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# Filtered Calculus and crossed products by $\mathbb{R}$ -actions

Clément Cren

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## Abstract

We show an isomorphism between the kernel of the  $C^*$  algebra of the tangent groupoid of a filtered manifold and the crossed product of the order 0 pseudodifferential operators in the associated filtered calculus by a natural  $\mathbb{R}$ -action. This isomorphism is constructed in the same way as in the classical pseudodifferential calculus by Debord and Skandalis. The proof however relies on a structure result for the  $C^*$ -algebra of graded nilpotent Lie groups which did not appear in the commutative case.

## Introduction

This paper deals with an alternate viewpoint of calculus on filtered manifolds. Filtered manifolds are manifolds equipped with a filtration of their tangent bundle by subbundles with a compatibility with the Lie bracket of vector fields. These manifolds have an alternate pseudodifferential calculus which allows the construction of parametrices for hypoelliptic but not elliptic differential operators. Classical examples gather contact [17, 3], CR [20] and Heisenberg manifolds [4] but also foliated manifolds [23, 9] and more recently, manifolds with parabolic geometries were also considered [13]. These new advances are all thanks to a new approach on the pseudodifferential calculus using groupoid methods [37, 1] (there is another approach relying on harmonic analysis techniques in [19]). We adapt a construction of Debord and Skandalis [15] to the filtered case. They showed for the classical calculus an isomorphism between the  $C^*$ -algebra of pseudodifferential operators (of order 0) on a manifold  $M$  and an algebra constructed from the tangent groupoid and its zoom action. Indeed if  $\mathbb{T}M$  is the tangent groupoid of  $M$  we can restrict it to  $\mathbb{T}^+M \rightrightarrows M \times \mathbb{R}_+$ . It then sits in the exact sequence:

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C^*(\mathbb{T}^+M) \longrightarrow \mathcal{C}_0(T^*M) \longrightarrow 0.$$

Now the zoom action  $(\alpha_\lambda)_{\lambda>0}$  is defined on  $\mathbb{T}^+M$  via:

$$\begin{aligned} \alpha_\lambda(x, y, t) &= (x, y, \lambda^{-1}t) & (x, y, t) &\in M \times M \times \mathbb{R}_+^* \\ \alpha_\lambda(x, \xi, 0) &= (x, \lambda\xi, 0) & (x, \xi) &\in TM. \end{aligned}$$

If we remove the zero section from the fiber at zero we get the exact sequence

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}_+^*, \mathcal{K}) \longrightarrow C_0^*(\mathbb{T}^+M) \longrightarrow \mathcal{C}_0(T^*M \setminus \{0\}) \longrightarrow 0.$$

Now the zoom action becomes free and proper. Taking the cross product yields the exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \longrightarrow C_0^*(\mathbb{T}^+M) \rtimes_{\alpha} \mathbb{R}_+^* \longrightarrow \mathcal{C}(\mathbb{S}^*M) \otimes \mathcal{K} \longrightarrow 0.$$

Debord and Skandalis related this sequence to the one of pseudodifferential operators:

$$0 \longrightarrow \mathcal{K} \longrightarrow \Psi^*(M) \longrightarrow \mathcal{C}(\mathbb{S}^*M) \longrightarrow 0$$

where  $\Psi^*(M)$  is the closure of order 0 operators in the algebra of bounded operators on  $L^2(M)$  and the arrow on the right is the principal symbol map. As an important step, they showed that any pseudodifferential operator of order  $m$  could be written as an integral of the form:

$$\int_0^{+\infty} t^{-m} f_t \frac{dt}{t}$$

where  $(f_t)_{t>0}$  is a function in a suitable ideal of the Schwartz algebra of the tangent groupoid. In the context of filtered manifold one could perform the same type of construction. We thus define the appropriate Schwartz algebra on the tangent groupoid and show that pseudodifferential operators can be written with an integral *à la* Debord-Skandalis. This time however, the identification at the level of symbols is not trivial at all. To do this we use Kirillov's theory and Pedersen's stratification of the coadjoint orbits to decompose the algebra  $\Sigma(G)$  of principal symbols of order 0 on a graded Lie group  $G$  into a finite sequence of nested ideals:

$$0 = \Sigma_0(G) \triangleleft \Sigma_1(G) \triangleleft \cdots \triangleleft \Sigma_r(G) = \Sigma(G),$$

such that for all  $i \geq 0$  we have  $\Sigma_i(G)/\Sigma_{i-1}(G) \cong \mathcal{C}(\Lambda_i/\mathbb{R}_+^*, \mathcal{K}_i)$ . Here  $\Lambda_i$  is a certain subset of the set of coadjoint orbits  $\mathfrak{g}^*/G$  so that  $\Lambda_i/\mathbb{R}_+^*$  is compact Hausdorff and  $\mathcal{K}_i$  is an algebra of compact operators. This decomposition generalizes the decomposition of symbols of Epstein and Melrose in the contact case (i.e. for the Heisenberg group) [17]. Combining this with the construction on the tangent groupoid gives an isomorphism:

$$\Psi_H^*(M) \rtimes \mathbb{R} \cong C_0^*(\mathbb{T}_H^+M).$$

Here  $\mathbb{T}_H^+M$  is the restriction over  $M \times \mathbb{R}_+$  of the tangent groupoid of the filtered manifold  $(M, H)$  and the  $\mathbb{R}$  action on the  $C^*$ -algebra of order 0 pseudodifferential operators in the filtered calculus is constructed so that the isomorphism is  $\mathbb{R}_+^*$ -equivariant (for the dual action on the left and the zoom action on the right). As a corollary of our construction, we derive a KK-equivalence between the algebras of pseudodifferential operators of order 0 for the classical and filtered calculus, extending the one for the symbols obtained in [29] from ideas of [3]. This equivalence shows that one cannot hope to obtain new K-theoretic invariants from the filtered calculus (other than the ones that were previously obtainable with the classical calculus).

## 1 The isomorphism for symbols

If a groupoid  $G$  is a continuous family of groups, we write  $C_0^*(G) = \bigcap_{x \in G^{(0)}} \ker(1_x) \subset C^*(G)$ , where  $1_x$  is the trivial representation of the group fiber  $G_x$ ,  $x \in G^{(0)}$ . We also

consider deformation groupoids  $\mathbb{G} \rightrightarrows M \times \mathbb{R}$  (i.e. continuous families of groupoids over  $M$  parameterized by  $\mathbb{R}$ ) e.g. the tangent groupoid of a manifold. If  $\mathbb{G}_0$  is as before, we denote by  $C_0^*(\mathbb{G})$  the pre-image of  $C_0^*(\mathbb{G}_0)$  under the canonical map  $C^*(\mathbb{G}) \rightarrow C^*(\mathbb{G}_0)$ . All the groupoids considered in this article will be amenable. Therefore we will not distinguish the reduced or maximal groupoid  $C^*$ -algebras.

## 1.1 Representation theory of graded nilpotent Lie groups

Let  $G$  be a graded nilpotent Lie group. This means that  $G$  is a simply connected Lie group and that its lie algebra  $\mathfrak{g}$  admits a decomposition  $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{g}_j$  compatible with the

Lie bracket in the following sense:  $\forall j, k, [\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$  (we make the convention that  $\mathfrak{g}_\ell = \{0\}$  whenever  $\ell > r$ ). Given such a group, one can define a family of inhomogeneous dilations  $\delta_\lambda \in \text{Aut}(G)$  for  $\lambda \in \mathbb{R}_+^*$ . At the level of the Lie algebra  $\mathfrak{g}$  they are linear maps given by:

$$\forall \lambda > 0, \forall 1 \leq k \leq r, \forall X \in \mathfrak{g}_k, d\delta_\lambda(X) = \lambda^k X.$$

The application  $d\delta_\lambda$  is well defined and is a Lie algebra automorphism and hence lifts to a Lie group automorphism of  $G$  denoted  $\delta_\lambda$ .

**Example 1.1.** *Let  $G = \mathbb{R}^n$  with the trivial decomposition, we get the usual dilations  $d\delta_\lambda(X) = \lambda X$  for every  $X \in \mathbb{R}^n$ .*

**Example 1.2.** *Let  $G = H_{2n+1}$  be the Heisenberg group of dimension  $2n + 1$ . Its Lie algebra is generated by elements  $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$  with the commutation relations  $[X_j, Y_k] = \delta_{j,k}Z$  and the other being trivial. The decomposition of its Lie algebra  $\mathfrak{g}$  is given by  $\mathfrak{g}_1 = \langle X_1, \dots, X_n, Y_1, \dots, Y_n \rangle$  and  $\mathfrak{g}_2 = \langle Z \rangle$ . The dilation is thus given by  $d\delta_\lambda(X_k) = \lambda X_k, d\delta_\lambda(Y_k) = \lambda Y_k, d\delta_\lambda(Z) = \lambda^2 Z$ .*

The action of  $\mathbb{R}_+^*$  on  $G$  gives by duality an action on  $\hat{G}$ . This action, still denoted by  $\delta_\lambda$ , stabilizes the trivial representation 1. The restricted action on  $\hat{G} \setminus \{1\}$  is free and proper. We wish to understand the structure of  $C_0^*(G) \rtimes \mathbb{R}_+^*$ . One could hope to write  $C_0^*(G)$  as some crossed product over  $\mathbb{R}, A \rtimes \mathbb{R}$ , for a  $\mathbb{R} - C^*$ -algebra  $A$  so that  $\delta_\lambda$  would become the dual action.

In the commutative case  $C_0^*(\mathbb{R}^n) = \mathcal{C}_0(\mathbb{R}^n \setminus \{0\})$ . We thus have  $C_0^*(\mathbb{R}^n) \rtimes \mathbb{R}_+^* = \mathcal{C}\left(\mathbb{R}^n \setminus \{0\} / \mathbb{R}_+^*\right) \otimes \mathcal{K}(L^2(\mathbb{R})) = \mathcal{C}(S^{n-1}) \otimes \mathcal{K}(L^2(\mathbb{R}))$ . In [15], Debord and Skandalis showed that this algebra naturally appears as the algebra of principal symbols of pseudodifferential operators (of order 0). We wish to do the same here in the general setting of graded nilpotent groups.

To better understand the structure of  $C_0^*(G)$  we study the representation theory of  $G$  through Kirillov's theory [24] and the stratification of its unitary dual  $\hat{G}$  constructed by Pukánszky [33].

Kirillov's orbit method gives a homeomorphism between  $\hat{G}$  equipped with the hull-kernel topology and the set of coadjoint orbits  $\mathfrak{g}^* / \text{Ad}^*(G)$  equipped with the quotient of the euclidean topology. This homeomorphism is  $\mathbb{R}_+^*$ -equivariant for the natural actions obtained by composition with  $\delta_\lambda$  on the representations and by the transpose of  $d\delta_\lambda$  on  $\mathfrak{g}^*$ . Consider  $(X_1, \dots, X_n)$ , a basis of  $\mathfrak{g}$  consisting of eigenvectors for the dilations, i.e.  $d\delta_\lambda(X_i) = \lambda^{q_i} X_i$ . Here  $q_i$  is the integer such that  $X_i \in \mathfrak{g}_{q_i}$ . If the  $q_i$  are in ascending order

then  $(X_1, \dots, X_n)$  is a Jordan-Hölder basis. Following Pukánszky [33] and Pedersen [32] (see also [11]), we obtain a decomposition of  $\mathfrak{g}^* \setminus \{0\}$  with the following properties:

**Theorem 1.3** (Pukánszky, Pedersen). *There exist an integer  $d$  and algebraic sets  $\Omega_i \subset \mathfrak{g}^* \setminus \{0\}$ ,  $1 \leq i \leq d$  that are  $G$ -invariant and  $\mathbb{R}_+^*$ -invariant such that:*

- the sets  $W_i = \bigcup_{1 \leq j \leq i} \Omega_j$  are Zariski open, invariant for the actions of  $G$  and  $\mathbb{R}_+^*$  and  $W_d = \mathfrak{g}^* \setminus \{0\}$ .
- for each  $i$  there exist complementary vector subspaces of  $\mathfrak{g}^*$   $V_i^S$  and  $V_i^T$  such that each  $G$ -orbit in  $\Omega_i$  meets  $V_i^T$  in exactly one point. The sets  $\Sigma_i = \Omega_i \cap V_i^T$  are  $\mathbb{R}_+^*$ -invariant algebraic subsets and define  $\mathbb{R}_+^*$ -cross-sections for the  $G$ -orbits in  $\Omega_i$ .
- there is a birational nonsingular bijection  $\psi_i: \Sigma_i \times V_i^S \rightarrow \Omega_i$  such that if  $p_i^T$  denotes the natural projection onto  $V_i^T$  then for each  $v \in \Sigma_i$  the map  $p_i^T \circ \psi_i(v, \cdot): V_i^S \rightarrow V_i^T$  is a polynomial whose graph is the orbit  $G \cdot v$ . The composition  $\text{pr}_1 \circ \psi_i^{-1}: \Omega_i \rightarrow \Sigma_i$  is  $G$ -invariant and its quotient through  $G$  is the aforementioned cross-section.

