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Damien Calaque

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Derived symplectic geometry

Damien Calaque

IMAG, Univ Montpellier, CNRS, Montpellier, France

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Abstract

Derived symplectic geometry studies symplectic structures on derived stacks. Derived stacks are the main players in derived geometry, the purpose of which is to deal with singular spaces, while symplectic structures are an essential ingredient of the geometric formalism of classical mechanics and classical field theory. In addition to providing an overview of a relatively young field of research, we provide a case study on Casson's invariant.

Keywords – Derived geometry. Shifted symplectic structure. Lagrangian morphism. Lagrangian intersection. Virtual class. Critical chart. Darboux theorem. Casson invariant. Donaldson–Thomas invariant. Topological field theory.

Key points/objectives

- Motivate the use of derived geometric techniques in symplectic geometry.
- Introduce shifted symplectic structures, with examples.
- Emphasize the importance of (-1) -symplectic structures for virtual count.
- Introduce lagrangian morphisms, and notice their relevance for symplectic geometry (moment maps, Weinstein's symplectic creed, ...) and virtual count.
- Provide a case study with the Casson invariant.

1 Introduction

Derived geometry One of the purposes of derived geometry is to deal in a satisfying way with spaces that are singular, which is the case for many spaces appearing in algebraic geometry (moduli spaces) and in classical¹ physics (spaces of solutions of equations of motion). The reader can consult Toën (2014) and Anel (2021) for extensive overviews of the history, results and main ideas of derived geometry. Foundations of derived algebraic geometry have been laid out by Toën & Vezzosi (2008) and Lurie (2004). For a treatment of (as well as references for) derived differential geometry we recommend the work of Steffens (2023).

Very roughly, modern derived geometry combines two ways of introducing more flexibility (using homotopy theory, or higher category theory) into the geometry of spaces:

- (a) It generalizes local/test objects (affine schemes, euclidean spaces, Stein manifolds, ...) by adding homological/homotopical data to them (for instance, in algebraic geometry, one can replace rings with simplicial rings, or connective commutative differential graded algebras). This line of thoughts can be seen as coming out of an attempt to give an actual geometric content to the intersection formula from Serre (1965), as explained for instance by Lurie (2004) and Toën (2014).
- (b) It allows to glue local objects up to specified identifications. This led to the concept of a stack, introduced by Giraud (1971), and ultimately to the higher stacks of Simpson (1996), who was somehow continuing the work of Grothendieck (1983). We refer to Mestrano & Simpson (2021) for a gentle introduction to stacks and higher stacks.

These two ways of introducing more flexibility aim to address two different sources of singularities/pathologies: (a) bad intersections (and, more generally, fiber products) and (b) bad quotients. Kontsevich (1995) is one of the first authors who combined both aspects to solve enumerative problems in algebraic geometry, using pre-derived geometric tools.

As a complement to the excellent introduction to the thesis of Lurie (2004), where derived (algebraic) geometry is motivated through the lens of Bézout's theorem (that is an intersection problem) and deformation theory, we also recommend the introduction to the paper of Steffens (2023), where the need for derived geometry is motivated by the study of solution spaces to elliptic PDEs and their local Kuranishi models.

Symplectic geometry Symplectic geometry is a natural geometric setting for the hamiltonian formulation of classical mechanics, as one can learn for instance from Souriau (1970) and Arnol'd (1989); most phase spaces appear to be symplectic manifolds (or variations of these, like Poisson manifolds).

¹Here, classical is understood as opposed to quantum.

Recall that a symplectic manifold is a smooth manifold X equipped with a 2-form $\omega \in \Omega^2(X)$ that is non-degenerate (meaning that the induced bundle map $T_X \rightarrow T_X^*$ is an isomorphism) and closed (meaning that $d_{dR}\omega = 0$). Observe that this definition makes sense in the algebro-geometric context only if X is a smooth algebraic variety. The cotangent bundle T^*M of a manifold M is an example of a symplectic manifold, with $\omega_{can} = d_{dR}\lambda$, where λ is the tautological 1-form on T^*M . Symplectic manifolds do not have local invariants: it follows from a theorem of Darboux (1882) that every symplectic manifold is locally symplectomorphic to $T^*\mathbb{R}^n$ equipped with ω_{can} .

