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Georges Gras. Exceptional case of the non semi-simple Real Abelian Main Conjecture. 2023. hal-04178551

HAL Id: hal-04178551

<https://hal.science/hal-04178551>

Preprint submitted on 8 Aug 2023

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EXCEPTIONAL CASE OF THE NON SEMI-SIMPLE REAL ABELIAN MAIN CONJECTURE

GEORGES GRAS

ABSTRACT. In the papers: “The Chevalley–Herbrand formula and the real abelian Main Conjecture (New criterion using capitulation of the class group), *J. Number Theory* **248** (2023)” and “On the real abelian main conjecture in the non semi-simple case, arXiv (2023)”, we consider real cyclic fields K and primes $\ell \equiv 1 \pmod{2p^N}$ totally inert in K , implying implicitly, $K \cap \mathbb{Q}(\mu_{p^\infty})^+ = \mathbb{Q}$. In this complementary work, we examine the non linearly disjoint case.

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1. INTRODUCTION

Let K/\mathbb{Q} be a real cyclic extension of Galois group $\mathcal{G}_K \simeq \gamma \oplus \Gamma$, with γ of prime-to- p order d and $\Gamma \simeq \mathbb{Z}/p^c\mathbb{Z}$, $c \geq 1$. Let \mathcal{H}_K be the p -class group of K as $\mathbb{Z}_p[\mathcal{G}_K]$ -module, and let E_K (resp. F_K) be the group of units (resp. of Leopoldt’s cyclotomic units) of K ; set $\mathcal{E}_K = E_K \otimes \mathbb{Z}_p$, $\mathcal{F}_K = F_K \otimes \mathbb{Z}_p$. We consider $\widehat{\mathcal{E}}_K = \widehat{E}_K \otimes \mathbb{Z}_p$, where \widehat{E}_K is the subgroup of E_K generated by the units of the strict subfields of K .

The “Real Abelian Main Conjecture” (RAMC for short), stated in [Gra1977], writes as follows in terms of arithmetic φ -objects (for irreducible p -adic character $\varphi = \varphi_0\varphi_p$ of K , φ_0 of prime-to- p order, φ_p of p -power order):

$$(1.1) \quad \#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\widehat{\mathcal{E}}_K \mathcal{F}_K)_{\varphi_0},$$

where $\mathcal{H}_{K,\varphi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/k}(x) = 1, \forall \kappa \subsetneq K\}_{\varphi_0}$ and where the semi-simple φ_0 -components are given by the idempotents $e_{\varphi_0} := \frac{1}{\#\gamma} \sum_{\tau \in \gamma} \varphi_0(\tau^{-1})\tau$.

Date: August 8, 2023.

1991 *Mathematics Subject Classification.* Primary 11R18, 11R29, 11R27 ; Secondary 11R37, 12Y05.

Key words and phrases. abelian fields; p -adic characters; class groups and units; Abelian Main Conjecture; class field theory.

We have proved in [Gra2023^a, Gra2023^b] that the RAMC (1.1) holds for K as soon as there exists a prime number $\ell \equiv 1 \pmod{2p^N}$, totally inert in K , such that \mathcal{H}_K capitulates in the auxiliary p -sub-extension L/K of $K(\mu_\ell)/K$. Obviously, when $\mathcal{C} := K \cap \mathbb{Q}(\mu_{p^\infty})^+$ is distinct from \mathbb{Q} , any $\ell \equiv 1 \pmod{2p^N}$ totally splits in \mathcal{C} and can not be totally inert in K (this was not made explicit in our papers); so we shall look at this particular case using a similar process of capitulation of the p -class group \mathcal{H}_K in suitable absolute abelian extensions.

One proves the RAMC, except in the case $K \subset \mathbb{Q}(\mu_{p^\infty})^+$ using Greenberg's conjecture, and in the "special case" $K \cap \mathbb{Q}(\mu_{p^\infty})^+ \subsetneq K$ with φ_0 character of $\mathbb{Q}(\mu_p)^+$, which remains open.

Some proofs, identical to that given in the two previous papers, will not be given again, as well as notations, definitions and references.

