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Total stability of equilibria motivates integral action in discrete-time nonlinear systems

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Abstract

Robustness guarantees are important properties to be looked for during control design. They ensure stability of closed-loop systems in face of uncertainties, unmodeled effects and bounded disturbances. Inspired by the continuous-time literature, in this article we investigate the total stability properties of dynamical discrete-time nonlinear systems. Such a property locally relates phase plots of dynamical systems of the same dimension which are “similar enough”. As a consequence, it enables the analysis of robustness properties via simple model difference in the discrete-time context. First, we study how existence of equilibria for a nominal model transfers to sufficiently similar ones. Then, we provide results on the propagation of stability guarantees to perturbed systems. Finally, we relate such properties to the constant output regulation problem by motivating the use of discrete-time integral action in discrete-time nonlinear systems for model robustness purposes.

1. Introduction

Stability of an equilibrium is of utter importance in the design of feedback controllers. Stability properties are typically inferred through the analysis of the system model in closed-loop. However, it is well known that uncertainties inherently exist in control applications, due to the presence of unmodeled effects and parameters mismatches. To tackle such an issue, robust control and robust stability analysis have become fundamental tools for control design [1]. The problem of synthesizing such robust controllers can be cast in an optimization framework, both in the linear and the nonlinear scenario, e.g. [2–6]. Nevertheless, a strong theoretical foundation turns out to be a fundamental design tool. While the theory on robust stability and regulation is well-developed and mature for continuous-time systems, see e.g., [7, 8], the discrete-time nonlinear scenario still misses some important results. In practice, control laws are often implemented via digital devices working at certain time instants. Moreover, many digital sensors provide measurements discontinuously in time, controlled by the frequencies of internal clocks. Thus, it can be effective to design controllers based on a discrete-time model of the plant, see, e.g., [9]. Although it is known that, for sufficiently small sampling times, continuous-time results are valid in the discrete framework (see, e.g., [10]), such an approach may be restrictive or inapplicable for some control applications.

In this article, we suppose that a discrete-time nonlinear model has already been derived and we focus on the robust stability properties of an equilibrium of such a model. In this context, some necessary local conditions linked to robustness appeared in [11]. Therein, the authors state that a necessary condition for local stability of nonlinear discrete-time autonomous systems comes from invertibility of the vector field at the equilibrium and close to it. The aforementioned condition can be seen as requiring the existence of a fixed point for a perturbed system that does not differ too much in norm from the initial one. Also, results on robust stability appeared in [12, Chapter 5]. Therein, it is shown that if the origin of the nominal system is locally stable and Lipschitz, then it is also locally robustly stable for bounded disturbances. More recently, in the context of converse Lyapunov theorems for discrete-time systems, [13, 14] proved that a necessary condition for robust stability is the existence of a smooth Lyapunov function. Successively, [15] extended the results to the stochastic framework. However, all the aforementioned works mainly focused on perturbations vanishing in the equilibrium. Moreover, when considering persistent constant disturbances (see e.g., [15, Section 7.1]) the authors only provide practical stability guarantees and do not characterize the system behavior inside the attractive set. This doesn’t guarantee the correct functioning of a tool such as the integral action, whose foundation lies in the existence of equilibria [8]. Indeed, to the best of the authors’ knowledge, there are no results mimicking well-established general results of “total stability” for continuous-time systems. The concept of total stability was early introduced in the works of Dubosin, Gorsin and Malkin [16, pp. 316] (see also [17] and references therein), and more recently

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studied in [8, 17–19]. In this context, robustness properties are analyzed directly via nonlinear unstructured models differences. This allows inferring the preservation of a stable equilibrium point for sufficiently similar plants by means of simple model comparison [8, Lemma 4, 5].

The goal of this paper is therefore to translate such results to the discrete-time scenario. We draw conclusions similar to the continuous case, yet under some fundamental differences, given by the discrete nature of the system. In particular, we show that stability properties of the equilibrium of a nominal model imply the existence and stability of an equilibrium (possibly different from the former) for any perturbed system sufficiently “close” to the nominal one. The result is proved under some regularity assumptions and bounded mismatches. Moreover, we provide a counterexample highlighting that some results from the continuous-time scenario may not apply in the discretized framework. This disproves some arguments of [11]. Finally, we link the obtained results on robust stability to the robust output regulation problem, building on recent forwarding techniques [20]. We justify the addition of an integral action for rejecting constant disturbances or tracking constant references. More specifically, we show that if the true model of the plant to be controlled is sufficiently close to one used for controller design, then output regulation is still achieved.

The rest of the paper is organized as follows: Section 2 presents the main result of the paper on total stability; Section 3 applies the result to the problem of constant output regulation; Section 4 comments and concludes the paper.

1.1. Notation

\mathbb{R} , resp. \mathbb{N} , denotes the set of real numbers, resp. non-negative integers. $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. In this work, we define a time-invariant nonlinear dynamical discrete-time system as $x^+ = f(x)$, where $x \in \mathbb{R}^n$ is the state solution evaluated at timestep $k \in \mathbb{N}$ with initial condition x_0 , and x^+ is the state solution at step $k+1$. Sets are denoted by calligraphic letters and, for a given set \mathcal{X} , we identify its boundary by $\partial\mathcal{X}$. The notation $\mathcal{X} \setminus \mathcal{G}$ identifies the intersection between \mathcal{X} and the complement of \mathcal{G} . When a set \mathcal{X} is strictly included in a set \mathcal{G} , we use the notation $\mathcal{X} \subsetneq \mathcal{G}$. We use $|\cdot|$ as the norm operator for matrices and vectors. Moreover, we denote as $d(x, \mathcal{X})$ a generic distance function between any point $x \in \mathbb{R}^n$ and a closed set $\mathcal{X} \subset \mathbb{R}^n$. For instance, one may choose $d(x, \mathcal{X}) = \inf_{z \in \mathcal{X}} |x-z|$. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{K} ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing. Similarly, we say $\alpha \in \mathcal{K}_{\infty}$ if $\alpha \in \mathcal{K}$ and $\lim_{s \rightarrow \infty} \alpha(s) = \infty$. Moreover, for a square matrix A we denote by $\lambda_{\max}(A)$ its maximum eigenvalue. Given a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, for any $A \in \mathbb{R}^{n \times n}$ and for any arbitrary scalar $r := r_1 r_2$ with $r_1, r_2 > 0$, the generalized Schur’s complement implies that

the following inequalities are equivalent

$$\begin{cases} A^{\top} P A - r P \preceq 0 \\ -P \prec 0 \end{cases} \iff \begin{pmatrix} -r_1 P & A^{\top} P \\ P A & -r_2 P \end{pmatrix} \preceq 0.$$

Given two topological spaces \mathcal{X}, \mathcal{Y} , a function $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be a homeomorphism if it is continuous, bijective (i.e. its inverse exists and it is unique), and its inverse is continuous. Moreover, \mathcal{X} and \mathcal{Y} are said to be homeomorphic if there exists a function $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ which is a homeomorphism.

2. Total stability results

In this section, we study how the stability properties of the origin of a given discrete-time autonomous nonlinear system

$$x^+ = f(x), \quad (1)$$

transfer to systems described by a sufficiently similar difference equation

$$x^+ = \hat{f}(x), \quad (2)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous functions. We propose two different results. The first one links the existence of an equilibrium for system (1) to the existence of an equilibrium for the perturbed system (2). Without loss of generality, we assume such an equilibrium for (1) to be the origin. We show that an equilibrium for (2) exists, provided that the two models are locally close enough. More precisely, the result holds if the functions f and \hat{f} are not too different in the C^0 norm, and system (1) presents a forward invariant set containing its equilibrium and which is homeomorphic to the unit ball. The second result considers the case where both the dynamics and the Jacobians of the two systems (1), (2) are sufficiently similar. Under such conditions, we show that the existence of a locally exponentially stable equilibrium for (1) implies the existence and uniqueness of a locally exponentially stable equilibrium for (2) close to it. Moreover, we present a lower bound on the size of the domain of attraction of the equilibrium for (2).

The definition of total stability historically involves quantitative properties of the solution initiated from an equilibrium point under constantly acting disturbances [16, pp. 303]. Inspired by these seminal works, we propose the following qualitative version of such a definition, which is related to the preservation of an equilibrium. In particular, we define local similarities in the phase plots between the vector fields of systems whose dynamics are close enough. A corresponding delta-epsilon condition is given by Theorem 1 in Section 2.4.