*Proof.* The proof can be found in [33] for an arbitrary nilpotent Lie group. In our context however, our choice of basis makes all the elements in the construction stable under the  $\mathbb{R}_+^*$ -action. For instance, the jump indices of an element in  $\mathfrak{g}^*$  are the same as the other elements of its orbit through the  $\mathbb{R}_+^*$ -action. Therefore the sets  $\Omega_i, W_i, \Sigma_i \dots$  are stable under the  $\mathbb{R}_+^*$ -action and the maps  $\psi_i$  are equivariant.  $\square$

**Corollary 1.3.1.** *With the same notations, for every  $1 \leq i \leq d$  the natural map  $\Sigma_i \rightarrow \Omega_i/G$  is a  $\mathbb{R}_+^*$ -equivariant homeomorphism. In particular the action  $\mathbb{R}_+^* \curvearrowright \Omega_i/G$  is free and proper.*

*Proof.* The map is a cross section of the action, therefore an homeomorphism. Its equivariance was also stated in the previous theorem. The action  $\mathbb{R}_+^* \curvearrowright \Sigma_i$  is the restriction of an action on the vector space  $V_i^T \setminus \{0\}$  (remember that 0 corresponds to the trivial representation that we have removed). This action is made of diagonal matrices of dilations by powers of  $\lambda > 0$ . It is therefore free and proper. The equivariant homeomorphism then transfers this result to  $\mathbb{R}_+^* \curvearrowright \Omega_i/G$ .  $\square$

**Corollary 1.3.2.** *There exists a filtration  $\emptyset = V_0 \subset V_1 \subset \dots \subset V_d = \hat{G} \setminus \{0\}$  by open subsets such that the each  $\Lambda_i = V_i \setminus V_{i-1}$  is Hausdorff,  $\mathbb{R}_+^*$ -invariant and the action  $\mathbb{R}_+^* \curvearrowright \Lambda_i$  is free and proper. In particular the sets  $\Lambda_i/\mathbb{R}_+^*$  are Hausdorff.*

*Proof.* Define  $V_i = W_i/G$ , it defines a filtration of  $\hat{G} \setminus \{0\}$  through Kirillov's map. Moreover the previous corollary gives a homeomorphism between  $\Lambda_i$  and  $\Sigma_i$  which is thus Hausdorff. Since the action  $\mathbb{R}_+^* \curvearrowright \Sigma_i$  is free and proper and the homeomorphism  $\Sigma_i \cong \Lambda_i$  is  $\mathbb{R}_+^*$ -equivariant then the action  $\mathbb{R}_+^* \curvearrowright \Lambda_i$  is also free and proper.  $\square$

*Remark 1.4.* In both cases, the last stratum corresponds to the elements in  $\mathfrak{g}^*$  whose corresponding unirep is one dimensional. All the other unirep are infinite dimensional because of Engel's theorem.

This description of Pedersen's fine stratification, without the details on the construction, does not show the differences with Pukánszky's coarse stratification. Without going into too much details, we will use the following properties of Pedersen's stratification:

**Theorem 1.5** (Pedersen [32] ; Lipsman, Rosenberg [27]). *Let  $\Omega_i \subset \mathfrak{g}^*$  be a stratum of Pedersen's fine stratification. Then there exists an integer  $n \in \mathbb{N}$  such that every representation in the corresponding set  $\Lambda_i = \Omega_i / G \subset \hat{G}$  can be realized on  $L^2(\mathbb{R}^n)$  with smooth vectors being the Schwartz functions  $\mathcal{S}(\mathbb{R}^n)$  and the elements of  $\mathfrak{g}$  are realized as differential operators with polynomial coefficients. These coefficients have a (non-singular) rational dependence when the representation varies in  $\Sigma_i$  (and its complexification).*

*Remark 1.6.* The integer  $n$  correspond to half the dimension of any orbit in  $\Omega_i$ .

**Example 1.7.** *For the Heisenberg group  $G = H_{2n+1}$ , we consider the dual coordinates  $(x_1, \dots, x_n, y_1, \dots, y_n, z)$  on  $\mathfrak{g}^*$  with respect to the usual basis on  $\mathfrak{g}$ . The stratification is then given by  $\Omega_1 = \{(x, y, z) / z \neq 0\}$  and  $\Omega_2 = \{(x, y, 0) / (x, y) \neq (0, 0)\}$ . When taking the quotient by the coadjoint action we get  $\Lambda_1 \cong \mathbb{R}^*$  and  $\Lambda_2 \cong \mathbb{R}^{2n}$ . The  $\mathbb{R}_+^*$  action is given by dilations on each components:  $\delta_\lambda|_{\Lambda_1} = \lambda^2$  and  $\delta_\lambda|_{\Lambda_2} = \lambda$ .*

We end this section with the definition and properties of quasi-norms on graded Lie groups. They will serve two purposes in the following. The first one will be to realize elements of  $C^*(G)$  as symbols for the filtered calculus. It will also provide a section for the quotient map  $\hat{G} \setminus \{0\} \rightarrow \hat{G} \setminus \{0\} / \mathbb{R}_+^*$ . It is the analog of a norm in the commutative case except now we want it to be homogeneous with respect to the inhomogeneous family of dilations.

**Definition 1.8.** A homogeneous quasi-norm on a Lie group  $G$  is a continuous function  $|\cdot|: G \mapsto \mathbb{R}_+$  such that:

- $|g| = 0$  if and only if  $g$  is the neutral element of  $G$
- $|\cdot|$  is homogeneous of degree 1, i.e.  $\forall \lambda > 0, \forall g \in G, |\delta_\lambda(g)| = \lambda|g|$ .

Such a function always exists and any two of them are equivalent. Now given a homogeneous quasi-norm  $|\cdot|$  we can induce a similar function on  $\mathfrak{g}$  through the exponential map. By duality we obtain a similar function on  $\mathfrak{g}^*$  for which we keep the same notation. We then obtain a map on  $\hat{G}$  through Kirillov homomorphism and setting  $|\text{Ad}^*(G) \cdot \xi| := \inf_{g \in G} |\text{Ad}^*(g)(\xi)|$ . The function  $|\cdot|: \hat{G} \rightarrow \mathbb{R}_+$  is still continuous. It is homogeneous of degree 1 with respect to the dilations on  $\hat{G}$  and vanishes only at the class of the trivial representation. We then obtain

**Lemma 1.9.** *Let  $|\cdot|$  be a homogeneous quasi-norm on  $G$ . The canonical map*

$$\{\pi \in \hat{G} / |\pi| = 1\} \rightarrow \hat{G} \setminus \{0\} / \mathbb{R}_+^*$$

*is a homeomorphism.*

## 1.2 Osculating group bundle of a filtered manifold

Calculus on graded nilpotent Lie groups appears naturally when one studies operators on filtered manifolds. In this section we describe the basic geometric features of such manifolds.

**Definition 1.10.** A filtered manifold is a manifold  $M$  equipped with a filtration of its tangent bundle by subbundles  $\{0\} = H^0 \subset H^1 \subset \dots \subset H^r = TM$  with the following condition on the Lie brackets:

$$\forall i, j, [\Gamma(H^i), \Gamma(H^j)] \subset \Gamma(H^{i+j})$$

with  $H^k = TM$  for  $k \geq r$ .

Given a filtered manifold  $(M, H)$  we can associate to it a bundle of graded nilpotent Lie groups called the osculating group bundle. It replaces the tangent bundle and captures its filtered structure. We start by defining the associated bundle of Lie algebras. Denote by  $\mathfrak{t}_H M = H^1 \oplus H^2/H^1 \oplus \dots \oplus H^r/H^{r-1}$ . We endow it with a Lie algebroid structure with trivial anchor and the following bracket: if  $X \in \Gamma(H^i), Y \in \Gamma(H^j)$  we have  $[X, Y] \in H^{i+j}$  and its class  $[X, Y] \bmod \Gamma(H^{i+j-1})$  only depends on the class of  $X \bmod \Gamma(H^{i-1})$  and the class of  $Y \bmod \Gamma(H^{j-1})$ . We thus have defined a Lie bracket

$$[\cdot, \cdot]: \Gamma(\mathfrak{t}_H M) \wedge \Gamma(\mathfrak{t}_H M) \rightarrow \Gamma(\mathfrak{t}_H M).$$

This Lie Bracket is  $\mathcal{C}^\infty(M)$ -bilinear, indeed let  $f \in \mathcal{C}^\infty(M), X \in \Gamma(H^i), Y \in \Gamma(H^j)$ , we have

$$[fX, Y] = f[X, Y] - df(Y)X \equiv f[X, Y] \bmod \Gamma(H^{i+j-1}).$$

This implies that the algebroid structure on  $\mathfrak{t}_H M$  is from a bundle of Lie algebras. These algebras are graded by construction.

**Definition 1.11.** Let  $(M, H)$  be a filtered manifold. We denote by  $T_H M$  the osculating group bundle. It is the bundle of (connected, simply connected) Lie groups integrating  $\mathfrak{t}_H M$  obtained by the Baker-Campbell-Hausdorff formula.

**Example 1.12.** For the trivial filtration  $H^1 = TM$  we get  $T_H M = TM$  as a bundle of abelian groups (locally trivial).

**Example 1.13.** Let  $M$  be a contact manifold with contact structure  $H \subset TM$ . Then on each fiber  $T_H M$  has the structure of a Heisenberg group, moreover this bundle is locally trivial (it is trivialized in Darboux coordinates).

Despite the apparent simplicity of these two examples, the general case can be more complicated : the group structure might change abruptly between fibers and even if all the fibers are isomorphic there might be no analog of the Darboux theorem. We can encompass these kind of bundles and graded Lie groups by dealing with arbitrary smooth bundles of graded Lie groups over smooth manifolds (non necessarily locally trivial). Despite the absence of local theory for such bundles, our strategy will be to reduce our study to graded Lie groups. Group bundles are an example of Lie groupoids. Associated to them there is thus a  $C^*$ -algebra. Because the fibers of the bundle are nilpotent Lie groups the bundles are amenable groupoids so we will not make any difference between the maximal and reduce  $C^*$ -algebras. The end of this section is devoted to the description of this kind of algebra in the light of our strategy.

**Example 1.14.** If  $G = G_0 \times M$  is a trivial Lie group bundle over  $M$  then  $C^*(G) = C^*(G_0) \otimes \mathcal{C}_0(M)$ .

**Proposition 1.15.** *Let  $G \rightarrow M$  be a bundle of graded Lie groups. The algebra  $C^*(G)$  is a continuous field of  $C^*$ -algebras over  $M$ , the fiber at  $x \in M$  is equal to the group  $C^*$ -algebra  $C^*(G_x)$ .*

*Proof.* This is an easy consequence of the fact that  $G$  is a smooth (hence continuous) family of group(oid)s over  $M$  in the sense of [25].  $\square$

For a graded Lie group we can use Pedersen's stratification to decompose the  $C^*$ -algebra. Let us denote by  $V_1 \subset \dots \subset V_r = \hat{G}$  Pedersen's stratification of the unitary dual, i.e.  $V_i = W_i / \hat{G}$  where  $W_i$  is the  $i$ -th stratum in Pedersen's stratification of  $\mathfrak{g}^*$ . We identify  $V_i \setminus V_{i-1}$  with the algebraic set  $\Lambda_i \subset \mathfrak{g}^*$ . Corresponding to these open subsets are ideals  $J_i := \bigcap_{\pi \in \hat{G} \setminus V_i} \ker(\pi)$  of  $C^*(G)$ . We get an increasing sequence of ideals:

$$\{0\} = J_0 \triangleleft J_1 \triangleleft \dots \triangleleft J_r = C^*(G)$$

and by construction  $\widehat{J_i / J_{i-1}} = \Lambda_i$ . The subquotients thus have Hausdorff spectrum and therefore are continuous fields of  $C^*$ -algebras over  $\Lambda_i$  according to [30]. Using the result of Lipsman-Rosenberg stated above we have:

**Corollary 1.15.1.** *For each  $1 \leq i \leq r$  we have:*

$$J_i / J_{i-1} \cong \mathcal{C}_0(\Lambda_i, \mathcal{K}_i).$$

Here  $\mathcal{K}_i$  denotes the algebra of compact operators on a separable Hilbert space (of dimension 1 for  $i = r$  and infinite dimensional otherwise).

### 1.3 Symbols in the filtered calculus

Let  $\mathcal{G}$  be a Lie groupoid. Denote by  $\Omega^{1/2} = \Omega^{1/2}(\ker(ds)) \otimes \Omega^{1/2}(\ker(dr))$  the bundle of half densities on  $\mathcal{G}$  and  $\mathcal{C}_c^\infty(\mathcal{G}, \Omega^{1/2})$  denotes the algebra of sections with the convolution product  $f * g(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2)$ . Equivalently, one can also replace half-densities by a choice of a Haar system for  $\mathcal{G}$  which would correspond to a trivialization of  $\Omega^{1/2}$ , both yield the same convolution algebra for  $\mathcal{G}$ .

For distributions:  $\mathcal{D}'(\mathcal{G}, \Omega^{1/2})$  (resp.  $\mathcal{E}'(\mathcal{G}, \Omega^{1/2})$ ) will denote the topological dual of  $\mathcal{C}_c^\infty(\mathcal{G}, \Omega^{-1/2} \otimes \Omega_{\mathcal{G}}^1)$  (resp.  $\mathcal{C}_c^\infty(\mathcal{G}, \Omega^{-1/2} \otimes \Omega_{\mathcal{G}}^1)$ ) where  $\Omega_{\mathcal{G}}^1$  is the bundle of 1-densities obtained from  $T\mathcal{G}$ . This choice is made so that  $\mathcal{C}_c^\infty(\mathcal{G}, \Omega^{1/2}) \hookrightarrow \mathcal{D}'(\mathcal{G}, \Omega^{1/2})$  (resp.  $\mathcal{C}_c^\infty(\mathcal{G}, \Omega^{1/2}) \hookrightarrow \mathcal{E}'(\mathcal{G}, \Omega^{1/2})$ ).