Lagrangian submanifolds play a crucial role in symplectic geometry: recall that a lagrangian submanifold $L \subset X$ in a symplectic manifold (X, ω) is lagrangian if $\omega|_L = 0$ and the induced map $T_L \rightarrow T_{L/X}^*$ is an isomorphism. Generalizing Darboux's theorem, Weinstein (1971) proved that in the neighborhood of a lagrangian submanifold L every symplectic manifold is symplectomorphic to a neighborhood of the zero section of T^*L . Thus lagrangian submanifolds can naturally be interpreted as generalized configurations of a classical mechanical system. Lagrangian submanifolds pop up everywhere: graphs of closed 1-forms, graphs of symplectomorphisms (an example of which is the time one flow of a hamiltonian vector field), conormal bundles, zero loci of moment maps, ... This led Weinstein (1981) to follow the symplectic creed claiming that "everything is a lagrangian submanifold", and envision a symplectic category whose objects are symplectic manifolds and morphisms are lagrangian correspondences (i.e. lagrangian submanifolds of a product $(X_1 \times X_2, p_1^*\omega_1 - p_2^*\omega_2)$). At this point, the need to deal with singular/pathological spaces in symplectic geometry should be obvious:

- The zero locus $\mu^{-1}(0)$ of a moment map μ might be singular.
- The above example is actually a lagrangian correspondence between the original symplectic manifold X and its symplectic reduction X_{red} , where X_{red} is a quotient of $\mu^{-1}(0)$ and thus could be even more singular.
- The composition in the symplectic category involves taking fiber products, that might not be well-behaved. A traditional way to deal with that is by applying a small geometric perturbation, but there are issues with this approach: (a) it cannot always be used in algebraic geometry and (b) geometric perturbations are not functorial.

A leitmotiv of derived geometry is to replace geometric perturbations with homological perturbations for computing fiber products. Homological perturbations can be made functorial (in a higher categorical sense), and make sense in the algebro-geometric context, resolving both issues.

Derived symplectic geometry The ancestors of shifted symplectic structures on derived stacks are the odd symplectic structures on super-manifolds and Q -manifolds appearing in the work of Schwarz (1993), Alexandrov et al. (1997) and in other mathematical physics publications on the geometry of the

Batalin–Vilkovisky formalism. These excellent works and also the beautiful treatment by Costello (2013), using L_∞ -spaces and elliptic moduli problems, have two drawbacks:

- (1) None of the two defining properties of a symplectic structure (closedness and nondegeneracy) are homotopy invariant.
- (2) The geometric objects they consider only capture infinitesimal symmetries.

Both issues are dealt with by the formalism of derived geometry: it indeed encompasses global symmetries (stacks have been invented for that purpose) and is homotopy invariant by definition.

Let us be a bit more specific about the homotopy invariance issue: all ways of doing pre-derived geometry somehow involve spaces having a commutative differential graded algebra of functions, and whose tangent at a point is a cochain complex rather than just a vector space. Therefore, differential forms have an internal cohomological degree in addition to the usual form degree, and there are two differentials at play: the internal differential δ and the de Rham differential d_{dR} . If we have an equivalence $f : X \xrightarrow{\sim} Y$ (e.g. a quasi-isomorphism of differential graded manifolds in the sense of Kapranov (2001), that is a morphism inducing an isomorphism at the level of the cohomology of the differential graded algebra of functions), then:

- The pull-back of a 2-form ω_Y on Y that is strictly non-degenerate (meaning that the induced map $T_Y \rightarrow T_Y^*$ is an isomorphism of complexes) might not be strictly non-degenerate.
- A strictly closed 2-form ω_X on X (meaning that $d_{dR}\omega_X = \delta\omega_X = 0$) may not be the pull-back of a strictly closed 2-form ω_Y on Y .

Pantev et al. (2013) introduce a very flexible notion of symplectic structure on derived stacks, that addresses all the above issues. As we see below, their non-degeneracy condition only requires a quasi-isomorphism, and the closedness is relaxed up to coherent homotopies (where all homotopies are actually part of the structure).