Definition 1 (Total stability). *The origin of system (1) is said to be totally stable if, for any \hat{f} close enough to f , we can associate an equilibrium x_e which is (locally) asymptotically stable for system (2).*

2.1. Existence of equilibrium

We now present the minimal assumption required to show the existence of an equilibrium for (2). To this end, we introduce the following notation. Given a positive function $V : \mathcal{A} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a positive real number $c > 0$, we denote the sublevel set of such a function as

$$\mathcal{V}_c(V) := \{x \in \mathcal{A} : V(x) \leq c\}. \quad (3)$$

Assumption 1. *Let \mathcal{A} be an open subset of \mathbb{R}^n . There exists a C^0 function $V : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $V(0) = 0$ and such that the following holds:*

1. *there exists a positive real number \bar{c} such that the set $\mathcal{V}_{\bar{c}}(V)$ is homeomorphic to the unit ball;*
2. *there exists $\rho \in (0, 1)$ such that*

$$V(f(x)) \leq \rho V(x), \quad \forall x \in \mathcal{V}_{\bar{c}}(V). \quad (4)$$

The first result is formalized by the following proposition. It states that, if the conditions of Assumption 1 hold, system (2) admits (at least) one equilibrium inside the set $\mathcal{V}_{\bar{c}}(V)$ if the functions f and \hat{f} are ‘‘close enough’’.

Proposition 1. *Let Assumption 1 hold. Then, for any positive $\underline{c} \leq \bar{c}$ there exists a positive real number δ such that, for any continuous function $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying*

$$|\hat{f}(x) - f(x)| < \delta, \quad \forall x \in \mathcal{V}_{\bar{c}}(V) \quad (5)$$

the corresponding system (2) admits an equilibrium point $x_e \in \mathcal{V}_{\underline{c}}(V)$. Moreover, such system has no other equilibrium in the set $\mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)$.

Proof. Consider $\underline{c} \leq \bar{c}$ and let $\hat{\rho}$ be any positive real number satisfying $\rho < \hat{\rho} < 1$. Since the set $\mathcal{V}_{\bar{c}}(V)$ is homeomorphic to the unit ball, it is bounded. Moreover, the function V being continuous, $\mathcal{V}_{\bar{c}}(V)$ is a compact subset. Next, we define the function $p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ as

$$p(s) = \max_{\substack{x \in \mathcal{V}_{\bar{c}}(V) \\ v \in \mathbb{R}^n : |v|=1}} \{V(f(x) + sv) - \bar{c}\}, \quad (6)$$

with s a positive real number. Recalling item 2 of Assumption 1, we obtain

$$\begin{aligned} p(0) &= \max_{\substack{x \in \mathcal{V}_{\bar{c}}(V) \\ v \in \mathbb{R}^n : |v|=1}} \{V(f(x)) - \bar{c}\} \\ &\leq \max_{\substack{x \in \mathcal{V}_{\bar{c}}(V) \\ v \in \mathbb{R}^n : |v|=1}} \{\rho V(x) - \bar{c}\} \leq (\rho - 1)\bar{c} < 0. \end{aligned} \quad (7)$$

Then, we define the function $q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ as

$$q(s) = \max_{\substack{x \in \mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V) \\ v \in \mathbb{R}^n : |v|=1}} \{V(f(x) + sv) - \hat{\rho}V(x)\}.$$

It satisfies

$$\begin{aligned} q(0) &= \max_{x \in \mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)} \{V(f(x)) - \hat{\rho}V(x)\}, \\ &\leq \max_{x \in \mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)} \{(\rho - \hat{\rho})V(x)\} \leq (\rho - \hat{\rho})\underline{c} < 0. \end{aligned} \quad (8)$$

As a consequence, since p, q are continuous functions satisfying (7), (8), there exists $\delta > 0$ such that

$$p(s) < 0, \quad q(s) < 0, \quad \forall s \in [0, \delta]. \quad (9)$$

Now, pick any continuous function \hat{f} satisfying (5). Note that for all x in $\mathcal{V}_{\bar{c}}(V)$ such that $f(x) \neq \hat{f}(x)$,

$$V(\hat{f}(x)) - \bar{c} = V(f(x) + sv) - \bar{c}$$

with $s = |f(x) - \hat{f}(x)|$, $v = \frac{f(x) - \hat{f}(x)}{|f(x) - \hat{f}(x)|}$. Consequently, for all x in $\mathcal{V}_{\bar{c}}(V)$ inequalities (9) and (5) imply

$$V(\hat{f}(x)) - \bar{c} \leq p(|f(x) - \hat{f}(x)|) < 0. \quad (10)$$

Hence, $\hat{f}(x) \in \mathcal{V}_{\bar{c}}(V)$. By assumption, $\mathcal{V}_{\bar{c}}(V)$ is homeomorphic to a unitary ball, which we denote as $\mathcal{B} = \{z \in \mathbb{R}^n : |z| \leq 1\}$. Hence, there exists two continuous mappings $T : \mathcal{V}_{\bar{c}}(V) \rightarrow \mathcal{B}$ and $T^{-1} : \mathcal{B} \rightarrow \mathcal{V}_{\bar{c}}(V)$ such that $T \circ T^{-1}(z) = z$ for all z in \mathcal{B} . Hence, the mapping $T \circ \hat{f} \circ T^{-1} : \mathcal{B} \rightarrow \mathcal{B}$ is a continuous function. Hence, by Brouwer’s fixed point theorem, there exists $z^* \in \mathcal{B}$ such that $T \circ \hat{f} \circ T^{-1}(z^*) = z^*$. Thus, it implies $\hat{f} \circ T^{-1}(z^*) = T^{-1}(z^*)$. We deduce that $x_e = T^{-1}(z^*)$ is a fixed point belonging to $\mathcal{V}_{\bar{c}}(V)$. Now, let us consider the set $\mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)$. As before, with the same definitions of s and v , inequalities (9) and (5) imply that, for all $x \in \mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)$, it holds

$$\begin{aligned} V(\hat{f}(x)) - \hat{\rho}V(x) &= V(f(x) + sv) - \hat{\rho}V(x) \\ &\leq q(s) < 0. \end{aligned}$$

Hence, $\hat{f}(x) \neq x$ for all $x \in \mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)$. Consequently, x_e belongs to $\mathcal{V}_{\underline{c}}(V)$ and this concludes the proof. \square

2.2. Comments about Assumption 1

We remark that Assumption 1 and Proposition 1 do not imply that equilibria of systems (1) and (2) are unique or attractive. Indeed, Assumption 1 is not requiring the nominal system (1) to be asymptotically stable. It solely assumes the existence of a forward invariant compact set that is homeomorphic to the unit ball. As a matter of fact, since V is not strictly positive outside of the origin (i.e. we are not asking the function V to be lower-bounded by a class \mathcal{K} function of the norm of x), V may have multiple local minima. This does not allow concluding asymptotic stability of the origin. In order to clarify this aspect, we give the following simple example. Consider a system of the form

$$x^+ = f(x) := \begin{cases} \frac{1}{2}(x + c) & \text{if } x \geq c \\ \frac{1}{2}(x - c) & \text{if } x \leq -c \\ x & \text{otherwise} \end{cases}$$

where $x \in \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a piecewise continuous linear function and c is a strictly positive constant. It can

be seen that all the points satisfying $|x| \leq c$ are equilibrium points, yet, none of them is attractive. Now, select V as

$$V(x) := \begin{cases} 0 & \text{if } |x| \leq c, \\ |x| - c & \text{otherwise.} \end{cases}$$

It is non-negative, continuous, and all level sets identified by $\bar{c} > 0$ are homeomorphic to the unit ball. Moreover, by the definition of f and since $c > 0$ we obtain

$$V(x^+) = \begin{cases} 0 \leq \frac{1}{2}V(x) & \text{if } |x| \leq c \\ \frac{1}{2}|x| - \frac{1}{2}c = \frac{1}{2}V(x) & \text{otherwise} \end{cases}$$

This shows that Assumption 1 holds and Proposition 1 applies. For instance, consider $\hat{f}(x) := f(x) + \delta$, satisfying (5). Pick $\bar{c} = 2c$ and $\underline{c} = c$. If $2|\delta| \leq c$ then \hat{f} admits a unique fixed point x_e satisfying $|x_e| = c + 2|\delta| \leq 2c$ and no other equilibrium exists inside the set $\mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V) := \{2c < |x_e| \leq 3c\}$. Finally, as a further example, by selecting any $0 < \delta < \frac{1}{2}c$ and

$$\hat{f}(x) = \begin{cases} \delta & \text{if } 0 \leq x \leq \delta \\ \frac{cx - \delta^2}{c - \delta} & \text{if } \delta \leq x \leq c \\ f(x) + \delta & \text{otherwise} \end{cases}$$

it can be verified that $x_e^* = \delta$ is an additional equilibrium point with respect to the previously defined x_e . This new equilibrium lies inside $\mathcal{V}_0(V)$.