Let us recall the definition of symbols in the context of filtered calculus. In the following,  $\pi: G \rightarrow M$  denotes a bundle of Carnot Lie groups. The family of inhomogeneous dilations  $(\delta_\lambda)_{\lambda>0}$  defines by pullback a family of algebra automorphisms  $(\delta_\lambda^*)_{\lambda>0}$  of  $\mathcal{C}_c^\infty(G, \Omega^{1/2})$ . Note that this is the pullback of half-densities which takes the jacobian into account. If a choice of Haar system is made, the corresponding action on  $\mathcal{C}_c^\infty(G)$  is given by  $(\lambda^n \delta_\lambda^*)_{\lambda>0}$  (the pullback of functions will not preserve the convolution product). Here  $n$  denotes the homogeneous dimension of  $G$ , i.e.  $n = \sum_{i \geq 1} i \operatorname{rk}(\mathfrak{g}_i)$ .

**Definition 1.16.** A symbol of order  $m \in \mathbb{C}$  in  $G$  is a distribution  $u \in \mathcal{D}'(G, \Omega^{1/2})$  satisfying



- $u$  is properly supported, i.e.  $\pi: \text{supp}(u) \rightarrow M$  is a proper map
- $u$  is transversal to  $\pi$  (in the sense of [2]), i.e.  $\pi_*(u) \in \mathcal{C}^\infty(M)$
- $\forall \lambda \in \mathbb{R}_+^*, \delta_{\lambda*}u - \lambda^m u \in \mathcal{C}_p^\infty(G)$

$S_p^m(G)$  denotes the set of these distributions and  $S_p^*(G) = \bigcup_{m \in \mathbb{Z}} S_p^m(G)$ . The subspace of compactly supported symbols will be denoted by  $S_c^m(G)$ . We will write  $S^m(G)$  when the statements apply for both  $S_p^m(G)$  and  $S_c^m(G)$ .

The second condition implies that  $u$  corresponds to a  $C^\infty$  family of distributions  $u_x \in \mathcal{D}'(G_x, \Omega^{1/2})$  (see [26] for a more precise statement). The first condition then implies that the support of each  $u_x$  is a compact subset of the fiber  $G_x$  for every  $x \in M$ , i.e.  $u_x \in \mathcal{E}'(G_x, \Omega^{1/2})$ . The third condition implies that  $u$  is smooth outside of the zero unit section  $M \subset G$ , it also gives the asymptotic behavior of  $u$  near  $M$ .

The symbols used here are called quasi-homogeneous. They correspond to principal symbols of pseudodifferential operators in the filtered calculus up to  $\mathcal{C}_p^\infty(G)$  functions. For compactly supported symbols, the quasi-homogeneity condition can equivalently be stated mod  $\mathcal{C}_c^\infty(G)$ .

The push-forward of distributions is obtained by duality from the pullback of half-densities. To understand what it does on functions seen as distributions first note that  $\Omega_G^1 \otimes \Omega^{-1/2} \cong (r^* \otimes s^*)(\Omega_M^{1/2})$ . On elements  $f \in \mathcal{C}^\infty(\mathcal{G}, \Omega^{-1/2} \otimes \Omega_G^1)$  seen as elements of  $\mathcal{D}'(\mathcal{G}, \Omega^{1/2})$  we then have:

$$\delta_{\lambda*}f = \delta_{\lambda^{-1}}^*f.$$

*Remark 1.17.* As we use properly supported distributions we do the same for functions:  $\mathcal{C}_p^\infty(\cdot)$  denotes the space of properly supported functions on a groupoid (if  $G$  is a groupoid,  $X \subset G$  is proper if  $s|_X$  and  $r|_X$  are proper maps). With these notations we see that  $\mathcal{C}_p^\infty(G, \Omega^{1/2})$  embeds into the space of properly supported fibered distributions on  $G$  (an arbitrary Lie groupoid).

We recall some useful results on filtered calculus, for more details and proofs we refer to [37, 29, 13] and their references:

**Lemma 1.18.** *Let  $u \in S^{m_1}(G), v \in S^{m_2}(G)$  then  $u * v \in S^{m_1+m_2}(G)$  and  $u^* \in S^{m_1}(G)$ . This result can be extended to  $m_1 = -\infty$  or  $m_2 = -\infty$  with  $S_{c/p}^{-\infty}(G) = \mathcal{C}_{c/p}^\infty(G, \Omega^{1/2})$ .*

Let  $u \in S^*(G)$ , for each  $x \in M$  we get a convolution operator

$$\text{Op}(u_x) : \begin{array}{ccc} \mathcal{C}_c^\infty(G_x, \Omega^{1/2}) & \longrightarrow & \mathcal{C}_c^\infty(G_x, \Omega^{1/2}) \\ f & \longmapsto & u_x * f \end{array} .$$

The transversality condition then allows us to "glue" these operators to obtain  $\text{Op}(u): \mathcal{C}_c^\infty(G, \Omega^{1/2}) \rightarrow \mathcal{C}_c^\infty(G, \Omega^{1/2})$  with  $\text{Op}(u)(f)|_{\pi^{-1}(x)} = \text{Op}(u_x)(f|_{\pi^{-1}(x)})$ .

**Lemma 1.19.** *Let  $u, v \in S^*(G)$  and  $f, g \in \mathcal{C}_c^\infty(G, \Omega^{1/2})$  then*

- $\text{Op}(u)(f * g) = \text{Op}(u)(f) * g$
- $\text{Op}(u * v) = \text{Op}(u) \circ \text{Op}(v)$

We now state the Rockland condition and its consequences on the symbolic calculus. Rockland condition replaces the usual ellipticity condition. Moreover, when the symbols yield operators on  $M$  (i.e. for  $G = T_H M$ , see [37]) the operators whose symbols satisfy Rockland's condition are hypoelliptic and admit parametrices (in the filtered calculus). This condition was first introduced by Rockland in [34] for differential operators on Heisenberg groups. It was then extended to manifolds with an osculating Lie group of rank at most two by Helffer and Nourrigat in [22] and Lie groups with dilations in [8]. See also the recent advances in [1] in a very broad setting generalizing the results of Helffer and Nourrigat.

**Definition 1.20.** A symbol  $u \in S^*(G)$  satisfies the Rockland condition at  $x \in M$  if there exist a compact set  $K \subset \widehat{G}_x$  on the dual space of irreducible representations such that for all  $\pi \notin K$ ,  $\pi(\text{Op}(u))$  and  $\pi(\text{Op}(u^*))$  are injective. The symbol satisfies the Rockland condition if it satisfies it at every point  $x \in M$  ( $u$  is then also said to be Rockland).

As in the classical case, Rockland condition corresponds exactly to the existence of parametrices. It was proved in [8] for trivial bundles and this result was used in [13] for the proof in the general case.

**Theorem 1.21** ([13]). *Let  $u \in S_p^k(G)$  then there is an equivalence :*

- 1  $u$  satisfies the Rockland condition
- 2 there exists  $v \in S_p^{-k}(G)$  such that  $u * v - 1, v * u - 1 \in \mathcal{C}_p^\infty(G, \Omega^{1/2})$

A proof can be found in [13], using arguments originating in [8] (which corresponds to the case of trivial bundles).

**Theorem 1.22** ([13]). *Let  $u \in S_c^k(G)$  then*

- If  $\Re(k) \leq 0$  then  $\text{Op}(u)$  extends to an element of the multiplier algebra  $\mathcal{M}(C^*(G))$
- If  $\Re(k) < 0$  then  $\text{Op}(u)$  extends to an element of  $C^*(G)$
- If  $k > 0$ ,  $M$  is compact and  $u$  satisfies the Rockland condition then  $\text{Op}(u)$  extends to an unbounded regular operator  $\overline{\text{Op}(u)}$  on  $C^*(G)$  (viewed as a  $C^*$ -module over itself) and  $\overline{\text{Op}(u)}^* = \overline{\text{Op}(u^*)}$

Once again see [13] for a detailed proof.

**Definition 1.23.** A symbol  $u \in S^*(T_H M)$  is Rockland (i.e "elliptic" in the filtered calculus) if it satisfies the Rockland condition.

Theorem 1.22 allows to extend the algebra of symbols. Let  $\bar{S}_0^0(G) \subset \mathcal{M}(C^*(G))$  be the  $C^*$ -closure of  $S_c^0(G)$  (we identify the algebra of order 0 symbols and its image by  $\overline{\text{Op}}$ ).

**Definition 1.24.** We denote by  $\Sigma_{p/c}^k(G) = S_{p/c}^k(G) / S_{p/c}^{-\infty}(G)$  of order  $k \in \mathbb{C}$  the set of principal symbols in the filtered calculus. For  $k = 0$  we also have  $\Sigma(G) = \bar{S}_0^0(G) / C^*(G)$  for the  $C^*$ -algebra of principal symbols. The algebra  $\Sigma(G)$  is a completion of  $\Sigma_c^0(G)$ .

**Example 1.25.** Let  $\mathcal{U}(\mathfrak{g})$  be the enveloping algebra of the Lie algebra bundle  $\mathfrak{g}$ . The inhomogeneous dilations  $(\delta_\lambda)_{\lambda \in \mathbb{R}_+^*}$  extend to algebra automorphisms of  $\mathcal{U}(\mathfrak{g})$ . Denote by  $\mathcal{U}_k(\mathfrak{g})$  the subset of elements homogeneous of order  $k$  for the dilations. Elements of the enveloping algebra decompose as sums of homogeneous elements, i.e.  $\mathcal{U}(\mathfrak{g}) = \bigoplus_{k \geq 0} \mathcal{U}_k(\mathfrak{g})$ .

Any element  $D \in \Gamma(M, \mathcal{U}_k(\mathfrak{g}))$  can be seen as a family of right-invariant differential operators on the fibers of  $G$  and thus defines an element  $D \in S_p^k(G)$ .

Let  $k > 0$  and  $D \in \Gamma(M, \mathcal{U}_k(\mathfrak{g}))$  be a Rockland differential operator. Assume  $D$  to be self-adjoint and non-negative and  $M$  to be a compact manifold. The third point of 1.22 allows to construct  $\overline{\text{Op}}(D)^{it} \in M(C^*(G))$  through functional calculus for  $t \in \mathbb{R}$ . One would hope to have some actual symbol  $D^{it} \in S_p^{ikt}(G)$  that would extend to the multiplier obtained through functional calculus. This is possible (even for the pseudodifferential operators) thanks to estimates on the heat kernels in the filtered calculus obtained in [12].

**Theorem 1.26** ([12]). Let  $G \rightarrow M$  be a Carnot bundle on a compact base. Let  $D \in \Gamma(M, \mathcal{U}_k(\mathfrak{g}))$  be Rockland, essentially self-adjoint and non-negative with  $k > 0$  an even number. There is a family of symbols  $D^{it} \in S_p^{ikt}(G)$  such that  $\text{Op}(D^{it})$  extends to the multiplier  $\overline{\text{Op}}(D)^{it}$  obtained from  $\overline{\text{Op}}(D)$  through functional calculus. These symbols are also compatible with the product:  $D^{it_1} D^{it_2} = D^{i(t_1+t_2)}$ .

We end this section by an analog of 1.15 for the algebra of principal symbols.

**Proposition 1.27.** Let  $G \rightarrow M$  be a bundle of graded Lie groups. The algebra of principal symbols  $\Sigma(G)$  is a continuous field of  $C^*$ -algebras over  $M$  with fiber at  $x \in M$  equal to  $\Sigma(G_x)$ .

*Proof.* It is clear from the fact that the distributions are fibered over  $M$  that  $\Sigma(G)$  is a  $\mathcal{C}_0(M)$ -algebra and the fiber at  $x \in M$  is equal to  $\Sigma(G_x)$ . The definition of the norm from 1.22 makes  $x \mapsto \|\sigma_x\|$  continuous for  $\sigma \in \Sigma(G)$  by 1.15.  $\square$

*Remark 1.28.* If  $G = G_0 \times M$  is a trivial bundle then  $\Sigma(G) = \Sigma(G_0) \otimes \mathcal{C}_0(M)$ .