2. RECALLS ABOUT REAL ABELIAN FIELDS

Let \mathbb{Q}^{ab} be the maximal abelian extension of \mathbb{Q} . We consider the set of even irreducible p -adic characters Φ , for a prime $p \geq 2$, and the set \mathcal{X} of irreducible rational characters of \mathbb{Q}^{ab} ; then the subsets of irreducible characters Φ_K, \mathcal{X}_K , of $K \subset \mathbb{Q}^{\text{ab}}$. Thus, K is the fixed field of a unique $\chi \in \mathcal{X}$ and we put $K = K_\chi$.

2.1. Arithmetic χ and φ -objects. Let k be the subfield of $K = K_\chi$ fixed by $\Gamma \simeq \mathbb{Z}/p^c\mathbb{Z}$. The field of values of $\psi \mid \chi$, of degree 1, is $\mathbb{Q}(\mu_{dp^c})$, direct compositum $\mathbb{Q}(\mu_d)\mathbb{Q}(\mu_{p^c})$; thus $\psi = \psi_0\psi_p$, ψ_0 of order d , ψ_p of order p^c and $\chi = \chi_0\chi_p$, $\chi_0 \in \mathcal{X}_k$ above ψ_0 and $\chi_p \in \mathcal{X}_{K^\gamma}$ above ψ_p . Similarly, in the direct compositum $\mathbb{Q}_p(\mu_d)\mathbb{Q}_p(\mu_{p^c})$, the irreducible p -adic characters $\varphi \mid \chi$ are of the form $\varphi = \varphi_0\varphi_p$, $\varphi_0 \mid \chi_0$, and where $\varphi_p = \chi_p$ since $\text{Gal}(\mathbb{Q}_p(\mu_{p^c})/\mathbb{Q}_p) \simeq \text{Gal}(\mathbb{Q}(\mu_{p^c})/\mathbb{Q})$.

If \mathcal{A} is a $\mathbb{Z}_p[\mathcal{G}_K]$ -module, $\mathcal{A}_{\varphi_0} := \mathcal{A}^{e_{\varphi_0}}$ denotes its semi-simple φ_0 -component given by the idempotent $e_{\varphi_0} := \frac{1}{\#\gamma} \sum_{\tau \in \gamma} \varphi_0(\tau^{-1})\tau \in \mathbb{Z}_p[\gamma]$.

Recall that for $K = K_\chi$ and $\varphi \mid \chi$:

$$(2.1) \quad \begin{cases} \mathcal{H}_{K,\chi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K\}, \\ \mathcal{H}_{K,\varphi}^{\text{ar}} := (\mathcal{H}_{K,\chi}^{\text{ar}})_{\varphi_0}. \end{cases}$$

The above definitions of χ and φ -objects lead to the fundamental formula [Gra2023^b, Corollary 2.3]:

Theorem 2.1. *For all $\rho \in \mathcal{X}$, we have the semi-simple decomposition $\mathcal{H}_{K,\rho}^{\text{ar}} = \bigoplus_{\varphi \mid \rho} \mathcal{H}_{K,\varphi}^{\text{ar}}$, and for all real cyclic extension K/\mathbb{Q} , we*

$$\text{have the global formula } \#\mathcal{H}_K = \prod_{\rho \in \mathcal{X}_K} \#\mathcal{H}_{K,\rho}^{\text{ar}} = \prod_{\rho \in \mathcal{X}_K} \prod_{\varphi \mid \rho} \#\mathcal{H}_{K,\varphi}^{\text{ar}}.$$

2.2. Analytic expression of $\#\mathcal{H}_{K,\chi}^{\text{ar}}$ and the RAMC. Since the RAMC is trivial for χ of p -power order, we assume $d > 1$, $\chi_0 \neq 1$. The interpretation of Leopoldt's analytic formulas yields the following result for $K = K_\chi$ real (see the survey [Gra2023^b, Theorem 3.2]):

Theorem 2.2. *Let $\mathcal{H}_{K,\chi}^{\text{ar}} := \{x \in \mathcal{H}_K, \mathbf{N}_{K/\kappa}(x) = 1, \forall \kappa \subsetneq K\}$. Then $\#\mathcal{H}_{K,\chi}^{\text{ar}} = (\mathcal{E}_K : \widehat{\mathcal{E}}_K \mathcal{F}_K)$ where $\widehat{\mathcal{E}}_K$ is the subgroup of \mathcal{E}_K generated by the \mathcal{E}_κ 's for all $\kappa \subsetneq K$ and \mathcal{F}_K is the group of Leopoldt's cyclotomic units.*

