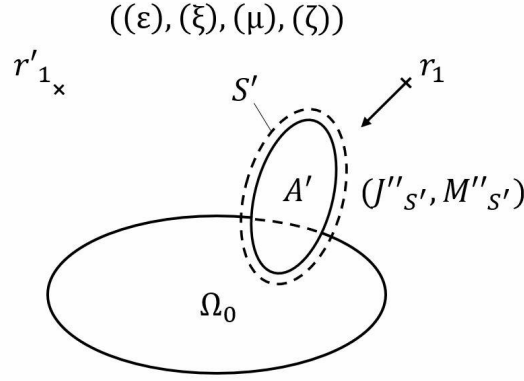


figure 6 : $J''_{S'} = (\hat{n} \wedge H_{1S'})$ and $M''_{S'} = -(\hat{n} \wedge E_{1S'})$ on S' for primary source at r_1

6.2.2) Expressions of $(E_{S,S'}, H_{S,S'})$ for $M'_1 = M_1 = 0$

Taking $M'_1 = M_1 = 0$ in (72), we can then write,

$$\begin{aligned} & J'_1 \cdot (E_2 - E_{1S'}) (J_1 \delta_{r_1}; r'_1) - J_1 \cdot (E_{1SC} - E_{0C}) (J'_1 \delta_{r'_1}; r_1) \\ &= \int_{S'} (E_{1SC} - E_{0C}) (J'_1 \delta_{r'_1}; r') \cdot J'_{es', J_1 \delta_{r_1}}(r') \\ & - (H_{1SC} - H_{0C}) (J'_1 \delta_{r'_1}; r') \cdot M'_{es', J_1 \delta_{r_1}}(r') dS_{r'} \end{aligned} \quad (73)$$

for r_1 and r'_1 strictly outside S and S' , where

$$\begin{aligned} & J'_{es'} = \hat{n} \wedge H_{1S'}|_{S'} \text{ and } M'_{es'} = -\hat{n} \wedge E_{1S'}|_{S'}, \\ & \text{or, } J'_{es'} = \hat{n} \wedge (H_{1S'} - H_0)|_{S'} \text{ and } M'_{es'} = -\hat{n} \wedge (E_{1S'} - E_0)|_{S'}. \end{aligned} \quad (74)$$

Using reciprocity, we have,

$$J_1 \cdot (E_{1SC} - E_{0C}) (J'_1 \delta_{r'_1}; r_1) = J'_1 \cdot (E_{1S} - E_0) (J_1 \delta_{r_1}; r'_1), \quad (75)$$

and thus, from (73) and (67), derive,

$$\begin{aligned} & J'_1 \cdot E_{S,S'} (J_1 \delta_{r_1}, r'_1) \\ &= \int_{S'} (E_{1SC} - E_{0C}) (J'_1 \delta_{r'_1}; r') \cdot J'_{es', J_1 \delta_{r_1}}(r') \\ & - (H_{1SC} - H_{0C}) (J'_1 \delta_{r'_1}; r') \cdot M'_{es', J_1 \delta_{r_1}}(r') dS_{r'}, \end{aligned} \quad (76)$$

for r_1 and r'_1 strictly outside S and S' . We can then write, using notations given in (63)-(64),

$$\begin{aligned}
& E_{S,S'}(J_1\delta_{r_1}, M_1\delta_{r_1} = 0; r'_1) \\
&= \int_{S'} [\underline{\mathcal{F}}_{c,SC}^1(r'_1, r')] \cdot Z_0 J'_{es', J_1\delta_{r_1}}(r') - [\underline{\mathcal{D}}_{c,SC}^1(r'_1, r')] \cdot M'_{es', J_1\delta_{r_1}}(r') dS_{r'} \\
&= \int_{S'} Z_0 J'_{es', J_1\delta_{r_1}}(r') \cdot [\underline{\mathcal{F}}_{c,S}^1(r', r'_1)] - M'_{es', J_1\delta_{r_1}}(r') [\underline{\mathcal{D}}_{m,S}^1(r', r'_1)] dS_{r'}, \tag{77}
\end{aligned}$$

and the associated magnetic field, following,

$$\begin{aligned}
& Z_0 H_{S,S'}(J_1\delta_{r_1}, M_1\delta_{r_1} = 0; r'_1) \\
&= \int_{S'} [\underline{\mathcal{D}}_{m,SC}^1(r'_1, r')] \cdot Z_0 J'_{es', J_1\delta_{r_1}}(r') + [\underline{\mathcal{F}}_{m,SC}^1(r'_1, r')] \cdot M'_{es', J_1\delta_{r_1}}(r') dS_{r'} \\
&= \int_{S'} Z_0 J'_{es', J_1\delta_{r_1}}(r') \cdot [\underline{\mathcal{D}}_{c,S}^1(r', r'_1)] + M'_{es', J_1\delta_{r_1}}(r') [\underline{\mathcal{F}}_{m,S}^1(r', r'_1)] dS_{r'}. \tag{78}
\end{aligned}$$

6.2.3) Expressions of $(E_{S,S'}, H_{S,S'})$ for $J'_1 = J_1 = 0$

In the same manner, taking $J'_1 = J_1 = 0$ in (72), we have,

$$\begin{aligned}
& -M'_1 \cdot (H_2 - H_{1S'}) (M_1\delta_{r_1}; r'_1) + M_1 \cdot (H_{1SC} - H_{0C}) (M'_1\delta_{r'_1}; r_1) \\
&= \int_{S'} (E_{1SC} - E_{0C}) (M'_1\delta_{r'_1}; r') \cdot J'_{es', M_1\delta_{r_1}}(r') \\
& - (H_{1SC} - H_{0C}) (M'_1\delta_{r'_1}; r') \cdot M'_{es', M_1\delta_{r_1}}(r') dS_{r'}, \tag{79}
\end{aligned}$$

and thus, from generalized reciprocity principle,

$$\begin{aligned}
& M'_1 \cdot (H_2(r_1, r'_1) - H_{1S'}(r_1, r'_1)) - M'_1 (H_{1S}(r_1, r'_1) - H_0(r_1, r'_1)) \\
&= - \int_{S'} (E_{1SC} - E_{0C}) (M'_1\delta_{r'_1}; r') \cdot J'_{es', M_1\delta_{r_1}}(r') \\
& - (H_{1SC} - H_{0C}) (M'_1\delta_{r'_1}; r') \cdot M'_{es', M_1\delta_{r_1}}(r') dS_{r'}, \tag{80}
\end{aligned}$$

which implies

$$\begin{aligned}
& M'_1 \cdot H_{S,S'}(J_1\delta_{r_1} = 0, M_1\delta_{r_1}; r'_1) \\
&= - \int_{S'} (E_{1SC} - E_{0C}) (M'_1\delta_{r'_1}; r') \cdot J'_{es', M_1\delta_{r_1}}(r') \\
& - (H_{1SC} - H_{0C}) (M'_1\delta_{r'_1}; r') \cdot M'_{es', M_1\delta_{r_1}}(r') dS_{r'}, \tag{81}
\end{aligned}$$

for r_1 and r'_1 strictly outside S and S' . We can then write, using the notations given in (63)-(64),

$$\begin{aligned}
& Z_0 H_{S,S'}(J_1 \delta_{r_1} = 0, M_1 \delta_{r_1}; r'_1) \\
&= \int_{S'} [\underline{\mathcal{D}_{m,SC}^1}(r'_1, r')] \cdot Z_0 J'_{es', M_1 \delta_{r_1}}(r') + [\underline{\mathcal{F}_{m,SC}^1}(r'_1, r')] \cdot M'_{es', M_1 \delta_{r_1}}(r') dS_{r'} \\
&= \int_{S'} Z_0 J'_{es', M_1 \delta_{r_1}}(r') \cdot [\underline{\mathcal{D}_{c,S}^1}(r', r'_1)] + M'_{es', M_1 \delta_{r_1}}(r') \cdot [\underline{\mathcal{F}_{m,S}^1}(r', r'_1)] dS_{r'}, \quad (82)
\end{aligned}$$

and the associated electric field, following,

$$\begin{aligned}
& E_{S,S'}(J_1 \delta_{r_1} = 0, M_1 \delta_{r_1}; r'_1) \\
&= \int_{S'} [\underline{\mathcal{F}_{c,SC}^1}(r'_1, r')] \cdot Z_0 J'_{es', M_1 \delta_{r_1}}(r') - [\underline{\mathcal{D}_{c,SC}^1}(r'_1, r')] \cdot M'_{es', M_1 \delta_{r_1}}(r') dS_{r'} \\
&= \int_{S'} Z_0 J'_{es', M_1 \delta_{r_1}}(r') \cdot [\underline{\mathcal{F}_{c,S}^1}(r', r'_1)] - M'_{es', M_1 \delta_{r_1}}(r') \cdot [\underline{\mathcal{D}_{m,S}^1}(r', r'_1)] dS_{r'}, \quad (83)
\end{aligned}$$

where $J'_{es'} = \hat{n} \wedge H_{1S'}|_{S'}$ and $M'_{es'} = -\hat{n} \wedge E_{1S'}|_{S'}$, or, $J'_{es'} = \hat{n} \wedge (H_{1S'} - H_0)|_{S'}$ and $M'_{es'} = -\hat{n} \wedge (E_{1S'} - E_0)|_{S'}$.

Remark : Considering a surface (S or S') partly included in a part of the unperturbed system where we let losses tend to infinity, the contribution of this part vanishes, making like a hole in the surface expression of couplings.