In what follows, whenever we don't give a specific reference for something, the reader shall attribute it to Pantev et al. (2013). Note that we also always work over a field of characteristic zero.

2 Shifted symplectic structures

The tangent space of a derived Artin stack X is no longer a vector bundle, but a perfect complex of \mathcal{O}_X -modules, denoted \mathbb{T}_X (whose dual is denoted \mathbb{L}_X). Here \mathcal{O}_X denotes the sheaf of functions on X . A 2-form of degree n on X is a cochain map $\omega_0 : \wedge^2 \mathbb{T}_X \rightarrow \mathcal{O}_X[n]$. Note that ω_0 is closed under the differential δ of the complex $\Omega^2(X) = \Gamma(X, \wedge^2 \mathbb{L}_X)$ of 2-forms on X ; in particular, it induces a morphism of complexes $\mathbb{T}_X \rightarrow \mathbb{L}_X[n]$. We say that ω_0 is non-degenerate if

this induced morphism $\mathbb{T}_X \rightarrow \mathbb{L}_X[n]$ is a quasi-isomorphism, meaning that it becomes an isomorphism at the level of cohomology.

Definition 1 *An n -shifted symplectic (or, simply, n -symplectic) structure on X is a sequence $\omega := (\omega_0, \omega_1, \omega_2, \dots)$, where ω_k belongs to the complex $\Omega^k(X)$ of k -forms on X , satisfying two conditions:*

- *The leading term ω_0 is a 2-form of degree n such that is non-degenerate (as explained above, this means that the induced morphism of complexes $\mathbb{T}_X \rightarrow \mathbb{L}_X[n]$ becomes an isomorphism at the level of cohomology).*
- *For every $k \geq 0$, $d_{dR}(\omega_k) + \delta(\omega_{k+1}) = 0$, meaning that ω_0 is closed under the de Rham differential up to coherent homotopies. In other words, ω is an n -cocycle in the truncated de Rham complex*

$$\left(\prod_{k \geq 2} \Omega^k(X), d_{dR} + \delta \right).$$

First examples Because of the non-degeneracy condition, a scheme equipped with a 0-symplectic structure must be smooth. Conversely, all symplectic algebraic varieties provide examples of 0-symplectic structures.

If G is an affine algebraic group, then any non-degenerate symmetric invariant pairing $c \in S^2(\mathfrak{g}^*)^G$ on the Lie algebra \mathfrak{g} of G defines a 2-symplectic structure on the classifying stack BG .

It was proven in Calaque (2019) that the n -shifted cotangent stack $\mathbf{T}^*[n]X$ of an Artin stack X is indeed n -symplectic. In particular, for every affine algebraic group G , $\mathbf{T}^*[1](BG) \simeq [\mathfrak{g}^*/G]$ is 1-symplectic.

Quotient stacks of quasi-symplectic groupoids, as defined by Xu (2004), are also 1-symplectic, according to Calaque (2021). In particular, for a reductive group G , $[G/G^{ad}]$ is 1-symplectic.

The moduli of objects, in the sense of Toën & Vaquié (2007), of a d -Calabi–Yau category, is $(2 - d)$ -symplectic, as witnessed by Toën (2014) and shown by Brav & Dyckerhoff (2021). This implies in particular that the derived moduli of coherent sheaves on a Calabi–Yau d -fold is $(2 - d)$ -symplectic.

The AKSZ construction Following Alexandrov et al. (1997), Pantev et al. (2013) developed a systematic way of producing new shifted symplectic structures on mapping stacks via transgression: if (X, ω) is an n -symplectic derived stack and Σ is a nice enough derived stack equipped with a fundamental d -class $[\Sigma]$ then there is a $(n - d)$ -symplectic structure on the mapping stack $\mathbf{Map}(\Sigma, X)$, given by $\int_{[\Sigma]} ev^* \omega$, where $ev : \Sigma \times \mathbf{Map}(\Sigma, X) \rightarrow X$ is the evaluation morphism.