We have formulate in the 1970's [Gra1977] the following RAMC:

Conjecture 2.3. *Let $\chi = \chi_0\chi_p$, $\chi_0 \neq 1$, be an even irreducible rational character and let $K = K_\chi$. For all irreducible p -adic character $\varphi \mid \chi$ (whence $\varphi_0 \mid \chi_0$, $\varphi_p = \chi_p$), we have $\#\mathcal{H}_{K,\varphi}^{\text{ar}} = \#(\mathcal{E}_K/\widehat{\mathcal{E}}_K \mathcal{F}_K)_{\varphi_0}$ (φ_0 -component of the χ -object $\mathcal{E}_K/\widehat{\mathcal{E}}_K \mathcal{F}_K$).*

As we have explained, the particular case $K \cap \mathbb{Q}(\mu_{p^\infty})^+ \neq \mathbb{Q}$ must be examined separately; this is coherent with the specific properties of p -adic L-functions regarding characters conductors powers of p .

Let $\mathcal{C} := K \cap \mathbb{Q}(\mu_{p^\infty})^+$ and $\mathcal{C}_0 := k \cap \mathbb{Q}(\mu_p)^+$ with $d = [k : \mathbb{Q}] > 1$, $\varphi_0 \mid \chi_0 \neq 1$ and $[K : k] = p^c > 1$. Note that for $p \leq 3$, $\mathcal{C}_0 = \mathbb{Q}$ and \mathcal{C} is a layer of the cyclotomic \mathbb{Z}_2 -extension \mathbb{Q}_∞ ; since $k \neq \mathbb{Q}$ and $\varphi_0 \neq 1$, this case will be solved in §4.2. So we can focus on the case $p > 3$ in the sequel.

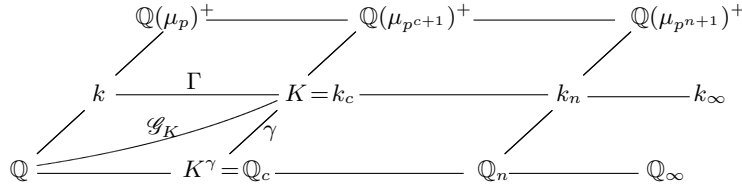
We will analyze the RAMC, under capitulation properties, in two steps corresponding to the following cases:

- (i) $\mathcal{C} = K \subset \mathbb{Q}(\mu_{p^\infty})^+$ (cf. Section 3),
- (ii) $\mathbb{Q} \subsetneq \mathcal{C} \subsetneq K$, with the two sub-cases $\mathcal{C}_0 \subsetneq k$ and the “special case” $\mathcal{C}_0 = k$ (cf. Section 4.1).

3. CASE $K \subset \mathbb{Q}(\mu_{p^\infty})^+$ AND GREENBERG’S CONJECTURE

Note that this framework makes sense only if Vandiver’s conjecture is false, otherwise all the subfields of $\mathbb{Q}(\mu_{p^\infty})^+$ are p -principal.

Thus, k is a subfield of $\mathbb{Q}(\mu_p)^+$ and K is the layer k_c of the cyclotomic \mathbb{Z}_p -extension k_∞ of k .



This case can not use an auxiliary totally inert prime $\ell \equiv 1 \pmod{2p^N}$ (since it is totally split in a non-trivial subfield of K), but Greenberg’s conjecture implying a capitulation principle in k_∞ (in some sense, p replaces ℓ and k_∞ replaces $L \subset K(\mu_\ell)$).

Assuming Greenberg’s conjecture [Gree1976, Theorem 2], we know that, for all $m \geq n \geq n_0$, the arithmetic norms $\mathbf{N}_{k_m/k_n} : \mathcal{H}_{k_m} \rightarrow \mathcal{H}_{k_n}$ are isomorphisms and that $\mathcal{H}_{k_m} = \mathcal{H}_{k_m}^{\text{Gal}(k_m/k_n)}$; so, ν denoting the algebraic norm, $\nu_{k_m/k_n}(\mathcal{H}_{k_m}) = \mathbf{J}_{k_m/k_n} \circ \mathbf{N}_{k_m/k_n}(\mathcal{H}_{k_m}) = \mathcal{H}_{k_m}^{p^{m-n}} = \mathbf{J}_{k_m/k_n}(\mathcal{H}_{k_n}) = 1$ for $m \gg n \geq n_0$. It follows that the p -class group of any layer k_n ($n \geq 0$) capitulates in k_∞ ; so, \mathcal{H}_K capitulates in k_∞ .