6.3) First order coupling tensors $[\underline{E_{S,S'}^{c(m)}}(r_1, r'_1)]$ and $[\underline{H_{S,S'}^{c(m)}}(r_1, r'_1)]$

6.3.1) Coupling tensors $[\underline{E_{S,S'}^c}(r_1, r'_1)]$ and $[\underline{H_{S,S'}^c}(r_1, r'_1)]$

We consider the dyadic tensors for the coupling of A' on A . Let us write in (77)-(78),

$$\begin{aligned}
& E_{S,S'}(J_1 \delta_{r_1}, M_1 \delta_{r_1} = 0; r'_1) = J_1 \cdot [\underline{E_{S,S'}^c}(r_1, r'_1)], \\
& Z_0 H_{S,S'}(J_1 \delta_{r_1}, M_1 \delta_{r_1} = 0; r'_1) = J_1 \cdot [\underline{Z_0 H_{S,S'}^c}(r_1, r'_1)], \\
& Z_0 J'_{es', J_1 \delta_{r_1}} = J_1 \cdot [\underline{Z_0 J'_{es',c}}(r_1, r')], \quad M'_{es', J_1 \delta_{r_1}} = J_1 \cdot [\underline{M'_{es',c}}(r_1, r')], \quad (84)
\end{aligned}$$

we then obtain,

$$\begin{aligned}
[\underline{E}_{S,S'}^c(r_1, r'_1)] &= \left(\int_{S'} ([\underline{Z}_0 \underline{J}'_{es',c}(r_1, r')]) \cdot {}^t[\underline{\mathcal{F}}_{c,SC}^1(r'_1, r')] \right. \\
&\quad \left. - [\underline{M}'_{es',c}(r_1, r')] \cdot {}^t[\underline{\mathcal{D}}_{c,SC}^1(r'_1, r')] \right) dS_{r'} \\
&= \left(\int_{S'} ([\underline{Z}_0 \underline{J}'_{es',c}(r_1, r')]) \cdot [\underline{\mathcal{F}}_{c,S}^1(r', r'_1)] \right. \\
&\quad \left. - [\underline{M}'_{es',c}(r_1, r')] \cdot [\underline{\mathcal{D}}_{m,S}^1(r'_1, r')] \right) dS_{r'} \tag{85}
\end{aligned}$$

and,

$$\begin{aligned}
[\underline{Z}_0 \underline{H}_{S,S'}^c(r_1, r'_1)] &= \int_{S'} ([\underline{Z}_0 \underline{J}'_{es',c}(r_1, r')]) \cdot {}^t[\underline{\mathcal{D}}_{m,SC}^1(r'_1, r')] \\
&\quad + [\underline{M}'_{es',c}(r_1, r')] \cdot {}^t[\underline{\mathcal{F}}_{m,SC}^1(r'_1, r')] dS_{r'} \\
&= \int_{S'} ([\underline{Z}_0 \underline{J}'_{es',c}(r_1, r')]) \cdot [\underline{\mathcal{D}}_{c,S}^1(r', r'_1)] \\
&\quad + [\underline{M}'_{es',c}(r_1, r')] \cdot [\underline{\mathcal{F}}_{m,S}^1(r', r'_1)] dS_{r'}. \tag{86}
\end{aligned}$$

6.3.2) Coupling tensors $[\underline{E}_{S,S'}^m(r_1, r'_1)]$ and $[\underline{H}_{S,S'}^m(r_1, r'_1)]$

In the same manner, noting in (82)-(83),

$$\begin{aligned}
Z_0 \underline{H}_{S,S'}(J_1 \delta_{r_1} = 0, M_1 \delta_{r_1}; r'_1) &= M_1 \cdot [\underline{Z}_0 \underline{H}_{S,S'}^m(r_1, r'_1)], \\
\underline{E}_{S,S'}(J_1 \delta_{r_1} = 0, M_1 \delta_{r_1}; r'_1) &= M_1 \cdot [\underline{E}_{S,S'}^m(r_1, r'_1)], \\
Z_0 \underline{J}'_{es', M_1 \delta_{r_1}} &= M_1 \cdot [\underline{Z}_0 \underline{J}'_{es',m}(r_1, r')], \quad \underline{M}'_{es', M_1 \delta_{r_1}} = M_1 \cdot [\underline{M}'_{es',m}(r_1, r')], \tag{87}
\end{aligned}$$

we obtain,

$$\begin{aligned}
[\underline{Z}_0 \underline{H}_{S,S'}^m(r_1, r'_1)] &= \int_{S'} [\underline{Z}_0 \underline{J}'_{es'}(r_1, r')] \cdot {}^t[\underline{\mathcal{D}}_{m,SC}^1(r'_1, r')] \\
&\quad + [\underline{M}'_{es'}(r_1, r')] \cdot {}^t[\underline{\mathcal{F}}_{m,SC}^1(r'_1, r')] dS_{r'}, \\
&= \int_{S'} [\underline{Z}_0 \underline{J}'_{es'}(r_1, r')] \cdot [\underline{\mathcal{D}}_{c,S}^1(r', r'_1)] \\
&\quad + [\underline{M}'_{es'}(r_1, r')] \cdot [\underline{\mathcal{F}}_{m,S}^1(r', r'_1)] dS_{r'}, \tag{88}
\end{aligned}$$

and

$$\begin{aligned}
[\underline{E}_{S,S'}^m(r_1, r'_1)] &= \int_{S'} [Z_0 \underline{J}'_{es'}(r_1, r')] \cdot {}^t[\underline{\mathcal{F}}_{c,SC}^1(r'_1, r')] \\
&\quad - [\underline{M}'_{es'}(r_1, r')] \cdot {}^t[\underline{\mathcal{D}}_{c,SC}^1(r'_1, r')] dS_{r'} \\
&= \int_{S'} [Z_0 \underline{J}'_{es'}(r_1, r')] \cdot [\underline{\mathcal{F}}_{c,S}^1(r', r'_1)] \\
&\quad - [\underline{M}'_{es'}(r_1, r')] \cdot [\underline{\mathcal{D}}_{m,S}^1(r', r'_1)] dS_{r'}.
\end{aligned} \tag{89}$$

Let us notice that, in practice, we can consider that (E_0, H_0) is given either by closed form expressions (for r_1, r'_1 at large distance and/or simple geometries) or numerically, while $(E_{1S'}, H_{1S'})$ and (E_{1S}, H_{1S}) are generally calculated numerically.

6.4) Reciprocity relations for coupling tensors

Concerning the coupling of A on A' , we can exchange A' and S' for A and S in previous equations. However, it is also possible to derive $E_{S',S}^c$ from $E_{S,S'}^c$ by some use of reciprocity. For this, we use (54)-(60) in (85) and notice that,

$$\begin{aligned}
J_1 \cdot [\underline{E}_{S,S'}^c(r_1, r'_1)] \cdot J_2 &= \\
&= C_0 \int_{S'} \int_S ((J_1 \cdot [\underline{Z}_0 \underline{J}'_{es',c}(r_1, r'_{S'})]) \cdot ([\underline{\mathcal{F}}_{he,t}^0(r'_{S'}, r'_S)]) \\
&\quad \cdot ({}^t[\underline{Z}_0 \underline{J}'_{esC,c}(r'_1, r'_S)] \cdot J_2) dS_{r'_s} dS_{r'_{s'}} \\
&\quad + C_1 \int_{S'} \int_S (J_1 \cdot [\underline{Z}_0 \underline{J}'_{es',c}(r_1, r'_{S'})]) \cdot ([\underline{\mathcal{D}}_{he,t}^0(r'_{S'}, r'_S)]) \\
&\quad \cdot ({}^t[\underline{M}'_{esC,c}(r'_1, r'_S)] \cdot J_2) dS_{r'_s} dS_{r'_{s'}} \\
&\quad + C_1 \int_{S'} \int_S (J_1 \cdot [\underline{M}'_{es',c}(r_1, r'_{S'})]) \cdot ([\underline{\mathcal{D}}_{eh,t}^0(r'_{S'}, r'_S)]) \\
&\quad \cdot ({}^t[\underline{Z}_0 \underline{J}'_{esC,c}(r'_1, r'_S)] \cdot J_2) dS_{r'_s} dS_{r'_{s'}} \\
&\quad + C_2 \int_{S'} \int_S (J_1 \cdot [\underline{M}'_{es',c}(r_1, r'_{S'})]) \cdot ([\underline{\mathcal{F}}_{eh,t}^0(r'_{S'}, r'_S)]) \\
&\quad \cdot ({}^t[\underline{M}'_{esC,c}(r'_1, r'_S)] \cdot J_2) dS_{r'_s} dS_{r'_{s'}},
\end{aligned} \tag{90}$$

where $[\underline{\mathcal{F}}_{he(eh),t}^0]$ et $[\underline{\mathcal{D}}_{he(eh),t}^0]$ are the Green's tensors corresponding to the unperturbed system, without A and A' , and the C_j are constants. We can then transpose the scalar expression in double integrals without changing the results in each integral of (90). Using $[\underline{\mathcal{F}}_{he(eh),t}^0(r, r')] = {}^t[\underline{\mathcal{F}}_{he(eh),tC}^0(r', r)]$ and $[\underline{\mathcal{D}}_{he(eh),t}^0(r, r')] = {}^t[\underline{\mathcal{D}}_{eh(he),tC}^0(r', r)]$, we change the order of integration, and obtain $J_2 \cdot [\underline{E}_{S',S}^c(r'_1, r_1)] \cdot J_1$ for adjoint characteristics.

This implies,

$$[\underline{E}_{S',S}^c(r'_1, r_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] = {}^t[\underline{E}_{S,S'}^c(r_1, r'_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}], \quad (91)$$

where $()|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}$ means that the system is considered with adjoint characteristics. In the same manner, we can obtain the reciprocity relations,

$$\begin{aligned} [\underline{Z}_0 \underline{H}_{S',S}^m(r'_1, r_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= {}^t[\underline{Z}_0 \underline{H}_{S,S'}^m(r_1, r'_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}], \\ [\underline{H}_{S',S}^c(r'_1, r_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= -{}^t[\underline{E}_{S,S'}^m(r_1, r'_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}], \\ [\underline{E}_{S',S}^m(r'_1, r_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= -{}^t[\underline{H}_{S,S'}^c(r_1, r'_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}]. \end{aligned} \quad (92)$$

We then note that the total coupling tensors of first order,

$$\begin{aligned} [\underline{E}_{ctot}^{c(m)}(r_1, r'_1)] &= [\underline{E}_{S,S'}^{c(m)}(r_1, r'_1)] + [\underline{E}_{S',S}^{c(m)}(r_1, r'_1)], \\ [\underline{H}_{ctot}^{m(c)}(r_1, r'_1)] &= [\underline{H}_{S,S'}^{m(c)}(r_1, r'_1)] + [\underline{H}_{S',S}^{m(c)}(r_1, r'_1)], \end{aligned} \quad (93)$$

satisfy the reciprocity property,

$$\begin{aligned} [\underline{E}_{ctot}^c(r_1, r'_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= {}^t[\underline{E}_{ctot}^c(r'_1, r_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}], \\ [\underline{H}_{ctot}^m(r_1, r'_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= {}^t[\underline{H}_{ctot}^m(r'_1, r_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}], \\ [\underline{H}_{ctot}^c(r_1, r'_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= -{}^t[\underline{E}_{ctot}^m(r'_1, r_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}], \\ [\underline{E}_{ctot}^m(r_1, r'_1)|_{((\epsilon),(\xi),(\mu),(\varsigma))}] &= -{}^t[\underline{H}_{ctot}^c(r'_1, r_1)|_{(t(\epsilon),-t(\varsigma),t(\mu),-t(\xi))}]. \end{aligned} \quad (94)$$

which is in accordance with the generalized reciprocity principle.