2.3. Continuous and discrete-time results

Proposition 1 establishes the conditions under which the existence of an equilibrium for the nominal model (1) implies the existence of an equilibrium for any perturbed system (2) sufficiently close to the nominal one. This result parallels the continuous-time one presented in [8, Lemma 4]. However, different assumptions are needed. In the continuous-time case, the origin of the nominal system is supposed to be asymptotically stable. In turn, for any forward invariant set, this ensures the existence of a Lyapunov function whose level sets are homeomorphic to a sphere, see [21, Theorem 1.2]. Unfortunately, in the discrete-time case this fact is in general not true. As a matter of fact, Lyapunov level sets may be non-homeomorphic to spheres, contrarily to what is stated in [11, Proof of Theorem 2.7]. As an example, see [22]. This phenomenon is due to the nature of such systems, as the presence of jumps doesn't allow an easy translation of continuous-time results. As a consequence, in Assumption 1 we ask for the existence of an invariant sublevel set $\mathcal{V}_{\bar{c}}(V)$ homeomorphic to a ball. At the same time, we do not require asymptotic stability of the origin.

In the following, we present an example showing the aforementioned behavior. In particular, we focus on a simple and stable linear system. We carefully craft a non-trivial Lyapunov function guaranteeing asymptotic stability of the origin. Successively, we exploit the structure of

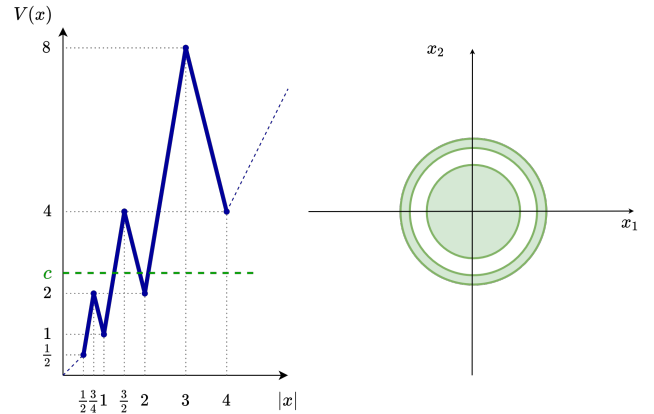


Figure 1: Lyapunov sublevel set for $x \in \mathbb{R}^2$

such a function to show that it has no sublevel set which is homeomorphic to a ball. Consider the linear system

$$x^+ = f(x) := \frac{1}{2}I_n x,$$

where $x \in \mathbb{R}^n$, I_n is the identity matrix of dimension n and the candidate Lyapunov function

$$V(x) = \begin{cases} 0 & x = 0 \\ 6|x| - 5 \cdot 2^i & 2^i \leq |x| < 2^i + 2^{i-1} \\ -4|x| + 5 \cdot 2^{i+1} & 2^i + 2^{i-1} \leq |x| < 2^{i+1} \end{cases}$$

where $i \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and it is uniquely defined by $|x|$. It can be verified that $V(x)$ is continuous. Consider the case where $2^i \leq |x| < 2^i + 2^{i-1}$ for some i in \mathbb{Z} . We have $2^{i-1} \leq |f(x)| < 2^{i-1} + 2^{i-2}$ and

$$\begin{aligned} V(f(x)) - V(x) &= 6|f(x)| - 5 \cdot 2^{i-1} - 6|x| + 5 \cdot 2^i \\ &= -3|x| + 5 \cdot 2^{i-1} < -2^{i-1} < 0. \end{aligned}$$

Similarly, for $2^i + 2^{i-1} \leq |x| < 2^{i+1}$ for some i in \mathbb{Z} we obtain $2^{i-1} + 2^{i-2} \leq |x| < 2^i$ and

$$\begin{aligned} V(f(x)) - V(x) &= -4|f(x)| + 5 \cdot 2^i + 4|x| - 5 \cdot 2^{i+1} \\ &= 2|x| - 5 \cdot 2^i < -2^i < 0. \end{aligned}$$

Hence, V is a continuous Lyapunov function for the system. However, it does not exist any $c > 0$ such that the corresponding sublevel set $\mathcal{V}_c(V)$ defined in (3) is homeomorphic to a ball, since each sublevel set is not path-connected. Figure 1 shows such a behavior for the planar case $x \in \mathbb{R}^2$.

Clearly, one could have picked a quadratic Lyapunov function for such a linear system. Yet, this example shows that discrete-time Lyapunov functions do not always possess the desired homeomorphicity property. As a consequence, we cannot guarantee that Lyapunov functions provided by converse Lyapunov theorems (e.g., [23]) satisfy such a property.

2.4. Existence of an exponentially stable equilibrium

We now present the main result on total stability of this paper, which is formalized in the next theorem. This result parallels [8, Lemma 5]. We exploit the asymptotic properties of the equilibrium for the nominal system (1) to prove existence, uniqueness and stability of an equilibrium for (2).

Theorem 1. *Assume the origin of the system (1) is asymptotically stable with domain of attraction \mathcal{A} and locally exponentially stable. Let $\underline{\mathcal{C}}$ be an arbitrary compact set satisfying $\{0\} \subsetneq \underline{\mathcal{C}} \subsetneq \mathcal{A}$ and suppose the function f is C^0 for all $x \in \mathcal{A}$ and C^1 for all $x \in \underline{\mathcal{C}}$. Then, for any forward invariant compact set $\bar{\mathcal{C}}$ verifying*

$$\{0\} \subsetneq \underline{\mathcal{C}} \subsetneq \bar{\mathcal{C}} \subsetneq \mathcal{A},$$

there exists a positive scalar $\delta > 0$ such that, for any function \hat{f} which is C^0 for all $x \in \mathcal{A}$ and C^1 for all $x \in \underline{\mathcal{C}}$ and satisfying

$$|\hat{f}(x) - f(x)| \leq \delta, \quad \forall x \in \bar{\mathcal{C}}, \quad (11)$$

$$\left| \frac{\partial \hat{f}}{\partial x}(x) - \frac{\partial f}{\partial x}(x) \right| \leq \delta, \quad \forall x \in \underline{\mathcal{C}}, \quad (12)$$

the corresponding system (2) admits an equilibrium point $x_e \in \underline{\mathcal{C}}$, which is asymptotically stable with a domain of attraction containing $\bar{\mathcal{C}}$ and locally exponentially stable. In other words, the origin of system (1) is totally stable.

Note that, differently from [8, Lemma 5], the nominal and perturbed models are required to be C^1 inside $\underline{\mathcal{C}}$ solely, while C^0 outside. This allows considering interesting continuous functions, such as saturation functions, which are not differentiable everywhere.

Proof. The proof is organized in several steps. We start by establishing the existence, uniqueness and local exponential stability of an equilibrium for the perturbed system (2). Then, we characterize the size of its domain of attraction by combining the converse Lyapunov theorems established in [23] and techniques similar to the ones used in the proof of Proposition 1.