An important source of examples is when $\Sigma = M_B$ is the Betti stack associated with a closed oriented d -manifold M , and G is a reductive group: then the derived stack $\mathbf{Loc}(M, G) := \mathbf{Map}(M_B, BG)$ of G -local systems on M is

$(2 - d)$ -symplectic². Whenever $M = S^1$, $\mathbf{Loc}(S^1, G) \simeq [G/G^{ad}]$ and we recover the 1-symplectic structure on $[G/G^{ad}]$, as shown by Safronov (2016).

Virtual count on (-1) -symplectic derived schemes Let X be a derived scheme equipped with a (-1) -shifted symplectic structure. The non-degeneracy condition for the symplectic structure imposes that the cotangent complex of X has amplitude $[-1, 0]$, meaning that X is quasi-smooth. Schürg et al. (2015) proved that for a quasi-smooth derived scheme X the cotangent complex \mathbb{L}_X induces a perfect obstruction theory in the sense of Behrend & Fantechi (1997) on the underived truncation $t_0(X)$, allowing to define a well-behaved virtual fundamental class.

Pantev et al. (2013) noticed that the non-degeneracy condition actually guarantees that the obstruction theory is symmetric in the sense of Behrend & Fantechi (2008): in this case Behrend (2009) proved that the virtual fundamental class does not depend on the choice of symmetric obstruction theory, and when $Y = t_0(X)$ is proper the virtual count (the degree of the virtual class) is obtained as a weighted Euler characteristic associated with a constructible function canonically defined on Y .

Finally, Brav et al. (2019) prove a Darboux theorem for (-1) -symplectic derived schemes, saying that they are locally equivalent to derived critical loci (see below).

The above discussion for derived schemes generalizes without problem to derived Deligne–Mumford stacks (i.e. derived orbifolds).

3 Lagrangian morphisms

In this section we present a far reaching generalization of the notion of a lagrangian submanifold that is well-suited for derived geometry (and encompasses various situations): the one of a lagrangian morphism (or, more accurately, lagrangian structure on a morphism).

Let $f : L \rightarrow X$ be a morphism of derived Artin stacks, and let ω be an n -symplectic structure on X .

Definition 2 *A lagrangian structure on f (relatively to ω) is a homotopy η between ω and 0 (meaning that $\omega = (d_{dR} + \delta)(\eta)$) in the truncated de Rham complex such that the map $\mathbb{T}_X \rightarrow \mathbb{L}_{L/X}[n - 1]$ induced by η_0 is a quasi-isomorphism.*

Note that the leading term η_0 of the homotopy is a homotopy between ω_0 and 0 in the complex of 2-forms: $\omega_0 = \delta\eta_0$. This means that the sequence $\mathbb{T}_L \rightarrow f^*\mathbb{L}_X[n] \rightarrow \mathbb{L}_L[n]$ is null-homotopic, leading to a map $\mathbb{T}_X \rightarrow \mathbb{L}_{L/X}[n - 1]$.

The derived symplectic creed “Everything is a lagrangian morphism”. The inclusion of a lagrangian subvariety into a symplectic variety is an example

²A similar result holds in derived differential geometry if one replaces the reductive group with a compact Lie group.

of a lagrangian morphism. Calaque (2015) provides more surprising examples of lagrangian morphisms:

- An n -symplectic structure on X is the same as a lagrangian structure on $X \rightarrow *$, where $*$ is equipped with the trivial $(n + 1)$ -symplectic structure.
- If G is an affine algebraic group, X is a symplectic G -variety, and $\mu : X \rightarrow \mathfrak{g}^*$ is a moment map, then the induced morphism $[X/G] \rightarrow [\mathfrak{g}^*/G]$ between quotient stacks carries a lagrangian structure.
- If G is reductive and X is a quasi-hamiltonian G -variety with group-valued moment map $\mu : X \rightarrow G$, then $[X/G] \rightarrow [G/G^{ad}]$ also carries a lagrangian structure.

Calaque (2015) proves that lagrangian correspondences (i.e. lagrangian morphisms with codomain a product $(X_1 \times X_2, p_1^*\omega_1 - p_2^*\omega_2)$) compose well. This allows to define a (shifted and derived) version of the symplectic category envisioned by Weinstein, whose objects are n -symplectic derived stacks and morphisms are lagrangian correspondences.