Remark : The reciprocity relations derived in (91)-(94) are more general than the ones commonly used in hybrid methods [9].

7) Application to the numerical removal of couplings

In this section, we study the problem of suppressing numerically the coupling between A and A' to recover the contribution of A considered alone in the initial system, i.e. without A' , when we know :

- the fields for the total system, with A and A' ,

- the fields for the system with A' but without A ,

i.e., for example, the case of the determination of the radiation of a scatterer A inserted in infinite plane when we know the results for a truncated plane with and without A , the scatterer A' being assumed as the ends of the plate. In what follows, we note,

$$V(r'_1) \equiv J.[V(r_1, r'_1)], \quad (95)$$

where $J\delta_{r_1}$ is the electric dipole source exciting the system, and $[V(r_1, r'_1)]$ is the dyadic tensor attached to the field $V(r'_1)$.

7.1) Expression of the total field

The dyadic tensor for the total field in presence of both scatterers A and A' respectively limited by S and S' , with the first order coupling terms between them satisfying (85), is given by,

$$\begin{aligned} [\underline{E}_{tot}(r_1, r'_1)] &\simeq [\underline{E}_{1S}(r_1, r'_1)] + [\underline{E}_{1S'}(r_1, r'_1)] \\ &- [\underline{E}_0(r_1, r'_1)] + [\underline{E}_{S,S'}^c(r_1, r'_1)] + [\underline{E}_{S',S}^c(r_1, r'_1)], \\ &= [\underline{E}_{1S}(r_1, r'_1)] + [\underline{E}_{1S'}(r_1, r'_1)] \\ &- [\underline{E}_0(r_1, r'_1)] + [\underline{E}_{S,S'}^c(r_1, r'_1)] + {}^t[\underline{E}_{S,S'}^c(r'_1, r_1)|_C] \end{aligned} \quad (96)$$

where E_{1S} (resp. $E_{1S'}$) is the field without A' (resp. without A), E_0 is the field of the system unperturbed (without A and A'), r_1 and r'_1 are the respective positions of dirac primary source and the observation, while the index C specifies the adjoint system (note : in a similar manner, we can express H_{tot} , replacing E by H in (96), and for isotropic cases, the index C can be suppressed).

7.2) Radiation of sources on S , its properties, and relation with coupling terms

We define E_S^f as the radiation of the sources $J_s = \hat{n} \wedge H_{tot}$ and $M_s = -\hat{n} \wedge E_{tot}$ on S , in presence of the unperturbed system, with the associated tensor verifying,

$$\begin{aligned} [\underline{E}_S^f(r_1, r'_1)] &= \int_S Z_0[J_s].([\underline{\mathcal{F}}_{he,t}^0(r', r'_1)]) - [M_s].([\underline{\mathcal{D}}_{eh,t}^0(r', r'_1)])dS_{r'} \\ &= \int_S [\underline{\mathcal{F}}_{he,tC}^0(r'_1, r')].Z_0[J_s] - [\underline{\mathcal{D}}_{he,tC}^0(r'_1, r')].[M_s]dS_{r'}, \end{aligned} \quad (97)$$

for r_1 and r'_1 outside S . Considering the total fields given by (96), and noticing that S' and S enclose disjoint domains, we can use (22) and then note that,

$$\begin{aligned} &\text{in the expressions of } J_s \text{ and } M_s \text{ with the total field,} \\ &\hat{n} \wedge (H_{1S'} + H_{S',S}^c) \text{ and } -n \wedge (E_{1S'} + E_{S',S}^c) \text{ on } S \\ &\text{do not contribute to } E_S^f \text{ from (22),} \end{aligned} \quad (98)$$

so that we can write,

$$\begin{aligned} [\underline{E}_S^f(r_1, r'_1)] &\simeq ([\underline{E}_{1S}(r_1, r'_1)] - [\underline{E}_0(r_1, r'_1)]) \\ &+ [\underline{E}_{S,S'}^c(r_1, r'_1)]. \end{aligned} \quad (99)$$

Exchanging r_1 for r'_1 , then using transpose, with the reciprocity relation (35)-(36) for E_0 , (63)-(64) for E_{1S} , and (91) for $E_{S,S'}^c$, this also gives us,

$$\begin{aligned} {}^t[\underline{E}_{SC}^f(r'_1, r_1)] &= \\ &= ([\underline{E}_{1S}(r_1, r'_1)] - [\underline{E}_0(r_1, r'_1)]) + {}^t[\underline{E}_{S,S'}^c(r'_1, r_1)|_C] \\ &= ([\underline{E}_{1S}(r_1, r'_1)] - [\underline{E}_0(r_1, r'_1)]) + [\underline{E}_{S',S}^c(r_1, r'_1)]. \end{aligned} \quad (100)$$

(note : in general, we note that, for bianisotropic system, ${}^t[\underline{E}_{SC}^f(r'_1, r_1)] \neq [\underline{E}_S^f(r_1, r'_1)]$.)

7.3) Efficient determination of the radiation of A without A'

Adding (99) and (100), then subtracting (96), we then note that we recover the field $(E_{1S} - E_0)$, scattered by A without A' , following,

$$\begin{aligned} ([\underline{E}_{1S}(r_1, r'_1)] - [\underline{E}_0(r_1, r'_1)]) &= \\ &= ([\underline{E}_S^f(r_1, r'_1)] + {}^t[\underline{E}_{SC}^f(r'_1, r_1)]) \\ &- ([\underline{E}_{tot}(r_1, r'_1)] - [\underline{E}_{1S'}(r_1, r'_1)]). \end{aligned} \quad (101)$$

Therefore, we conclude that, if we know the fields,

- for the complete system with both scatterers A and A' , for the calculus of $[\underline{E}_S^f(r_1, r'_1)]$, ${}^t[\underline{E}_{SC}^f(r'_1, r_1)]$ et $[\underline{E}_{tot}(r_1, r'_1)]$,
- for the system with A' but without A , for the calculus of $[\underline{E}_{1S'}(r_1, r'_1)]$,

then it is possible, from (101), to know a second order expression of the field diffracted by A without the presence of A' , i.e. without the direct contribution of A' and the first order coupling with it.

It is worth noticing that, in the literature, $([\underline{E}_{tot}(r_1, r'_1)] - [\underline{E}_{1S'}(r_1, r'_1)])$ or $[\underline{E}_S^f(r_1, r'_1)]$ are commonly used to evaluate the field diffracted by A without the presence of A' , but, most often, it gives poor results (in particular at grazing incidence for the case of the removal of the contribution of the ends of a plate).

It seems to us remarkable that, for the same cost, the simple combination of them, given in (101), permits to remove direct and first order couplings for much better results.

Remark : A similar result can be obtained for the magnetic field, following,

$$\begin{aligned}
& ([\underline{H}_{1S}(r_1, r'_1)] - [\underline{H}_0(r_1, r'_1)]) = \\
& = ([\underline{H}_S^f(r_1, r'_1)] + {}^t[\underline{H}_{SC}^f(r'_1, r_1)]) \\
& - ([\underline{H}_{tot}(r_1, r'_1)] - [\underline{H}_{1S'}(r_1, r'_1)]),
\end{aligned} \tag{102}$$

where,

$$[Z_0 \underline{H}_S^f(r_1, r'_1)] = \int_S Z_0 [J_s] \cdot [\underline{\mathcal{D}}_{he,t}^0(r', r'_1)] + [M_s] \cdot [\underline{\mathcal{F}}_{eh,t}^0(r', r'_1)] dS_{r'}. \tag{103}$$

remark : Let notice that, for isotropic media, in monostatic case when $r_1 = r'_1$, we have,

$$\begin{aligned}
& \hat{w} \cdot ([\underline{E}_S^f(r_1, r_1)] + {}^t[\underline{E}_S^f(r_1, r_1)]) \cdot \hat{w} \\
& = 2\hat{w} \cdot [\underline{E}_S^f(r_1, r_1)] \cdot \hat{w},
\end{aligned} \tag{104}$$

and that, for isotropic A , grounded in a PEC plane [6], limited by a plane aperture S ,

$$E_S^f(r_1, r) = -2\text{curl} \int_S M_s(r') \frac{e^{-ik|r-r'|}}{4\pi|r-r'|} dS_{r'} \text{ with } M_s = -n \wedge E_{tot}. \tag{105}$$

8) Numerical results

We discuss in this section numerical removal of couplings from (101). For that, we consider the RCS of a semicircular groove of radius $\lambda/10$ in a ground 2D plate of total length L , and the removal of couplings with the ends of the plate, as in [1]. The aim is then to recover the contribution of the groove on an infinite plane, and thus to suppress the perturbation due to the ends of the plate.

For this, we calculate the fields at infinity, with and without the groove, and on the aperture, for the calculus of E_S^f . We can then compare our results, when applying our postprocess (101), or when simply subtracting the field for the plate without the groove, with the exact result given in [10], for different plate lengths $L = 1\lambda, 3\lambda, 6\lambda, 12\lambda$. The usual method of subtraction gives poor results, while our postprocess gives excellent ones on a domain of angles that increases with L . The calculus of E_S^f and thus of (101) is very simple and fast, and the cost of our calculation is appromatively the same than for the simple subtraction method.