## 1.4 Construction of the isomorphism for symbols

Let  $G \rightarrow M$  denote a bundle of graded Lie groups over a compact base. We want to construct an isomorphism:

$$\Phi: \Sigma(G) \rtimes \mathbb{R} \rightarrow C_0^*(G).$$

We first need to construct the  $\mathbb{R}$  action on  $\Sigma(G)$  underlying the crossed product. For that let  $\Delta_0 \in \mathcal{U}(\mathfrak{g})$  be an essentially self-adjoint, non-negative Rockland symbol of positive even order  $m$ . Thanks to 1.26 we can construct the complex powers of  $\Delta_0$  and get a family of symbols  $\Delta_0^{it/m} \in \Sigma^{it}(G)$ ,  $t \in \mathbb{R}$  which is a group under composition law for symbols. Since  $\Delta_0$  is self-adjoint they also are unitaries. Let  $\sigma \in \Sigma_c^0(G)$  and  $t \in \mathbb{R}$ , the symbol  $\text{Ad}(\Delta_0^{it/m})\sigma := \Delta_0^{it/m} \sigma \Delta_0^{-it/m}$  is of order 0. We hence obtain an action of  $\mathbb{R}$  on  $\Sigma_c^0(G)$ . This action is continuous and thus defines an action on the  $C^*$  completion  $\Sigma(G)$ . We now want to define the morphism  $\Phi$  sending the crossed product to  $C_0^*(G)$ . To do

this we need a (strongly continuous) family of unitary multipliers  $u_t \in M_u(C_0^*(G))$  and a morphism  $\varphi: \Sigma(G) \rightarrow M(C_0^*(G))$  with the property that

$$\forall t \in \mathbb{R}, \sigma \in \Sigma(G), \varphi \left( \text{Ad}(\Delta_0^{it/m})\sigma \right) = u_t \varphi(\sigma) u_t^*. \quad (1)$$

These two constructions are similar and rely on a lemma due to Taylor. This lemma's purpose is to represent each class of principal symbols by a particular distribution on  $G$  which will be homogeneous on the nose. To use this lemma however we need to use tempered distributions on the fiber. To do this, recall that the Schwartz class of functions on a vector space does not depend on the choice of basis (nor on the norm). Therefore on a bundle of graded Lie groups, we denote by  $\mathcal{S}(G)$  the space of smooth sections that are of Schwartz type on the fibers (we identify  $G$  and  $\mathfrak{g}$  as bundles over  $M$ ). To use the groupoid convolution we extend the definition to half-densities. Denote by  $\mathcal{S}(G, \Omega^{1/2})$  the space of sections which, in any trivialization of  $\Omega^{1/2}$ , become an element of  $\mathcal{S}(G)$ . This definition indeed does not depend on the choice of trivialization (it only changes the Schwartz semi-norms to other equivalent ones). The space  $\mathcal{S}(G, \Omega^{1/2})$  contains  $\mathcal{C}_c^\infty(G, \Omega^{1/2})$ . Moreover the groupoid convolution extends to a continuous product on  $\mathcal{S}(G, \Omega^{1/2})$ , see [21]. This makes the Schwartz algebra a subalgebra of  $C^*(G)$  (it is also stable under the involution).

**Definition 1.29.** Let  $s \in \mathbb{C}$  be a complex number, denote by  $\mathcal{K}^s(G) \subset \mathcal{S}'(G, \Omega^{1/2})$  the set of fibered tempered distributions on  $G$  that have singular support contained in the unit section and are homogeneous of degree  $s$  with respect to the family of dilations  $(\delta_\lambda)_{\lambda>0}$ .

On the space of Schwartz sections, we define a Fourier transform. Identifying  $G$  with  $\mathfrak{g}$ , it becomes:

$$\begin{aligned} \mathcal{F}: \mathcal{S}(G, \Omega^{1/2}) &\rightarrow \mathcal{S}(\mathfrak{g}^*) \\ f &\mapsto \left( (x, \eta) \mapsto \int_{\xi \in \mathfrak{g}_x} e^{i\langle \eta, \xi \rangle} f(x, \xi) \right) \end{aligned}$$

Consider the dual action of  $\mathbb{R}_+^*$  on  $\mathfrak{g}^*$  given by  $({}^t d\delta_\lambda)_{\lambda>0}$ . The Fourier transform has the following equivariance property:

$$\forall \lambda > 0, \mathcal{F} \circ \delta_\lambda^* = {}^t d\delta_\lambda^* \circ \mathcal{F}$$

Among the Schwartz sections, denote by  $\mathcal{S}_0(G, \Omega^{1/2})$  those for which the Fourier transform vanishes at infinite order on the zero section of  $\mathfrak{g}^*$ . It is a closed subalgebra of the Schwartz algebra (for the topology induced by the Schwartz semi-norms).

**Lemma 1.30** (Taylor [35]). *Let  $k \in \mathbb{C}$ ,  $u \in S_p^k(G)$ . Then there exists a unique smooth function  $v \in \mathcal{C}^\infty(\mathfrak{g}^* \setminus 0)$  such that:*

- *$v$  is homogeneous of degree  $k$ :  $\forall \lambda > 0, v \circ {}^t d\delta_\lambda = \lambda^k v$*
- *if  $\chi \in \mathcal{C}_c^\infty(\mathfrak{g}^*)$  is equal to 1 on a neighborhood of the zero section then  $\hat{u} - (1 - \chi)v \in \mathcal{S}(\mathfrak{g}^*)$ .*

*Conversely if a function  $v \in \mathcal{C}^\infty(\mathfrak{g}^* \setminus 0)$  is homogeneous of degree  $k$  then one can find a symbol  $u \in S_c^k(G)$  such that the second property is satisfied.*

**Corollary 1.30.1.** *There is a linear isomorphism  $\Theta: \Sigma_p^s(G) \rightarrow \mathcal{K}^s(G)$  for each  $s \in \{z \in \mathbb{C} / \Re(z) > -n\}$  which is compatible with the convolution and adjoints. Here  $n$  denotes the homogeneous dimension of  $G$ .*

*Proof.* Let  $u \in S_p^s(G)$ . Define  $v$  as in Taylor's lemma. The condition on the order  $\Re(s) > -n$  ensures that  $v \in L_{loc}^1(\mathfrak{g}^*)$ . The function  $v$  thus extends to a tempered distribution

$$v \in \mathcal{S}'(\mathfrak{g}^*).$$

Its inverse Fourier transform  $w$  is in  $\mathcal{S}'(G, \Omega^{1/2})$  and is homogeneous of degree  $s$ , i.e.  $w \in \mathcal{K}^s(G)$ . This construction gives a linear map  $S_c^s(G) \rightarrow \mathcal{K}^s(G)$  which is surjective. By the uniqueness of  $v$  in Taylor's lemma, the kernel of this map consists of smoothing symbols. It is thus equal to  $S_p^{-\infty}(G)$ . The resulting quotient map  $\Theta: \Sigma_p^s(G) \rightarrow \mathcal{K}^s(G)$  is then an isomorphism.  $\square$

**Lemma 1.31** (Christ, Geller, Głowacki, Polin [8]). *Let  $u \in \mathcal{K}^s(G)$ . The convolution by  $u$  is a continuous map from  $\mathcal{S}_0(G)$  to itself.*

**Proposition 1.32.** *Let  $k \in \mathcal{K}^s(G)$  be a homogeneous distribution of degree  $s \in \mathbb{C}$ . If  $\Re(s) \leq 0$  then  $k$  extends to an element of  $M(C_0^*(G))$ . Moreover*

$$\|k\| = \sup_{x \in M} \|k_x\|_{L^2(G_x)} = \sup_{x \in M, \pi \in \hat{G}_x} \|\pi(k_x)\|.$$

**Proposition 1.33.** *The composition of  $\Theta$  with the injection  $\mathcal{K}^0(G) \hookrightarrow M(C_0^*(G))$  extends to a \*-homomorphism  $\varphi: \Sigma(G) \rightarrow M(C_0^*(G))$ .*

*Proof.* It remains to show that this composition is continuous but this is a consequence of the previous proposition since for  $u \in S_c^0(G), x \in M, \pi \in \hat{G}_x$  we have:

$$\|\pi(\Theta(u)_x)\| \leq \|u\|_{L^2(G)}.$$

Indeed,  $\beta := u - \Theta(u) \in \mathcal{S}(G, \Omega^{1/2})$  and by the Plancherel formula:

$$\|\pi \circ \delta_\lambda(\beta)\| \leq \text{Tr}(\pi \circ \delta_\lambda(\beta^* * \beta)) = \int_{\mathfrak{t}_{\delta_\lambda(\mathcal{O}_\pi)}^*} \widehat{\beta^* * \beta}(x, \xi) \, d\xi \xrightarrow{\lambda \rightarrow +\infty} 0.$$

The Plancherel formula can be found in [11], here  $\mathcal{O}_\pi$  denotes the coadjoint orbit in  $\mathfrak{g}^*$  corresponding to the representation  $\pi$  through Kirillov's orbit method. Since  $\Theta(u)$  is homogeneous of degree 0 we have  $\|\pi(\Theta(u)_x)\| = \|\pi \circ \delta_\lambda(\Theta(u)_x)\|$  for every  $\lambda > 0$ . We thus get  $\|\pi(\Theta(u)_x)\| \leq \lim_{\lambda \rightarrow +\infty} \|\pi(u_x)\| \leq \|u\|$ .  $\square$

In the same way, let us denote by  $u_t$  the extension as an element of  $M(C_0^*(G))$  of  $\Theta(\Delta_0^{it/m})$ . Since  $\Theta$  preserves the adjoints then  $u_t$  is a unitary multiplier and since it preserves the convolution then  $u_t u_s = u_{t+s}$  and we get the relation (1). We have thus constructed from  $\varphi$  and  $(u_t)_{t \in \mathbb{R}}$  a \*-homomorphism  $\Phi: \Sigma(G) \rtimes \mathbb{R} \rightarrow C_0^*(G)$ . The rest of this section is devoted to the proof that  $\Phi$  is an isomorphism.

To show that  $\Phi$  is an isomorphism, we need a better understanding of the representations of  $\Sigma(T_H M)$ . We first reduce our study to the bundle of osculating Lie group  $T_H M$  to a single graded Lie group. This follows from the following result:

**Proposition 1.34.** *The algebras  $\Sigma(G)$  and  $C_0^*(G)$  are bundles of  $C^*$ -algebras over  $M$  with respective fibers at  $x \in M$  equal to  $\Sigma(G_x)$  and  $C_0^*(G_x)$ . Moreover the action of  $\mathbb{R}$  on  $\Sigma(G)$  preserves the fibers and  $\Phi$  restricts to a map from  $\Sigma(G_x) \rtimes \mathbb{R}$  to  $C_0^*(G_x)$ .*

We can now consider a graded Lie group  $G$  and prove that  $\Phi: \Sigma(G) \rtimes \mathbb{R} \rightarrow C_0^*(G)$  is an isomorphism.

Since  $\varphi$  maps  $\Sigma(G)$  to  $M(C_0^*(G))$ , every non-trivial irreducible representation of  $G$  induces a representation of  $\Sigma(G)$ . Moreover, since the image of  $\varphi$  consists of invariant elements under the inhomogeneous  $\mathbb{R}_+^*$ -action then  $\pi \circ \varphi = \delta_\lambda(\pi) \circ \varphi$  for all  $\lambda > 0$  and  $\pi \in \widehat{G} \setminus \{1\}$ . The following theorem due to Fermanian-Kammerer and Fischer asserts that these representations of  $\Sigma(G)$  are irreducible and allow to recover the whole spectrum of  $\Sigma(G)$ .

**Theorem 1.35** (Fermanian-Kammerer, Fischer). *Let  $G$  be a graded Lie group. For  $\pi \in \widehat{G} \setminus \{1\}$  denote by  $[\pi]$  its class in  $(\widehat{G} \setminus \{1\})/\mathbb{R}_+^*$ . For every  $\pi \in \widehat{G} \setminus \{1\}$ , we have  $\pi \circ \varphi \in \widehat{\Sigma(G)}$  and the map*

$$R: (\widehat{G} \setminus \{1\})/\mathbb{R}_+^* \rightarrow \widehat{\Sigma(G)}$$

$$[\pi] \mapsto \pi \circ \varphi$$

is a homeomorphism.

*Idea of proof.* Let  $|\cdot|$  be a homogeneous quasi-norm on  $G$ . To  $f \in \mathcal{C}_c^\infty(G, \Omega^{1/2})$  we associate  $\sigma_f: g \rightarrow \delta_{|g|^*} f(g) = \delta_{|g|-1} f(g)$  defined on  $G \setminus \{e\}$ . The function  $\sigma_f$  is  $\mathcal{C}^\infty$  and homogeneous of order 0 by construction. The construction  $f \rightarrow \sigma_f$  thus defines a mapping  $\mathcal{C}_c^\infty(G) \rightarrow \mathcal{K}^0(G)$ . The composition with  $\Theta^{-1}: \mathcal{K}^0(G) \rightarrow \Sigma^0(G)$  gives a  $*$ -homomorphism  $\mathcal{C}_c^\infty(G) \rightarrow \Sigma^0(G)$ . Let  $\rho$  be a representation of  $\Sigma(G)$ ,  $\pi_\rho: f \rightarrow \rho(\sigma_f)$  extends to a continuous representation of  $C^*(G)$ . From this construction, Fermanian-Kammerer and Fischer proceed to show that if  $\rho$  is irreducible then  $\pi_\rho$  is irreducible, non-trivial and  $\pi_\rho \circ \varphi = \rho$ . This defines a right inverse to  $R$  which is thus surjective. They also show that  $R$  is injective. The fact that  $R$  is a homeomorphism is then implied by the compactness of  $(\widehat{G} \setminus \{1\})/\mathbb{R}_+^*$ .  $\square$

**Corollary 1.35.1.** *Let  $(M, H)$  be a filtered manifold then the spectrum of  $\Sigma(T_H M)$  is homeomorphic to  $(\widehat{T_H M} \setminus \{1\})/\mathbb{R}_+^*$ .*

*Proof.* Thanks to 1.34 the result reduces to each fiber over the points of  $M$ , we can then apply 1.35.  $\square$

We are now ready to prove the main theorem of this section.

**Theorem 1.36.** *Let  $G$  be a graded Lie group, then*

$$\Phi: \Sigma(G) \rtimes \mathbb{R} \rightarrow C_0^*(G),$$

is an isomorphism of  $C^*$ -algebras.

**Corollary 1.36.1.** *Let  $(M, H)$  be a filtered manifold, then*

$$\Phi: \Sigma(T_H M) \rtimes \mathbb{R} \rightarrow C_0^*(T_H M),$$

is an isomorphism of  $C^*$ -algebras.