Step 1: Local analysis. Let Π be a positive definite symmetric matrix and $a \in (0, 1)$ a real scalar satisfying

$$\frac{\partial f^\top}{\partial x}(0)\Pi\frac{\partial f}{\partial x}(0) \preceq a\Pi.$$

Since the origin is a locally exponentially stable equilibrium for system (1), its linearization around the origin is stable. Hence, the existence of Π is guaranteed by discrete-time Lyapunov inequality. In the following, given any $c > 0$, we denote with $\mathcal{V}_c(\Pi)$ the subset of \mathbb{R}^n defined as

$$\mathcal{V}_c(\Pi) := \{x \in \mathbb{R}^n : x^\top \Pi x \leq c\} \subset \mathbb{R}^n. \quad (13)$$

Now, note that the quadratic Lyapunov function defined by Π is a local Lyapunov function for system (1). Moreover, note that for any $a \in (0, 1)$ we have $a \leq \frac{1+a}{2} < 1$. Then, by continuity there exists a real number $\varepsilon > 0$ such that $\mathcal{V}_\varepsilon(\Pi) \subseteq \underline{\mathcal{C}}$ and

$$\frac{\partial f^\top}{\partial x}(x)\Pi\frac{\partial f}{\partial x}(x) \preceq \frac{1+a}{2}\Pi \quad \forall x \in \mathcal{V}_\varepsilon(\Pi).$$

Equivalently (refer to Section 1.1) it holds

$$\Psi(x) := \begin{pmatrix} -\frac{1+a}{2}\Pi & \frac{\partial f^\top}{\partial x}(x)\Pi \\ \Pi\frac{\partial f}{\partial x}(x) & -\Pi \end{pmatrix} \preceq 0 \quad (14)$$

for all $x \in \mathcal{V}_\varepsilon(\Pi)$. Consider the candidate Lyapunov function

$$V(x) = x^\top \Pi x, \quad V(x^+) = f(x)^\top \Pi f(x).$$

By defining a function $F : \mathbb{R} \rightarrow \mathbb{R}^n$ as $F(s) := f(sx)$, and since we assumed the origin to be an equilibrium point for f , we have

$$f(x) = f(x) - f(0) = F(1) - F(0) = \int_0^1 \frac{\partial f}{\partial x}(sx)x ds.$$

Then, recalling that $\frac{1+a}{2} < 1$ and via the addition and subtraction of $V(x^+)$, we compute

$$\begin{aligned} V(x^+) - \frac{1+a}{2}V(x) &= -\frac{1+a}{2}x^\top \Pi x \\ &\quad + 2 \int_0^1 x^\top \frac{\partial f^\top}{\partial x}(sx)\Pi f(x) ds - f(x)^\top \Pi f(x) \\ &= -\frac{1+a}{2}x^\top \Pi x \int_0^1 ds + \int_0^1 x^\top \frac{\partial f^\top}{\partial x}(sx)\Pi f(x) ds \\ &\quad + f(x)^\top \Pi \int_0^1 \frac{\partial f}{\partial x}(sx)x ds - f(x)^\top \Pi f(x) \int_0^1 ds \\ &= \int_0^1 \left[-\frac{1+a}{2}x^\top \Pi x + x^\top \frac{\partial f^\top}{\partial x}(sx)\Pi f(x) \right. \\ &\quad \left. + f(x)^\top \Pi \frac{\partial f}{\partial x}(sx)x - f(x)^\top \Pi f(x) \right] ds \\ &= (x^\top \quad f(x)^\top) \int_0^1 \Psi(sx) ds \begin{pmatrix} x \\ f(x) \end{pmatrix} \leq 0 \end{aligned} \quad (15)$$

for all $x \in \mathcal{V}_\varepsilon(\Pi)$ and $s \in [0, 1]$, where in the last step we used the definition of Ψ in (14).

Step 2: Existence of an equilibrium for the perturbed system. Now, let \hat{f} satisfy

$$|\hat{f}(x) - f(x)| \leq \delta_1 := \sqrt{\frac{\varepsilon(1-a)^2}{8\lambda_{\max}(\Pi)(3+a)}}, \quad (16)$$

for all $x \in \mathcal{V}_{\frac{\varepsilon}{3}}(\Pi)$. In the following, we show that the existence of such a bound and the local stability of system

(1) imply the existence of an equilibrium point $x_e \in \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ for the perturbed system (2). To this aim, we leverage on Brouwer's fixed point theorem. Hence, we need to show that the set $\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ is forward invariant for system (2) and that it is homeomorphic to a unit ball.

Let us choose $\bar{x} \in \mathbb{R}^n$ satisfying $\bar{x} \in \partial\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ and note that for all $x \in \partial\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$

$$\begin{aligned} \hat{f}(x)^\top \Pi \hat{f}(x) &= \left[\hat{f}(x) - f(x) \right]^\top \Pi \left[\hat{f}(x) - f(x) \right] \\ &\quad + f(x)^\top \Pi f(x) + 2 \left[\hat{f}(x) - f(x) \right]^\top \Pi f(x). \end{aligned}$$

Then, using the generalized Young's inequality $2\alpha\beta \leq \nu^{-1}\alpha^2 + \nu\beta^2$ with $\nu = \frac{1-a}{2(1+a)}$ on the last term, we obtain

$$\begin{aligned} \hat{f}(x)^\top \Pi \hat{f}(x) &\leq \frac{3+a}{1-a} \left[\hat{f}(x) - f(x) \right]^\top \Pi \left[\hat{f}(x) - f(x) \right] \\ &\quad + \frac{3+a}{2(1+a)} f(x)^\top \Pi f(x) \\ &\leq \frac{1-a}{4} \bar{x}^\top \Pi \bar{x} + \frac{3+a}{4} x^\top \Pi x \\ &\leq \frac{1-a}{4} \bar{x}^\top \Pi \bar{x} + \frac{3+a}{4} \bar{x}^\top \Pi \bar{x} \leq \bar{x}^\top \Pi \bar{x}, \end{aligned}$$

where we used inequality (16) in the second step for bounding the first term on the right-hand side of the inequality.

Thus, $\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ is forward invariant for the system (2). Moreover, since $\partial\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ is the level surface of a quadratic Lyapunov function, it is homeomorphic to a sphere. Hence, the set $\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ is homomorphic to a closed unit ball. Following the proof of Proposition 1 and employing Brouwer's fixed point theorem, there exists a point $x_e \in \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ satisfying $\hat{f}(x_e) = x_e$.

Step 3: Local exponential stability. Now we show that $x_e \in \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ is locally exponentially stable for (2). In particular, we show that if \hat{f} satisfies

$$\left| \frac{\partial \hat{f}}{\partial x}(x) - \frac{\partial f}{\partial x}(x) \right| \leq \delta_2 := \frac{1-a}{2\sqrt{10+6a}}, \quad (17)$$

for all $x \in \mathcal{V}_\varepsilon(\Pi)$, then, by denoting

$$\begin{aligned} \tilde{x} &:= x - x_e \\ \bar{f}(\tilde{x}) &:= \hat{f}(\tilde{x} + x_e) - \hat{f}(x_e) = \hat{f}(x) - x_e, \end{aligned}$$

the candidate Lyapunov function $\hat{V}(\tilde{x}) = \tilde{x}^\top \Pi \tilde{x}$ has to satisfy

$$\hat{V}(\tilde{x}^+) = \bar{f}(\tilde{x})^\top \Pi \bar{f}(\tilde{x}) \leq \frac{3+a}{4} \hat{V}(\tilde{x}) \leq \hat{V}(\tilde{x}) \quad (18)$$

for all $\tilde{x} \in \tilde{\mathcal{V}}_\varepsilon(\Pi)$, where

$$\tilde{\mathcal{V}}_\varepsilon(\Pi) = \{\tilde{x} \in \mathbb{R}^n : (x_e + s\tilde{x}) \in \mathcal{V}_\varepsilon(\Pi), \forall s \in [0, 1]\}.$$

To show such a property, first note that by defining a function $G : \mathbb{R} \rightarrow \mathbb{R}^n$ as $G(s) = \hat{f}(x_e + s\tilde{x})$, it holds

$$\bar{f}(\tilde{x}) = G(1) - G(0) = \int_0^1 \frac{\partial \hat{f}}{\partial x}(x_e + s\tilde{x}) ds \tilde{x}.$$