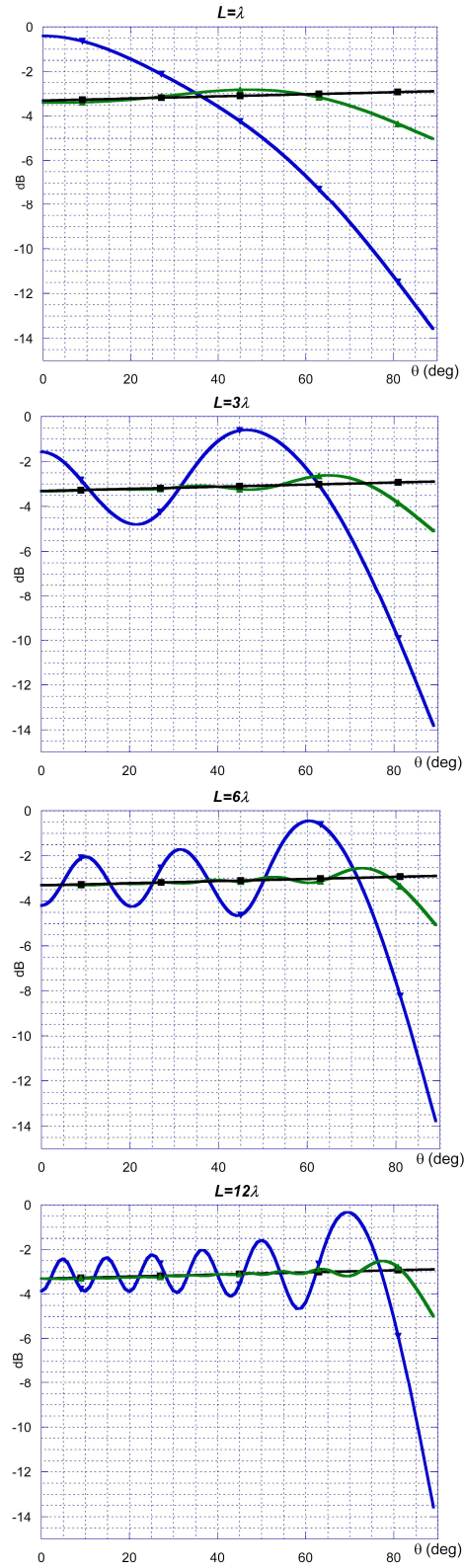


figure 7 : comparison of results, when we classically subtract the field of the plate without the groove truncated at L (curves in blue), when we apply our postprocess (101) (curves in green), when we apply the exact calculus for the groove in a infinite plane given in [10] (curves in black)

Appendix A : Note on some identities for conjugate fields

We can consider (1) and conjugate of (2). In a manner similar to the one used for (8), we then obtain equalities, with, this time, the conjugate of (E_{0C}, H_{0C}) , following,

$$\begin{aligned}
 \int_{\partial\Omega} H_{0C}^* \cdot (\hat{n} \wedge E) - E_{0C}^* \cdot (\hat{n} \wedge H) dS &= \int_{\partial\Omega} (E_{0C}^* \wedge H + E \wedge H_{0C}^*) \cdot \hat{n} dS \\
 &= \int_{\Omega} i\omega E_{0C}^* \cdot ((\epsilon) - (\epsilon_0)^*) E dV + \int_{\Omega} i\omega H_{0C}^* \cdot ((\mu) - (\mu_0)^*) H dV \\
 &+ \int_{\Omega} i\omega H_{0C}^* \cdot ((\varsigma) + (\varsigma_0)^*) E dV + \int_{\Omega} i\omega E_{0C}^* \cdot ((\xi) + (\xi_0)^*) H dV \\
 &+ \int_{\Omega} (J_{0C}^* \cdot E + M_{0C}^* \cdot H) dV + \int_{\Omega} (E_{0C}^* \cdot J + H_{0C}^* \cdot M) dV, \tag{106}
 \end{aligned}$$

where (E_{0C}, H_{0C}) and (E, H) respectively correspond to $({}^t(\epsilon_0), -{}^t(\varsigma_0), {}^t(\mu_0), -{}^t(\xi_0))$ and $((\epsilon), (\xi), (\mu), (\varsigma))$ characteristics, and \hat{n} is the inward normal to $\partial\Omega$. Considering the real part of this equation, we can obtain a generalization of the well-known integral equality for the Poynting vector [6], when we consider some choices of parameters giving $(E_{0C}, H_{0C}) = (E, H)$ on $\partial\Omega$.

Appendix B : Note on some identities for surface integrals

B.1) On the finite part (FP) of a surface integral expression

B.1.1) FP by dropping some divergent terms

The Hadamard finite part (also called Hadamard regularization, or FP) is a method of regularizing divergent integral by dropping some divergent terms and keeping the finite part. In our case, our integrals depends of a parameter r , and the FP part is not necessarily continuous relatively to it. For example, for integrals defined on a surface S excluding a small domain belonging to the ball centered at r of radius ϵ , the continuity up to S is verified only if the limit when $r \rightarrow S$ and the one as $\epsilon \rightarrow 0$ can be interchanged without changing the result (note : for example, the integral expression of the indicator function 1_Ω of the domain Ω is bounded but not continuous up to its frontier included).

We now consider an example where S is a closed surface composed of piecewise continuous open part $S_i, i = 1, \dots, n$, and Ω a domain exterior to S . Letting,

$V = E$ or H ,

$$f(r') = \operatorname{div}(u_0 \frac{f_1(r')}{|r' - r|}) = u_0 \cdot \operatorname{grad}(\frac{f_1(r')}{|r' - r|}) \text{ with } f_1(r') = O(1),$$

$$S_\epsilon \text{ is the intersection of } S \text{ and a ball } \Omega'_\epsilon \text{ of radius } \epsilon \text{ centered at } r \in \overline{\Omega}, \quad (107)$$

we assume that $\hat{n}(r').E(r')$ and $\hat{n}(r').H(r')$ is continuous on S_i except possibly on its contour \mathcal{L}_i , while $\hat{t}(r).E(r')$ and $\hat{t}(r).H(r')$ is continuous across \mathcal{L}_i , $\hat{t}(r')$ being the unit tangential vector along \mathcal{L}_i . Our aim is to extract the finite part of the integral expression,

$$\begin{aligned} \mathcal{F}(r) &= \int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge V(r')).\operatorname{grad}_{r'}(f) dS_{r'} \\ &= - \int_{S_i \setminus S_\epsilon} \hat{n}(r').(\operatorname{grad}_{r'}(f) \wedge V(r')) dS_{r'}, \end{aligned} \quad (108)$$

for $r \in \overline{\Omega}$. We can use that,

$$\begin{aligned} \operatorname{grad}_{r'}(f) \wedge V &= \operatorname{curl}_{r'}(fV) - f \operatorname{curl}_{r'}(V), \\ \int_{S_i \setminus S_\epsilon} \hat{n}(r').\operatorname{curl}_{r'}(fV) dS_{r'} &= \left(\int_{\mathcal{L}_i} - \int_{\mathcal{L}_\epsilon} \right) fV.dr', \end{aligned} \quad (109)$$

and obtain,

$$\begin{aligned} \mathcal{F}(r) &= - \int_{S_i \setminus S_\epsilon} \hat{n}(r').\operatorname{curl}_{r'}(fV) dS_{r'} + \int_{S_i \setminus S_\epsilon} \hat{n}(r').\operatorname{curl}_{r'}(V) f dS_{r'} \\ &= - \left(\int_{\mathcal{L}_i} - \int_{\mathcal{L}_\epsilon} \right) fV.dr' + \int_{S_i \setminus S_\epsilon} \hat{n}(r').\operatorname{curl}_{r'}(V) f dS_{r'}. \end{aligned} \quad (110)$$

where \hat{n} is the outward normal to $S_{i(\epsilon)}$, and $\mathcal{L}_{i(\epsilon)}$ is positively oriented. The term $\int_{\mathcal{L}_\epsilon} fV.dr'$ vanishes as $\epsilon \rightarrow 0$ when $r \in \overline{\Omega} \setminus S_i$, and is generally divergent as $\epsilon \rightarrow 0$ for any point of S_i . Dropping this term in $\mathcal{F}(r)$, the result unchanges when $r \in \overline{\Omega} \setminus S_i$, and becomes finite on $\overline{\Omega} \setminus \mathcal{L}_i$. We then obtain the finite part (FP) of $\mathcal{F}(r)$, following,

$$\begin{aligned} \operatorname{FP}(\mathcal{F}(r)) &= \lim_{\epsilon \rightarrow 0} \left(\int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge V(r')).\operatorname{grad}_{r'}(f) dS_{r'} - \int_{\mathcal{L}_\epsilon} f(r')V(r').dr' \right) \\ &= pv \int_{S_i} \hat{n}(r').\operatorname{curl}_{r'}(V(r')) f(r') dS_{r'} - \int_{\mathcal{L}_i} f(r')V(r').dr', \end{aligned} \quad (111)$$

where pv denotes the principal value of Cauchy, and \mathcal{L}_i is positively oriented.

B.1.2) FP by the theory of distributions

It is worth noticing that the previous result is strongly connected with the theory of distributions. Indeed, noting Ω'_ϵ the ball of small radius ϵ centered at r , and letting,

$$\begin{aligned} 1_{R^3 \setminus \Omega'_\epsilon} &= 1 \text{ as } r' \in R^3 \setminus \overline{\Omega'_\epsilon}, = 0 \text{ as } r' \notin R^3 \setminus \Omega'_\epsilon, \\ \bar{f} &\text{ with } \frac{1_{R^3 \setminus \Omega'_\epsilon}}{|r' - r|} \text{ instead of } f \text{ with } \frac{1}{|r' - r|}, \\ \bar{f}(r') &= 1_{R^3 \setminus \Omega'_\epsilon} u_0 \cdot \text{grad} \left(\frac{f_1(r')}{|r' - r|} \right) + \hat{n}'_\epsilon \cdot u_0 \delta_{\partial \Omega'_\epsilon}, \end{aligned} \quad (112)$$

where \hat{n}'_ϵ is the outward normal to Ω'_ϵ , it is sufficient to replace $f(r')$ by $\bar{f}(r')$ in the initial expression of $\mathcal{F}(r)$ to obtain directly the finite part (FP) of $\mathcal{F}(r)$, following,

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \int_{S_i} (\hat{n}(r') \wedge V(r')) \cdot \text{grad}_{r'}(\bar{f}) dS_{r'} \\ &= \lim_{\epsilon \rightarrow 0} (\mathcal{F}(r) + \int_{S_i} f V \cdot (\hat{n}'_\epsilon \wedge \hat{n}) \delta_{\partial \Omega'_\epsilon} dS_{r'}) \\ &= \lim_{\epsilon \rightarrow 0} (\mathcal{F}(r) - \int_{\mathcal{L}_\epsilon} f V \cdot dr') = \text{FP}(\mathcal{F}(r)), \end{aligned} \quad (113)$$

where we have used that

$$\int_{S_i} f V \cdot (\hat{n}'_\epsilon \wedge \hat{n}) \delta_{\partial \Omega'_\epsilon} dS_{r'} = \int_{\mathcal{L}_\epsilon} f V \cdot (\hat{n}'_\epsilon \wedge \hat{n}) dl_{r'} \sim - \int_{\mathcal{L}_\epsilon} f V \cdot dr' \text{ as } \epsilon \rightarrow 0. \quad (114)$$

Remark : We note that the components of $V(r')$ along the \mathcal{L}_i are generally continuous across this frontier. Using $S \equiv \sum_i S_i$, we can then write, up to S (even on \mathcal{L}_i),

$$\begin{aligned} \sum_i \int_{\mathcal{L}_i} f(r') V(r') \cdot dr' &= 0, \\ \sum_i \text{FP} \int_{S_i} (\hat{n}(r') \wedge V(r')) \cdot \text{grad}_{r'}(f(r')) dS_{r'} &= O(1). \end{aligned} \quad (115)$$

Remark : The fields component along the edge due to radiation of 'one face' of a wedge (composed of simple and double layers contributions) is singular at the edge, and it is, by taking account of radiations of 'both faces' of the wedge, that it becomes regular at the edge.