Since p is totally ramified in k_∞/\mathbb{Q} , the Chevalley–Herbrand formula [Che1933, pp. 402-406] yields $\#\mathcal{H}_{k_n}^{\text{Gal}(k_n/k)} = \#\mathcal{H}_k$, and in the same way as for [Gra2023^b, Corollary 4.5], we have:

$$\mathcal{H}_{k_n,\varphi_0}^{\text{Gal}(k_n/k)} \simeq (\mathcal{E}_k/\mathbf{N}_{k_n/k}(\mathcal{E}_{k_n}))_{\varphi_0},$$

of order $\#\mathcal{H}_{k,\varphi_0}$, as soon as $\mathcal{H}_{k,\varphi_0}$ capitulates in k_n .

In terms of cyclotomic units \mathcal{F}_{k_n} , we have the fundamental norm relation $\mathbf{N}_{k_m/k_n}(\mathcal{F}_{k_m}) = \mathcal{F}_{k_n}$, for all $m \geq n \geq 0$; the principle of computation giving [Gra2023^b, Theorem 4.4] holds, which leads to the proof of the non semi-simple RAMC 2.3, for $K \subset \mathbb{Q}(\mu_{p^\infty})^+$.

The $\mathbb{Z}_p[\gamma]$ -modules $\Omega_K = \{(s_i)_{i=1}^r \in G^r, \prod_{i=1}^r s_i = 1\}$ and $\Omega_{\widehat{K}} = \{(\widehat{s}_i)_{i=1}^r \in \widehat{G}^r, \prod_{i=1}^r \widehat{s}_i = 1\}$, where $r = [\mathcal{C} : \mathbb{Q}]$, are isomorphic. Then, from (4.1), (4.2) giving two expressions of $\mathcal{H}_{L,\varphi_0}^G$ and $\mathcal{H}_{\widehat{L},\varphi_0}^G$:

$$(4.3) \quad \begin{cases} \#\mathcal{H}_{K,\varphi_0} = \frac{\#\mathcal{H}_{L,\varphi_0}^{\text{ram}} \times (\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{\#\Omega_{K,\varphi_0}}, \\ \#\mathcal{H}_{\widehat{K},\varphi_0} = \frac{\#\mathcal{H}_{\widehat{L},\varphi_0}^{\text{ram}} \times (\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0}}{\#\Omega_{\widehat{K},\varphi_0}}, \end{cases}$$

One sees easily that $\mathcal{H}_{L,\varphi_0}^{\text{ram}} = \widehat{\mathbf{J}}(\mathcal{H}_{L,\varphi_0}^{\text{ram}})$ since prime ideals over ℓ are inert in L/\widehat{L} . We have the exact sequence defining $\mathcal{H}_{K,\varphi}^{\text{ar}}$:

$$1 \rightarrow (\mathcal{H}_{K,\chi}^{\text{ar}})_{\varphi_0} = \mathcal{H}_{K,\varphi}^{\text{ar}} \longrightarrow \mathcal{H}_{K,\varphi_0} \xrightarrow{\widehat{\mathbf{N}}} \mathcal{H}_{\widehat{K},\varphi_0} \rightarrow 1;$$

then, from (4.3), and the fact that $\#\Omega_{K,\varphi_0} = \#\Omega_{\widehat{K},\varphi_0}$, we get, under capitulations, the fundamental relation:

$$(4.4) \quad \#\mathcal{H}_{K,\varphi}^{\text{ar}} = \frac{\#\widehat{\mathbf{J}}(\mathcal{H}_{L,\varphi_0}^{\text{ram}})}{\#\mathcal{H}_{L,\varphi_0}^{\text{ram}}} \times \frac{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{(\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0}},$$

to be compared with $X_{\varphi_0} := (\mathcal{E}_K : \mathcal{E}_{\widehat{K}}\mathcal{F}_K)_{\varphi_0}$. So, the proof of the RAMC is essentially based on $\mathbf{N}(\mathcal{F}_L)$ regarding $\mathbf{N}(\mathcal{E}_L)$.