Then, similarly to the previous part of the proof, we compute

$$\begin{aligned} \hat{V}(\tilde{x}^+) - \frac{3+a}{4} \hat{V}(\tilde{x}) &= -\frac{3+a}{4} \tilde{x}^\top \Pi \tilde{x} - \bar{f}(\tilde{x})^\top \Pi \bar{f}(\tilde{x}) \\ &\quad + 2\bar{f}(\tilde{x})^\top \Pi \int_0^1 \frac{\partial \hat{f}}{\partial x}(x_e + s\tilde{x}) \tilde{x} ds \\ &= \left(\tilde{x}^\top \quad \bar{f}(\tilde{x})^\top \right) \int_0^1 \hat{\Psi}(x_e + s\tilde{x}) ds \begin{pmatrix} \tilde{x} \\ \bar{f}(\tilde{x}) \end{pmatrix}, \quad (19) \end{aligned}$$

where in the last step we defined

$$\hat{\Psi}(x) := \begin{pmatrix} -\frac{3+a}{4} \Pi & \frac{\partial \hat{f}^\top}{\partial x}(x) \Pi \\ \Pi \frac{\partial \hat{f}}{\partial x}(x) & -\Pi \end{pmatrix}. \quad (20)$$

Recalling Section 1.1 and since $a \in (0, 1)$, if

$$\Phi(x) := \begin{pmatrix} -\frac{5+3a}{8} \Pi & \frac{\partial \hat{f}^\top}{\partial x}(x) \Pi \\ \Pi \frac{\partial \hat{f}}{\partial x}(x) & -\frac{2(3+a)}{5+3a} \Pi \end{pmatrix} \preceq 0$$

then $\hat{\Psi}(x) \preceq 0$. Thus, we can study the sign semi-definiteness of $\Phi(x)$ to conclude the sign semi-definiteness of $\hat{\Psi}(x)$. By adding and subtracting $\Psi(x)$ to $\Phi(x)$ and recalling inequality (14), we obtain,

$$\Phi(x) = \Phi(x) - \Psi(x) + \Psi(x) \preceq \Phi(x) - \Psi(x)$$

Then,

$$\Phi(x) \preceq \begin{pmatrix} \frac{a-1}{8} \Pi & \frac{\partial \hat{f}^\top}{\partial x}(x) \Pi \\ \Pi \frac{\partial \hat{f}}{\partial x}(x) & \frac{a-1}{5+3a} \Pi \end{pmatrix},$$

with $\frac{\partial \hat{f}}{\partial x}(x) := \frac{\partial \hat{f}}{\partial x}(x) - \frac{\partial f}{\partial x}(x)$. Via generalized Schur's complement and bound (17), by following the lines of Step 2 and since $a \in (0, 1)$, we conclude that the matrix on the right-hand side is negative semi-definite. Consequently, it holds $\hat{\Psi}(x) \preceq 0$ for all $x \in \Omega_\varepsilon$. As a consequence, we conclude $\hat{\Psi}(x_e + s\tilde{x}) \preceq 0$ for all $\tilde{x} \in \tilde{\Omega}$ showing (18). We proved that x_e is locally exponentially stable for (2) with a domain of attraction including the set $\{\tilde{x} \in \tilde{\mathcal{V}}_\varepsilon(\Pi)\}$. Note that by the definition of $\tilde{\mathcal{V}}_\varepsilon(\Pi)$ and since $\mathcal{V}_\varepsilon(\Pi)$ is convex, the point $x_e + s\tilde{x} = (1-s)x_e + sx$ belongs to the set $\mathcal{V}_\varepsilon(\Pi)$ for all $(x, x_e, s) \in \mathcal{V}_\varepsilon(\Pi) \times \mathcal{V}_\varepsilon(\Pi) \times [0, 1]$. Hence, the domain of attraction of $x_e \in \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$ contains the set $\{x \in \mathcal{V}_\varepsilon(\Pi)\}$.

Step 4: Domain of attraction. We now provide a stronger lower bound on the size of the domain of attraction of the equilibrium point x_e for the perturbed system (2). In particular, we show that x_e has a domain of attraction that includes the set $\bar{\mathcal{C}}$. First, by picking

$$\delta_3 = \min\{\delta_1, \delta_2\} \quad (21)$$

where δ_1 comes from (16) and δ_2 is defined in (17), we obtain that the point x_e is a locally exponentially stable equilibrium point for (2). Moreover, it is contained in $\underline{\mathcal{C}}$ and its domain of attraction includes $\mathcal{V}_\varepsilon(\Pi)$.

Now, since $\bar{\mathcal{C}}$ is forward invariant and since system (1) is time-invariant and described by a continuous function on \mathcal{A} , we can leverage on [23, Theorem 1] to claim the existence of smooth Lyapunov functions $V_0 : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ and $V_{\bar{\mathcal{C}}} : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{A}$ it holds

$$\begin{aligned} \alpha_1(|x|) &\leq V_0(x) \leq \alpha_2(|x|), \\ V_0(f(x)) &\leq \rho_0 V_0(x), \\ V_0(x) = 0 &\iff x = 0, \end{aligned}$$

and

$$\begin{aligned} \alpha_1(d(x, \bar{\mathcal{C}})) &\leq V_{\bar{\mathcal{C}}}(x) \leq \alpha_2(d(x, \bar{\mathcal{C}})), \\ V_{\bar{\mathcal{C}}}(f(x)) &\leq \rho_{\bar{\mathcal{C}}} V_{\bar{\mathcal{C}}}(x), \\ V_{\bar{\mathcal{C}}}(x) = 0 &\iff x \in \bar{\mathcal{C}}, \end{aligned}$$

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $(\rho_0, \rho_{\bar{\mathcal{C}}}) \in (0, 1) \times (0, 1)$. Note that the two Lyapunov functions can be bounded by different \mathcal{K}_∞ functions. Yet, we can select the minimum (maximum) between those and obtain the same lower (upper) bound for the two. Consider now the set $\bar{\mathcal{C}}$. Since it is compact and included in \mathcal{A} , there exists a strictly positive real number \bar{d} such that the set $\mathcal{D} := \{x \in \mathcal{A} : d(x, \bar{\mathcal{C}}) \in (0, \bar{d}]\}$ is a subset of \mathcal{A} , namely $\mathcal{D} \subsetneq \mathcal{A}$. Let

$$\mathbf{V}(x) = V_{\bar{\mathcal{C}}}(x) + \sigma V_0(x), \quad (22)$$

with

$$\sigma = \frac{\alpha_1(\bar{d})}{2\nu}, \quad \nu = \sup_{x \in \mathcal{A} : d(x, \bar{\mathcal{C}}) \leq \bar{d}} V_0(x).$$

By picking $\rho = \max\{\rho_0, \rho_{\bar{\mathcal{C}}}\}$, for all $x \in \mathcal{A}$ it satisfies

$$\begin{aligned} \alpha_3(|x|) &\leq \mathbf{V}(x) \leq \alpha_4(|x|), \\ \mathbf{V}(f(x)) &\leq \rho \mathbf{V}(x), \\ \mathbf{V}(x) = 0 &\iff x = 0 \end{aligned} \quad (23)$$

with $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$. This implies that \mathbf{V} is a Lyapunov function for system (1) on \mathcal{A} . Define the following sets

$$\begin{aligned} \partial\mathcal{V}_{\bar{d}}(\mathbf{V}) &:= \{x \in \mathcal{A} : \mathbf{V}(x) = \alpha_1(\bar{d})\}, \\ \mathcal{V}_{\bar{d}}(\mathbf{V}) &:= \{x \in \mathcal{A} : \mathbf{V}(x) \leq \alpha_1(\bar{d})\}. \end{aligned} \quad (24)$$

Due to the definitions of $V_{\bar{\mathcal{C}}}$ in (22) and its lower bound, we have

$$\alpha_1(d(x, \bar{\mathcal{C}})) \leq V_{\bar{\mathcal{C}}}(x) \leq \mathbf{V}(x).$$

Hence, $d(x, \bar{\mathcal{C}}) \leq \bar{d}$ for all $x \in \mathcal{V}_{\bar{d}}(\mathbf{V})$. As a consequence, for all $x \in \mathcal{V}_{\bar{d}}(\mathbf{V})$, we have $\nu \geq V_0(x)$ and by (22) it holds

$$V_{\bar{\mathcal{C}}}(x) = \alpha_1(\bar{d}) \left(1 - \frac{V_0(x)}{2\nu}\right) > 0.$$