Virtual count for lagrangian intersections In particular, composing two lagrangian morphisms $L_1 \rightarrow X \leftarrow L_2$, with X being n -shifted symplectic, gives a lagrangian structure on $L_1 \times_X L_2 \rightarrow *$, and thus a $(n - 1)$ -symplectic structure on $L_1 \times_X L_2$. For $n = 0$ we get a (-1) -symplectic structure on the derived intersection of two lagrangian subvarieties, and the virtual count gives the intersection number of L_1 and L_2 in X . An even more specific situation is when $X = T^*M$, L_1 is the zero section, and L_2 is the graph of $d_{dR}f$ for a function $f \in \mathcal{O}(M)$: in this case the derived lagrangian intersection is the so-called derived critical locus of f , and the virtual count is given as the Euler characteristic of the hypercohomology of (a shift by $\dim(M)$ of) the twisted de Rham complex $(\Omega_M, d_{dR} - d_{dR}f \wedge)$, that is a perverse sheaf on the reduced ordinary critical locus. Sabbah & Saito (2014) prove that, up to a shift by $\dim(M)$, the twisted de Rham complex is equivalent to the sheaf of vanishing cycles associated with f , which plays a crucial role for motivic extensions and categorifications of the virtual count.

The AKSZ construction as a TFT A relative version of the AKSZ construction is proven by Calaque (2015) and leads, for every $d \geq 0$ and every n -symplectic derived stack X , to an oriented d -dimensional Topological Field Theory (TFT), given by $\mathbf{Map}((-)_B, X)$, and taking values in the symplectic category (with shift $n - d + 1$). Calaque et al. (2022) prove that this oriented TFT is fully extended in the sense of Lurie (2009).

4 Case study: Casson invariant

We refer to Akbulut & McCarthy (1990) for a detailed exposition, following Casson's original proposal from his talk at MSRI in 1985. Let M be a com-

compact oriented 3-manifold. We consider the moduli space of irreducible $SU(2)$ -representations of the fundamental group of M :

$$R^{irr}(M) := \frac{Hom^{irr}(\pi_1(M), SU(2))}{SU(2)}.$$

Irreducible representations are the ones with centralizer $\mathcal{Z}(SU(2)) = \mathbb{Z}/2\mathbb{Z}$. For simplicity, we assume that M is an integral homology sphere (this implies in particular that the only reducible representation is the trivial one).

Definition of the Casson invariant Let now $M = H_1 \amalg_{\Sigma} H_2$ be a Heegaard splitting of M : H_1 and H_2 are handlebodies with boundary Σ , a closed oriented surface. It turns out that $R^{irr}(H_1)$ and $R^{irr}(H_2)$ are lagrangian submanifolds of $R^{irr}(\Sigma)$, that is a symplectic manifold of dimension $6g - 6$, where g is the genus of Σ . Even though $R^{irr}(M)$ is not a manifold, it is the intersection of $R^{irr}(H_1)$ and $R^{irr}(H_2)$, leading to define the Casson invariant $\lambda(M)$ as $((-1)^g$ times) half the intersection number of $R^{irr}(H_1)$ and $R^{irr}(H_2)$ in $R^{irr}(\Sigma)$.

Derived symplectic nature of the Casson invariant Computing the intersection number instead of looking at the naive intersection $R^{irr}(M)$ suggests one should consider the derived intersection. Also note that the Casson invariant is defined as half the intersection number, which is due to the hidden stacky nature of the moduli space: every irreducible representation carries a trivial $\mathbb{Z}/2\mathbb{Z}$ symmetry.

The Casson invariant is in fact a virtual count on the connected component of the derived stack $\mathbf{Loc}(M, SU(2))$ that does not contain the trivial local system. Indeed, the underived truncation of $\mathbf{Loc}(M, SU(2))$ is $[R^{irr}(M)/(\mathbb{Z}/2\mathbb{Z})] \amalg [*/SU(2)]$. The (-1) -symplectic structure on $\mathbf{Loc}(M, SU(2))$ therefore induces a symmetric obstruction theory on $[R^{irr}(M)/(\mathbb{Z}/2\mathbb{Z})]$, allowing for a virtual count.