*Proof.* Let  $V_0 \subset V_1 \subset \dots \subset V_r = \hat{G} \setminus \{1\}$  be a fine stratification of  $\hat{G}$  as in Theorem 1.3.2. Recall that every  $V_i$  is open and  $\mathbb{R}_+^*$ -invariant, that the spaces  $\Lambda_i := V_i \setminus V_{i-1}$  are Hausdorff and that the action  $\mathbb{R}_+^* \curvearrowright \Lambda_i$  is free and proper. This stratification induces a filtration of  $\Sigma(G)$  and  $C_0^*(G)$  into sequences of increasing ideals:

$$\begin{aligned} \{0\} &= J_0 \triangleleft J_1 \triangleleft \dots \triangleleft J_r = C_0^*(G) \\ \{0\} &= \Sigma_0 \triangleleft \Sigma_1 \triangleleft \dots \triangleleft \Sigma_r = \Sigma(G) \end{aligned}$$

with  $J_i = \bigcap_{\pi \in \hat{G} \setminus V_i} \ker(\pi)$  and  $\Sigma_i = \bigcap_{\pi \in \hat{G} \setminus V_i} \ker(\pi \circ \varphi)$ . By construction the subsets  $\Sigma_i$  are  $\mathbb{R}$ -invariant and  $\Phi$  restricts to maps  $\Phi: \Sigma_i \times \mathbb{R} \rightarrow J_i$ . The spectrum of  $J_i/J_{i-1}$  is  $\widehat{J_i/J_{i-1}} = \widehat{J_i} \setminus \widehat{J_{i-1}} = V_i \setminus V_{i-1} = \Lambda_i$  for every  $i$ . Recall that the quotient algebra  $J_i/J_{i-1}$  is a continuous field of  $C^*$ -algebras over its spectrum  $\Lambda_i$  and by Corollary 1.15.1 (see also [5]) the fiber at  $\lambda \in \Lambda_i$  is  $\mathcal{K}(\mathcal{H}_i)$  where  $\mathcal{H}_i$  is a Hilbert space (of dimension 1 for  $i = r$  and of infinite dimension otherwise). Since the action of  $\mathbb{R}_+^*$  on  $\Lambda_i$  is free and proper,  $\Lambda_i/\mathbb{R}_+^*$  is also a Hausdorff space and each  $\Sigma_i/\Sigma_{i-1}$  is a continuous field of  $C^*$ -algebras over  $\Lambda_i/\mathbb{R}_+^*$ . The fiber at  $[\lambda] \in \Lambda_i/\mathbb{R}_+^*$  is the image of  $\lambda \circ \varphi$ . It is a sub-algebra of  $\mathcal{B}(\mathcal{H}_i)$ , we will denote it by  $A_\lambda$ .

**Lemma 1.37.** *For every  $\lambda \in \hat{G} \setminus \{1\}$ ,  $A_\lambda$  is a simple algebra.*

*Proof.* The spectrum of a continuous field of  $C^*$ -algebra is in bijection with the direct sum of the spectra of the fibers. Here the spectrum of the continuous field is exactly its base hence the fibers have trivial spectrum and are thus simple. Indeed having a trivial spectrum means having no non-trivial primitive ideal but since every ideal is an intersection of primitive ideals then an algebra with no non-trivial primitive ideals is simple.  $\square$

**Lemma 1.38.** *For every  $\lambda \in \hat{G} \setminus \{1\}$ ,  $A_\lambda$  contains  $\mathcal{K}(\mathcal{H}_i)$  (where  $\lambda \in \Lambda_i$ ).*

*Proof.* As in the proof of 1.35, let us choose a homogeneous quasi-norm on  $G$ . This induces a  $*$ -homomorphism  $\mathcal{S}(G) \rightarrow \Sigma(G)$  and if  $\lambda \in \hat{G}$  is of norm 1 then the previous morphism composed with  $\lambda$  seen as a representation of  $\Sigma(G)$  extends to  $\lambda$  seen as a representation of  $C_0^*(G)$ . Thus  $A_\lambda$  contains  $\lambda(C^*(G)) = \mathcal{K}(\mathcal{H}_\lambda)$ .  $\square$

**Corollary 1.38.1.** *The algebras  $\Sigma_i/\Sigma_{i-1}$  are trivial fields of  $C^*$ -algebras:*

$$\Sigma_i/\Sigma_{i-1} \cong \mathcal{C} \left( \Lambda_i/\mathbb{R}_+^*, \mathcal{K}(\mathcal{H}_i) \right).$$

**Corollary 1.38.2.** *For every  $i \geq 0$ , the action of  $\mathbb{R}$  on  $\Sigma_i/\Sigma_{i-1}$  is inner and  $\Phi: \Sigma_i/\Sigma_{i-1} \times \mathbb{R} \rightarrow J_i/J_{i-1}$  is an isomorphism.*

*Proof.* Let us fix a number  $t \in \mathbb{R}$ . Since  $\Delta_0^{it/m}$  acts as a multiplier  $u_t$  of  $J_i/J_{i-1}$  then for each representation  $\lambda \in \Lambda_i$ ,  $\lambda(u_t) \in \mathcal{B}(\mathcal{H}_i)$ . The operator  $\Delta_0^{it/m}$  thus preserves each fiber

of the continuous field of  $C^*$  algebras defining  $\Sigma_i/\Sigma_{i-1}$ . We then need to show that its norm is bounded but being a multiplier of  $J_i/J_{i-1}$  we have:

$$\sup_{\lambda \in \Lambda_i} \|\lambda(\Delta_0^{it/m})\|_{\mathcal{B}(\mathcal{H}_\lambda)} < +\infty$$

Therefore we have:

$$\Delta_0^{it/m} \in \mathcal{C}_b(\Lambda_i/\mathbb{R}_+^*, \mathcal{K}(\mathcal{H}_i)) \subset \mathcal{M}(\Sigma_i/\Sigma_{i-1}).$$

The crossed product  $\Sigma_i/\Sigma_{i-1} \rtimes \mathbb{R}$  is therefore trivial and we have:

$$\Sigma_i/\Sigma_{i-1} \rtimes \mathbb{R} \cong \mathcal{C}_0(\Lambda_i, \mathcal{K}(\mathcal{H}_i)).$$

The identification  $\Lambda_i/\mathbb{R}_+^* \times \mathbb{R}_+^* \cong \Lambda_i$  is made by the choice of quasi-norm on  $G$ . □

We have thus far showed that the maps:

$$\Phi: \Sigma_i/\Sigma_{i-1} \rtimes \mathbb{R} \rightarrow J_i/J_{i-1},$$

were isomorphisms of  $C^*$ -algebras. We can then use the respective exact sequences to show inductively that each

$$\Phi: \Sigma_i \rtimes \mathbb{R} \rightarrow J_i,$$

is an isomorphism (starting from  $i = 0$ ). The result for  $i = r$  then concludes the proof. □

*Remark 1.39.* Although we have shown that  $\Delta_0^{it/m}$  was a multiplier of each  $\Sigma_i/\Sigma_{i-1}$  this does not mean that it is a multiplier of  $\Sigma_i$ . This would mean that the crossed product is trivial which only seem to hold in the commutative case.

As a corollary of the proof we obtain a decomposition result for the symbol algebra:

**Theorem 1.40.** *Let  $G$  be a graded Lie group<sup>1</sup>,  $\emptyset = V_0 \subset V_1 \subset \dots \subset V_r = \hat{G} \setminus \{1\}$  the Pedersen stratification of its unitary dual. Denote by  $\Sigma_i(G)$  the ideal corresponding to the subspace  $V_i/\mathbb{R}_+^*$  of the spectrum  $\widehat{\Sigma(G)}$ . This gives a nested sequence of ideals:*

$$\{0\} = \Sigma_0(G) \triangleleft \Sigma_1(G) \triangleleft \dots \triangleleft \Sigma_r(G) = \Sigma(G)$$

and we have isomorphisms:

$$\Sigma_i(G)/\Sigma_{i-1}(G) \cong \mathcal{C}(\Lambda_i/\mathbb{R}_+^*, \mathcal{K}_i).$$

Here  $\Lambda_i = V_i \setminus V_{i-1}$ ,  $\Lambda_i/\mathbb{R}_+^*$  is a compact Hausdorff space and  $\mathcal{K}_i$  is the algebra of compact operators on a separable Hilbert space (of dimension 1 if  $i = r$  and infinite dimensional otherwise).

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<sup>1</sup>Or a locally trivial bundle of graded Lie groups.



This is a generalization of Epstein and Melrose decomposition of the symbol algebra in the contact case [17]. They showed that if  $(M, H)$  is a compact manifold then there is an exact sequence:

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow \Sigma(T_H M) \longrightarrow \mathcal{C}(\mathbb{S}H^*) \longrightarrow 0.$$

Regarding our result this corresponds to the Pedersen stratification of the Heisenberg group associated to  $H_x$  for each  $x \in M$ . Recall that this consists of  $\Lambda_1 = \mathbb{R}^*$  and  $\Lambda_2 = H_x^*$  and the  $\mathbb{R}_+^*$ -action on each component is a regular dilation. In this case we can extend the decomposition to the whole algebra (rather than the fibers over each point of  $M$ ) because  $T_H M$  is locally trivial (and our result clearly extends to locally trivial bundles of graded Lie groups). In general it is not clear if such a decomposition is possible since even the number of strata in Pedersen's stratification might vary from point to point.

## 2 The isomorphism for operators

### 2.1 Calculus on filtered manifolds

Let us now go back to the case of a filtered manifold  $(M, H)$ . There is an analog of Connes' tangent groupoid [10] denoted by  $\mathbb{T}_H M$ . Algebraically it has the form:

$$M \times M \times \mathbb{R}^* \sqcup T_H M \times \{0\} \rightrightarrows M \times \{0\}.$$

Its algebroid is given by the following algebra of sections:

$$\Gamma(\text{Lie}(\mathbb{T}_H M)) = \{X \in \mathfrak{X}(M \times \mathbb{R}) / \forall k \geq 0, \partial_t^k X|_{t=0} \in \Gamma(H^k)\}$$

which is almost injective hence integrable by a theorem of Debord [14]. The smooth structure is detailed in [36, 7, 28]. Recall that  $\mathbb{T}_H M$  is endowed with the zoom action  $\alpha$  of  $\mathbb{R}_+^*$ :

$$\begin{aligned} \alpha_\lambda(x, y, t) &= (x, y, \lambda^{-1}t) \\ \alpha_\lambda(x, \xi, 0) &= (x, \delta_\lambda(\xi), 0) \end{aligned}$$

where  $\delta$  is the inhomogeneous action of  $\mathbb{R}_+^*$  on  $T_H M$  used in the previous section. We briefly recall the definition of pseudodifferential operators in this context, following [37]:

**Definition 2.1.** A pseudodifferential operator in the filtered calculus on  $M$  of order  $m \in \mathbb{C}$  is a properly supported distribution  $P \in \mathcal{D}(M \times M, \Omega^{1/2})$  such that there exists  $\mathbb{P} \in \mathcal{D}'(\mathbb{T}_H M, \Omega^{1/2})$  with the following properties:

- $\mathbb{P}$  is properly supported
- $\mathbb{P}_1 = P$  where  $\mathbb{P}_t = \text{ev}_{t*} \mathbb{P}$  and  $\text{ev}_t$  is the evaluation map on the fiber at time  $t \in \mathbb{R}$  on  $\mathbb{T}_H M$ .
- $\mathbb{P}$  is quasi-homogeneous w.r.t the zoom action i.e.  $\forall \lambda > 0, \alpha_{\lambda*} \mathbb{P} - \lambda^m \mathbb{P} \in \mathcal{C}_p^\infty(\mathbb{T}_H M)$ .

We denote by  $\Psi_H^m(M)$  the set of such distributions and  $\Psi_H^m(M)$  the set of their extensions  $\mathbb{P}$  to  $\mathbb{T}_H M$ .