These last relations imply $\bar{\mathcal{C}} \subsetneq \mathcal{V}_{\bar{d}}(\mathbf{V}) \subseteq \mathcal{D} \subsetneq \mathcal{A}$. Now, let us identify by $\underline{v} > 0$ a scalar such that if $\mathbf{V}(x) \leq \underline{v}$ then $x \in \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$, namely $x^\top \Pi x \leq \frac{\varepsilon}{2}$ for all $x \in \mathcal{A} : \mathbf{V}(x) \leq \underline{v}$. Define

$$\delta_4 := \min \left\{ \inf_{(y', y'') \in \partial\mathcal{V}_{\bar{d}}(\mathbf{V}) \times \partial\bar{\mathcal{C}}} |y' - y''|, \frac{(1 - \rho)\underline{v}}{\sup_{x \in \mathcal{V}_{\bar{d}}(\mathbf{V})} \left| \frac{\partial \mathbf{V}}{\partial x}(x) \right|} \right\} \quad (25)$$

and consider (11) with (25). We can define the function $p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ as

$$p(s) = \max_{x \in \bar{\mathcal{C}}, |v|=1} \{V(f(x) + sv) - \alpha_1(\bar{d})\}.$$

Then, by the first argument of (25) and (24), for all $s \in [0, \delta_4]$ we have $p(s) < 0$. Note that for all x in $\bar{\mathcal{C}}$ such that $f(x) \neq \hat{f}(x)$,

$$\mathbf{V}(\hat{f}(x)) - \alpha_1(\bar{d}) = V(f(x) + sv) - \alpha_1(\bar{d}) \quad (26)$$

with $s = |\hat{f}(x) - f(x)|$, $v = \frac{\hat{f}(x) - f(x)}{|\hat{f}(x) - f(x)|}$. Consequently, for all x in $\bar{\mathcal{C}}$,

$$\mathbf{V}(\hat{f}(x)) - \alpha_1(\bar{d}) \leq p(|\hat{f}(x) - f(x)|) \leq 0. \quad (27)$$

Thus, $\hat{f}(x) \in \mathcal{V}_{\bar{d}}(\mathbf{V})$ for all $x \in \bar{\mathcal{C}}$ and

$$f(x) + s(\hat{f}(x) - f(x)) \in \mathcal{V}_{\bar{d}}(\mathbf{V}), \quad \forall (x, s) \in \bar{\mathcal{C}} \times [0, 1].$$

Then, by picking $s \in [0, 1]$ and a function $H : \mathbb{R} \rightarrow \mathbb{R}$ as $H(s) = V(f(x) + s(\hat{f}(x) - f(x)))$, we have

$$\begin{aligned} \mathbf{V}(\hat{f}(x)) - \mathbf{V}(f(x)) &= H(1) - H(0) \\ &= \int_0^1 \frac{\partial \mathbf{V}}{\partial x} \left(f(x) + s(\hat{f}(x) - f(x)) \right) ds [\hat{f}(x) - f(x)]. \end{aligned}$$

Consequently, for all $x \in \mathcal{V}_{\bar{d}}(\mathbf{V})$, we obtain

$$\begin{aligned} \mathbf{V}(\hat{f}(x)) &\leq \mathbf{V}(f(x)) + \int_0^1 \sup_{x \in \mathcal{V}_{\bar{d}}(\mathbf{V})} \left| \frac{\partial \mathbf{V}}{\partial x}(x) \right| ds |\hat{f}(x) - f(x)| \\ &< \rho \mathbf{V}(x) + \sup_{x \in \mathcal{V}_{\bar{d}}(\mathbf{V})} \left| \frac{\partial \mathbf{V}}{\partial x}(x) \right| \delta < \rho \mathbf{V}(x) + (1 - \rho)\underline{v}. \end{aligned}$$

Since $\rho \in (0, 1)$, it holds

$$\begin{cases} \mathbf{V}(\hat{f}(x)) < \mathbf{V}(x) & \forall x \in \bar{\mathcal{C}} \setminus \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi) \\ \mathbf{V}(\hat{f}(x)) < \underline{v} & \forall x \in \mathcal{V}_{\frac{\varepsilon}{2}}(\Pi). \end{cases} \quad (28)$$

Therefore, we may pick any function \hat{f} satisfying conditions (11), (12) with $\delta < \min\{\delta_3, \delta_4\}$. The analysis shows

that bound (25) ensures trajectories of the perturbed system (2) initialized in $\bar{\mathcal{C}}$ converge to $\mathcal{V}_{\frac{\varepsilon}{2}}(\Pi)$. Moreover, we previously proved $\mathcal{V}_{\varepsilon}(\Pi)$ is included in the exponentially stable domain of attraction of x_e . Hence, x_e is an asymptotically stable equilibrium for (2) and it is locally exponentially stable with domain of attraction including $\bar{\mathcal{C}}$. \square

Theorem 1 states that, in the presence of bounded mismatches, existence of a stable equilibrium is preserved for the perturbed system (2). This new equilibrium may not coincide with the original one, yet it is guaranteed to be in its neighborhood. As a particular example, we may study the case in which $x^+ = f(x) + \delta$ with δ being a constant small perturbation. In such scenario, the equilibrium of the perturbed system slightly shifts (it is actually computed as the solution x_e to $f(x_e) + \delta = x_e$), and we recover the result proposed in [11, Theorem 2.7]. Hence, even if we highlight that the proof proposed by the authors is either incorrect or missing some arguments, we believe that the general statement may be correct.

Finally, we highlight that the existence of a (possibly shifted) equilibrium may be of great value in some applications. A direct example is the control of systems via integral-action, see e.g., [8, 24, 25], as detailed in the next section.

3. Application to integral action control

As stated at the end of the previous section, the results of Theorem 1 can be of interest for the output regulation problem and, in particular, in the context of integral-action controllers, see e.g. [26]. Such control schemes are commonly employed for (constant) perturbation rejection or (constant) reference tracking. The most trivial examples are PI and PID laws. In particular, consider the following system

$$\begin{aligned} w^+ &= w, \\ \xi^+ &= F(w, \xi, u), \quad e = H(w, \xi), \end{aligned} \quad (29)$$

where $\xi \in \mathbb{R}^q$ is the state of the plant, $u \in \mathbb{R}^m$ is the control input and $e \in \mathbb{R}^p$ represents the output to be regulated to zero (without loss of generality), with $p \leq m$. For instance, in a tracking problem, one can set $e = y - y_{\text{ref}}$ where y is an output that must converge to a desired reference y_{ref} . In the representation (29), $w \in \mathbb{R}^{n_w}$ are constant signals representing references to be tracked or (unknown) perturbations to be rejected. Following the output regulation theory and the celebrated internal-model principle, asymptotic regulation of the output e can be achieved robustly with respect to model uncertainties of the functions F and H only if the controller contains an integral action (able to perfectly compensate unknown constant perturbations), see, e.g. [7, 8, 24] for the continuous-time case. We take inspiration from the results of the previous section, where constant perturbations can be included in a

more general context of perturbed functions (e.g. instead of considering $f(x) + w$ with $f(0) = 0$ we can simply consider a new function $\hat{f}(x)$ possibly with $\hat{f}(0) \neq 0$). Then, we set a nominal value for the model (29) and for the signals w . We describe the nominal system, possibly after a change of coordinate, in the form

$$\xi^+ = \varphi(\xi) + g(\xi, u), \quad e = h(\xi)$$

with $\varphi(0) = 0$, $g(\xi, 0) = 0$ and $h(\xi) = 0$, so that the origin represents the nominal desired equilibrium on which the regulation objective $e = 0$ is satisfied. Our objective is to show that the use of an integral action allows to robustly preserve the desired asymptotic regulation property $e = 0$ in the presence of model uncertainties in the nominal functions φ, g, h (that may come from different values of w or uncertainties in F, H). For this reason, the explicit use of the variable w in the state dynamics is not used anymore. Still, one can keep in mind that all the functions may depend on some nominal value w^* . Then, consider the following extended system

$$\begin{cases} \xi^+ = \varphi(\xi) + g(\xi, u), & e = h(\xi), \\ z^+ = z + k(\xi, e) \end{cases} \quad (30)$$

where $z \in \mathbb{R}^p$ represents a generalized integral discrete-time action with k being a C^1 function satisfying the following set of conditions

$$\begin{aligned} k(\xi, e) = 0 & \iff e = 0 \\ \begin{cases} |k(\xi, y_a) - k(\xi, y_b)| & \leq L_1(\xi) |y_a - y_b| \\ \left| \frac{\partial k}{\partial \xi}(\xi, y_a) - \frac{\partial k}{\partial \xi}(\xi, y_b) \right| & \leq L_2(\xi) |y_a - y_b| \end{cases} \end{aligned} \quad (31)$$

for all $\xi \in \mathbb{R}^q$, for all $(y_a, y_b) \in \mathbb{R}^p \times \mathbb{R}^p$ and for some continuous functions $L_1, L_2 : \mathbb{R}^q \rightarrow \mathbb{R}_{\geq 0}$. It can be easily seen that by selecting $k(\xi, e) = e$ we recover a standard discrete-time integrator

$$z^+ = z + e$$

that trivially satisfies the previous requirements with $L_1(\xi) = 1$, $L_2(\xi) = 0$. Under a feedback controller $u = \alpha(\xi, z)$, we can define the extended dynamics for $x = (\xi^\top, z^\top)^\top$ as

$$x^+ = f(x) := \begin{pmatrix} \varphi(\xi) + g(\xi, \alpha(\xi, z)) \\ z + k(\xi, h(\xi)) \end{pmatrix}. \quad (32)$$