An important result is that $\lambda(M)$ does not depend on the choice of Heegaard splitting $M = H_1 \amalg_{\Sigma} H_2$. This is a direct consequence of the fact that $\mathbf{Loc}(-, SU(2))$ defines an oriented 3-dimensional TFT with values in the derived version of Weinstein's symplectic category, as recalled in the previous section. The original definition of Casson for $\lambda(M)$ becomes a computation, that uses the fact that the (-1) -symplectic moduli can be obtained as a lagrangian intersection (thanks to the excision property of the TFT).

Gauge theoretic approach Taubes (1990) gives a gauge theoretic definition of the Casson invariant, for which we provide a derived symplectic interpretation. A first observation is that there is a morphism $M_{dR} \rightarrow M_B$ of 3-oriented derived stacks that leads to an equivalence of (-1) -symplectic derived stacks between $\mathbf{Loc}(M, G) := \mathbf{Map}(M_B, BG)$ and $\mathbf{Flat}(M, G) := \mathbf{Map}(M_{dR}, BG)$, for $G = SU(2)$.

Then one identifies $\mathbf{Flat}(M, G)$ with the derived critical locus of the Chern-Simons functional S that is defined on the (infinite dimensional) moduli stack

$\mathbf{Conn}(M, G)$ of G -connections:

$$S(A) := \int_M \text{tr} \left(d_{dR}A \wedge A + \frac{2}{3}A \wedge A \wedge A \right).$$

In order to be fully accurate, one shall say that S is not well-defined on this moduli space, as it is not gauge invariant: there exists a constant c such that for a gauge $g \in C^\infty(M, G)$, $S(A^g) = S(g^{-1}d_{dR}g + g^{-1}Ag) = S(A) + c\mathbb{Z}$. Hence, even though S is not defined on the moduli space, its exterior derivative $d_{dR}S$ defines a closed 1-form on $\mathbf{Conn}(G)$, so that we can still talk about the derived critical locus of S (that is the derived intersection of the zero section with the graph of $d_{dR}S$).

Because of infinite dimensional issues, it is not easy at all to define correctly a (possibly virtual) count for the derived critical locus of S . This is where analytic techniques come into the game: Taubes (1990) uses elliptic regularity and Fredholm theory to reduce locally to finite dimension. In fact, he constructs local potentials and finds critical charts (sometimes called Chern–Simons charts) in the sense of Brav et al. (2019), whose Darboux theorem in fact already provides such critical charts.

It is Tu (2015) who first gave a virtual count formulation of the gauge theoretic approach to the Casson invariant, using L_∞ -spaces and Kuranishi charts.

5 Conclusion

An important topic of current research is the quantization of shifted symplectic structures. The study of their geometric quantization has been initiated by Safronov (2020), while the one of their deformation quantization has been initiated by Calaque et al. (2017). The latter required to develop a whole new theory of shifted Poisson structure, which was independently done by Pridham (2017).

The virtual count on (-1) -symplectic derived schemes can already be seen as a quantization process. General quantization principles suggest that a (-1) -symplectic structures shall be quantized by a cochain complex, whose Euler characteristic would give back the virtual count. Brav et al. (2015) prove that if a scheme X_0 admits local critical charts with a global choice of appropriate signs for their glueing (such an X_0 is called an oriented d-critical locus), then one can construct a perverse sheaf on X_0 that is locally equivalent to the sheaf of vanishing cycles of the local potential; the Euler characteristic of the hypercohomology of this sheaf gives back the virtual count on X_0 .

Examples of such oriented d-critical loci are given as underived truncations of (-1) -symplectic derived schemes equipped with a square root of the canonical sheaf (such a square root exists for instance whenever there is a lagrangian foliation structure on the (-1) -symplectic derived scheme). Pridham (2019) provides a detailed explanation of the relation between the above and deformation quantization of (-1) -symplectic structures.

This circle of ideas can be applied to recover a holomorphic version of the Casson invariant, originally introduced by Thomas (2000), that “counts” sheaves on a Calabi–Yau 3-fold, and that is the starting point of the theory of Donaldson–Thomas invariants.

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