B.2) On the principal value (pv) of a surface integral

In many case, finite part (FP) reduces to take principal value (pv) of the integral. This principal value remains finite, but can be discontinuous. An interesting example consists

to study,

$$pv \int_{S_i} W_0(r') \cdot \text{grad}_{r'}(g(r', r)) dS_{r'},$$

$$\text{with } W_0(r') = \hat{n}(r') \wedge W(r') + w(r') \hat{n}(r'), \quad (116)$$

when we assume that,

$$g(r', r)|r' - r| = h(r') \text{ is a smooth function on } S,$$

$$W(r') - W(r) = O(|r - r'|), w(r') - w(r) = O(|r - r'|). \quad (117)$$

B.2.1) The part of integral with $\hat{n}(r') \wedge W(r')$ is finite and continuous

For this case, we begin to write,

$$pv \int_{S_i} (\hat{n}(r') \wedge W(r')) \cdot \text{grad}_{r'}(g(r')) dS_{r'} =$$

$$\lim_{\epsilon \rightarrow 0} \int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge (W(r') - W(r))) \cdot \text{grad}_{r'}(g(r')) dS_{r'}$$

$$+ \lim_{\epsilon \rightarrow 0} \int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge W(r)) \cdot \text{grad}_{r'}(g(r')) dS_{r'}. \quad (118)$$

Using that,

$$\int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge W(r)) \cdot \text{grad}_{r'}(g(r')) dS_{r'}$$

$$= - \int_{S_i \setminus S_\epsilon} \hat{n}(r') \cdot \text{curl}_{r'}(g(r')W(r)) dS_{r'} = - \left(\int_{\mathcal{L}_i} - \int_{\mathcal{L}_\epsilon} \right) g(r')W(r) \cdot dr',$$

$$\text{with } \lim_{\epsilon \rightarrow 0} \int_{\mathcal{L}_\epsilon} g(r')W(r) \cdot dr' = h(r)W(r) \lim_{\epsilon \rightarrow 0} \int_{\mathcal{L}_\epsilon} \frac{1}{|r' - r|} \cdot dr' = 0, \quad (119)$$

and noting that,

$$\lim_{\epsilon \rightarrow 0} \int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge (W(r') - W(r))) \cdot \text{grad}_{r'}(g(r')) dS_{r'} = O(1), \quad (120)$$

we have,

$$pv \int_{S_i} (\hat{n}(r') \wedge W) \cdot \text{grad}_{r'}(f) dS_{r'} =$$

$$\lim_{\epsilon \rightarrow 0} \int_{S_i \setminus S_\epsilon} (\hat{n}(r') \wedge (W(r') - W(r))) \cdot \text{grad}_{r'}(f) dS_{r'} - \int_{\mathcal{L}_i} g(r')W(r) \cdot dr', \quad (121)$$

which is continuous at any r' up to S_i included, except possibly on \mathcal{L}_i (note : the restriction on \mathcal{L}_i does no more exist if we consider the closed surface $S \equiv \sum_i S_i$).

B.2.2) The part of integral with $w(r')\hat{n}(r')$ is finite but discontinuous

In the same manner, we can study,

$$pv \int_{S_i} w(r')\hat{n}(r').\text{grad}_{r'}(g(r'))dS_{r'}, \quad (122)$$

and note that it is finite, but can be discontinuous. For that, we remark that,

$$\lim_{\epsilon \rightarrow 0} \int_{S_i \setminus S_\epsilon} \hat{n}(r').\text{grad}_{r'}\left(\frac{1}{|r' - r|}\right)dS_{r'}, \quad (123)$$

is finite, but discontinuous as r crosses S_i .

Remark : let S'_i (resp. $S'_{\epsilon'}$) be a disk of radius a (resp. ϵ') at $z = 0$, centered at the origin. We note that,

$$\begin{aligned} \mathcal{D}_{\epsilon'}(r) &= - \int_{S_i \setminus S'_{\epsilon'}} \hat{z}.\text{grad}_{r'}\left(\frac{-1/4\pi}{|r' - r|}\right)dS_{r'} = \int_0^{2\pi} \int_{\epsilon'}^a \frac{\partial\left(\frac{1/4\pi}{\sqrt{\rho^2 + (z - z')^2}}\right)}{\partial z'} \rho d\rho d\varphi \\ &= \int_{\epsilon'^2}^{a^2} \frac{z/4}{(\sqrt{v + z^2})^3} dv|_{z'=0} = \frac{z/2}{\sqrt{\epsilon'^2 + z^2}} - \frac{z/2}{\sqrt{a^2 + z^2}}, \end{aligned} \quad (124)$$

and thus,

$$\lim_{\epsilon' \rightarrow 0} \mathcal{D}_{\epsilon'}(r) = \text{sign}(z)/2 - \frac{z/2}{\sqrt{a^2 + z^2}}, \quad (125)$$

is finite, but discontinuous at $z = 0$.

B.3) On properties of the surface divergence operator Div

The term $\hat{n} \cdot \text{curl}_{r'}(V)$ on S_i only depends of tangential derivatives of tangential components, and we note that,

$$\begin{aligned} \int_{S_i} -\hat{n} \cdot \text{curl}_{r'}(V) dS_{r'} &= \int_{\mathcal{L}_i} \hat{n} \wedge (\hat{n} \wedge V) \cdot dr' \\ &= - \int_{\mathcal{L}_i} (\hat{n} \wedge V) \cdot \hat{n} \wedge dr' = \int_{\mathcal{L}_i} (\hat{n} \wedge V) \cdot \hat{\nu} ds', \end{aligned} \quad (126)$$

where $\hat{\nu}$ is the outward normal to \mathcal{L}_i . Letting,

$$\text{Div}_{r'}(\hat{n} \wedge V) = -(\hat{n} \cdot \text{curl}_{r'})(V)|_S, \quad (127)$$

in these integral equalities, we can then define the surface operator Div, as Jones in [7], following,

$$\text{Div}(\hat{n} \wedge V) = \lim_{A \rightarrow 0} \frac{1}{A} \int_{\mathcal{L}_i} (\hat{n} \wedge V) \cdot \hat{\nu} ds', \quad (128)$$

where A is the area of S_i . These equalities are particularly useful, in particular when $V = E$ or H , and $\text{curl}_{r'}(V)$, are given by Maxwell equations.

Appendix C : Note on distributions and some integrals expressions

C.1) Elementary equalities for distributions

Let us consider $\Omega' \setminus \Omega'_\epsilon$, with Ω'_ϵ a ball of small radius ϵ centered at r , and Ω' a domain with a piecewise smooth frontier. We note that, in the sense of distributions [6],

$$\begin{aligned} \text{curl}_{r'}(1_{\Omega' \setminus \Omega'_\epsilon} V) &= 1_{\Omega' \setminus \Omega'_\epsilon} \text{curl}_{r'}(V) - (\hat{n}' \wedge V) \delta_{\partial\Omega'} + (\hat{n}'_\epsilon \wedge V) \delta_{\partial\Omega'_\epsilon}, \\ \text{grad}_{r'}(1_{\Omega' \setminus \Omega'_\epsilon} g) &= 1_{\Omega' \setminus \Omega'_\epsilon} \text{grad}_{r'}(g) - (\hat{n}' g) \delta_{\partial\Omega'} + (\hat{n}'_\epsilon g) \delta_{\partial\Omega'_\epsilon}, \\ \text{div}_{r'}(1_{\Omega' \setminus \Omega'_\epsilon} V) &= 1_{\Omega' \setminus \Omega'_\epsilon} \text{div}_{r'}(V) - (\hat{n}' \cdot V) \delta_{\partial\Omega'} + (\hat{n}'_\epsilon \cdot V) \delta_{\partial\Omega'_\epsilon}, \\ (\Delta_{r'} + k_0^2)(1_{R^3 \setminus \Omega'_\epsilon}(r') \frac{-e^{-ik|r'-r|}}{4\pi|r'-r|}) &\rightarrow \delta(r' - r) \text{ as } \epsilon \rightarrow 0, \end{aligned} \quad (129)$$

where $1_{\Omega' \setminus \Omega'_\epsilon}$ is the indicator function of $\Omega' \setminus \Omega'_\epsilon$, with $1_{\Omega' \setminus \Omega'_\epsilon}(r') = 1$ as $r' \in \Omega' \setminus \Omega'_\epsilon$, $= 0$ as $r' \notin \overline{\Omega' \setminus \Omega'_\epsilon}$. The unit vector \hat{n}' (resp. \hat{n}'_ϵ) is the outward normal to $\partial\Omega'$ (resp. $\partial\Omega'_\epsilon$), while $1_{R^3 \setminus \Omega'_\epsilon}$ is $1_{\Omega' \setminus \Omega'_\epsilon}$ as $\Omega' \rightarrow R^3$, and $\Delta \equiv \text{div}(\text{grad}(\cdot))$ (note : $1_{R^3 \setminus \Omega'_\epsilon} \frac{-e^{-ik|r'-r|}}{4\pi|r'-r|}$ is also called $fp(\frac{-e^{-ik|r'-r|}}{4\pi|r'-r|})$ as $\epsilon \rightarrow 0$).