As previously explained, variations of the nominal value of the signals w and model uncertainties of the functions f, g, h can be fully captured and represented with a closed-loop perturbed model

$$x^+ = \hat{f}(x) := \begin{pmatrix} \hat{\varphi}(\xi) + \hat{g}(\xi, \alpha(\xi, z)) \\ z + k(\xi, \hat{h}(\xi, \alpha(\xi, z))) \end{pmatrix}. \quad (33)$$

Thus, the goal is to study the robustness properties of the closed-loop $x^+ = f(x)$ and, in particular, to establish

conditions under which the existence of an equilibrium is preserved. As a matter of fact, the properties of the generalized integral action (31) allow concluding the desired regulation property $e = 0$ is still preserved at the equilibrium in the presence of model uncertainties. To this end, we introduce the following compact notation that will be used in the following sections:

$$\begin{aligned}
\Delta_\xi(x) &:= |\hat{\varphi}(\xi) - \varphi(\xi) + \hat{g}(\xi, \alpha(\xi, z)) - g(\xi, \alpha(\xi, z))|, \\
\Delta_e(x) &:= |\hat{h}(\xi, \alpha(\xi, z)) - h(\xi)|, \\
\Delta_z(x) &:= |k(\xi, \hat{h}(\xi, \alpha(\xi, z))) - k(\xi, h(\xi))|, \\
\Delta_{\partial\xi}(x) &:= \left| \frac{\partial\hat{\varphi}}{\partial\xi}(\xi) - \frac{\partial\varphi}{\partial\xi}(\xi) \right. \\
&\quad \left. + \frac{\partial\hat{g}}{\partial\xi}(\xi, \alpha(\xi, z)) - \frac{\partial g}{\partial\xi}(\xi, \alpha(\xi, z)) \right|, \\
\Delta_{\partial u}(x) &:= \left| \frac{\partial\hat{g}}{\partial u}(\xi, \alpha(\xi, z)) - \frac{\partial g}{\partial u}(\xi, \alpha(\xi, z)) \right|, \\
\Delta_{\partial e}(x) &:= \left| \frac{\partial\hat{h}}{\partial\xi}(\xi, \alpha(\xi, z)) - \frac{\partial h}{\partial\xi}(\xi) \right| + \left| \frac{\partial\hat{h}}{\partial u}(\xi, \alpha(\xi, z)) \right|.
\end{aligned} \tag{34}$$

3.1. Existence of equilibria

We first study the existence of equilibria where regulation is achieved for the perturbed system (33). Hence, we assume φ, g, α, h to be continuous functions.

Proposition 2. *Let Assumption 1 hold for the closed-loop system (32). Moreover, assume $h(0) = 0, g(\xi, 0) = 0$ for all $\xi \in \mathbb{R}^q$ and let k satisfy (31). Then, for any positive $\underline{c} \leq \bar{c}$ there exists a positive real number $\bar{\delta} > 0$ such that, for any continuous functions $\hat{\varphi} : \mathbb{R}^q \rightarrow \mathbb{R}^q, \hat{g} : \mathbb{R}^q \times \mathbb{R}^m \rightarrow \mathbb{R}^q, \hat{h} : \mathbb{R}^q \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ satisfying*

$$\Delta_\xi(x) + \Delta_e(x) \leq \bar{\delta}, \quad \forall x \in \mathcal{V}_{\bar{c}}(V), \tag{35}$$

with Δ_ξ, Δ_e defined in (34), the corresponding system (33) admits an equilibrium point $(\xi_e, z_e) \in \mathcal{V}_{\bar{c}}(V)$ on which output regulation is achieved, i.e. $\hat{h}(\xi_e, \alpha(\xi_e, z_e)) = 0$. Furthermore, system (33) has no other equilibrium in the set $\mathcal{V}_{\bar{c}}(V) \setminus \mathcal{V}_{\underline{c}}(V)$.

Proof. The proof relies on Proposition 1. It can be easily checked that (35) implies (11) for the extended system (32). Since $\mathbb{R}^q \times \mathbb{R}^p$ is a finite dimensional space, all norms are equivalent. Hence, there exist a strictly positive constant ℓ such that $|x| \leq \ell \sum_{i=1}^n |x_i|$ where x_i denotes the i -th component of x . Hence, recalling the compact notation (34), we have $|\hat{f}(x) - f(x)| \leq \ell(\Delta_\xi(x) + \Delta_z(x))$ for all $x \in \mathbb{R}^{q+p}$. By defining $L := \sup_{\xi \in \mathcal{V}_{\bar{c}}(V)} L_1(\xi) \geq 0$, we obtain $|\Delta_z(x)| \leq L|\Delta_e(x)|$ for all $x \in \mathcal{V}_{\bar{c}}(V)$ and therefore equation (31) yields $|\hat{f}(x) - f(x)| \leq \ell(1+L)(\Delta_\xi(x) + \Delta_e(x))$ for all $x \in \mathcal{V}_{\bar{c}}(V)$. Hence, (35) implies (5) if $\bar{\delta} < \frac{\delta}{\ell(1+L)}$. Then, Proposition 1 guarantees the existence of an equilibrium $(\xi_e, z_e) \in \mathcal{V}_{\bar{c}}(V)$. In such an equilibrium, $z_e^+ = z_e$ and consequently $k(\xi, \hat{h}(\xi_e, \alpha(\xi_e, z_e))) = 0$. By (31), the

last relation implies $\hat{h}(\xi_e, \alpha(\xi_e, z_e)) = 0$ and this concludes the proof. \square

Proposition 2 is a direct application of Proposition 1 to the extended system (32). Hence, it guarantees the existence of (at least one) equilibrium for the perturbed closed-loop dynamics (33). In turn, due to the presence of the integral action, in such equilibrium the outputs e are identically 0, even in presence of model mismatches of the function φ, g . Thus, Proposition 2 shows that the addition of discrete-time generalized integrator dynamics to discrete-time nonlinear systems guarantees the existence of equilibria where (constant) reference tracking and (constant) disturbance rejection is achieved.

In other words, if the controller (designed for a nominal system) is able to make the trajectories of the perturbed model converge to an equilibrium, then asymptotic regulation $e = 0$ is achieved.

3.2. Robust regulation

Proposition 2 does not provide any characterization of the attractivity properties of the equilibria. In other words, it doesn't characterize the asymptotic regulation objective when starting from an initial condition different from any of the equilibria. As a matter of fact, the generation of attractive and stable equilibria is often the main objective of control designs. Hence, we build on Theorem 1 to show that the addition of a discrete-time integral component to stabilizing controllers allows for robust asymptotic (constant) reference tracking and (constant) disturbance rejection.

Assumption 2. *There exists an open set $\mathcal{A} \subseteq \mathbb{R}^q \times \mathbb{R}^p$ and a C^1 function $\alpha : \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ such that the control law $u = \alpha(\xi, z)$ makes the origin of system (32) asymptotically stable with domain of attraction \mathcal{A} and locally exponentially stable.*

Under Assumption 2, we can infer robust setpoint-tracking properties for the extended system (32) by means of the results in the previous section.