From this setup one can derive the usual properties of a pseudodifferential calculus, namely the algebra structure, the existence of a symbol map, ellipticity criterion, parametrices... The symbol of an operator  $P \in \Psi_H^m(M)$  is here obtained by extending it to some quasi-homogeneous  $\mathbb{P}$  and considering  $\mathbb{P}_0 \in S^m(T_H M)$ . This will depend on the choice of  $\mathbb{P}$  but the class  $[\mathbb{P}_0] \in \Sigma_c^m(T_H M)$  only depends on  $P$ . We get the exact sequence:

$$0 \longrightarrow \Psi_H^{m-1}(M) \longrightarrow \Psi_H^m(M) \longrightarrow \Sigma_c^m(T_H M) \longrightarrow 0.$$

As for the symbols, operators of non-positive order extend to bounded operators on  $L^2(M)$  and negative order operators to compact operators. Let us denote by  $\Psi_H^*(M)$  the closure of  $\Psi_H^0(M)$  in  $\mathcal{B}(L^2(M))$ . The previous exact sequence for  $m = 0$  extends to the following one:

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi_H^*(M) \longrightarrow \Sigma(T_H M) \longrightarrow 0.$$

The usual ellipticity condition is here replaced by the Rockland condition on symbols, we will say that a pseudodifferential operator satisfies the Rockland condition if its symbol does. Van Erp and Yuncken proved in [37] that this condition gives the existence of a parametrix for the operator in this calculus: if  $P \in \Psi_H^m(M)$  is Rockland if and only if there exists  $Q \in \Psi_H^{-m}(M)$  such that  $PQ - 1, QP - 1 \in \mathcal{C}^\infty(M \times M)$ . The last thing we will need is the existence of complex powers for some positive operators:

**Theorem 2.2** (Dave, Haller [13]). *Let  $M$  be a compact filtered manifold. Let  $P$  be a differential operator of positive even order  $d$  which satisfies the Rockland condition and is positive. Then the complex powers  $P^z, z \in \mathbb{C}$ , obtained through functional calculus are pseudodifferential operators of respective order  $dz$ . Moreover these operators form a holomorphic family of pseudodifferential operators.*

## 2.2 The Schwartz algebra

We now define the Schwartz algebra of the tangent groupoid. Our approach is analogous to the one of Carillo-Rouse in the unfiltered case [6]. A similar approach has been used by Ewert in [18] but she used the coordinates of Choi and Ponge [7] while we will use the ones of Van Erp and Yuncken [36] (from which we take some terminology for the constructions in this section). Both approaches yield the same algebra. The idea is to define the algebra on exponential charts and show that these algebras can be glued on the whole tangent groupoid. This Schwartz algebra will essentially contain two kinds of functions. The first are the smooth functions on the open subgroupoid  $\mathbb{T}_H M|_{\mathbb{R}^*}$  which are compactly supported at each  $t \neq 0$  and have a Schwartz decay at infinity and 0. The other are the Schwartz type functions on exponential charts (in particular their restriction on  $T_H M$  will be Schwartz). In order to describe these last functions, we fix  $\nabla$  a graded connection on  $\mathfrak{t}_H M$ , compatible with the dilations and a splitting  $\psi: \mathfrak{t}_H M \rightarrow TM^2$ . Recall that this induces a vector bundle isomorphism  $\Psi: \mathfrak{t}_H M \times \mathbb{R} \rightarrow \mathfrak{t}_H M$ . We fix  $\mathcal{U} \subset \mathfrak{t}_H M$  a domain of injectivity which gives us an open subset  $\mathbb{U} \subset \mathfrak{t}_H M \times \mathbb{R}$  such that  $\exp^{\nabla, \psi} \circ \Psi: \mathbb{U} \rightarrow \mathbb{T}_H M$  is a diffeomorphism onto its image  $\mathbb{V}$ . We have  $\mathbb{V} = (T_H M \times \{0\}) \sqcup (\exp^{\psi \circ \nabla \circ \psi^{-1}}(\mathcal{U}) \times \mathbb{R}^*)$  so  $\mathbb{V}$  is both a neighborhood of the unit section of  $\mathbb{T}_H M$  and of the zero-fiber  $T_H M$ . We thus have that  $\mathbb{T}_H M = \mathbb{V} \cup (M \times M \times \mathbb{R}^*)$ . We define the Schwartz algebra on  $\mathbb{V}$  by defining one on  $\mathbb{U}$  and pushing it forward with the exponential map.

<sup>2</sup>This means that for every  $j, \psi|_{H^j/H^{j-1}} \rightarrow H^j$  is a section of the quotient map.

**Definition 2.3.** The algebra  $\mathcal{S}(\mathbb{V})$  is defined as  $(\exp^{\nabla, \psi} \circ \Psi)_*(\mathcal{S}(\mathbb{U}))$ . A function  $f \in \mathcal{C}^\infty(\mathbb{U})$  is in  $\mathcal{S}(\mathbb{U})$  if there is a compact set  $K \subset \mathcal{U}$  and  $T > 0$  such that if  $(x, \delta_t \xi, t) \notin K \times [-T, T]$  then  $f(x, \xi, t) = 0$  and also  $f$  is Schwartz as a function on the bundle of nilpotent Lie algebras  $\mathfrak{t}_H M \times \mathbb{R} \rightarrow M \times \mathbb{R}$ . A function  $f \in \mathcal{C}^\infty(\mathbb{T}_H M)$  is a Schwartz type function if there exists an exponential chart  $\exp^{\nabla, \psi} \circ \Psi: \mathbb{U} \rightarrow \mathbb{V}$  such that

$$f \in \mathcal{S}(\mathbb{V}) + \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M)).$$

We denote by  $\mathcal{S}(\mathbb{T}_H M)$  the space of such functions.

Remember that  $\delta_0 = 0$  so in particular the whole  $\mathfrak{t}_H M$ , seen as the zero fiber of  $\mathfrak{t}_H M \times \mathbb{R}$ , can be included in the support the functions in  $\mathcal{S}(\mathbb{U})$ . For  $t \neq 0$  however the support of  $f_t = f(\cdot, \cdot, t)$  is contained in  $\delta_t^{-1}(K)$  which is compact. In particular a smooth function on  $\mathbb{U}$  is Schwartz if and only if it has the aforementioned support condition and is Schwartz at  $t = 0$  (asking it on  $\mathfrak{t}_H M \times \mathbb{R}$  is redundant).

**Proposition 2.4.** *Let  $f = (f_t)_{t \in \mathbb{R}} \in \mathcal{S}(\mathbb{T}_H M)$  then:*

- $f_0 \in \mathcal{S}(T_H M)$
- for every  $t \neq 0$ ,  $f_t \in \mathcal{C}_c^\infty(M \times M)$  moreover the support of each  $f_t$  is contained in a compact set of  $M \times M$  independent of  $t$
- the function  $t \mapsto f_t$  has rapid decay when  $t$  goes to  $\pm\infty$ .

*Proof.* These properties follow from the definition if the function is in the subalgebra  $\mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M))$ . If  $f \in \mathcal{S}(\mathbb{V})$  where  $\mathbb{V}$  is the range of an exponential chart then it follows directly from the definition of  $\mathcal{S}(\mathbb{U})$  and the explicit computation of the exponential map  $\exp^{\psi \circ \nabla \circ \psi^{-1}} \circ \Psi$ . For the first point we have  $\Psi|_{t=0} = \text{Id}_{\mathfrak{t}_H M}$  so we recover the definition of the Schwartz class on a graded Lie group bundle.  $\square$

**Theorem 2.5.** *The definition of  $\mathcal{S}(\mathbb{T}_H M)$  does not depend on the choice of connection, splitting and exponential chart. That means that for any connection  $\nabla$ , splitting  $\psi$ , domain of injectivity  $\mathcal{U}$  and corresponding exponential chart  $\mathbb{U} \xrightarrow{\sim} \mathbb{V} \subset \mathbb{T}_H M$  then:*

$$\mathcal{S}(\mathbb{T}_H M) = \mathcal{S}(\mathbb{V}) + \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M)).$$

*Proof.* If  $\mathbb{V}' \subset \mathbb{V}$  we can use a function  $\varphi$  compactly supported in  $\mathbb{V}$  with value 1 near the zero fiber so that if  $f \in \mathcal{S}(\mathbb{V})$  then  $f = \varphi f + (1 - \varphi)f$ . Since  $(1 - \varphi)f \in \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M))$  we can just show that  $\varphi f \in \mathcal{S}(\mathbb{V}')$ . Therefore we can restrict ourselves to the case where  $\mathbb{V}$  and  $\mathbb{V}'$  are defined from the same domain of injectivity  $\mathcal{U} \subset \mathfrak{t}_H M$  and show that if  $\Psi, \Psi': \mathfrak{t}_H M \times \mathbb{R} \xrightarrow{\sim} \mathfrak{t}_H M$  are the respective isomorphisms used to construct the exponential charts then:

$$(\Psi^{-1} \circ \Psi')^*(\mathcal{S}(\mathbb{U})) = \mathcal{S}(\mathbb{U}').$$

Notice that  $\Psi^{-1} \circ \Psi' = (\delta_t^{-1} \circ \psi^{-1} \circ \psi' \circ \delta_t)_{t \in \mathbb{R}}$ , the value at  $t = 0$  is  $\text{Id}$ . It is thus a vector bundle isomorphism of  $\mathfrak{t}_H M \times \mathbb{R}$  and therefore preserves the Schwartz class. We now need to show that the condition on the support is preserved under the transformation given by  $\Psi^{-1} \circ \Psi'$ . For  $K \subset \mathcal{U}$  and  $T > 0$ , we define:

$$K_T := \{(x, \xi, t) \in \mathfrak{t}_H M \times \mathbb{R}, t \in [-T; T] \& (x, \delta_t(\xi)) \in K\}.$$

By definition a function in  $\mathcal{S}(\mathbb{U})$  has a support contained in a set of the form  $K_T$  for some  $T > 0$  and  $K \subset \mathcal{U}$  compact. The computation of  $\Psi^{-1} \circ \Psi'$  shows directly that:

$$\Psi^{-1} \circ \Psi'(K_T) = (\Psi^{-1} \circ \Psi'(K))_T,$$

hence the condition on the support is preserved by the transformation  $\Psi^{-1} \circ \Psi'$ .  $\square$

In order to deal with convolution we define a Schwartz class for half-densities denoted by  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$ . They are the sections of the half-density bundle for which in one (hence any) zoom-invariant trivialization of the half-density bundle<sup>3</sup> they become Schwartz functions as defined before. To see that this definition is not ambiguous, let us consider two different zoom-invariant trivializations. They differ by a multiplication by a positive smooth function. By zoom invariance this function is zoom-invariant. In particular it is bounded and thus preserves the growth conditions at infinity used to define the Schwartz algebra.

**Theorem 2.6.** *The groupoid convolution and adjoint extend to the space  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  making it a  $*$ -algebra. Moreover we have the inclusions:*

$$\mathcal{C}_c^\infty(\mathbb{T}_H M, \Omega^{1/2}) \subset \mathcal{S}(\mathbb{T}_H M, \Omega^{1/2}) \subset C^*(\mathbb{T}_H M).$$

*Proof.* From what we have written thus far we can easily see that the convolution of two functions in the Schwartz class is well defined fiberwise. It thus remains to show that the result of the convolution is still in the Schwartz class. Let us fix a splitting  $\psi$ , an equivariant connection  $\nabla$ , consider a domain of injectivity  $\mathcal{U} \subset \mathfrak{t}_H M$  and let  $\mathbb{U} \subset \mathfrak{t}_H M \times \mathbb{R}$  the corresponding open subset and  $\exp^{\nabla, \psi}: \mathbb{U} \xrightarrow{\sim} \mathbb{V} \subset \mathbb{T}_H M$  the resulting exponential chart. Since the convolution will force us to consider products in  $\mathbb{T}_H M$  we choose an open subset  $\mathcal{U}' \subset \mathcal{U}$  such that  $m_{\mathbb{T}_H M}(\mathbb{V}' \times_r \mathbb{V}') \subset \mathbb{V}$ . Let  $f, g \in \mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$ , we decompose them as  $f = f_1 + f_2, g = g_1 + g_2$  with  $f_1, g_1 \in \mathcal{S}(\mathbb{V}', \Omega^{1/2})$  and  $f_2, g_2 \in \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$ . We have:

$$f * g = f_1 * g_1 + f_1 * g_2 + f_2 * g_1 + f_2 * g_2,$$

and we easily see that  $f_1 * g_2, f_2 * g_1, f_2 * g_2 \in \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$ . We now show that  $f_1 * g_1 \in \mathcal{S}(\mathbb{V}, \Omega^{1/2})$ . At any non-zero fiber the restrictions  $f_1$  and  $g_1$  are compactly supported and so is the restriction of  $f_1 * g_1$ . The same goes for the zero fiber since the Schwartz class on a nilpotent Lie group is preserved under convolution. Thus the only remaining property to verify is the support condition for  $f_1 * g_1$ . Let  $K, C \subset \mathcal{U}'$  be compact subsets and  $T > 0$  such that we have the inclusions  $\text{supp}(f_1) \subset \exp^{\nabla, \psi}(K_T), \text{supp}(g_1) \subset \exp^{\nabla, \psi}(C_T)$ . Define  $K \tilde{\oplus} C \subset \mathfrak{t}_H M$  as:

$$\begin{aligned} \exp^{\psi \circ \nabla \circ \psi^{-1}}(\psi(K \tilde{\oplus} C)) = & \left\{ \left( \exp_{\exp_x^{\psi \circ \nabla \circ \psi^{-1}}(\psi(\xi))}^{\psi \circ \nabla \circ \psi^{-1}}(\psi(\eta)), x \right), \right. \\ & \left. (x, \xi) \in K \ \& \ (\exp_x^{\psi \circ \nabla \circ \psi^{-1}}(\psi(\xi)), \eta) \in C \right\}. \end{aligned}$$

This set is constructed so that  $\exp^{\nabla, \psi}((K \tilde{\oplus} C)_T)$  would contain all possible products (in  $\mathbb{T}_H M$ ) of elements in  $\text{supp}(f_1)$  and  $\text{supp}(g_1)$  hence it would contain  $\text{supp}(f_1 * g_1)$ . By

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<sup>3</sup>This means that taking the pullback by  $\alpha_\lambda$  scales the measure by a factor  $\lambda^{-d_H}$  where  $d_H$  is the homogeneous dimension of  $M$ .

construction the set  $\exp^{\psi \circ \nabla \circ \psi^{-1}}(\psi(K \tilde{\oplus} C))$  is contained in  $\exp^{\nabla, \psi}(\mathbb{U})$  so that  $K \tilde{\oplus} C$  is well defined, compact and hence  $f_1 * g_1 \in \mathcal{S}(\mathbb{V})$ .