C.2) Some integrals concerning the definition of $1_{\Omega'}(r)$ up to $\partial\Omega'$

We analyse the properties of integrals on Ω' in relation with,

$$(\operatorname{div}(\operatorname{grad}(.)) + k^2) \frac{-1_{R^3 \setminus \Omega'_\epsilon} e^{-ik|r'-r|}}{4\pi|r'-r|} = \delta(r' - r), \quad (130)$$

for $k = 0$ and for $\operatorname{Re}(ik) > 0$ in (130). In our analysis, $\partial\Omega'$ is piecewise smooth, and we consider integrals related to the definition of $1_{\Omega'}(r)$ for r up to $\partial\Omega'$ included.

C.2.1) The case $k = 0$

Let us consider $g(r')$ an analytic function with $\operatorname{grad}(g(r')) = O(1)$ in R^3 . We can write,

$$\begin{aligned} I_\epsilon(r) &= \int_{R^3} 1_{\Omega'} g(r') \operatorname{div}(\operatorname{grad}(\frac{-1_{R^3 \setminus \Omega'_\epsilon}}{4\pi|r'-r|})) dV_{r'} \\ &= \int_{R^3} 1_{\Omega'} g(r') \operatorname{div}(1_{R^3 \setminus \Omega'_\epsilon} \operatorname{grad}_{r'}(\frac{-1}{4\pi|r'-r|}) - \frac{\widehat{n}'_\epsilon \delta_{\partial\Omega'_\epsilon}}{4\pi|r'-r|}) dV \\ &= \int_{\partial\Omega'_\epsilon} \frac{\operatorname{grad}(1_{\Omega'} g(r')) \widehat{n}'_\epsilon}{4\pi|r'-r|} dS + \int_{\Omega'} g(r') \operatorname{div}(1_{R^3 \setminus \Omega'_\epsilon} \operatorname{grad}_{r'}(\frac{-1}{4\pi|r'-r|})) dV \\ &= \int_{\partial\Omega'_\epsilon} \frac{\operatorname{grad}(1_{\Omega'} g(r')) \widehat{n}'_\epsilon}{4\pi|r'-r|} dS + \int_{\Omega' \setminus \Omega'_\epsilon} \frac{-g(r')}{4\pi} \operatorname{div}(\operatorname{grad}_{r'}(\frac{1}{|r'-r|})) dV \\ &\quad + \int_{\Omega'} g(r') \delta_{\partial\Omega'_\epsilon} (\widehat{n}'_\epsilon \cdot \operatorname{grad}_{r'}(\frac{-1}{4\pi|r'-r|})) dV \\ &= \int_{\partial\Omega'_\epsilon} \frac{\operatorname{grad}(1_{\Omega'} g(r')) \widehat{n}'_\epsilon}{4\pi|r'-r|} dS + 0 + \int_{\partial\Omega'_\epsilon} \frac{1_{\Omega'} g(r') \widehat{n}'_\epsilon \cdot \operatorname{grad}_{r'}(\frac{-1}{|r'-r|})}{4\pi} dS \\ &= \int_{\partial\Omega'_\epsilon \cap \partial\Omega'} \frac{-\widehat{n}' \cdot \widehat{n}'_\epsilon \frac{g(r')}{|r'-r|}}{4\pi} dl + \int_{\partial\Omega'_\epsilon} \frac{1_{\Omega'} \frac{\operatorname{grad}(g(r'))}{|r'-r|} \widehat{n}'_\epsilon dS}{4\pi} + \int_{\partial\Omega'_\epsilon} \frac{1_{\Omega'} \frac{g(r')}{|r'-r|^2} dS}{4\pi}, \quad (131) \end{aligned}$$

Since $\widehat{n}' \cdot \widehat{n}'_\epsilon = 0$ on points (if any) of $\partial\Omega'_{\epsilon \rightarrow 0} \cap \partial\Omega'$, we then obtain,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} I_\epsilon(r) &= g(r) \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi\epsilon^2} \int_{\Omega' \cap \partial\Omega'_\epsilon} dS = g(r) 1_{\Omega'}(r), \\ \text{and thus, } \operatorname{div}(\operatorname{grad}(\frac{-1_{R^3 \setminus \Omega'_\epsilon}(r')}{4\pi|r'-r|})) &= \delta(r' - r). \quad (132) \end{aligned}$$

So defined, $1_{\Omega'}(r)$ is $= 1$ in $\Omega' \setminus \partial\Omega'$, $= 0$ outside $\overline{\Omega'}$, and is fractional on $\partial\Omega'$ (note : we have, on smooth part of $\partial\Omega'$, $1_{\Omega'} = \frac{1}{2}$, while, at the tip of a circular cone of half angle θ_c , $1_{\Omega'} = \frac{1 - \cos\theta_c}{2}$, and at the edge of a wedge of half angle $\pi - \Phi$, $1_{\Omega'} = \frac{\Phi}{\pi}$).

In other respects, we can use another development, following,

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} I_\epsilon(r') &= \lim_{\epsilon \rightarrow 0} \int_{R^3} \text{grad}(1_{\Omega'} g(r')) \cdot \text{grad}\left(\frac{1_{R^3 \setminus \Omega'_\epsilon}}{4\pi|r' - r|}\right) dV \\
&= \lim_{\epsilon \rightarrow 0} \int_{\Omega' \setminus \Omega'_\epsilon} \text{grad}(g(r')) \cdot \text{grad}\left(\frac{1}{4\pi|r' - r|}\right) dV \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_{\partial\Omega'_\epsilon} 1_{\Omega'} \text{grad}(g(r')) \cdot \left(\frac{\widehat{n}'_\epsilon}{4\pi|r' - r|}\right) dS \\
&\quad - \lim_{\epsilon \rightarrow 0} \int_{\partial\Omega'} 1_{R^3 \setminus \Omega'_\epsilon} \widehat{n}' g(r') \cdot \text{grad}\left(\frac{1}{4\pi|r' - r|}\right) dS \\
&= \lim_{\epsilon \rightarrow 0} \int_{\Omega' \setminus \Omega'_\epsilon} \text{grad}(g(r')) \text{grad}\left(\frac{1}{4\pi|r' - r|}\right) dV \\
&\quad + pv \int_{\partial\Omega'} g(r') (-\widehat{n}') \cdot \text{grad}\left(\frac{1}{4\pi|r' - r|}\right) dS. \tag{133}
\end{aligned}$$

C.2.2) The case $\text{Re}(ik) > 0$ in comparison to the case $k = 0$

Let us consider,

$$J_\epsilon(r') = \int_{R^3} 1_{\Omega'} g(r') (\text{div}(\text{grad}(\cdot)) + k^2) \left(\frac{-1_{R^3 \setminus \Omega'_\epsilon}}{4\pi|r' - r|}\right) dV_{r'}, \tag{134}$$

If we develop the integral in a manner similar to (131), the influence of k is negligible, and we obtain, like in (132),

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} J_\epsilon(r) &= g(r) \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi\epsilon^2} \int_{\Omega' \cap \partial\Omega'_\epsilon} dS, \\
\text{and, } (\text{div}(\text{grad}(\cdot)) + k^2) &\frac{-1_{R^3 \setminus \Omega'_\epsilon} e^{-ik|r' - r|}}{4\pi|r' - r|} = \delta(r' - r). \tag{135}
\end{aligned}$$

Now, developing the integral in a manner similar to (133), the parameter k clearly

influences the expression, although its value is the same for any k . So, we can write,

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} J_\epsilon(r') &= \int_{R^3} 1_{\Omega'} g(r') (\text{div}(\text{grad}(\cdot)) + k^2) \frac{-1_{R^3 \setminus \Omega'_\epsilon}}{4\pi|r' - r|} dV \\
&= \lim_{\epsilon \rightarrow 0} \int_{\Omega' \setminus \Omega'_\epsilon} \text{grad}(g(r')) \cdot \text{grad}\left(\frac{e^{-ik|r' - r|}}{4\pi|r' - r|}\right) dV \\
&\quad + pv \int_{\partial\Omega'} g(r') (-\hat{n}') \cdot \text{grad}\left(\frac{e^{-ik|r' - r|}}{4\pi|r' - r|}\right) dS \\
&\quad + pv \int_{\Omega'} k^2 g(r') \frac{-1_{R^3 \setminus \Omega'_\epsilon} e^{-ik|r' - r|}}{4\pi|r' - r|} dV.
\end{aligned} \tag{136}$$

and remark that the integrals on $\partial\Omega'$ vanish as $\Omega' \rightarrow R^3$ when $\text{Re}(ik) > 0$, which is not the case for $k = 0$, while the one on Ω' vanish for $k = 0$, which is not the case for $k \neq 0$. However, the sum of integrals is the same for $k = 0$ and for $\text{Re}(ik) > 0$. For that, let us consider the example of $g(r') = 1$ and $\Omega' \rightarrow R^3$. After noticing that,

$$-\int_0^\infty k^2 e^{-ik|r|} |r| d|r| = 1 \text{ and } \lim_{|r| \rightarrow \infty} e^{-ik|r|} = 0 \text{ when } \text{Re}(ik) > 0, \tag{137}$$

we note that,

$$\begin{aligned}
\lim_{\Omega' \rightarrow R^3} \int_{\partial\Omega'} (-\hat{n}') \cdot \text{grad}\left(\frac{e^{-ik|r' - r|}}{4\pi|r' - r|}\right) dS &= \begin{cases} =0 & \text{if } \text{Re}(ik) > 0 \\ =1 & \text{if } k=0 \end{cases}, \\
\lim_{\Omega' \rightarrow R^3} pv \int_{\Omega'} k^2 \frac{-e^{-ik|r' - r|}}{4\pi|r' - r|} dV &= \begin{cases} =1 & \text{if } \text{Re}(ik) > 0 \\ =0 & \text{if } k=0 \end{cases},
\end{aligned} \tag{138}$$

and, as expected, the sum of both terms $= 1$, for both cases $k = 0$ and $\text{Re}(ik) > 0$.