Proposition 3. *Let Assumption 2 hold and let $\underline{\mathcal{C}}$ be an arbitrary compact subset of \mathcal{A} including the origin. Moreover, let functions φ, g, h be C^0 for all $(\xi, z) \in \mathcal{A}$ and C^1 for all $(\xi, z) \in \underline{\mathcal{C}}$. Finally, assume $h(0) = 0, g(\xi, 0) = 0$ for all $\xi \in \mathbb{R}^q$ and let k satisfy (31). Then, for any forward invariant compact set $\bar{\mathcal{C}}$ verifying $\{0\} \subsetneq \underline{\mathcal{C}} \subsetneq \bar{\mathcal{C}} \subsetneq \mathcal{A}$, there exists a strictly positive scalar $\bar{\delta}$ such that, for functions $\hat{\varphi}, \hat{g}, \hat{h}$ that are C^0 for all $(\xi, z) \in \mathcal{A}$ and C^1 for all $(\xi, z) \in \underline{\mathcal{C}}$, and satisfy*

$$\Delta_\xi(x) + \Delta_e(x) \leq \bar{\delta}, \quad \forall x \in \bar{\mathcal{C}}, \tag{36}$$

$$\Delta_{\partial\xi}(x) + \Delta_{\partial u}(x) + \Delta_{\partial e}(x) \leq \bar{\delta}, \quad \forall x \in \underline{\mathcal{C}}, \tag{37}$$

with $\Delta_\xi, \Delta_e, \Delta_{\partial\xi}, \Delta_{\partial u}, \Delta_{\partial e}$ defined in (34), the corresponding system (33) admits an equilibrium point

$(\xi_e, z_e) \in \underline{\mathcal{C}}$ which is locally exponentially stable and asymptotically stable with domain of attraction containing $\bar{\mathcal{C}}$. Moreover, on such an equilibrium, $\hat{h}(\xi_e, \alpha(\xi_e, z_e)) = 0$.

Proof. To prove the existence of an asymptotically stable equilibrium for (33) with a locally exponential behavior, we rely on Theorem 1. As in the proof of Proposition 2, it can be easily checked that (36) implies (11) for the extended system (32). By following the same steps, we define $L := \sup_{\xi \in \bar{\mathcal{C}}} L_1(\xi) \geq 0$. With the same definition of ℓ as in the proof of Proposition 2, (36) implies (11) for all $x \in \bar{\mathcal{C}}$ if $\delta < \frac{\min\{\delta_3, \delta_4\}}{\ell(1+L)}$, with δ_3, δ_4 defined in Step 4 of the proof of Theorem 1. Similarly, we can check that (37) implies (12) for the extended system (32). Let us define

$$L_\alpha := \sup_{(\xi, z) \in \underline{\mathcal{C}}} \left\{ \left| \frac{\partial \alpha}{\partial \xi}(\xi, z) \right|, \left| \frac{\partial \alpha}{\partial z}(\xi, z) \right| \right\},$$

$$L_k := \sup_{\xi \in \underline{\mathcal{C}}} \{L_1(\xi), L_2(\xi)\} \geq L.$$

To show this, let $\bar{\ell} > 0$ such that $|A| \leq \bar{\ell} \sum_{i=1}^n |A_i|$ where A is any (rectangular) matrix and A_i denotes the i -th row. Then, by recalling the notation introduced in (34) and using the previous inequalities, we obtain

$$\begin{aligned} & \left| \frac{\partial \hat{f}}{\partial x}(x) - \frac{\partial f}{\partial x}(x) \right| \\ & \leq \bar{\ell} \Delta_{\partial \xi}(x) + 2\bar{\ell} L_\alpha \Delta_{\partial u}(x) + 2\bar{\ell} L_\alpha L_k \left| \frac{\partial \hat{h}}{\partial u}(\xi, \alpha(\xi, z)) \right| \\ & \quad + 2\bar{\ell} L_k \left| \frac{\partial \hat{h}}{\partial \xi}(\xi, \alpha(\xi, z)) - \frac{\partial h}{\partial \xi}(\xi) \right| \\ & \leq \bar{\ell} (1 + 2(L_\alpha + L_k + L_\alpha L_k)) (\Delta_{\partial \xi} + \Delta_{\partial u} + \Delta_{\partial e}) \end{aligned}$$

for all $x \in \underline{\mathcal{C}}$. Then, by picking

$$\mu = \max \{ \ell(1+L), \bar{\ell}(1+2(L_\alpha + L_k + L_\alpha L_k)) \}$$

and $\bar{\delta} < \frac{\min\{\delta_3, \delta_4\}}{\mu}$. Theorem 1 guarantees the existence of an asymptotically stable equilibrium $x_e = (\xi_e, z_e)$ which is close to the origin and locally exponentially stable. In such an equilibrium, $z_e^+ = z_e$ and consequently $k(\xi_e, \hat{h}(\xi_e, \alpha(\xi_e, z_e))) = 0$. By (31), $\hat{h}(\xi_e, \alpha(\xi_e, z_e)) = 0$ and this concludes the proof. \square

Proposition 3 states that discrete-time stabilizing controllers exploiting integral actions achieve robust output regulation to constant setpoints. As an intuitive example, we may consider controllers designed for a nonlinear model, whose constant parameters are identified through data. Then, Proposition 3 guarantees that, if the approximation error is sufficiently small, the controller will still stabilize the system and the output will reach the desired constant setpoint.

We finally remark that Proposition 3 holds for small model perturbations, i.e. for small variations of the value of w from its nominal value w^* , when considering the original problem (29). Such a result parallels the works [8] in continuous-time. If one wants to guarantee output regulation in the presence of large variations of the values of w , it is generally needed to rely on incremental properties, e.g., [24].

3.3. About Assumption 2 and Forwarding approach

It is not straightforward to see when and how Assumption 2 can be satisfied, namely how to design the feedback law $\alpha(x, z)$ for the extended dynamics (30). An elegant solution comes from discrete-time forwarding technique [20]. If the origin of the autonomous system $\xi^+ = \varphi(\xi)$ is globally asymptotically stable and locally exponentially stable, the origin of the extended system (32) can be stabilized via a feedback controller. By considering the SISO scenario $m = p = 1$, the result in [20, Theorem 4.2] provides a feedback law $u = \alpha(\xi, z)$ achieving global asymptotic and local exponential stability of the origin of the extended system (32).

In particular, suppose that $W : \mathbb{R}^q \rightarrow \mathbb{R}_{\geq 0}$ is a Lyapunov function for system (1) which is locally quadratic. Assume also that the linearization of (32) around the origin is stabilizable and suppose to know a mapping $M : \mathbb{R}^q \rightarrow \mathbb{R}$ satisfying $M(\xi^+) = M(\xi) + k(\xi, h(\xi))$. Then, we can perform a change of coordinates $\eta = z - M(\xi)$ and consider $\zeta = (\xi^\top, \eta^\top)^\top$. This leads to the stabilizing controller $u = \bar{\alpha}(\zeta)$ given by the (implicit) solution to

$$u = -\frac{1}{u} \int_0^u \frac{\partial V}{\partial \zeta} G(\zeta^+(v), v) dv$$

with

$$V(\zeta) = W(\xi) + \eta^2, \quad G(\zeta, u)^\top = \left(\frac{\partial g}{\partial u}(\xi, u), -\frac{\partial M}{\partial \xi} \frac{\partial g}{\partial u}(\xi, u) \right).$$

In the original coordinates, we obtain $\alpha(\xi, z) = \bar{\alpha}(\xi, z - M(\xi))$. The function V above defined is a Lyapunov function for the closed-loop systems. Moreover, note that if the function W has the desired homeomorphicity properties (see in particular Section 2.3), so has V by construction. We refer to [20] and references therein for more details about the existence of such a mapping M , and the extension to the MIMO scenario.

4. Conclusion

In this paper, we proved that stability properties of a discrete-time nonlinear system transfer to plants described by sufficiently similar models. The results are independent of the structural properties of the system. We applied these results to the robust output regulation problem, showing the rejection of constant disturbances via integral action for nonlinear systems. The application to the problem of robust output regulation in the presence of more complex exogenous signals is subject of ongoing research. As a future research direction, it would be also interesting to study whether the total stability property of a continuous-time nonlinear system [8] is preserved under sampling.

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