Now the fact that  $\mathcal{C}_c^\infty(\mathbb{T}_H M, \Omega^{1/2}) \subset \mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  is rather obvious. Moreover since we also have  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2}) \subset L^1(\mathbb{T}_H M, \Omega^{1/2})$  then every continuous representation of  $\mathbb{T}_H M$  extends continuously to  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  and thus  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2}) \subset C^*(\mathbb{T}_H M)$ .  $\square$

**Corollary 2.6.1.** *The subalgebra  $\mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  is an ideal of  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$ .*

## 2.3 The ideal $\mathcal{J}$ of Debord and Skandalis

In this section we introduce the analog of the ideal  $\mathcal{J}(G)$  of Debord and Skandalis (see [15]) for  $G = M \times M$  (although the idea directly generalizes to groupoids with filtered algebroid). Roughly speaking this ideal of  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  corresponds, in the commutative case, to the functions that vanish at infinite order on the zero section of  $TM$  seen as the zero fiber. In the noncommutative case we need to replace evaluation at zero by the trivial representation of the fiber. Elements of  $\mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  already vanish with infinite order on  $T_H M$  so they will satisfy this condition. Therefore we need to declare which functions of  $\mathcal{S}(\mathbb{V})$  will be in the ideal for an exponential chart  $\exp^{\nabla, \psi}: \mathbb{U} \xrightarrow{\sim} \mathbb{V}$ . Using the exponential map we can define the functions on the ideal working on  $\mathbb{U}$ . This allows us to use the Fourier transform  $\mathcal{F}: \mathcal{S}(\mathfrak{t}_H M \times \mathbb{R}, \Omega^{1/2}) \xrightarrow{\sim} \mathcal{S}(\mathfrak{t}_H M^* \times \mathbb{R})$ . Now the trivial representation of a fiber corresponds to the evaluation at 0 for the Fourier transform<sup>4</sup>. Consequently we can reformulate vanishing on the trivial representation by the vanishing of the Fourier transform at zero.

**Definition 2.7.** Let  $\exp^{\nabla, \psi}: \mathbb{U} \xrightarrow{\sim} \mathbb{V}$  be an exponential chart for  $\mathbb{T}_H M$ . A function  $f \in \mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  written as  $f = f_0 + f_1 \in \mathcal{S}(\mathbb{V}) + \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  is in the subspace  $\mathcal{J}_H(M)$  if  $\mathcal{F}(f \circ \exp^{\nabla, \psi} \circ \Psi)$  vanishes at infinite order on the zero section of  $\mathfrak{t}_H M^* \times \mathbb{R}$ .

**Proposition 2.8.** *A function  $f = (f_t)_{t \in \mathbb{R}} \in \mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  is in  $\mathcal{J}_H(M)$  if and only if for every  $g \in \mathcal{C}_c^\infty(M \times M, \Omega^{1/2})$  the function  $t \mapsto f_t * g$  extends to a function that belongs to  $\mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$ .*

*Proof.* Since  $\mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  is an ideal we can assume that  $f$  lies in an exponential chart

$$\exp^{\nabla, \psi} \circ \Psi: \mathbb{U} \xrightarrow{\sim} \mathbb{V}.$$

Writing the Taylor series at 0 of  $t \mapsto \mathcal{F}(f_t \circ \exp^{\nabla, \psi} \circ \Psi)(x, {}^t d\delta_t \eta)$  we see that every  $t \mapsto f_t * g$  vanishes at order  $k$  at  $t = 0$  if and only if the Taylor coefficients up to order  $k$  vanish as well.  $\square$

*Remark 2.9.* The proof actually shows that the functions  $t \mapsto f_t * g$  have the same order of annulation as the one of the Fourier transform of  $f$  in local charts.

**Corollary 2.9.1.** *The set  $\mathcal{J}_H(M)$  is a  $*$ -ideal of  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$ .*

*Proof.* This directly follows from the previous proposition and the fact that the subalgebra  $\mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  is a  $*$ -ideal of  $\mathcal{S}(\mathbb{T}_H M, \Omega^{1/2})$  itself.  $\square$

<sup>4</sup>Here we use the scalar Fourier transform. In the non-commutative setup a representation valued Fourier transform is often more suitable, see [27, 19]. However, since we are only interested in the trivial representation (which corresponds to a single point through the orbit method) the scalar valued transform is more manageable.

## 2.4 Pseudodifferential operators as integrals

We now adapt the main result of [15] to the non-commutative case. To prove that a certain distribution on  $M \times M$  is a pseudodifferential operator we will want to find a quasi-homogeneous extension to  $\mathbb{T}_H M$ . This is not very natural however and the natural extension is exactly homogeneous. We thus need a global version of Taylor's lemma 1.30. Its proof is similar to the previous one that can be found in [35], this global version can be found in [1] (proposition 3.4).

**Lemma 2.10** (Global Taylor's lemma). *Let  $\exp^{\nabla, \psi}: \mathbb{U} \xrightarrow{\sim} \mathbb{V} \subset \mathbb{T}_H M$  be an exponential neighborhood. Let  $\mathbb{P} \in \Psi_H^m(M)$  with support in  $\mathbb{V}$ . There is a (unique) smooth function  $v \in \mathcal{C}^\infty((\mathfrak{t}_H M \times \mathbb{R}) \setminus (M \times \{(0, 0)\}))$  such that:*

*i)  $v$  is homogeneous of degree  $m$ :  $\forall \lambda > 0, v(x, {}^t\delta_\lambda(\eta), t\lambda) = \lambda^m v(x, \eta, t)$ .*

*ii) If  $\chi \in \mathcal{C}_c^\infty(\mathfrak{t}_H M^* \times \mathbb{R})$  equal to 1 in the neighborhood of the zero section, then the map:*

$$(x, \xi, t) \mapsto \exp^{\nabla, \psi^*} \mathbb{P}(x, \xi, t) - \int_{\eta \in \mathfrak{t}_{H,x} M^*} e^{i\eta(\xi)} (1 - \chi)(x, \eta, t) v(x, \eta, t)$$

*is a smooth section of  $\Omega^1(\mathfrak{t}_H M)$  over  $\mathfrak{t}_H M \times \mathbb{R}$ . It has support included in  $\mathfrak{t}_H M \times [-T; T]$  for some  $T > 0$ , and all the derivatives w.r.t  $x$  and  $t$  are Schwartz in the  $\xi$  direction uniformly in  $x$  and  $t$ .*

*Conversely if a function  $v$  satisfies i) then one can find  $\mathbb{P} \in \Psi_H^m(M)$  such that ii) is satisfied.*

**Theorem 2.11.** *Let  $m \in \mathbb{C}$ ,  $f \in \mathcal{J}_H(M)$ . The convolution operator by  $\int_0^{+\infty} t^{-m} f_t \frac{dt}{t}$  on  $M \times M$  corresponds to a pseudodifferential operator of order  $m$  in the filtered calculus. Its principal symbol is given as an element of  $\mathcal{K}^m(T_H M)$  by  $\int_0^{+\infty} t^{-m} \delta_{t^*} f_0 \frac{dt}{t}$ .*

*Proof.* If  $f \in \mathcal{S}(\mathbb{R}^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  then the integral converges absolutely. Proposition 2.8 then gives meaning to the integral  $\int_0^{+\infty} t^{-m} f_t \frac{dt}{t}$  as a convolution operator. Moreover we can assume that  $f$  is supported in an exponential chart and work on  $\mathfrak{t}_H M \times \mathbb{R}$ . Using the converse of Taylor's Lemma we need to find a homogeneous extension of the Fourier transform of the operator on  $\mathbb{U}$  where  $\mathbb{U} \xrightarrow{\sim} \mathbb{V}$  is the exponential neighborhood where  $f$  is supported. We can rewrite the operator as

$$\int_0^{+\infty} s^{-m} \alpha_{s^*} f_1 \frac{ds}{s},$$

and thus consider the natural extension:

$$\mathbb{P}_t := \int_0^{+\infty} s^{-m} \alpha_{s^*} f_t \frac{ds}{s}.$$

This extension is exactly homogeneous of degree  $m$  for the push-forward by the zoom action. Since  $\alpha$  restricts to  $\delta$  on the zero fiber we obtain that  $\mathbb{P}_0 = \int_0^{+\infty} s^{-m} \delta_{s^*} f_0 \frac{ds}{s}$ . In order to use Taylor's lemma we consider  $f$  as a function in  $\mathcal{S}(\mathbb{U})$ , we now need to show

that the Fourier transform of  $\mathbb{P}$  defines a smooth function. We have  $f_t \in \mathcal{S}(\mathfrak{t}_H M)$  so its Fourier transform is well defined. Recall that under the exponential map, the zoom action on  $\mathfrak{t}_H M \times \mathbb{R}$  becomes  $\beta$  with:

$$\beta_\lambda(x, \xi, t) = (x, d\delta_\lambda(\xi), \lambda^{-1}t).$$

The action on  $\mathfrak{t}_H M^* \times \mathbb{R}$  is then  ${}^t\beta$  where:

$${}^t\beta_\lambda(x, \eta, t) = (x, {}^t d\delta_\lambda(\eta), \lambda t).$$

If  $\mathcal{F}: \mathcal{S}(\mathfrak{t}_H M \times \mathbb{R}) \rightarrow \mathcal{S}(\mathfrak{t}_H M^* \times \mathbb{R})$  denotes the Fourier transform in the  $\mathfrak{t}_H M$  directions then we have:

$$\mathcal{F} \circ \beta_{\lambda*} = {}^t\beta_\lambda^* \circ \mathcal{F}.$$

This result yields:

$$\hat{\mathbb{P}}(x, \eta, t) = \int_0^{+\infty} s^{-m} \hat{f}_{ts}(x, {}^t d\delta_s(\eta)) \frac{ds}{s}.$$

Now  $s \mapsto \hat{f}_{ts}(x, {}^t d\delta_s(\eta))$  is in  $\mathcal{S}(\mathbb{R}_+^*)$  by assumption thus the integral converges (we can factor out a high enough power of  $s$  to compensate for the  $s^{-(m+1)}$  and get the convergence at 0). This convergence is uniform for  $(x, \eta, t)$  in any compact subset of  $\mathfrak{t}_H M^* \times \mathbb{R} \setminus (M \times \{0\})$ . Therefore the function  $\hat{\mathbb{P}}$  is smooth on  $\mathfrak{t}_H M^* \times \mathbb{R} \setminus (M \times \{0\})$  and homogeneous by construction. Therefore the operator given by Taylor's lemma gives a quasi homogeneous extension to  $\mathbb{P}_1$  (it is equal to  $\mathbb{P}_1$  at  $t = 1$  modulo a smoothing operator).  $\square$

*Remark 2.12.* We actually see that to obtain the convergence of the integral we only need  $\hat{f}$  to vanish on the zero section at order superior or equal to  $\Re(m) + 1$ . This will be useful later when using the factorisation lemma.

**Lemma 2.13.** *Let  $f_0 \in \mathcal{S}(T_H M)$  such that  $\hat{f}_0$  vanishes on  $M \times \{0\}$  at order  $k \in \mathbb{N} \cup \{+\infty\}$ , then there exists  $f \in \mathcal{S}(\mathbb{T}_H M)$  such that, in an exponential chart,  $(\hat{f}_t)_{t \in \mathbb{R}}$  vanishes on  $M \times \{0\} \subset \mathfrak{t}_H M \times \mathbb{R}$  at order  $k$  and  $f_0 = f(\cdot, 0)$ .*

*Proof.* Let us consider a function  $\varphi \in \mathcal{C}_c^\infty(\mathfrak{t}_H M)$  such that  $\varphi(x, 0) = 1$  and  $\lim_{t \rightarrow +\infty} \hat{\varphi} \circ {}^t d\delta_t = \delta_M$  where  $\delta_M$  is the distribution corresponding to integrating along the zero section of  $\mathfrak{t}_H M$  (it corresponds to the Dirac distribution at zero on each fiber of  $\mathfrak{t}_H M$ ). Then we set:

$$\tilde{f}(x, \xi, t) = f_0(x, \xi) \varphi(x, d\delta_t(\xi)).$$

We have  $\tilde{f}_t(x, \eta) = \widehat{f_0}(x, \eta) *_{ab} \varphi(x, {}^t d\delta_{t^{-1}}(\eta))$  where  $*_{ab}$  is the convolution product on  $\mathfrak{t}_H^* M$  seen as a bundle of abelian groups and we obtain the vanishing of  $\tilde{f}$  at the right order. However the function  $\tilde{f}$  is not in the Schwartz class as  $\tilde{f}(x, 0, t) = f_0(x, 0)$  is constant in  $t$  hence not rapidly decreasing in general. This can be corrected by multiplying  $\tilde{f}$  by a cutoff function  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  that is constant equal to 1 in a neighborhood of 0.  $\square$

**Proposition 2.14.** *Every pseudodifferential operator can be written as in theorem 2.11: if  $P \in \Psi_H^m(M)$  there exists  $f \in \mathcal{J}_H(M)$  such that*

$$P = \int_0^{+\infty} t^{-m} f_t \frac{dt}{t}.$$



*Proof.* The result is trivial if  $P \in \mathcal{C}_c^\infty(M \times M, \Omega^{1/2})$ . If  $P$  is a pseudodifferential operator of order  $m$  then let  $\sigma^m(P) \in \Sigma^m(T_H M)$  be its principal symbol, its associated full symbol  $v \in \mathcal{C}^\infty(\mathfrak{t}_H M^* \setminus (M \times \{0\}))$  obtained by 2.10 is homogeneous of order  $m$  and can thus be written as  $\int_0^{+\infty} t^{-m} \delta_t^* \hat{f}_0 \frac{dt}{t}$  for a function  $f_0 \in \mathcal{S}_0(\mathbb{T}_H M, \Omega^{1/2})$ . We then use lemma 2.13 to extend  $f_0$  to a function  $f \in \mathcal{J}_H(M)$ . Then  $P$  and  $P_f := \int_0^{+\infty} t^{-m} f_t \frac{dt}{t}$  have the same principal symbol and thus belong to  $\Psi_H^{m-1}(M)$ . We then iterate the argument and approach  $P$  with an asymptotic series<sup>5</sup>.  $\square$