Appendix D : Integral expressions and equations when S is piecewise smooth

D.1) General integral expressions for piecewise smooth S

Let us consider an initial system with bianisotropic media, perturbed by a scatterer bounded by a closed surface $S = \sum_i S_i$, with geometric discontinuities along the frontiers ∂S_i . The ∂S_i , when they are positively oriented, are denoted \mathcal{L}_i . Considering the regularity of the field, we now assume the continuity across \mathcal{L}_i of fields components along it (note : this condition is related to canonical wedge cases and the uniqueness theorem of Levine).

Applying (47), we can write,

$$u_0.(1_\Omega^c E - E_0)(r) = \sum_i FP \int_{S_i} (\hat{n} \wedge H).E_{0C}(J_{0C}; r') \\ + (\hat{n} \wedge E).H_{0C}(J_{0C}; r') dS_{r'}, \quad (139)$$

where \hat{n} is the outward normal to S , and $u_0.E$ is continuous at $r \in \Omega \equiv R^3 \setminus \Omega_S$, outside and on S . The integrals are considered for their finite part (FP) (see details in previous appendix). The fields E and E_0 are respectively the field radiated in presence of the perturbed and the unperturbed system, while (E_{0C}, H_{0C}) are the fields attached to radiation of primary source $J_{0C} = u_0 \delta(r' - r)$ in presence of the unperturbed system with adjoint (complementary) characteristics.

For the points along \mathcal{L}_i , we can write,

$$u_0 = \hat{u}_e(r) \text{ tangent to the discontinuities along } \mathcal{L}_i, \\ \hat{u}_e(r).(1_\Omega^c E - E_{inc})(r) = \sum_i FP \int_{S_i} (\hat{n} \wedge H).E_{0C}(J_{0C}; r') \\ + (\hat{n} \wedge E).H_{0C}(J_{0C}; r') dS_{r'}, \quad (140)$$

taking account of the continuity across \mathcal{L}_i of the component of E along them. In a similar manner, for $J_0 = J = 0$, $M_{0C} = -u_0 \delta(r' - r)$, we have, from (49),

$$u_0.(1_\Omega^c H - H_0)(r) = \sum_i FP \int_{S_i} (\hat{n} \wedge H).E_{0C}(M_{0C}; r') \\ + (\hat{n} \wedge E).H_{0C}(M_{0C}; r') dS_{r'}, \quad (141)$$

where $u_0.H$ is continuous at $r \in \Omega \equiv R^3 \setminus \Omega_S$, while H and H_0 are respectively the field radiated in presence of the perturbed and the unperturbed system. We can take account of the continuity across the \mathcal{L}_i of the component of H along them, as previously for E .

D.2) Generalities when the domain $\Omega \equiv R^3 \setminus \Omega_S$ holds the isotropic free space

Let us consider the case where the domain $\Omega \equiv R^3 \setminus \Omega_S$, exterior to S , is filled by isotropic homogenous medium. Different choices of (E_{0C}, H_{0C}) are possible with this constraint, in particular the one where the unperturbed system in R^3 is the free space. In this case, considering previous equations, we can write,

$$\begin{aligned}
u_0.(1_\Omega H - H_{inc})(r) &= \sum_i FP \int_{S_i} ((\hat{n} \wedge H).E_{0C}(M_{0C}; r')) \\
&+ (\hat{n} \wedge E).H_{0C}(M_{0C}; r')) dS_{r'}, \\
\text{with } M_{0C} &= -u_0 \delta(r' - r), \quad E_{0C} = \text{curl}_{r'}(u_0 \frac{e^{-ikR}}{4\pi R}) = -u_0 \wedge \text{grad}_{r'}(\frac{e^{-ikR}}{4\pi R}), \\
\text{and } H_{0C} &= i\omega\varepsilon u_0 \frac{e^{-ikR}}{4\pi R} - \text{grad div} \frac{u_0}{i\omega\mu} \frac{e^{-ikR}}{4\pi R}, \tag{142}
\end{aligned}$$

and,

$$\begin{aligned}
u_0.(1_\Omega E - E_{inc})(r) &= \sum_i FP \int_{S_i} ((\hat{n} \wedge H).E_{0C}(J_{0C}; r')) \\
&+ (\hat{n} \wedge E).H_{0C}(J_{0C}; r')) dS_{r'}, \\
\text{with } J_{0C} &= u_0 \delta(r' - r), \quad H_{0C} = \text{curl}_{r'}(u_0 \frac{e^{-ikR}}{4\pi R}) = -u_0 \wedge \text{grad}_{r'}(\frac{e^{-ikR}}{4\pi R}), \\
\text{and } E_{0C} &= -i\omega\mu u_0 \frac{e^{-ikR}}{4\pi R} + \text{grad div} \frac{u_0}{i\omega\varepsilon} \frac{e^{-ikR}}{4\pi R}, \tag{143}
\end{aligned}$$

with $R = |r' - r|$ and $\text{div}(\frac{u_0 e^{-ikR}}{4\pi R}) = u_0 \cdot \text{grad}(\frac{e^{-ikR}}{4\pi R})$. We note that, considering a previous appendix, we have for $r \in \Omega \setminus \mathcal{L}_i$,

$$\begin{aligned}
&- FP \int_{S_i} (\hat{n} \wedge E). \text{grad div} \frac{u_0}{i\omega\mu} \frac{e^{-ikR}}{4\pi R} dS_{r'} \\
&= FP \int_{S_i} \hat{n}(r').((\text{grad div} \frac{u_0}{i\omega\mu} \frac{e^{-ikR}}{4\pi R}) \wedge E) dS_{r'} \\
&= -pv \int_{S_i} \hat{n}(r'). \text{curl}_{r'}(E) \text{div}_{r'}(\frac{\frac{u_0}{i\omega\mu} e^{-ikR}}{4\pi R}) dS_{r'} + \int_{\mathcal{L}_i} \text{div}_{r'}(\frac{\frac{u_0}{i\omega\mu} e^{-ikR}}{4\pi R}) E \cdot dr' \\
&= pv \int_{S_i} (\hat{n}(r').H(r')) u_0 \cdot \text{grad}_{r'}(\frac{e^{-ikR}}{4\pi R}) dS_{r'} + \int_{\mathcal{L}_i} \frac{u_0 \cdot \text{grad}(\frac{e^{-ikR}}{4\pi R})}{i\omega\mu} E(r') \cdot dr', \tag{144}
\end{aligned}$$

and

$$\begin{aligned}
&FP \int_{S_i} (\hat{n} \wedge H). \text{grad div} \frac{u_0}{i\omega\varepsilon} \frac{e^{-ikR}}{4\pi R} dS_{r'} \\
&= -FP \int_{S_i} \hat{n}.((\text{grad div} \frac{u_0}{i\omega\varepsilon} \frac{e^{-ikR}}{4\pi R}) \wedge H) dS_{r'} \\
&= pv \int_{S_i} \hat{n}(r'). \text{curl}_{r'}(H) \text{div}_{r'}(\frac{u_0}{i\omega\varepsilon} \frac{e^{-ikR}}{4\pi R}) dS_{r'} - \int_{\mathcal{L}_i} (\text{div}(\frac{u_0}{i\omega\varepsilon} \frac{e^{-ikR}}{4\pi R})) H \cdot dr' \\
&= pv \int_{S_i} (\hat{n}(r').E) u_0 \cdot \text{grad}(\frac{e^{-ikR}}{4\pi R}) dS_{r'} - \int_{\mathcal{L}_i} (\frac{u_0}{i\omega\varepsilon} \cdot \text{grad}(\frac{e^{-ikR}}{4\pi R})) H \cdot dr', \tag{145}
\end{aligned}$$

where $-\hat{n}.\text{curl}_{r'}(E)|_S = \text{Div}(\hat{n} \wedge E)$ and $-\hat{n}.\text{curl}_{r'}(H)|_S = \text{Div}(\hat{n} \wedge H)$.

D.3) Discontinuities and developments of integrals expressions

Assuming that all lines of discontinuities are along \mathcal{L}_i , we can use the continuity, across the oriented contours \mathcal{L}_i , of the components parallel to them. This implies that, when S is a closed surface,

$$\begin{aligned} \sum_i \int_{\mathcal{L}_i} \left(\frac{u_0}{i\omega\mu} \cdot \text{grad} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) E \cdot dr' &= 0, \\ \sum_i \int_{\mathcal{L}_i} \left(\frac{u_0}{i\omega\varepsilon} \cdot \text{grad} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) H \cdot dr' &= 0. \end{aligned} \quad (146)$$

We have,

$$\begin{aligned} -(\hat{n}.\text{curl}_{r'})(E)|_S &= \text{Div}_{r'}(\hat{n} \wedge E), \\ -(\hat{n}.\text{curl}_{r'})(H)|_S &= \text{Div}_{r'}(\hat{n} \wedge H), \end{aligned} \quad (147)$$

as defined in previous appendix, and we can develop the expressions of fields, outside and on S , following,

$$\begin{aligned} u_0.(1_\Omega H - H_{inc})(r) &= \sum_i \left(pv \int_{S_i} -(\hat{n} \wedge H).(u_0 \wedge \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right)) \right. \\ &+ (\hat{n} \wedge E).i\omega\varepsilon u_0 \frac{e^{-ikR}}{4\pi R} + \text{Div}(\hat{n} \wedge E) \left(\frac{u_0}{i\omega\mu} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) dS_{r'} \\ &\left. + \int_{\mathcal{L}_i} \left(\frac{u_0}{i\omega\mu} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) E \cdot dr' \right), \end{aligned} \quad (148)$$

and,

$$\begin{aligned} u_0.(1_\Omega E - E_{inc})(r) &= - \sum_i \left(pv \int_{S_i} \text{Div}(\hat{n} \wedge H) \left(\frac{u_0}{i\omega\varepsilon} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) \right. \\ &+ ((\hat{n} \wedge H).i\omega\mu u_0 \frac{e^{-ikR}}{4\pi R} + (\hat{n} \wedge E).(u_0 \wedge \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right)) dS_{r'} \\ &\left. + \int_{\mathcal{L}_i} \left(\frac{u_0}{i\omega\varepsilon} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) H \cdot dr' \right). \end{aligned} \quad (149)$$

where we note the following identity,

$$(\hat{n} \wedge V).(u_0 \wedge \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right)) = ((u_0.\hat{n})V - (u_0.V)\hat{n}).\text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right). \quad (150)$$