We have $\mathbf{N}(\mathcal{F}_L) = \mathcal{F}_K^{1-\tau_\ell^{-1}}$, where τ_ℓ is the Artin automorphism of ℓ in K/\mathbb{Q} ; it generates $\text{Gal}(K/\mathcal{C})$ and is of p -power order if and only if $\mathcal{C}_0 = k$, which defines the ‘‘special case’’.

4.2. Proof of the RAMC when $\mathcal{C}_0 \not\subseteq k$. In this case, $(1 - \tau_\ell^{-1})e_{\varphi_0}$ is invertible in $\mathbb{Z}_p[\mathcal{G}_K]e_{\varphi_0}$ [Gra2023^b, Proposition 3.4]. Recall that this applies for $p \leq 3$ since $\mathcal{C}_0 = \mathbb{Q}$ and $k \neq \mathbb{Q}$.

The computations [Gra2023^b, Proof of Theorem 4.4] leads in this case to the relation:

$$(4.5) \quad X_{\varphi_0} = \frac{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{(\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0}} \times Z_{\varphi_0},$$

where:

$$Z_{\varphi_0} := \frac{(\mathcal{E}_{\widehat{K}}\mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_{\widehat{L}})\mathcal{F}_K)_{\varphi_0}}{(\mathcal{E}_{\widehat{K}}\mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} \times (\mathbf{N}(\mathcal{E}_{\widehat{L}})\widehat{\mathcal{F}}_K : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0},$$

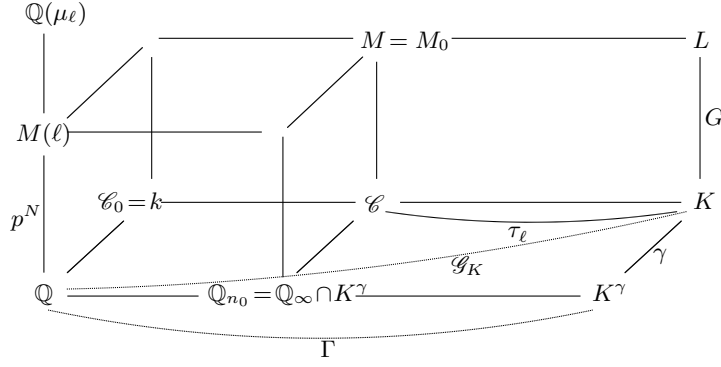
with $\widehat{\mathcal{F}}_K := \mathcal{F}_K \cap \mathcal{E}_{\widehat{K}}$. Since $\mathbf{N}(\mathcal{E}_{\widehat{L}})\mathcal{F}_K \subseteq \mathbf{N}(\mathcal{E}_L)$, Z_{φ_0} is an integer and we obtain from the relations (4.4) and (4.5) under capitulations:

$$X_{\varphi_0} = \#\mathcal{H}_{K,\varphi}^{\text{ar}} \times \frac{\#\mathcal{H}_{L,\varphi_0}^{\text{ram}}}{\#\widehat{\mathbf{J}}(\mathcal{H}_{L,\varphi_0}^{\text{ram}})} \times Z_{\varphi_0} \geq \#\mathcal{H}_{K,\varphi}^{\text{ar}};$$

in other words, one gets the supplementary factor $\#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0}$ which was trivial in the case ℓ totally inert in K and $\varphi_0 \neq 1$. Whence the equality as usual implying the supplementary interesting relations:

- $\mathcal{F}_{K,\varphi_0} \cap \mathcal{E}_{\widehat{K},\varphi_0} \subseteq \mathbf{N}(\mathcal{E}_{\widehat{L},\varphi_0})$,
- $\mathbf{N}(\mathcal{E}_{\widehat{L},\varphi_0})\mathcal{F}_{K,\varphi_0} = \mathbf{N}(\mathcal{E}_{L,\varphi_0})$,
- $\widehat{\mathbf{J}}(\mathcal{H}_{L,\varphi_0}^{\text{ram}}) \simeq \mathcal{H}_{L,\varphi_0}^{\text{ram}}$.