**Lemma 2.15.** *Let  $f, g \in \mathcal{J}_H(M)$  then  $u \mapsto (f_t * g_{tu})_{t \in \mathbb{R}} \in \mathcal{S}(\mathbb{R}_+, \mathcal{J}_H(M))$ .*

*Proof.* This is the same type of reasoning as in 2.8 but applied to the groupoid  $\mathbb{T}_H M$  instead of  $M \times M$  (recall our construction is valid for an arbitrary groupoid). Here  $\mathbb{T}_H M$  has its algebroid naturally filtered by the filtration on  $TM$  and on  $\mathfrak{t}_H M$  (the  $j$ -th stratum of  $\mathfrak{t}_H M$  is  $\bigoplus_{i=1}^j H^i / H^{i-1}$ ).  $\square$

**Proposition 2.16.** *Let  $f = (f_t)_{t \in \mathbb{R}} \in \mathcal{J}_H(M)$  and  $P \in \Psi_H^s(M)$  with  $\Re(s) \leq 0$ , then  $(f_t * P)_{t \in \mathbb{R}^*}$  extends to an element of  $\mathcal{J}_H(G)$ . Moreover its value at  $t = 0$  is  $f_0 * \sigma^s(P)$  with  $\sigma^s(P) \in \mathcal{K}^s(T_H M)$  seen as a multiplier of  $\mathcal{S}_0(T_H M)$ .*

*Proof.* Write  $P$  as an integral  $\int_0^{+\infty} g_t \frac{dt}{t}$  then use the previous lemma with  $f$  and  $g$ .  $\square$

## 2.5 Completion and crossed product of the pseudodifferential algebra

We now consider the  $C^*$ -completion of the previous algebras. From now on we restrict the tangent groupoid over  $\mathbb{R}_+$  and denote it by

$$\mathbb{T}_H^+ M := \mathbb{T}_H M|_{M \times \mathbb{R}_+}.$$

We also denote by  $\mathcal{J}_H^+(M)$  the space of restriction of functions in  $\mathcal{J}_H(M)$  to  $\mathbb{T}_H^+ M$ . The respective completions of the algebras  $\mathcal{S}(\mathbb{T}_H^+ M, \Omega^{1/2})$ ,  $\mathcal{J}_H^+(M)$  and  $\mathcal{S}(\mathbb{R}_+, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$  are the  $C^*$ -algebras  $C^*(\mathbb{T}_H^+ M)$ ,  $C_0^*(\mathbb{T}_H^+ M)$  and  $\mathcal{C}_0(\mathbb{R}_+, \mathcal{K}(L^2(M)))$ . They sit in the obvious exact sequences<sup>6</sup>

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}_+, \mathcal{K}(L^2(M))) \longrightarrow C_0^*(\mathbb{T}_H^+ M) \longrightarrow C_{(0)}^*(T_H M) \longrightarrow 0.$$

We also denote by  $\Psi_H^*(M)$  the  $C^*$ -completion of  $\Psi_H^0(M)$ .

**Proposition 2.17.** *Let  $P \in \Psi_H^0(M)$ , the action of on  $\mathcal{J}_H^+(M)$  defined in proposition 2.16 extends to a multiplier of  $C_0^*(\mathbb{T}_H M^+)$ . Moreover the resulting morphism  $\Psi_H^0(M) \rightarrow \mathcal{M}(C_0^*(\mathbb{T}_H M^+))$  is continuous and if moreover  $P \in \Psi_H^{-1}(M)$ , then it preserves the ideal  $\mathcal{C}_0(\mathbb{R}_+, \mathcal{K}(L^2(M)))$ .*

<sup>5</sup>Recall from [37] that the pseudodifferential filtered calculus is asymptotically complete.

<sup>6</sup>The (0) means that the sequences are exact with or without the 0.



*Proof.* Recall that  $C_0^*(\mathbb{T}_H^+M)$  is a continuous field of  $C^*$ -algebras over  $\mathbb{R}_+$  with fiber at  $t \neq 0$  equal to  $\mathcal{K}(L^2(M)) = C^*(M \times M)$  and its fiber at  $t = 0$  is  $C_0^*(T_H M)$ . We thus have for each  $t \neq 0$ ,  $\|(P * f)_t\| = \|P\|_{\Psi^*} \|f_t\|$  and  $\|(P * f)_0\| \leq \|\sigma(P)\| \|f_0\| \leq \|P\|_{\Psi^*}$ . Here  $\|\cdot\|_{\Psi^*}$  denotes the norm of  $\Psi_H^*(M)$ , i.e. the norm of the operator  $P^*$  acting on  $L^2(M)$ . The last inequality follows from the continuity of the principal symbol map. From these inequalities it follows that:

$$\forall f \in \mathcal{J}_H^+(M), \|P * f\| \leq \|P\|_{\Psi^*} \|f\|.$$

Therefore the operator  $(f_t)_{t \geq 0} \mapsto (P * f_t)_{t \geq 0}$  extends continuously to an operator on  $C_0^*(\mathbb{T}_H M)$ . The morphism  $\Psi_H^0(M) \rightarrow \mathcal{M}(C_0^*(\mathbb{T}_H M))$  is continuous and extends to

$$\Psi_H^*(M) \rightarrow \mathcal{M}(C_0^*(\mathbb{T}_H M)).$$

The last statement follows from the fact that the restriction of  $P * f$  to the zero fiber is given by the action of the principal symbol (of order 0) of  $P$  on  $f_0$ . This action is thus zero if  $P \in \Psi_H^{-1}(M)$ .  $\square$

*Remark 2.18.* More precisely if  $P \in \Psi_H^{-1}(M)$  and  $f \in \mathcal{J}_H^+(M)$  then one can prove that  $P * f \in \mathcal{S}(\mathbb{R}_+^*, \mathcal{C}_c^\infty(M \times M, \Omega^{1/2}))$ .

Let  $\Delta \in \Psi_H^q(M)$  be a positive Rockland differential operator (for some even  $q > 0$ ). Using theorem 2.2 we have a continuous family of operators  $\Delta^{it/q} \in \Psi_H^{it}(M)$ . These operators act on  $\mathcal{J}_H^+(M)$  as the order zero operators. Analogously to the previous proposition, this action extends to unitary multipliers of  $C_0^*(\mathbb{T}_H M^+)$ . The operators also give an  $\mathbb{R}$ -action on  $\Psi_H^*(M)$  with  $\text{Ad}(\Delta^{it/q})$ . We thus get a  $*$ -homomorphism:

$$\Phi: \Psi_H^*(M) \rtimes \mathbb{R} \rightarrow C_0^*(\mathbb{T}_H^+ M).$$

**Theorem 2.19.** *The morphism  $\Phi$  preserves the respective exact sequences of  $\Psi_H^*(M)$  and  $C_0^*(\mathbb{T}_H M^+)$ , i.e. there is a commutative diagram:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M))) & \longrightarrow & \Psi_H^*(M) \rtimes \mathbb{R} & \longrightarrow & \Sigma(G) \rtimes \mathbb{R} \longrightarrow 0 \\ & & \parallel & & \downarrow \Phi & & \downarrow \Phi_0 \\ 0 & \longrightarrow & \mathcal{C}_0(\mathbb{R}_+^*, \mathcal{K}(L^2(M))) & \longrightarrow & C_0^*(\mathbb{T}_H^+ M) & \longrightarrow & C_0^*(T_H M) \longrightarrow 0. \end{array}$$

Here  $\Phi_0: \Sigma(G) \rtimes \mathbb{R} \rightarrow C_0^*(T_H M)$  is the isomorphism of the previous section. In particular  $\Phi$  is an isomorphism.

*Proof.* This is a direct consequence of the previous proposition. The fact that the crossed product is trivial on the ideal in the first row is because each  $\Delta^{it/q}$  extends to a multiplier of  $\mathcal{K}(L^2(M)) = \Psi_H^{-1}(M)$ . The fact that  $\Phi_0$  is the isomorphism of the previous section is deduced from proposition 2.16. The fact that  $\Phi$  is an isomorphism follows from the short five lemma.  $\square$

From this result we can state another one similar to Debord and Skandalis original approach:

**Corollary 2.19.1.** *There is a canonical bimodule yielding a Morita equivalence between the algebras  $\Psi_H^*(M)$  and  $C_0^*(\mathbb{T}_H M) \rtimes_{\alpha} \mathbb{R}_+^*$  preserving their respective exact sequences.*

*Proof.* The zoom action  $\alpha$  is dual to the one defined with  $(\Delta^{it/q})_{t \in \mathbb{R}}$  thus this is just an application of Takai duality.  $\square$

This result allows for some index theoretic consequences:

**Proposition 2.20.** *Let  $(M, H)$  be a (compact) filtered manifold then the algebras of symbols  $\Sigma(T_H M)$  and  $\Sigma(TM)$  are KK-equivalent. The pseudodifferential algebras  $\Psi_H^*(M)$  and  $\Psi^*(M)$  are KK-equivalent as well.*

*Proof.* We use the last corollary to get a Morita equivalence

$$\Sigma(T_H M) \sim_{mor} C_0^*(T_H M) \rtimes \mathbb{R}_+^*.$$

The same goes in the commutative case with  $\Sigma(TM) \sim_{mor} C_0^*(TM) \rtimes \mathbb{R}_+^*$  (which is just the isomorphism  $\mathcal{C}_0(S^*M) \cong \mathcal{C}_0(T^*M \setminus 0) \rtimes \mathbb{R}_+^*$ ). Now we can use the Connes-Thom isomorphism for both groups bundles  $T_H M$  and  $TM$  to get a KK-equivalence  $C^*(TM) \sim_{KK} C^*(T_H M)$  which restricts to the kernels of trivial representations in a  $\mathbb{R}_+^*$ -equivariant way, we thus get  $C_0^*(TM) \sim_{KK, \mathbb{R}_+^*} C_0^*(T_H M)$  and thus the desired KK-equivalence by composition (remember a Morita equivalence induces a KK-equivalence).

For the pseudodifferential algebras we also use the Morita equivalence  $\Psi_H^*(M) \sim_{mor} C_0^*(\mathbb{T}_H^+ M)$ . We now need to show a KK-equivalence:

$$C_0^*(\mathbb{T}_H^+ M) \sim_{KK} C_0^*(\mathbb{T}^+ M).$$

Using the (classical) adiabatic groupoid of  $\mathbb{T}_H M$ <sup>7</sup> and restricting it over  $[0; 1]^2$ , we obtain a groupoid  $\mathbb{G} \rightrightarrows M \times [0; 1]^2$ . It has the following restrictions with coordinates  $(s, t) \in [0; 1]^2$ :

- the restriction to a fiber for  $s \neq 0$  of  $\mathbb{G}$  is  $\mathbb{T}_H M|_{[0; 1]}$
- the restriction to the fiber  $s = 0$  of  $\mathbb{G}$  is  $TM \times [0; 1]$  (after a choice of splitting for  $\mathfrak{t}_H M$ ).
- the restriction to a fiber for  $t \neq 0$  of  $\mathbb{G}$  is  $\mathbb{T}M|_{[0; 1]}$
- the restriction to the fiber  $t = 0$  of  $\mathbb{G}$  is  $T_H M|_{[0; 1]}^{ad}$

Now using the KK-equivalences obtained by the exact sequences induces by the row and columns of the square we get the KK-equivalences:

$$\begin{aligned} C^*(\mathbb{T}_H M|_{[0; 1]}) &\sim C^*(TM \times [0; 1]) \\ &\sim C^*(TM) \\ &\sim C^*(T_H M|_{[0; 1]}^{ad}) \\ &\sim C^*(\mathbb{T}M|_{[0; 1]}) \end{aligned}$$

Now this KK-equivalence commutes with the evaluation maps on the same fiber and thus restricts to a KK-equivalence between the groupoid algebras  $C^*(\mathbb{T}_H M|_{[0; 1]})$  and  $C^*(\mathbb{T}M|_{[0; 1]})$ . We can now use a homeomorphism between  $[0; 1)$  and  $\mathbb{R}_+$  to get a KK-equivalence  $C^*(\mathbb{T}_H^+ M) \sim C^*(\mathbb{T}^+ M)$ . By construction this equivalence restricts to a KK-equivalence between  $C_0^*(\mathbb{T}_H^+ M)$  and  $C_0^*(\mathbb{T}^+ M)$ <sup>8</sup> in an  $\mathbb{R}_+^*$  equivariant way. We thus obtain

<sup>7</sup>One could also use the filtered adiabatic groupoid of  $\mathbb{T}M$  for the filtration given by the  $H^i \times \mathbb{R}$ . This results in an isomorphic groupoid under the flip of coordinates  $(s, t) \mapsto (t, s)$  on  $\mathbb{R}^2$ .

<sup>8</sup>The same exact sequences used to obtain the KK-equivalences can be written for the  $C_0^*$  algebras.

the KK-equivalence between their crossed products  $C_0^*(\mathbb{T}_H^+M) \rtimes \mathbb{R}_+^*$  and  $C_0^*(\mathbb{T}^+M) \rtimes \mathbb{R}_+^*$  and thus between  $\Psi_H^*(M)$  and  $\Psi^*(M)$ . □

Roughly speaking, the last proposition shows that one could not hope to obtain further index theoretic invariants from the filtered calculus, other than the ones that could previously be obtained from the classical pseudodifferential calculus. This however does not help when one wants to compute the index of an actual operator. In this the difficulty lies in computing the inverse of the Connes-Thom isomorphism that was used here (see [3, 29]).

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