D.4) Integral equations for impedance boundary conditions on S

D.4.1) Impedance, continuity, and discontinuities

We now consider impedance boundary conditions on each S_i composing S , following,

$$(\hat{n} \wedge E)|_{S_i} = (Z)_i(\hat{n} \wedge (\hat{n} \wedge H))|_{S_i}, \quad (151)$$

for $i = 1, \dots, n$. All lines of discontinuities are along the oriented contours \mathcal{L}_i , the surface S is closed, and we have, from continuity of tangential component across \mathcal{L}_i ,

$$\begin{aligned} \sum_i \int_{\mathcal{L}_i} \left(\frac{u_0}{i\omega\mu} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) E \cdot dr' &= 0, \\ \sum_i \int_{\mathcal{L}_i} \left(\frac{u_0}{i\omega\varepsilon} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) H \cdot dr' &= 0, \end{aligned} \quad (152)$$

From impedance conditions, we remark that,

concerning the tangential components of fields on S ,
the boundness of parallel fields components along \mathcal{L}_i implies
also the ones of perpendicular ones when $(Z)_i$ is invertible,
but in contrary to components parallel to \mathcal{L}_i that are continuous across it,
the perpendicular components can be discontinuous across it. (153)

In what follows, we note :

$$\begin{aligned} \hat{u}_e^i(r) &\text{ is the unit tangent to } \mathcal{L}_i \text{ (positively oriented) as } r \in \mathcal{L}_i, \\ \hat{\nu}_e^i(r) &= -\hat{n} \wedge \hat{u}_e^i(r) \text{ is the outward normal to } \mathcal{L}_i. \end{aligned} \quad (154)$$

D.4.2) For $r \in \Omega \setminus \mathcal{L}_i$, i.e. on and outside S , except at discontinuities

For $r \in \Omega \setminus \mathcal{L}_i$, $i = 1, 2, \dots, n$, and $(Z)_i$ is piecewise smooth, we can write,

$$\begin{aligned} (1_\Omega H - H_{inc})(r) &= \sum_i \left(pv \int_{S_i} ((\hat{n} \wedge H) \wedge \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right)) + \right. \\ &\quad + \text{Div}((Z)_i(\hat{n} \wedge (\hat{n} \wedge H))) \frac{1}{i\omega\mu} \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \\ &\quad \left. + (Z)_i(\hat{n} \wedge (\hat{n} \wedge H)) i\omega\varepsilon \frac{e^{-ikR}}{4\pi R} \right) dS_{r'}, \end{aligned} \quad (155)$$

and

$$(1_\Omega E - E_{inc})(r) = - \sum_i \left(pv \int_{S_i} \text{Div}(\hat{n} \wedge H) \frac{1}{i\omega\epsilon} \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right. \\ \left. + (\hat{n} \wedge H) i\omega\mu \frac{e^{-ikR}}{4\pi R} + (Z)_i (\text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \wedge (\hat{n} \wedge (\hat{n} \wedge H))) dS_{r'} \right). \quad (156)$$

We can then obtain integral equations on tangential fields components for $r \in S_i \setminus \mathcal{L}_i$, $i = 1, 2, \dots, n$, considering, first integral equation,

$$\text{from (155) as } r \in S_i \setminus \mathcal{L}_i \text{ (reduce to surface MFIE for } (Z)_i = 0), \quad (157)$$

and second integral equation from the combination of (155) and (156) following,

$$(\hat{n} \wedge E)|_{S_i} - (Z)_i (\hat{n} \wedge (\hat{n} \wedge H))|_{S_i} = 0 \text{ as } r \in S_i \setminus \mathcal{L}_i \\ \text{(reduce to surface EFIE for } (Z)_i = 0), \quad (158)$$

with $J_s = (\hat{n} \wedge H)|_{S_i}$, $(H \cdot \hat{u}_e^i)$ and $(H \cdot \hat{\nu}_e^i)$ on \mathcal{L}_i , $i = 1, 2, \dots, n$, as unknowns.

D.4.3) For $r \in \mathcal{L}_i$ along lines of discontinuities on S

For $r \in \mathcal{L}_i$, we take $u_0 \equiv u_0(r)$ as $r \in \mathcal{L}_i$, following,

$$u_0(r) \equiv \hat{u}_e^i(r) \text{ is the unit tangent to } \mathcal{L}_i \text{ positively oriented as } r \in \mathcal{L}_i, \quad (159)$$

The components of E and H along $\hat{u}_e^i(r)$ are components that are continuous across discontinuities on \mathcal{L}_i , and we can write for $r \in \mathcal{L}_i$,

$$u_0(r) \cdot 1_\Omega H(r) = u_0(r) \cdot H_{inc}(r) \\ \sum_i \left(pv \int_{S_i} (- (\hat{n} \wedge H(r')) \cdot (u_0(r) \wedge \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right)) + \right. \\ \left. + \text{Div}_{r'}((Z)_i (\hat{n} \wedge (\hat{n} \wedge H(r')))) \left(\frac{u_0(r)}{i\omega\mu} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) \right. \\ \left. + (Z)_i (\hat{n} \wedge (\hat{n} \wedge H(r'))) \cdot i\omega\epsilon u_0(r) \frac{e^{-ikR}}{4\pi R} \right) dS_{r'}, \quad (160)$$

and

$$\begin{aligned}
u_0(r).1_\Omega E(r) &= u_0(r).E_{inc}(r) \\
&- \sum_i \left(pv \int_{S_i} \text{Div}(\hat{n} \wedge H(r')) \left(\frac{u_0(r)}{i\omega\varepsilon} \cdot \text{grad}_{r'} \left(\frac{e^{-ikR}}{4\pi R} \right) \right) \right. \\
&+ (\hat{n} \wedge H(r')).i\omega\mu u_0(r) \frac{e^{-ikR}}{4\pi R} \\
&\left. + (Z)_i(\hat{n} \wedge (\hat{n} \wedge H(r'))).(u_0(r) \wedge \text{grad} \left(\frac{e^{-ikR}}{4\pi R} \right)) dS_{r'} \right). \quad (161)
\end{aligned}$$

Therefore, those equations on \mathcal{L}_i can complete the system of integral equations given in (157) and (158) on $S \setminus \mathcal{L}_i$.

D.5) Integral expressions when S is a perfectly conducting surface in free space

Let us consider the example of a perfectly conducting surface S in free space, with a discontinuity line l . The electric tangential field E vanishes on S , and we can now write,

$$\begin{aligned}
u_0.(1_\Omega H - H_{inc})(r) &= pv \int_S (\hat{n} \wedge H(r')).\text{curl}_{r'} \left(u_0 \frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right) dS_{r'}, r \in \Omega \setminus l \\
\hat{u}_e.(1_\Omega H - H_{inc})(r) &= pv \int_S (\hat{n} \wedge H(r')).\text{curl}_{r'} \left(\hat{u}_e(r) \frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right) dS_{r'}, r \in l, \quad (162)
\end{aligned}$$

where $\hat{u}_e(r)$ is the unit vector tangent to l . In these equations, we note that,

$$\begin{aligned}
(n \wedge H).\text{curl}_{r'} \left(u_0 \frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right) &= u_0.((\hat{n} \wedge H) \wedge \text{grad}_{r'} \left(\frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right)), \\
(\hat{n} \wedge H).\text{curl}_{r'} \left(\hat{u}_e(r) \frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right) &= \text{grad}_{r'} \left(\frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right).(\hat{u}_e(r) \wedge (\hat{n} \wedge H)) \\
&= \text{grad}_{r'} \left(\frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right).((\hat{u}_e(r).H(r'))\hat{n}(r') - (\hat{u}_e(r).\hat{n}(r'))H(r')), \quad (163)
\end{aligned}$$

where the component $\hat{u}_e(r).H(r')$ is continuous across l , and $(\hat{u}_e(r).\hat{n}) \rightarrow 0$, when $r' \rightarrow r \in l$. Therefore, we obtain,

$$\begin{aligned}
(1_\Omega H - H_{inc})(r) &= pv \int_S (\hat{n} \wedge H) \wedge \text{grad}_{r'} \left(\frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right) dS_{r'}, r \in \Omega \setminus l, \\
1_\Omega \hat{u}_e(r).H(r) &= \hat{u}_e(r).H_{inc}(r) + pv \int_S \text{grad}_{r'} \left(\frac{e^{-ik|r'-r|}}{4\pi|r'-r|} \right).((\hat{u}_e(r).H(r'))\hat{n}(r') \\
&- (\hat{u}_e(r).\hat{n}(r'))H(r')) dS_{r'}, r \in l. \quad (164)
\end{aligned}$$

Remark : If we want to consider the system of a perfectly conducting plane, perturbed by a scatterer above it, with impedance boundary conditions, we can choose to change the free space Green functions for the one of the perfectly conducting plane. For that, we have to add the contributions of the image sources on each side of equations (155), (156), (160), and (161), which corresponds to make modifications following,

$$M_0 = [-u_0\delta(r' - r)] \rightarrow M_0 = [-u_0\delta(r' - r)] + [-(u_0 - 2(u_0.\hat{z})\hat{z})\delta(r' - r_{im})],$$

$$u_0 \frac{e^{-ikR}}{4\pi R} \rightarrow u_0 \frac{e^{-ikR}}{4\pi R} + (u_0 - 2(u_0.\hat{z})\hat{z}) \frac{e^{-ikR_{im}}}{4\pi R_{im}}, \quad (165)$$

and

$$J_0 = [u_0\delta(r' - r)] \rightarrow J_0 = [u_0\delta(r' - r)] + [-(u_0 - 2(u_0.\hat{z})\hat{z})\delta(r' - r_{im})],$$

$$u_0 \frac{e^{-ikR}}{4\pi R} \rightarrow u_0 \frac{e^{-ikR}}{4\pi R} - (u_0 - 2(u_0.\hat{z})\hat{z}) \frac{e^{-ikR_{im}}}{4\pi R_{im}}, \quad (166)$$

with $r_{im} = r - 2(r.\hat{z})\hat{z}$, $R = |r' - r|$ and $R_{im} = |r' - r_{im}|$.

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