4.3. **The special case** $\mathcal{C}_0 = k$. So, τ_ℓ generates $\text{Gal}(K/\mathcal{C})$, of p -power order, and $(1 - \tau_\ell^{-1})e_{\varphi_0}$ is not invertible in $\mathbb{Z}_p[\mathcal{G}_K]e_{\varphi_0}$.



Under capitulations, we still have the fundamental relation (4.4) for which we have to prove the inequality:

$$(\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathcal{F}_K)_{\varphi_0} \geq \# \mathcal{H}_{K,\varphi}^{\text{ar}} = \frac{\#\widehat{\mathbf{J}}(\mathcal{H}_{L,\varphi_0}^{\text{ram}})}{\#\mathcal{H}_{L,\varphi_0}^{\text{ram}}} \times \frac{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}{(\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0}},$$

whence to prove, since the first factor is $(\#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0})^{-1}$, that:

$$\#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0} \times \frac{(\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0} (\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathcal{F}_K)_{\varphi_0}}{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} \geq 1.$$

We have the exact sequence:

$$1 \rightarrow \mathcal{E}_{\widehat{K}} \mathbf{N}(\mathcal{E}_L) / \mathbf{N}(\mathcal{E}_L) = \mathcal{E}_{\widehat{K}} / \mathcal{E}_{\widehat{K}} \cap \mathbf{N}(\mathcal{E}_L) \rightarrow \mathcal{E}_K / \mathbf{N}(\mathcal{E}_L) \rightarrow \mathcal{E}_K / \mathcal{E}_{\widehat{K}} \mathbf{N}(\mathcal{E}_L) \rightarrow 1$$

and the inclusions $\mathbf{N}(\mathcal{E}_{\widehat{L}}) \subseteq \mathcal{E}_{\widehat{K}} \cap \mathbf{N}(\mathcal{E}_L) \subseteq \mathcal{E}_{\widehat{K}}$, giving:

$$\begin{aligned} & \#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0} \times \frac{(\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0} (\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathcal{F}_K)_{\varphi_0}}{(\mathcal{E}_K : \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} \\ &= \#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0} \times \frac{(\mathcal{E}_{\widehat{K}} : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0} (\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathcal{F}_K)_{\varphi_0}}{(\mathcal{E}_{\widehat{K}} : \mathcal{E}_{\widehat{K}} \cap \mathbf{N}(\mathcal{E}_L))_{\varphi_0} (\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} \\ &= \#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0} \times \frac{(\mathcal{E}_{\widehat{K}} \cap \mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0} (\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathcal{F}_K)_{\varphi_0}}{(\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathbf{N}(\mathcal{E}_L))_{\varphi_0}}. \end{aligned}$$

We have $\mathcal{F}_K^{1-\tau_\ell^{-1}} = \mathbf{N}(\mathcal{F}_L) \subseteq \mathbf{N}(\mathcal{E}_L)$, which is not necessarily true for \mathcal{F}_K and prevents to conclude, contrary to the case ℓ totally inert.

If this case of RAMC is true, because of the product formula of Theorem (2.1), one should obtain, under capitulation of \mathcal{H}_K in L :

$$(4.6) \quad \#\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0} \times \frac{(\mathcal{E}_{\widehat{K}} \cap \mathbf{N}(\mathcal{E}_L) : \mathbf{N}(\mathcal{E}_{\widehat{L}}))_{\varphi_0} (\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathcal{F}_K)_{\varphi_0}}{(\mathcal{E}_K : \mathcal{E}_{\widehat{K}} \mathbf{N}(\mathcal{E}_L))_{\varphi_0}} = 1.$$

Numerical investigations will be necessary to study the behavior of each factor, but the relations obtained for the case $\mathcal{C}_0 \not\subseteq k$ (§ 4.2):

- $\mathbf{N}(\mathcal{E}_{L,\varphi_0}) \mathcal{F}_{K,\varphi_0} = \mathbf{N}(\mathcal{E}_{L,\varphi_0})$,
- $\text{Ker}(\widehat{\mathbf{J}})_{\varphi_0} = 1$,

with the supplementary one:

- $\mathcal{E}_{\widehat{K},\varphi_0} \cap \mathbf{N}(\mathcal{E}_{L,\varphi_0}) = \mathbf{N}(\mathcal{E}_{\widehat{L},\varphi_0})$,

satisfies the relation (4.6), but this is only speculation.

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